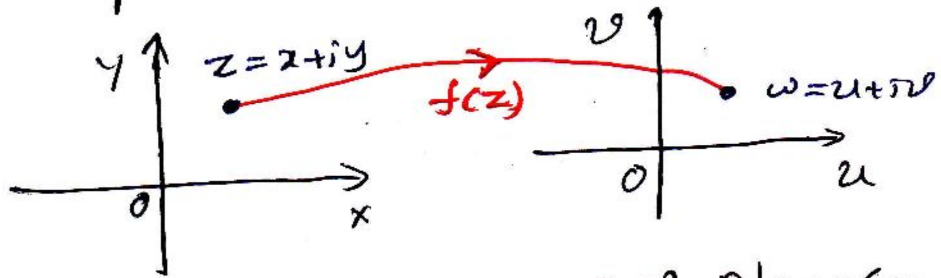
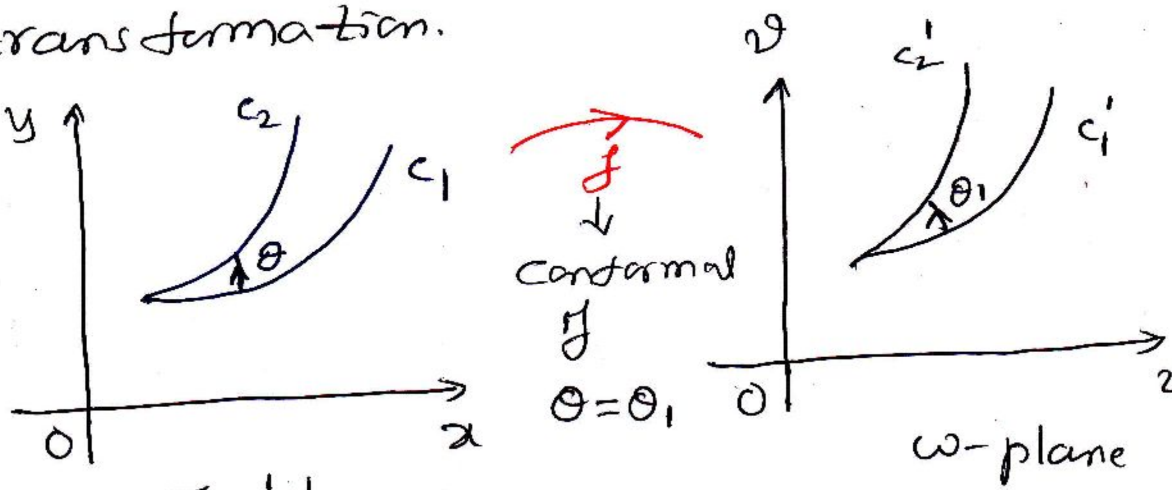
Conformal Mapping (Transformation)

A complex function $\omega = f(z)$ transforms (maps) a point $z = x+iy$ on xy plane (z-plane) to a point $\omega = u+iv$ on uv plane (ω-plane).



xy-plane (z-plane) uv-plane (ω-plane)

If a transformation preserves the angle between any two curves both in magnitude and sense then it is called a conformal transformation.



Z-plane ω-plane

Transformation: $W = z^2$

$$w = z^2 \Rightarrow u + iv = (x+iy)^2 = (x^2-y^2) + i(2xy)$$

$$\Rightarrow u = x^2 - y^2 \ \& \ v = 2xy \quad \rightarrow (1)$$

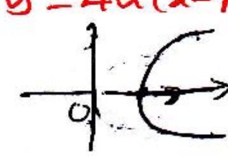
Case 1: Line parallel to y-axis ie $x = c_1$

From (1) $u = c_1^2 - y^2 \ \& \ v = 2c_1 y$

$$\Rightarrow y = v/2c_1 \quad \therefore u = c_1^2 - v^2/4c_1^2$$

$$\Rightarrow v^2/4c_1^2 = c_1^2 - u \Rightarrow v^2 = -4c_1^2(u - c_1^2)$$

$y^2 = 4a(x-h)$



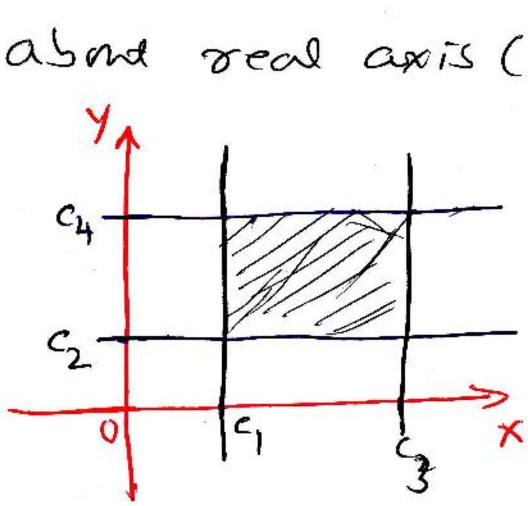
which is a parabola with vertex $(c_1^2, 0)$ and symmetrical about real axis (u-axis)

