LCM and GCD

§1 Definitions

Definition 1.1. A **multiple** of some number n is another number $m = nk$ for some integer k. So the multiples of n are

$$\dots, -3n, -2n, -n, 0, n, 2n, 3n, \dots$$

Definition 1.2. An integer n **divides** another integer m if m is a multiple of n. We denote this as $n \mid m$, and n is a **divisor** of m.

Definition 1.3. The **lowest common multiple** of two numbers a, b, denoted as $\operatorname{lcm}(a, b)$, is the smallest positive integer c such that $a \mid c$ and $b \mid c$.

Definition 1.4. The **greatest common divisor** of two numbers a, b, denoted as $\gcd(a, b)$, is the largest positive integer c such that $c \mid a$ and $c \mid b$.

Definition 1.5. Two numbers are **relatively prime** if their GCD is 1 (i.e. they don't share any prime factors).

Exercise. Find:

1. $\operatorname{lcm}(35, 42)$, $\operatorname{lcm}(1, 665)$, $\operatorname{lcm}(1434, 5)$
2. $\gcd(35, 42)$, $\gcd(1, 665)$, $\gcd(20, 2000)$, $\gcd(27632763, 27632764)$*

Exercise. When will $\gcd(a, b) = a$, and when will $\operatorname{lcm}(a, b) = a$?

Exercise. Alice and Bob both pick integers $a, b < 100$, and are trying to maximize $\gcd(a, b)$. What values of a, b should they pick? What if they are trying to maximize $\operatorname{lcm}(a, b)$?

§2 Applications

To get a better hold on gcd and lcm, we commonly consider the prime factorization of n, and it's usually written as

$$n = p_1^{e_1} p_2^{e_2} p_3^{e_3} \cdots p_k^{e_k}$$

which is ugly but necessary.

Corollary 2.1

Given two integers m, n defined as

$$n = p_1^{e_1} p_2^{e_2} p_3^{e_3} \cdots p_k^{e_k}, \; m = p_1^{f_1} p_2^{f_2} p_3^{f_3} \cdots p_k^{f_k},$$

we can redefine divisibility in this way:

- $n \mid m$ iff $e_i \le f_i$ for all i.
- $\gcd(n, m)$ is equal to the product of $p_i^{\min(e_i, f_i)}$ for all i.
- $\operatorname{lcm}(n, m)$ is equal to the product of $p_i^{\max(e_i, f_i)}$ for all i.

Example 2.2 (Well-known)

For any two integers a, b, prove that

$$\gcd(a, b) \times \operatorname{lcm}(a, b) = ab.$$

Example 2.3 (AIME I 2020/10)

Let m and n be positive integers satisfying the conditions

- $\gcd(m + n, 210) = 1$,
- m^m is a multiple of n^n, and
- m is not a multiple of n.

Find the least possible value of $m + n$.

See Problem 3.6 for the Euclidean Algorithm.

§3 Problem Set A

Problem 3.1. For how many (not necessarily positive) integer values of n is the value of $4000 \cdot \left(\frac{2}{5}\right)^n$ an integer?

Problem 3.2. What is the smallest number of marbles that could either be divided up into bags of 18 marbles or into bags of 42 marbles with no marbles leftover in that case?

Problem 3.3. What is $\operatorname{lcm}(25, 35, 45)$?

Problem 3.4. What is the greatest prime divisor of $N = 1 + 2 + 3 + \cdots + 70$?

Problem 3.5. What is the result when we decrease the greatest common divisor of 6432 and 132 by 8?

Problem 3.6. The **Euclidean Algorithm** is a method for computing the GCD of two integers.

a) Explain why $\gcd(m, n) = \gcd(m - n, n)$ for all positive integers $m > n$.

b) Find $\gcd(10, 15)$ with the Euclidean Algorithm.

c) Find $\gcd(2001, 25001)$.

d) Prove that the fraction $\frac{21n+4}{14n+3}$ is irreducible for every natural number n.

Problem 3.7. Find the number of positive integers that are divisors of at least one of $10^{10}, 15^7, 18^{11}$.

Problem 3.8. Consider the sequence $(a_k)_{k\geq 1}$ of positive rational numbers defined by $a_1 = \frac{2020}{2021}$ and for $k \geq 1$, if $a_k = \frac{m}{n}$ for relatively prime positive integers m and n, then

$$a_{k+1} = \frac{m + 18}{n + 19}.$$

Determine the sum of all positive integers j such that the rational number a_j can be written in the form $\frac{t}{t+1}$ for some positive integer t.

§4 Problem Set B

Problem 4.1. Find the smallest positive integer greater than 1 that leaves a remainder of 1 when divided by $2, 3, \ldots, 8$, and 9.

Problem 4.2. Ben, Fang, and Venetia are playing a game in which a card numbered $2, 3, 4, 5, 6, 7$, or 8 are stuck to each of their foreheads, so that each player can see the other two numbers but not their own. Joey walks in and observes that the three numbers are not all different, and that the product of the three numbers is a perfect square. How many of the three players can now deduce the numbers on their foreheads?

Problem 4.3. Suppose a, b, c are positive integers such that

$$a + b + c = 23$$

and

$$\gcd(a, b) + \gcd(b, c) + \gcd(c, a) = 9.$$

What is the sum of all possible distinct values of $a^2 + b^2 + c^2$?

Problem 4.4. Bernardo and Silvia play the following game. Silvia picks a two digit number, and Bernardo can either copy the number down twice, or flip its digits. (So if Silvia picks 16, Bernardo can write 1616 or 61.) How many numbers can Sylvia pick such that Bernardo is able to write down a number with 4 divisors?

Problem 4.5. There exists a prime number p such that $16p+1$ is the cube of a positive integer. Find p.

Problem 4.6. For any positive integers a, b, n, prove that

a) $\gcd(na, nb) = n\gcd(a, b)$ and $\operatorname{lcm}(na, nb) = n\operatorname{lcm}(a, b)$

b) $\gcd(a^n, b^n) = (\gcd(a, b))^n$.

Problem 4.7. There are positive integers x and y that satisfy the system of equations

$$\log_{10} x + 2\log_{10}(\gcd(x, y)) = 60$$
$$\log_{10} y + 2\log_{10}(\operatorname{lcm}(x, y)) = 570.$$

Let m be the number of (not necessarily distinct) prime factors in the prime factorization of x, and let n be the number of (not necessarily distinct) prime factors in the prime factorization of y. Find $3m + 2n$.

Problem 4.8. The Fibonacci numbers are defined by $F_0 = 0$, $F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for all $n \geq 1$. Prove that consecutive Fibonacci numbers are relatively prime.