§1 2015 AIME I

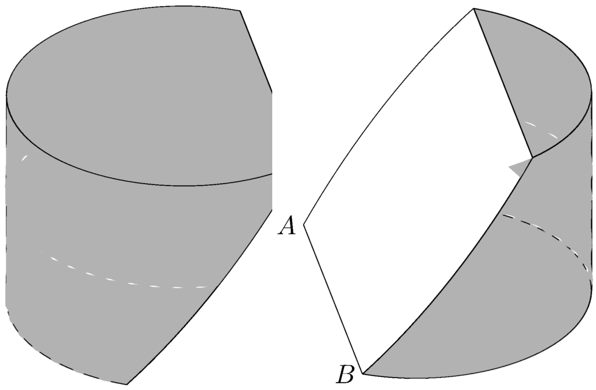
Problem 1.1. Triangle ABC has positive integer side lengths with $AB = AC$. Let I be the intersection of the bisectors of $\angle B$ and $\angle C$. Suppose $BI = 8$. Find the smallest possible perimeter of $\triangle ABC$.

Problem 1.2. Consider all 1000-element subsets of the set $\{1, 2, 3, ..., 2015\}$. From each such subset choose the least element. The arithmetic mean of all of these least elements is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.

Problem 1.3. With all angles measured in degrees, the product $\prod_{k=1}^{45} \csc^2(2k-1)^\circ = m^n$, where m and n are integers greater than 1. Find $m + n$.

Problem 1.4. For each integer $n \geq 2$, let $A(n)$ be the area of the region in the coordinate plane defined by the inequalities $1 \leq x \leq n$ and $0 \leq y \leq x \left\lfloor \sqrt{x} \right\rfloor$, where $\left\lfloor \sqrt{x} \right\rfloor$ is the greatest integer not exceeding $\sqrt{x}$. Find the number of values of n with $2 \leq n \leq 1000$ for which $A(n)$ is an integer.

Problem 1.5. A block of wood has the shape of a right circular cylinder with radius 6 and height 8, and its entire surface has been painted blue. Points A and B are chosen on the edge of one of the circular faces of the cylinder so that arc AB on that face measures 120°. The block is then sliced in half along the plane that passes through point A, point B, and the center of the cylinder, revealing a flat, unpainted face on each half. The area of one of these unpainted faces is $a \cdot \pi + b\sqrt{c}$, where a, b, and c are integers and c is not divisible by the square of any prime. Find $a + b + c$.



§2 2015 AIME II

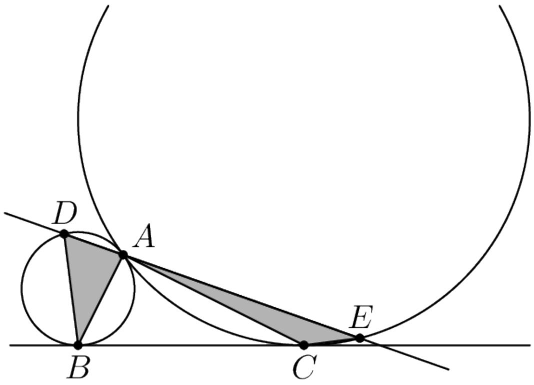
Problem 2.1. The circumcircle of acute $\triangle ABC$ has center O. The line passing through point O perpendicular to $\overline{OB}$ intersects lines AB and BC at P and Q, respectively. Also $AB = 5$, $BC = 4$, $BQ = 4.5$, and $BP = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 2.2. There are $2^{10} = 1024$ possible 10-letter strings in which each letter is either an A or a B. Find the number of such strings that do not have more than 3 adjacent letters that are identical.

Problem 2.3. Define the sequence $a_1, a_2, a_3, \ldots$ by $a_n = \sum\limits_{k=1}^{n} \sin k$, where k represents radian measure. Find the index of the 100th term for which $a_n < 0$.

Problem 2.4. Let x and y be real numbers satisfying $x^4y^5 + y^4x^5 = 810$ and $x^3y^6 + y^3x^6 = 945$. Evaluate $2x^3 + (xy)^3 + 2y^3$.

