UNIT II

CHAPTER-2

APPLICATIONS OF FIRST ORDER DIFFERENTIAL EQUATIONS

2.1 INTRODUCTION

Differential equations are of great importance in engineering, because many physical laws and relations appear mathematically in the form of differential equations.

In the first chapter we discussed methods to solve ordinary differential equations of the first order and first degree. In this chapter, we shall show how these methods enable us to study many interesting problems, such as orthogonal trajectories, decay of radio- active materials, cooling or heating up of bodies, rectilinear motion, motion under gravity, analysis of electrical circuits, simple harmonic motion, vibrating string, heat conduction along a pipe or spherical shells and some problems related to chemical engineering.

It will be quite interesting to see, how methods discussed in first chapter are useful to solve the above mentioned problems of practical importance. Taking into account the importance of differential equations in mathematical modeling, we briefly discuss the technique of modeling.

2.2 TECHNIQUE OF MATHEMATICAL MODELLING

Mathematical modelling essentially consists of translating real world problems into mathematical problems, solving the mathematical problems and interpreting these solutions in the language of the real world. i.e.

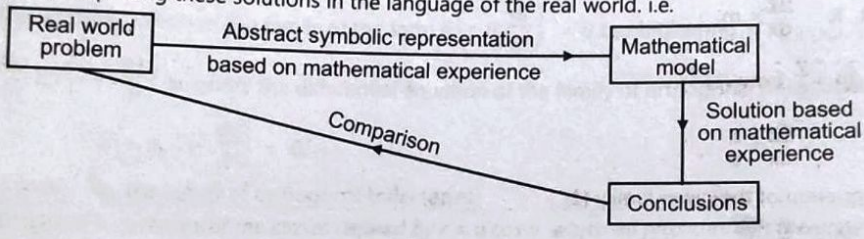

Fig. 2.1

Differential equations arise in many engineering and other applications as mathematical models of various physical and other systems.

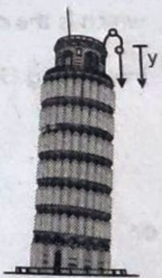

Fig. 2.2

For example, if we drop a stone then its acceleration $y'' = \frac{d^2y}{dt^2}$ is equal to the acceleration of gravity g (a constant). Hence the model of this problem of "free fall" is $y'' = g$ (neglecting air resistance). We have velocity $y' = \frac{dy}{dt} = gt + v_0$, where v_0 is the initial velocity with which the motion is started (e.g. $v_0 = 0$).

We get the distance traveled $y = \frac{g}{2}t^2 + v_0 \; t + y_0$, where y_0 is the distance from 0 at the beginning (e.g. $y_0 = 0$).

We shall consider physical problems which lead to a differential equation of first order and first degree and that of second order which reduces to first order.

2.3 ORTHOGONAL TRAJECTORIES

1. **Trajectory :** A curve which cuts every member of a given family of curves according to some definite law is called a *trajectory* of the family.
2. **Orthogonal Trajectory :** A curve which cuts every member of a given family of curves at *right angles* is called as *orthogonal trajectory* of the family.
3. **Orthogonal Trajectories :** Two families of curves are said to be orthogonal if every member of the either family cuts each member of the other family at right angles.

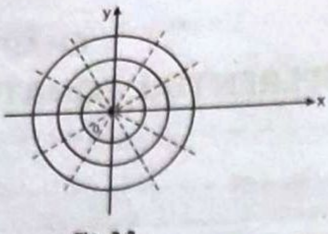

Fig. 2.3

Thus, if the given family consists of straight lines $y = mx$ (m = constant) representing family of straight lines all passing through the origin (shown by dotted lines in Fig. 2.3), then the family of circles $x^2 + y^2 = a^2$, (a is a parameter), with centres at (0, 0) represents a family of orthogonal trajectories to the family $y = mx$ (Refer Fig. 2.3).

2.4 WORKING RULE TO FIND THE EQUATION OF ORTHOGONAL TRAJECTORIES

I. For Rectangular Cartesian Co-ordinates :

Step 1 : Given $f(x, y, a) = 0$, where a is a variable parameter.

Step 2 : Differentiate $f(x, y, a) = 0$ w.r.t. x and eliminate 'a'. We thus form a differential equation of the family of the form $\phi\left(x, y, \frac{dy}{dx}\right) = 0$.

Step 3 : Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$. Then the differential equation of the family of orthogonal trajectories will be : $\phi\left(x, y, -\frac{dx}{dy}\right) = 0$

Step 4 : The solution of step 3 is the family of orthogonal trajectories.

