Qudit Codes

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April 2022

Abstract

Higher-dimensional quantum codes can be formed as an extension of qubit-codes, which themselves are often analogues of classical linear codes. In this paper, we provide a preliminary introduction to generalized modular-qudit stabilizer codes by defining elementary operations and Clifford group elements.We compare the code construction of qubits, prime-dimensional qudits, and composite-dimensional qudits using the [[5,1,3]], highlighting the ease of extending qubits to prime-dimensional qudits, the non-integer nature of modular-qudit codes of composite dimension, and an explanation of Galois-qudit codes of composite dimension. Then, we discuss a method to measure syndromes in prime-dimensional qudit systems using the discrete Fourier Transform. The 3-qutrit code is taken as an example of a qudit-CSS code.

1 Introduction

A qudit is the generalization of a qubit to a d-dimensional system. Experimentally, qudits may be implemented by using different degrees of freedom of a photon, or the energy levels of atoms or trapped ions. Quantum error correction schemes using qubits employ linear codes which are extended from classical error correcting codes. Much like classical theory makes use of redundancy, quantum error correction with qubits encodes quantum information of a qubit, which maps to a Hilbert space H_2, into a larger Hilbert space H_D for $D > 2$, wherein $H_D = H_2^{\bigotimes n}$, where n refers to the number of physical qubits used in the code. This is also known as block encoding.

However, by using a d-dimensional variation of the qubit (qudits), we could potentially encode quantum information into space $C : H_2 \longrightarrow H_D \neq H_2^{\bigotimes n}$, the limit being that of an infinite D, referred to as a Bosonic Mode. Thus, quantum error correction with qudits should allow us to fully utilize the space to encode quantum information without requiring as many redundant physical qubits, thus reducing the resources needed to error-correct [8]. In this paper, we focus on explaining stabilizer and CSS codes in higher dimensions, and the use of the discrete Fourier Transform to extract eigenvalues from a syndrome.

The stabilizer formalism, which is widely used in quantum error correction with qubits, can be extended to d-dimensions, as can other quantum linear codes (such as the Calderbank-Shor-Steane code), which themselves have been derived from classical error correction. Prime-dimensional qudit codes have an advantage over composite-dimensional qudit codes, as they extend more easily from the qubit analogues. Composite-dimensional qudit codes may encode non-integer number of qudits, and modular composite-dimensional qudit codes have non-unique generators, which makes construction of useful quantum error correcting codes in composite dimensions difficult.

Galois-qudit codes of composite dimensions, however, can be easily extended from their underlying prime-dimensional codes. A characteristic of odd prime-dimensional codes (dimension p) is the equivalence of elements of the Pauli group acting on n qudits, which are all modulo-p. This results in syndrome measurements which are not constrained to ± 1 eigenvalues; instead, measurement results in an eigenvalue in powers of ω. We can use the Discrete Fourier Transform to deduce the syndrome of a qudit in an odd prime-dimensional system.

2 Qubits to Qudits

The stabilizer formalism put [5] has a construction analogous to classical linear codes and extends classical linear codes to the quantum domain and allows a more compact description of quantum error-correcting codes than state vector descriptions of the codes.

The stabilizer group S is some Abelian subgroup of the group G_n generated by the elementary operations I,X,Y,Z such that $S|\psi\rangle = |\psi\rangle$. These operations comprise the Pauli group, which is generated by X, Z, and their tensor products, where n is the number of qubits the operators are acting on. Its codes are part of the joint eigenspace of the commuting operators that form the stabilizer group. For an $[[n, k, d]]$ code, the stabilizer code $C(S)$ will have dimension 2^k, with S having 2^{n-k} elements. We commonly look at stabilizer groups that commute under Pauli operations, but other extensions for qubits such as XS and XP stabilizer codes also exist. We will only be looking at stabilizer codes that commute under the Pauli group in this paper.

Errors occur when an operator in G anticommutes with an operator in the stabilizer group S, taking the codeword(s) from the codespace to some space orthogonal to the codespace. By measuring the stabilizers, we get the syndromes, which we map back to the errors and by applying the same erroneous operator, we bring the codeword back into the codespace, thus correcting the error. This is the gist of how quantum error correction occurs in 2 dimensional systems using the stabilizer formalism.