Problem 2.5. Circles $\mathcal{P}$ and $\mathcal{Q}$ have radii 1 and 4, respectively, and are externally tangent at point A. Point B is on $\mathcal{P}$ and point C is on $\mathcal{Q}$ such that BC is a common external tangent of the two circles. A line ℓ through A intersects $\mathcal{P}$ again at D and intersects $\mathcal{Q}$ again at E. Points B and C lie on the same side of ℓ, and the areas of $\triangle DBA$ and $\triangle ACE$ are equal. This common area is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.



§3 2016 AIME I

Problem 3.1. Let $P(x)$ be a nonzero polynomial such that $(x-1)P(x+1) = (x+2)P(x)$ for every real x, and $(P(2))^2 = P(3)$. Then $P(\frac{7}{2}) = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problem 3.2. Find the least positive integer m such that $m^2 - m + 11$ is a product of at least four not necessarily distinct primes.

Problem 3.3. Freddy the frog is jumping around the coordinate plane searching for a river, which lies on the horizontal line $y = 24$. A fence is located at the horizontal line $y = 0$. On each jump Freddy randomly chooses a direction parallel to one of the coordinate axes and moves one unit in that direction. When he is at a point where $y = 0$, with equal likelihoods he chooses one of three directions where he either jumps parallel to the fence or jumps away from the fence, but he never chooses the direction that would have him cross over the fence to where $y < 0$. Freddy starts his search at the point $(0, 21)$ and will stop once he reaches a point on the river. Find the expected number of jumps it will take Freddy to reach the river.

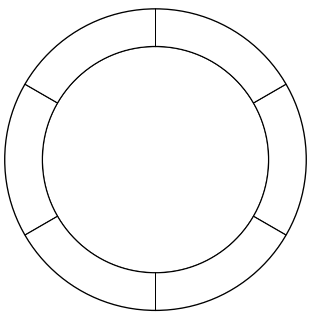
Problem 3.4. Centered at each lattice point in the coordinate plane are a circle radius $\frac{1}{10}$ and a square with sides of length $\frac{1}{5}$ whose sides are parallel to the coordinate axes. The line segment from $(0, 0)$ to $(1001, 429)$ intersects m of the squares and n of the circles. Find $m + n$.

Problem 3.5. Circles ω_1 and ω_2 intersect at points X and Y. Line ℓ is tangent to ω_1 and ω_2 at A and B, respectively, with line AB closer to point X than to Y. Circle ω passes through A and B intersecting ω_1 again at $D \neq A$ and intersecting ω_2 again at $C \neq B$. The three points C, Y, D are collinear, $XC = 67$, $XY = 47$, and $XD = 37$. Find AB^2.

§4 2016 AIME II

Problem 4.1. For positive integers N and k, define N to be k-nice if there exists a positive integer a such that a^k has exactly N positive divisors. Find the number of positive integers less than 1000 that are neither 7-nice nor 8-nice.

Problem 4.2. The figure below shows a ring made of six small sections which you are to paint on a wall. You have four paint colors available and you will paint each of the six sections a solid color. Find the number of ways you can choose to paint the sections if no two adjacent sections can be painted with the same color.



Problem 4.3. Beatrix is going to place six rooks on a 6×6 chessboard where both the rows and columns are labeled 1 to 6; the rooks are placed so that no two rooks are in the same row or the same column. The "value" of a square is the sum of its row number and column number. The "score" of an arrangement of rooks is the least value of any occupied square. The average score over all valid configurations is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$.

Problem 4.4. Equilateral $\triangle ABC$ has side length 600. Points P and Q lie outside the plane of $\triangle ABC$ and are on opposite sides of the plane. Furthermore, $PA = PB = PC$, and $QA = QB = QC$, and the planes of $\triangle PAB$ and $\triangle QAB$ form a 120° dihedral angle (the angle between the two planes). There is a point O whose distance from each of A, B, C, P, and Q is d. Find d.

