Intimidating Problems

RubixMaster21

November 17, 2022

Thanks to Evan Chen for `evan.sty`, and Dennis Chen for the `scrambledenvs` package.

Whenever you see a problem you really like, store it (and the solution) in your mind like a cherished memory... The point of this is that you will see problems which will remind you of that problem despite having no obvious relation. You will not be able to say concretely what the relation is, but think a lot about it and give a name to the common aspect of the two problems. Eventually, you will see new problems for which you feel like could also be described by that name.

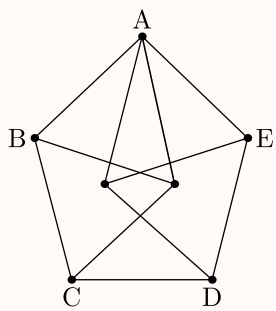
Do this enough, and you will have a very powerful intuition that cannot be described easily concretely (and in particular, that nobody else will have).

– from 'On Reading Solutions'

§1 Illustrative example

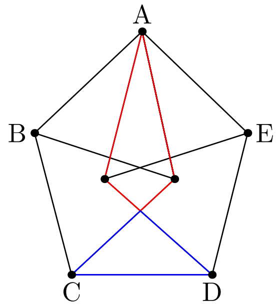
Example 1 (AMC 10B Spring 2021/20)

The figure is constructed from 11 line segments, each of which has length 2. The area of pentagon $ABCDE$ can be written as $\sqrt{m} + \sqrt{n}$, where m and n are positive integers. What is $m + n$?



I saw this problem in contest, and didn't solve it. Maybe it was because of the placement at #20, or because the $\sqrt{m} + \sqrt{n}$ answer extraction looked formidable, but I was not very confident in attempting it.

Naturally, I then thought the diagram was more complicated than it actually is; I was set on seeing the diagram like this:

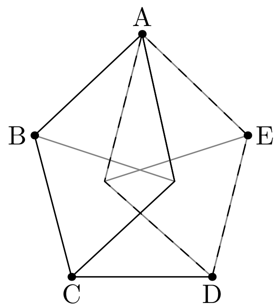


with extra emphasis on the red kite and blue triangle. And if you see it this way it means you're going to start thinking about solving for the area like this:

$$[ABCDE] = 4\sqrt{3} + \text{blue} - \text{red}.$$

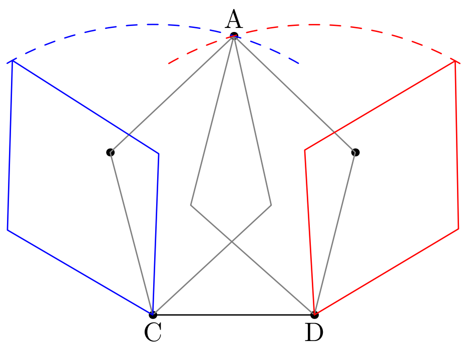
Doing this is not really feasible.

I saw this a second time recently (while prepping for the AMCs). This time, less deterred by any trepidation, I tried seeing the diagram as two diamonds pushed together (and ignored whatever mess was happening in the middle).

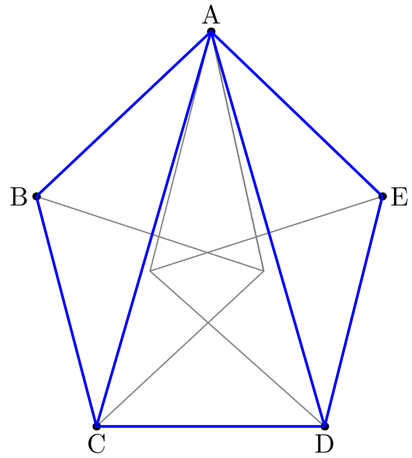


Then I tried to imagine how there was only one possible diagram; it was nice that there's only one possible diamond, so the diagram was already pretty restricted.

So I imagined this picture:



This would explain what is happening; there is exactly one diagram that works because $\triangle ACD$ is isosceles with sides $2, 2\sqrt{3}, 2\sqrt{3}$. So in fact this was happening behind the scenes:



and then I just had to extract the answer to get $\sqrt{11} + 2 \cdot \sqrt{3} \implies 23$.

§2 Roadblocks

I first encountered Problem 4 a very long while ago, and in addition to it: "Apparently this was an infamously hard problem at the time." As a result, I read the problem and did not do anything but wonder at how hard it was. I went from having a *small but positive* chance of solving it to an *essentially zero* chance of solving it, just because I was intimidated by it.

The idea of hard/soft techniques would have aided me very well. Instead of outright trying to solve the problem (and having to wrangle with the confusing prompt), I could have at least drawn a square with a point on my piece of paper.

Along with that advice, here's the standard laundry list of things to do when completely stuck:

- **Do *something*.** Draw a diagram, get some easy facts, etc. At least, don't just stare at the problem statement in confusion.
- **Simplify.** As in Example 1, the problem is much easier when you simplify the diagram.
- **Examine small/extreme/convenient cases.** For example, this year's 10A #23 which you could cheese by putting both bases of the trapezoid on a line...
- **Work backwards.**
- **Use wishful thinking.** See Problem 1.
- **Do stupid things.** See Problem 3, Problem 5, or Problem 8. It's not stupid if it actually works.

(The hard/soft techniques blog post linked above is also very good.)

In addition, 100% true facts are way more useful than mere suspicions[1]; take your ideas and run with them, instead of being hesitant.