Ex. 1 : *Find the orthogonal trajectories of the family of straight lines $y = mx$.*

Sol. : Given $y = mx$ ... (1)

Differentiate (1) w.r.t. x, $\frac{dy}{dx} = m$

Eliminating m using $m = \frac{y}{x}$ from (1)

$$\therefore \quad \frac{dy}{dx} = \frac{y}{x} \quad \text{... (2)}$$

which is the differential equation of the given family (1).

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in (2), we get

$$-\frac{dx}{dy} = \frac{y}{x} \quad \text{or} \quad x\,dx + y\,dy = 0 \quad \text{... (3)}$$

which is the differential equation of the orthogonal trajectories.

Integrating (3), $\int x\,dx + \int y\,dy = b$

$$\frac{x^2}{2} + \frac{y^2}{2} = b$$

or $\boxed{x^2 + y^2 = c^2}$

which is the equation of the required orthogonal trajectories of (1).

Ex. 2 : *Find the orthogonal trajectories of the curves given by $x^2 + 2y^2 = c^2$.*

Sol. : Given $x^2 + 2y^2 = c^2$ ... (1)

Differentiating (1) w.r.t. x, $x + 2y\frac{dy}{dx} = 0$ ... (2)

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in (2), we obtain differential equation of orthogonal trajectories as

$$x + 2y\left(-\frac{dx}{dy}\right) = 0 \quad \text{or} \quad 2\frac{dx}{x} = \frac{dy}{y} \quad \text{... (3)}$$

Integrating (3), $2 \log x = \log y + \log k$ or $x^2 = ky$

Thus, $\boxed{x^2 = ky}$ is the required orthogonal trajectories of (1).

***Ex. 3 :** Show that the family of confocal conics $\frac{x^2}{C}+\frac{y^2}{C-\lambda} = 1$, where C is arbitrary constant, is self orthogonal.*

Sol. : Given $\quad \frac{x^2}{C}+\frac{y^2}{C-\lambda}=1 \qquad \text{... (1)}$

Differentiating (1) w.r.t. x, $\quad \frac{x}{C}+\frac{y\,(dy/dx)}{C-\lambda}=0 \qquad \text{... (2)}$

Simplifying (2), $\quad -\frac{x}{C}=\frac{y\,(dy/dx)}{C-\lambda}=\frac{x+y\,(dy/dx)}{-\lambda} \qquad \text{... (3)}$

From (3), $\quad C=\frac{\lambda x}{x+y\,(dy/dx)}$ and $C-\lambda=\frac{-\lambda y\,(dy/dx)}{x+y\,(dy/dx)} \qquad \text{... (4)}$

To obtain the differential equation for the family of curves (1), we eliminate C from (1) using C, C – λ from (4), thus

$$x^2\,\frac{[x+y\,(dy/dx)]}{\lambda x}+y^2\,\frac{[x+y\,(dy/dx)]}{-\lambda y\,(dy/dx)}=1$$

or $\quad [x+y\,(dy/dx)]\,[x\,(dy/dx)-y]-\lambda\,(dy/dx)=0 \qquad \text{... (5)}$

Replacing dy/dx by (–dx/dy) in (5), we obtain differential equation of orthogonal family of curves as

$$[x-y\,(dx/dy)]\,[-x\,(dx/dy)-y]+\lambda\,(dx/dy)=0$$

which on simplification gives the differential equation as

$$[x+y\,(dy/dx)]\,[x\,(dy/dx)-y]-\lambda\,(dy/dx)=0$$

which is same as (5). ... (6)

Thus orthogonal family of curves of system given by (6) is the same as given by (1). Hence the confocal conics (1) is self orthogonal.

II. For Polar Co-ordinates :

Step 1 : Given f (r, θ, a) = 0, where a is a variable parameter.

Step 2 : Form a differential equation of the family of the form $\phi\left(r,\theta,\frac{dr}{d\theta}\right)=0$ by eliminating 'a'.

Step 3 : Replace $\frac{dr}{d\theta}$ by $\left(-r^2\frac{d\theta}{dr}\right)$ whereby the differential equation of the family of orthogonal trajectories become

$$\phi\left(r,\theta,-r^2\frac{d\theta}{dr}\right)=0.$$

Step 4 : Solve step 3 which is the family of orthogonal trajectories.

