# CHAPTER - 1
# MEAN VALUE THEOREM

## 1.1 INTRODUCTION

Rolle's Theorem (concerning derivatives) is one of the Fundamental Theorems in differential calculus which forms the logical base for Taylor's Theorem. In this chapter, we shall discuss Rolle's Theorem, Lagrange's Mean Value Theorem and Cauchy's Mean Value Theorem. Consequences of these theorems are also discussed in this chapter.

## 1.2 ROLLE'S THEOREM

**Statement :** If a function f (x) is

(i) continuous in the closed interval [a, b] (i.e. continuous in $a \le x \le b$).

(ii) differential in the open interval (a, b) (i.e. $f'(x)$ exists and is unique in $a < x < b$).

(iii) f (a) = f (b).

then there exists at least one point x = c in (a, b) (i.e. for $a < c < b$) such that $f'(c) = 0$

**Proof :** Since f (x) is continuous in the closed interval [a, b], therefore it is bounded and attains its bounds (maxima and minima). Let M and m be the maximum and minimum values (least upper bound and greatest lower bound) of f (x) respectively and let c and d be two numbers in the interval [a, b] such that

$$f(c) = M \text{ and } f(d) = m$$

Now two cases arise :

**Case I :** If M = m (i.e. if maximum value coincides with minimum value).

Then $\quad f(x) = M = m$ for all x in [a, b]

i.e. $\quad$ f (x) reduces to a constant for every x in [a, b]

$\therefore \quad f'(x) = 0$ for all x in [a, b]

Hence the theorem is true for any value c in the open interval (a, b).

**Case II :** If $M \neq m$ (i.e. if maximum value and minimum value are unequal).

Then, since f(a) = f(b), either M or m must be different from f (a) and f (b).

Let us suppose that M = f (c) be different from each of f (a) and f (b).

The number 'c' being different from a and b lies within the open interval (a, b). $\left\{ \begin{array}{l} \because \; f(c) \neq f(a) \quad c \neq a \\ \text{and } f(c) \neq f(b) \quad c \neq b \end{array} \right.$

As the function f (x) is differentiable in the open interval (a, b), so it is also differentiable at x = c (i.e. $f'(c)$ exists and is unique).

We shall now show that $f'(c) = 0$.

By definition,
$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \qquad \dots (1)$$

exists and is the same when $h \to 0$ through positive and negative values.

Since f (c) = M, is maximum value (greatest value) of function f (x) in (a, b), we have

$$f(c+h) \le f(c)$$

whatever positive negative values h has. Thus

$$\frac{f(c+h)-f(c)}{h} \leq 0 \text{ for } h > 0$$

$$\frac{f(c+h)-f(c)}{h} \geq 0 \text{ for } h < 0$$

From (2) and (3), in the limit, when $h \to 0$ through positive and negative values of h, Right Hand Derivative and Left Hand Derivatives have opposite signs. But from second condition of the theorem, f'(c) being unique, we must have

$$f'(c) = 0 \text{ for } a < c < b.$$

Similar argument can be used when $m = f(d) \neq f(a)$ or $f(b)$. This proves the theorem.

**Remark 1 :**

There may exist more than one real number $c \in (a, b)$ at which $f'(x)$ vanishes.

**Remark 2 :**

The converse of Rolle's theorem is not true i.e. f (x) may vanish at point $c \in (a, b)$, without satisfying the three conditions of Rolle's theorem for f (x).

In other words, the three conditions of Rolle's theorem are the sufficient (but not necessary) condition for $f'(x) = 0$ for some $x \in (a, b)$.

*For example,* consider the function

$$f(x) = \frac{1}{x} + \frac{1}{1-x} \text{ in } [0, 1]$$

Here

$$f'(c) = 0 \text{ at } c = \frac{1}{2}, \text{ but } f(x) \text{ is not continuous at } x = 1.$$

**Remark 3 :**

The conclusion of Rolle's theorem may not *hold good* for a function which does not satisfy any of the conditions. If either

(i) f (x) is discontinuous at some point in [a, b] or

(ii) $f'(x)$ does not exists at any point in (a, b) or

(iii) $f(a) \neq f(b)$.

*For example,* consider the function given by

$$y = f(x) = |x| \text{ in the interval } [-1, 1]$$

Here

(i) $f(x) = |x|$ is continuous in [–1, 1]

(ii) $f(x) = |x|$ is not differentiable at x = 0

(iii) $f(-1) = f(1)$

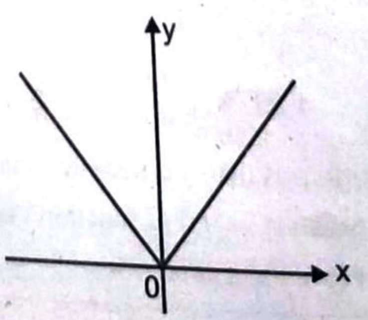


**Fig. 1.1**

Thus, not all the three conditions of Rolle's theorem are satisfied. Hence the conclusion is not valid in that $f'(x) \neq 0$ for all $x \in (-1, 1)$ (or $f'(x)$ vanishes for no value of x).

In what follows, we shall show that $f(x) = |x|$ is not differentiable at $x = 0$. For

$$\text{L.H.D.} = f'(0-) = \lim_{h \to 0-} \frac{f(0-h) - f(0)}{h} = \lim_{h \to 0-} \frac{|h| - 0}{h}$$

$$= \lim_{h \to 0-} \frac{|h|}{h} = \lim_{h \to 0} \frac{h}{-h} = -1 \text{ (Note)} \quad \text{... (1)}$$

and

$$\text{R.H.D.} = f'(0+) = \lim_{h \to 0+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0+} \frac{|h| - 0}{h}$$

$$= \lim_{h \to 0+} \frac{|h|}{h} = \lim_{h \to 0+} \frac{h}{h} = 1 \quad \text{... (2)}$$

From (1) and (2), $f'(0)$ does not exist

**Remark 4 : Geometrical Interpretation of Rolle's Theorem :**

If the graph of the function f (x) be drawn between x = a and x = b and if it is a continuous curve between x = a and x = b having a unique tangent at all points in the above interval and f (a) = f (b), then there exists at least one point P on the curve (corresponding to x = c between x = a and x = b), such that the tangent a which is parallel to x – axis i.e. $f'(c) = 0$. (See Figs. 1.2 and 1.3).

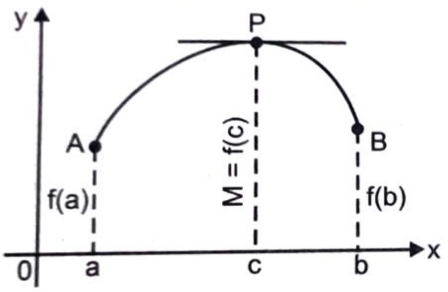


**Fig. 1.2**

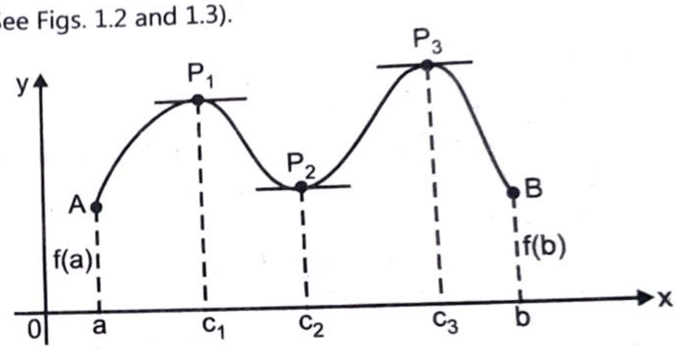


**Fig. 1.3**

**Remark 5 : Algebraic Interpretation of Rolle's Theorem :**

If f (x) be a polynomial in x (i.e. f (x) = 0) satisfies conditions of Rolle's theorem and x = a, x = b be the two roots of the equation f (x) = 0, then by Rolle's theorem at least one root of the equation $f'(x) = 0$ lies between a and b.

In other words, if f (x) = 0 be an equation, where f (x) satisfies conditions of Rolle's theorem and $x_1, x_2, \ldots x_n$ be the roots of the equation. Then by Rolle's theorem the roots $c_1, c_2, \ldots c_{n-1}$ of the equation $f'(x) = 0$ lie between the roots of f (x) = 0 i.e.

$$x_1 \leq c_1 \leq x_2 \leq \ldots\ldots\ldots \leq x_{n-1} \leq c_{n-1} \leq x_n$$

**Remark 6 : Another statement of Rolle's Theorem :**

If a function f (x) is such that

(i) it is continuous in the closed interval [a, a + h]

(ii) it is differentiable in the open interval (a, a + h)

(iii) f (a) = f (a + h)

then there exists at least one real number θ such that

$$f'(a + \theta h) = 0 \text{ for } 0 < \theta < 1$$

## 1.3 ILLUSTRATIONS ON ROLLE'S THEOREM

**Type I :**

**Ex. 1 :** Verify Rolle's Theorem for the following functions :

(i) $f(x) = x^2$ in $[-1, 1]$ (ii) $f(x) = 2 + (x-1)^{2/3}$ in $[0, 2]$

(iii) $f(x) = e^x \sin x$ in $[0, \pi]$ (iv) $f(x) = x(x+3)\, e^{-x/2}$ in $[-3, 0]$

(v) $f(x) = x^2 (1-x)^2$ in $[0, 1]$

**Sol. : (i) $f(x) = x^2, x \in [-1, 1]$**

The given function is *continuous* in $[-1, 1]$ and *differentiable* in $(-1, 1)$.

Also
$$f(-1) = (-1)^2 = +1 \text{ and } f(1) = (1)^2 = 1$$

$\therefore$
$$f(-1) = f(1)$$

Hence all the three conditions of Rolle's theorem are satisfied.

Therefore, the derivative of f (x) must vanish for at least one value of $x \in (-1, 1)$.

Directly, we see that the derivative $f'(x) = 2x$ vanishes for $x = 0$ which belongs to $(-1, 1)$. Hence the verification.

**(ii) $f(x) = 2 + (x-1)^{2/3}, x \in [0, 2]$**

We note that the derivative of given function f (x),

$$f'(x) = \frac{2}{3}(x-1)^{-1/3} = \frac{2}{3(x-1)^{1/3}}$$

does not exist (i.e. it is not finite) at $x = 1 \in (0, 2)$ and so $f'(x)$ does not exist at every point of $(0, 2)$ i.e. f (x) is *not differentiable* in the open interval $(0, 2)$.

Hence the Rolle's theorem is *not applicable* to f (x) in the interval $[0, 2]$.

**(iii) $f(x) = e^x \sin x, x \in [0, \pi]$**

We note the following :

(a) Since $e^x$ and $\sin x$ are both continuous for every value of x, therefore their product $f(x) = e^x \sin x$ is also continuous for every value of x and in particular f (x) is *continuous* in the closed interval $[0, \pi]$.

(b) $f'(x) = e^x \cos x + e^x \sin x = e^x (\cos x + \sin x)$

and this does not become infinite or indeterminate for any value of x in the open interval $(0, \pi)$, it follows that f (x) is *differentiable* in the open interval $(0, \pi)$.

(c) $f(0) = e^0 \sin 0 = 1 \times 0 = 0$ and $f(\pi) = e^\pi \sin \pi = e^\pi \times 0 = 0$

$\therefore$ $f(0) = 0 = f(\pi)$

Hence f (x) satisfies all the three conditions of Rolle's theorem. Therefore there exists at least one value of $x \in (0, \pi)$ such that $f'(x) = 0$.

$$\therefore \quad f'(x) = e^x (\cos x + \sin x) = 0$$

$$\therefore \quad \cos x + \sin x = 0 \text{ or } \tan x = -1 \qquad \{\because e^x \neq 0 \text{ for any finite value of } x\}$$

$$\therefore \quad \tan x = -\tan\left(\frac{\pi}{4}\right) = \tan\left(-\frac{\pi}{4}\right)$$

$$\therefore \quad x = n\pi + \left(-\frac{\pi}{4}\right), n \in I \qquad \{\because \text{if } \tan\theta = \tan\alpha, \text{ then } \theta = n\pi + \alpha, n \in I\}$$

Now on putting $n = 0, 1, 2, 3, \ldots\ldots$, we get

$$x = -\frac{\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \ldots$$

Of these values of x, clearly the value $x = \frac{3\pi}{4}$ belongs to open interval $(0, \pi)$, which verifies Rolle's theorem.

**(iv) $f(x) = x(x+3)e^{-x/2}$, $x \in [-3, 0]$**

We note the following :

(a) Since $x(x+3)$ being a polynomial function is continuous for all x and $e^{-x/2}$ is also continuous function for all x, therefore their product $f(x) = x(x+3)e^{-x/2}$ is also continuous for every value of x and in particular f (x) is *continuous* in the closed interval $[-3, 0]$.

(b)
$$\begin{aligned} f'(x) &= (2x+3)e^{-x/2} + (x^2+3x)e^{-x/2}(-1/2) \\ &= e^{-x/2}[2x+3-(x^2+3x)/2] \\ &= e^{-x/2}[(6+x-x^2)/2] \end{aligned}$$

which does not become infinite or indeterminate at any point of the interval $(-3, 0)$ and thus f (x) is *differentiable* in the open interval $(-3, 0)$.

(c) $f(-3) = 0 = f(0)$

Hence all the three conditions of Rolle's theorem are satisfied. Therefore there must exist at least one value of $x \in (-3, 0)$ for which $f'(x) = 0$.

i.e. $$f'(x) = e^{-x/2}[(6+x-x^2)/2] = 0$$

$\therefore$ $$-x^2+x+6 = 0 \text{ or } x^2-x-6=0$$

$\therefore$ $$(x-3)(x+2) = 0 \qquad \{\because e^{-x/2} \text{ is not zero for any finite value of } x\}$$

$\therefore$ $$x = -2, 3$$

Of these two values of x for which $f'(x) = 0$, $x = -2$ belongs to open interval (3, 0) under consideration. Hence the verification.

**(v) $f(x) = x^2(1-x)^2$, $x \in [0, 1]$**

We note the following :

(a) The given function is *continuous* in [0, 1], being polynomial function in x.

(b) $f'(x) = 2x(1-x)^2 - 2x^2(1-x)$

which does not become infinite or indeterminate at any point of the interval (0, 1), it follows that f (x) is *differentiable* in the open interval (0, 1).

(c) $f(0) = 0 = f(1)$

Hence the conditions of Rolle's theorem are satisfied. Therefore, there exists at least one point x = c in (0, 1) such that $f'(c) = 0$.

$\therefore$ $$f'(c) = 2c(1-c)^2 - 2c^2(1-c) = 0$$

$\therefore$ $$2c(1-c)(1-c-c) = 0$$

$\therefore$ $$c = 0,\ c = 1,\ c = \frac{1}{2}$$

Of these three values of c for which $f'(c) = 0$, $c = 1/2$ lies in the open interval (0, 1), which verifies Rolle's theorem.

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**Ex. 2 :** Discuss the applicability of Rolle's theorem to the function :

$$f(x) = \log\left[\frac{x^2+ab}{x(a+b)}\right] \text{ in } [a, b],\ a > 0,\ b > 0.$$

**Sol. :** We note the following :

(a) $f(x) = \log(x^2+ab) - \log x - \log(a+b)$ being composite function of continuous functions in [a, b] is a *continuous* function of x in [a, b].

(b) $f'(x) = \frac{2x}{x^2 + ab} - \frac{1}{x}, x \neq 0 \{\because x \in [a, b], a > 0, b > 0\}$

which does not become infinite or indeterminate for $a < x < b$ and so f (x) is *differentiable* in the interval (a, b).

(c)
$$f(a) = \log \frac{a^2 + ab}{a(a + b)} = \log 1 = 0$$
$$f(b) = \log \frac{b^2 + ab}{b(a + b)} = \log 1 = 0$$

Thus f (x) satisfies all the three conditions of Rolle's theorem and therefore there must exists at least point $x = c$ in $a < x < b$ such that $f'(c) = 0$.

$$\therefore \quad f'(c) = 0 \Rightarrow \frac{2c}{c^2 + ab} - \frac{1}{c} = 0$$

or
$$\frac{2c^2 - c^2 - ab}{c^2 + ab} = 0 \text{ or } c^2 = ab$$

$$\therefore \quad c = \pm\sqrt{ab}$$

Of these two values of c, clearly $c = +\sqrt{ab}$ lies between a and b (being geometric mean of a and b). Hence Rolle's theorem is applicable to f (x) in the interval [a, b].

**Ex. 3 :** Verify Rolle's theorem for the function $f(x) = (x - a)^m (x - b)^n$, where m and n are positive integers; and $x \in [a, b]$

**Sol. :** We note the following :

(a) As m and n are positive integers, $f(x) = (x - a)^m (x - b)^n$ will be a polynomial in x on expansion by binomial theorem. Since every polynomial is a continuous function of x for every value of x, therefore f (x) is *continuous* in the closed interval [a, b].

(b)
$$\begin{aligned} f'(x) &= m(x - a)^{m-1}(x - b)^n + n(x - a)^m (x - b)^{n-1} \\ &= (x - a)^{m-1}(x - b)^{n-1}[m(x - b) + n(x - a)] \\ &= (x - a)^{m-1}(x - b)^{n-1}[(m + n)x - (mb + na)] \end{aligned}$$

which exists for all $x \in (a, b)$ i.e. $f'(x)$ has a unique and finite value for all x in (a, b). Thus f (x) is *derivable* in the open interval (a, b).

(c) Also, $f(a) = 0 = f(b)$

Hence, f (x) satisfies all the three conditions of Rolle's theorem.

