calc4 sec03 HW3

16.5

33

32–34 Let $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and $r = |\mathbf{r}|$.

33. Verify each identity.

(a) $\nabla r = \mathbf{r}/r$

(b) $\nabla \times \mathbf{r} = \mathbf{0}$

(c) $\nabla(1/r) = -\mathbf{r}/r^3$

(d) $\nabla \ln r = \mathbf{r}/r^2$

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \text{ and } r = |\mathbf{r}|$$

(a)

$$r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$$

$$\begin{aligned} \nabla r &= \langle r_x, r_y, r_z \rangle \\ &= \left\langle \frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}} \right\rangle \\ &= \frac{1}{\sqrt{x^2+y^2+z^2}} \langle x, y, z \rangle \\ &= \frac{1}{r} \cdot \mathbf{r} \\ &= \mathbf{r}/r \text{ [Q.E.D.]} \end{aligned}$$

(b)

$$\begin{aligned} \nabla \times \mathbf{r} &= \text{curl } \mathbf{r} \\ &= \langle z_y - y_z, x_z - z_x, y_x - x_y \rangle \\ &= \mathbf{0} \text{ [Q.E.D.]} \end{aligned}$$

(c)

$$\begin{aligned}
\nabla(\tfrac{1}{r}) &= \left\langle (\tfrac{1}{r})_x, (\tfrac{1}{r})_y, (\tfrac{1}{r})_z \right\rangle \\
&= -(x^2+y^2+z^2)^{\frac{-3}{2}} \langle x, y, z \rangle \\
&= -(\sqrt{x^2+y^2+z^2})^{-3} \langle x, y, z \rangle \\
&= -r^{-3}\mathbf{r} \\
&= -\mathbf{r}/r^3 \text{ [Q.E.D.]}
\end{aligned}$$

(d)

$$\begin{aligned}
\nabla \ln r &= \left\langle (\ln r)_x, (\ln r)_y, (\ln r)_z \right\rangle \\
&= \tfrac{1}{x^2+y^2+z^2} \langle x, y, z \rangle \\
&= \tfrac{1}{r^2}\mathbf{r} \\
&= \mathbf{r}/r^2 \text{ [Q.E.D.]}
\end{aligned}$$

38

38. Use Green's first identity to show that if f is harmonic on D, and if $f(x, y) = 0$ on the boundary curve C, then $\iint_D |\nabla f|^2 \, dA = 0$. (Assume the same hypotheses as in Exercise 35.)

Green's first identity:

$$\iint_D f(\nabla^2 g) dA = \oint_C f(\nabla g) \cdot \mathbf{n} ds - \iint_D \nabla f \cdot \nabla g dA$$

$\iff$

$$\iint_D \nabla f \cdot \nabla g dA = \oint_C f(\nabla g) \cdot \mathbf{n} ds - \iint_D f(\nabla^2 g) dA$$

Let $f = g = f$

$$f \text{ is harmonic on } D \text{ and } f(x,y) = 0 \forall (x,y) \in C$$

$$\begin{aligned}
\textstyle\iint_D |\nabla f|^2 dA &= \textstyle\iint_D \nabla f \cdot \nabla f dA \\
&= \textstyle\oint_C f(\nabla f) \cdot \mathbf{n} ds - \iint_D f(\nabla^2 f) dA \\
&= \textstyle\oint_C f(\nabla f) \cdot \mathbf{n} ds \; [\nabla^2 f = 0] \\
&= 0 \; [f(P) = 0 \forall P \in C] \text{ [Q.E.D.]}
\end{aligned}$$

40

40. Maxwell's equations relating the electric field $\mathbf{E}$ and magnetic field $\mathbf{H}$ as they vary with time in a region containing no charge and no current can be stated as follows:

$$\operatorname{div} \mathbf{E} = 0 \qquad \operatorname{div} \mathbf{H} = 0$$

$$\operatorname{curl} \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} \qquad \operatorname{curl} \mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}$$

where c is the speed of light. Use these equations to prove the following:

(a) $\nabla \times (\nabla \times \mathbf{E}) = -\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}$

(b) $\nabla \times (\nabla \times \mathbf{H}) = -\frac{1}{c^2}\frac{\partial^2 \mathbf{H}}{\partial t^2}$

(c) $\nabla^2 \mathbf{E} = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}$ [*Hint:* Use Exercise 31.]

