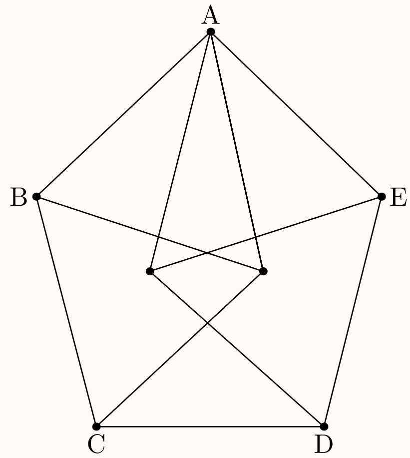
Intimidating Problems

§1 Illustrative example

Example 1 (AMC 10B Spring 2021/20)

The figure is constructed from 11 line segments, each of which has length 2. The area of pentagon $ABCDE$ can be written as $\sqrt{m}+\sqrt{n}$, where m and n are positive integers. What is $m+n$?



§2 Roadblocks

- **Do *something*.** Draw a diagram, get some easy facts, etc. At least, don't just stare at the problem statement in confusion.
- **Simplify.** As in Example 1, the problem is much easier when you simplify the diagram.
- **Examine small/extreme/convenient cases.** For example, this year's 10A #23 which you could cheese by putting both bases of the trapezoid on a line...
- **Work backwards.**
- **Use wishful thinking.** See Problem 1.
- **Do stupid things.** See Problem 3, **??**, or Problem 7. It's not stupid if it actually works.

In addition, 100% true facts are way more useful than mere suspicions[1]; take your ideas and run with them, instead of being hesitant.

[1] In that certain problems become easier if you are told they are easy, or it's much easier to solve a problem when you always know you're correct along the way.

§3 Problem Set A

Please use hints sparingly, and try to solve Problem 3 and Problem 4.

Problem 1 (AIME I 2014/14)**.** Let m be the largest real solution to the equation

$$\frac{3}{x-3}+\frac{5}{x-5}+\frac{17}{x-17}+\frac{19}{x-19}=x^2-11x-4.$$

There are positive integers a, b, c such that $m = a+\sqrt{b+\sqrt{c}}$. Find $a+b+c$. **Hints:** 17 18

Problem 2 (AIME I 2012/13)**.** Equilateral $\triangle ABC$ has side length $\sqrt{111}$. There are four distinct triangles AD_1E_1, AD_1E_2, AD_2E_3, and AD_2E_4, each congruent to $\triangle ABC$, with $BD_1 = BD_2 = \sqrt{11}$. Find $\sum_{k=1}^{4}(CE_k)^2$. **Hint:** 11

Problem 3 (OMMC Year 2 P23)**.** A 39-tuple of real numbers $(x_1, x_2, \dots, x_{39})$ satisfies

$$2\sum_{i=1}^{39}\sin(x_i)=\sum_{i=1}^{39}\cos(x_i)=-34.$$

The ratio between the maximum of $\cos(x_1)$ and the maximum of $\sin(x_1)$ over all tuples $(x_1, x_2, \dots, x_{39})$ satisfying the condition is $\frac{a}{b}$ for coprime positive integers a, b (these maxima aren't necessarily achieved using the same tuple of real numbers). Find $a+b$.
Hints: 13 16 5

Problem 4 (AMC 10A 2011/25)**.** Let R be a square region and $n \geq 4$ an integer. A point X in the interior of R is called *n-ray* partitional if there are n rays emanating from X that divide R into n triangles of equal area. How many points are 100-ray partitional but not 60-ray partitional? **Hints:** 8 7

Problem 5 (AMC 10A 2020/25, featuring Jason)**.** Jason rolls three fair standard six-sided dice. Then he looks at the rolls and chooses a subset of the dice (possibly empty, possibly all three dice) to reroll. After rerolling, he wins if and only if the sum of the numbers face up on the three dice is exactly 7. Jason always plays to optimize his chances of winning. What is the probability that he chooses to reroll exactly two of the dice?
Hint: 6

Problem 6 (USEMO 2022/1, Holden Mui)**.** A *stick* is defined as a $1\times k$ or $k\times 1$ rectangle for any integer $k \geq 1$. We wish to partition the cells of a 2022×2022 chessboard into m non-overlapping sticks, such that any two of these m sticks share at most one unit of perimeter. Determine the smallest m for which this is possible. **Hint:** 4

Problem 7 (2000 Shortlist A2)**.** Let a, b, c be positive integers satisfying the conditions $b > 2a$ and $c > 2b$. Show that there exists a real number λ with the property that all the three numbers $\lambda a, \lambda b, \lambda c$ have their fractional parts lying in the interval $\left(\frac{1}{3}, \frac{2}{3}\right]$. **Hints:** 15 3

§4 Problem Set B

Problem 8. Quadratic polynomials $P(x)$ and $Q(x)$ have leading coefficients 2 and -2, respectively. The graphs of both polynomials pass through the two points $(16, 54)$ and $(20, 53)$. Find $P(0) + Q(0)$.

