Physics-Informed Neural Networks (PINNs) in Finance

Miquel Noguer i Alonso
Julián Antolín Camarena

Artificial Intelligence Finance Institute

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Abstract

Physics-Informed Neural Networks (PINNs) provide a framework to embed the Heston model dynamics directly into the learning process. This ensures that predictions are not only data-consistent but also obey the underlying stochastic differential equations. Physics-Informed Neural Networks offer an innovative way to embed financial rules and physics directly into the learning process of neural networks. By doing so, PINNs not only provide accurate predictions but also ensure that these predictions are consistent with known financial rules and structures. We test the architecture with Black-Scholes and the Heston models as parametric models. The architecture seems to learn them correctly.

1 Introduction

Finance is a field that often relies on differential equations, particularly partial and stochastic differential equations (PDEs and SDEs, respectively), to model various phenomena such as options pricing. Physics-Informed Neural Networks (PINNs) provide a promising methodology to solve these PDEs by embedding the physics directly into the architecture and training process.The architecture was invented by [Raissi et al., 2019].

2 Physics-Informed Neural Networks

2.1 Theoretical considerations

Physics-Informed Neural Networks (PINNs) are a type of neural network that incorporates physical knowledge, often in the form of differential equations, into the learning process. This approach allows the neural network to be guided by known physical laws, which can improve the accuracy and generalization of the model, especially when training data is sparse.

The main idea behind PINNs is to use the neural network to approximate a function that satisfies a given differential equation. During training, in addition to minimizing the error between the network's predictions and the available data, the network also minimizes the error between its predictions and the known physical laws represented by the differential equations. The main objective of this paper is to show that PINNs can be used in mathematical finance to solve stochastic differential equations (SDEs) by using the Heston model as a test bed.

A Physics-Informed Neural Network (PINN) is defined by a neural network architecture and a loss function. The loss function consists of two terms: a data mismatch term and a physics-informed regularization term:

$$L = L_{\text{data}} + L_{\text{physics}} \tag{1}$$

Where L_{data} measures the mismatch between the neural network predictions and observed data, and L_{physics} ensures the predictions satisfy the Black-Scholes PDE.

Given a neural network $f_\theta(S,t)$ with parameters θ, our prediction for the option price is $V(S,t) = f_\theta(S,t)$. The physics-informed regularization term can be computed as:

$$L_{\text{physics}} = \left| \frac{\partial f_\theta}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f_\theta}{\partial S^2} + rS\frac{\partial f_\theta}{\partial S} - rf_\theta \right|^2 \tag{2}$$

Training involves minimizing the total loss L with respect to the network parameters θ.

2.2 Bayesian Interpretation of PINNs

The PINN architecture has a Bayesian interpretation. The loss, L can be considered as the negative log-likelihood of a posterior distribution for f_θ, where the data mismatch comes from the likelihood function and the physics-informed loss is the prior information available on f_θ. Thus, in a Bayesian framework, for data X, we have:

$$\overset{\text{posterior}}{p(f_\theta|X)} = \frac{\overset{\text{likelihood}}{p(X|f_\theta)}\overset{\text{prior}}{p(f_\theta|\mathcal{D})}p(\mathcal{D})}{\underset{\text{evidence}}{p(X)}}, \tag{3}$$

where the prior is a distribution over differential operators $\mathcal{D}$ known to describe the dynamics of the function f_θ, i.e., that satisfy

$$\mathcal{D}f_\theta = 0, \quad L_{\text{physics}} = |\mathcal{D}f_\theta|^2 . \tag{4}$$

For many dynamical problems, the differential operator in question is known in general form, it is the parameters that specify it that need to be fit from data. In the case that we know the operator exactly, $p(\mathcal{D}) = \delta(\mathcal{D})$ (Dirac delta distribution), i.e. there is certainty about the specific operator.

In the more general case we do not precisely know the parameters, λ, that specify the operator - hyperparameters from the Bayesian perspective. This means that the operator itself becomes stochastic as there is now need for a distribution over λ to describe our uncertainty in the dynamics of the system. For such cases, we may write

$$p(f_\theta, \mathcal{D}_\lambda, \lambda) = p(f_\theta|\mathcal{D}_\lambda, \lambda)p(\mathcal{D}_\lambda|\lambda)p(\lambda). \tag{5}$$

The advantage afforded by PINNs in this case is that operator or system parameters can be fit with data. Thus PINNs allow dynamics to be learned in a data-driven fashion.

Lastly, note that in the Bayesian view, the prior imposes a regularization constraint on estimation since it will penalize parameters that stray far from what is already known about the parameters and that don't satisfy the data via the likelihood function. In this sense, the well-known fact about PINNs that imposing physical dynamics as a constraint acts as a regularization term for the neural network arises naturally . This term, represented by L_{physics}, will penalize functions that poorly satisfy the dynamics, but that also satisfy the data model poorly. Thus, a regularized function will be forced to satisfy both constraints.