(d) $\nabla^2 \mathbf{H} = \frac{1}{c^2}\frac{\partial^2 \mathbf{H}}{\partial t^2}$

(a)

$$\begin{aligned}
\nabla \times (\nabla \times \mathbf{E}) &= \operatorname{curl} \operatorname{curl} \mathbf{E} \\
&= \operatorname{curl}\left(-\tfrac{1}{c}\tfrac{\partial \mathbf{H}}{\partial t}\right) \\
&= -\tfrac{1}{c}\operatorname{curl}\left(\tfrac{\partial \mathbf{H}}{\partial t}\right) \\
&= -\tfrac{1}{c}\tfrac{\partial}{\partial t}\operatorname{curl} \mathbf{H} \\
&= -\tfrac{1}{c}\tfrac{\partial}{\partial t}\left(-\tfrac{1}{c}\tfrac{\partial \mathbf{H}}{\partial t}\right) \\
&= -\tfrac{1}{c^2}\tfrac{\partial}{\partial t}\left(\tfrac{\partial \mathbf{E}}{\partial t}\right) \\
&= -\tfrac{1}{c^2}\tfrac{\partial^2 \mathbf{E}}{\partial t^2} \text{ [Q.E.D.]}
\end{aligned}$$

(b)

$$\begin{aligned}\nabla\times(\nabla\times\mathbf{H}) &= \text{curl curl }\mathbf{H}\\ &= \text{curl }\left(\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}\right)\\ &= \frac{1}{c}\frac{\partial}{\partial t}\text{curl }\mathbf{E}\\ &= \frac{1}{c}\frac{\partial}{\partial t}\left(-\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t}\right)\\ &= -\frac{1}{c^2}\frac{\partial^2\mathbf{H}}{\partial t^2}\ [\text{Q.E.D.}]\end{aligned}$$

(c)

$$\begin{aligned}\nabla^2\mathbf{E} &= \nabla(\text{div }\mathbf{E})-\text{curl curl }\mathbf{E}\ [\text{Exercise 31.}]\\ &= \nabla 0-\left(-\frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}\right)\ [\text{(a)}]\\ &= \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}\ [\text{Q.E.D.}]\end{aligned}$$

(d)

$$\begin{aligned}\nabla^2\mathbf{H} &= \nabla(\text{div }\mathbf{H})-\text{curl curl }\mathbf{H}\ [\text{Exercise 31.}]\\ &= \nabla 0-\left(-\frac{1}{c^2}\frac{\partial^2\mathbf{H}}{\partial t^2}\right)\ [\text{(b)}]\\ &= \frac{1}{c^2}\frac{\partial^2\mathbf{H}}{\partial t^2}\ [\text{Q.E.D.}]\end{aligned}$$

16.6

42

39–50 Find the area of the surface.

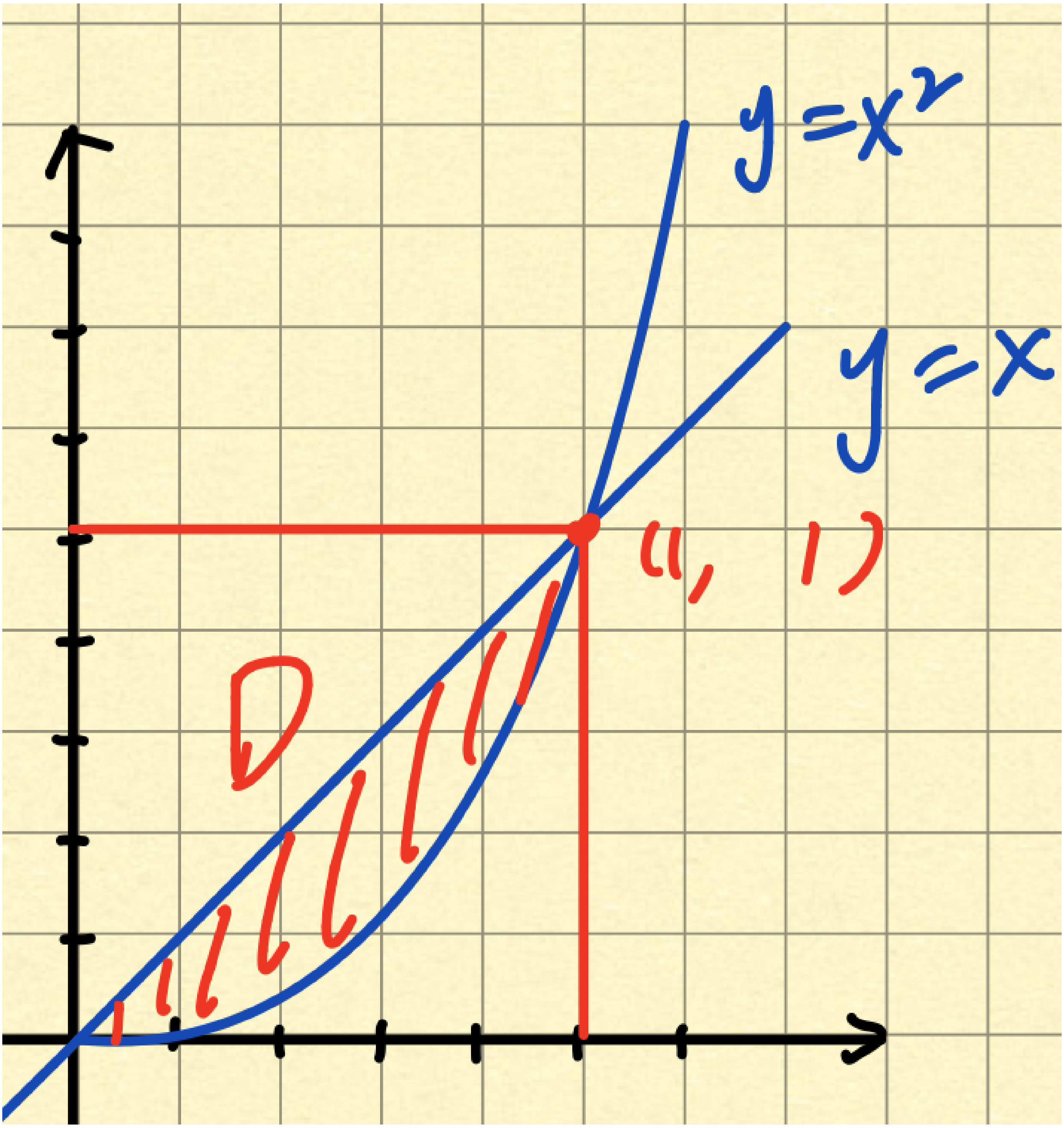
42. The part of the cone $z=\sqrt{x^2+y^2}$ that lies between the plane $y=x$ and the cylinder $y=x^2$