Problem 4.5. For $1 \le i \le 215$ let $a_i = \dfrac{1}{2^i}$ and $a_{216} = \dfrac{1}{2^{215}}$. Let $x_1, x_2, ..., x_{216}$ be positive real numbers such that $\sum_{i=1}^{216} x_i = 1$ and $\sum_{1 \le i < j \le 216} x_i x_j = \dfrac{107}{215} + \sum_{i=1}^{216} \dfrac{a_i x_i^2}{2(1-a_i)}$. The maximum possible value of $x_2 = \dfrac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

§5 2017 AIME I

Problem 5.1. Consider arrangements of the 9 numbers $1, 2, 3, \dots, 9$ in a 3×3 array. For each such arrangement, let a_1, a_2, and a_3 be the medians of the numbers in rows 1, 2, and 3 respectively, and let m be the median of $\{a_1, a_2, a_3\}$. Let Q be the number of arrangements for which $m = 5$. Find the remainder when Q is divided by 1000.

Problem 5.2. Call a set S product-free if there do not exist $a, b, c \in S$ (not necessarily distinct) such that $ab = c$. For example, the empty set and the set $\{16, 20\}$ are product-free, whereas the sets $\{4, 16\}$ and $\{2, 8, 16\}$ are not product-free. Find the number of product-free subsets of the set $\{1, 2, 3, 4, \dots, 7, 8, 9, 10\}$.

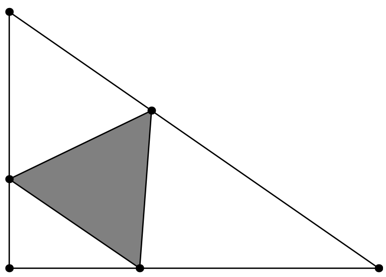
Problem 5.3. For every $m \ge 2$, let $Q(m)$ be the least positive integer with the following property: For every $n \ge Q(m)$, there is always a perfect cube k^3 in the range $n < k^3 \le mn$. Find the remainder when

$$\sum_{m=2}^{2017} Q(m)$$

is divided by 1000.

Problem 5.4. Let $a > 1$ and $x > 1$ satisfy $\log_a(\log_a(\log_a 2) + \log_a 24 - 128) = 128$ and $\log_a(\log_a x) = 256$. Find the remainder when x is divided by 1000.

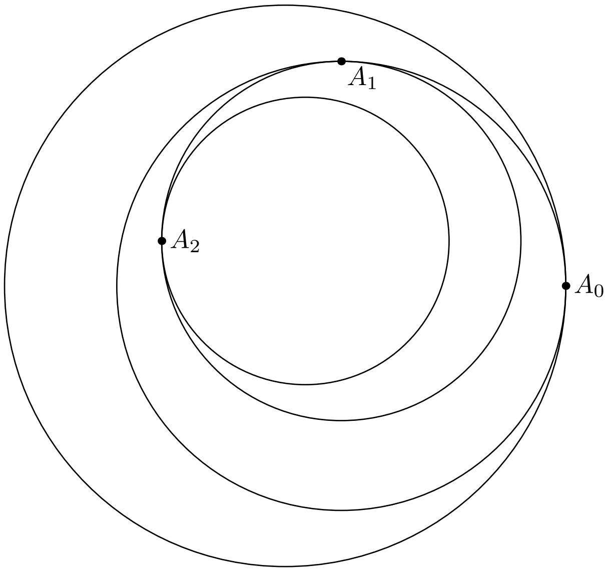
Problem 5.5. The area of the smallest equilateral triangle with one vertex on each of the sides of the right triangle with side lengths $2\sqrt{3}$, 5, and $\sqrt{37}$, as shown, is $\frac{m\sqrt{p}}{n}$, where m, n, and p are positive integers, m and n are relatively prime, and p is not divisible by the square of any prime. Find $m+n+p$.



§6 2017 AIME II

Problem 6.1. Five towns are connected by a system of roads. There is exactly one road connecting each pair of towns. Find the number of ways there are to make all the roads one-way in such a way that it is still possible to get from any town to any other town using the roads (possibly passing through other towns on the way).

Problem 6.2. Circle C_0 has radius 1, and the point A_0 is a point on the circle. Circle C_1 has radius $r < 1$ and is internally tangent to C_0 at point A_0. Point A_1 lies on circle C_1 so that A_1 is located 90° counterclockwise from A_0 on C_1. Circle C_2 has radius r^2 and is internally tangent to C_1 at point A_1. In this way a sequence of circles $C_1, C_2, C_3, \ldots$ and a sequence of points on the circles $A_1, A_2, A_3, \ldots$ are constructed, where circle C_n has radius r^n and is internally tangent to circle C_{n-1} at point A_{n-1}, and point A_n lies on C_n 90° counterclockwise from point A_{n-1}, as shown in the figure below. There is one point B inside all of these circles. When $r = \frac{11}{60}$, the distance from the center C_0 to B is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.