Therefore, there exists at least one value c of x in (a, b) such that $f'(c) = 0$ i.e.

$$f'(c) = 0 \Rightarrow (c - a)^{m-1}(c - b)^{n-1}[(m + n)c - (mb + na)] = 0$$

or
$$(m + n)c - (mb + na) = 0$$

$$\therefore \quad c = \frac{mb + na}{m + n}$$

which is a point within the open interval (a, b) because it divides a and b internally in the ratio m : n. the Rolle's theorem is verified.

**Ex. 4 :** Verify Rolle's theorem for the following functions :

(i) $f(x) = 2x^3 + x^2 - 4x - 2$ in $[-\sqrt{2}, \sqrt{2}]$

(ii) $f(x) = 3x^4 - 4x^2 + 5$ in $[-1, 1]$

(iii) $f(x) = \begin{cases} x^2 + 1 & \text{for } 0 \le x \le 1 \\ 3 - x & \text{for } 1 \le x \le 2 \end{cases}$

**Sol. : (i)** $f(x) = 2x^3 + x^2 - 4x - 2,\ x \in [-\sqrt{2}, \sqrt{2}]$

Since f (x) is a polynomial in x, so that it is continuous in $[-\sqrt{2}, \sqrt{2}]$ and derivable in $(-\sqrt{2}, \sqrt{2})$.

Also, $$f(-\sqrt{2}) = 0 = f(\sqrt{2})$$

All the conditions of Rolle's theorem are satisfied. Therefore, the derivative of f (x) must vanish for at least one value of $x \in (-\sqrt{2}, \sqrt{2})$ i.e.

$$f'(x) = 6x^2 + 2x - 4 = 0 \quad \text{or} \quad (3x - 2)(x + 1) = 0$$

$$\therefore \quad x = \frac{2}{3},\ x = -1$$

Since both the values of x lie in the open interval $(-\sqrt{2}, \sqrt{2})$, Rolle's theorem is verified.

**(ii)** $f(x) = 3x^4 - 4x^2 + 5,\ x \in [-1, 1]$

Since f (x) is a polynomial in x, therefore it is continuous in [–1, 1] and derivable in (–1, 1).

Also $$f(-1) = 4 = f(1).$$

All the conditions of Rolle's theorem are satisfied. Therefore, there must exists at least one value of x for which $f'(x) = 0$ i.e.

$$f'(x) = 12x^3 - 8x = 0 \quad \text{or} \quad 4x(3x^2 - 2) = 0$$

$$\therefore \quad x = 0,\ x = \pm\sqrt{\frac{2}{3}}$$

Since all these values of x lie in (– 1, 1), Rolle's theorem is verified.

**(iii)** $f(x) = \begin{cases} x^2 + 1\ ; & \text{for } 0 \le x \le 1 \\ 3 - x\ ; & \text{for } 1 \le x \le 2 \end{cases}$

Here $f_1(x) = x^2 + 1$ for $0 \le x \le 1$, which being a polynomial is continuous and derivable in [0, 1].

And $f_2(x) = 3 - x$ for $1 \le x \le 2$, which being a polynomial is continuous and derivable in [1, 2].

Since the domain of definition is partitioned at x = 1 while defining f (x), therefore we shall check the continuity and derivability of f (x) at x = 1.

Now $$\text{L.H.L} = f(1 - 0) = \lim_{x \to 1-0} f(x) = \lim_{x \to 1-0} (x^2 + 1) = \lim_{h \to 0} [(1 - h)^2 + 1] = 2$$

$$\text{R.H.L} = f(1 + 0) = \lim_{x \to 1+0} f(x) = \lim_{x \to 1+0} (3 - x) = \lim_{h \to 0} [3 - (1 + h)] = 2$$

$\therefore$ f (x) is continuous at x = 1 in the closed interval [0, 2].

And L.H.D. = $f'(1-) = \lim_{h\to0-} \frac{f(1-h)-f(1)}{h} = \lim_{h\to0-} \frac{\{(1-h)^2+1\}-2}{h}$

$= \lim_{h\to0-} \frac{h^2-2h}{h} = \lim_{h\to0-} \frac{h(h-2)}{-h}$

$= \lim_{h\to0} (2-h) = 2$

R.H.D. = $f'(1+) = \lim_{h\to0+} \frac{f(1+h)-f(1)}{h} = \lim_{h\to0+} \frac{\{3-(1+h)\}-2}{h}$

$= \lim_{h\to0+} \frac{-h}{h} = \lim_{h\to0} (-1) = -1$

We note that L.H.D. ≠ R.H.D. and so f (x) is not derivable at x = 1 in the open interval. Hence Rolle's theorem is not applicable to the given function f (x) in (0, 2).

**Remark :**

$\because \quad f'(x) = \begin{cases} 2x ; \text{ for } 0 \le x \le 1 \\ -1 ; \text{ for } 1 \le x \le 2 \end{cases}$

$\therefore$ f (x) is differentiable in (0, 2) except perhaps at x = 1.

**TYPE II :**

**Ex. 5 :** By considering the function f (x) = (x – 2) log x, show that the equation x log x = 2 – x is satisfied by at least one value of x lying between 1 and 2.

**Sol. :** Let $f(x) = (x-2)\log x$

(a) Since the functions (x – 2) and log x are continuous in 1 ≤ x ≤ 2, therefore their product f (x) = (x – 2) log x is also *continuous* in the closed interval [1, 2].

(b) $f'(x) = (x-2)\frac{1}{x} + \log x$

which is neither infinite nor indeterminate for any value of x in the open interval (1, 2), so f (x) is *differentiable* in the open interval (1, 2).

(c) Also f (1) = 0 = f (2)

Hence f (x) satisfies all the three conditions of Rolle's theorem. Therefore f' (x) must be zero for at least one value of x in (1, 2).

$\therefore \quad f'(x) = \frac{x-2}{x} + \log x = 0$ for at least one value of x in (1, 2)

or $x \log x = 2 - x$ for at least one value of x in (1, 2)

i.e. the equation x log x = 2 – x is satisfied by at least one value of x lying between 1 and 2. Hence the result.

**Ex. 6 :** Prove that if $a_0, a_1, a_2, \ldots, a_n$ are real numbers such that

$$\frac{a_0}{n+1} + \frac{a_1}{n} + \ldots + \frac{a_{n-1}}{2} + a_n = 0,$$

then there exists at least one real number x between 0 and 1 such that

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n = 0.$$

**Sol. :** Consider the function f (x) defined as

$$f(x) = \frac{a_0}{n+1} x^{n+1} + \frac{a_1}{n} x^n + \ldots + \frac{a_{n-1}}{2} x^2 + a_n x, \ x \in [0, 1]$$

f (x) being polynomial satisfies the following conditions :

(a) f (x) is continuous in [0, 1]

(b) f (x) is derivable in (0, 1)

(c) Also, $f(0) = 0$

and

$$f(1) = \frac{a_0}{n+1} + \frac{a_1}{n} + \ldots\ldots + \frac{a_{n-1}}{2} + a_n = 0 \qquad \text{(given)}$$

∴ $f(0) = f(1)$

Hence by Rolle's theorem, there exists at least one value of x in (0, 1) such that $f'(x) = 0$.

i.e.

$$f'(x) = a_0 x^n + a_1 x^{n-1} + \ldots\ldots + a_{n-1} x + a_n = 0$$

i.e. $a_0 x^n + a_1 x^{n-1} + \ldots\ldots + a_{n-1} x + a_n$ vanishes for at least one real number x between 0 and 1.

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**Ex. 7 :** Use Rolle's theorem to prove that the equation $ax^2 + bx = \frac{a}{3} + \frac{b}{2}$ has a root between 0 and 1.

**Sol. :** Consider the function f(x) defined as

$$f(x) = a\frac{x^3}{3} + b\frac{x^2}{2} - \frac{a}{3}x - \frac{b}{2}x$$

f(x) being polynomial satisfies the following conditions :

(a) f(x) is continuous in [0, 1]

(b) f(x) is derivable in (0, 1)

(c) Also $f(0) = 0$

and $f(1) = 0$

Hence by Rolle's theorem, there exists at least one value of x in (0, 1) such that $f'(x) = 0$

$$f'(x) = ax^2 + bx - \frac{a}{3} - \frac{b}{2} = 0$$

i.e. $ax^2 + bx = \frac{a}{3} + \frac{b}{2}$ vanish for at least one real number x between 0 and 1.

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**Remark 1 :** For Examples 6 and 7, we construct a function f(x) whose derivative is the given equation.

**Remark 2 : Ex. 6 :** Let $f(x) = \int \left(a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_n\right) dx$

**Ex. 7 :** Let $f(x) = \int \left(ax^2 + bx - \frac{a}{3} - \frac{b}{2}\right) dx.$

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**Ex. 8 :** Show that there is no real number K for which the equation $x^3 - 3x + K = 0$ has two distinct roots [0, 1].

**Sol. :** Consider, $f(x) = x^3 - 3x + K = 0$ and let α and β be two distinct roots of $f(x) = 0$ where, $\alpha < \beta < 1$.

$f(x)$ being a polynomial in x is continuous and derivable for all values of x.

$\therefore$ f(x) is continuous in $[\alpha, \beta]$ and derivable in $(\alpha, \beta)$.

Also $$f(\alpha) = 0 = f(\beta) \quad \{\because \alpha \text{ and } \beta \text{ are roots of } f\ldots$$

$\therefore$ By Rolle's theorem, there must exists at least one value c of x in $(\alpha, \beta)$ such that $f'(c) = 0$

i.e. $$f'(c) = 3c^2 - 3 = 0 \quad \{\because f'(x) = \ldots$$

$\therefore$ $$c = \pm 1$$

which contradicts the fact that $0 < \alpha < c < \beta < 1$. Hence there is no real number K for which the equation has two distinct roots $\alpha$ and $\beta$ in [0, 1].

**Ex. 9 :** If f, $\phi$, $\psi$ are continuous functions in [a, b] and derivable in (a, b), then show that there is a value c lying between a and b such that

$$\begin{vmatrix} f(a) & f(b) & f'(c) \\ \phi(a) & \phi(b) & \phi'(c) \\ \psi(a) & \psi(b) & \psi'(c) \end{vmatrix} = 0$$

**Sol. :** Consider
$$F(x) = \begin{vmatrix} f(a) & f(b) & f(x) \\ \phi(a) & \phi(b) & \phi(x) \\ \psi(a) & \psi(b) & \psi(x) \end{vmatrix}$$

Expanding F (x) (i.e. determinant) in the form
$$F(x) = A\, f(x) + B\, \phi(x) + C\, \psi(x)$$

where A, B, C are constants.

Since f (x), $\phi$ (x), $\psi$ (x) are continuous functions in [a, b] and derivable in (a, b), therefore F (x) is continuous in [a, b] and differentiable in [a, b].

Also $$F(a) = 0 = F(b) \quad \left\{\begin{array}{l}\because \text{ on putting } x = a, b \text{ the two rows become identical} \\ \text{and so the determinant vanishes}\end{array}\right.$$

Hence F (x) satisfies all the three conditions of Rolle's theorem and so there must exists at least one value c of x such that $F'(c) = 0$.

Now
$$F'(x) = A\, f'(x) + B\, \phi'(x) + C\, \psi'(x)$$
$$= \begin{vmatrix} f(a) & f(b) & f'(c) \\ \phi(a) & \phi(b) & \phi'(c) \\ \psi(a) & \psi(b) & \psi'(c) \end{vmatrix}$$

$\therefore$
$$F'(c) = 0 \Rightarrow \begin{vmatrix} f(a) & f(b) & f'(c) \\ \phi(a) & \phi(b) & \phi'(c) \\ \psi(a) & \psi(b) & \psi'(c) \end{vmatrix} = 0$$

which proves the required result.

**Ex. 10 :** If $f(x) = x(x + 1)(x + 2)(x + 3)$, then show that $f'(x)$ has three zeros.

**Sol. :** Consider the given polynomial function in the closed intervals [– 3, – 2], [– 2, – 1], [– 1, 0] and note that :

(a) f (x) is continuous in [– 3, – 2], [– 2, –1], [– 1, 0]

(b) f (x) is derivable in (–3, – 2) (–2, – 1) (– 1, 0)

(c) F (– 3) = F (– 2) = f (– 1) = f (0) = 0

Hence all the conditions of Rolle's theorem are satisfied by f(x) in closed intervals [−3, −2], [−2, −1], [−1, 0]. Therefore, there exist $c_1 \in (-3, -2)$; $c_2 \in (-2, -1)$; $c_3 \in (-1, 0)$ such that $f'(c_1) = 0$; $f'(c_2) = 0$; $f'(c_3) = 0$ respectively. Thus $c_1, c_2, c_3$ are roots of $f'(x) = 0$. Hence the required result.

**Ex. 11 :** Prove that between any two real roots of $e^x \sin x = 1$, there is at least one root of

$$e^x \cos x + 1 = 0.$$

**Sol. :** Consider the function $f(x) = e^{-x} - \sin x$

Let α and β be any two real and distinct roots of f (x) = 0 where α < β.

f (x) being composite function of continuous and differentiable functions is continuous and derivable for all values of x.

∴ f (x) is continuous in [α, β] and differentiable in (α, β).

Also
$$f(\alpha) = f(\beta) = 0 \qquad \{\because \alpha \text{ and } \beta \text{ are roots of } f(x) = 0\}$$

Therefore by Rolle's theorem, there must exists at least one value c of x, in (α, β) such that f ′(c) = 0

i.e.
$$-e^{-c} - \cos c = 0 \text{ or } e^{-c} + \cos c = 0$$

i.e.
$$1 + e^c \cos c = 0$$

i.e. c satisfies the equation $e^x \cos x + 1 = 0$. Hence the required result.

## 1.4 LAGRANGE'S MEAN VALUE THEOREM

**Statement :** If a function f (x) is

(i) continuous in the closed interval [a, b]

(ii) differentiable in the open interval (a, b)

then there exists at least one point x = c in (a, b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

**Proof :** To prove the theorem, we construct a new function F (x) involving f (x) such that it satisfy the conditions of Rolle's theorem.

Consider the function

$$F(x) = f(x) + Ax, \; x \in [a, b] \qquad \dots (1)$$

where, A is a constant to be determined such that F (a) = F (b).

Thus
$$f(a) + Aa = f(b) + Ab$$

$$\therefore \quad A = -\frac{f(b) - f(a)}{b - a} \qquad \dots (2)$$

Substituting the value of A from (2) in (1), we have

$$F(x) = f(x) - \frac{f(b) - f(a)}{b - a} x \qquad \dots (3)$$

Now, we note that

(i) Since f (x) is continuous in [a, b] and also x is continuous function for all finite values of x, therefore from (3), F (x) is continuous in [a, b].

(ii) Since f (x) is differentiable in (a, b) and x being differentiable in every interval, it is differentiable in (a, b) and hence from (3), F (x) is differentiable in (a, b).

(iii) Also F (a) = F (b).

Hence F (x) satisfies all the three conditions of Rolle's theorem. Therefore, there exists at least one point $c \in (a, b)$ such that $F'(c) = 0$, so that from (1), we have

$$F'(x) = f'(x) + A = 0$$

$$\therefore \quad F'(c) = f'(c) + A = 0 \text{ or } f'(c) = -A$$

$$\therefore \quad \frac{f(b) - f(a)}{b - a} = f'(c)$$

which proves the Lagrange's Mean Value Theorem (LMVT).

**Alternative Statement of Lagrange's Mean Value Theorem**

If we write $b - a = h$ (i.e. if h denote the length of the closed interval [a, b]), then the closed interval [a, b] can be rewritten as [a, a + h] and by the mean value theorem result (4), we have

$$\frac{f(b) - f(a)}{h} = f'(c), \text{ where, } a < c < b$$

Also, since c lies between a and b, we have

$$a < c < b$$

i.e. $$0 < c - a < b - a$$

i.e. $$0 < \frac{c - a}{b - a} < 1$$

i.e. $$0 < \theta < 1, \text{ where, } \theta = \frac{c - a}{b - a} = \frac{c - a}{h}$$

$$\therefore \quad c = a + \theta h, \quad 0 < \theta < 1$$

*In other words, the point c which lies in (a, b) i.e. in (a, a + h) is greater than a and less than a + h, so that we may write $c = a + \theta h$, where $\theta$ is some number between 0 and 1.*

Substituting the values of $b = a + h$ and $c = a + \theta h$ in result (5), the *alternative form of the Lagrange mean value theorem (LMVT) is*

$$\frac{f(a + h) - f(a)}{h} = f'(a + \theta h) \quad \text{where } 0 < \theta < 1$$

$$\therefore \quad f(a + h) = f(a) + h f'(a + \theta h) \quad \text{where } 0 < \theta < 1$$

**Remark 1 :**

Rolle's theorem is a special case (when f(b) = f(a)) of Lagrange's Mean Value Theorem (LMVT).

**Remark 2 : Geometrical Interpretation of Lagrange's Mean Value Theorem :**

Let the graph of the function f (x) is continuous curve, represented by APB. The geometric interpretation is that if the curve APB has tangent at all points on it between A [a, f (a)] and B [b, f (b)], then there exists at least one point P [c, f (c)] on the curve between A and B such that the tangent at P is parallel to the chord joining extremities of the curve (See Figs. 1.4 and 1.5).

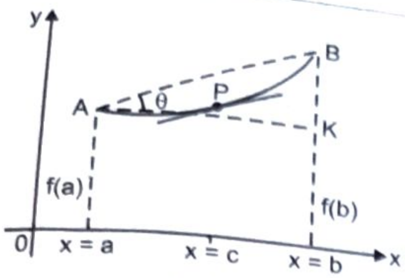


Fig. 1.4

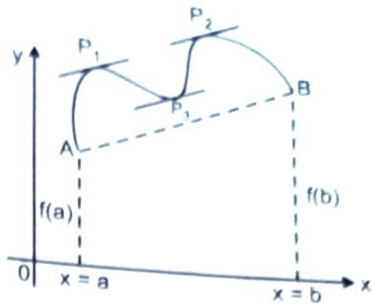


Fig. 1.5

Thus, if θ be the inclination of chord AB to x–axis then, we have

$$\text{Slope of chord AB} = \tan\theta = \frac{BK}{AK} = \frac{f(b) - f(a)}{b - a}$$

and slope of the tangent at P [c, F (c)] is $f'(c)$.

