Intro to Counting and Probability

Just a quick reminder:

$$n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot n, \; \binom{n}{k} = \frac{n!}{k!(n-k)!},$$

$\binom{n}{k}$ is the number of ways to pick k out of n objects, **regardless of order.**

§1 Problems

Problem 1.1. In how many ways can I order six distinct books A, B, C, D, E, F on a bookshelf if I do not want to put book E next to book F?

Problem 1.2. What is the integer closest to $\frac{2022!-2020!}{2021!}$?

Problem 1.3. How many perfect cubes lie between $2^8 + 1$ and $2^{18} + 1$, inclusive?

Problem 1.4. The Smith family has 4 sons and 3 daughters. In how many ways can they be seated in a row of 7 chairs such that at least 2 boys are next to each other?

Problem 1.5. A teacher randomly assigns 20 students into pairs. What is the chance that two students, Camellia and Cameron, are paired together?

Problem 1.6. How many squares of any size can be drawn on a 5×5 grid of dots?

Problem 1.7. Al, Bill, and Cal will each randomly be assigned a whole number from 1 to 10, inclusive, with no two of them getting the same number. What is the probability that Al's number will be a whole number multiple of Bill's and Bill's number will be a whole number multiple of Cal's?

Problem 1.8. Why does $n\binom{n}{k-1} = k\binom{n}{k}$?

Problem 1.9. Two different 2-digit numbers are randomly chosen and multipliued together. What is the probability that the resulting product is even?

§2 Harder problems

Intended for people who have some experience.

Problem 2.1. An urn contains 4 green balls and 6 blue balls. A second urn contains 16 green balls and N blue balls. A single ball is drawn at random from each urn. The probability that both balls are of the same color is 0.58. Find N.

Problem 2.2. How many sequences of four letters are in alphabetical order? (Don't bother multiplying out the result)

Problem 2.3. Four distinct circles are drawn in a plane. What is the maximum number of points where at least two of the circles intersect?

Problem 2.4. An ordered pair (m, n) of non-negative integers is called "simple" if the addition $m + n$ in base 10 requires no carrying. Find the number of simple ordered pairs of non-negative integers that sum to 1492.

Problem 2.5. The increasing sequence $2, 3, 5, 6, 7, 10, 11, \dots$ consists of all positive integers that are neither the square nor the cube of a positive integer. Find the 500th term of this sequence.

Problem 2.6. Adam, Benin, Chiang, Deshawn, Esther, and Fiona have internet accounts. Some, but not all, of them are internet friends with each other, and none of them has an internet friend outside this group. Each of them has the same number of internet friends. In how many different ways can this happen?

§3 Walkthroughs

These two problems are much harder. I find them to be great examples of how there are only so many fundamentals you can learn before needing to apply them cleverly.

Example (1985 AIME)

In a tournament each player played exactly one game against each of the other players. In each game the winner was awarded 1 point, the loser got 0 points, and each of the two players earned $\frac{1}{2}$ point if the game was a tie. After the completion of the tournament, it was found that exactly half of the points earned by each player were earned in games against the ten players with the least number of points. (In particular, each of the ten lowest scoring players earned half of her/his points against the other nine of the ten). What was the total number of players in the tournament?

For this problem (and in general) you should always be clear on what exactly you are doing. Pushing numbers around mindlessly generally will get you confused.

Walkthrough.

(a) Start by setting w as the number of people not in the bottom ten. How many points does this group of people earn?

(b) Show that we must have

$$\binom{w+10}{2} = 2\binom{w}{2} + 90,$$

and conclude the answer from this. (You should rule out a certain value of w.)

Example (2020 AMC 10)

Jason rolls three fair standard six-sided dice. Then he looks at the rolls and chooses a subset of the dice (possibly empty, possibly all three dice) to reroll. After rerolling, he wins if and only if the sum of the numbers face up on the three dice is exactly 7. Jason always plays to optimize his chances of winning. What is the probability that he chooses to reroll exactly two of the dice?

This problem is intimidating, but being careful and writing out cases should help tremendously.

Walkthrough.

(a) When should Jason reroll zero die?

(b) Here's an example. Say Jason rolls $1, 1, 4$. What is the chance that he wins rerolling one die? What is the chance that he wins rerolling two, three, or zero? Thus, how many dice will he reroll in this case?

(c) What is the chance that Jason wins rerolling three, two, one dice?

(d) Explicitly state Jason's optimal strategy (of the form "he will reroll this many dice if this condition is satisfied), and then solve the problem.