Problem 6.3. For each integer $n \geq 3$, let $f(n)$ be the number of 3-element subsets of the vertices of a regular n-gon that are the vertices of an isosceles triangle (including equilateral triangles). Find the sum of all values of n such that $f(n+1) = f(n) + 78$.

Problem 6.4. A $10 \times 10 \times 10$ grid of points consists of all points in space of the form (i, j, k), where i, j, and k are integers between 1 and 10, inclusive. Find the number of different lines that contain exactly 8 of these points.

Problem 6.5. Tetrahedron $ABCD$ has $AD = BC = 28$, $AC = BD = 44$, and $AB = CD = 52$. For any point X in space, define $f(X) = AX + BX + CX + DX$. The least possible value of $f(X)$ can be expressed as $m\sqrt{n}$, where m and n are positive integers, and n is not divisible by the square of any prime. Find $m + n$.

§7 2018 AIME I

Problem 7.1. Find the least positive integer n such that when 3^n is written in base 143, its two right-most digits in base 143 are 01.

Problem 7.2. For every subset T of $U = \{1, 2, 3, \ldots, 18\}$, let $s(T)$ be the sum of the elements of T, with $s(\emptyset)$ defined to be 0. If T is chosen at random among all subsets of U, the probability that $s(T)$ is divisible by 3 is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find m.

Problem 7.3. Let $\triangle ABC$ have side lengths $AB = 30$, $BC = 32$, and $AC = 34$. Point X lies in the interior of $\overline{BC}$, and points I_1 and I_2 are the incenters of $\triangle ABX$ and $\triangle ACX$, respectively. Find the minimum possible area of $\triangle AI_1I_2$ as X varies along $\overline{BC}$.

Problem 7.4. Let $SP_1P_2P_3EP_4P_5$ be a heptagon. A frog starts jumping at vertex S. From any vertex of the heptagon except E, the frog may jump to either of the two adjacent vertices. When it reaches vertex E, the frog stops and stays there. Find the number of distinct sequences of jumps of no more than 12 jumps that end at E.

Problem 7.5. David found four sticks of different lengths that can be used to form three non-congruent convex cyclic quadrilaterals, $A,\ B,\ C$, which can each be inscribed in a circle with radius 1. Let φ_A denote the measure of the acute angle made by the diagonals of quadrilateral A, and define φ_B and φ_C similarly. Suppose that $\sin \varphi_A = \frac{2}{3}$, $\sin \varphi_B = \frac{3}{5}$, and $\sin \varphi_C = \frac{6}{7}$. All three quadrilaterals have the same area K, which can be written in the form $\dfrac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

§8 2018 AIME II

Problem 8.1. Find the number of permutations of $1, 2, 3, 4, 5, 6$ such that for each k with $1 \leq k \leq 5$, at least one of the first k terms of the permutation is greater than k.

Problem 8.2. Let $ABCD$ be a convex quadrilateral with $AB = CD = 10$, $BC = 14$, and $AD = 2\sqrt{65}$. Assume that the diagonals of $ABCD$ intersect at point P, and that the sum of the areas of triangles APB and CPD equals the sum of the areas of triangles BPC and APD. Find the area of quadrilateral $ABCD$.

Problem 8.3. Misha rolls a standard, fair six-sided die until she rolls 1-2-3 in that order on three consecutive rolls. The probability that she will roll the die an odd number of times is $\dfrac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

Problem 8.4. The incircle ω of triangle ABC is tangent to $\overline{BC}$ at X. Let $Y \neq X$ be the other intersection of $\overline{AX}$ with ω. Points P and Q lie on $\overline{AB}$ and $\overline{AC}$, respectively, so that $\overline{PQ}$ is tangent to ω at Y. Assume that $AP = 3$, $PB = 4$, $AC = 8$, and $AQ = \dfrac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 8.5. Find the number of functions f from $\{0, 1, 2, 3, 4, 5, 6\}$ to the integers such that $f(0) = 0$, $f(6) = 12$, and

$$|x - y| \leq |f(x) - f(y)| \leq 3|x - y|$$

for all x and y in $\{0, 1, 2, 3, 4, 5, 6\}$.

