Cluster State Quantum Computation

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April 25, 2022

Abstract

Cluster state quantum computation is one type of measurement-based quantum computation, also known as one-way quantum computation. This paper discusses the methodologies first described by Raussendorf and Briegel in their cluster state approach to quantum computation, describing two-qubit gates, the equivalence of the cluster state to the circuit model, the graphical representation of circuits, and possible physical implementations of the model.

1 Introduction

Raussendorf and Briegel's 'Cluster State' quantum computation [RB01] is one type of measurement-based quantum computation that differs from the standard quantum circuit model by only using single-qubit transformations and measurements to perform quantum computation. The sole coherent resource used in this form of quantum computation is memory, which stores the initially formed cluster resource state. The emergence of cluster state computing allowed us to get more insight into the requirements of quantum computation. Until its conceptualization, conventional wisdom dictated that quantum computation could only be performed using coherent, superposition-preserving unitary operations [NC11].

A cluster state is an entangled state of multiple qubits which is created prior to performing the quantum computation and makes up the entire resource used in the computation. As the cluster state is acted upon by one-qubit measurements, all entanglement gets destroyed, such that the cluster state may only be used once in performing the quantum computation, and a new cluster state must be created for multiple rounds of the computation. Thus, this is also known as "one-way" quantum computing.

Since the cluster state is the resource into which information is written, and then processed and read-out, it is commonly referred to as the "substrate" for quantum computation.

The single-qubit transforms and measurements on the input qubits feedforward the input state through the graph to the end qubit, destroying the previous state held by preceding qubits. This way, the information contained in the input qubit at the beginning of a graph state is iterated through the intermediate qubits, and output at the end qubit after being rotated and transformed.

In terms of error correction, the cluster state itself is a stabilizer state, existing as the +1 eigenstate of the Pauli operators X and Z related by the relation given in (3). Additionally, due to its graphical representation, method of removal of qubits upon measurement, cluster state quantum computation is easily translated to surface codes used in quantum error correction. Furthermore, fault tolerant cluster state quantum computation is possible with higher dimensional cluster states [RHG06].

2 Discussion and Methodology

The quantum circuit model is analogous to the circuit model in classical computing, using quantum gates acting on qubits. The wires represent the qubits, with gates acting upon them. The circuit diagram represents the passage of time, and the gate operations are carried out sequentially. Thus, the quantum circuit model employs unitary gates that perform operations on the qubits. Since all gates used are unitary, the circuit as a whole performs a joint unitary transformation on the n input qubits. In such circuits, the number of gates required to synthesize such a generic unitary transform on the qubits scale exponentially with the number of qubits the unitary acts on.

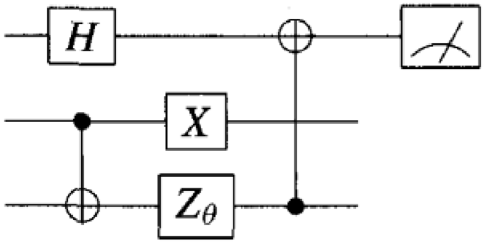

Figure 1: An example of a quantum circuit diagram [Nie06]

Cluster states are created between two-state particles in any system that allows Ising-type nearest-neighbor interactions and may be realized in a 2D or a 3D lattice structure. The 2D lattice structure is that of a square lattice, described by the graph state $G(C, E_c)$, whereas the 3D lattice structure is a cubic lattice with the set of edges E_c described by $\{(a,b)|a,b \subset C, b \subset nbgh(a)\}$. [RB01]

Here, the two states $|0>=|0>_{z,a}\ and|1>=|1>_{z,a}$ are the eigenstates of the Z operator and form the computational basis. The cluster state is characterized by a set of eigenvalue equations:

$$\sigma_x^a \bigotimes_{a' \in ngbh(a)} \sigma_z^{a'} \left|\phi\right\rangle_C = \pm \left|\phi\right\rangle_C \tag{1}$$

$$K^(a) \left|\phi_{\{K\}}\right\rangle_C = (-1)^{K_a} \left|\phi_{\{K\}}\right\rangle_C \tag{2}$$

$$K^{(a)} = \sigma_x^{(a)} \bigotimes_{b \in ngbh(a)} \sigma_z^{(b)} \tag{3}$$

Where (3) is the correlation operator for the set of eigenvalue equations given in (1).

