MODELS THAT DESCRIBE THE DEGENERATE MATTER IN STARS

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5 Models were compared to the TOV mass limit of neutron stars. Model 3 has been TOV corrected. It switches its equation of state EoS for central pressures larger than (1.0976606±0.0000001) $\cdot 10^{35}$ Pa. Model 4 is the combined state of the non-relativistic and relativistic states, that has a continuous EoS transition. It's mass is (0.9585±0.0005)M⊙ and its curve fit constants are A_{NR}=6247.4472±0.04 & $A_{R\_}$=0.00085±0.00007. Model five includes nuclear potentials from the binding energies of the fermi neutron gases and has an approximate maximum mass of 2.2 M⊙. The nuclear compressibility and purity of the neutron star were varied to see their effects of maximum mass.

1. Introduction

This report was written because neutron stars exist. We exist because of them. New studies show that neutron star collisions are the main driver for the nuclear fusion of heavier elements, which we are made of [1]. Neutron stars are very dense, small, and massive stars. They consist of mostly neutrons and have an approximate radius and mass of 10km and 1.2 solar masses, M⊙[2]. A neutron star is formed from the dead core of a massive star, with a mass between 1-3M⊙, which reached the end of its fusion cycle and caused a supernova. The neutron was predicted by Ernest Rutherford in 1920 and was first observed in 1932 by James Chadwick. Two years later Walter Baade and Fritz Zwicky predicted the existence of neutron stars. In 1967 the first neutron star was observed by Jocelyn Bell Burnell[3]. A neutron star provides an environment, that is not achievable in labs, to investigate extremely dense matter. Neutrons stars are stabilized by the hydrostatic equilibrium between their enormous gravity and neutron degeneracy pressure. This results in a minimum distance its particles can be compressed. In this report several models will be discussed and compared. The hydrostatic equilibrium eq(1) and mass eq(2) equations of stellar structure were used to determine the internal structure of neutron stars.

$$\frac{dP}{dr} = -\frac{G\epsilon(r)M(r)}{c^2r^2} \quad eq(1), \qquad \frac{dM}{dr} = \frac{4\pi r^2\epsilon(r)}{c^2} \quad eq(2)$$

Current research suggests that the neutron star has a very complex structure with distinct layers. The different layers of the star are due to the different states within the star. The energy density equations, addressed later within this report, where integrated with the equations above to find the properties of neutron stars.

2. Theory

A neutron is a spin-1/2 particle (fermion). A particle's spin is the intrinsic angular momentum of that particle. Fermions are modelled statistically by the Fermi-Dirac distribution. It is constrained by the Pauli exclusion principle**.** A simple model is used to explain these principles. The model is a large number of spin-1/2 trapped in a "box" of side length L[4]. Because there are lots of fermions in this box, we call the system a fermi gas. There are quantized energy levels of the fermi gas, stated in Math Appendix 1. The sum of individual particle states is found from these quantized energy levels and is used to find the density of states, g(E). The lowest energy configuration of the particles fills the ground state to the fermi energy E_F, by the Pauli exclusion principle. This low energy state is defined at T = 0K. The total number of particles, N in the system in this low energetic configuration is given by

$$N = \int_0^{E_F} g(E)dE \qquad eq(3)$$

Using the derivation in Math Appendix 1. The fermi energy E_F and the fermi momentum k_F are defined as functions of number density (the number of particles which occupy a given volume)

$$E_F = \frac{\hbar^2}{2m}(3n\pi^2)^{\frac{2}{3}} \qquad \text{eq(4)}$$

$$k_F = (3\pi^2 n)^{\frac{1}{3}} \qquad \text{eq(5)}$$

Using eq(4), and Math Appendix 1 and 2, the average kinetic energy of a neutron in this low energy state is given in eq(6).

$$\langle E_{T=0K}\rangle = \frac{3}{5}E_F \qquad \text{eq(6)}$$

The neutron star's enormous gravitational forces neutrons close to the core, where there is a higher pressure, to be bound in the lowest energy state. The neutrons form a degenerate fermi gas, similar to the "box" filled with fermions at T = 0K mentioned before. The non-relativistic expression for the kinetic energy $\langle E_{KE}^0\rangle$ of the system at this point, in the neutron star, is given as

$$\langle E_{KE}^0\rangle = \frac{3}{5}\langle E_F^0\rangle \qquad \text{eq(7)}$$

Where $\langle E_F^0\rangle$ is the fermi energy for this low energy state. The derivation is similar to the derivation for the fermi gas box. In nature the star is not a pure neutron fermi gas, it has a very complicated structure which can be broken down into 5 parts[5]: the atmosphere, surface, outer crust, inner crust, and the core. The atmosphere of a neutron star has atoms, ions and electrons. A high fraction of the ions and atoms are Fe^{56} because they are the remnants of the star that went supernova to form the neutron star. Fe^{56} is the last element of fusion cycle that produces energy.

The height of the atmosphere is a few cm. The density of the star gets higher closer to the core. At the surface of the star the density is very large. At these large densities the nuclei have a small separation, and the strong force repulsion prevents the ions from moving past each other. This is symmetric (acts in all directions) so it forces the ions into a crystalline lattice, a frozen plasma. Below the surface is the outer crust. Here Equations of state are introduced. The Equation of state, EoS or a Polytrope describes the physical state of the system at a given pressure P.

$$P = K\epsilon^{\Upsilon} \qquad \text{eq(8)}$$

Where ϒ is the polytopic index, ϵ is the energy density and K is the polytope constant. The EoS in this part of the star is non-relativistic,

$$K_{non-rel} = \frac{\hbar^2}{15\pi^2 m_n}\left(\frac{3\pi^2}{m_n c^2}\right)^{\frac{5}{3}} eq(9), \qquad \gamma = \frac{5}{3}$$

For a fermi momentum larger than $kF_{-switch}$ (= $m_n c$)
The polytrope has switched into the relativistic EoS,

$$K_{rel} = \frac{1}{3}\ eq(10), \quad \gamma = \frac{4}{3}$$

This more energetic state allows electron capture.