§9 2019 AIME I

Problem 9.1. In $\triangle ABC$, the sides have integer lengths and $AB = AC$. Circle ω has its center at the incenter of $\triangle ABC$. An "excircle" of $\triangle ABC$ is a circle in the exterior of $\triangle ABC$ that is tangent to one side of the triangle and tangent to the extensions of the other two sides. Suppose that the excircle tangent to $\overline{BC}$ is internally tangent to ω, and the other two excircles are both externally tangent to ω. Find the minimum possible value of the perimeter of $\triangle ABC$.

Problem 9.2. Given $f(z) = z^2 - 19z$, there are complex numbers z with the property that z, $f(z)$, and $f(f(z))$ are the vertices of a right triangle in the complex plane with a right angle at $f(z)$. There are positive integers m and n such that one such value of z is $m + \sqrt{n} + 11i$. Find $m + n$.

Problem 9.3. Triangle ABC has side lengths $AB = 4$, $BC = 5$, and $CA = 6$. Points D and E are on ray AB with $AB < AD < AE$. The point $F \neq C$ is a point of intersection of the circumcircles of $\triangle ACD$ and $\triangle EBC$ satisfying $DF = 2$ and $EF = 7$. Then BE can be expressed as $\frac{a+b\sqrt{c}}{d}$, where a, b, c, and d are positive integers such that a and d are relatively prime, and c is not divisible by the square of any prime. Find $a + b + c + d$.

Problem 9.4. Find the least odd prime factor of $2019^8 + 1$.

Problem 9.5. Let $\overline{AB}$ be a chord of a circle ω, and let P be a point on the chord $\overline{AB}$. Circle ω_1 passes through A and P and is internally tangent to ω. Circle ω_2 passes through B and P and is internally tangent to ω. Circles ω_1 and ω_2 intersect at points P and Q. Line PQ intersects ω at X and Y. Assume that $AP = 5$, $PB = 3$, $XY = 11$, and $PQ^2 = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

§10 2019 AIME II

Problem 10.1. Triangle ABC has side lengths $AB = 7, BC = 8$, and $CA = 9$. Circle ω_1 passes through B and is tangent to line AC at A. Circle ω_2 passes through C and is tangent to line AB at A. Let K be the intersection of circles ω_1 and ω_2 not equal to A. Then $AK = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 10.2. For $n \geq 1$ call a finite sequence $(a_1, a_2, \ldots, a_n)$ of positive integers ¡i¿progressive¡/i¿ if $a_i < a_{i+1}$ and a_i divides a_{i+1} for $1 \leq i \leq n - 1$. Find the number of progressive sequences such that the sum of the terms in the sequence is equal to 360.

Problem 10.3. Regular octagon $A_1A_2A_3A_4A_5A_6A_7A_8$ is inscribed in a circle of area 1. Point P lies inside the circle so that the region bounded by $\overline{PA_1}, \overline{PA_2}$, and the minor arc $\widehat{A_1A_2}$ of the circle has area $\frac{1}{7}$, while the region bounded by $\overline{PA_3}, \overline{PA_4}$, and the minor arc $\widehat{A_3A_4}$ of the circle has area $\frac{1}{9}$. There is a positive integer n such that the area of the region bounded by $\overline{PA_6}, \overline{PA_7}$, and the minor arc $\widehat{A_6A_7}$ of the circle is equal to $\frac{1}{8} - \frac{\sqrt{2}}{n}$. Find n.

Problem 10.4. Find the sum of all positive integers n such that, given an unlimited supply of stamps of denominations $5, n$, and $n+1$ cents, 91 cents is the greatest postage that cannot be formed.

Problem 10.5. In acute triangle ABC, points P and Q are the feet of the perpendiculars from C to $\overline{AB}$ and from B to $\overline{AC}$, respectively. Line PQ intersects the circumcircle of $\triangle ABC$ in two distinct points, X and Y. Suppose $XP = 10$, $PQ = 25$, and $QY = 15$. The value of $AB \cdot AC$ can be written in the form $m\sqrt{n}$ where m and n are positive integers, and n is not divisible by the square of any prime. Find $m + n$.