Since the chord is parallel to tangent at P

$$\therefore \quad \frac{f(b) - f(a)}{b - a} = f'(c), \; a < c < b$$

**Remark 3 : Algebraic Interpretation of Lagrange's Mean Value Theorem :**

We note that f (b) – f (a) is the change in the function f (x) from a to b so that $\frac{[f(b) - f(a)]}{(b - a)}$ is the average rate of change of the function over the closed interval [a, b]. Also $f'(c)$ is the actual rate of change of the function for x = c. Thus, *the theorem states that the average rate of change of a function over an interval is also the actual rate of change of the function at some point of the interval.*

This interpretation of theorem justifies the name Mean Value for the theorem.

**Remark 4 : Solution of f(x) = 0 using Lagrange's Mean Value Theorem :**

Using LMV theorem, approximate solution of f(x) = 0 can be obtained by Newton's methods as follows :

$$0 = f(a + h) = f(a) + h f'(a + \theta h), \; 0 < \theta < 1$$

Therefore, $h = -\frac{f(a)}{f'(a)}$. Thus starting at a guess value a, h (correction) can be calculated approximately by iteration a better root can be obtained.

**Remark 5 : Deductions from Mean Value Theorem (Meaning of the Sign of Derivative) :**

Consider a function f (x) which satisfies the conditions of the Lagrange's mean value theorem in [a, b]. Let $x_1$, $x_2$ be any two points of [a, b] such that $x_1 < x_2$.

Applying the Lagrange's mean value theorem to $[x_1, x_2]$, we see that there exists $\xi \in (x_1, x_2)$ such that

$$f(x_2) - f(x_1) = (x_2 - x_1) f'(\xi) \qquad \text{... (1)}$$

Now we consider the following cases :

**Case I :** If $f'(x) = 0$ for every $x \in (a, b)$, then f (x) is constant in that interval.

From (1), we get

$$f(x_2) - f(x_1) = 0 \qquad \text{... (2)}$$

$$\therefore \quad f(x_2) = f(x_1)$$

Since $x_1$ and $x_2$ are any two members of [a, b], it follows that f (x) has the same value for every x in [a, b] and so it is a constant function.

We note that this is the converse of the theorem, "Derivative of a constant function is the zero function".

**Corollary** : If two functions f (x) and $\phi$ (x) have the same derivative for every $x \in$ (a, b), then they differ only by a constant.

Given that $\quad f'(x) = \phi'(x)$

Let $\quad F(x) = f(x) - \phi(x)$.

$F'(x) = f'(x) - \phi'(x) = 0$ for every $x \in (a, b)$.

Derivative of $[f(x) - \phi(x)]$ is zero for every $x \in (a, b)$.

$\therefore$ F (x) is a constant function.

**Case II** : If $f'(x) > 0$ for every $x \in (a, b)$, then f (x) is strictly increasing function *(monotonically increasing function)* in [a, b].

From (1), we get $\qquad x_2 - x_1 > 0$ and $f'(\xi)$ is positive

$$f(x_2) - f(x_1) > 0$$

$$\therefore \quad f(x_2) > f(x_1)$$

Thus, f (x) is strictly increasing function *(monotonically increasing function)* in [a, b].

**Case III** : If $f'(x) < 0$ for every $x \in (a, b)$ then f (x) is strictly decreasing function *(monotonically decreasing function)* in [a, b].

From (1), we get

$$f(x_2) - f(x_1) < 0 \qquad \text{for } x_2 - x_1 > 0 \text{ and } f'(\xi) \text{ is negative} \qquad \ldots (4)$$

$$\therefore \quad f(x_2) < f(x_1)$$

Thus, f (x) is strictly decreasing function *(monotonically decreasing function)* in [a, b].

## 1.5 ILLUSTRATIONS ON LAGRANGE'S MEAN VALUE THEOREM

### TYPE (I) PROBLEMS ON VERIFICATION OF ROLLE'S THEOREM

**Ex. 1 :** Verify the Lagrange's Mean Value theorem for the following functions :

(i) $f(x) = \log x$ in [1, e] $\qquad$ (ii) $f(x) = x^3$ in [– 2, 2]

(iii) $f(x) = x(x-1)(x-2)$ in [0, 1/2] $\qquad$ (iv) $f(x) = lx^2 + mx + n$ in [a, b]

**Sol. : (i) f (x) = log x, x ∈ [1, e]**

The given function f (x) = log x is continuous in [1, e] and differentiable in (1, e). Hence all the conditions of the Lagrange's Mean Value Theorem (LMVT) are satisfied. Therefore there exists at least one point c in (1, e) such that

$$\frac{f(e) - f(1)}{e - 1} = f'(c) \qquad \left\{\because \frac{f(b) - f(a)}{b - a} = f'(c)\right.$$

$$\therefore \quad \frac{\log e - \log 1}{e - 1} = \frac{1}{c} \qquad \left\{\begin{array}{l} \because f'(x) = \frac{1}{x} \\ \therefore f'(c) = \frac{1}{c} \end{array}\right.$$

or $$\frac{1 - 0}{e - 1} = \frac{1}{c}$$

$$\therefore \quad c = e - 1$$

Since 2 < e < 3, the value of c = e – 1 lies in the open interval (1, e). Therefore, LMVT is verified for the given function in the given interval.

**(ii) $f(x) = x^3, x \in [-2, 2]$**

We note the following :

(a) $f(x) = x^3$ being polynomial in x is continuous in $[-2, 2]$.

(b) $f'(x) = 3x^2$ is finite and unique for all x. Hence all the conditions of LMVT are satisfied. Therefore, there exists at least one point x = c in (– 2, 2) such that

$$\frac{f(2) - f(-2)}{2 - (-2)} = f'(c)$$

$$\therefore \quad \frac{(2)^3 - (-2)^3}{4} = 3c^2 \qquad \left\{ \begin{array}{l} f'(x) = 3x^2 \\ \therefore f'(c) = 3c^2 \end{array} \right.$$

$$\therefore \quad c^2 = \frac{4}{3} \text{ or } c = \pm \frac{2}{\sqrt{3}}$$

Clearly, these values of c lie in (– 2, 2). Hence the theorem is verified.

**Remark :**

Let $c_1 = -\frac{2}{\sqrt{3}}$ and $c_2 = \frac{2}{\sqrt{3}}$, then tangents to the curve of $f(x) = x^3$ at $c_1$ and $c_2$ are parallel to chord joining (– 2, – 8) and (2, 8). [See Fig. 1.6]

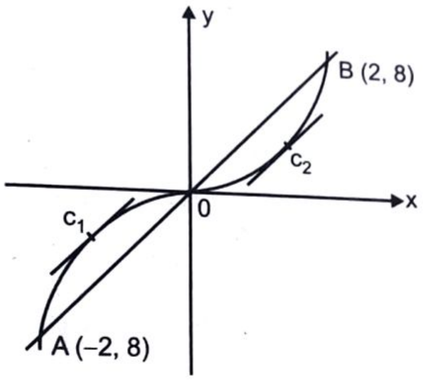


**Fig. 1.6**

**(iii) $f(x) = x(x-1)(x-2), x \in [0, 1/2]$**

We note the following :

(a) $f(x) = x(x-1)(x-2) = x^3 - 3x^2 + 2x$ being polynomial in x is continuous in [0, 1/2].

(b) $f'(x) = 3x^2 - 6x + 2$ is always unique and definite for every value of x in (0, 1/2). Hence f (x) is derivable in (0, 1/2). Hence f (x) satisfies all the conditions of LMVT and therefore there exists at least one point c in (0, 1/2) such that

$$\frac{f(1/2) - f(0)}{1/2 - 0} = f'(c)$$

$$\therefore \quad \frac{3/8 - 0}{1/2} = 3c^2 - 6c + 2 \qquad \left\{ \begin{array}{l} \because f'(x) = 3x^2 - 6x + 2 \\ \therefore f'(c) = 3c^2 - 6c + 2 \end{array} \right.$$

$$\text{or} \quad 3c^2 - 6c + 2 = 3/4 \text{ or } 12c^2 - 24c + 5 = 0$$

$$\therefore \quad c = \frac{6 + \sqrt{21}}{6} \text{ and } \frac{6 - \sqrt{21}}{6}$$

The value of $c = \frac{6 - \sqrt{21}}{6}$ lies between 0 and 1/2. Hence the LMVT is verified.

**(iv) $f(x) = lx^2 + mx + n, x \in [a, b]$**

We note the following :

(a) $f(x) = lx^2 + mx + n$ being polynomial in x is continuous function in [a, b].

(b) $f'(x) = 2lx + m$ which exists finitely in (a, b).

Hence f (x) satisfies all the conditions of LMVT. Therefore there must exists at least one point $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

$$\therefore \quad \frac{(lb^2 + mb + n) - (la^2 + ma + n)}{b - a} = 2lc + m$$

$$\therefore \quad \frac{l(b^2 - a^2) + m(b - a)}{(b - a)} = 2lc + m$$

$$\therefore \quad l(b + a) + m = 2lc + m$$

$$\therefore \quad c = \frac{a + b}{2}$$

Clearly, $c = \frac{a+b}{2} \in (a, b)$. LMVT is verified.

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**Ex. 2 :** Discuss the applicability of Lagrange's Mean Value theorem to the following functions :

(i) $f(x) = \frac{1}{x}$ in $[-1, 1]$ (ii) $f(x) = x^{1/3}$ in $[-1, 1]$

(iii) $f(x) = |x|$ in $[-1, 2]$ (iv) $f(x) = x - x^3$ in $[-2, 1]$

**Sol. : (i) $f(x) = \frac{1}{x}$, $x \in [-1, 1]$**

We note that f (x) is not finite at x = 0 which is a point of closed interval [–1, 1]

Also L.H.L. = $-\infty$ and R.H.L. = $\infty$

Therefore, Lagrange's Mean Value Theorem (LMVT) is not applicable.

**(ii) $f(x) = x^{1/3}$, $x \in [-1, 1]$**

We note that

$$f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$$

does not exist at x = 0 which is a point of (– 1, 1). Hence f (x) is not differentiable in (–1, 1).

**Remark :** We observe that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

$$\therefore \quad \frac{f(1) - f(-1)}{1 - (-1)} = f'(c)$$

$$\therefore \quad \frac{1 - (-1)}{2} = \frac{1}{3c^{2/3}} \text{ or } c = \left(\frac{1}{3}\right)^{3/2}$$

$$\therefore \quad c = \frac{1}{3\sqrt{3}} \text{ and } \frac{1}{3\sqrt{3}} \in (-1, 1)$$

Hence the hypothesis of LMVT is not valid. Thus the two conditions of LMVT are sufficient but not necessary.

**(iii)** $f(x) = |x|, x \in [-1, 2]$

Here we note that $f(x) = |x|$ is continuous in the closed interval $[-1, 2]$. Also $f(x)$ is differentiable at every point of the open interval $(-1, 2)$ except at $x = 0$.

Thus $f(x)$ is not differentiable in the open interval $(-1, 2)$ and hence the second condition of Lagrange's Mean Value Theorem is not satisfied and therefore LMVT is not applicable to $f(x)$ in $[-1, 2]$.

**(iv)** $f(x) = x - x^3, x \in [-2, 1]$

We note that

(a) $f(x)$ being polynomial function in x is continuous function in $[-2, 1]$

(b) $f'(x) = 1 - 3x^2$ which exists finitely in $(-2, 1)$.

Hence all the conditions of LMVT are satisfied.

Therefore there exists at least one point $c \in (-2, 1)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

$$\therefore \quad \frac{f(1) - f(-2)}{1 - (-2)} = f'(c)$$

$$\therefore \quad \frac{0 - 6}{3} = 1 - 3c^2 \qquad \left\{ \begin{array}{l} \because f'(x) = 1 - 3x^2 \\ \therefore f'(c) = 1 - 3c^2 \end{array} \right.$$

$$\therefore \quad c = \pm 1$$

Clearly the values of $c = -1$ lie in $(-2, 1)$. Hence the LMVT is verified.

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**Ex. 3 :** Using Lagrange's Mean Value Theorem, find 'c' such that $a < c < b$ if

$$f(x) = (x-1)(x-2)(x-3) \; ; a = 0, b = 4.$$

**Sol. :** We have $f(4) = 6$ and $f(0) = -6$

$$\therefore \quad \frac{f(4) - f(0)}{4 - 0} = \frac{6 - (-6)}{4} = 3$$

Also

$$f'(x) = (x-2)(x-3) + (x-3)(x-1) + (x-1)(x-2)$$

$$= 3x^2 - 12x + 11$$

By Lagrange's Mean Value Theorem, we have

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

$$\therefore \quad 3 = 3c^2 - 12c + 11$$

$$\therefore \quad c = \frac{6 + 2\sqrt{3}}{8}, \frac{6 - 2\sqrt{3}}{8}$$

Taking $\sqrt{3} = 1.732$ ... we may see that both these values of c belong to the open interval $(0, 4)$, so that we have two numbers for c in this case. Hence the answer.

## TYPE II : PROBLEMS ON ALTERNATIVE FORM OF LMVT

**Ex. 4 :** Prove that for any quadratic function $px^2 + qx + r$ in $[a, a + h]$, the value of $\theta$ in Lagrange's theorem is always $\frac{1}{2}$ whatever p, q, r, a h may be.

**Sol. :** Let

$$f(x) = px^2 + qx + r, x \in [a, a + h]$$

Since f (x) is a polynomial, it satisfies the conditions of Lagrange's Mean Value Theorem and hence there exists $\theta$ $(0 < \theta < 1)$ satisfying

$$f(a + h) - f(a) = hf'(a + \theta h)$$

$$\therefore \quad \{p(a + h)^2 + q(a + h) + r\} - \{pa^2 + qa + r\} = h\{2p(a + \theta h) + q\}$$

$$\therefore \quad 2aph + ph^2 + qh = 2aph + 2p\theta h^2 + qh$$

$$\therefore \quad ph^2 = 2ph^2\theta$$

$$\therefore \quad \theta = \frac{1}{2}$$

Hence the answer.

**Ex. 5 :** Using Lagrange's Mean Value Theorem, prove that,

$$\log_{10}(x + 1) = \frac{(x \log_{10} e)}{(1 + \theta x)}, \quad \text{where, } 0 < \theta <$$

**Sol. :** Consider the function

$$f(x) = \log_{10} x, x \in [1, 1 + x], x > 0$$

We note that

(a) $f(x) = \log_{10} x$ is continuous function in $[1, 1 + x]$

(b) $f'(x) = \frac{d}{dx}[\log_e x/\log_e 10] = \frac{1}{x \log_e 10}$ exists and finite in $(1, 1 + x)$.

Hence by using alternative form of LMVT, we have

$$f(a + h) = f(a) + hf'(a + \theta h), \quad \text{where, } 0 < \theta <$$

$$\therefore \quad \log_{10}(1 + x) = \log_{10}(1) + x\frac{1}{(1 + \theta x)\log_e 10}, \quad \text{where, } 0 < \theta <$$

$$\therefore \quad \log_{10}(1 + x) = 0 + \frac{x \log_{10} e}{1 + \theta x}, \quad \text{where, } 0 < \theta <$$

$$\therefore \quad \log_{10}(x + 1) = \frac{x \log_{10} e}{1 + \theta x}, \quad \text{where, } 0 < \theta <$$

Hence the required result.

**Ex. 6 :** Let f (x) be defined and continuous in $[a - h, a + h]$ and differentiable in $(a - h, a + h)$. Prove that there is a real number $\theta$ between 0 and 1 such that

$$f(a + h) - f(a - h) = h[f'(a + \theta h) + f'(a - \theta h)]$$

**Sol. :** Define a function $\phi(x)$ such that

$$\phi(x) = f(a + hx) - f(a - hx), x \in [0, 1]$$

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As x varies in [0, 1], a + hx varies in [a, a + h] and a − hx varies in [a − h, a]. Thus $\phi(x)$ is continuous in [0, 1] and differentiable in (0, 1). Hence by alternative form of LMVT, there exists $\theta$ $(0 < \theta < 1)$ such that;

$$\frac{\phi(1) - \phi(0)}{1 - 0} = \phi'(\theta), \ 0 < \theta < 1$$

$$\therefore \quad f(a + h) - f(a - h) = h\,[f'(a + \theta h) + f'(a - \theta h)]$$

Hence the required result.

## TYPE III : PROBLEMS ON INEQUALITY

**Ex. 7 :** Prove that if $0 < a < b$

$$\frac{b - a}{1 + b^2} < (\tan^{-1} b - \tan^{-1} a) < \frac{b - a}{1 + a^2}$$

and deduce that (i) $\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}$ (ii) $\frac{5\pi + 4}{20} < \tan^{-1} 2 < \frac{\pi + 2}{4}$

**Sol. :** Let $f(x) = \tan^{-1} x$, $x \in [a, b]$

$$\therefore \quad f'(x) = \frac{1}{1 + x^2}$$

By Lagrange's Mean Value Theorem, there exists at least one point $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

$$\therefore \quad \frac{\tan^{-1} b - \tan^{-1} a}{b - a} = \frac{1}{1 + c^2}$$

Since $a < c$ then $a^2 < c^2$ and $1 + a^2 < 1 + c^2$, therefore

$$\frac{1}{1 + a^2} > \frac{1}{1 + c^2}$$

$$\therefore \quad \frac{\tan^{-1} b - \tan^{-1} a}{b - a} < \frac{1}{1 + a^2}$$

$$\therefore \quad \tan^{-1} b - \tan^{-1} a < \frac{b - a}{1 + a^2} \qquad \text{... (1)}$$

Again, since $c < b$ then $c^2 < b^2$ and $1 + c^2 < 1 + b^2$, therefore

$$\frac{1}{1 + c^2} > \frac{1}{1 + b^2}$$

$$\therefore \quad \frac{\tan^{-1} b - \tan^{-1} a}{b - a} > \frac{1}{1 + b^2}$$

$$\therefore \quad \tan^{-1} b - \tan^{-1} a > \frac{b - a}{1 + b^2} \qquad \text{... (2)}$$

From (1) and (2), we obtain

$$\frac{b - a}{1 + b^2} < (\tan^{-1} b - \tan^{-1} a) < \frac{b - a}{1 + a^2} \qquad \text{... (3)}$$

which is the required result.

