Introduction to Options

A *financial derivative* is a security whose value depends on the value of some other quantity, called the *underlying*. The underlying can be a traded entity, like a stock, bond, or commodity. It could also be other non-traded entities, like a stock index, e.g., the Standard and Poors 500, an interest rate, another derivative, or the voltility of a security. Also the value of a derivative can depend on the values of a number of underlyings.

Perhaps the simplest derivative is the *forward* which basically is a contract to purchase (or sell) an underlying at a future date for a prescribed price. Related to the forward is the *future* which is essentially a forward that is traded on an exchange is subject to daily (or more frequent) settlement of the contracted purchase price.

While futures and forwards are contracts that *r*equire the holder to purchase or sell, an *option* is a contract in which one party, the holder, has the *right*, but not obligation, to purchase or sell the underlying under circumstance that are described in the contract. The holder generally will *exercise* the right if it makes financial sense. The seller, or writer, of the option contract receives an initial payment for the right sold. The writer has the *obligation* to honor the holder's exercise by selling or purchasing the underlying as the case may be. In some cases, for example, when the underlying is an index, cash is exchanged on exercise. Major investment banks regularly offer *structured securities* to their customers. Often these contain embedded options of the simplest (so called "Vanilla" type).

Vanilla Options

In fairly broad terms an OPTION on a traded UNDERLYING is a contract in which the seller of the contract (the WRITER) agrees to give to the purchaser of the contract (the HOLDER) either an amount of stock in the underlying or an amount of money, in a certain time frame, with the amount of being exchanged determined by a function (the PAYOFF) of the underlying and/or its price history. The holder pays the writer a *PREMIUM* for the contract.

There are two basic examples of options on publicly traded stocks: the CALL OPTION and the PUT OPTION. A Call option gives the holder the right to purchase, or "call", from the writer a number of shares (usually 100) of the underlying at a price prescribed by the payoff. In a VANILLA CALL the purchase price is a fixed price, called the STRIKE PRICE. A Put option gives the holder the right to sell to the writer a number of shares of the underlying at a prescribed price. As with vanilla calls, in a VANILLA PUT the sales price is fixed and is again called the STRIKE. So the holder of a call has the right to "call away" the underlying from the writer and the holder of a put has the right to "put it to" the holder.

The time frame in which one can enforce (or EXERCISE) an option is prescribed in the option contract. Usually, options have a finite life with a prescribed EXPIRATION DATE. If exercise is permitted on only the date of expiration of the option, the option is said to be of EUROPEAN TYPE. If exercise can occur at any time up to expiration, the option is called AMERICAN. An option that can be exercised on only a fixed set of dates is called BERMUDAN. Most exchange traded options on stocks are American. Some options on indices, for example, the Standard and Poor's 500 (SPX), the Russell 2000 (RUT), and the Dow Jones Industrial Average (DJX) are European-style. With index options the settlement is made in cash. The interesting settlement rules[1] are posted on the site of the CBOE. (see http://www.cboe.com/products)

Many variations are possible on the put and call themes. For example, one might have a fixed strike, but the settlement price might be the average of the closing prices (of an exchange traded underlying) on the last several days before the option expires. If the average is an arithmetic average $A_{AVE} = (s_1 + s_2 + \cdots + s_N)/N$ then the call payoff is $P(s_1, \ldots, s_n) = \max(A_{AVE} - K, 0)$ This would be called a "Asian tail" option or, more generally, a discretely sampled Asian option.

It is also possible to write the contract so that the option comes into or goes out of existence depending on the behavior of the underlying. For example, an option (call or put) that becomes worthless if the underlying exceeds a value M would be called an "up-and-out" BARRIER option with barrier M. One can similarly define up-and-in, down-and-out, down-and-in options. We will look more deeply at these in one of our assignments.

It is pretty clear that when one introduces the path of the underlying into the option contract, things can get complicated and exact formulas for the option value might not exist. For this reason numerical methods are indispensable in the modeling of dynamics of options and their pricing. The most important numerical methods are Monte-Carlo Simulation, Numerical Methods for the Partial Differential Equations (PDE) that some option pricing functions satisfy, and tree methods (which share some of the properties of both Monte-Carlo and PDE methods).

We will focus, at least initially, on Vanilla European and American Calls and Puts and their pricing under the assumptions that Black and Scholes made in their seminal paper. Here is the some of the notation we will use

- t denotes time. The present is usually normalized as $t = 0$.
- T denotes the expiration time.
- S denotes the spot price of the underlying. This is really a function of t.
- K denotes the strike price.
- r denotes the "risk-free" interest rate.
- q denotes the "dividend yield" of the underlying.