Let $\mathbf{r}$ the parametric equation of the cone

$z=f(x,y)=\sqrt{x^2+y^2}$

$$\mathbf{r}(x,y)=\left\langle x,y,\sqrt{x^2+y^2}\right\rangle$$

$|\mathbf{r}_x\times\mathbf{r}_y|=\sqrt{1+(z_x)^2+(z_y)^2}=\sqrt{1+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}}=\sqrt{2}$



Let $D = \{(x, y) | 0 \leq x \leq 1 \wedge x^2 \leq y \leq x\}$

Let A the surface area of the cone between the cylinders.

$$\begin{aligned} A &= \iint_D \sqrt{2} dA \\ &= \sqrt{2} \int_0^1 \int_{x^2}^x dy dx \\ &= \sqrt{2} \int_0^1 x - x^2 dx \\ &= \sqrt{2} [\tfrac{1}{2} x^2 - \tfrac{1}{3} x^3]_0^1 \\ &= \frac{\sqrt{2}}{6} \end{aligned}$$

63

63. Find the area of the part of the sphere $x^2 + y^2 + z^2 = a^2$ that lies inside the cylinder $x^2 + y^2 = ax$.

Let S the surface of the sphere that lies inside the cylinder

S consists of two surfaces, S_1, the upper one, and S_2, the lower one.

Let $\mathbf{r}$ the parametric equation of S_1

$z = f(x,y) = \sqrt{a^2 - x^2 - y^2}$

$$\mathbf{r} = \left\langle x, y, \sqrt{a^2 - x^2 - y^2} \right\rangle$$

$|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{1 + (z_x)^2 + (z_y)^2} = \frac{a}{z}$

Let $D = \{(x,y) | x^2 + y^2 \le ax\}$

Express $\mathbf{r}$ in polor coordinates

$$\mathbf{r} = \left\langle r\cos\theta, r\sin\theta, \sqrt{a^2 - r^2} \right\rangle$$

$|\mathbf{r}_r \times \mathbf{r}_\theta| = \frac{a}{\sqrt{a^2-r^2}}$

$D = \{(r,\theta) | 0 \le r \le a\cos\theta \wedge -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\}$

$A(S) = 2A(S_1)$

$$\begin{aligned} A(S) &= 2\iint_D \frac{a}{\sqrt{a^2-x^2-y^2}} dA \\ &= 2a\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{a\cos\theta} \frac{r}{\sqrt{a^2-r^2}} dr d\theta \end{aligned}$$

Let $u = a^2 - r^2 \implies du = -2rdr$
$(u(0), u(a\cos\theta)) = (a^2, a^2\sin^2\theta)$