In D-dimensions, the stabilizer group S is still an Abelian subgroup of the Pauli group of operators acting on n-qudits [4] . Generalized X and Z operators acting on a single qubit for **any** dimension $D > 2$ can be given by (1), with addition being modulo-D [3].

$$Z = \sum_{j=0}^{D-1} \omega^j |j\rangle \langle j|, \qquad X = \sum_{j=0}^{D-1} \omega^j |j\rangle \langle j+1| \tag{1}$$

where X and Z operators thus defined satisfy the following properties:

$$X^D = Z^D = I, \qquad XZ = \omega ZX, \qquad \omega = e^{2\pi i/D} \tag{2}$$

When acting on n qudits, the elementary operations will be Z_i and X_i, with i as the indexing term used to specify the qudit. The Pauli group P_n may be formed under multiplication of operators acting on n qudits with the equation (3):

$$\omega^\lambda X^x Z^z := \omega^\lambda X_1^{x^1} Z_1^{z^1} \bigotimes X_2^{x^2} Z_2^{z^2} \bigotimes ... \bigotimes X_n^{x^n} Z_n^{z^n} \tag{3}$$

The Pauli group P_n thus defined is not Abelian; any two elements of this group commute up to a phase $\omega^{\lambda_{1,2}}$.

In order to perform operations in higher-dimensional systems, we also require a Clifford group, which is a group of operators that map Pauli operators to the Pauli group upon conjugation. The qudit-equivalent of the Hadamard gate is the Fourier gate given by [3] [4]:

$$F := \frac{1}{\sqrt{D}} \sum_{j=0}^{D-1} \omega^{jk} \left| j \right\rangle \left\langle k \right| \tag{4}$$

The Controlled-NOT gate, acting on any arbitrary qudits a, b, generalizes to the SUM gate in higher dimensions.

$$CNOT_{ab} := \sum_{j=0}^{D-1} \left| j \right\rangle \left\langle j \right|_a \bigotimes X_b^j = \bigotimes_{j,k=0}^{D-1} \left| j \right\rangle \left\langle j \right|_a \bigotimes \left| k \right\rangle \left\langle k + kj \right|_b \tag{5}$$

Furthermore, the SWAP and CPhase gates are defined as:

$$SWAP_{ab} = CNOT_{ab}(CNOT_{ab})^{\dagger} CNOT_{ab}(F_a^2 \bigotimes I_b) \tag{6}$$

$$CP_{ab} = \sum_{j=0}^{D-1} \left| j \right\rangle \left\langle j \right|_a \bigotimes Z_b^j = \sum_{j,k=0}^{D-1} \omega^{jk} \left| j \right\rangle \left| j \right\rangle_a \bigotimes \left| k \right\rangle \left\langle k \right|_b \tag{7}$$

2.1 Prime vs. Composite Dimensions

Most work in higher-dimensional quantum error correction has been in prime dimensions since the derivation of higher-dimensional quantum codes from their classical analogues is pretty straightforward. On the other hand, composite dimensions bring forth a difficulty in maintaining the stabilizer formalism, by requiring additional generator elements to maintain commutation relations. Here, we take the example of the [[5,1,3]] code and compare it between the qubit case (D=2), prime-dimensional qudit case, and *odd* composite dimensional qudit case (d=9). The d=9 code is an example of a Galois-qudit code, in which the composite-dimensional code, where the composite dimension q may be written as $q = p^m$, inherits the properties of the prime-dimensional (p) code [2]. An odd-composite dimension that may be written as a power of a prime dimension factor $q = p^m$ has a similar code structure to the p-dimensional code.

For an $[[n, k, d]]$ qubit code, n physical qubits encode k logical ones, and there are r = n - k stabilizer generators. The other elements of the stabilizer group S are products of generators. The rate of this code is k/n.

For the $[[5, 1, 3]_2$ code, the stabilizer generators are given by:

$$\begin{bmatrix} X & Z & Z & X & I \\ I & X & Z & Z & X \\ X & I & X & Z & Z \\ Z & X & I & X & Z \end{bmatrix} \tag{8}$$

The code space is the simultaneous +1 eigenvalues of the elements of the stabilizer group, which is why an erroneous operation, which anticommutes with an element of the stabilizer group, and gives us the eigenvalue of -1. This way, since the eigenvalues are always ± 1, the error syndromes in a qubit code can also be represented as a binary vector of length r, that is, the error syndrome $\in (Z_2)^r$

In prime dimensions (> 2) p, the phase ω is equivalent to the other elements of the Pauli group by also being modulo-p. This is in contrast to the qubit-case, in which phase $P|x\rangle = i^x|x\rangle$, which is not modulo-4 like the other elements of the qubit Pauli group. The phase for the prime-qudit code can be given by $P|x\rangle = \omega^{x(x-1)}|x\rangle$, making the phase quadratic.