A cluster state of a system of qubits may be prepared by following the set of eigenvalue equations as described in (1), since doing so would create a complete set $\|C\|$ of independent and commuting variables. The distribution of the qubits in the lattice will determine the eigenvalues according to the equations. Here, nbgh(a) specifies the sites of the other qubits interacting with the qubit a belong to C.

A more implementable approach is to first prepare a product state of $\left|+\right\rangle_C = \bigotimes_{b \in C} \left|+\right\rangle_a$ after which the unitary transform given in (4) is carried out on the $\left|+\right\rangle$ state. Thus, a genuine resource state is created wherein this cluster state can be used to perform any quantum computation.

If, instead of initializing all qubits in the $\left|+\right\rangle$ state, the first qubit was encoded with information, thus placing it in the $\left|\psi\right\rangle_{in}$ state, the cluster state formed after transformation by (4) would not be a genuine resource, but rather an information-initialized resource which would only be used for that specific computation. That is to say, creating a cluster state in the former method allows us to entangle input any information post-creation of the cluster, whereas creating the cluster state using the second scheme only allows us to compute on the information already entangled with the rest of the qubits. This way, the second scheme is not as powerful as the first one [RB01].

$$S^{(C)} = \prod_{a,b \in C | b-a \in \gamma_d} S^{ab} \tag{4}$$

Processing the quantum information depends on the order in which the qubits are measured. Qubits horizontally adjacent to each other represent measurement, whereas vertical connections represent two qubit gates such as CNOT or CPhase.

Measuring a subset of qubits in the cluster will project any two qubits a, a' $\in C$ into the Bell basis. The basis of measurement of a qubit depends on the basis used in the preceding measurements. If the qubits are initialized in the $|+\rangle$ basis, first an H transform will be applied to the qubits before a Z measurement to yield results in the computational basis. As measurements are carried out, the amount of entanglement present in the entire process decreases. The final output is reached upon successive measurements and qubit gates applied to the cluster, which destroys the entanglement present in the system.

2.1 Simulating quantum circuits using cluster-states

The main idea behind proving quantum circuits can be simulated using the cluster-state model is using 'half-teleportation' or a one-qubit teleportation to carry out the computation.

As shown in Fig.2., since the U_z rotation commutes with the CPhase gate, they commute and we can thus perform the U_z rotation first. This is equivalent to performing an X measurement on a rotated input state. Here, the measurement on qubit 1 $= U_z^\dagger X U_z$ is any arbitrary rotation, which consists of an X rotation conjugated with rotation in the Z-axis. This produces the rotation $U_z = e^{i\alpha Z}$.

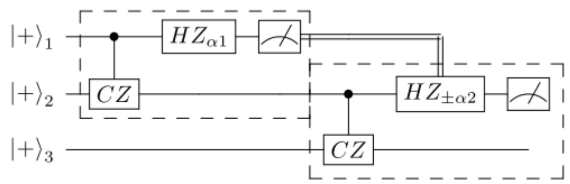

Figure 2: The half-teleportation circuits can be 'stacked', building the cluster state as the operations are carried out [ND05]. This models the gate application via teleportation.

Although the first qubit gets measured, there is no loss of quantum information, since the state of the second qubit is related by a known unitary transform to the input $|\psi\rangle$.

Thus, by this we prove that changing the angle of measurement changes the basis of measurement, which varies the unitary transform that affects the second qubit, without destroying any quantum information encoded into the first qubit.

We can write any quantum circuit comprising of gates of the form HZ, or CPhase or SWAP gates,and one which has all inputs in the $|+\rangle$ state into the equivalent circuit shown in Fig.2. Then, we can commute the operators of the form X^m completely to the left part of the circuit, and use the feedforward technique to define the signs for the $Z_{\pm\alpha}$ term such that the resulting circuit has Z_α rotations [ND05].

This way, such a quantum circuit with HZ or CPhase or SWAP gates can be simulated using cluster states.

2.2 Visualization and Experimental Implementation

2D clusters states can be represented as a lattice of qubits, with label numbers indicating the order of measurements, and the labels themselves indicating the basis of measurement. The empty circles on the right in Fig. 3. represent the output qubits [Nie04].

