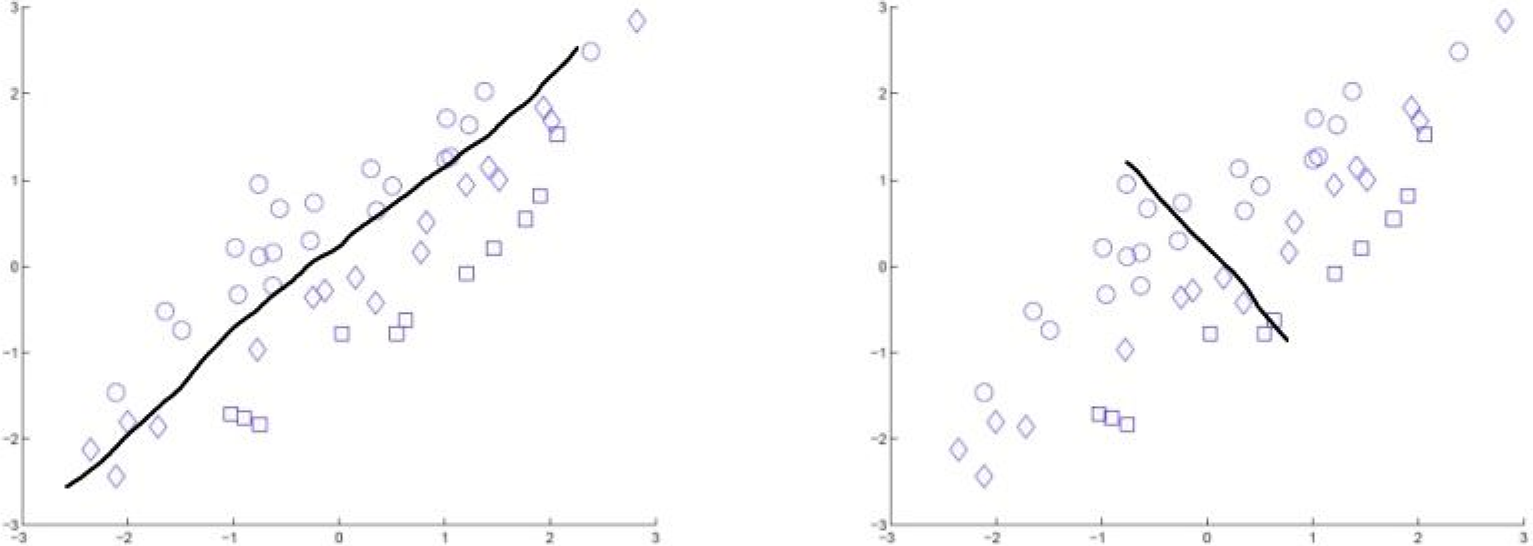
3
2
1
0
-1
-2
-3
-3 -2 -1 0 1 2 3

Assignment 2

Q2) 1. In logistic regression method, please derive the derivative of the negative logarithm of the likelihood function with respect to parameter w. Show the detailed steps to obtain the following results.

$$\nabla_w E(w) = \sum_{n=1}^{N} (f(x_n) - y_n) x_n$$

→ For a given dataset ~~the~~ (x_n, y_n) with $n = 1, 2 \ldots N$ the likelihood function can be written as

$P(C_1|x) = \sigma(w^T x + w_0) = f(x)$

$$L(w) = \prod_{n=1}^{N} p(C_1|x_n)^{y_n} (1 - p(C_1|x_n))^{1-y_n}$$

$$= \prod_{n=1}^{N} f(x_n)^{y_n} (1 - f(x_n))^{1-y_n} \ldots i$$

Now, the derivative of the logistic sigmoid function $\sigma(a)$ can be written as

$$\frac{\partial \sigma(a)}{\partial w} = \frac{\partial}{\partial a} \frac{1}{1+e^{-a}} = \frac{e^{-a}}{(1+e^{-a})^2} = \frac{1}{1+e^{-a}} \cdot \frac{e^{-a}}{(1+e^{-a})}$$

$$= \frac{1}{1+e^{-a}} \left(1 - \frac{1}{1+e^{-a}}\right) = \sigma(a)(1 - \sigma(a)) \ldots ii$$