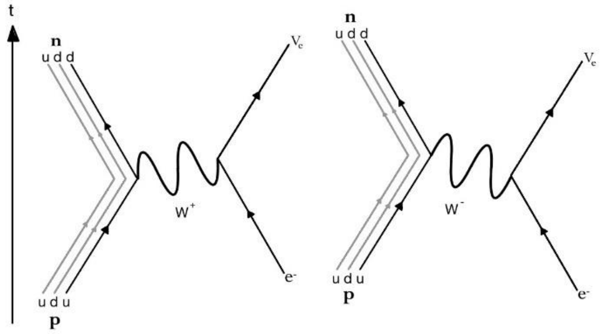


Figure 1: Two Feynman diagrams showing the electron capture process via different weak interactions.

Fe^{56} is no longer the most stable nucleus at this high fermi momentum and more neutron-rich nuclei are favoured. Some of the electrons are captured into the iron nuclei forming neutron rich nuclei. These neutron rich nuclei formed would not be stable on earth, however they are in a neutron star. This is due to the high fermi momentum stabilizing these nuclei. A few 100m into the neutron star elements are observed which cannot exist outside of these extreme conditions. A neutron has a decay time of 15 minuntes[6] on earth however it is stabilized in this high fermi momentum and the presence of protons and electrons. Because of the weak force equilibrium, the Pauli exclusion principle prevents the decay from taking place. All the available low-energy levels for the decay proton are already filled by the protons present. The number density of electrons and protons gets smaller closer to the core due to electron capture figure 1 and neutron number density increases. The neutrons that have leaked to form a fermi gas are in the lowest energy configuration. There is a weak force equilibrium between electron capture figure 1 and neutron decay figure 2, as neutron number density for any region in the neutron star is fixed in that region.

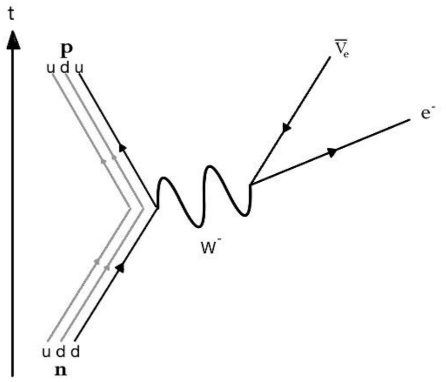


Figure 2: A Feynman diagram showing neutron decay.

Roughly 500m deep, in the inner core, neutron drip occurs. The wavefunctions of the neutrons from the neutrons-rich Nuclei leak between the decreasing gap between these nuclei. The exact point when neutron drip happens is not known. As this effect increases, the gap between the nuclei is filled with a neutron fermi gas. It is difficult to localise leaked neutrons in nuclei, because their wavefunctions are much larger than the nuclei. The number densities and fermi momentums of protons and electrons are equal, because of charge neutrality. The fraction of protons to neutrons is determined by their chemical potentials. The most extreme conditions in the modern universe are located in the core of a neutron star. The neutrons form a frictionless superfluid because the spin-1/2 neutrons form cooper pairs. There are also super rare cooper pair protons. Like electrical superconductivity the neutron pair forms a boson with spin 0 or 1. These particles are now modelled by Bose-Einstein statistics and do not obey the Pauli exclusion principle. The entropy of the star decreases here. As we approach the centre of the star a debate starts, What is the core made of?

3.Results/Discussion

Throughout this paper the energy density is converted to a pressure to solve the equations of stellar state, here we will investigate how the nuclear properties of the star effect the energy density, as a nuclear potential. The electron contribution to the nuclear potential is small in comparison the nucleon's contributions, because the strong force dominates at these high densities. So, we will ignore their contributions. Another assumption in our nuclear models is that the neutrons and protons have the same rest mass m_n. [The number of protons Z, the number of neutrons N and the number of nucleons A = N + Z]
The average energy per nucleon is[7]

$$\frac{E(n)}{A} = \frac{\epsilon(n)}{n} \qquad eq(11)$$

As mentioned in the theory section, the neutron star consists of a small fraction of protons due to the weak force equilibrium. The fraction of neutrons and protons is quantized by $\alpha \in [-1,1]$,

$$n_n = \frac{1+\alpha}{2} n \quad eq(12) \quad , \quad n_p = \frac{1-\alpha}{2} n \quad eq(13)$$

The nucleon number density n is the sum of proton and neutron number densities ($n = n_p + n_n$).
The kinetic energy of the system is given by

$$\epsilon_{KE}(n,\alpha) = \frac{3}{5}\frac{k_{F,n}^2}{2m_n} + \frac{3}{5}\frac{k_{F,p}^2}{2m_n} \qquad eq(14)$$

The difference in kinetic energy in the non-symmetric case and the symmmetric case is given by

$$\Delta\epsilon_{KE}(n,\alpha) = \epsilon_{KE}(n,\alpha) - \epsilon_{KE}(n,0) \qquad eq(15)$$

And the total energy per nucleon is given by

$$E(n,\alpha) = E(n,0) + \alpha^2 S(n) \qquad eq(16)$$

The difference between the non-symmetrical case and the symmetrical case is proportional to the α^2. Which is interesting as the fraction of neutron and protons squared is proportional to the symmetry breaking of the system when the purity of the neutron star is changed.
The S(n) is an energy function which corrects the symmetric nuclear matter energy equation, by contributing a change in kinetic energy and bulk symmetry.

$$S(u) = \frac{\Delta\epsilon_{KE}(n,\alpha)}{n}\left(u^{\frac{2}{3}} - F(u)\right) + S_0 F(u) \quad eq(17)$$

This derivation is in Math Appendix 3.
Where the bulk symmetry energy S_0 = 30Mev [7]. When u = 0 there is no matter, so the S(u) function vanishes.