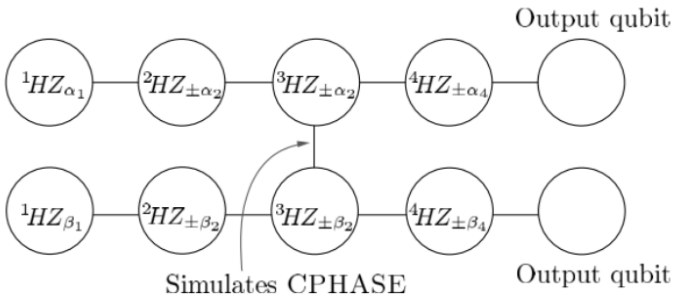

Figure 3: Graphical representation of a cluster state with labels specifying the order of measurements [Nie04]

Cluster state quantum computation makes use of Ising-type interactions between the particles that comprise the qubits in the scheme. Possible implementations of the one-way scheme are using neutral atoms stored in micropotentials, which have Ising-type interactions brought about by controlled collisions between atoms, or arrays of capacitively coupled quantum dots [RB01].

In [Wal+05], the authors use nonlinear optics to directly produce a 4-photon cluster state using mode and polarization entangled outputs of nonlinear spontaneous parametric down conversion (SPDC). The group was able to perform a two-qubit Grover search algorithm, thus proving that cluster state quantum computation is possible and ideally suited for optical systems. The challenge in realizing cluster states in optical systems is the lack of Ising-type interactions between photons. While [Wal+05] used nonlinear components as well as linear optical components, the Knill Laflamme Milburn protocol lays out non-deterministic methods to implement CZ gates in a cluster-state system. The probabilistic nature of linear optical gates, however, makes cluster state preparation highly inefficient in a photonic system.

2.3 Cluster state as a substrate for quantum computation

In Fig.4. we have assumed each site in this lattice is occupied by a qubit. The shaded region represents a network N of unmeasured particles which are thus, still part of the initialized entangled cluster state. The unshaded region with the circles and the arrows represent the measured qubits which are a subset of the cluster C\N.

This way, the state $|\phi\rangle_C$ is projected into a tensor product of the unmeasured and the measured qubits $|\mu\rangle_{C\setminus N} \bigotimes |\phi'\rangle_N$ is related to the cluster state of unmeasured qubits $|\phi\rangle_N$ depending on the the set of measurement results mu of the measured qubits [RBB03].

Thus, for a sub-cluster N,$|\phi\rangle_N$ still satisfies the set of eigenvalue equations previously described. This is an important result, since it shows that non-unitary one-qubit measurements carried out on a subset of qubits part of the cluster will form an independent

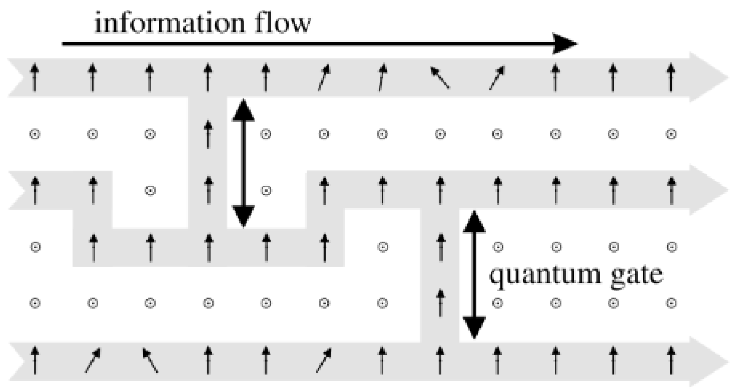

Figure 4: The shaded region shows the information flow in the cluster state.The vertical connections represent 2-qubit gates (here, CNOT), whereas the smaller arrows pointing upwards represent X rotations. The circles with the dots are Z rotations, which remove the qubit upon measurement. The angular arrows represent arbitrary rotations [RB01].

set of measured qubits that are essentially "removed" from the cluster, keeping the remaining sub-cluster still entangled. This sub-cluster of qubits still follow the eigenvalue equations that lay out their commuting properties and their interactions with each other as described [RBB03].

For example, a quantum logic circuit can be implemented on a cluster state in the following way: A cluster state is formed by preparing a chain of qubits in the $|+\rangle$ state, with the first qubit prepared in the $|\psi\rangle_{in}$ state. This chain of interacting qubits are then entangled using the unitary entangling operation S. This way, the state encoded in qubit 1 is now delocalized over the cluster of the total qubits.

