* Poisson distribution

Let $x(t)$ be no. of taxis arriving at a particular location in time t.

Let i be the point of carpool/ride-sharing position.

- Necessary conditions of taxi sharing:

1.) $x(0) = 0$

2.) P(having taxi in $t \to (t+h)$) is independent of t i.e $x(t)$ is independent stationary increment process

3.) $x(t)$ satisfies:

$$P[x(t+h) - x(t) = 1] = \lambda h + o(h)$$

$$P[x(t+h) - x(t) \geq 2] = o(h)$$

where $o(h)$ is limit applicable function where its value is less than h when h tends to 0.

Now, probability that there are n taxis at car pool position in time t will be,

$$P_n(t) = P\{X(t)=n\} = P\{X(t)-X(0)=n\}$$

From (2),(3);

$$P_0(t+h) = P\{X(t+h)=0\}$$
$$= P\{X(t)-X(0)=0\} \cdot P\{X(t+h)-X(t)=0\}$$
$$= P_0(t)\left[1-\lambda h+2o(h)\right]$$

As $h \to 0$,

$$P_0(t+h) = P_0(t)\left[1-\lambda h\right]$$

$$\therefore P_0(t+h) - P_0(t) = -\lambda P_0(t) h$$

$$\therefore \frac{P_0(t+h) - P_0(t)}{h} = -\lambda P_0(t)$$

$$\therefore \frac{dP_0(t)}{dt} = -\lambda P_0(t)$$

$$\therefore \frac{dP_0(t)}{P_0(t)} = -\lambda\, dt$$

Integrating both sides,

$\therefore \log_e(P_0(t)) = -\lambda t$

$\therefore P_0(t) = e^{-\lambda t}$

Similarly, $P_1(t) = \lambda t \, e^{-\lambda t}$

Now, we'll apply principle of mathematical induction, we'll have;

$$\boxed{\therefore P_k = \frac{e^{-\lambda x} (\lambda x)^k}{k!}} \longrightarrow \text{Poisson process}$$