Case 2: Line parallel to x-axis ie $y = c_2$

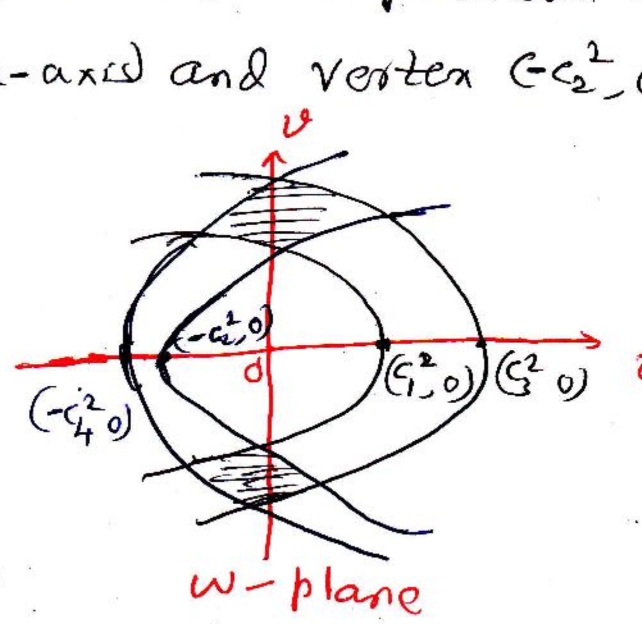
From (1) $u = x^2 - c_2^2 \ \& \ v = 2c_2 x$

$$\Rightarrow x = v/2c_2 \quad \therefore u = v^2/4c_2^2 - c_2^2 \Rightarrow \frac{v^2}{4c_2^2} = u + c_2^2$$

$\Rightarrow v^2 = 4c_2^2(u + c_2^2)$, which is a parabola symmetrical about real axis (u-axis) and vertex $(-c_2^2, 0)$



z- plane



w- plane

Transformation: $w=e^z$

$w=e^z \Rightarrow u+iv=e^{x+iy}=e^x(\cos y+i\sin y)$

$\Rightarrow u=e^x\cos y,\ v=e^x\sin y$

$\Rightarrow u^2+v^2=e^{2x}$ — (1) & $v/u=\tan y$ → (2)

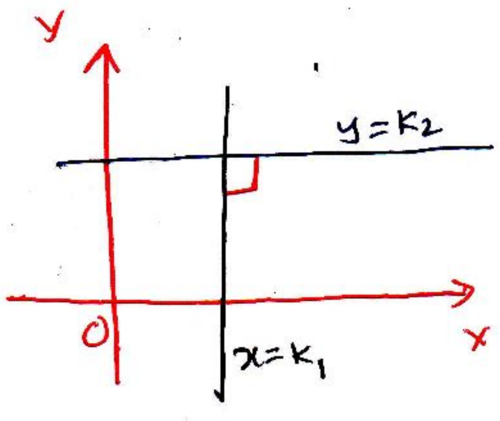
Case (I) line parallel to y-axis i.e. $x=k_1$

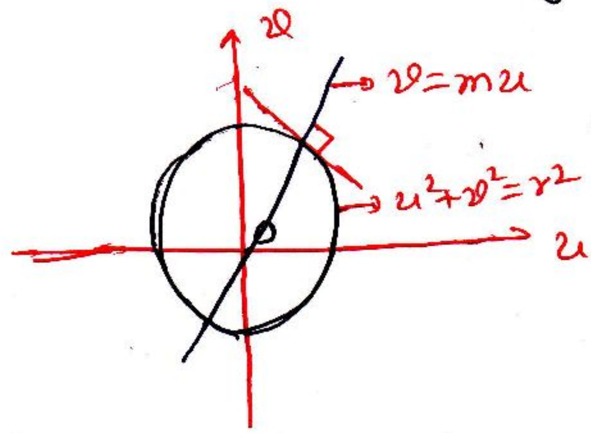
From (1) $u^2+v^2=e^{2k_1}=r^2$ (say) which is a circle with center origin & radius r

Case (II) line parallel to x-axis i.e. $y=k_2$

From (2) $v/u=\tan k_2 \Rightarrow v=(\tan k_2)u \Rightarrow v=mu$ | $y=mx$

which is a straight line passing thro' origin.