No matter implies no energy. Thus the F(u) function must satisfy F(1) = 1 and F(0) = 0. It has no other constraints, so it can be represented as

$$F(u) = u^b, b \neq 0 \qquad \text{eq(18)}$$

The E(n,0) term represents the energy from the symmetric case N=Z. The N=Z binding energy is modelled by eq(18). Where the binding energy, BE is energy that holds nucleons together via the strong force.
The binding energy has a minimum when

$$\frac{d}{dn}\left(\frac{E(n)}{A}\right) = \frac{d}{dn}\left(\frac{\epsilon(n)}{n}\right) = 0 \quad , n = n_0 \qquad \text{eq(19)}$$

$n_0 = 0.16\ \text{fm}^{-3}$ is the equilibrium number density, and $u = n/n_0$.

$$\frac{\epsilon(n)}{n} - m_n = \frac{3\hbar^2 k_F^2}{10 m_n} + \frac{A}{2}u + \frac{B}{\sigma+1}u^\sigma \qquad \text{eq(20)}$$

Eq(20) is solved in Math Appendix 4 and gives

$$\sigma = \frac{K_0 + 2\langle E_F^0\rangle}{3\langle E_F^0\rangle - 9BE} \qquad \text{eq(21)}$$

$$B = \frac{\sigma+1}{\sigma-1}\left[\frac{\langle E_F^0\rangle}{3} - BE\right] \qquad \text{eq(22)}$$

$$A = BE - \frac{5}{3}\langle E_F^0\rangle - BE \qquad \text{eq(23)}$$

Where K_0 is the nuclear incompressibility constant which has a literature value between 200-400MeV [7][8]. The $\langle E_F^0\rangle$ is the average kinetic energy term for symmetric nuclear matter, it is the first term on the right hand side of the equality in eq(20). In Math appendix 5 a table of values is given for these constants for different K_0 values. The errors in these constants are negligible. The F(u) function, K_0, and purity of the neutron star were investigated against the standard configuration. The standard configuration has a nuclear incompressibility modulus of K_0=240MeV[8]; α = 0.9 which gives a 95% purity[9] and b = 1.

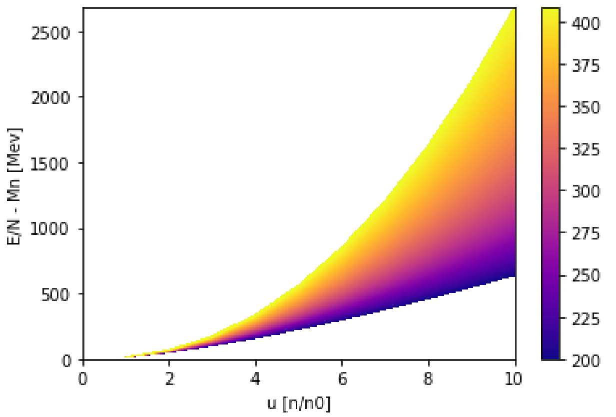


Figure 3: shows the BE against u. The colour bar is the nuclear incompressibility which varies from 200-400 MeV. For α = 0.9 and F(u) = u, [The Step size = 10 * 210 = number of u * number of K_0 values plotted].

The binding energy of nucleons increases, for a higher nuclear incompressibility modulus. This effect is more prevalent for larger densities. The σ is an exponent in eq(21). This increases for larger K_0, By eq(21) and Math Appendix 5. This implies that a model which includes nuclear interactions will have a higher maximum mass for a higher incompressibility modulus.

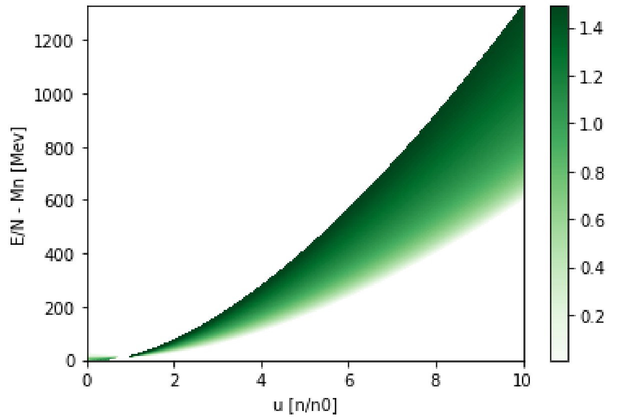


Figure 9: shows the BE against u. The colour bar is the b exponent in the F(u), eq(18). For K_0 = 240MeV and α = 0.9. [Step size = 1000*150]

Figure 9 shows that for a higher exponent for the F(u) function you will get a higher BE. Using the same logic when analysing figure 8, a higher exponent for F(u) will give a higher maximum mass for a neutron star which incorporates nuclear potential in its model.

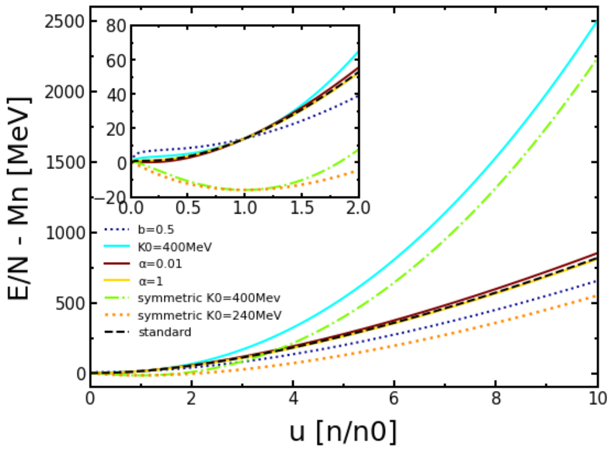


Figure 10: shows the binding energy of a nucleon for different α, K_0 and b, stated in the legend, from the standard. [Stepsize = 7, 7 different models]

The symmetric matter models dip below the x-axis and the trough is at a minimum when u = 1, this is the minimum binding energy of the neutron star (BE = -16MeV). There is a little bump from the origin for all the models because of cavitation (besides α=1).
The neutron star forms complex structures within its crust. A region with nuclear density close to u = 0, can be visualised as having a similar interior to Swiss cheese. This region is close to the surface as the number density, u increases closer to the core. There are near empty bubbles that lie within the degenerate neutron fermi gas in this region. The weak force interactions between neutrons and protons makes these fermions compete for space in the neutron star and this creates near matter-less bubbles. The true name for this phenomena is nuclear pasta[10]. Figure 10 shows that the fraction of protons and neutron has a small effect on the binding energy of the nucleons, in the neutron star, for large u. Therefore, the shapes of the nuclear pasta become more dense closer to the core where there are less protons. This supports the theory as a frictionless superfluid of neutron stars is predicted for extremely high densities.

From the graph the biggest contribution to maximum mass in models, which include nuclear potential, is the nuclear incompressibility modulus.
There are lots of types of neutron stars, the first 4 models, stated below, assume that the neutron star is a pure neutron degenerate fermi gas[11].The largest contribution to the final mass and final radius of the neutron star, with this assumption, is the crust and core. Therefore, the surface and atmosphere will be ignored in our models.