~~The de~~ The negative log of the likelihood function can be written as

$$E(w) = -\ln L(w) = -\ln \prod_{n=1}^{N} p(C_1|x_n)^{y_n} (1 - p(C_1|x_n))^{1-y_n}$$

$$= -\sum_{n=1}^{N} (y_n \ln f(x_n) + (1 - y_n) \ln(1 - f(x_n))) \ldots iii$$

Now taking the derivative of eq iii we get

$$\frac{\partial E(w)}{\partial (w)} = \frac{\partial}{\partial w} \left[-\ln \prod_{n=1}^{N} p(C_1|x_n)^{y_n} (1 - p(C_1|x_n))^{1-y_n} \right]$$

$$-\left[\frac{\partial}{\partial w}\left(\ln p(C_1|x_n)^{y_n}\right) + \frac{\partial}{\partial w}\left(\ln (1-p(C_1|x_n))^{1-y_n}\right)\right]$$

$$= -\left[\frac{y_n}{f(x(n))} - \frac{1-y_n}{1-f(x(n))}\right]\frac{\partial (f(x_n))}{\partial w}$$

$$= -\left[\frac{y_n + y_n f(x(n)) - f(x(n)) - y_n f(x(n))}{f(x(n))(1-f(x(n)))}\right]\frac{\partial}{\partial w} \sigma(w^T x + w_0)$$

$$= -\left[\frac{y_n - f(x_n)}{\sigma(w^T x + w_0)(1-\sigma(w^T x + w_0))}\right] \cdot \sigma(w^T x)(1-\sigma(w^T x)) x_n$$

$$= -(y_n - f(x_n)) \cdot x_n$$

$$= (f(x_n) - y_n) \cdot x_n$$

$$\therefore \nabla_w E(w) = \sum_{n=1}^{N} (f(x_n) - y_n) \cdot x_n$$

Q3) 2 Consider 3 datasets in the 2D space: (2,2), (0,0), (-2,-2). Please answer the following

a. Calculate the first principal component by calculating the eigenvalue and eigenvector of the covariance matrix. Provide the actual vector of the first principal component

Feature	datapoint 1	datapoint 2	datapoint 3
x	2	0	-2
y	2	0	-2

No of features, $n = 2$

No of samples, $N = 3$

$\bar{x} = 0$, $\bar{y} = 0$

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Covariance matrix can be calculated as

$$\mathrm{cov}(x,x) = \frac{1}{N-1}\sum_{n=1}^{N}(x_n-\bar{x})^2$$

$$= \frac{1}{3-1}\left[(2-0)^2 + 0^2 + (-2-0)^2\right]$$

$$= 4$$

$$\mathrm{cov}(x,y) = \frac{1}{N-1}\sum_{n=1}^{N}(x_n-\bar{x})(y_n-\bar{y})$$

$$= \frac{1}{3-1}\sum\left[(2-0)(2-0) + 0 + (-2-0)(-2-0)\right]$$

$$= \frac{1}{2}[8] = 4$$

$$\mathrm{cov}(y,x) = \mathrm{cov}(x,y) = 4$$

$$\mathrm{cov}(y,y) = \frac{1}{N-1}\sum_{n=1}^{N}(y_n-\bar{y})^2$$

$$= \frac{1}{3-1}\left[(2-0)^2 + 0^2 + (-2-0)^2\right]$$

$$= 4$$

$$\therefore \text{covariance matrix} = \begin{bmatrix} \mathrm{cov}(x,x) & \mathrm{cov}(x,y) \\ \mathrm{cov}(y,x) & \mathrm{cov}(y,y) \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Eigen values can be calculated by formula

$$|S - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = 0$$

$(1-\lambda)^2 \cancel{8} - 1 = 0$

$\lambda^2 + 4 - 2\lambda - 4 = 0$

$\lambda(\lambda - 2) = 0$

$\lambda_1 = 0, \lambda_2 = 2$

since $\lambda_2 > \lambda_1$

$\lambda = 2$ is our first principal component

Eigen vector for λ_1

$(S - \lambda_2 I)(u) = 0$

$$\begin{bmatrix} 1-\lambda_2 & 1 \\ 1 & 1-\lambda_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0$$

$(1-\lambda_2) u_1 + u_2 = 0 \quad \ldots \; i$

$u_1 + (1-\lambda_2) u_2 = 0 \quad \ldots \; ii$

consider i

$u_1 = \frac{-u_2}{1-\lambda_2} = t$

when $t = 1$

$u_1 = 1$

$u_2 = \lambda_2 - 1 = 1$

$\therefore$ eigen vector u_2 for $\lambda_2 = 5$ $= \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Normalize the eigen vector