$$\begin{bmatrix} X & Z & Z^{-1} & X^{-1} & I \\ I & X & Z & Z^{-1} & X^{-1} \\ X^{-1} & I & X & Z & Z^{-1} \\ Z^{-1} & X^{-1} & I & X & Z \end{bmatrix} \tag{9}$$

In the prime-dimensional qudit case, the stabilizer codes are still an Abelian subgroup of the Pauli group acting on n qudits, but the errors do not commute or anticommute with the stabilizer elements. Instead, the errors commute to some power of the phase ω. Thus, the stabilizer generators also contain powers of ω.To ensure the generators form an Abelian group, inverse X and Z operations are required. This kind of 'modular' qudit code detects up to d-1 errors and corrects erasure errors on up to d-1 qudits.

For a qudit code of a composite dimension q, the stabilizer group is defined under the Heisenberg-Weyl group instead of the Pauli group [3]. Unlike in the odd-prime dimensional case, some Pauli operators acting on the stabilizer group have different orders, and thus are not equivalent. For example, for a 6-dimensional qudit, the order of some Pauli elements may be 2 or 3, making elements nonequivalent and not modulo-q. Because of this reason, the number of encoded qudits k is no longer equal to the difference between the number of physical qudits and the number of generators. In fact, the number of generators is no longer a unique property of the stabilizer group, as the composite dimension may be factored into a prime dimension p of power m($q = p^m$).

This way, $k \neq n - r$, and r is not necessarily an integer. By the relation between q, p, and m given above, k = n - r/m is the number of logical qudits one can attain in a composite-dimensional code. If r is not an integer, the number of logical qudits will also not be an integer number. Due to this, the composite-dimensional $[[n, k, d]]_q$ notation is often written instead as $[[n, K, d]]_q$, with $k = log_q K$.

In the $[[5, 1, 3]]_9$ example, the generators are:

$$\begin{bmatrix} X & Z & Z^{-1} & X^{-1} & I \\ X^{\alpha} & Z^{\alpha} & Z^{-\alpha} & X^{-\alpha} & I \\ I & X & Z & Z^{-1} & X^{-1} \\ I & X^{\alpha} & Z^{\alpha} & Z^{-\alpha} & X^{-\alpha} \\ X^{-1} & I & X & Z & Z^{-1} \\ X^{-\alpha} & I & X^{\alpha} & Z^{\alpha} & Z^{-\alpha} \\ Z^{-1} & X^{-1} & I & X & Z \\ Z^{-\alpha} & X^{-\alpha} & I & X^{\alpha} & Z^{\alpha} \end{bmatrix} \tag{10}$$

Thus, in this case we need additional generators of powers of α as the dimension 9 = 3^2. The error syndrome here is not a vector over GF(q=9), but a vector of length 'r' over $\mathbb{Z}_p$. This way, composite dimensional codes which may be written as a even powers of prime dimensions are merely a manifestation of the underlying prime-dimensional code.

The comparison of the [[5,1,3]] code for qubits, prime-dimensional qudits, and composite-dimensional qudits shows the dissimilarities in the structure of the stabilizers formed for these higher-dimensions. Furthermore, it highlights the ease with which odd, prime-dimensional qudit codes can be extended from classical n-ary (and their quantum analogue qubit codes), as well as the lack of structure of composite-dimensional codes.

2.2 Eigenvalue extraction using DFT

[3] described elementary operators for qudits of **any** dimension. Considering only prime-dimensional qudits now, a Discrete Fourier Transform can be used to transform a state $|0\rangle$ to a superposition of all basis states of the qudit code:

$$DFT\,|0\rangle = \frac{1}{\sqrt{p}} \sum_{l \in \mathbb{F}_p} |l\rangle_p \tag{11}$$

[7] describe using DFT and $DFT^\dagger$ on a stabilizer whose eigenvalue we want to obtain for an erroneous state to transfer the eigenvalue-based information to a syndrome subqudit. Here, a syndrome subqudit s_I is generated by applying the DFT on a state $|0\rangle_p$, which has a form described by (11). The state of the erroneous state with the syndrome subqudit becomes:

$$E\,|\psi\rangle\,|s_I\rangle = (I_p^{\bigotimes mn} \bigotimes DFT_p)(E\,|\psi\rangle\,|0\rangle_p) \tag{12}$$

The syndrome is computed using the operator described in (12) where S_1^j acts on the codeword subqudits and $|j\rangle_p \langle j|$ acts on the syndrome subqudit. This way, the eigenvalue ω^{ij} is obtained as $S_1^j E\,|\psi\rangle = \omega^{ij} E\,|\psi\rangle$. The eigenvalue is some power of ω for prime-dimensional qudits as described in the previous section.