(i) Now let $b = \frac{4}{3}$ and $a = 1$ in result (3), we get

$$\frac{4/3-1}{1+(4/3)^2} < \tan^{-1}\frac{4}{3} - \tan^{-1} 1 < \frac{4/3-1}{1+(1)^2}$$

$$\therefore \quad \frac{3}{25} < \tan^{-1}\frac{4}{3} - \frac{\pi}{4} < \frac{1}{6}$$

$$\therefore \quad \frac{3}{25} + \frac{\pi}{4} < \tan^{-1}\frac{4}{3} < \frac{1}{6} + \frac{\pi}{4}$$

which is the required deduction.

(ii) Next let $a = 1, b = 2$ in result (3), we get,

$$\frac{2-1}{1+4} < \tan^{-1} 2 - \tan^{-1} 1 < \frac{2-1}{1+1}$$

$$\frac{1}{5} < \tan^{-1} 2 - \frac{\pi}{4} < \frac{1}{2}$$

Adding $\pi/4$,

$$\frac{5\pi+4}{20} < \tan^{-1} 2 < \frac{\pi+2}{4}$$

**Ex. 8 :** Prove that if $a < 1$, $b < 1$ and $a < b$, then

$$\frac{b-a}{\sqrt{1-a^2}} < \sin^{-1} b - \sin^{-1} a < \frac{b-a}{\sqrt{1-b^2}}$$

and hence show that

$$\frac{\pi}{6} - \frac{1}{2\sqrt{3}} < \sin^{-1}\frac{1}{4} < \frac{\pi}{6} - \frac{1}{\sqrt{15}}$$

**Sol. :** Let $f(x) = \sin^{-1} x, \quad x \in [a, b]$

$$\therefore \quad f'(x) = \frac{1}{\sqrt{1-x^2}}$$

By Lagrange's Mean Value Theorem, we have

$$\frac{\sin^{-1} b - \sin^{-1} a}{b-a} = \frac{1}{\sqrt{1-c^2}} \quad \text{... (1)}$$

Since $a < c < b$ then $a^2 < c^2 < b^2$

$$\therefore \quad 1-a^2 > 1-c^2 > 1-b^2$$

$$\therefore \quad \frac{1}{\sqrt{1-a^2}} < \frac{1}{\sqrt{1-c^2}} < \frac{1}{\sqrt{1-b^2}} \quad \text{... (2)}$$

Using (2), result (1) can be expressed as

$$\frac{1}{\sqrt{1-a^2}} < \frac{\sin^{-1} b - \sin^{-1} a}{b-a} < \frac{1}{\sqrt{1-b^2}}$$

$$\therefore \quad \frac{b-a}{\sqrt{1-a^2}} < \frac{\sin^{-1} b - \sin^{-1} a}{b-a} < \frac{b-a}{\sqrt{1-b^2}} \quad \text{... (3)}$$

which is the required result.

Now let $b = \frac{1}{2}$ and $a = \frac{1}{4}$

$$\therefore \quad \frac{1/2 - 1/4}{\sqrt{1-(1/4)^2}} < \sin^{-1}\frac{1}{2} - \sin^{-1}\frac{1}{4} < \frac{1/2 - 1/4}{\sqrt{1-(1/2)^2}}$$

$$\therefore \quad \frac{1}{\sqrt{15}} < \frac{\pi}{6} - \sin^{-1}\frac{1}{4} < \frac{1}{2\sqrt{3}}$$

$$\therefore \quad -\frac{1}{\sqrt{15}} > -\frac{\pi}{6} + \sin^{-1}\frac{1}{4} > -\frac{1}{2\sqrt{3}}$$

$$\therefore \quad \frac{\pi}{6} - \frac{1}{2\sqrt{3}} < \sin^{-1}\frac{1}{4} < \frac{\pi}{6} - \frac{1}{\sqrt{15}}$$

which is the required deduction.

**Remark :** If we let b = 3/5 and a = 1/2 in (3), we obtain the result

$$\frac{1}{5\sqrt{3}} < \sin^{-1}\frac{3}{5} - \frac{\pi}{6} < \frac{1}{8}$$

---

**Ex. 9 :** If f (0) = 2, $f'(x) = \frac{1}{5-x^2}$, estimate f (1).

**Sol. :** We have by Lagrange's Mean Value Theorem,

$$\frac{f(b) - f(a)}{b-a} = f'(c)$$

$$\therefore \quad \text{minimum of } f'(c) \leq \frac{f(b)-f(a)}{b-a} \leq \text{maximum of } f'(c)$$

Let a = 0, b = 1 and $f'(c) = \frac{1}{5-c^2}$

$$\therefore \quad \text{minimum } \frac{1}{5-c^2} \leq \frac{f(1)-f(0)}{b-a} \leq \text{maximum } \frac{1}{5-c^2}$$

Now $f'(c) = \frac{1}{5-c^2}$ is smallest when c = 0 and largest when c = 1

$$\therefore \quad \frac{1}{5-0} \leq \frac{f(1)-2}{1-0} \leq \frac{1}{5-1} \qquad \{\because f(0) = 2\}$$

$$\therefore \quad 2 + \frac{1}{5} \leq f(1) \leq 2 + \frac{1}{4}$$

$$\therefore \quad 2.2 \leq f(1) \leq 2.25$$

Hence f (1) lies between 2.2 and 2.25.

---

**Ex. 10 :** Applying Lagrange's Mean Value Theorem to function $e^x$, determine the value of θ in terms of a and h and deduce that $0 < \frac{1}{x}\log\left(\frac{e^x - 1}{x}\right) < 1$

**Sol. :** Let $f(x) = e^x, x \in [a, a+h]$

$$\therefore \quad f'(x) = e^x$$

By Lagrange's Mean Value Theorem

$$\frac{f(a+h)-f(a)}{h} = f'(a+\theta h), \qquad 0<\theta<1$$

$$\therefore \quad \frac{e^{a+h}-e^{a}}{h} = e^{a+\theta h}, \qquad 0<\theta<1$$

$$\therefore \quad \frac{e^{a}(e^{h}-1)}{h} = e^{a}e^{\theta h}, \qquad 0<\theta<1$$

$$\therefore \quad \frac{e^{h}-1}{h} = e^{\theta h}, \qquad 0<\theta<1$$

$$\therefore \quad \theta h = \log\left(\frac{e^{h}-1}{h}\right), \qquad 0<\theta<1$$

$$\therefore \quad \theta = \frac{1}{h}\log\left(\frac{e^{h}-1}{h}\right), \qquad 0<\theta<1$$

For the interval (a, a + x),

$$\theta = \frac{1}{x}\log\left(\frac{e^{x}-1}{x}\right), \qquad 0<\theta<1$$

$$\therefore \quad 0 < \frac{1}{x}\log\left(\frac{e^{x}-1}{x}\right) < 1$$

which is the required deduction.

### TYPE IV : PROBLEMS ON SIGNS OF DERIVATIVE

**Ex. 11 :** Show that $f(x) = x^3 - 3x^2 + 3x + 2$ is strictly (monotonically) increasing in every interval.

**Sol. :** We have given $\quad f(x) = x^3 - 3x^2 + 3x + 2$

$$\therefore \quad f'(x) = 3x^2 - 6x + 3 = 3(x-1)^2$$

Thus, $f'(x) > 0$ for every value of x except at one point x = 1 ; at which $f'(x) = 0$.

Hence the given function is strictly (monotonically) increasing in every interval.

**Ex. 12 :** Separate the intervals in which $f(x) = 2x^3 - 15x^2 + 36x + 1$ is increasing or decreasing.

**Sol. :** We have given $\quad f(x) = 2x^3 - 15x^2 + 36x + 1$

$$\therefore \quad f'(x) = 6x^2 - 30x + 36 = 6(x-2)(x-3)$$

Thus $\quad f'(x) > 0 \quad$ if $x < 2$ ; $\quad$ [ i.e. $f'(x)$ is positive in $(-\infty, 2)$]

$\quad f'(x) < 0 \quad$ if $2 < x < 3$ ; $\quad$ [ i.e. $f'(x)$ is negative in (2, 3)]

$\quad f'(x) > 0 \quad$ if $x > 3$ ; $\quad$ [ i.e. $f'(x)$ is positive in $(3, \infty)$]

Also $\quad f'(x) = 0 \quad$ if x = 2 and 3.

Hence the following result follows.

The given function f (x) is strictly (monotonically) increasing in $(-\infty, 2)$ and $(3, \infty)$; and strictly (monotonically) decreasing in (2, 3).

**Ex. 13 :** (i) If $f(x) = x \sin x + \cos x + \cos^2 x$, then show that $\frac{\pi}{2} < f(x) < 2$.

(ii) If $F(x) = \phi(x) + \phi(2-x)$ and $F''(x) > 0$ in (0, 2), then show that F (x) decreases in (0, 1) and increases in (1, 2).

**Sol. :** (i) We have given $f(x) = x \sin x + \cos x + \cos^2 x$

To prove the required result, we first find the interval on which f (x) is defined.

Since $f(0) = 0 + 1 + 1 = 2$ and $f(\pi/2) = \pi/2 + 0 + 0 = \pi/2$, we choose $(0, \pi/2)$ as the interval for given function f (x).

Now,
$$f'(x) = x \cos x + \sin x - \sin x - 2 \cos x \sin x$$
$$= x \cos x \left(1 - 2\frac{\sin x}{x}\right) \quad \text{...(1)}$$

We note that $x \cos x$ is positive in $(0, \pi/2)$ and

$$\left(1 - 2\frac{\sin x}{x}\right)_{x=0} = 1 - 2 = -1 < 0;$$

$$\left(1 - 2\frac{\sin x}{x}\right)_{x=\pi/2} = 1 - \frac{4}{\pi} < 0$$

Thus from (1), $f'(x) < 0$ for $x \in (0, \pi/2)$.

Hence f (x) is decreasing function $(0, \pi/2)$. i.e. $f(0) > f(x) > f(\pi/2)$

Hence the required result.

(ii) We have given,

$$F(x) = \phi(x) + \phi(2 - x)$$

$\therefore$
$$F'(x) = \phi'(x) - \phi'(2 - x)$$

Also
$$F''(x) = \phi''(x) + \phi''(2 - x)$$

Since $F''(x) > 0$ in (0, 2), therefore $F'(x)$ is increasing function of x in (0, 2).

Also
$$F'(0) = \phi'(0) - \phi'(2)$$

and
$$F'(1) = \phi'(1) - \phi'(1)$$
$$F'(2) = \phi'(2) - \phi'(0)$$

Now since $F'(1) = 0$ and $F'(x)$ is increasing function,

$\therefore F'(x) < 0$ in (0, 1) and $F'(x) > 0$ in (1, 2).

It follows that F (x) decreases in (0, 1) and increases in (1, 2).

**Ex. 14 :** Show that

(i) $\frac{x}{1+x} < \log(1 + x) < x$ for all $x > 0$.

(ii) $x - \frac{x^2}{2} < \log(1 + x) < x - \frac{x^2}{2(1 + x)}$ for all $x > 0$.

(iii) $x - \frac{x^2}{2} < \log(1 + x) < x - \frac{x^2}{2} + \frac{x^3}{3}$ for all $x > 0$.

**Sol :** (i) Consider $f(x) = \log(1 + x) - \frac{x}{1+x}, x > 0$

$\therefore$
$$f'(x) = \frac{1}{1+x} - \frac{1}{1+x} + \frac{x}{(1+x)^2}$$

Since $f'(x) > 0$ for all $x > 0$ and is 0 for $x = 0$, therefore, $f(x)$ is strictly (monotonically) increasing in $(0, \infty)$ and also $f(0) = \log 1 - 0 = 0$.

It follows that $\quad f(x) > f(0) = 0 \quad$ for all $x > 0$

$\therefore \quad \log(1 + x) > \dfrac{x}{1 + x} \quad$ for all $x > 0$ ...(1)

Again consider $\quad F(x) = x - \log(1 + x),\quad x > 0$

$\therefore \quad F'(x) = 1 - \dfrac{1}{1 + x} = \dfrac{x}{1 + x}$

Since $F'(x) > 0$ for all $x > 0$ and is 0 for $x = 0$, therefore, $F(x)$ is strictly (monotonically) increasing in $(0, \infty)$ and also $F(0) = 0 - \log 1 = 0$.

Thus, $\quad F(x) > F(0) = 0 \quad$ for all $x > 0$

$\therefore \quad x > \log(1 + x) \quad$ for all $x > 0$ ...(2)

From (1) and (2), we get the required result.

(ii) Consider $\quad f(x) = \log(1 + x) - x + \dfrac{x^2}{2},\ x > 0$

$\therefore \quad f'(x) = \dfrac{1}{1 + x} - 1 + x = \dfrac{x^2}{1 + x}$

Here since $f'(x) > 0$ for all $x > 0$ and is 0 for $x = 0$, therefore, $f(x)$ is an (monotonically) increasing function in $(0, \infty)$ and also $f(0) = \log 1 - 0 + 0 = 0$.

It follows that $\quad f(x) > f(0) = 0$

$\therefore \quad \log(1 + x) > x - \dfrac{x^2}{2}$ for all $x > 0$ ...(1)

Similarly, by considering the function,

$$F(x) = x - \frac{x^2}{2(1 + x)} - \log(1 + x),\ x > 0$$

We can show that

$$\log(1 + x) < x - \frac{x^2}{2(1 + x)} \qquad \text{for all } x > 0 \qquad ...(2)$$

From (1) and (2), we get the required result.

(iii) Considering $f(x) = \log(1 + x) - x + \dfrac{x^2}{2}$ and using result (1) in the example (ii) above, we obtain

$$x - \frac{x^2}{2} < \log(1 + x) \qquad \text{for all } x > 0 \qquad ...(1)$$

Again consider $\quad F(x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \log(1 + x),\ x > 0$

$\therefore \quad F'(x) = 1 - x + x^2 - \dfrac{1}{1 + x} = \dfrac{x^3}{1 + x}$

Here since $F'(x) > 0$ for all $x$ and is 0 for $x = 0$, therefore, $F(x)$ is strictly (monotonically) increasing function in $(0, 1)$ and also $F(0) = 0$.

It follows that, $\quad F(x) > F(0) = 0$

$\therefore \quad x - \dfrac{x^2}{2} + \dfrac{x^3}{3} > \log(1 + x)$ for all $x > 0$ ...(2)

From (1) and (2), we get the required result.

## Miscellaneous Problems

**Ex. 15 :** Apply Rolle's theorem to $f(x) = \sin x \sqrt{\cos 2x}$ in $\left[0, \frac{\pi}{4}\right]$ and find the value of c such that $f'(c) = 0$. Also show that $0 < \sin x \sqrt{\cos 2x} < \frac{\sqrt{2}}{4}$ for $0 < x < \frac{\pi}{6}$.

**Sol. :** Here $f(x) = \sin x \sqrt{\cos 2x}$

(a) Since f (x) being product of two continuous functions, there it is continuous in $[0, \pi/4]$.

(b) $$f'(x) = \cos x \sqrt{\cos 2x} + \sin x \frac{1}{2\sqrt{\cos 2x}} (-2 \sin 2x)$$

$$= \frac{\cos x \cos 2x - \sin x \sin 2x}{\sqrt{\cos 2x}} = \frac{\cos 3x}{\sqrt{\cos 2x}}$$

which exist and finite in $(0, \pi/4)$.

(c) Also $f(0) = 0 = f(\pi/4)$

Hence all the conditions of Rolle's theorem are satisfied. Therefore, there exists at least one point $x = c$ in $(0, \pi/4)$ such that $f'(c) = 0$.

i.e. $$\frac{\cos 3c}{\sqrt{\cos 2c}} = 0 \text{ or } \cos 3c = 0$$

$\therefore$ $$3c = \frac{\pi}{2} \text{ or } c = \frac{\pi}{6}$$

which lies in $(0, \pi/4)$.

To show that $0 < \sin x < \sqrt{\cos 2x} < \frac{\sqrt{2}}{4}$ in $(0, \pi/6)$, we consider sines of derivative for increasing or decreasing function.

We have, $$f(x) = \sin x \sqrt{\cos 2x}$$

$\therefore$ $$f'(x) = \frac{\cos 3x}{\sqrt{\cos 2x}}$$

and $$f'(0) = 1 \text{ and } f'(\pi/6) = 0.$$

Since $f'(x) > 0$ for all x in $(0, \pi/6)$, therefore, f (x) is increasing function of x in $(0, \pi/6)$.

i.e. $$f(0) < f(x) < f(\pi/6) \quad \text{for all x in } (0, \pi/6)$$

Now since $f(0) = 0$ and $f(\pi/6) = \sin \pi/6 \sqrt{\cos \pi/3} = \frac{\sqrt{2}}{4}$, we obtain

$$0 < \sin x \sqrt{\cos 2x} < \frac{\sqrt{2}}{\pi}$$

Hence the required result.