As a test bed for PINNs, we consider the Heston model, which is a stochastic volatility model describing the joint evolution of an asset price and its volatility. We aim to utilize PINNs to find solutions that both fit data and obey the model's dynamics.

Here are some key references on the topic: [Raissi et al., 2019], [Raissi and Karniadakis, 2018], [Raissi et al., 2017], and [Patrick Kidger and Lyons, 2021]. WE will use two benchmark models in finance Black-Scholes [Black and Scholes, 1973] and the Heston Model [Heston, 1993].

3 Dynamical Systems in Finance

As noted in the introduction, quantitative descriptions of finance rely heavily on PDEs and SDEs. These are natural choices as asset prices can depend on several variables and change over time randomly.

We provide two well-kon examples below.

3.1 The Black-Scholes Model

Consider the Black-Scholes equation, which is a PDE used to describe the price of a European call option:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{6}$$

where $V(S,t)$ is the option price, S is the stock price, σ is the volatility, and r is the risk-free interest rate.

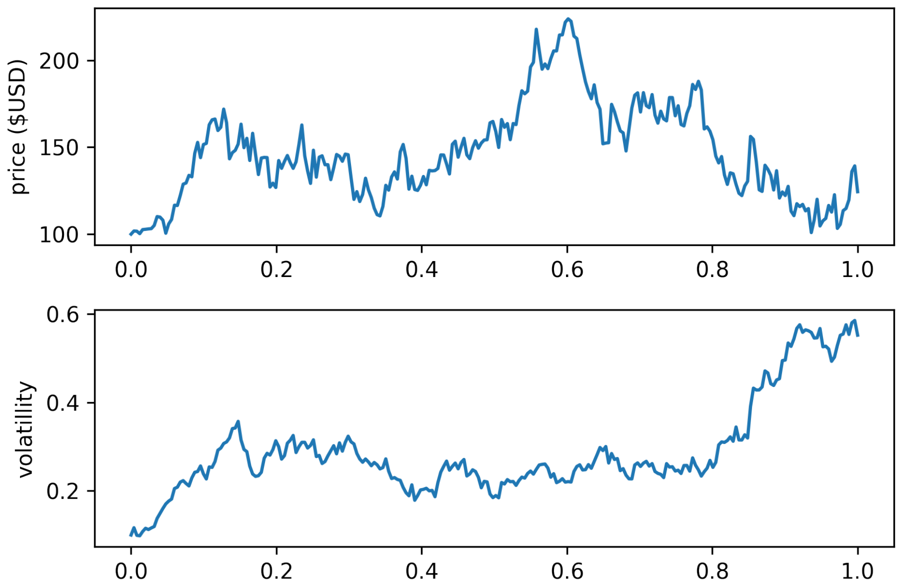

Figure 1: Typical simulation trajectory of the Heston model. Top panel: Simulation of stock price S_t under instantaneous volatility ν_t. Bottom panel: Instantaneous volatility ν_t.

3.2 The Heston Model

In the previous section, we described how PINNs can be used to solve a well-known PDE in mathematical finance.

The main objective of this paper is to show that PINNs can be used in mathematical finance to solve SDEs. We will utilize the celebrated Heston model for asset volatility as a test bed.

The Heston model is given by:

$$dS_t = rS_t dt + \sqrt{v_t} S_t dW_t^S$$
$$dv_t = \kappa(\theta - v_t)dt + \xi\sqrt{v_t} dW_t^v,$$

where the Feller condition

$$2\kappa\theta > \xi^2 \tag{7}$$

is required for positivity of the processes. Figure 1 displays a typical trajectory of a stock price and its instantaneous volatility.

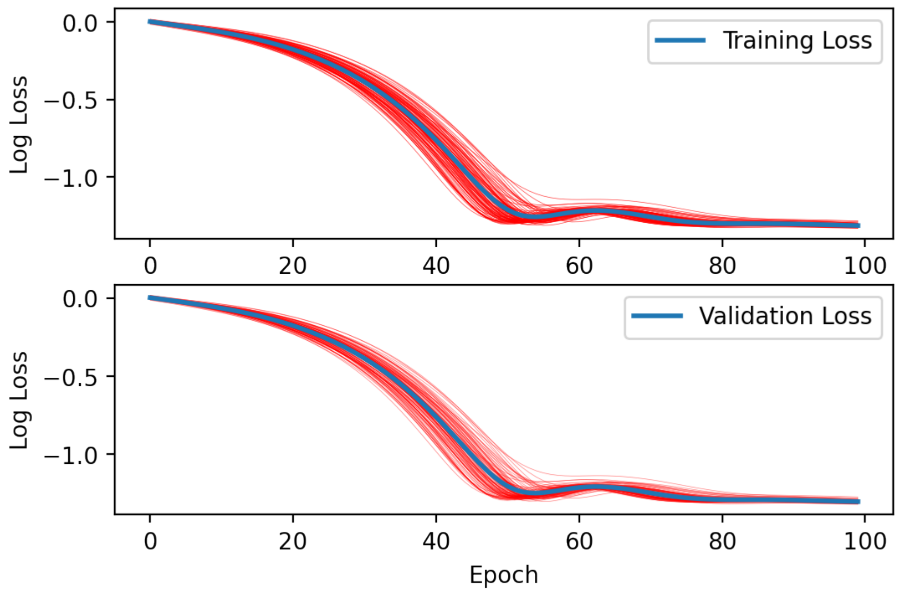

Figure 2: Log-loss curves of training and validation losses.