$$S_1'(E\,|\psi\rangle\,|s_I\rangle) = E\,|\psi\rangle \bigotimes \frac{1}{\sqrt{p}} \sum_{j \in \mathbb{F}_p} \omega^{ij}\,|j\rangle_p \tag{13}$$

The operation $DFT\,|i\rangle$ brings it to a superposition of the $|j\rangle$ states with the eigenvalue as the coefficient. Thus, from (12), we obtain:

$$S_1'(E\,|\psi\rangle\,|s_I\rangle) = E\,|\psi\rangle \bigotimes DFT_p\,|i\rangle_p \tag{14}$$

Applying $DFT^\dagger$ on the syndrome transfers the eigenvalue based information (ω^i) to the syndrome subqudit $|i\rangle_p$:

$$(I_p^{mn} \bigotimes DFT_p^\dagger) S_1'(E\,|\psi\rangle\,|s_I\rangle) = E\,|\psi\rangle \bigotimes |i\rangle_p \tag{15}$$

We obtain the value of i by using standard basis measurements on the qudit that comprises the syndrome qudit. This way, we use DFT to get the syndrome of a qudit in prime-dimensional systems.

Upon evaluating the syndrome, error deduction and recovery could be induced by either applying the inverse operation of the error by first classically deducing the error or by using control-X or control-Z operations to perform the inverse of the error on the syndrome.

2.3 Qudit CSS Codes

In Calderbank-Shor-Steane (CSS) codes, encoding is based on two related linear codes C_1 and C_2 such that code $C_1^\perp \subseteq C_2$. In qubit-system, we can have $C_X^\perp \subseteq C_Z$, which allows us to correct Z and X errors separately due to their parity check matrix having either Z-type or X-type rows, i.e. stabilizer generators having either X or Z type Pauli strings.

$$H = \begin{pmatrix} 0 & H_Z \\ H_X & 0 \end{pmatrix} \tag{16}$$

For higher-dimensions, CSS codes can conduct encoding based on two p-ary linear codes: $[[n, k_X, d_X]]_p$ and $[[n, k_Z, d_Z]]_p$, with the former being code C_X and the latter, code C_Z. These linear codes also satisfy the above property, $C_X^\perp \subseteq C_Z$. The basis states for the code will be [6] [1]:

$$|\psi_\omega\rangle := \frac{1}{\sqrt{C_X^\perp}} \sum_{c \in C_X^\perp} |c + w\rangle \tag{17}$$

For example, the $[[3, 1, 2]]_3$ code is a CSS code of prime-dimension, having stabilizer generators ZZZ and XXX. This code is used in a quantum secret-sharing scheme with its codewords being [1]:

$$|\bar{0}\rangle = \frac{1}{\sqrt{3}}(|000\rangle + |111\rangle + |222\rangle),$$
$$|\bar{1}\rangle = \frac{1}{\sqrt{3}}(|012\rangle + |120\rangle + |201\rangle),$$
$$|\bar{2}\rangle = \frac{1}{\sqrt{3}}(|021\rangle + |102\rangle + |210\rangle)$$

Here, the quantum information is split into n=3 shares and can be reconstructed using k encoded qudits. There is no qubit analogue of this scheme. It is known as the 'secret sharing' scheme since the reduced density matrix of any single qutrit used in this code is maximally mixed, which means no form of eavesdropping can extract information from any qutrit. You need a pair of qutrits in order to learn the quantum information. This is done by performing a unitary transform $U_{ij} \bigotimes I$ on any two qutrits, with the identity acting on the third qutrit. As the codewords of this code are cyclic, the unitary transform permutes the codewords and recovers the information by storing it in one of the two qutrits acted upon by the unitary.

3 Resource Consideration

As mentioned in [8], the non-utilized higher-dimensional Hilbert space essentially requires a higher number of qubits than needed when

To achieve fault-tolerant quantum computation, we require a fault-tolerant set of gates, the transversal set of gates. However, it there are many restraints on realizing transversal gates

4 Conclusion

We have seen how the stabilizer formalism can be extended to d-dimensions. We highlighted the disadvantages of composite-dimensional qudit codes and marked the significance of Galois qudit codes of composite dimensions, using the $[[5, 1, 3]]_9$ code as an example. A method to evaluate the syndrome of a prime-dimensional qudit code was discussed, marking the importance of the DFT operation used in the process. CSS codes for qudits were defined, along with the example of the quantum secret sharing scheme using three qutrits.

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