Model	Description
Model 1	Classical
Model 2	Relativistic switch
Model 3	TOV switch
Model 4	Combined Polytrope
Model 5	Nucleon Interactions

Table 1: the models for the neutron stars

The models where integrated with a negligible initial mass and central pressures from 10^{30} to 10^{48} Pa[12]. The first model is a classical model, only the non-relativistic polytope eq(9) was used. In the second model there is a switch in polytrope from the non-relativistic to the relativistic EoS eq(10), at fermi momentums greater than the fermi switch momentum ($k_{F-switch} = m_n c$). By eq(5) and Math Appendix 3 the fermi momentum, k_F can expressed as

$$k_F = \hbar \left(\frac{3\pi^2 \epsilon}{m_n c^2}\right)^{\frac{1}{3}} \qquad eq(24)$$

This model contributes the changes in the fermi momentum of the "leaked" neutrons when it enters a higher energetic state. The more massive a stellar object the more spacetime is curved around it. A better model for more massive stars would include General Relativistic (GR) corrections[7] , which are made in the third model. The hydrostatic equilibrium equation of stellar state eq(1), is corrected to the TOV equation.

$$\frac{dP}{dr} = -\frac{G\epsilon(r)M(r)}{c^2 r^2} \cdot \left[1 + \frac{P(r)}{\epsilon(r)}\right] \cdot \left[1 + \frac{4\pi r^3 P(r)}{M(r)c^2}\right] \cdot \left[1 - \frac{2GM(r)}{c^2 r}\right]^{-1} \qquad eq(25)$$

The non-relativistic polytope constants remain the same, however the new relativistic constants are

$$K_{rel} = \frac{1}{3} eq(26), \qquad \gamma = 1$$

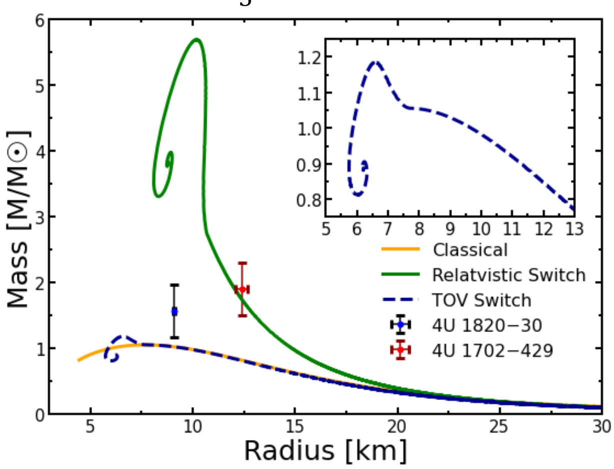


Figure 11 and 12: This is a plot of final mass vs final radius for 3000 stars (3000steps) for models 1,2 and 3 in Table 1, with the range of central pressures from 10^{30} to 10^{48} Pascals. The subplot in the top right corner is the TOV switch curve zoomed in, so its spiral can be more distinct. The legend states the model and the stars that was plotted.

Neutron stars generally have a smaller radius when they are more massive, because the higher gravitational force lowers the nucleon separation. When there is a switch in polytope there is a spiral shape. The final masses and radii provide solutions for the hydrostatic equilibrium[4]. This spiral symbolises the complexity of this equilibrium.

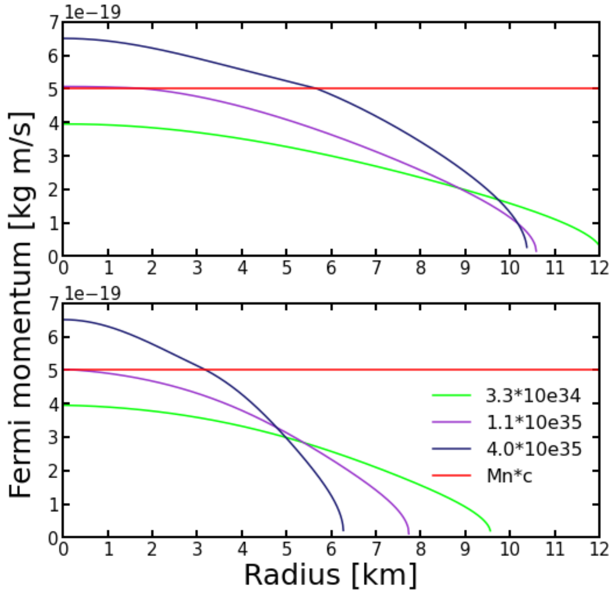


Figure 13 and 14: Graphs of fermi momentum against final radius for 3 different stars with their central pressures in the legend in Pascals for different models. The first graph is for the relativistic switch model (model 2) and the second graph is for the TOV switch model (model 3). The red line is the $k_{F\text{-switch}}$, the minimum fermi momentum for a switch in polytrope equal to $m_n c$. [Step size = 3000, 3000 stars per model]

Model	Maximum Mass	Fractional error
Model 1	$(1.05589 \pm 0.00007)M_\odot$	0.00007
Model 2	$(5.695 \pm 0.001)M_\odot$	0.0002
Model 3	$(1.1859 \pm 0.0004)M_\odot$	0.0003

Table 2: shows the maximum masses for neutron stars modelled by the specified model stated in Table 1.