---

**Ex. 16 :** If $\frac{\sin \theta}{\theta} = \cos x\theta$, be equation in x, show that it has one root lying between 0 and 1 and deduce that

$$\cos \theta < \frac{\sin \theta}{\theta} < 1$$

Sol. : Let $f(x) = \cos x\theta - \frac{\sin\theta}{\theta}$

Construct a function f (x) such that $f'(x) = \phi(x)$

i.e.
$$f(x) = \int \phi(x)\,dx = \int\left(\cos x\theta - \frac{\sin\theta}{\theta}\right) dx$$
$$= \int \cos x\theta\,dx - \int \frac{\sin\theta}{\theta}\,dx$$
$$= \frac{\sin x\theta}{\theta} - \frac{\sin\theta}{\theta}\,x$$

We note the following :

(a) f (x) is continuous in [0, 1].

(b) $f'(x) = \phi(x)$ is exist and finite in (0, 1)

(c) also f (0) = 0 = f (1).

Hence by Rolle's theorem, there exists at least one point x = c in (0, 1) such that $f'(c) = 0$.

Thus c is a root of $f'(x) = 0$ i.e. of $\phi(x) = 0$ which lies between 0 and 1.

To show that required deduction, we have,
$$f(x) = \frac{\sin x\theta}{\theta} - \frac{\sin\theta}{\theta}\,x$$
$$\therefore \quad f'(x) = \cos x\theta - \frac{\sin\theta}{\theta}$$
Also,
$$f'(0) = 1 - \frac{\sin\theta}{\theta} > 0 \text{ and } f'(1) = \cos\theta - \frac{\sin\theta}{\theta} < 0$$
$$\therefore \quad \cos\theta < \frac{\sin\theta}{\theta} < 1.$$

Hence the required deduction.

---

**Ex. 17 :** Using Lagrange's mean value theorem, prove that $\sqrt{101}$ lies between 10 and 10.05.

**Sol. :** Let $f(x) = x^{1/2}$ be defined in [100, 101]
$$\therefore \quad f'(x) = \frac{1}{2}x^{-1/2}$$

Applying LMVT to f(x) in [100, 101],
$$\frac{f(101) - f(100)}{101 - 100} = f'(c), \text{ where } 100 < c < 101$$
or
$$f(101) = f(100) + f'(c)$$
or
$$\sqrt{101} = \sqrt{100} + \frac{1}{2}c^{-1/2}$$
$$< 10 + \frac{1}{2}\,\frac{1}{10} \qquad \left\{\because c > 100 \therefore \frac{1}{c} < \frac{1}{100} \Rightarrow \frac{1}{\sqrt{c}} < \frac{1}{\sqrt{100}} = \frac{1}{10}\right.$$
$$= 10 + \frac{1}{20}$$
$$= 10.05$$
Again
$$100 < 101 \text{ or } \sqrt{100} < \sqrt{101} \text{ or } 10 < \sqrt{101}$$
Thus
$$10 < \sqrt{101} < 10.05$$

**Ex. 18 :** Calculate approximately the root of equation $x^4 - 12x + 7 = 0$ near 2.

**Sol. :** Choose $f(x) = x^4 - 12x + 7$ so that $f'(x) = 4x^3 - 12$

By LMVT
$$f(a + h) = f(a) + h f'(a + \theta h)$$
$$h = -\frac{f(a)}{f'(a)}$$
$$f(2) = 16 - 24 + 7 = -1, \quad f'(2) = 32 - 12 = 20$$
$$\therefore \quad h = -\frac{(-1)}{20} = 0.05$$

An approximate root $x = a + h = 2 + 0,.05 = 2.05$

Applying LMVT agin
$$h = -\frac{f(2.05)}{f'(2.05)} = -\frac{(2.05)^4 - 12(2.05) + 7}{4(2.05)^3 - 12} = -\frac{0.061}{22.46}$$
$$= -0.0027$$

A better approximated root, $x = a + h = 2.05 - 0.0027 = 2.0473$

## 1.6 CAUCHY'S MEAN VALUE THEOREM

**Statement :** If f (x) and g (x) are any two functions of x such that they are

(a) continuous in the closed interval [a, b].

(b) derivable in the open interval (a, b).

(c) $g'(x) \neq 0$ for any value of x in (a, b).

then there exists at least one point x = c in (a, b) such that
$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

**Proof :** [*Note :* We note that $g(b) - g(a) \neq 0$, for if g (b) = g (a), then the function g (x) satisfies all the conditions of Rolle's theorem and so there must exists at least one value of x in (a, b) for which $g'(x) = 0$ which contradicts the hypothesis (condition (c) of the theorem) that $g'(x) \neq 0$ for any value of x in (a, b)].

Consider the function F (x) defined by,
$$F(x) = f(x) + A\,g(x) \quad \text{...(1)}$$

where, A is a constant to be determined, such that F (a) = F (b). Thus,
$$f(a) + A\,g(a) = f(b) + A g(b)$$
$$\therefore \quad A = \frac{f(b) - f(a)}{g(a) - g(b)} \quad \text{...(2)}$$
$$\therefore \quad F(x) = f(x) + \frac{f(b) - f(a)}{g(a) - g(b)}\, g(x) \quad \text{...(3)}$$

Since F (a) = F (b) and also f (x), g (x) are continuous in [a, b] and derivable in (a, b), therefore F (x) satisfies all conditions of Rolle's theorem. Here there must exists at least one value c of x in (a, b) such that $F'(c) = 0$. Thus from (3), we get,
$$F'(c) = f'(c) + \frac{f(b) - f(a)}{g(a) - g(b)}\, g'(c) = 0.$$
$$\therefore \quad \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)} \quad \text{where, } a < c < b \text{ ...(4)}$$

Hence the theorem.

**Another (alternative) form of Cauchy's Mean Value Theorem (CMVT) :**

If two functions f (x) and g (x) are derivable in a closed interval [a, a + h] and $F'(x) \neq 0$, then there exists at least one number θ lying between 0 and 1 (i.e. $0 < \theta < 1$) such that

$$\frac{f(a+h) - f(a)}{g(a+h) - g(a)} = \frac{f'(a+\theta h)}{g'(a+\theta h)} \quad \text{where, } 0 < \theta < 1$$

The equivalence of the two statements can be easily seen in the case of the Lagrange's Mean Value *Theorem.*

**Remark (1)** : If we take g (x) = x, it can easily be seen that Lagrange's Mean Value Theorem is only particular case of Cauchy's Mean Value Theorem.

**Remark (2) : Physical Interpretation :** Result (4) can be expressed as

$$\frac{[f(b) - f(a)]/b - a}{[g(b) - g(a)]/b - a} = \frac{f'(c)}{g'(c)}$$

Since we know that $\frac{f(b) - f(a)}{b - a}$ represents the ratio of increment of the function to that of independent variable i.e. the expression represents mean (average) rate of increase of the function in the interval (a, Therefore from result (6), the Cauchy's Mean Value Theorem (CMVT) interprets that the ratio of actual rate of increase of f (x) and g (x) at x = c is equal to the ratio of the mean (average) rate of increase of functions in the interval (a, b).

## 1.7 ILLUSTRATIONS ON CAUCHY'S MEAN VALUE THEOREM

**Ex. 1 :** Verify the Cauchy's Mean Value Theorem for the following functions :

(i) sin x and cos x in [0, π/2]

(ii) $x^2$ and $x^3$ in [1, 2].

**Sol. :** (i) Let f (x) = sin x and g (x) = cos x in [0, π/2].

We note that :

(a) f (x) and g (x) are continuous in [0, π/2]

(b) $f'(x) = \cos x$ and $g'(x) = -\sin x$ exist and are finite in (0, π/2)

(c) Also $g'(x) = -\sin x \neq 0$ for any value of x in (0, π/2).

Hence all the conditions of CMVT are satisfied.

Therefore, there exists at least one value c of x in (0, π/2) such that;

$$\frac{f(\pi/2) - f(0)}{g(\pi/2) - g(0)} = \frac{f'(c)}{g'(c)}$$

$\left\{ \text{Since by CMVT, } \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)} \text{ Here, } a = 0, b = \pi/2 \right.$

$$\therefore \quad \frac{\sin \pi/2 - \sin 0}{\cos \pi/2 - \cos 0} = \frac{\cos c}{-\sin c}$$

$$\text{or} \quad \frac{1-0}{0-1} = -\cot c \quad \text{or} \quad \cot c = 0$$

$$\therefore \quad c = \frac{\pi}{4}, \text{ which lies in } [0, \pi/2].$$

Hence the theorem is verified.

(ii) Let $f(x) = x^2$ and $g(x) = x^3$ in [1, 2].

We note that :

(a) $f(x)$ and $g(x)$ are continuous in [1, 2].

(b) $f'(x) = 2x$ and $g'(x) = 3x^2$ exist and are finite in (1, 2).

(c) Also $g'(x) = 3x^2 \neq 0$ for any value of $x \in (1, 2)$. Hence f and g satisfy the required conditions of CMVT. Consequently, there exists a point $c \in (1, 2)$ such that

$$\frac{f(2) - f(1)}{g(2) - g(1)} = \frac{f'(c)}{g'(c)}$$

$$\therefore \quad \frac{(2)^2 - (1)^2}{(2)^3 - (1)^3} = \frac{2c}{3c^2} \qquad \left\{ \because f(x) = x^2 \text{ and } g(x) = x^3 \right.$$

$$\text{or} \quad \frac{4-1}{8-1} = \frac{2c}{3c^2} \quad \text{or} \quad \frac{3}{7} = \frac{2c}{3c^2}$$

$$\therefore \quad 9c^2 - 14c = 0$$

$$\therefore \quad c = 0 \text{ or } \frac{14}{9}$$

Here $c = 0 \notin (1, 2)$, whereas $c = \frac{14}{9} \in (1, 2)$ which verifies the theorem.

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**Ex. 2 :** If in the Cauchy's Mean Value Theorem, $f(x) = e^x$ and $g(x) = e^{-x}$, show that c is the arithmetic mean between a and b.

**Sol. :** We have by CMVT,

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}, \qquad a < c < b.$$

$$\therefore \quad \frac{e^b - e^a}{e^{-b} - e^{-a}} = \frac{e^c}{-e^{-c}} \qquad \left\{ \because f'(x) = e^x \text{ and } g'(x) = -e^{-x} \right.$$

$$\text{or} \quad \frac{(e^b - e^a)}{\left(\dfrac{e^a - e^b}{e^b e^a}\right)} = -e^{2c}$$

$$\text{or} \quad e^{(a+b)} = e^{2c}$$

$$\therefore \quad a + b = 2c$$

$$\therefore \quad c = \frac{a+b}{2}$$

Hence, c is the arithmetic mean between a and b.

---

**Ex. 3 :** For the functions $f(x) = x^2 + 2$ and $g(x) = x^3 - 1$, test whether Cauchy's Mean Value Theorem holds in the interval (1, 2) and if so, find c.

**Sol. :** Here we note that

(a) $f(x)$ and $g(x)$ are continuous in [1, 2]

(b) $f'(x) = 2x$ and $g'(x) = 3x^2$ exist and are finite in (1, 2).

(c) $g'(x) = 3x^2 \neq 0$ for any value of x in (1, 2).

$$\therefore \quad \frac{\sqrt{1-c^2}}{1+c} = \frac{\log(1+x)}{\sin^{-1}x}$$

$$\therefore \quad \sqrt{\frac{1-c^2}{(1+c)^2}} = \frac{\log(1+x)}{\sin^{-1}x}$$

$$\therefore \quad \sqrt{\frac{(1-c)(1+c)}{(1+c)(1+c)}} = \frac{\log(1+x)}{\sin^{-1}x}$$

$$\therefore \quad \sqrt{\frac{1-c}{1+c}} = \frac{\log(1+x)}{\sin^{-1}x}$$

Now, since $0 < c < x < 1$, we have,

$$c < x < 1$$

$$\therefore \quad \frac{1}{c} < \frac{1}{x} > \frac{1}{1}$$

$$\therefore \quad \frac{1-c}{1+c} > \frac{1-x}{1+x} > \frac{1-1}{1+1}$$

... Taking componendo-dividendo throughout inequal...

$$\therefore \quad \sqrt{\frac{1-c}{1+c}} > \sqrt{\frac{1-x}{1+x}} > 0$$

... Taking positive square-ro...

$$\therefore \quad \sqrt{\frac{1-c}{1+c}} > \sqrt{\frac{1-x}{1+x}}$$

Further, $\quad \sqrt{\frac{1-c}{1+c}} < 1$

Thus, $\quad \sqrt{\frac{1-x}{1+x}} < \sqrt{\frac{1-c}{1+c}} < 1$

Using (II) this becomes,

$$\sqrt{\frac{1-x}{1+x}} < \frac{\log(1+x)}{\sin^{-1}x} < 1.$$

---

**Ex. 8 :** If f(x), g(x), h(x) are three functions differentiable in the interval (a, b), then show that there exists point c in (a, b) such that $\begin{vmatrix} f'(c) & g'(c) & h'(c) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} = 0.$

Hence deduce Lagrange's and Cauchy's mean value theorems.

**Sol. :** In this example, the construction of a new function is very important.

Let us consider a new function,

$$F(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix}$$

We see from this determinant, that

$$F(a) = \begin{vmatrix} f(a) & g(a) & h(a) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} = 0 \quad \text{as } R_1 = R_2$$

and

$$F(b) = \begin{vmatrix} f(b) & g(b) & h(b) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} = 0 \quad \text{as } R_1 = R_3$$

$\therefore$ $$F(a) = F(b) = 0$$

Further, F(x) is continuous in [a, b] because it is a composite of continuous functions f(x), g(x), h(x) in (a, b). F(x) is differentiable in (a, b) as f(x), g(x) and h(x) are differentiable in (a, b).

$\therefore$ F(x) satisfies all the three conditions of Rolle's theorem in [a, b].

$\therefore$ There exists a number c in (a, b) such that

$$F'(c) = 0 \quad \text{... (II)}$$

But

$$F'(c) = \begin{vmatrix} f'(c) & g'(c) & h'(c) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} \quad \text{... (III)}$$

$\therefore$ From (II) and (III), we have,

$$\begin{vmatrix} f'(c) & g'(c) & h'(c) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} = 0 \quad \text{... (IV)}$$

Hence the required result.

**Deduction of Lagrange's mean value theorem :** In above determinant, if we take $g(x) = x$, $h(x) = 1$.

Then, $g'(x) = 1$, $h'(x) = 0 \Rightarrow g'(c) = 1$ and $h'(c) = 0$.

$$g(a) = a, \; h(a) = 1$$
$$g(b) = b, \; h(b) = 1$$

Then, the determinant (IV) becomes,

$$\begin{vmatrix} f'(c) & 1 & 0 \\ f(a) & a & 1 \\ f(b) & b & 1 \end{vmatrix} = 0$$

Expanding $R_1$ – wise, we get,

$$f'(c)\,[a - b] - 1\,[f(a) - f(b)] = 0$$

$\therefore$ $$f'(c)\,(a - b) = f(a) - f(b)$$

$\therefore$ $$f'(c)\,(b - a) = f(b) - f(a)$$

$\therefore$ $$f'(c) = \frac{f(b) - f(a)}{b - a}$$ which is a Lagrange's mean value theorem.

**Deduction of Cauchy's mean value theorem :** If we take $h(x) = 1$ and keeping $f(x)$ and $g(x)$ as ... have,

$$h'(x) = 0 \quad \text{... Independent of variable (constant function)}$$

$$\left.\begin{array}{l} h(a) = 1 \\ h(b) = 1 \end{array}\right\}$$

Then, the determinant (IV) becomes,

$$\begin{vmatrix} f'(c) & g'(c) & 0 \\ f(a) & g(a) & 1 \\ f(b) & g(b) & 1 \end{vmatrix} = 0$$

Expanding the determinant $R_1$ - wise, we get,

$$f'(c)\{g(a) - g(b)\} - g'(c)\{f(a) - f(b)\} = 0$$

$$\therefore \quad f'(c)\{g(a) - g(b)\} = g'(c)\{f(a) - f(b)\}$$

$$\therefore \quad \frac{f'(c)}{g'(c)} = \frac{f(a) - f(b)}{g(a) - g(b)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

$$\therefore \quad \frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$ which is Cauchy's mean value theorem.

## EXERCISE 1.1

1. State whether Rolle's theorem is applicable for the following functions and if so, find the appropriate value of c :

   (i) $f(x) = x(x-2)e^{-x}$ in $[0, 2]$  (ii) $f(x) = \sqrt{(x-a)(x-b)}$ in $[a, b]$

   (iii) $f(x) = 1 - (x-1)^{3/2}$ in $[0, 2]$  (iv) $f(x) = x^3(1-x)^2$ in $[0, 1]$

   (v) $f(x) = (x-2)^{2/3}$ in $[0, 4]$.  (vi) $f(x) = |x|$ in $[-1, 1]$

   **Ans.** (i) $c = 2 - \sqrt{2}$ (ii) $c = \frac{a \ldots}{\ldots}$

   (iii) Not applicable ; since f (x) is not differentiable in (...

   (iv) $c = \frac{3}{5}$ (v) Not applicable ; since $f'(x)$ is not finite in (...

   (vi) Not applicable; since $f'(x)$ is not differentiable at $x = 0$ in (-1...

2. Verify Rolle's theorem for the following functions :

   (i) $f(x) = \frac{\sin x}{e^x}$ in $[0, \pi]$  (ii) $f(x) = e^x(\sin x - \cos x)$ in $\left[\frac{\pi}{2}, \frac{5\pi}{4}\right]$

   (iii) $f(x) = (x-1)(x-2)(x-3)$ in $[0, 3]$

   (iv) $f(x) = x^2 - 6x + 8$ in $[2, 4]$

3. The equation $e^x = 1 + x$ has a root $x = 0$. Show that the equation cannot have any other real root.

4. If k is a real constant, show that the equation $x^3 - 12x + k = 0$ cannot have two distinct roots in the interval $[0, 4]$.