$e_2 = \begin{bmatrix} 1/\sqrt{1^2+1^2} \\ 1/\sqrt{1^2+1^2} \end{bmatrix} = \begin{bmatrix} 0.707 \\ 0.707 \end{bmatrix}$

b If we project the three data points into the 1D subspace by the principal component obtained in (a), what are the new coordinates of the three data points in the 1D subspace? what is the variance of the data after projection

The first principal component in 1D space can be calculated as

$$P_{11} = e_2^T \begin{bmatrix} x_1 - \bar{x} \\ y - \bar{y} \end{bmatrix}$$

$$= [0.707 \quad 0.707] \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$= 1.414 + 1.414 = 2.828$$

$$P_{12} = e_2^T \begin{bmatrix} x_2 - \bar{x} \\ y_2 - \bar{y} \end{bmatrix}$$

$$= [0.707 \quad 0.707] \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$= 0$$

$$P_{13} = e_2^T \begin{bmatrix} x_3 - \bar{x} \\ y_3 - \bar{y} \end{bmatrix}$$

$$= \cancel{\begin{bmatrix} -2 \\ -2 \end{bmatrix}} [0.707 \quad 0.707] \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$

$$= -1.414 - 1.414 = -2.828$$

$\therefore$ 1D for first principal component

$PC_1 \quad P_{11} = 2.828 \qquad P_{12} = 0 \qquad P_{13} = -2.828$

variance of data after projection can be calculated as

$$Var(x) = \frac{1}{N-1} \sum_{n=1}^{N} (x_n - \bar{x})^2$$

$$= \frac{1}{3-1} \left[(2.828 - 0)^2 + 0^2 + (-2.828 - 0)^2 \right]$$

$$= \frac{1}{2} \times 16 = 8$$

c What is the cumulative explained variance of the first principal component? Is there any variance that is not captured by it?

→ The cumulative explained variance of the first principal component is

$$\frac{\lambda_1}{\lambda_1+\lambda_2} = \frac{8}{8+0} = 1$$

Hence, we can conclude that the first principal component captures the complete variance

4. Given 10 points in Table 1, along with their classes and their Lagranian multipliers (α_i) answer the following questions

Data	x_{i1}	x_{i2}	y	α_i
x_1	4	2.9	1	0.414
x_2	4	4	1	0
x_3	1	2.5	-1	0
x_4	2.5	1	-1	0.018
x_5	4.9	4.5	1	0
x_6	1.9	1.9	-1	0
x_7	3.5	4	1	0.018
x_8	0.5	1.5	-1	0
x_9	2	2.1	-1	0.414
x_{10}	4.5	2.5	1	0

1. What is the equation of SVM hyperplane $h(x)$. Draw the hyperplane with the 10 points

Consider the data points with $\alpha > 0$

∴ we get

Data	x_{i1}	x_{i2}	y	α_i
x_1	4	2.9	1	0.414
x_4	2.5	1	-1	0.018
x_7	3.5	4	1	0.018
x_9	2	2.1	-1	0.414

The weight vector can be calculated using the formula $w = \sum_{i, \alpha_i > 0} \alpha_i y_i x_i$

$w_1 = 0.414(1)(4) + 0.018(-1)(2.5) + 0.018(3.5)(1) + 0.414(-1)(2)$

$= 1.656 - 0.045 + 0.063 - 0.828$

$= 0.846$

$w_2 = 0.414(1)(2.9) + 0.018(-1)(1) + 0.018(1)(4) + 0.414(-1)(2.1)$

$= 1.2006 - 0.018 + 0.072 - 0.8694$

$= 0.3852$

$\therefore$ The weight vector $w = \begin{bmatrix} 0.846 \\ 0.3852 \end{bmatrix}$

The bias can be calculated using formula
$b_i = y_i - w^T x_i$

$b_1 = 1 - \begin{bmatrix} 0.846 & 0.3852 \end{bmatrix} \begin{bmatrix} 4 \\ 2.9 \end{bmatrix}$

$= 1 - 4.501$

$= -3.501$

$b_2 = -1 - \begin{bmatrix} 0.846 & 0.3852 \end{bmatrix} \begin{bmatrix} 2.5 \\ 1 \end{bmatrix}$