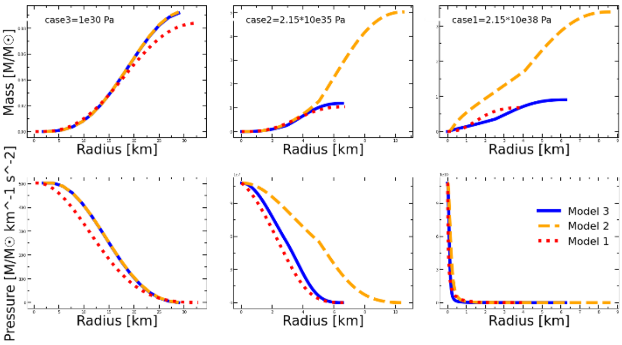
A functional approach was taken to calculate error in the maximum masses in Table 2[13]. The equations of stellar state and the parameters for the polytopes were maximised and minimised using the constants $\hbar, m_n, c$ and G and their errors[16]. The fermi momentum is very important in these models, because only stars with fermi momentums greater than $k_{F\text{-switch}}$ can enter the higher energy relativistic EoS. By Math Appendix 6 the theoretical minimum central pressure for a star to switch state is P_{switch}

$$P_{switch} = \frac{m_n^4 c^5}{15\pi^2 \hbar^3} \qquad eq(27)$$

$$P_{switch} = (1.0976606 \pm 0.0000001) \cdot 10^{35} \text{ Pa}$$

In figures 13 and 14 the purple line in the graph shows a central pressure equal to this polytrope switching central pressure, P_{switch}. All central pressures that lie below the red lines cannot switch polytropes. The TOV equation amplifies the gravity by contributing general relativistic corrections. This results in a lower degenerate pressure needed to support the star, and hence it produces a star was a smaller maximum mass than model 2 (as shown in figure 11 and table 2). Model 1 has the smallest maximum mass,

because the degenerate fermi gas neutrons do not switch into the higher energy relativistic polytrope.
The relativistic switch model breaks away from the classical model for lower a final radius than TOV corrected switch model. This is shown in figures 13-20. A higher sufficient (enough to switch state) central pressure causes a smaller final radius and a higher fermi momentum. For this same sufficient central pressure model 3 achieves a larger final radius. The three models agree for predicting the final mass and final radii for low central density neutron stars, because the “leaked” neutrons will remain in the low energy non-relativistic state.
Both neutron stars in figure 11 are X-ray binary stars. That is a neutron in a binary system with a main sequence star that it accretes the mass from (takes this mass). Model 2 supports the star with the red marker right now. As its mass grows it will eventually exceed the TOV limit[17] and it will become a black hole. Therefore model 2 is flawed because it produces masses which are too large. Model 3’s maximum mass is too small, as neutron stars with higher masses have been observed.



Figures 15-20: The vertical columns show the mass vs radius and the pressure vs radius relationships for an individual neutron star with central pressure shown in the top entry. The figures are labelled from the top row from left to right and the bottom row from left to right.The legend in the figure 12 identifies each of the models used.

There is a bump in model 3’s spiral. Model 4 is introduced as a combined polytrope state, with TOV corrections. The data from model 3 was curve fitted with

$$\epsilon(P) = A_{NR}P^{3/5} + A_R P \qquad \text{eq(28)}$$

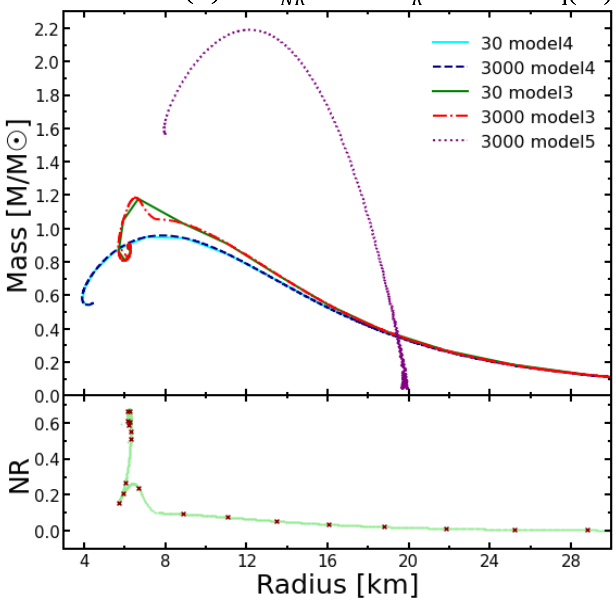


Figure 21: This is a plot of final mass vs final radius for Model 3, 4 and 5 in **Table 1**, with the range of central pressures from 10^{30} to 10^{48} Pascals. The legend states the model and the number of stars, it has modelled. Below the graph are the normalised residuals between the curve fitted data and the original data. (model 3 and model 4)

Model 4 is useful to estimate the final masses and radii of model 3 for low central pressure stars. Because for low central pressures the models are in ≈1% agreement, however for higher central pressures they have a high percentage error of≈60%. The fractional uncertainty of the A constants from model 4 are not effected by step size (the number of stars). Math Appendix 6 shows this. However, the higher the step size the more accurate the A value. The largest contribution to the A errors were the error in the values of $\hbar, c, G, m_n$ [13][14].
The combined function makes the TOV correction spiral smoother. This result is significant. Model 4 is more realistic than model 3. A way to visualise this is to think about the transition in state for model 3 vs model 4. In model 3 the fermi momentum gas in one region is in a low energy non-relativistic state, then suddenly one step later in the integration it is now in the high energy relativistic state. This is not realistic as the transition is continuous. This is supported by nuclear pasta theory, shown in figure 10, were the fermi gases experiences less cavitation closer to the core. Therefore the smoothness of the model 4 implies it is significantly a better model. This same logic is applied for the analysis of the fractional errors in Table 2. After curve-fitting 3000 stars modelled by the TOV switch model the A constants found are

$$A_{NR} = 6247.4472 \pm 0.04\,, A_R = 0.00085 \pm 0.00007$$

Taking the small pressure limit of the combined function eq(28) the A_{NR} value dominates and for the larger pressure limit the A_R dominates. The larger the pressure the higher the fermi momentum. This model has the same relationship with states of the systems as model 3, but is superior as it treats the state transition as continuous rather than discrete. Other curve fitted values for A_{NR} and A_R for different numbers of stars are located in Math appendix 5
Model 5 includes the nuclear potential of the system.
Model 5 is superior to the other models because it not only includes a continuous EoS, but also the shapes of the structures the degenerate fermi neutron gas produces.
This equation of state can be modelled by

$$\epsilon(P) = A_{GR}P^{\frac{1}{2}} \qquad \text{eq(29)}$$

The value for constant A_{GR} was estimated from[7]. This model was included because it provides closure to the main point of this report.
The maximum mass produce by model 4 was

$$M_{model\,4} = (0.9585 \pm 0.0005)\mathrm{M}\odot$$

And the maximum mass produced by model 5 is approximately 2.3M⊙, which approaches the TOV limit.