   **[Hint :** $f'(k) = 3k(k-4) = 0$; $k = 0$ and $4$ are outside the open interval (0, 4...

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5. Show that ...
   [Hint : Ap...

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5. Show that between any two roots of $e^x \cos x = 1$, there exists at least one root of $e^x \sin x - 1 = 0$.
[**Hint :** Apply Rolle's theorem to $f(x) = e^{-x} - \cos x$.]

6. By considering the function $(x - 3) \log x$, show that the equation $x \log x = 3 - x$ is satisfied by at least one value of x lying between 1 and 3.

7. Prove that the equation $a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_n = \frac{a_0}{n+1} + \frac{a_1}{n} + \frac{a_2}{n-1} + \ldots + \frac{a_{n-1}}{2} + a_n$ has a root between 0 and 1.

[**Hint :** Construct a function f (x) such that its derivative yields the given equation (i.e. by integrating all the terms of given equation).

$$f(x) = a_0 \frac{x^{n+1}}{n+1} + a_1 \frac{x^n}{n} + a_2 \frac{x^{n-1}}{n-1} + \ldots\ldots\ldots + a_n x$$
$$-\left(\frac{a_0}{n+1} + \frac{a_1}{n} + \frac{a_2}{n-1} + \ldots\ldots + \frac{a_{n-1}}{2} + a_n\right) x = 0.]$$

8. If f and F are continuous in [a, b] and derivable in (a, b) with $F'(x) \neq 0 \ \forall \ x \in (a, b)$, prove that there exists $c \in (a, b)$ such that

$$\frac{f'(c)}{F'(c)} = \frac{f(b) - f(a)}{F(b) - F(a)}$$

[**Hint :** Consider the function $\phi(x) = f(x)[F(b) - F(a)] - [f(x) - f(a)][F(x) - F(a)]$]

9. If a function f (x) is such that its derivative f' (x) is continuous in [a, b] and derivable in (a, b), then prove that there exists a number c (a < c < b) such that

$$f(b) = f(a) + (b - a) f'(a) + \frac{1}{2}(b - a)^2 f''(c).$$

[**Hint :** Consider a function $\phi(x) = f(b) - f(x) - (b - x) f'(x) - (b - x)^2 A$ and determine constant $A = \frac{1}{2} f''(c)$ using Rolle's theorem.]

10. Assuming f" (x) to be continuous in [a, b], show that

$$f(c) - \frac{b-c}{b-a} f(a) - \frac{c-a}{b-a} f(b) = \frac{1}{2}(c-a)(c-b) f''(d),$$ where c and d both lie in (a, b).

[**Hint :** Consider a function $F(x) = (b - a) f(x) - (b - a) f(a) - (x - a) f(b) - (x - a)(x - b)(b - a) A$ and determine $A = \frac{1}{2} f''(d)$ using Rolle's theorem.]

11. Verify the Lagrange's Mean Value Theorem for

(i) $f(x) = \log_e x$ in [1/2, 2] (ii) $f(x) = x^{2/3}$ in [– 8, 8]. (iii) $f(x) = \sin^{-1} x$ in [0, 1]

(iv) $f(x) = x^2 + 3x + 2$ in [1, 2] (v) $f(x) = 4 - (6 - x)^{2/3}$ in [5, 7]

**Ans.** (i) c = 1.08 **Hint :** $\log_e 2 = \frac{0.3010}{0.4343}$ (ii) Not applicable ; since f' (x) is not finite at x = 0.

(iii) $c = \frac{\sqrt{\pi^2 - 4}}{\pi} = 0.7712.$

(iv) $c = \frac{3}{2}$ (v) Not applicable since $f'(x) = \frac{2}{3}\frac{1}{(6-x)^{1/3}}$ is not define at x = 6.

12 Find 'c' so that $f'(c) = \frac{[f(b) - f(a)]}{b - a}$ in the following cases :

(i) $f(x) = x^2 - 3x - 1$ in $[-11/7, 13/7]$

(ii) $f(x) = \sqrt{x^2 - 4}$ in [2, 3]

(iii) $f(x) = e^x$ in [0, 1]

(iv) $\tan^{-1} x$ in [0, 1].

**Ans.** (i) $c = \frac{1}{7}$ (ii) $c = \sqrt{5}$ (iii) $\log(e - 1)$ (iv) $\sqrt{\frac{4-\pi}{\pi}}$

13. Prove that (i) $1 - \frac{a}{b} < \log\frac{b}{a} < \frac{b}{a} - 1$ and hence show that $\frac{1}{6} < \log\frac{6}{5} < \frac{1}{5}$.

[**Hint :** (i) Let f (x) = log x in [a, b] and let a = 5, b = 6 in the inequality

(ii) $\sqrt{\frac{1-x}{1+x}} < \frac{\log(1+x)}{\sin^{-1} x} <$

14. Use Lagrange's Mean Value Theorem to show that $\sin(x + h) - \sin x = h \cos c$, $x < c < x + h$.

15. If $F(x) = \phi(x) + \phi(1 - x)$, and $\phi''(x) < 0$ in (0, 1), then show that F (x) increases in (0, 1/2) and decreases in (1/2, 1). By taking logarithms or otherwise, show that the function

$$\frac{\sin \pi x}{x(1-x)} = \left[2\frac{\sin \pi x/2}{x} \cdot \frac{\sin \pi(1-x)/2}{1-x}\right] \text{ increases in } (0, 1/2).$$

Deduce that $\pi < \frac{\sin \pi x}{x(1-x)} < 4$ in (0, 1).

[**Hint :** $F'(x) = \phi'(x) - \phi'(1-x)$ and $F''(x) = \phi''(x) + \phi''(1-x)$ and $\phi''(x) < 0$ $\therefore$ F'(x) is decreasing function in (0, 1) and hence shows that $F'(x) > 0$ in (0, 1/2) and $F'(x) < 0$ in (1/2, 1).

For second part, we write $F(x) = \frac{\sin \pi x}{x(1-x)}$

and let $\psi(x) = \log\frac{F(x)}{2}\left[\log \sin\frac{\pi x}{2} - \log x\right] + \left[\log \sin\frac{\pi(1-x)}{2} - \log(1-x)\right]$

16. If x > 0, prove that $\tan^{-1} x > \frac{x}{1 + x^2/3}$.

[**Hint :** $0 < \tan^{-1} x < \frac{\pi}{2}$ $\therefore 0 < x < \infty$, $f(x) = \tan^{-1} x - \frac{x}{1 + x^2/3}$, $f'(x) = \frac{(4/9)x^4}{(1 + x^2)(1 + x^2/3)^2} > 0$ for all $x \in (0, \infty)$, f(x) is strictly increasing. Also f(0) = 0 $\therefore$ f(x) > f(0) in $(0, \infty)$. Hence $\tan^{-1} x - \frac{x}{1 + x^2/3} > 0$.]

17. If $f''(x) > 0$, then show that $f\left[\frac{1}{2}(x_1 + x_2)\right] \le \frac{1}{2}[f(x_1) + f(x_2)]$.

[**Hint :** Let $x_1 < x_2$ and since f" (x) exists, f (x) satisfies all conditions of LMVT. Applying LMVT to f(x) in $\left[x_1, \frac{x_1 + x_2}{2}\right]$ and $\left[\frac{x_1 + x_2}{2}, x_2\right]$. If $c_1 < c_2$ and since $f''(x) > 0$, f'(x) is increasing function $f'(c_1) < f'(c_2)$.]



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18. Show that $\dfrac{f(a+h)+f(x-h)-2f(x)}{h^2} = f''(a+\theta h)$, where $0 < \theta < 1$.

[**Hint :** Construct a function $F(x) = f(x) + [a+h-x] f'(x) + A(a+h-x)^2$, where A is constant to be so chosen that $F(a) = F(a+h)$. It satisfies all conditions of Rolle's theorem. $F'(c) = 0$. $\therefore A = \dfrac{f''(c)}{2}$

$\therefore f(a+h) = f(a) + h f'(a) + \dfrac{h^2}{2} f''(c) < f(a-h) = f(a) - h f'(a) + \dfrac{h^2}{2} f''(c)$. Adding we get the result.]

19. Using geometrical interpretation of LMVT for the segment of a parabola $y = x^2$ in the interval (1, 3) to find the point, the tangent at which is parallel to the chord of the segment.

[**Hint :** Given $y = f(x) = x^2$ in $(1, 3) = (a, b)$. $f'(x) = 2x$. By LMVT $\dfrac{f(3)-f(1)}{3-1} = f'(c) = 2c \Rightarrow c = 2 \in (1, 3)$. Hence at $x = 2$, the tangent to the curve $y = x^2$ in (1, 3) is parallel to the segment joining A (1, 1) and B (3, 9) at $x = 1$ and $x = 3$.]

20. Apply LMVT to prove that the chord on the parabola $f(x) = x^2 + 2ax + b$ joining points at $x = \alpha$ and $x = \beta$ is parallel to its tangent at the point $x = \dfrac{1}{2}(\alpha + \beta)$.

[**Hint :** $f(\alpha) = \alpha^2 + 2a\alpha + b$ and $f(\beta) = \beta^2 + 2a\beta + b$, $f'(x) = 2x + 2a$. By LMVT : $c = \dfrac{\alpha+\beta}{2}$]

21. Apply Lagrange's Mean Value Theorem to the function log x in [a, a + h] and determine θ in terms of a and h. Hence deduce $0 < \dfrac{1}{\log(1+x)} - \dfrac{1}{x} < 1$.

[**Hint :** By LMVT $\dfrac{\log(a+h) - \log a}{(a+h)-a} = \dfrac{1}{c}$, $a < c < a+h \Rightarrow \theta = \dfrac{c-a}{h}$, $0 < \theta < 1$.]

**Use :** $(a, a+x) = (1, 1+x)$ for $a = 1$ and $h = x$.

22. Calculate approximately $\sqrt[5]{245}$ by using LMVT.

[**Hint :** Choose $f(x) = x^{1/5}$, $a = 243$, $b = 245$ then $f'(x) = \dfrac{1}{5}x^{-4/5}$. By LMVT : $f(a+h) = f(a) + h f'(c)$.

Take $c = 243$ $\therefore h = 2$ $\therefore (245)^{1/5} = (243)^{1/5} + 2\dfrac{1}{5}(243)^{-4/5} = 3.0049$]

23. Find an approximate value of the root of the equation $x^3 - 2x - 5 = 0$ in (2, 3).

[**Hint :** $h = -\dfrac{f(2)}{f'(2)} = 0.1$ $\therefore$ root $= 2 + 0.1 = 2.1$

Again $h = -\dfrac{f(2.1)}{f'(2.1)} = - - 0.00543$ $\therefore$ root $= 2.1 - 0.00543 = 2.0946$]

24. Separate the intervals in which the following polynomials are increasing or decreasing :

(i) $x^3 - 6x^2 - 36x + 7$ (ii) $-x^3 + 7x^2 - 8x + 8$

(iii) $x^3 + 8x^2 + 5x - 2$ (iv) $x^4 - 13x^2 + 36$

(v) $-2x^4 + 3x^2 - 1$ (vi) $(x-2)^2(x+1)$

**Ans.** (i) Increasing for $x > 6$ and for $x < -2$; decreasing in $(-2, 6)$;

(ii) Decreasing in $(-\infty, \infty)$.

(iii) Increasing in $[\infty, -5]$ and $\left[-\frac{1}{3}, \infty\right]$; decreasing in $\left[-5, -\frac{1}{3}\right]$

(iv) Decreasing in $\left[-\infty, -\frac{\sqrt{3}}{2}\right]$ and $\left[0, \frac{\sqrt{3}}{2}\right]$; increasing in $\left[-\frac{\sqrt{3}}{2}, 0\right]$ and $\left[\frac{\sqrt{3}}{2}, \infty\right]$

(v) Increasing in $[-\infty, -\sqrt{3}/2]$ and $[0, \sqrt{3}/2]$ ; decreasing in $[-\sqrt{3}/2, 0]$ and $[\sqrt{3}/2, \infty]$

(vi) Increasing in $[-\infty, 0]$ and $[2, \infty]$ ; decreasing in $[0, 2]$.

25. Show that

(i) $x < -\log(1-x) < x(1-x)^{-1}$ in $(0, 1)$

(ii) $2x < \log\frac{1+x}{1-x} < 2x\left(1+\frac{1}{3}\frac{x^2}{1-x^2}\right)$ in $(0, 1)$

(iii) $x-1 > \log x > (x-1)x^{-1}$ in $(1, \infty)$

(iv) $x^2-1 > 2x\log x > 4(x-1) - 2\log x$ in $(1, \infty)$.

[**Hint :** (iii) Consider $f(x) = x - 1 - \log x$ and $F(x) = \log x - \ldots$

(iv) Consider $f(x) = x^2 - 1 - 2x\log x$ and $\phi(x) = 2x\log x - 4(x-1) + 2\log\ldots$

26. Show that $3x^4 + 5x^3 - 24x^2 - 48x + 112$ is positive for $x > 2$.

27. Using Cauchy's Mean Value Theorem, show that $\frac{\sin\alpha - \sin\beta}{\cos\beta - \cos\alpha} = \cot\theta, 0 < \alpha < \theta < \beta < \frac{\pi}{2}$.

[**Hint :** Let $f(x) = \sin x$ and $g(x) = \cos x$, $x \in [\alpha\ldots$

**Remark :** We can also prove the above result using Lagrange's Mean Value Theore...

28. Verify Cauchy's Mean Value Theorem for the functions $x^2$ and $x^4$ in the interval $[a, b]$ ; a, b be... positive.

29. Find the interval in which $f(x) = x + \frac{1}{x}$ is strictly increasing or decreasing.

**Ans.** $f'(x) > 0$; $|x| > 1$, hence f(x) is monotonically increasing in $(1, \infty)$, $(-\infty, \ldots$

$f'(x) < 0$; $|x| < 1$; hence f(x) is monotonically decreasing in $(-1, \ldots$

Question Paper : Phase -

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# APPENDIX
## MULTIPLE CHOICE QUESTIONS

### Chapter 1 : Rolle's Theorem and Mean Value Theorems

Marks

(1) = f(a) Rolle's theorem is a special case of ...... (1)

(A) Cauchy's mean value (B) Lagrange's theorem

(C) Taylor's theorem (D) None of these

Lagrange's mean value theorem is a special case of ...... (1)

(A) Rolle's theorem (B) Taylor's theorem

(C) Cauchy's mean value theorem (D) None of these

Which of the following statement(s) is/are correct ? (1)

(a) Rolle's theorem ensures that there is a print on the curve, the tangent at which is parallel to the x-axis.

(b) Lagrange's mean value theorem ensures that there is a point on the curve, the tangent at which is parallel to the y-axis.

(c) Cauchy's mean value theorem can be deduced from Lagrange's mean value theorem.

(d) Rolle's theorem can be deduced from Lagrange's mean value theorem.

(A) (a) and (d) (B) (b) and (d) (C) (a) alone (d) (a), (b) and (c)

f(a) = f(b) in Lagrange's mean value theorem, then it becomes ...... (1)

(A) Leibnitz's theorem (B) Rolle's theorem

(C) Taylor's series (D) Cauchy mean value theorem

Mean value theorem is also known as ...... (1)

(A) Rolle's theorem (B) Lagrange's theorem

(C) Taylor expansion (d) Leibnitz's theorem

f(x) = 3 sin 2x, is continuous over interval $[0, \pi]$ and differentiable over interval $(0, \pi)$ then by Rolle's theorem the value of c is ...... (2)

(A) $\pi$ (B) 2 (C) $\frac{\pi}{4}$ (D) $\frac{\pi}{8}$

f(x) = $x^\alpha \log x$ and f(0) = 0 then the value of $\alpha$ for which Rolle's theorem can be applied in [0, 1] ...... (2)

(A) - 2 (B) − 1 (C) 0 (D) $\frac{1}{2}$

For which interval does the function $\frac{x^2 - 3x}{x + 1}$ satisfy all the condition of Rolle's theorem ? (2)

(A) [0, 3] (B) [−3 ,0] (C) [1.5, 3] (D) for no interval

The value of c in the mean value theorem $f(b) - f(a) = (b - a) f'(c)$ for $a_1x^2 + a_2x + a_3$ in (a, b) is ...... (2)

(A) b + a (B) b − a (C) $\frac{a + b}{2}$ (D) $\frac{b - a}{2}$

f(x) = $4x^2$, then the value of c in the (−1, 3) for which $\frac{f(3) - f(-1)}{4} = f'(c)$ is ...... (2)

(A) 0 (B) 1 (C) 2 (D) 3

The abscissa of the points of the curve $y = x^3$ in the interval [− 2, 2], where the slope of the tangent can be obtain by mean value theorem for the interval [−2, 2] ...... (2)

(A) $\pm\frac{2}{\sqrt{3}}$ (B) $\pm\sqrt{3}$ (C) $\pm\sqrt{\frac{3}{2}}$ (D) 0

on.