4 The Physics-Informed Neural Network Approach

4.1 Setup

Let $f_\theta^S(t)$ and $f_\theta^v(t)$ be neural network representations for S_t and v_t respectively. The PINN loss function is:

$$L = L_{\text{data}} + L_{\text{physics}}$$

Where:

$$L_{\text{physics}} = \left| \frac{df_\theta^S}{dt} - r f_\theta^S - \sqrt{f_\theta^v} f_\theta^S \right|^2 + \left| \frac{df_\theta^v}{dt} - \kappa(\theta - f_\theta^v) - \xi \sqrt{f_\theta^v} \right|^2$$

4.2 Experiments

The PINN was trained on 10,000 trajectories of simulated asset price-volatility pairs of length 252 days. This is to simulate one year of market data, assuming 252 trading days per year. The Wiener process was simulated with correlated Gaussian noise with correlation coefficient $\rho = 0.5$, drift $\mu = 0.01$, asset price standard deviation $\sigma_S = 2.0$, volatility standard deviation $\xi = 0.1$, long-term mean of volatility of $\theta = 0.2$, mean reversion rate of $\kappa = 0.075$. The experiment was repeated 100 times for statistically meaningful results.

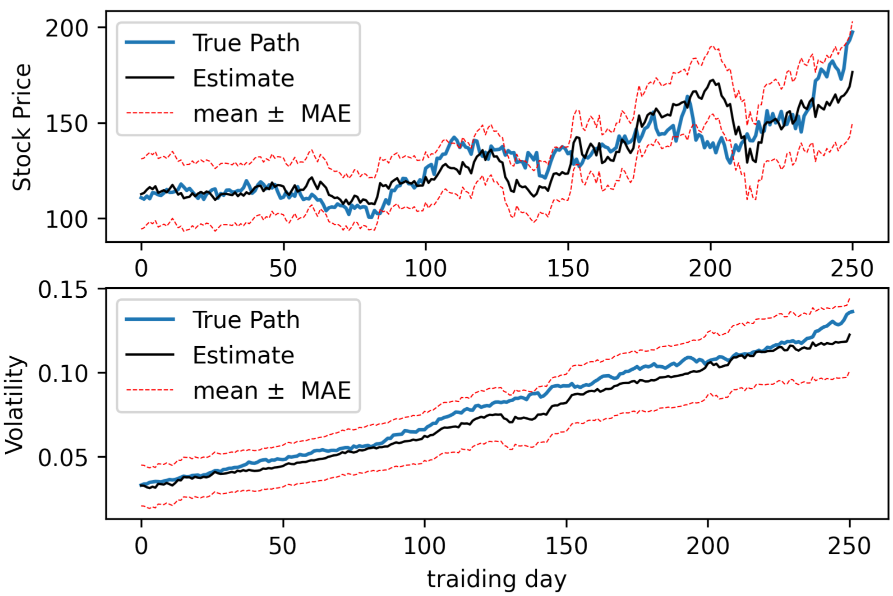

Figure 3: Numerical results of Heston PINN. **Top panel:** . Stock price over a trading year in blue, with the mean PINN prediction in black and the confidence bands at one daily standard deviation. **Bottom panel:** Volatility of the stock price over a trading year in blue, with the mean PINN prediction in black and the confidence bands at one daily standard deviation.

Quantity	Value	Ensemble average	Relative MAE
Stock price	\$ 16.17	\$ 101.9	0.16
Volatility	0.016	0.072	0.22

Table 1: Results over 10,000 trajectories of length 252 days each.

4.3 Results

We find that the PINNs train the Heston model adequately, as evidenced in Fig. 2. The estimated mean is largely within one daily standard deviation of the ground truth, indicating that the Heston model is being learned relatively well. This is largely due to the simple structure of the Heston model, but also alludes to the power of the PINN approach.

Figure 3 shows a typical realization of the Heston price-volatility pair (top and bottom panels, respectively). The ground truth is shown in blue, the PINN estimate is shown in black, and the red dotted curves show the mean absolute error (MAE) averaged over all trajectories at each time step. Table 1 shows quantitatively these results.

5 Conclusion

Physics-Informed Neural Networks (PINNs) provide a framework to embed the Heston model dynamics directly into the learning process. This ensures that predictions are not only data-consistent but also obey the underlying stochastic differential equations. Physics-Informed Neural Networks offer an innovative way to embed financial rules and physics directly into the learning process of neural networks. By doing so, PINNs not only provide accurate predictions but also ensure that these predictions are consistent with known financial rules and structures.We test the architecture with Black-Scholes and the Heston models as parametric models. The architecture seems to correctly learn them.

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