4. Conclusion

The Maximum mass produced by a model is important. The flaws in models can be adjusted, by adding new physics and assumptions, to get it closer to the TOV mass limit. Model 3 is better than model 2 and model 1. For these models the neutron star is treated as a pure neutron degenerate fermi gas. Model 1 models all the fermions to

the low energy, non-relativistic state. Its maximum mass, stated in Table 2, is too small.
Model 2 provides a discrete transition to the higher relativistic polytrope, if the central pressure of the star exceeds P_{switch}. This maximum mass is incorrect because it exceeds the TOV limit. Model 3 also has a discrete transition to the relativistic polytrope but adds general relativistic corrections. Because neutron stars can be massive stellar objects, which curve the spacetime around them. Model 3 is flawed because it has a discrete transition between states. This can be seen in its bump in figure 12. Model 4 corrects model 3 by creating a combined EoS that produces a continuous transition between states and a smooth curve in Figure 21. However, both models have maximum masses much smaller than recorded values of neutron stars. Finally, the nuclear potential is introduced in model 5. This correction involves the shape of the fermi gas. The Topology is an important factor in the maximum mass for a neutron star. This increases the mass to approximately to 2.2 M⊙ which is much closer to the TOV limit.
Next time the maximum masses for model 5 will be collected for different K_0, F(u), and α. Figures 8 and 9 show that for higher values of K_0 and b there are higher binding energies. This will increase the maximum mass of the star. In figure 10 the purity is shown to have no noticeable effects on the binding energy, so it effects for the maximum mass will not be as noticeable.
For low number densities in figure 10 display the phenomena cavitation, which is when small matter-less air bubbles form in the fermi neutron gas. This is due to the presence of protons.
The models where successfully built up to the next by addressing the previous models flaws and correcting them. It is very interesting to see that the matter closer to the core is the most important factor for finding the maximum mass of a neutron star.
This brings us back to the original question in the Theory section, what is the core made of?
This topic becomes highly theoretical. We may find baryons which have dissolved into a quark gluon plasma, or hyperon particles containing heavy strange quarks, because of the extremely high pressure. The only time matter has existed like this in the universe was within a fraction of a second after the Big Bang[9]. A more complete model would include different types of cores, and involve the surface structure of the neutron stars, such as plate tectonics.

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Math Appendix

Math Appendix 1: fermion properties

The allowed energies of this fermi gas inside of this box is given by

$$E_n = \frac{\hbar^2\pi^2}{2mL^2}w^2 \quad eq(30)$$

Where w is the quantum number ($w \in \mathbb{N}$)

For energies up to E, the sum of individual particle states, N_s is

$$N_s = 2\frac{1}{8}\frac{4}{3}\pi n^3 = \frac{1}{3}\pi n^3 \qquad \text{eq(31)}$$

The factor of 1/8 is there as this sum is evaluated in the 1st octant (in cartesian coordinates) and the factor of 2 is there because of the spin half fermions. Using eq(30)

$$\rightarrow N_s = \frac{1}{3\pi^2}\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}} VE^{\frac{3}{2}} \qquad \text{eq(32)}$$

The density of states g(E) is given by

$$g(E) = \frac{dN_s}{dE}$$

$$= \frac{1}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}} VE^{\frac{1}{2}} \qquad eq(33)$$

This density of states is effected by temperature

The total number of particles, N in the system is

$$N = \int_0^{E_F} g(E)dE \qquad \text{eq(3)}$$

Using eq(61)

$$N = \frac{1}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}} V\int_0^{E_F} E^{\frac{1}{2}}dE = \frac{1}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}} VE_F^{\frac{3}{2}} \quad eq(34)$$

Math Appendix 2: average particle energy

And the total energy is for the system of spin-1/2 particles in a box is

$$E_{tot} = \int_0^{E_F} Eg(E)dE = \frac{3}{5}NE_F \qquad \text{eq(35)}$$

Therefore the average particle energy at T = 0K is given by

$$\langle E_{T=0K}\rangle = \frac{E_{tot}}{N} = \frac{3}{5}E_F \qquad eq(6)$$

Math Appendix 3

For N≠Z, α≠ 0

The kinetic energy term is written as

$$\epsilon_{KE}(n,\alpha) = \frac{3}{5}\frac{k_{F,n}^2}{2m_n} + \frac{3}{5}\frac{k_{F,p}^2}{2m_n} = n\langle E_F\rangle\frac{[(1+\alpha)^{\frac{5}{3}}+(1-\alpha)^{\frac{5}{3}}]}{2} \text{eq(36)}$$

By eq(5) and Math appendix 1 and 2

$$\langle E_F\rangle = \frac{3}{5}\frac{\hbar^2}{2m_n}\left(\frac{3\pi^2 n}{2}\right)^{\frac{2}{3}} \qquad \text{eq(37)}$$

This is the mean kinetic energy for non -symmetric nuclear matter at number densities n.

The excess kinetic energy can be written as

$$\Delta\epsilon_{KE}(n,\alpha) = \epsilon_{KE}(n,\alpha) - \epsilon_{KE}(n,0) \qquad \text{eq(15)}$$

The total energy per nucleon is

$$E(n,\alpha) = E(n,0) + \alpha^2 S(n) \qquad \text{eq(16)}$$

Math Appendix 4: energy constants

for u = 1.

$$BE = \langle E_F^0\rangle + \frac{A}{2} + \frac{B}{\sigma+1} \qquad \text{eq(20)}$$

The next constraint comes from the minimum binding energy. Because there is a minimum at u = 1, eq(20) becomes

$$\frac{d}{dn}\left(\frac{E(n)}{A}\right) = \frac{d}{dn}\left(\frac{\epsilon(n)}{n}\right) = 0 \qquad eq(19)$$

$$\frac{2}{3}\langle E_F^0\rangle + \frac{A}{2} + \frac{B\sigma}{\sigma+1} = 0 \qquad eq(38)$$

The final constraint comes from the nuclear incompressibility equation.

$$\frac{K(n_0)}{9} = \frac{dp(n)}{dn} = \left[n^2\frac{d^2}{dn^2}\left(\frac{\epsilon}{n}\right) + 2n\frac{d}{dn}\left(\frac{\epsilon}{n}\right)\right] \text{eq(39)}$$

This implies that for u = 1.