§11 2020 AIME I

Problem 11.1. For integers a, b, c and d, let $f(x) = x^2 + ax + b$ and $g(x) = x^2 + cx + d$. Find the number of ordered triples (a, b, c) of integers with absolute values not exceeding 10 for which there is an integer d such that $g(f(2)) = g(f(4)) = 0$.

Problem 11.2. Let n be the least positive integer for which $149^n - 2^n$ is divisible by $3^3 \cdot 5^5 \cdot 7^7$. Find the number of positive integer divisors of n.

Problem 11.3. Point D lies on side $\overline{BC}$ of $\triangle ABC$ so that $\overline{AD}$ bisects $\angle BAC$. The perpendicular bisector of $\overline{AD}$ intersects the bisectors of $\angle ABC$ and $\angle ACB$ in points E and F, respectively. Given that $AB = 4, BC = 5$, and $CA = 6$, the area of $\triangle AEF$ can be written as $\frac{m\sqrt{n}}{p}$, where m and p are relatively prime positive integers, and n is a positive integer not divisible by the square of any prime. Find $m + n + p$.

Problem 11.4. Let $P(x)$ be a quadratic polynomial with complex coefficients whose x^2 coefficient is 1. Suppose the equation $P(P(x)) = 0$ has four distinct solutions, $x = 3, 4, a, b$. Find the sum of all possible values of $(a + b)^2$.

Problem 11.5. Let $\triangle ABC$ be an acute triangle with circumcircle ω, and let H be the intersection of the altitudes of $\triangle ABC$. Suppose the tangent to the circumcircle of $\triangle HBC$ at H intersects ω at points X and Y with $HA = 3, HX = 2$, and $HY = 6$. The area of $\triangle ABC$ can be written in the form $m\sqrt{n}$, where m and n are positive integers, and n is not divisible by the square of any prime. Find $m + n$.

§12 2020 AIME II

Problem 12.1. Let $P(x) = x^2 - 3x - 7$, and let $Q(x)$ and $R(x)$ be two quadratic polynomials also with the coefficient of x^2 equal to 1. David computes each of the three sums $P + Q$, $P + R$, and $Q + R$ and is surprised to find that each pair of these sums has a common root, and these three common roots are distinct. If $Q(0) = 2$, then $R(0) = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 12.2. Let m and n be odd integers greater than 1. An $m \times n$ rectangle is made up of unit squares where the squares in the top row are numbered left to right with the integers 1 through n, those in the second row are numbered left to right with the integers $n + 1$ through $2n$, and so on. Square 200 is in the top row, and square 2000 is in the bottom row. Find the number of ordered pairs (m, n) of odd integers greater than 1 with the property that, in the $m \times n$ rectangle, the line through the centers of squares 200 and 2000 intersects the interior of square 1099.

Problem 12.3. Convex pentagon $ABCDE$ has side lengths $AB = 5$, $BC = CD = DE = 6$, and $EA = 7$. Moreover, the pentagon has an inscribed circle (a circle tangent to each side of the pentagon). Find the area of $ABCDE$.

Problem 12.4. For real number x let $\lfloor x \rfloor$ be the greatest integer less than or equal to x, and define $\{x\} = x - \lfloor x \rfloor$ to be the fractional part of x. For example, $\{3\} = 0$ and $\{4.56\} = 0.56$. Define $f(x) = x\{x\}$, and let N be the number of real-valued solutions to the equation $f(f(f(x))) = 17$ for $0 \leq x \leq 2020$. Find the remainder when N is divided by 1000.

Problem 12.5. Let $\triangle ABC$ be an acute scalene triangle with circumcircle ω. The tangents to ω at B and C intersect at T. Let X and Y be the projections of T onto lines AB and AC, respectively. Suppose $BT = CT = 16$, $BC = 22$, and $TX^2 + TY^2 + XY^2 = 1143$. Find XY^2.