Transformation: $w = z + a^2/z \quad (z \neq 0)$

put $z = re^{i\theta}$

$w = z + a^2/z \Rightarrow u + iv = re^{i\theta} + \frac{a^2}{r} e^{-i\theta}$

$\Rightarrow u + iv = r(\cos\theta + i\sin\theta) + \frac{a^2}{r}(\cos\theta - i\sin\theta)$

$\Rightarrow \quad u = \left(r + a^2/r\right)\cos\theta, \quad v = \left(r - a^2/r\right)\sin\theta \longrightarrow (1)$

Case (i) Let $r = K_1$ (circle with radius K_1 & center origin)

From (1)

$$\frac{u}{K + a^2/K} = \cos\theta \quad \& \quad \frac{v}{K - a^2/K} = \sin\theta$$

$$\Rightarrow \frac{u^2}{(K + a^2/K)^2} + \frac{v^2}{(K - a^2/K)^2} = 1 \Rightarrow \frac{u^2}{A^2} + \frac{v^2}{B^2} = 1$$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

which is an ellipse on w-plane with foci

$(\pm\sqrt{A^2 - B^2}, 0) = (\pm 2a, 0)$

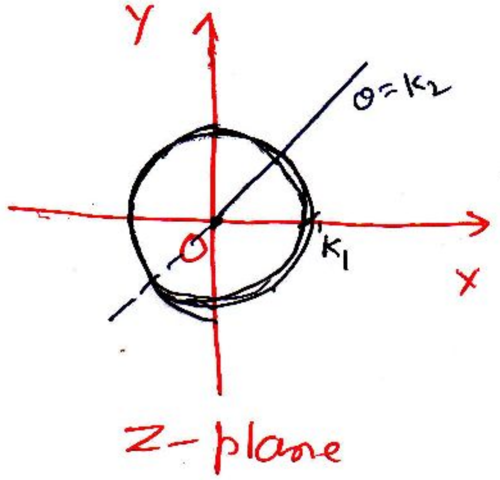
Case (ii) Let $\theta = K_2$ (straight line passing thro' origin)

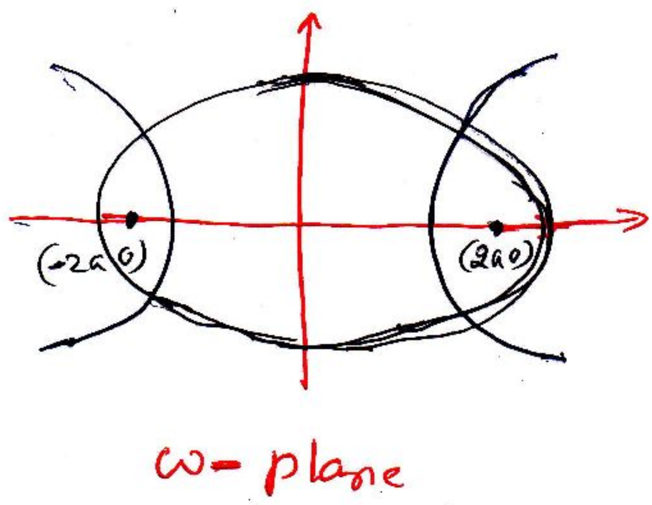
From (1)

$$\frac{u}{\cos K_2} = r + a^2/r \quad \& \quad \frac{v}{\sin K_2} = r - a^2/r$$

$$\Rightarrow \frac{u^2}{\cos^2 K_2} - \frac{v^2}{\sin^2 K_2} = 4a^2 \Rightarrow \frac{u^2}{(2a\cos K_2)^2} - \frac{v^2}{(2a\sin K_2)^2} = 1$$

$\Rightarrow \frac{u^2}{A^2} - \frac{v^2}{B^2} = 1$, which is an hyperbola on w-plane with foci $(\pm\sqrt{A^2 + B^2}, 0) = (\pm 2a, 0)$





Bilinear Transformation

The transformation $\omega = \dfrac{az+b}{cz+d}$ where a, b, c, d are real/complex constants such that $ad-bc \neq 0$ is called bilinear transformation.

Note: If z_1, z_2, z_3 are mapped to $\omega_1, \omega_2, \omega_3$ under bilinear transformation $\omega = \dfrac{az+b}{cz+d}$ then we have

$$\frac{(\omega-\omega_1)(\omega_2-\omega_3)}{(\omega_1-\omega_2)(\omega_3-\omega)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

Examples on bilinear transformation

Ex ① Find the bilinear transformation which maps $1, i, -1$ onto $i, 0, -i$

Sol^n Let $z_1 = 1,\ z_2 = i,\ z_3 = -1,\ \omega_1 = i,\ \omega_2 = 0,\ \omega_3 = -i$

we have

$$\frac{(\omega-\omega_1)(\omega_2-\omega_3)}{(\omega_1-\omega_2)(\omega_3-\omega)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\Rightarrow \frac{(\omega-i)(0+\cancel{i})}{(i-\cancel{0})(-i-\omega)} = \frac{(z-1)(i+1)}{(1-i)(-1-z)}$$

$$\Rightarrow (\omega-i)(1-i)(-1-z) = (z-1)(i+1)(-i-\omega)$$

$$\Rightarrow (\omega-i)(-1-z+i+iz) = (z-1)(1-i\omega-i-\omega)$$

$$\Rightarrow -\omega-\cancel{\omega z}+\cancel{i\omega}+iz\omega+\cancel{i}+iz+1+\cancel{z}$$
$$= \cancel{z}-i\omega z-iz-\cancel{z\omega}-1+\cancel{i\omega}+\cancel{i}+\omega$$

$$\Rightarrow -\omega+iz\omega+iz+i\omega z+iz+2-\omega = 0$$

$$\Rightarrow -2\omega+2iz\omega+2iz+2 = 0$$

$$\Rightarrow \omega(2iz-2) = -2-2iz$$

$$\Rightarrow \omega = \frac{-2(1+iz)}{2(iz-1)} \qquad \therefore \boxed{\omega = \frac{1+iz}{1-iz}}$$

Ex ② Find the bilinear transformation which maps the points $1, i, -1$ onto $2, i, -2$. Also find invariant points of the transformation.