$$\frac{K_0}{9} = \frac{10}{9}\langle E_F^0\rangle + A + B\sigma \qquad \text{eq(40)}$$

To solve equations 16, 17 and 19 first add equations 16 and 17, then rearrange for A.

$$\frac{5}{3}\langle E_F^0\rangle + A + B = BE \qquad \text{eq(41)}$$

$$\rightarrow A = BE - B - \frac{5}{3}\langle E_F^0\rangle \qquad \text{eq(23)}$$

Next minus equation 16 from 17 and rearrange for B.

$$-\frac{1}{3}\langle E_F^0\rangle + \frac{B(\sigma-1)}{\sigma+1} = -BE \qquad \text{eq(42)}$$

$$\rightarrow B = \frac{\sigma+1}{\sigma-1}\left[\frac{1}{3}\langle E_F^0\rangle - BE\right] \qquad eq(23)$$

The final step is to plug in the values for A and B and rearrange for σ.

$$B(\sigma-1) = \frac{K_0}{9} - BE + \frac{5}{9}\langle E_F^0\rangle \qquad \text{eq(43)}$$

$$\rightarrow \sigma = \frac{K_0 + 2\langle E_F^0\rangle}{3\langle E_F^0\rangle - 9BE} \qquad eq(21)$$

Math Appendix 5: the constants for the symmetric case

K_0 [MeV]	σ	A [MeV]	B [MeV]
200	1.161	-366.112	313.279
204	1.18	-335.515	282.681
208	1.199	-310.759	257.926
212	1.218	-290.318	237.485
216	1.237	-273.154	220.321
220	1.256	-258.538	205.705
224	1.275	-245.941	193.108
228	1.294	-234.973	182.139
232	1.313	-225.335	172.502
236	1.332	-216.801	163.968
240	1.351	-209.191	156.357
244	1.37	-202.362	149.529
248	1.389	-196.2	143.367
252	1.408	-190.612	137.779
256	1.427	-185.522	132.688
260	1.447	-180.865	128.031
264	1.466	-176.588	123.755
268	1.485	-172.648	119.814
272	1.504	-169.005	116.171
276	1.523	-165.627	112.794
280	1.542	-162.486	109.653
284	1.561	-159.559	106.726
288	1.58	-156.824	103.99
292	1.599	-154.262	101.429
296	1.618	-151.858	99.025
300	1.637	-149.598	96.765
304	1.656	-147.469	94.636

308	1.675	-145.46	92.627
312	1.694	-143.561	90.728
316	1.713	-141.764	88.931
320	1.732	-140.06	87.226
324	1.751	-138.442	85.609
328	1.77	-136.904	84.071
332	1.789	-135.441	82.607
336	1.808	-134.046	81.213
340	1.827	-132.715	79.882
344	1.846	-131.445	78.611
348	1.865	-130.23	77.396
352	1.884	-129.067	76.234
356	1.903	-127.954	75.12
360	1.922	-126.886	74.053
364	1.941	-125.862	73.028
368	1.96	-124.878	72.044
372	1.979	-123.932	71.099
376	1.998	-123.022	70.189
380	2.017	-122.147	69.313
384	2.036	-121.303	68.47
388	2.055	-120.49	67.657
392	2.074	-119.706	66.873
396	2.093	-118.949	66.116

Table 3: values for symmetric nuclear matter constants for the given nuclear incompressibility constants.

Math Appendix 6: minimum central pressure for a switch in state

$$number\ density, n = \frac{k_F^3}{3\pi^2\hbar^3} \qquad eq(44)$$

Using

$$mass\ density, \rho = nm_n\frac{A}{Z} \qquad eq\ (45)$$

Where m_n is the mass of a neutron, A is the number of nucleons in the atoms and Z is the charge of the nucleus of the atoms.

$$k_F = \hbar(\frac{3\pi^2}{m_N}\cdot\frac{M}{\frac{4}{3}\pi r^3}\cdot\frac{Z}{A})^{\frac{1}{3}} \qquad eq(46)$$

For neutron stars Z/A is 1.

With e = mc^2 equation

$$k_F = \hbar(\frac{3\pi^2}{m_N}\cdot\frac{\varepsilon}{c^2})^{\frac{1}{3}} \qquad eq(47)$$

And the same as before using EoS equation

$$k_F = \hbar(\frac{3\pi^2}{m_N c^2})^{\frac{1}{3}}\cdot(\frac{P}{K})^{\frac{1}{3\gamma}} \qquad eq(48)$$

the fermi momentum for a switch between the non-relativistic polytrope to the relativistic polytrope is

$$k_{F-switch} = m_N \cdot c$$

From above rearranging for pressure gives you

$$\left(\frac{P}{K_{non-rel}}\right)^{\frac{1}{5}} = \frac{m_n c}{\hbar}\left(\frac{m_n c^2}{3\pi^2}\right)^{\frac{1}{3}} \qquad eq(49)$$

Now with a bit of rearranging

$$P = K_{non-rel}\left(\frac{m_n c}{\hbar}\right)^5\left(\frac{3\pi^2}{m_n c^2}\right)^{\frac{5}{3}} \qquad eq(50)$$

Now substituting the K value in

$$P_{switch} = \frac{m_n^4 c^5}{15\pi^2\hbar^3} \qquad eq(27)$$

As $\Upsilon = 5/3$ for the non-relativistic case.

Math appendix 7: curve fit parameters.