[1] In that certain problems become easier if you are told they are easy, or it's much easier to solve a problem when you always know you're correct along the way.

§3 Problems

Please use hints sparingly, and try to solve Problem 3 and Problem 4.

Problem 1 (AIME I 2014/14)**.** Let m be the largest real solution to the equation

$$\frac{3}{x-3}+\frac{5}{x-5}+\frac{17}{x-17}+\frac{19}{x-19}=x^2-11x-4.$$

There are positive integers a,b,c such that $m=a+\sqrt{b+\sqrt{c}}$. Find $a+b+c$. **Hints:** 4 1

Problem 2 (AIME I 2012/13)**.** Equilateral $\triangle ABC$ has side length $\sqrt{111}$. There are four distinct triangles AD_1E_1, AD_1E_2, AD_2E_3, and AD_2E_4, each congruent to $\triangle ABC$, with $BD_1=BD_2=\sqrt{11}$. Find $\sum_{k=1}^{4}(CE_k)^2$. **Hint:** 2

Problem 3 (OMMC Year 2 P23)**.** A 39-tuple of real numbers $(x_1,x_2,\dots,x_{39})$ satisfies

$$2\sum_{i=1}^{39}\sin(x_i)=\sum_{i=1}^{39}\cos(x_i)=-34.$$

The ratio between the maximum of $\cos(x_1)$ and the maximum of $\sin(x_1)$ over all tuples $(x_1,x_2,\dots,x_{39})$ satisfying the condition is $\frac{a}{b}$ for coprime positive integers a,b (these maxima aren't necessarily achieved using the same tuple of real numbers). Find $a+b$.
Hints: 12 10

Problem 4 (AMC 10A 2011/25)**.** Let R be a square region and $n\ge 4$ an integer. A point X in the interior of R is called *n-ray* partitional if there are n rays emanating from X that divide R into n triangles of equal area. How many points are 100-ray partitional but not 60-ray partitional? **Hints:** 6 7

Problem 5 (AMC 12A 2021/25)**.** Let $d(n)$ denote the number of positive integers that divide n, including 1 and n. For example, $d(1)=1, d(2)=2$, and $d(12)=6$. (This function is known as the *divisor function.*) Let

$$f(n)=\frac{d(n)}{\sqrt[3]{n}}.$$

There is a unique positive integer N such that $f(N)>f(n)$ for all positive integers $n\ne N$. What is the sum of the digits of N? **Hints:** 13 8

Problem 6 (AMC 10A 2020/25, featuring Jason)**.** Jason rolls three fair standard six-sided dice. Then he looks at the rolls and chooses a subset of the dice (possibly empty, possibly all three dice) to reroll. After rerolling, he wins if and only if the sum of the numbers face up on the three dice is exactly 7. Jason always plays to optimize his chances of winning. What is the probability that he chooses to reroll exactly two of the dice?
Hint: 3

Problem 7 (USEMO 2022/1, Holden Mui)**.** A *stick* is defined as a $1\times k$ or $k\times 1$ rectangle for any integer $k\ge 1$. We wish to partition the cells of a 2022×2022 chessboard into m non-overlapping sticks, such that any two of these m sticks share at most one unit of perimeter. Determine the smallest m for which this is possible. **Hint:** 11

Problem 8 (2000 Shortlist A2)**.** Let a, b, c be positive integers satisfying the conditions $b > 2a$ and $c > 2b$. Show that there exists a real number λ with the property that all the three numbers $\lambda a, \lambda b, \lambda c$ have their fractional parts lying in the interval $\left(\frac{1}{3}, \frac{2}{3}\right]$. **Hints:** 5 14

Problem 9 (By me)**.** Let $f(x) = \frac{\cos(2\pi x)+2x+1}{2}$, $g(x) = 17e^{h(x)} + f(0)$, where $h(x)$ is defined as

$$h(x) = \frac{(\ln 4)x^2 + e^4 x + \log_{\ln 4} e^4}{[(\ln 17)f(x) + e]\,[(\log_{17} e)f(f(x)) + f(\pi)]}.$$

Find

$$\lim_{n\to\infty} g\left(\lim_{x\to 0} f^n(x)\right),$$

provided that n is an integer and f^n is f applied n times. **Hint:** 9

§4 Hints

1. What is $\frac{3}{x-3}+1$?
2. Honestly, the scariest thing about this problem is the weird $\sum_{k=1}^{4}(CE_k)^2$ answer extraction, which is stupid to worry about anyways.
3. How many dice will Jason reroll if he gets $(1,4,3)$?
4. Out of all the things, the -4 on the right hand side is the weirdest.
5. This isn't really a geometry problem, but, uhh, try drawing a diagram anyways.
6. Draw a diagram. Why are there no 3-ray partitional points?
7. Don't pay attention to what it means to be 100-ray partitional but not 60-ray partitional. Just figure out when a point is n-ray partitional.
8. You only need to find f on prime powers, and then you're free to do stupid size estimates.
9. Yeah, it's ridiculously contrived.
10. Complex numbers: instead look at $z_i = \cos(x_i) + i\sin(x_i)$, and use the given condition accordingly.
11. Draw lots of small cases that seem intuitively minimal and then stare at them. (This strategy worked out somewhat for me, at least.)
12. What is a good way to *not* deal with $\sin$ and $\cos$?
13. Big spoiler: $f(n)$ is multiplicative.
14. The solution ends up being kind of Pidegonhole-y.