The *risk-free interest rate* is the rate at which a safe deposit grows. It is common in option pricing to assume continuous compounding. In the standard model for option pricing it is assumed that this is also the rate of interest one pays on loans. The existence of such a rate is one of the assumptions of Black and Scholes. The fact that this does not reflect reality does not really invalidate this approach. The notion of *dividend yield* is sometimes used to treat the case of underlyings that pay dividends. While the dividend payments for stocks snd indices are discrete events, it is sometimes comvenient to model the dividends as being paid continuously. This is particulary convenient in the case of broad-based indices, like the S and P 500.

With the above notation, we can write down the formula for the value,V, of a vanilla option at expiration ($t = T$):

CALL OPTION: $V = V(S,T) = max\{S - K, 0\} = (S - K)_+$
PUT OPTION: $V = V(S,T) = max\{K - S, 0\} = (K - S)_+$

[1]It turns out that the settlement price, i.e.,the number that is compared with the strike in order to close the deal, does not have to be a value that the index ever takes on in the time frame in which the settlement price is determined.

These formulas are more or less obvious. If you own a call option, it will be worthless at expiration if the price of the underlying is smaller than the strike price. If the price of the underlying is greater than the strike price, the option will be worth the difference between the spot and the strike. The idea is that you can purchase the underlying for K and sell it for S. A similar argument works for the put. The holder of a put profits only it the spot price is less that the strike.

The above formulas are only valid at expiration; what one really wants is a formula for the fair value of the contract at any time between inception and expiration. Before discussing this problem we consider the relationship between the prices of a call and a put having common expiration and strike.

PUT-CALL PARITY. Consider a portfolio consisting of one share of stock, whose value is S on which the owner has sold one European call option, whose value is denoted by C; and one European put option, whose value is P. Assume that the options have a common strike, K. At time t the value of this portfolio is

$$\Pi(t) = S(t) - C(t) + V(t)$$

Since the owner sold the call option, it is a liability that might cost the owner something at expiration. That's why there is a $-$ sign in fron of $C(t)$. In some sense the portfolio uses the money from the call to purchase the put. It is not hard to verify that at expiration the value of the portfolio is exactly K: there is no uncertainty concerning the value of the portfolio at expiration.

So at time t, that is $T - t$ years from expiration, the portfolio is equivalent to a Certificate of Deposit that will mature to value K in $T - t$ years. Assuming an interest rate of r, compounded continuously, one sees that $K = \Pi(t)e^{r(T-t)}$ or $\Pi(t) = Ke^{-r(T-t)}$ Putting this in terms of S, C, and P, we get the **Put-Call Parity** relationship:

$$C - P = S - Ke^{-r(T-t)} \tag{1}$$

This is of some limited practical use, but it gives little insight into the dynamics of option prices as the time passes and the underlying varies. With appropriate assumptions made on the dynamics of the underlying security, there is a formula—the Black-Scholes Formula—for the pricing of European-style vanilla Call and Put options on the underlying. The formula is typically applied to options on stocks and stock indices. While most exchange traded equity options are of the American type, a number of widely traded options on stock indexes are of the European type. Check the web site of the CBOE (http://www.cboe.com/)for more information. Also, a vanilla American Call option on a stock that does not pay a dividend is equivalent to a European option. The reason is that there is no point in putting out the money to execise a call unless there is a sizeable dividend to be captured.

The movement of the price of the underlying from $t = 0$ to $t = T$ appears to be random to anyone who observes it Let's call the value of the option at expiration the payoff and denote its dependence on S by $P(S)$. It is natural to consider $S(T)$ and $P(S)$ as being random variables. Assuming the distribution of values of $S(T)$ is given by a function $F(S)$, we have that the average value (or expected value) of $P(S)$ at expiration is

$$\int P(S)F(S)ds$$

With this, it makes sense to say that under this model[2], the value of the option at time t is the discounted expected value of the expiration values of the option:

$$e^{T-t}\int P(S)F(S)dS$$

Because this argument is so general, it is of limited practical use. Black and Scholes proposed a reasonable model for the dynamics of $S(t)$ that yields a very computable integral for the expected value of calls and puts.