$= -1 - 2.5002$

$= -3.5002$

$b_3 = 1 - \begin{bmatrix} 0.846 & 0.3852 \end{bmatrix} \begin{bmatrix} 3.5 \\ 4 \end{bmatrix}$

$= 1 - 4.5018$

$= -3.5018$

$b_4 = 1 - \begin{bmatrix} 0.846 & 0.3852 \end{bmatrix} \begin{bmatrix} 2 \\ 2.1 \end{bmatrix}$

$= 1 - 2.50092$

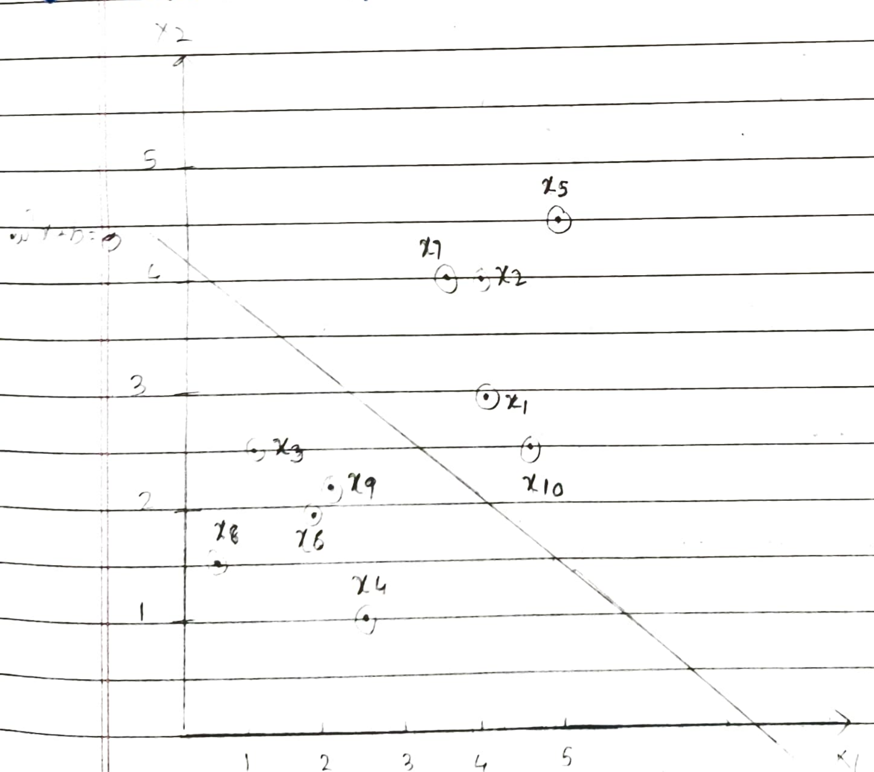
$= -3.50092$

$b = avg(b_i) = -3.5$

∴ The equation of svm hyperplane becomes

$h(x) = w^T x + b = \begin{bmatrix} 0.846 & 0.3852 \end{bmatrix} x - 3.5 = 0$

Q.2. ~~What is the distance of x6 from the~~



b What is the distance of x6 from the hyperplane?
Is it within the margin of the classifier?

Distance of x_6 from the hyperplane can be calculated as

$$\frac{|w^T x + b|}{\|w\|}$$

$$\frac{\left| \begin{bmatrix} 0.846 & 0.3852 \end{bmatrix} \begin{bmatrix} 1.9 \\ 1.9 \end{bmatrix} - 2.5 \right|}{\sqrt{(0.846)^2 + (0.3852)^2}}$$

$$= \frac{|1.6074 + 0.73188 - 2.5|}{0.928}$$

$$= \cancel{0.173}\ 1.25$$

The margin of the classifier is $\frac{1}{\|w\|} = \frac{1}{0.928} =$

Since ~~...~~ $1.25 > 1.077$ we can conclude that the point x_6 lies ~~within~~ outside the margin of classifier

3. Classify the point $z = (3,3)^T$ using $h(x)$ from above

$$h(x) = \begin{bmatrix} 0.846 & 0.3852 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix} - 2.5$$

$$= 2.538 + 1.1556 - 2.5$$

$$= 3.69 - 2.5$$

$$= 0.19$$

Since 0.19 is positive, the datapoint $(3,3)^T$ belongs to the positive class label