Sol^n Let $z_1 = 1$, $z_2 = i$, $z_3 = -1$,

$w_1 = 2$, $w_2 = i$, $w_3 = -2$

we have

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\Rightarrow \frac{(w-2)(i+2)}{(2-i)(-2-w)} = \frac{(z-1)(i+1)}{(1-i)(-1-z)}$$

$$\Rightarrow \frac{w-2}{-(w+2)} = \frac{z-1}{-(z+1)} \frac{(i+1)(2-i)}{(1-i)(i+2)}$$

$$\Rightarrow \frac{w-2}{w+2} = \frac{z-1}{z+1} \frac{(i+3)}{(3-i)}$$

$$\Rightarrow (w-2)(z+1)(3-i) = (z-1)(i+3)(w+2)$$

$$\Rightarrow (w-2)(3z-iz+3-i) = (z-1)(iw+2i+3w+6)$$

$$\Rightarrow 3\cancel{zw} - izw + 3w - \cancel{iw} - 6z + 2\cancel{iz} - \cancel{6} + 2i$$

$$= izw + 2\cancel{iz} + 3\cancel{wz} + 6z - \cancel{iw} - 2i - 3w - \cancel{6}$$

$$\Rightarrow -2izw + 6w - 12z + 4i = 0$$

$$\Rightarrow 2w(3-iz) = 12z - 4i$$

$$\Rightarrow \boxed{w = \frac{6z-2i}{3-iz}}$$

For invarient points, put $w = z$, we get

$$z = \frac{6z-2i}{3-iz} \Rightarrow -iz^2 + 3z = 6z - 2i$$

$$\Rightarrow -iz^2 - 3z + 2i = 0 \Rightarrow z = \frac{+3 \pm \sqrt{9-8}}{-2i} = -2/i, -1/i$$

$\therefore$ Invariant points: $2i$, i

(25)

Ex (3) Find the bilinear transformation that maps the points $0, i, \infty$ onto the points $1, -i, -1$ respectively.

Soln Let $z_1 = 0, \; z_2 = i, \; z_3 = \infty$

$\omega_1 = 1, \; \omega_2 = -i, \; \omega_3 = -1$

we have

$$\frac{(\omega-\omega_1)(\omega_2-\omega_3)}{(\omega_1-\omega_2)(\omega_3-\omega)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\Rightarrow \frac{(\omega-\omega_1)(\omega_2-\omega_3)}{(\omega_1-\omega_2)(\omega_3-\omega)} = \frac{(z-z_1)\cancel{z_3}(z_2/z_3-1)}{(z_1-z_2)\cancel{z_3}(1-z/z_3)}$$

$$\Rightarrow \frac{(\omega-1)(-i+1)}{(1+i)(-1-\omega)} = \frac{(z-0)(0-1)}{(0-i)(1-0)}$$

$$\Rightarrow \frac{\omega-1}{-(1+\omega)} = \frac{-z}{-i}\,\frac{(1+i)}{(1-i)} = \frac{z}{i}\,\frac{1+i}{1-i}\times\frac{1+i}{1+i}$$

$$\Rightarrow \frac{\omega-1}{\omega+1} = -\frac{z}{i}\,\frac{2i}{2} = -z$$

$$\Rightarrow (\omega-1) = -z(\omega+1) \Rightarrow \omega-1+z\omega+z=0$$

$$\Rightarrow \omega(1+z) = 1-z$$

$$\Rightarrow \boxed{\omega = \frac{1-z}{1+z}}$$

Ex (4) Find the bilinear transformation that maps $z_1 = i$, $z_2 = 1$, $z_3 = -1$ onto $w_1 = 1$, $w_2 = 0$, $w_3 = \infty$

Sol^n. we have

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\Rightarrow \frac{(w-w_1)(w_2/w_3-1)}{(w_1-w_2)(1-w/w_3)} = \frac{(z-i)\,2}{(i-1)(-1-z)}$$

$$\Rightarrow \frac{(w-1)(0-1)}{(1-0)(1-0)} = \frac{2(z-i)}{(1-i)(z+1)}$$

$$\Rightarrow (1-w)(1-i)(z+1) = 2(z-i)$$

$$\Rightarrow (1-w)(z+1-iz-i) = 2z-2i$$

$$\Rightarrow z+1-iz-i-w(z+1-iz-i)-2z+2i=0$$

$$\Rightarrow -w\{(1-i)z+(1-i)\} = z+iz+i-1$$

$$\Rightarrow -w(1-i)(z+1) = z(1+i)-(i+1)$$

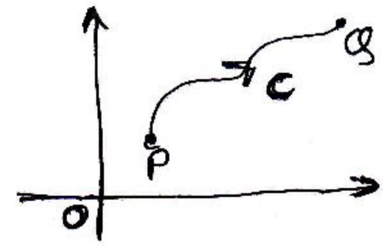
$$\Rightarrow -w = \frac{(z-1)(i+1)}{(z+1)(1-i)} = \frac{z-1}{z+1}\,\frac{i+1}{1-i} \times \frac{1+i}{1+i} = \frac{z-1}{z+1}\,\frac{2i}{2}$$