Black and Scholes. Black and Scholes published a paper in the early 1970's that readjusted how people thought about modeling option prices. They presented an idealized framework in which fair prices could be calculated. They assume that there is a constant risk-free interest rate at which one could lend and borrow, that there is infinite liquidity, that one could buy and short fractional numbers of shares, that trading is continuous. They assumed that the random movement of stock prices could be modeled by Geometric Brownian Motion with a single constant control parameter based on the randomness.

The assumption of Geometric Brownian Motion sort of says that the infinitesimal returns of the stock are normally distributed with some mean μ and a variance which when properly scaled equals a constant called volatility and denoted by σ. It is easier to think of it in terms of discrete changes in time. Think of time as having discrete increments of size Δt, with corresponding increments in the underlying being ΔS. The return from time t to time $t + \Delta t$ is then

$$\frac{\Delta S}{S}$$

and the discretized Geometric Brownian Motion assumption is that

$$\frac{\Delta S}{S} = \mu\Delta t + \sigma\sqrt{\Delta t}Z$$

where the quantity Z is a sample from a standard normal distribution; we write this as $Z \sim N(0,1)$. The assumption is that the values of Z are chosen independently at each time step.

While none of these assumptions is rigorously true, the framework and the resulting formulas were good enough to give reasonable insight into option prices and one can always force the Black-Scholes formula to be tied to the market by manipulating one of its unobservable parameters—the volatility. Furthermore, even though the market place discovers prices on exchange traded entities, the Black-Scholes framework provides a setting in which the prices of exotic, or rarely traded, financial instruments might be set or proposed in way that is consistent with publicly traded instruments.

By using a result of the mathematician Ito, one can derive a related model of the movement of an option. This allows one to analyse the dynamics of a portfolio consisting of some shares of stock and an option on the stock. Then, one can investigate conditions under which the portfolio is risk-free. It turns out that one can make the portfolio risk-free, by adjusting the number of shares in the portfolio. The final step in the Black-Scholes development is an argument that a risk-free portfolio must grow at the idealized risk-free interest rate. After all of this, one is left with a relationship among various partial derivatives of the option price (viewed

[2]The model that assumes that the distribution of expiration values of S is given by $F(S)$.

as a function of time and the spot price of hte underlying). The relationship is called the Black-Scholes Partial Differential Equation (PDE), or simply the **Black-Scholes Equation**. With S denoting the spot of the underlying, t denoting time, and $V(S,t)$ denoting the price of the option, the Black-Scholes PDE is

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

The Black-Scholes Equation as stated has infinitely many solution. The mathematical theory of PDEs requires additional conditions, called boundary conditions, to specify a unique solution to the PDE. It is through the boundary conditions that the type of option (Call, Put, whatever) is specified. The general form of the Black-Scholes Equation is well-known to engineers and mathematicians. Moreover, it can be transformed into the well-studied "Heat Equation" through a change of variables that is well understood by experts in the area. Lucky for Black and Scholes (and us) there is a formula for the solution of the Heat Equation. In the case of vanilla calls and puts, and a few other options, the formula for the Heat Equation can be exploited to give a solution of the Black-Scholes PDE. This is how the Black-Scholes formulas were first discovered. We will give more details of the development and numerical solution of the Black-Scholes PDE, as well as of other PDEs that arise in Finance, later in the course. Right now all we need are the Black-Scholes formulas which are introduced in the next section.

The Formulas. Here are the parameters that affect the current price of an option:

S_0 — The current, or spot, price of the underlying stock or option.

K — The strike price of the option.

T — The time to expiration of the option in years. (so we are taking $t = 0$).

r — The (assumed constant) risk free annual interest rate over the life of the option.

σ — The (assumed constant) volatility of the underlying, expressed in percent per year.

q — The dividend yield of the underlying over the life of the option. This is often 0 or so small that it can be ignored.

COMMENTS.

- Sometimes one uses the quantity $T - t$ to denote the time to expiration. In this case time is viewed as existing on a number line with the current time denoted by t and the expiration time denoted by T. This is convenient if one thinks of the current time as $t = 0$ and selects values of t from future times of interest.
- The more challenging problem is to model option prices when the dividends are discrete. This is usually the case with stocks even though the modeling of the dividend as a continuous yield is commonly used. However, I will often take $q = 0$.

Let $N(z)$ denote the cumulative Standard Normal Distribution function. That is, if z is any real number, then $N(z)$ is the area under the familiar bell-shaped curve

$$y = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

for $-\infty < x < z$.