Performing sigma x measurements on qubits 1 to n-1 (thus, in the x eigenbasis) will transfer the state psi in to the site n of the last qubit, which is the output to be read out. This way, the resulting state becomes a tensor product of all the qubits with the first n-1 being measurement states and the last one having the psi in state initially encoded to the first qubit.

This way, the state $|\psi\rangle_{out}$, at qubit n, is related to the input state $|\psi\rangle_{in}$ by unitary transformations carried out on the qubits 1 to n-1. Thus, the state has been transformed due to the measurements carried out on the cluster, which has also allowed it to be transferred from qubit 1 to qubit n.

2.4 Proof of Universality

In the quantum circuit model, universality is proved by the implementation of single qubit rotations, two-qubit entangling gates such as CNOT and CPhase, and the Hadamard gate. Similarly, here we show the implementation of these gates in the cluster-state model, thus proving the universality of cluster state quantum computation.

2.4.1 Single Qubit Rotations

An arbitrary rotation gate $U_{Rotation}$ can be realized using a chain of qubits. Taking n=5 qubits for example, we can consider the Euler representation of the rotation gate, where the rotations are about the X and Z axes, as given in (5).

$$U_{Rot}[\xi,\eta,\zeta] = U_x[\zeta]U_z[\eta]U_x[\xi] \tag{5}$$

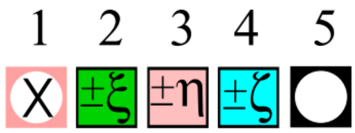

Figure 5: An arbitrary single qubit rotation can be specified using the angles ξ, η, ζ to change the basis of measurement on qubits 2, 3, 4, resulting in an arbitrary rotation A which is measured at qubit 5 [RBB03].

The cluster state is first prepared by initializing all the qubits in the $|+\rangle$ state, and then entangling them using the S unitary operation described in (4). Then, the input state $|\psi\rangle_{in}$ can be rotated by performing measurements on the first four qubits, with the state $|\psi\rangle_{in}$ being swapped with qubit 5 simultaneously. Here, the measurement occurs with the first four qubits being measured in the bases (6) where the measurement outcomes $s_j \in \{0,1\}$for j from 1 to 4. When $s_j = 0$, qubit j is projected into the first basis state described by (6), $B_j(\phi_j)$, where the measurement angle ϕ_j is the angle between the measurement direction for qubit j and the positive x-axis, since all the basis states of (6) lie on the x-y plane on the Bloch sphere. Changing this angle changes the basis state, which changes the basis the measurement is performed in.

$$B_j(\phi_j) = \{\frac{|0\rangle_j + e^{i|\phi\rangle_j}|1\rangle_j}{2}, \frac{|0\rangle_j - e^{i|\phi\rangle_j}|1\rangle_j}{2}\} \tag{6}$$

This way, upon measurement of the first four qubits in the bases described by iterating over j given in the equation above, the rotation transforms to (7).

$$U'_{Rot}[\xi,\eta,\zeta] = U_x[\zeta]U_z[\eta]U_x[\xi] \tag{7}$$

Equation (8) describes the extra rotations that occur on the measured qubits at the output state. Thus, since the basis of measurement of a qubit follows the basis of measurement used for the preceding qubits, the extra rotations can be accounted for by adjusting the measurement basis used for the final readout.

$$U_{\sum,Rot} = \sigma_x^{s_2+s_4}\sigma_z^{s_1+s_3} \tag{8}$$

2.4.2 CNOT Gate

A CNOT gate can be realized using a cluster of 15 qubits as shown in the figure.

We first prepare the input state $|\psi\rangle$ as given in (9), after which the entangling unitary transform S (4) is performed.

$$|\psi_{in}\rangle_{C_{15}} = |\psi_{in}\rangle_{1,9} \bigotimes (\bigotimes_{i\in C_{15}} |+\rangle_i) \tag{9}$$

Other than the output qubits 7 and 15, measurements of the other qubits can be performed simultaneously, with qubits 1, 9, 10, 11, 13, 14 measured in the x basis and

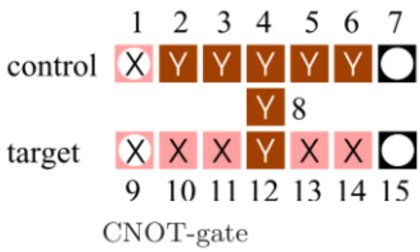

Figure 6: A CNOT gate can be performed using 15 qubits. However,this CNOT gate cannot be performed on non-nearest neighbours. A non-nearest neighbour CNOT gate is further discussed in [RBB03].

qubits 2-6,8,12 measured in the y basis. These measurements allow a CNOT operation to be realized as follows:

$$U'_{CNOT} = U_{\sum,CNOT} CNOT(c,t) \tag{10}$$

The byproduct operator $U_{\sum,CNOT}$ is the product of rotations about the x and z axes depending on the measurements.Qubits 2,3,4,5,6,7,8,and 12 are measured in the Y basis, as a product of X and Z rotations. Qubits 1, 9, 10, 11, 13, 14 are measured in the X basis and form the target of the CNOT.