12. The function f defined by $f(x) = (x + 2) e^{-x}$ is ......
   (A) decreasing for all x
   (B) decreasing in $(-\infty, -1)$ and increasing $(1, -\infty)$ (2)
   (C) increasing for all
   (d) decreasing in $(-1, \infty)$ and increasing in $(-\infty, -1)$

13. The function $f(x) = -2x^3 - 9x^2 - 12x + 1$ is an increasing function in the interval ......
   (A) $-2 < x < -1$ (B) $-2 < x < 1$ (C) $-1 < x < 2$ (D) $1 < x < 2$ (2)

14. The value of $\theta$ $(0 < \theta < 1)$ in the mean value theorem $f(h) = f(0) + h f'(\theta h)$ where $f(x) = \frac{1}{1+x}$ is ......
   (A) $\frac{1}{\sqrt{3}}$ (B) $\frac{1}{3}$ (C) $\frac{2}{3}$ (D) none of these (2)

15. Let f(x) and g(x) be differentiable for [0, 2] such that $f((0) = 4$, $f(2) = 8$, $g(0) = 0$ and $f'(x) = g'(x)$ for all x in [0, 2] then the value of g(2) must be ......
   (A) 2 (B) − 2 (C) 4 (D) − 4 (2)

**Answers**

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1. (B) | 2. (C) | 3. (A) | 4. (B) | 5. (B) | 6. (C) | 7. (D) | 8. (A) | 9. (C) | 10. (B) |
| 11. (A) | 12. (D) | 13. (A) | 14. (B) | 15. (C) | | | | | |

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## Chapter 2 : Taylor's and Maclaurin's Theorems

### Type I : Maclaurin's Theorem and Expansion of Functions :

**Marks**

1. Expansion of f(x) in ascending powers of x by Maclaurin's theorem is ...... (1)
   (A) $f(x) + xf'(x) + \frac{x^2}{2!} f''(x) + \dots$ (B) $1 + x + \frac{x^2}{2!} + \dots$
   (C) $f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \dots$ (D) $f(x) - xf'(x) + \frac{x^2}{2!} f''(x) - \frac{x^3}{3!} f''(x) + \dots$

2. Expansion of sin x in ascending powers of x is ...... (1)
   (A) $x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$ (B) $x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$
   (C) $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$ (D) $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

3. Expansion of cos x in ascending powers of x is ...... (1)
   (A) $1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots$ (B) $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$
   (C) $x + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots$ (D) $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

4. Expansion of tan x in ascending powers of x is ...... (1)
   (A) $1 + x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \dots$ (B) $x - \frac{1}{3}x^3 + \frac{2}{15}x^5 - \dots$
   (C) $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$ (D) $x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \dots$

5. Expansion of $e^x$ in ascending powers of x is ...... (1)
   (A) $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$ (B) $1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots$
   (C) $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$ (D) $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

6. Expansion of $e^{-x}$ in ascending powers of x is ...... (1)

(A) $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

(B) $1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots$

(C) $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

(D) $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

7. Expansion of sinh x in ascending powers of x is ...... (1)

(A) $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

(B) $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

(C) $1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots$

(D) $x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots$

8. Expansion of cosh x in ascending powers of x is ...... (1)

(A) $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

(B) $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

(C) $1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots$

(D) $x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots$

9. Expansion of tanh x in ascending powers of x is ...... (1)

(A) $1 + x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \dots$

(B) $x - \frac{1}{3}x^3 + \frac{2}{15}x^5 - \dots$

(C) $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

(D) $x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \dots$

10. Expansion of log (1 + x) in ascending powers of x is ...... (1)

(A) $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$

(B) $-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \dots$

(C) $1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots$

(D) $x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots$

11. Expansion of log (1 − x) in ascending powers of x is ...... (1)

(A) $-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots$

(B) $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$

(C) $1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots$

(D) $x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots$

12. Expansion of $\frac{1}{(1-x)}$ in ascending powers of x is ...... (1)

(A) $-1 - x - x^2 - x^3 - \dots$

(B) $1 - x + x^2 - x^3 + \dots$

(C) $1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots$

(D) $1 + x + x^2 + x^3 + \dots$

13. Expansion of $\frac{1}{(1+x)}$ in ascending powers of x is ...... (1)

(A) $-1 - x - x^2 - x^3 - \dots$

(B) $1 - x + x^2 - x^3 + \dots$

(C) $1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots$

(D) $1 + x + x^2 + x^3 + \dots$

14. Expansion of $(1 + x)^n$ in ascending powers of x is ...... (1)

(A) $1 - nx + \frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + \dots$

(B) $1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 + \dots$

(C) $1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \dots$

(D) $1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$

**15.** The limit of the series $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$ as x approaches to $\frac{\pi}{2}$ is ...... (2)

(A) 0 (B) $\frac{\pi}{2}$ (C) 1 (D) −1

**16.** First two terms in expansion of log $(1 + e^x)$ by Maclaurin's theorem is ...... (2)

(A) $\log 2 + \frac{1}{2}x + \ldots$ (B) $\log 2 - \frac{1}{2}x + \ldots$ (C) $x - \frac{x^2}{2} + \ldots$ (D) $x + \frac{x^2}{2} + \ldots$

**17.** First two terms in expansion of sec x by Maclaurin's theorem is ...... (2)

(A) $1 - \frac{x^2}{2!} + \ldots$ (B) $x - \frac{x^3}{3!} + \ldots$ (C) $1 + \frac{x^2}{2!} + \ldots$ (D) $x + \frac{x^3}{3!} + \ldots$

**18.** First two terms in expansion of $e^x$ sec x by Maclaurin's theorem is ...... (2)

(A) $x + x^2 + \ldots$ (B) $x - x^2 + \ldots$ (C) $1 + x + \ldots$ (D) $1 - x + \ldots$

**19.** First two terms in expansion of $\tan^{-1}(1 + x)$ by Maclaurin's theorem is ...... (2)

(A) $\frac{\pi}{4} + \frac{x}{2} - \ldots$ (B) $\frac{\pi}{4} - \frac{x}{2} - \ldots$ (C) $x - \frac{x^3}{3!} + \ldots$ (D) $x + \frac{x^3}{3!} + \ldots$

**20.** Expansion of $\sin\left(\frac{x}{2}\right) + \cos\left(\frac{x}{2}\right)$ in ascending powers of x is ...... (2)

(A) $1 - \frac{x}{2} + \frac{x^2}{8} + \frac{x^3}{48} - \frac{x^4}{384} + \ldots$ (B) $1 + \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{48} + \frac{x^4}{384} + \ldots$

(C) $1 + \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{24} + \frac{x^4}{120} + \ldots$ (D) $\frac{x^2}{8} - \frac{x^3}{48} + \frac{x^4}{384} + \ldots$

**21.** Expansion of $\log(1 - x^4) - \log(1 - x)$ in ascending powers of x is ...... (2)

(A) $-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{3}{4}x^4 + \ldots$ (B) $x + \frac{x^2}{2} + \frac{x^3}{3} - \frac{3}{4}x^4 + \ldots$

(C) $x + \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{3}{4!}x^4 + \ldots$ (D) $-x - \frac{x^2}{2!} - \frac{x^3}{3!} - \frac{3}{4!}x^4 + \ldots$

**22.** Expansion of $\log(1+ x)^{1/x}$ in ascending powers of x is ...... (2)

(A) $1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \ldots$ (B) $-1 - \frac{x}{2} - \frac{x^2}{3} - \frac{x^3}{4} - \ldots$

(C) $1 - \frac{x}{2!} + \frac{x^2}{3!} - \frac{x^3}{4!} + \ldots$ (D) $-1 - \frac{x}{2!} - \frac{x^2}{3!} - \frac{x^3}{4!} - \ldots$

**23.** Expansion of $\log(1 + x)^x$ in ascending powers of x is ...... (2)

(A) $x^2 + \frac{x^3}{2} + \frac{x^4}{3} + \frac{x^5}{4} + \ldots$ (B) $x^2 - \frac{x^3}{2!} + \frac{x^4}{3!} - \frac{x^5}{4!} + \ldots$

(C) $1 + x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} - \frac{x^5}{5} + \ldots$ (D) $x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^5}{4} + \ldots$

**24.** Expansion of $\cos^2 x$ in ascending powers of x is ...... (2)

(A) $\frac{1}{2}\left\{1 + \left(1 - \frac{2^2 x^2}{2!} + \frac{2^4 x^4}{4!} - \ldots\right)\right\}$ (B) $\frac{1}{2}\left\{1 - \left(1 - \frac{2^2 x^2}{2!} + \frac{2^4 x^4}{4!} - \ldots\right)\right\}$

(C) $\frac{1}{2}\left\{1 + \left(2x - \frac{2^3 x^3}{3!} + \frac{2^5 x^5}{5!} - \ldots\right)\right\}$ (D) $\frac{1}{2}\left\{1 - \left(2x - \frac{2^3 x^3}{3!} + \frac{2^5 x^5}{5!} - \ldots\right)\right\}$

**25.** Expansion of sin x cos x in ascending powers of x is ...... (2)

(A) $\frac{1}{2}\left(1 - \frac{2^2 x^2}{2!} + \frac{2^4 x^4}{4!} - \ldots\right)$ (B) $\frac{1}{2}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right)$

(C) $\frac{1}{2}\left(2x - \frac{2^3 x^3}{3!} + \frac{2^5 x^5}{5!} - \ldots\right)$ (D) $\frac{1}{2}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right)$

26. Expansion of sin 2x cos 3x in ascending powers of x is ...... (2)

(A) $\frac{1}{2}\left[\left(5x - \frac{5^3 x^3}{3!} + \frac{5^5 x^5}{5!} - \ldots\right) - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right)\right]$

(B) $\frac{1}{2}\left[\left(5x - \frac{5^3 x^3}{3!} + \frac{5^5 x^5}{5!} - \ldots\right) + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right)\right]$

(C) $\frac{1}{2}\left[\left(1 - \frac{5^2 x^2}{2!} + \frac{5^4 x^4}{4!} - \ldots\right) - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right)\right]$

(D) $\frac{1}{2}\left[\left(1 - \frac{5^2 x^2}{2!} + \frac{5^4 x^4}{4!} - \ldots\right) + \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right)\right]$

27. Expansion of $\tan^{-1} x$ in ascending powers of x is ...... (2)

(A) $x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$ (B) $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$ (C) $x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$ (D) $x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots$

28. Simplified expression of $1 + \left(x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \ldots\right) + \frac{1}{2}\left(x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \ldots\right)^2 + \ldots$ on neglecting $x^5$ and higher powers of x is ...... (2)

(A) $1 + x^2 + \frac{x^3}{2} + \frac{5x^4}{6} + \ldots$ (B) $1 + x^2 - \frac{x^3}{2} - \frac{x^4}{6} - \ldots$

(C) $x^2 - \frac{x^3}{2} + \frac{5x^4}{6} - \ldots$ (D) $1 + x^2 - \frac{x^3}{2} + \frac{5x^4}{6} - \ldots$

29. By using substitution x = tan θ, simplified form of $\sin^{-1}\left(\frac{2x}{1 + x^2}\right)$ is ...... (2)

(A) $\tan^{-1} x$ (B) $2 \cot^{-1} x$ (C) $2 \tan^{-1} x$ (D) none of these

30. By using substitution x = tan θ, simplified form of $\cos^{-1}\left(\frac{x - x^{-1}}{x + x^{-1}}\right)$ is ...... (2)

(A) $\frac{\pi}{2} + 2 \tan^{-1} x$ (B) $\pi - 2 \tan^{-1} x$ (C) $2 \tan^{-1} x$ (D) none of these

31. If x = log (1 + y), then expansion of y in ascending powers of x is (2)

(A) $x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ (B) $x - \frac{x^2}{2!} + \frac{x^3}{3!} - \ldots$ (C) $x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots$ (D) $-x - \frac{x^2}{2!} - \frac{x^3}{3!} - \ldots$

**Type II : Taylor's Theorem and Expansion of Functions :**

32. The Taylor's series expansion of f(x + h) in ascending powers of h is ...... (1)

(A) $f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \ldots$ (B) $-f(x) - hf'(x) - \frac{h^2}{2!} f''(x) - \ldots$

(C) $f(0) + hf'(0) + \frac{h^2}{2!} f''(0) + \ldots$ (D) $f(x) - hf'(x) + \frac{h^2}{2!} f''(x) - \frac{h^3}{3!} f'''(x) + \ldots$

33. The Taylor's series expansion of f(x + h) in ascending powers of x is ...... (1)

(A) $f(h) - xf'(h) + \frac{x^2}{2!} f''(h) - \frac{x^3}{3!} f'''(h) + \ldots$ (B) $f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \ldots$

(C) $f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \ldots$ (D) $f(h) + xf'(h) + \frac{x^2}{2!} f''(h) + \ldots$

34. The Taylor's series expansion of f(a + h) in ascending powers of h is ......

(A) $f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots$ (B) $f(h) + af'(h) + \frac{a^2}{2!} f''(h) + \ldots$

(C) $f(0) + hf'(0) + \frac{h^2}{2!} f''(0) + \ldots$ (D) $f(a) - hf'(a) + \frac{h^2}{2!} f''(a) - \frac{h^3}{3!} f'''(a) + \ldots$

**35.** Expansion of f(x) in ascending powers of (x – a) by Taylor's theorem is ...... (1)

(A) $f(x) + af'(x) + \frac{a^2}{2!} f''(x) + \ldots$

(B) $f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \ldots$

(C) $f(0) - (x-a) f'(0) + \frac{(x-a)^2}{2!} f''(0) - \frac{(x-a)^3}{3!} f'''(0) + \ldots$

(D) $f(a) - (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) - \frac{(x-a)^3}{3!} f'''(a) + \ldots$

**36.** First two terms in expansion of log sec x by Taylor's theorem in ascending powers of $\left(x - \frac{\pi}{4}\right)$ is ...... (2)

(A) $\frac{1}{2}\log 2 - \left(x - \frac{\pi}{4}\right) + \ldots$ (B) $\frac{1}{2}\log 2 + \left(x - \frac{\pi}{4}\right)\frac{1}{2} + \ldots$

(C) $\frac{1}{2}\log 2 + \left(x - \frac{\pi}{4}\right) + \ldots$ (D) $\frac{1}{2}\log 2 - \left(x - \frac{\pi}{4}\right)\frac{1}{2} \ldots$

**37.** First two terms in expansion of $\sqrt{x+h}$ by Taylor's theorem in ascending powers of h is ...... (2)

(A) $\sqrt{x} + h\frac{1}{\sqrt{x}} + \ldots$ (B) $\sqrt{x} - \frac{h}{2}\frac{1}{\sqrt{x}} + \ldots$ (C) $\frac{1}{\sqrt{x}} + \frac{h}{2}\frac{1}{\sqrt{x}} + \ldots$ (D) $\sqrt{x} + \frac{h}{2}\frac{1}{\sqrt{x}} + \ldots$

**38.** First two terms in expansion of log cos $\left(x + \frac{\pi}{4}\right)$ by Taylor's theorem in ascending powers of x is ... (2)

(A) $\log\frac{1}{\sqrt{2}} - x + \ldots$ (B) $\log\frac{1}{\sqrt{2}} + x + \ldots$ (C) $\log\frac{\sqrt{3}}{2} - x + \ldots$ (D) $\log\frac{\sqrt{3}}{2} + x + \ldots$

**39.** First two terms in expansion of $(x+2)^5 + 3(x+2)^4$ by Taylor's theorem in ascending powers of x is ...... (2)

(A) 48 + 98x + ... (B) 80 + 176x + ... (C) 80 + 98x + ... (D) 48 + 176x + ...

**40.** First two terms in expansion of $(x-1)^5 + 2(x-1)^4$ by Taylor's theorem in ascending powers of x is ...... (2)

(A) 3 – 13x + ... (B) 1 + 13x + ... (C) 1 – 3x + ... (D) 3 – 3x + ...

**41.** First two terms in expansion of sinh (x + a) by Taylor's theorem in ascending powers of x is ...... (2)

(A) sinh a + x cosh a + ... (B) sinh a – x cosh a + ...

(C) cosh a + x sinh a + ... (D) none of these

**42.** First two terms in expansion f (x + 2) + 3 $(x+2)^3 + (x+2)^4$ by Taylor's theorem in ascending powers of x is ...... (2)

(A) 42 + 68x + ... (B) 42 + 66x + ... (C) 42 + 69x + ... (D) 40 + 69x + ...