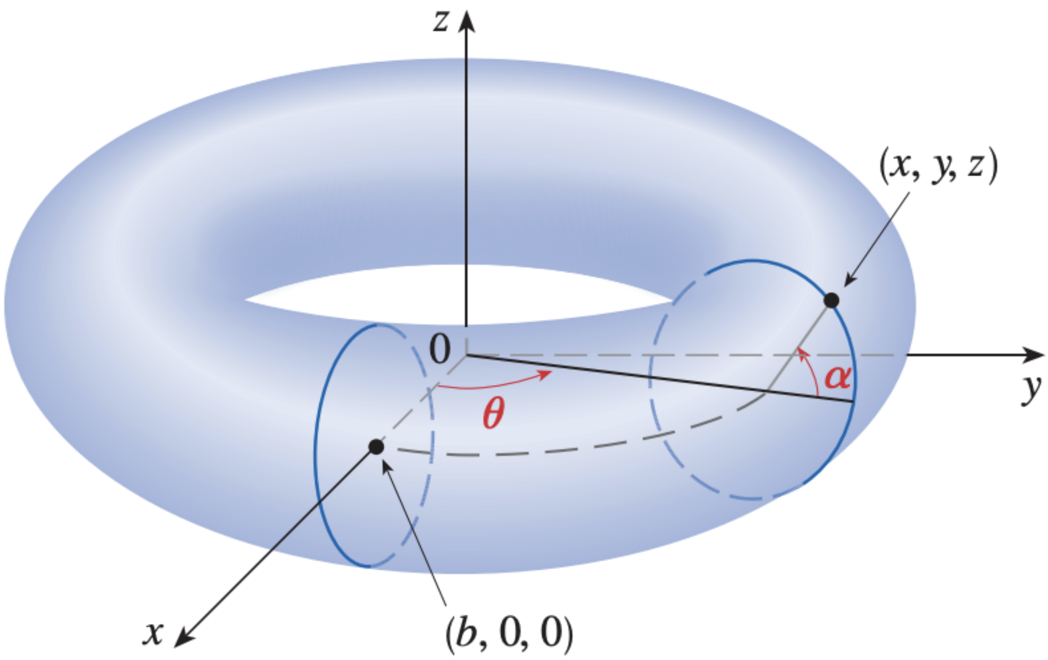
$$\begin{aligned} A(S) &= -a\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{a^2}^{a\sin^2\theta} \frac{1}{\sqrt{u}} du d\theta \\ &= -a\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\sqrt{u}\Big|_{a^2}^{a^2\sin^2\theta} d\theta \\ &= -2a^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta - 1 d\theta \\ &= -2a^2[\sin\theta - \theta]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\ &= 2a^2(\pi - 2) \end{aligned}$$

64(a)(c)

64. (a) Find a parametric representation for the torus obtained by rotating about the z-axis the circle in the xz-plane with center $(b, 0, 0)$ and radius $a < b$. [*Hint:* Take as parameters the angles θ and α shown in the figure.]

(b) Use the parametric equations found in part (a) to graph the torus for several values of a and b.

(c) Use the parametric representation from part (a) to find the surface area of the torus.



(a)

Let $\mathbf{r}_1$ the parametric function of the small circle

$$\mathbf{r}_1 = \langle a\cos\theta + b, 0, a\sin\theta\rangle$$

where $0 \le \theta \le 2\pi$

Let $\mathbf{r}_2$ the parametric function of the torus

$$\mathbf{r} = \langle (a\cos\theta + b)\cos\phi, (a\cos\theta + b)\sin\phi, a\sin\theta\rangle$$

where $0 \le \theta \le 2\pi$ and $0 \le \phi \le 2\pi$

(c)

Let $D = \{(\theta, \phi) | 0 \le \theta \le 2\pi \wedge 0 \le \phi \le 2\pi\}$

Let A the surface area of the torus

$\mathbf{r}_\theta = \langle -a\cos\phi\sin\theta, -a\sin\phi\sin\theta, a\cos\theta\rangle$

$\mathbf{r}_\phi = \langle -(a\cos\theta + b)\sin\phi, (a\cos\theta + b)\cos\phi, 0\rangle$

$$\begin{array}{rcc} A & = & \iint_D |\mathbf{r}_\theta \times \mathbf{r}_\phi| dA \\ & = & \iint_D a|a\cos\theta + b| d\theta d\phi \\ & = & a\int_0^{2\pi}\int_0^{2\pi} |a\cos\theta + b| d\theta d\phi \\ & = & 2a\pi\int_0^{2\pi} |a\cos\theta + b| d\theta \\ & = & 2a\pi\int_0^{2\pi} [a\cos\theta + b] d\theta \text{ [a<b]} \\ & = & 2a\pi[a\sin\theta + b\theta]_0^{2\pi} \\ & = & 2a\pi(2b\pi) \\ & = & 4ab\pi^2 \end{array}$$