2.4.3 Hadamard Gate

As shown in Fig.7, Hadamard gate can be realized using 5 qubits. Here, the state $|\psi_{in}\rangle$ is initialized with the input qubit being in the $|\psi_{in}\rangle$ state and the other four qubits being in the $|+\rangle$ state. The qubits are entangled using S given in (4), and the first four qubits are measured with the first qubit being measured in the x eigenbasis and qubits 2-4 being measured in the y eigenbasis.

Figure 7: A Hadamard gate is performed using 5 qubits, with qubits 2-4 measured in the y eigenbasis, and qubit 1 being measured in the x eigenbasis [RBB03].

Upon measuring the qubits 2-4 in the y-eigenbasis, the measurement outcomes of $s_{2-4} \in \{0,1\}$ are thus obtained which results in the eigenvalue equations:

$$\sigma_x^{(1)} \sigma_z^{(5)} |\phi\rangle_{C(H)} = (-1)^{s_3+s_4} |\phi\rangle_{C(H)} \tag{11}$$

and,

$$\sigma_z^{(1)} \sigma_x^{(5)} |\phi\rangle_{C(H)} = (-1)^{s_2+s_3} |\phi\rangle_{C(H)} \tag{12}$$

2.4.4 $\pi/2$ Phase Gate

This gate can also be realized with 5 qubits, as shown in Fig.8. Similarly to how the Hadamard gate was prepared, first the state $|\psi_{in}\rangle$ is initialized with the input qubit being in the $|\psi_{in}\rangle$ state and the other four qubits being in the $|+\rangle$ state. Here, however, qubits 1,2, and 4 are measured in the x eigenbasis and qubit 3 is measured in the y eigenbasis.

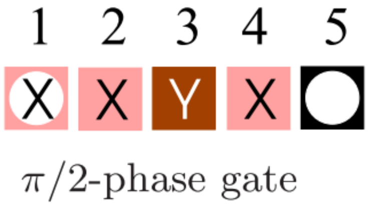

Figure 8: A $\pi/2$ Phase gate is performed using 5 qubits. [RBB03].

This way, we prove the universality of the cluster state model of quantum computation, as it is possible to realize single and two qubit transforms (CNOT and CPhase), as well as the Hadamard transform.

3 Resource Consumption

To run a specific quantum algorithm on a cluster state, the cluster needs to be of a certain size. Thus, the spatial resources (S_R) used in one-way quantum computation are the number of qubits required to form a cluster state $|\phi\rangle_C$, i.e. $S_R = \|C\|$. Because all operations are derived from single-qubit measurements, it makes up one unit of operational resource. Thus, the operational resources for cluster state quantum computation are $O \leq S_R$. The interaction time T of Ising-type interaction between qubits is independent of the cluster size, and thus, clusters of any size take the same amount of time to be realized [RBB03].

The CNOT gate described in the universality proof is is not feasible for universal quantum computation because it cannot act on non-neighbouring qubits. [RBB03] describes a non-neighbouring qubit CNOT operation which also encompasses a CZ gate. Furthermore, since the measurement bases for the Hadamard, $\pi/2$ phase, and CNOT (neighbouring) employ $\sigma_y and \sigma_x$ measurements only, they can be simultaneously realized in the first measurement round, regardless of their location in the network. Thus, circuits containing only Clifford operations can be realized in a single step. This reduces the temporal requirement to execute a circuit [RBB03].

4 Conclusion

The conceptualization of cluster state quantum computation provided a valuable insight into quantum computation – that it is possible to quantum compute by using only non-unitary measurements and single-qubit transformations, which contradicted the conventional wisdom that computation could only be performed using coherent, superposition-preserving unitary transforms.

Through this review, we have summarized the way of attaining universality in cluster state computation, as well as possible physical implementations of the method.

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