$$\Rightarrow w = \frac{-i(z-1)}{z+1}$$

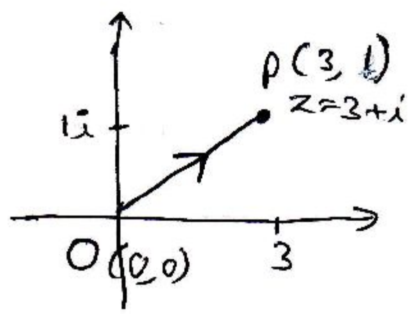
$$\Rightarrow \boxed{w = \frac{-iz+i}{z+1}}$$

Complex Line Integral

Consider a continuous function $f(z)$ defined over a curve C, the line integral of $f(z)$ over C is given by $\int_C f(z)\,dz$



Ex ① Evaluate $\int_C z^2\,dz$ along a straight line from $z=0$ to $z=3+i$



Sol^n Eq^n of OP: $y-y_1=m(x-x_1)$

$$\Rightarrow y-0=\frac{1-0}{3-0}(x-0) \Rightarrow y=x/3$$

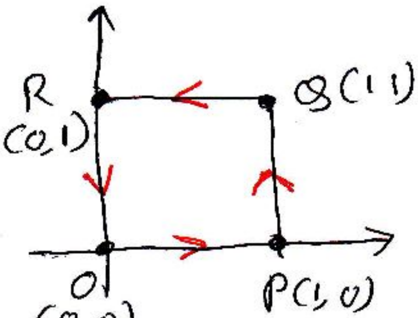
$$\int_C z^2\,dz=\int_C (x+iy)^2(dx+i\,dy)=\int_{x=0}^{3}(x+ix/3)^2(dx+i/3\,dx)$$

$$=(1+i/3)^3\int_0^3 x^2\,dx=(1+i-1/3-i/27)\left[x^3/3\right]_0^3$$

$$=\left(2/3+\frac{26}{27}i\right)(9-0)=6+\frac{26}{3}i$$

Ex ② Evaluate $\int_C |z|^2\,dz$ where C is the square with vertices $(0,0)$ $(1,0)$, $(1,1)$ $(0,1)$

Sol^n $z = x+iy$, $|z|^2 = x^2+y^2$
$dz = dx + i\,dy$



Along $\overrightarrow{OP}$: $y=0 \therefore dy=0$, $x \to 0$ to 1
Along $\overrightarrow{PQ}$: $x=1 \therefore dx=0$, $y \to 0$ to 1
Along $\overrightarrow{QR}$: $y=1 \therefore dy=0$ $x \to 1$ to 0
Along $\overrightarrow{RO}$: $x=0 \therefore dx=0$ $y \to 1$ to 0

$$\therefore \int_C |z|^2 dz = \int_{OP} |z|^2 dz + \int_{PQ} |z|^2 dz + \int_{QR} |z|^2 dz + \int_{RO} |z|^2 dz$$

$$= \int_{x=0}^{1} x^2\,dx + \int_{y=0}^{1} (1+y^2)\,i\,dy + \int_{x=1}^{0} (x^2+1)\,dx + \int_{y=1}^{0} y^2\,i\,dy$$

$$= \left[\frac{x^3}{3}\right]_0^1 + i\left[y+y^3/3\right]_0^1 + \left[x^3/3+x\right]_1^0 + i\left[y^3/3\right]_1^0$$

$$= 1/3 + i(1+1/3) + (0-4/3) + i(0-1/3)$$

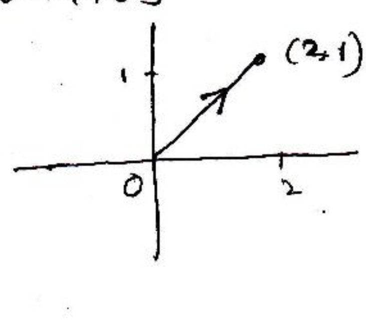
$$= -1+i$$

Ex③ Evaluate $\int_0^{2+i} (\bar{z})^2 dz$ along

ⓐ the line $x=2y$ ⓑ the real axis upto 2 and then vertically to $2+i$.