#steps	A_{NR}	Error in A_{NR}	A_R	Error in A_R
30	6247.42332	0.00631	0.00056	0.0001
60	6247.44361	0.00663	0.00053	0.00025
90	6247.434	0.01116	0.00116	0.00055
120	6247.47757	0.00938	-.00111	0.00052
150	6247.44857	0.00429	0.00055	0.00026
180	6247.44752	0.00479	0.00065	0.00031
210	6247.44593	0.00107	0.00078	7e-05
240	6247.46945	0.00561	-0.0008	0.00038
270	6247.47001	0.00582	-.00086	0.0004
300	6247.47014	0.00588	-.00089	0.00042
330	6247.46796	0.00489	-.00074	0.00035
360	6247.43943	0.00862	0.00133	0.00063
390	6247.48406	0.01246	-.00194	0.00092
420	6247.43387	0.01122	0.00177	0.00083
450	6247.44582	0.0056	0.00089	0.00042
480	6247.44142	0.0077	0.00122	0.00058
510	6247.4701	0.00588	-.00094	0.00044
540	6247.46944	0.00555	-0.0009	0.00042
570	6247.44605	0.00145	0.00089	0.00011
600	6247.47246	0.00179	-.00114	0.00014
630	6247.48653	0.00631	-.00223	0.00049
660	6247.44555	0.00572	0.00094	0.00044
690	6247.44431	0.0063	0.00104	0.00049
720	6247.46151	0.00012	-0.0003	1e-05
750	6247.46843	0.00507	-.00084	0.0004
780	6247.47262	0.00707	-.00117	0.00055
810	6247.44659	0.00523	0.00087	0.00041
840	6247.44047	0.00811	0.00135	0.00064
870	6247.4331	0.01158	0.00193	0.00091
900	6247.47306	0.00188	-.00121	0.00015
930	6247.46927	0.00109	-.00092	9e-05
960	6247.43851	0.00905	0.00152	0.00072
990	6247.47426	0.00219	-.00131	0.00017
1020	6247.44521	0.00115	0.00099	9e-05
1050	6247.44069	0.00803	0.00135	0.00064
1080	6247.46773	0.00076	-0.0008	6e-05
1110	6247.44435	0.00114	0.00106	9e-05
1140	6247.47102	0.00634	-.00106	0.0005
1170	6247.47032	0.00596	-.00101	0.00048
1200	6247.46798	0.00486	-.00082	0.00039
1230	6247.45353	0.00012	0.00033	1e-05
1260	6247.4611	9e-05	-.00027	1e-05
1290	6247.48434	0.01257	-.00214	0.00101
1320	6247.47352	0.00747	-.00127	0.0006
1350	6247.45771	0.0	-0.0	0.0
1380	6247.4463	0.00106	0.00091	8e-05
1410	6247.44838	0.00441	0.00075	0.00036
1440	6247.44939	0.0039	0.00067	0.00031
1470	6247.48472	0.00667	-.00218	0.00054
1500	6247.44153	0.00761	0.0013	0.00061
1530	6247.46974	0.00571	-.00097	0.00046
1560	6247.47273	0.00179	-.00121	0.00014
1590	6247.47216	0.00685	-.00117	0.00055
1620	6247.47544	0.00839	-.00143	0.00068
1650	6247.43845	0.00906	0.00155	0.00073
1680	6247.4442	0.00635	0.00109	0.00051
1710	6247.46177	0.00013	-.00033	1e-05
1740	6247.42217	0.01136	0.00287	0.00092

1770	6247.44559	0.00116	0.00098	9e-05
1800	6247.44096	0.00788	0.00135	0.00064
1830	6247.48214	0.01154	-.00198	0.00093
1860	6247.47008	0.00585	-.00101	0.00047
1890	6247.44449	0.00621	0.00107	0.0005
1920	6247.4458	0.00566	0.00096	0.00046
1950	6247.45142	0.00031	0.00051	3e-05
1980	6247.46982	0.00576	-.00099	0.00047
2010	6247.44367	0.00662	0.00114	0.00054
2040	6247.47131	0.00643	-.00111	0.00052
2070	6247.44611	0.00544	0.00094	0.00044
2100	6247.47273	0.00159	-.00122	0.00013
2130	6247.46973	0.0057	-.00098	0.00046
2160	6247.48146	0.01121	-.00193	0.00091
2190	6247.44244	0.00718	0.00124	0.00058
2220	6247.43383	0.01125	0.00194	0.00091
2250	6247.44673	0.00518	0.00089	0.00042
2280	6247.46944	0.00138	-.00096	0.00011
2310	6247.44951	0.00384	0.00066	0.00031
2340	6247.44995	0.00367	0.00063	0.0003
2370	6247.46964	0.00143	-.00098	0.00012
2400	6247.44544	0.0009	0.001	7e-05
2430	6247.42327	0.01062	0.0028	0.00087
2460	6247.47128	0.00642	-.00111	0.00052
2490	6247.44313	0.00221	0.00119	0.00018
2520	6247.47223	0.00686	-.00119	0.00056
2550	6247.47459	0.00799	-.00138	0.00065
2580	6247.47633	0.00882	-.00152	0.00072
2610	6247.4329	0.01167	0.00202	0.00095
2640	6247.45269	0.00019	0.00041	2e-05
2670	6247.44671	0.0052	0.00089	0.00043
2700	6247.44757	0.00075	0.00082	6e-05
2730	6247.43652	0.00998	0.00173	0.00081
2760	6247.44425	0.00634	0.0011	0.00052
2790	6247.47768	0.00943	-.00164	0.00077
2820	6247.44296	0.00693	0.0012	0.00057
2850	6247.47071	0.00618	-.00107	0.0005
2880	6247.47148	0.00652	-.00113	0.00053
2910	6247.44538	0.0009	0.001	7e-05
2940	6247.47029	0.00596	-.00103	0.00049
2970	6247.43262	0.0118	0.00205	0.00097
3000	6247.44723	0.00086	0.00085	7e-05

Table 2: shows the A_{NR} and A_R constants and their errors for given step size (the amount of stars from the model 3 fit)

Scientific Summary for a General Audience

The world around us consists of small particles called protons, neutrons and electrons. Our sun is a star. It produces the light we see from fusion reactions in its core. A fusion reaction occurs when small elements are fused into larger ones. When in you're in the country side and you look up to the sky you can see distant stars. Some of these distant stars are much larger than our sun. Eventually a star will run out of fuel to fuse and it will collapse under its own gravity. The causes an explosion 100000000000000000000 times more powerful than our deadliest nuclear bombs on earth. A white dwarf, neutron star or black hole is what is left of the core. Neutron stars consist of mostly neutrons.

Neutron stars have a diameter roughly 10km across and the mass roughly equal to our sun. This implies it has a density larger than the mass of all humans on earth contained within a volume the size of a sugar cube.

If you got too close to the neutron star then the particles from your body would spaghettified in a long line. Your spagthetti form would hit the star at speeds close to light and your life would flash within an instant.

Most of the heavier elements on earth are thought to originate from the collision of a neutron star with another neutron star. Because humans and most life on planet earth contain those heavier elements within our body, it is safe to say that we are made of dead stars.