§13 2021 AIME I

Problem 13.1. Let $ABCD$ be a cyclic quadrilateral with $AB = 4, BC = 5, CD = 6$, and $DA = 7$. Let A_1 and C_1 be the feet of the perpendiculars from A and C, respectively, to line BD, and let B_1 and D_1 be the feet of the perpendiculars from B and D, respectively, to line AC. The perimeter of $A_1B_1C_1D_1$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 13.2. Let $A_1A_2A_3 \dots A_{12}$ be a dodecagon (12-gon). Three frogs initially sit at A_4, A_8, and A_{12}. At the end of each minute, simultaneously, each of the three frogs jumps to one of the two vertices adjacent to its current position, chosen randomly and independently with both choices being equally likely. All three frogs stop jumping as soon as two frogs arrive at the same vertex at the same time. The expected number of minutes until the frogs stop jumping is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 13.3. Circles ω_1 and ω_2 with radii 961 and 625, respectively, intersect at distinct points A and B. A third circle ω is externally tangent to both ω_1 and ω_2. Suppose line AB intersects ω at two points P and Q such that the measure of minor arc $\widehat{PQ}$ is 120°. Find the distance between the centers of ω_1 and ω_2.

Problem 13.4. For any positive integer a, $\sigma(a)$ denotes the sum of the positive integer divisors of a. Let n be the least positive integer such that $\sigma(a^n) - 1$ is divisible by 2021 for all positive integers a. Find the sum of the prime factors in the prime factorization of n.

§14 2021 AIME II

Problem 14.1. A teacher was leading a class of four perfectly logical students. The teacher chose a set S of four integers and gave a different number in S to each student. Then the teacher announced to the class that the numbers in S were four consecutive two-digit positive integers, that some number in S was divisible by 6, and a different number in S was divisible by 7. The teacher then asked if any of the students could deduce what S is, but in unison, all of the students replied no.

However, upon hearing that all four students replied no, each student was able to determine the elements of S. Find the sum of all possible values of the greatest element of S.

Problem 14.2. A convex quadrilateral has area 30 and side lengths $5, 6, 9$, and 7, in that order. Denote by θ the measure of the acute angle formed by the diagonals of the quadrilateral. Then $\tan\theta$ can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 14.3. Find the least positive integer n for which $2^n + 5^n - n$ is a multiple of 1000.

Problem 14.4. Let ΔABC be an acute triangle with circumcenter O and centroid G. Let X be the intersection of the line tangent to the circumcircle of ΔABC at A and the line perpendicular to GO at G. Let Y be the intersection of lines XG and BC. Given that the measures of $\angle ABC, \angle BCA$, and $\angle XOY$ are in the ratio $13 : 2 : 17$, the degree measure of $\angle BAC$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 14.5. Let $f(n)$ and $g(n)$ be functions satisfying

$$f(n) = \begin{cases} \sqrt{n} & \text{if } \sqrt{n} \text{ is an integer} \\ 1 + f(n+1) & \text{otherwise} \end{cases}$$

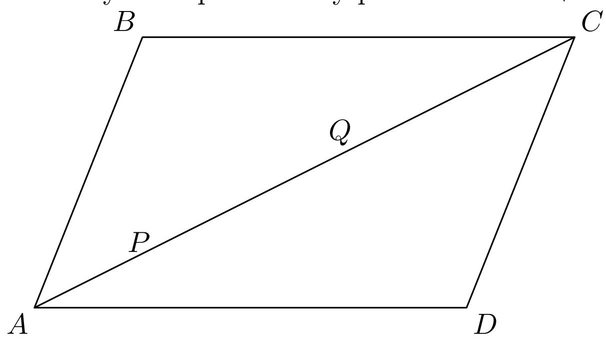
and

$$g(n) = \begin{cases} \sqrt{n} & \text{if } \sqrt{n} \text{ is an integer} \\ 2 + g(n+2) & \text{otherwise} \end{cases}$$

for positive integers n. Find the least positive integer n such that $\frac{f(n)}{g(n)} = \frac{4}{7}$.

§15 2022 AIME I

Problem 15.1. Let $ABCD$ be a parallelogram with $\angle BAD < 90^\circ$. A circle tangent to sides $\overline{DA}$, $\overline{AB}$, and $\overline{BC}$ intersects diagonal $\overline{AC}$ at points P and Q with $AP < AQ$, as shown. Suppose that $AP = 3$, $PQ = 9$, and $QC = 16$. Then the area of $ABCD$ can be expressed in the form $m\sqrt{n}$, where m and n are positive integers, and n is not divisible by the square of any prime. Find $m+n$.