Sol^n $z = x+iy$, $\bar{z} = x-iy$, $dz = dx+i\,dy$

ⓐ Along $x=2y$: $dx = 2\,dy$
$y \to 0$ to 1



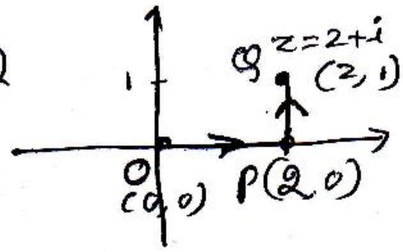
$$\int_0^{2+i} (\bar{z})^2 dz = \int_0^{2+i} (x-iy)^2 (dx+idy)$$

$$= \int_{y=0}^{1} (2y-iy)^2 (2dy+idy) = (2-i)^2(2+i)\int_0^1 y^2 dy$$

$$= (3-4i)(2+i)\left[\frac{y^3}{3}\right]_0^1 = (10-5i)\left(\tfrac{1}{3}-0\right) = \frac{5}{3}(2-i)$$

(b) Along OP: $y=0$, $dy=0$, $x \to 0$ to 2

Along PQ: $x=2$, $dx=0$, $y \to 0$ to 1



$$\therefore \int_0^{2+i} (\bar{z})^2 dz = \int_{OP} (\bar{z})^2 dz + \int_{PQ} (\bar{z})^2 dz$$

$$= \int_{x=0}^{2} x^2 dx + \int_{y=0}^{1} (2-iy)^2\, i\, dy$$

$$= \left(\frac{x^3}{3}\right)_0^2 + i\int_0^1 (4-4iy-y^2)\, dy$$

$$= \left(\frac{8}{3}-0\right) + i\left[4y - 4i\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1$$

$$= \frac{8}{3} + i\left(4-2i-\frac{1}{3}\right)$$

$$= \frac{8}{3} + \frac{11}{3}i + 2$$

$$= \frac{14}{3} + \frac{11}{3}i = \frac{1}{3}(14+11i)$$

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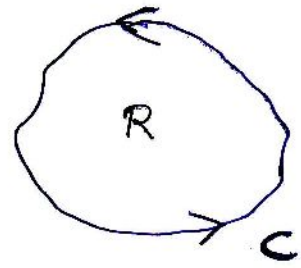
Cauchy's Theorem:

If $f(z)$ is analytic at all points inside and on a simple closed curve C then

$$\int_C f(z)\,dz = 0$$



Proof Let $f(z) = u + iv$, $dz = dx + idy$

$$\int_C f(z)\,dz = \int_C (u+iv)(dx+i\,dy)$$

$$= \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy) \quad \text{Green's Theorem}$$

$$= \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$

$\downarrow$ CR $\qquad\qquad \downarrow$ CR

$$= \iint_R \left(-\cancel{\frac{\partial v}{\partial x}} + \cancel{\frac{\partial v}{\partial x}}\right) dx\,dy + i\iint_R \left(\cancel{\frac{\partial v}{\partial y}} - \cancel{\frac{\partial v}{\partial y}}\right) dx\,dy$$

$$= 0$$

Green's Thm.

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

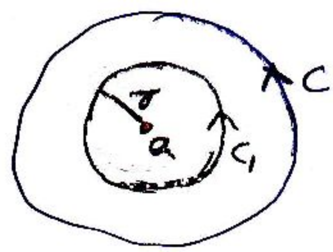
Corollary If C_1 & C_2 are two simple closed curves such that C_2 lies entirely within C_1 and if $f(z)$ is analytic on C_1, C_2 & the region bounded by C_1 & C_2

then $$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

Cauchy's Integral Formula

If $f(z)$ is analytic inside & on a closed curve 'c' and if 'a' is any point within 'c' then

$$f(a) = \frac{1}{2\pi i} \int_c \frac{f(z)}{z-a}\, dz$$



Proof Let c_1 be a circle $|z-a| = r$.

The function $\frac{f(z)}{z-a}$ is analytic on c, c_1 and in the region bounded by c & c_1

$$\therefore \int_c \frac{f(z)}{z-a}\, dz = \int_{c_1} \frac{f(z)}{z-a}\, dz$$

$|z-a| = r$
$\Rightarrow z-a = re^{i\theta}$
$0 \le \theta \le 2\pi$

$$= \int_0^{2\pi} \frac{f(a+re^{i\theta})}{re^{i\theta}}\, ire^{i\theta}\, d\theta$$

$$= i\int_0^{2\pi} f(a+re^{i\theta})\, d\theta$$

This holds for any $r>0$, hence as $r \to 0$, we get

$$= i\int_0^{2\pi} f(a)\, d\theta$$

$$= i f(a) \left[\theta\right]_0^{2\pi} = 2\pi i f(a)$$

$$\therefore \boxed{f(a) = \frac{1}{2\pi i} \int_c \frac{f(z)}{z-a}\, dz}$$

Note: Differentiate ① w.r.t 'a', we get

$$f'(a) = \frac{1}{2\pi i} \int_C \frac{\partial}{\partial a}\left(\frac{f(z)}{z-a}\right) dz$$

$$\Rightarrow f'(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)^2}\, dz$$

similarly,

$$f''(a) = \frac{2!}{2\pi i} \int_C \frac{f(z)}{(z-a)^3}\, dz$$

$$f'''(a) = \frac{3!}{2\pi i} \int_C \frac{f(z)}{(z-a)^4}\, dz$$

$$f^n(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}}\, dz$$

Remember

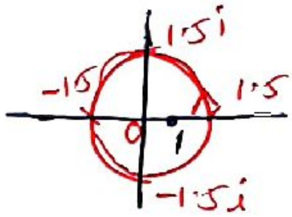
① $\int_C f(z)\, dz = 0$, if $f(z)$ is analytic on & wit