Problem 9. Find the three-digit positive integer $\underline{a}\,\underline{b}\,\underline{c}$ whose representation in base nine is $\underline{b}\,\underline{c}\,\underline{a}_{\text{nine}}$, where a, b, and c are (not necessarily distinct) digits.

Problem 10. In isosceles trapezoid $ABCD$, parallel bases $\overline{AB}$ and $\overline{CD}$ have lengths 500 and 650, respectively, and $AD = BC = 333$. The angle bisectors of $\angle A$ and $\angle D$ meet at P, and the angle bisectors of $\angle B$ and $\angle C$ meet at Q. Find PQ.

Problem 11. Let $w = \dfrac{\sqrt{3}+i}{2}$ and $z = \dfrac{-1+i\sqrt{3}}{2}$, where $i = \sqrt{-1}$. Find the number of ordered pairs (r, s) of positive integers not exceeding 100 that satisfy the equation $i \cdot w^r = z^s$.

Problem 12. A straight river that is 264 meters wide flows from west to east at a rate of 14 meters per minute. Melanie and Sherry sit on the south bank of the river with Melanie a distance of D meters downstream from Sherry. Relative to the water, Melanie swims at 80 meters per minute, and Sherry swims at 60 meters per minute. At the same time, Melanie and Sherry begin swimming in straight lines to a point on the north bank of the river that is equidistant from their starting positions. The two women arrive at this point simultaneously. Find D.

Problem 13 (With apologies to those who have not learned logarithms)**.** In a Martian civilization, all logarithms whose bases are not specified as assumed to be base b, for some fixed $b \geq 2$. A Martian student writes down

$$3\log(\sqrt{x}\log x) = 56$$
$$\log_{\log x}(x) = 54$$

and finds that this system of equations has a single real number solution $x > 1$. Find b.
Hint: 10

Problem 14. Find the sum of all positive integers n such that $\sqrt{n^2 + 85n + 2017}$ is an integer. **Hint:** 14

Problem 15. For a positive integer p, define the positive integer n to be p-safe if n differs in absolute value by more than 2 from all multiples of p. For example, the set of 10-safe numbers is $\{3, 4, 5, 6, 7, 13, 14, 15, 16, 17, 23, \ldots\}$. Find the number of positive integers less than or equal to $10,000$ which are simultaneously 7-safe, 11-safe, and 13-safe.
Hint: 12

Problem 16. A car travels due east at $\frac{2}{3}$ mile per minute on a long, straight road. At the same time, a circular storm, whose radius is 51 miles, moves southeast at $\frac{1}{2}\sqrt{2}$ mile per minute. At time $t = 0$, the center of the storm is 110 miles due north of the car. At time $t = t_1$ minutes, the car enters the storm circle, and at time $t = t_2$ minutes, the car leaves the storm circle. Find $\frac{1}{2}(t_1 + t_2)$.

Problem 17. Point D lies on side $\overline{BC}$ of $\triangle ABC$ so that $\overline{AD}$ bisects $\angle BAC$. The perpendicular bisector of $\overline{AD}$ intersects the bisectors of $\angle ABC$ and $\angle ACB$ in points E and F, respectively. Given that $AB = 4, BC = 5$, and $CA = 6$, the area of $\triangle AEF$ can be written as $\frac{m\sqrt{n}}{p}$, where m and p are relatively prime positive integers, and n is a positive integer not divisible by the square of any prime. Find $m + n + p$.[2] **Hints:** 1 9 2

[2]There are many different solutions and the hints just point to the solution I found.

§5 Hints

1. Draw a really good diagram.
2. Find a pair of similar triangles.
3. The solution ends up being kind of Pidegonhole-y.
4. Draw lots of small cases that seem intuitively minimal and then stare at them. (This strategy worked out somewhat for me, at least.)
5. How can you use the fact that $38^2 + 1^2 = 34^2 + 17^2$?
6. How many dice will Jason reroll if he gets $(1, 4, 3)$?
7. Don't pay attention to what it means to be 100-ray partitional but not 60-ray partitional. Just figure out when a point is n-ray partitional.
8. Draw a diagram. Why are there no 3-ray partitional points?
9. What is AM? What else is then needed for the area of $\triangle AEF$?
10. What is the most complicated thing to deal with?
11. Honestly, the scariest thing about this problem is the weird $\displaystyle\sum_{k=1}^{4}(CE_k)^2$ answer extraction, which is stupid to worry about anyways.
12. The solution requires Chinese Remainder Theorem.
13. What is a good way to *not* deal with $\sin$ and $\cos$?
14. Complete the square.
15. This isn't really a geometry problem, but, uhh, try drawing a diagram anyways.
16. Complex numbers: instead look at $z_i = \cos(x_i) + i\sin(x_i)$, and use the given condition accordingly.
17. Out of all the things, the -4 on the right hand side is the weirdest.
18. What is $\frac{3}{x-3} + 1$?