Problem 15.2. For any finite set X, let $|X|$ denote the number of elements in X. Define

$$S_n = \sum |A \cap B|,$$

where the sum is taken over all ordered pairs (A, B) such that A and B are subsets of $\{1, 2, 3, \ldots, n\}$ with $|A| = |B|$. For example, $S_2 = 4$ because the sum is taken over the pairs of subsets

$$(A, B) \in \{(\emptyset, \emptyset), (\{1\}, \{1\}), (\{1\}, \{2\}), (\{2\}, \{1\}), (\{2\}, \{2\}), (\{1, 2\}, \{1, 2\})\},$$

giving $S_2 = 0 + 1 + 0 + 0 + 1 + 2 = 4$. Let $\frac{S_{2022}}{S_{2021}} = \frac{p}{q}$, where p and q are relatively prime positive integers. Find the remainder when $p + q$ is divided by 1000.

Problem 15.3. Let S be the set of all rational numbers that can be expressed as a repeating decimal in the form $0.\overline{abcd}$, where at least one of the digits a, b, c, or d is nonzero. Let N be the number of distinct numerators obtained when numbers in S are written as fractions in lowest terms. For example, both 4 and 410 are counted among the distinct numerators for numbers in S because $0.\overline{3636} = \frac{4}{11}$ and $0.\overline{1230} = \frac{410}{3333}$. Find the remainder when N is divided by 1000.

Problem 15.4. Given $\triangle ABC$ and a point P on one of its sides, call line ℓ the *splitting line* of $\triangle ABC$ through P if ℓ passes through P and divides $\triangle ABC$ into two polygons of equal perimeter. Let $\triangle ABC$ be a triangle where $BC = 219$ and AB and AC are positive integers. Let M and N be the midpoints of $\overline{AB}$ and $\overline{AC}$, respectively, and suppose that the splitting lines of $\triangle ABC$ through M and N intersect at 30°. Find the perimeter of $\triangle ABC$.

Problem 15.5. Let x, y, and z be positive real numbers satisfying the system of equations:

$$\begin{aligned} \sqrt{2x - xy} + \sqrt{2y - xy} &= 1 \\ \sqrt{2y - yz} + \sqrt{2z - yz} &= \sqrt{2} \\ \sqrt{2z - zx} + \sqrt{2x - zx} &= \sqrt{3} \end{aligned}$$

. Then $[(1-x)(1-y)(1-z)]^2$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

§16 2022 AIME II

Problem 16.1. Let $ABCD$ be a convex quadrilateral with $AB = 2$, $AD = 7$, and $CD = 3$ such that the bisectors of acute angles $\angle DAB$ and $\angle ADC$ intersect at the midpoint of $\overline{BC}$. Find the square of the area of $ABCD$.

Problem 16.2. Let $a, b, x,$ and y be real numbers with $a > 4$ and $b > 1$ such that

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - 16} = \frac{(x-20)^2}{b^2 - 1} + \frac{(y-11)^2}{b^2} = 1.$$

Find the least possible value of $a + b$.

Problem 16.3. There is a polynomial $P(x)$ with integer coefficients such that

$$P(x) = \frac{(x^{2310}-1)^6}{(x^{105}-1)(x^{70}-1)(x^{42}-1)(x^{30}-1)}$$

holds for every $0 < x < 1$. Find the coefficient of x^{2022} in $P(x)$.

Problem 16.4. For positive integers a, b, and c with $a < b < c$, consider collections of postage stamps in denominations a, b, and c cents that contain at least one stamp of each denomination. If there exists such a collection that contains sub-collections worth every whole number of cents up to 1000 cents, let $f(a, b, c)$ be the minimum number of stamps in such a collection. Find the sum of the three least values of c such that $f(a, b, c) = 97$ for some choice of a and b.

Problem 16.5. Two externally tangent circles ω_1 and ω_2 have centers O_1 and O_2, respectively. A third circle Ω passing through O_1 and O_2 intersects ω_1 at B and C and ω_2 at A and D, as shown. Suppose that $AB = 2$, $O_1O_2 = 15$, $CD = 16$, and ABO_1CDO_2 is a convex hexagon. Find the area of this hexagon.