② $\int_C \frac{f(z)}{z-a}\, dz = 2\pi i\, f(a)$, if $a \in C$

③ $\int_C \frac{f(z)}{(z-a)^n}\, dz = \frac{2\pi i}{(n-1)!} f^{n-1}(a)$

Ex ① Evaluate $\int_C \frac{z^2+z-1}{z-1}\,dz$, where C is the circle (i) $|z|=1.5$ (ii) $|z|=0.5$

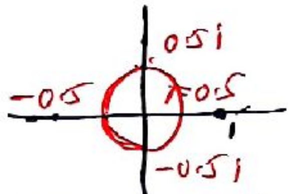
Solⁿ (i) $|z|=1.5$, $a=1$



Let $f(z)=z^2+z-1$

$$\therefore \int_C \frac{z^2+z-1}{z-1}\,dz = \int_C \frac{f(z)}{z-1}\,dz = 2\pi i\, f(1) = 2\pi i$$

(ii) $|z|=0.5$, $a=1$

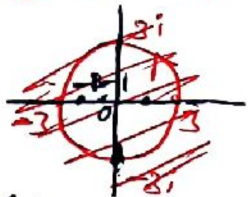


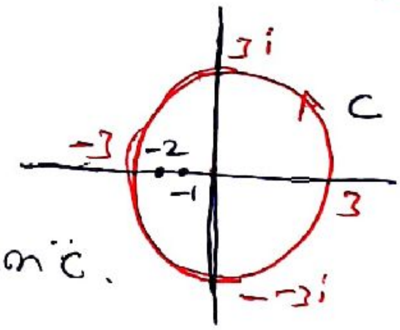
Here $a=1$ lies outside the circle $|z|=0.5$

$$\therefore \int_C \frac{z^2+z-1}{z-1}\,dz = 0 \text{ (by Cauchy's theorem)}$$

Ex ② Evaluate $\int_C \frac{e^{2z}}{(z+1)(z+2)}\,dz$ where $C: |z|=3$.

Solⁿ $|z|=3 \Rightarrow$





Here $a=-1$ & -2 both lies within C.

Consider $\frac{1}{(z+1)(z+2)} = \frac{1}{z+1} - \frac{1}{z+2}$

Also let $f(z)=e^{2z}$

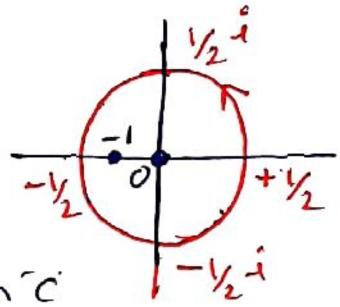
$$\therefore \int_C \frac{e^{2z}}{(z+1)(z+2)}\,dz = \int_C \frac{f(z)}{z+1}\,dz - \int_C \frac{f(z)}{z+2}\,dz$$

$$= 2\pi i\, f(-1) - 2\pi i\, f(-2)$$

$$= 2\pi i\,(e^{-2} - e^{-4})$$

Ex ③ Evaluate $\int_C \frac{2z+1}{z^2+2z}\, dz, \quad C : |z| = \frac{1}{2}$

$\underline{\text{Sol}^n}$ $\quad \frac{2z+1}{z^2+2z} = \frac{2z+1}{z(z+1)} \qquad |z| = \frac{1}{2} \Rightarrow$



Here $a = 0$ & $a = -1$ both lies within C

$\therefore \frac{1}{z(z+1)} = \frac{1}{z} - \frac{1}{z+1}$ & $f(z) = 2z+1$

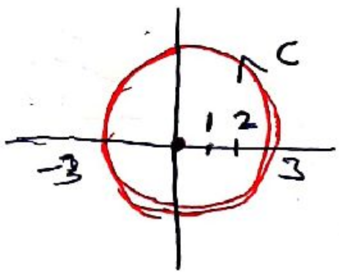
$$\therefore \int_C \frac{2z+1}{z^2+2z}\, dz = \int_C \frac{f(z)}{z}\, dz - \int_C \frac{f(z)}{z+1}\, dz$$

$$= 2\pi i\, f(0) - 2\pi i\, f(-1)$$

$$= 2\pi i\,(1+1) = 4\pi i \; //$$

Ex ④ Evaluate $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\, dz, \quad C : |z| = 3$