**43.** First two terms in expansion of $e^x$ by Taylor's theorem in ascending powers of (x – 2) is ...... (2)

(A) $e^{-2} - e^2(x-2) + \ldots$ (B) $e^{-2} + e^{-2}(x-2) + \ldots$

(C) $e^2 - e^2(x-2) + \ldots$ (D) $e^2 + e^2(x-2) + \ldots$

**44.** First two terms in expansion of $\tan^{-1} x$ by Taylor's theorem in ascending powers of (x – 1) is ...... (2)

(A) $\frac{\pi}{4} - \frac{1}{2}(x-1) + \ldots$ (B) $\frac{\pi}{4} + \frac{1}{2}(x-1) + \ldots$ (C) $1 + \frac{1}{2}(x-1) + \ldots$ (D) $1 - \frac{1}{2}(x-1) + \ldots$

**45.** First two terms in expansion of sin x by Taylor's theorem in ascending powers of $\left(x - \frac{\pi}{2}\right)$ is ...... (2)

(A) $\left(x - \frac{\pi}{2}\right) - \frac{1}{3!}\left(x - \frac{\pi}{2}\right)^3 + \ldots$ (B) $1 + \frac{1}{2!}\left(x - \frac{\pi}{2}\right)^2 + \ldots$

(C) $\left(x - \frac{\pi}{2}\right) + \frac{1}{3!}\left(x - \frac{\pi}{2}\right)^3 + \ldots$ (D) $1 - \frac{1}{2!}\left(x - \frac{\pi}{2}\right)^2 + \ldots$

46. First two terms in expansion of log cos x by Taylor's theorem in ascending powers of $\left(x-\frac{\pi}{4}\right)$ is ...... (2)

(A) $\log\frac{1}{2}-\left(x-\frac{\pi}{4}\right)+\dots$ (B) $\log\frac{1}{\sqrt{2}}+\left(x-\frac{\pi}{4}\right)+\dots$

(C) $\log\frac{1}{\sqrt{2}}-\left(x-\frac{\pi}{4}\right)+\dots$ (D) $\log\frac{1}{2}+\left(x-\frac{\pi}{4}\right)+\dots$

47. First two terms in expansion of $\sin^{-1}x$ by Taylor's theorem in ascending powers of $\left(x-\frac{1}{2}\right)$ is ... (2)

(A) $\frac{\pi}{6}+\left(x-\frac{1}{2}\right)\frac{2}{\sqrt{3}}+\dots$ (B) $\frac{\pi}{6}-\left(x-\frac{1}{2}\right)\frac{2}{\sqrt{3}}+\dots$

(C) $\frac{\pi}{6}+\left(x-\frac{1}{2}\right)\frac{1}{\sqrt{2}}+\dots$ (D) $\frac{\pi}{6}-\left(x-\frac{1}{2}\right)\frac{1}{\sqrt{2}}+\dots$

48. First two terms in expansion of $x^{1/3}$ by Taylor's theorem in ascending powers of $(x-8)$ is ...... (2)

(A) $2-(x-8)\frac{1}{12}+\dots$ (B) $2+(x-8)\frac{1}{12}+\dots$ (C) $2+(x-8)\frac{1}{24}+\dots$ (D) $2-(x-8)\frac{1}{24}+\dots$

49. First two terms in expansion of $\sqrt{x+2}$ by Taylor's theorem in ascending powers $(x-2)$ is ...... (2)

(A) $2+(x-2)\frac{1}{4}+\dots$ (B) $2-(x-2)\frac{1}{4}+\dots$ (C) $2+(x-2)\frac{1}{8}+\dots$ (D) $2-(x-2)\frac{1}{8}+\dots$

50. In the Taylor's series expansion of $e^x+\sin x$ about the point $x=\pi$, the coefficient of $(x-\pi)^2$ is ... (2)

(A) $e^\pi$ (B) $e^\pi+1$ (C) $e^\pi-1$ (D) $\frac{1}{2}e^\pi$

51. Which of the following function will have only odd powers of x in its Taylor's series expansion about the point $x = 0$? (2)

(A) $\sin(x^2)$ (B) $\sin(x^3)$ (C) $\cos(x^2)$ (D) $\cos(x^3)$

**Answers**

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1. (C) | 2. (C) | 3. (D) | 4. (D) | 5. (A) | 6. (B) | 7. (D) | 8. (C) | 9. (B) | 10. (A) |
| 11. (A) | 12. (D) | 13. (B) | 14. (D) | 15. (C) | 16. (A) | 17. (C) | 18. (C) | 19. (A) | 20. (B) |
| 21. (B) | 22. (A) | 23. (D) | 24. (A) | 25. (C) | 26. (A) | 27. (C) | 28. (D) | 29. (C) | 30. (B) |
| 31. (A) | 32. (A) | 33. (D) | 34. (A) | 35. (B) | 36. (C) | 37. (D) | 38. (A) | 39. (B) | 40. (C) |
| 41. (A) | 42. (C) | 43. (D) | 44. (B) | 45. (D) | 46. (C) | 47. (A) | 48. (B) | 49. (A) | 50. (D) |
| 51. (B) | | | | | | | | | |

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# Chapter 3 : Indeterminate Forms

**Marks**

### Type I : Indeterminate Forms $\left(\frac{0}{0}, \frac{\infty}{\infty}, 0\times\infty, \infty-\infty\right)$

1. If f(x) and g(x) be functions such that $f(a) = 0$ and $g(a) = 0$ then $\lim_{x\to a}\frac{f(x)}{g(x)}$ is equal to ...... (1)

(A) $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ (B) $\lim_{x\to a}\frac{g'(x)}{f'(x)}$ (C) $\frac{f(a)}{g(a)}$ (D) none of these

2. If f(x) and g(x) be functions such that $f(a) = 0$, $g(a) = 0$ and $f'(a) = 0$, $g'(a) = 0$ then $\lim_{x\to a}\frac{f(x)}{g(x)}$ is equal to ...... (1)

(A) $\frac{f'(a)}{g'(a)}$ (B) $\lim_{x\to a}\frac{g'(x)}{f'(x)}$ (C) $\lim_{x\to a}\frac{f''(x)}{g''(x)}$ (D) none of these

3. If f(x) and g(x) be functions such that f(a) = ∞ and g(a) = ∞ then $\lim_{x \to a} \frac{f(x)}{g(x)}$ is equal to ...... (1)

(A) $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ (B) $\lim_{x \to a} \frac{g'(x)}{f'(x)}$ (C) $\frac{f(a)}{g(a)}$ (D) none of these

4. $\lim_{x \to \pi/2} \frac{1 - \sin x}{\cos x}$ is equal to ...... (1)

(A) 1 (B) 0 (C) $\frac{1}{2}$ (D) –1

5. $\lim_{x \to 0} \frac{\sin x}{x}$ is equal to ...... (1)

(A) 2 (B) 0 (C) –1 (D) 1

6. $\lim_{x \to 0} \frac{\tan x}{x}$ is equal to ...... (1)

(A) 2 (B) 1 (C) $\frac{\pi}{2}$ (D) $\frac{3}{2}$

7. $\lim_{x \to 0} \frac{\sin^{-1} x}{x}$ is equal to ...... (1)

(A) 1 (B) –1 (C) $\frac{1}{2}$ (D) $\frac{\pi}{2}$

8. $\lim_{x \to 0} (1 + x)^{1/x}$ is equal to ...... (1)

(A) 1 (B) $e^2$ (C) $\frac{1}{e}$ (D) e

9. $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$ is equal to ...... (1)

(A) 1 (B) e (C) $\frac{1}{e}$ (D) $e^2$

10. $\lim_{x \to 0} \frac{e^x - 1}{x}$ is equal to ...... (1)

(A) 2 (B) $\frac{1}{2}$ (C) 1 (D) none of these

11. $\lim_{x \to 0} \frac{a^x - 1}{x}$ is equal to ...... (1)

(A) a (B) – log a (C) log a (D) 1

12. $\lim_{\theta \to 0} \frac{\sin\left(\frac{\theta}{2}\right)}{\theta}$ is equal to ...... (2)

(A) 1 (B) 2 (C) $\frac{1}{2}$ (D) not defined

13. $\lim_{x \to 0} \frac{\sin^2 x}{x}$ is equal to ...... (2)

(A) –1 (B) 1 (C) 0 (D) not defined

14. $\lim_{x \to 0} \frac{1 - \cos x}{x}$ is equal to ...... (2)

(A) 0 (B) 1 (C) –1 (D) 2

15. $\lim_{x \to 3} \dfrac{2x^2 - 7x + 3}{5x^2 - 12x - 9}$ is equal to ...... (2)

(A) $-\dfrac{1}{3}$ (B) $\dfrac{2}{5}$ (C) $\dfrac{5}{18}$ (D) 0

16. $\lim_{x \to 0} \dfrac{a^x - b^x}{x}$ is equal to ...... (2)

(A) 0 (B) 1 (C) $\log \dfrac{b}{a}$ (D) $\log \dfrac{a}{b}$

17. $\lim_{x \to 0} \dfrac{\sin^2 x}{x \cos x}$ is equal to ...... (2)

(A) 0 (B) 1 (C) $-1$ (D) 2

18. $\lim_{x \to 0} \dfrac{e^{ax} - e^{-ax}}{\log(1 + bx)}$ is equal to ...... (2)

(A) $\dfrac{a}{2b}$ (B) 0 (C) $\dfrac{b}{2a}$ (D) $\dfrac{2a}{b}$

19. $\lim_{x \to 0} \dfrac{(1 + x)^n - 1}{x}$ is equal to ...... (2)

(A) n (B) 1 (C) e (D) 0

20. $\lim_{x \to 0} \dfrac{2^x - 1}{\sqrt{(1 + x)} - 1}$ is equal to ...... (2)

(A) log 2 (B) $\dfrac{1}{2}\log 2$ (C) 0 (D) 2 log 2

21. $\lim_{x \to 0} \dfrac{\sqrt{(1 + x)} - \sqrt{(1 - x)}}{x}$ is equal to ...... (2)

(A) 0 (B) $-1$ (C) 1 (D) 2

22. If $\lim_{x \to 0} \dfrac{\sin 2x + p \sin x}{x^3}$ is finite then value of p is equal to ...... (2)

(A) $-2$ (B) 2 (C) 1 (D) $-1$

23. If $\lim_{x \to 0} \dfrac{a \sinh x - 5 \sin x}{x^3}$ is finite then value of a is equal to ...... (2)

(A) $-5$ (B) 5 (C) 0 (D) 10

24. If $\lim_{x \to 0} \dfrac{a \sin 2x + \tan x}{x^3}$ is finite then value of a is equal to ...... (2)

(A) $-2$ (B) 2 (C) $-\dfrac{1}{2}$ (D) $\dfrac{1}{2}$

25. If $\lim_{x \to 0} \dfrac{2 \cos x - 2 + bx^2}{x^4}$ is finite then value of b is equal to ...... (2)

(A) 2 (B) 0 (C) 1 (D) $-1$

26. $\lim_{x \to \pi/4} \dfrac{1 - \tan x}{1 - \sqrt{2} \sin x}$ is equal to ...... (2)

(A) 2 (B) 0 (C) 1 (D) $-2$

27. $\lim_{x \to \infty} \dfrac{\log x}{x^n}$ is equal to ...... (2)

(A) 2 (B) $-2$ (C) 1 (D) 0

28. $\lim\limits_{x \to \infty} \dfrac{\log(1 + e^{3x})}{x}$ is equal to ......

(A) 9 (B) 3 (C) $\frac{1}{3}$ (D) 0 (2)

29. $\lim\limits_{x \to 0} x \log x$ is equal to ......

(A) 2 (B) −1 (C) 1 (D) 0 (2)

30. $\lim\limits_{x \to \infty} x \sin \frac{1}{x}$ is equal to ......

(A) 2 (B) 0 (C) 1 (D) −1 (2)

31. $\lim\limits_{x \to 1} (1 - x) \tan \frac{\pi x}{2}$ is equal to ......

(A) $\frac{2}{\pi}$ (B) $\frac{\pi}{2}$ (C) $\pi$ (D) 0 (2)

32. $\lim\limits_{x \to \pi/2} (1 - \sin x) \tan x$ is equal to ......

(A) 1 (B) −1 (C) $\pi$ (D) 0 (2)

33. $\lim\limits_{x \to \pi/2} (\sec x - \tan x)$ is equal to ......

(A) 1 (B) −1 (C) $\pi$ (D) 0 (2)

34. $\lim\limits_{x \to \pi/2} \left(x \tan x - \frac{\pi}{2} \sec x\right)$ is equal to ......

(A) 1 (B) −1 (C) $\pi$ (D) 0 (2)

35. $\lim\limits_{x \to \infty} \left[x - x^2 \log\left(1 + \frac{1}{x}\right)\right]$ is equal to ......

(A) 1 (B) $-\frac{1}{2}$ (C) $\frac{1}{2}$ (D) 0 (2)

36. $\lim\limits_{x \to 0} \left[\frac{1}{x} - \frac{1}{x^2} \log(1 + x)\right]$ is equal to ......

(A) 1 (B) $-\frac{1}{2}$ (C) $\frac{1}{2}$ (D) 0 (2)

37. $\lim\limits_{x \to 0} \left[\frac{1}{x} - \frac{1}{\sin x}\right]$ is equal to ......

(A) 1 (B) $-\frac{1}{2}$ (C) $\frac{1}{2}$ (D) 0 (2)

38. $\lim\limits_{x \to 0} \left[\frac{1}{x} - \frac{1}{e^x - 1}\right]$ is equal to ......

(A) 1 (B) $-\frac{1}{2}$ (C) $\frac{1}{2}$ (D) 0 (2)

39. $\lim\limits_{x \to \pi/2} \left[\tan x - \frac{2x \sec x}{\pi}\right]$ is equal to ......

(A) $\frac{2}{\pi}$ (B) $-\frac{2}{\pi}$ (C) $\frac{\pi}{2}$ (D) 0 (2)

40. $\lim\limits_{x \to 1} \left[\frac{x}{\log x} - \frac{1}{\log x}\right]$ is equal to ......

(A) −1 (B) 1 (C) $\frac{1}{2}$ (D) 0 (2)

**Type II : Indeterminate Forms ($0^0, \infty^0, 1^\infty$) :**

41. $\lim_{x \to \infty} \left(\frac{1}{x}\right)^{1/x}$ is equal to ...... (2)

(A) e (B) $-\frac{1}{2}$ (C) $\frac{1}{2}$ (D) 1

42. $\lim_{x \to 0} (\sin x)^{\tan x}$ is equal to ...... (2)

(A) 1 (B) e (C) –1 (D) $\frac{1}{e}$

43. $\lim_{x \to \pi/2} (\cos x)^{\cos x}$ is equal to ...... (2)

(A) –1 (B) e (C) 1 (D) $\frac{1}{e}$

44. $\lim_{x \to 0} (x)^x$ is equal to ...... (2)

(A) e (B) 1 (C) –1 (D) none of these

45. $\lim_{x \to \infty} (x)^{1/x}$ is equal to ...... (2)

(A) e (B) –1 (C) 1 (D) none of these

46. $\lim_{x \to \pi/2} (\sec x)^{\cot x}$ is equal to ...... (2)

(A) e (B) 1 (C) $\frac{1}{e}$ (D) does not exist

47. $\lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^x$ is equal to ...... (2)

(A) $e^{-a}$ (B) $e^a$ (C) 1 (D) none of these

48. $\lim_{x \to 1} (x)^{\frac{1}{x-1}}$ is equal to ...... (2)

(A) e (B) 1 (C) –1 (D) none of these

49. $\lim_{x \to 0} (\cos x)^{1/x}$ is equal to ...... (2)

(A) e (B) 1 (C) –1 (D) none of these

50. $\lim_{x \to 0} (\cos x)^{\cot x}$ is equal to ...... (2)

(A) 1 (B) e (C) –1 (D) none of these

51. $\lim_{x \to 1} (1 - x^2)^{\frac{1}{\log(1-x)}}$ is equal to ...... (2)

(A) e (B) $\frac{1}{e}$ (C) 1 (D) $e^2$

52. $\lim_{x \to 0} \left(\frac{a+x}{a-x}\right)^{1/x}$ is equal to ...... (2)

(A) $e^{2/a}$ (B) $e^{1/2a}$ (C) 1 (D) $e^{a/2}$

53. $\lim_{x \to 0} (1 + \sin x)^{\cot x}$ is equal to ...... (2)

(A) $e^{1/2}$ (B) $e^2$ (C) 1 (D) e

54. $\lim_{x \to \pi/2} (\operatorname{cosec} x)^{\tan x}$ is equal to ...... (2)

(A) $e^{-1}$ (B) $e^2$ (C) 1 (D) e

**Answers**

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1. (A) | 2. (C) | 3. (A) | 4. (B) | 5. (D) | 6. (B) | 7. (A) | 8. (D) | 9. (B) | 10. (C) |
| 11. (C) | 12. (C) | 13. (C) | 14. (A) | 15. (C) | 16. (D) | 17. (A) | 18. (D) | 19. (A) | 20. (D) |
| 21. (C) | 22. (A) | 23. (C) | 24. (C) | 25. (C) | 26. (A) | 27. (D) | 28. (B) | 29. (D) | 30. (C) |
| 31. (A) | 32. (D) | 33. (D) | 34. (B) | 35. (C) | 36. (C) | 37. (D) | 38. (C) | 39. (A) | 40. (B) |
| 41. (D) | 42. (A) | 43. (C) | 44. (B) | 45. (C) | 46. (B) | 47. (B) | 48. (A) | 49. (B) | 50. (A) |
| 51. (A) | 52. (A) | 53. (D) | 54. (C) | | | | | | |

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# Chapter 4 : Fourier Series

**Fourier Series and Harmonic Analysis :**

**Marks**

1. A function f(x) is said to be periodic of period T if ...... (1)

(A) $f(x + T) = f(x)$ for all x (B) $f(x + T) = f(T)$ for all x

(C) $f(-x) = f(x)$ for all x (D) $f(-x) = -f(x)$ for all x

2. If $f(x + nT) = f(x)$ where n is any integer then the fundamental period of f(x) is ...... (1)

(A) 2T (B) $\frac{T}{2}$ (C) T (D) 3T

3. If f(x) is a periodic function with period T then f(ax), $a \neq 0$ is periodic function with fundamental period ...... (1)

(A) T (B) $\frac{T}{a}$ (C) aT (D) $\pi$

4. Fundamental period of sin 2x is ...... (1)

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{2}$ (C) $2\pi$ (D) $\pi$

5. Fundamental period of cos 2x is ...... (1)

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{2}$ (C) $\pi$ (D) $2\pi$

6. Fundamental period of tan 3x is ...... (1)

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{3}$ (C) $\pi$ (D) $\frac{\pi}{4}$

7. Fourier series representation of periodic function f(x) with period $2\pi$ which satisfies the Dirichlet's conditions is ...... (1)

(A) $\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ (B) $\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\pi x + b_n \sin n\pi x)$

(C) $\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx)(b_n \sin nx)$ (D) $\frac{a_0}{2} + (a_n \cos nx + b_n \sin nx)$

8. Fourier series representation of periodic function f(x) with period 2L which satisfies the Dirichlet's conditions is ...... (1)

(a) $\frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos (n\pi x) + b_n \sin (n\pi x)]$ (B) $\frac{a_0}{2} + \sum^{\infty} \left[ a_1 \cos\left(\frac{n\pi x}{L}\right) + b_1 \sin\left(\frac{n\pi x}{L}\right)\right]$