That is,

$$N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{x^2}{2}} dx.$$

With all of this notation the value of a vanilla European call option is

$$V_{call} = S_0 e^{-qT} N(d_1) - K e^{-rT} N(d_2)$$

and the value of a vanilla European put option is

$$V_{put} = -S_0 e^{-qT} N(-d_1) + K e^{-rT} N(-d_2)$$

where

$$d_1 = \frac{\ln(S_0/K) + (r - q + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

and

$$d_2 = \frac{\ln(S_0/K) + (r - q - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

If there is no dividend yield or if it is being ignored, the e^{-qT}-term is missing. The fact that $N(x) + N(-x) = 1$ reveals the "put-call parity" relationship

$$V_{call} - V_{put} = S_0 e^{-qT} - K e^{-rT}$$

Approximating the Normal Distribution. In the calculations with the Black-Scholes formula one needs values of $N(x)$. In the old days one used tables and linear interpolation to get these. Today, many spreadsheets have a formula built in for it. If we are working from scratch, the following formula gives about 6 decimal places of accuracy.

If $x \geq 0$:

$$N(x) = 1 - \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2}) R(z)$$

where

$$z = \frac{1}{1 + 0.2316419x},$$

$$R(z) = a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5$$

and

$$\begin{aligned} a_1 &= 0.319381530 \\ a_2 &= -0.356563782 \\ a_3 &= 1.781477937 \\ a_4 &= -1.821255978 \\ a_5 &= 1.330274429 \end{aligned}$$

If $x < 0$:
$N(x) = 1 - N(-x)$.

Comments on Programming Arithmetic. Never calculate if you don't have to. If possible, arrange the order of calculations so you can reuse results of previous calculations. This is particularly important for calculations performed in loops. You should always ask if it is possible to perform some of the calculations or partial calculations outside of the loop. However, in all of this be careful that you don't create bugs while trying to make things more efficient. Test your code.

When evaluating polynomial expressions use Horner's Rule which is just a way of arranging the algebra so that fewer multiplications are needed. For example, one can see that

$$a_0 + a_1x + a_2x^2 + a_3x^3 = x(x(a_3x + a_2) + a_1) + a_0$$

but the form on the left requires 3 multiplications without even counting those required to raise x to the powers, while the form on the right requires only 3 to do the complete evaluation!

Exercises.

(1) Write a function or functions to evaluate the Black-Scholes functions for Put and Call. Test it by verifying the Put-Call parity relationship. Perform the test on a variety of different values of r, σ, S_0, and $T - t$. (Take $q = 0$.) Write the program so that it takes as input actual dates. Note that most exchange traded options expire on the third Friday of the month.

(2) There are three variables of interest: $T - t$, S, σ. If $T - t$ is held fixed, one can investigate the near term behavior of the option price as S and σ vary. If σ is held fixed, one can view the theoretical future dynamics of the option prices. Each of these studies can be made by creating grids of values that can be imported into a spreadsheet and charted. Plan a program to create these types of data.

(3) Implement the plan of Problem 2 for the case of an option portfolio consisting of one European Call that expires in November and a short position in a European call that expires in October. Both option have the same underlying and the same strike. (This position is called a "Call Calendar Spread".)

Creating Other Payoffs

By combining call and put options one can create a variety of more complicated payoff functions. For example, if you own a call with strike K_1 and sell a call having the same expiration with strike $K_2 > K_1$, then at expiration, if the underlying lies in the range, $K_1 \leq S \leq K_2$, the option is worth $S - K_1$. But, if the underlying goes above K_2, the value of the option is stuck at $K_2 - K_1$. The profit is "capped". This is an example of a "Bull call spread".

More generally, one can approximate *any* continuous function by a combination of calls (or puts). To see this, consider the "Butterfly" spread: select a middle strike K and a width parameter h. Then, buy call options with strikes $K - h$ and $K + h$ and sell 2 call options with strike K. If you sketch the graph of the payoff, it will look like a tent that peaks at K and is zero if $|S - K| > h$. That is, at expiration the payoff has a positive value only if $K - h < S < K + h$; the payoff is largest if $S = K$. Now, given any continuous function, say f, one can select h and the location of the strikes $K_0 < K_1 < \cdots < K_N$, with $K_i - K_{i-1} = h$, so that the graph of the f is very close to the curve obtained by drawing straight lines between consecutive points $(K_i, f(K_i))$. It is not hard to see that the curve made by these straight line segments can be expressed as a weighted sum of butterfly spreads.

This construction ignores the transaction costs of creating the positions; however, it does reveal the power of vanilla options to create or replicate a potentially large number of payoff functions.