$\underline{\text{Sol}^n}$ $\quad |z| = 3 \Rightarrow$



Here $a = 1$ & $a = 2$ lies within C

Let $f(z) = \sin \pi z^2 + \cos \pi z^2$,

Also, $\frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2}$

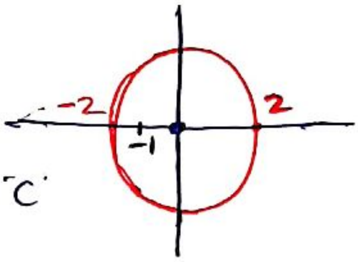
$$\therefore \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\, dz = \int_C \frac{f(z)}{z-2}\, dz - \int_C \frac{f(z)}{z-1}\, dz$$

$$= 2\pi i\, f(2) - 2\pi i\, f(1)$$

$$= 4\pi i$$

Ex ⑤ Evaluate $\int_C \frac{e^{2z}}{(z+1)^4}\, dz$, $C: |z|=2$

Sol^n $|z|=2 \Rightarrow$



Here $a=-1$ lies within 'C'

Let $f(z) = e^{2z}$

$$\therefore \int_C \frac{e^{2z}}{(z+1)^4}\, dz = \frac{2\pi i}{3!} f'''(-1)$$

$f'(z) = 2e^{2z}$

$f''(z) = 4e^{2z}$

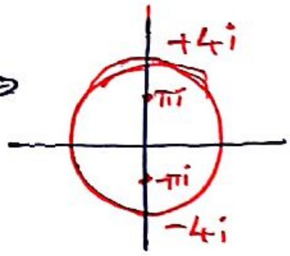
$f'''(z) = 8e^{2z}$

$$= \frac{2\pi i}{6} \times 8 e^{-2}$$

$$= \frac{8\pi i}{3} e^{-2} \quad //$$

Ex ⑥ Evaluate $\int_C \frac{e^{z}}{z^2+\pi^2}\, dz$, $C: |z|=4$

Sol^n $\int_C \frac{e^z}{z^2+\pi^2} dz = \int_C \frac{e^z}{(z+\pi i)(z-\pi i)} dz$ $\quad |z|=4 \Rightarrow$



Here $a = \pi i$ & $-\pi i$ belong to C.

Let $f(z) = e^z$ & also $\frac{1}{(z+\pi i)(z-\pi i)} = \frac{1/2\pi i}{z-\pi i} - \frac{1/2\pi i}{z+\pi i}$

$$\therefore \int_C \frac{e^z}{(z+\pi i)(z-\pi i)} dz = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-\pi i} dz - \frac{1}{2\pi i}\int_C \frac{f(z)}{z+\pi i}$$

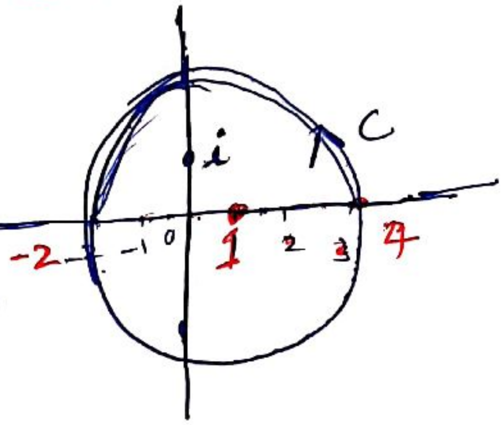
$$= \frac{1}{2\pi i} \cdot 2\pi i f(\pi i) - \frac{1}{2\pi i} 2\pi i\, f(-\pi i)$$

$$= e^{i\pi} - e^{-i\pi} = (\cos\pi + i\sin\pi) - (\cos\pi - i\sin\pi)$$

$$= 0$$

Ex ⑦ Evaluate $\int_C \frac{z^3-2z+1}{(z-i)^2}\,dz$; $C : |z-1|=3$

Sol^n $|z-1|=3 \Rightarrow$



Here $a=i$ lies within the C

Let $f(z) = z^3-2z+1$

$\therefore \int_C \frac{z^3-2z+1}{(z-i)^2}\,dz = \int_C \frac{f(z)}{(z-i)^2}\,dz$

$f'(z) = 3z^2-2$

$f'(i) = -3-2 = -5$

$= \frac{2\pi i}{1!} f'(i)$

$= 2\pi i \times -5$

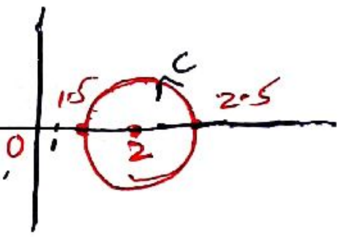
$= -10\pi i$

Ex ⑧ Evaluate $\int_C \frac{z}{z^2-3z+2}\,dz$, $C : |z-2| = 1/2$

Sol^n $\int_C \frac{z}{z^2-3z+2}\,dz = \int_C \frac{z}{(z-1)(z-2)}\,dz$

$|z-2| = 1/2 \Rightarrow$

Here $a=2$ lies within 'c'



Let $f(z) = z/(z-1)$

$\therefore \int_C \frac{z}{(z-1)(z-2)}\,dz = \int_C \frac{f(z)}{z-2}\,dz = 2\pi i f(2) = 2\pi i \times 2 = 4\pi i$

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