Ex. 1 :** Find the orthogonal trajectories of the circles defined by r = a cos θ which all pass through the origin and have their centres on the initial line, a being the variable diameter.* ***(May 2005)

Sol. : Given

$$r = a\cos\theta$$

$$\log r=\log a+\log\cos\theta \qquad \text{... (1)}$$

$$\frac{1}{r}\frac{dr}{d\theta}=0+\frac{1}{\cos\theta}\,(-\sin\theta)$$

$$\frac{1}{r}\frac{dr}{d\theta}=-\tan\theta$$

which is the differential equation of the given family (1). ... (2)

Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ in (2), we get

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right)=-\tan\theta$$

$$\therefore \quad r\frac{d\theta}{dr}=\tan\theta \quad \therefore \quad \frac{dr}{r}=\frac{d\theta}{\tan\theta}$$

$$\frac{dr}{r}=\cot\theta\cdot d\theta$$

which is the differential equation of the family of orthogonal trajectories. ... (3)

Integrating (3), $\quad \int\frac{dr}{r}=\int\cot\theta\,d\theta+\log C$

$$\log r=\log\sin\theta+\log C$$

$$\log r=\log\,(C\sin\theta)$$

$$\boxed{r = C\cdot\sin\theta}$$

which is the required equation of orthogonal trajectories of (1).

***Ex. 2 :** Find orthogonal trajectories of the family of curve $r^2 = a \sin 2\theta$.*

Sol. : Given : $r^2 = a \sin 2\theta$ ... (1)

Differentiating (1), w.r.t. θ, $2r\dfrac{dr}{d\theta} = 2a\cos 2\theta$ ... (2)

Eliminating a, by putting $a = \dfrac{r^2}{\sin 2\theta}$ in (2), we get

$$\frac{dr}{d\theta} = r \cot 2\theta \qquad \text{... (3)}$$

Replacing $\dfrac{dr}{d\theta}$ by $-r^2\dfrac{d\theta}{dr}$ in (3), we obtain

$$-r^2\frac{d\theta}{dr} = r\cot 2\theta$$

or $$\frac{dr}{r} = -\tan 2\theta\, d\theta \quad \text{... (4)}$$

which is the differential equation of the family of orthogonal trajectories.

Integrating (4), $\log r = \frac{1}{2}\log\cos 2\theta + \frac{1}{2}\log C$ (Note : $C_1 = \frac{1}{2}\log C$)

or $\boxed{r^2 = C\cos 2\theta.}$

which is the required equation of orthogonal trajectories of (1).

EXERCISE 2.1

Find the orthogonal trajectories of the family of

1. $xy = c$ **(May 2008)**
   Ans. : $x^2 - y^2 = c^2$
2. $2x^2 + y^2 = cx$
   Ans. : $x^2 = -y^2 \log(cy)$
3. $y^2 = 4ax$ **(Dec. 2010, May 07)**
   Ans. : $2x^2 + y^2 = c$
4. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2+\lambda} = 1$, λ is a parameter
   Ans. : $x^2 + y^2 = 2a^2 \lambda o \gamma\, x + c$
5. $e^x + e^{-y} = c$
   Ans. : $e^y - e^{-x} = c$
6. $x^2 + cy^2 = 1$ **(May 2009, Nov. 2014)**
   Ans. : $x^2 = c\, e^{x^2 + y^2}$
7. $r = a(1 - \cos\theta)$ **(Dec. 2008)**
   Ans. : $r = c(1 + \cos\theta)$
8. $r = \dfrac{2a}{1 + \cos\theta}$
   Ans. : $r = \dfrac{2c}{1 - \cos\theta}$
9. $r^2 = a^2 \cos 2\theta$
   Ans. : $r^2 = c^2 \sin 2\theta$
10. $r = a\cos^2\theta$
    Ans. : $r^2 = c\sin\theta$
11. $y = ax^2$
    Ans. : $\dfrac{x^2}{(\sqrt{2}c)^2} + \dfrac{y^2}{c^2} = 1$
12. $r = a(1 + \cos\theta)$
    Ans. : $r = a(1 - \cos\theta)$

2.5 RATE OF DECAY OF RADIOACTIVE MATERIALS

This law states that disintegration at any instant, is proportional to the amount of material present.

If u is the amount of material at any time t, then $\boxed{\dfrac{du}{dt} = -ku}$ *where k is a constant.*

***Ex. 1 :** Uranium disintegrates at a rate proportional to the amount that is present at any instant. If M_1 and M_2 grams of uranium are present at times T_1 and T_2 respectively, find the half-life of uranium.* **(May 2004)**

Sol. : Let the mass of uranium at any time t be m grams.

Then the equation of disintegration of uranium is

$$\frac{dm}{dt} = -\mu m, \text{ where } \mu \text{ is a constant}$$

Integrating, we get $\int \frac{dm}{m} = -\mu \int dt + c$

or $\log m = c - \mu t$

Initially, when t = 0, m = M (say) so that c = log M ... (i)

$\therefore$ (i) becomes $\mu t = \log M - \log m$

Also, when $t = T_1$, $m = M_1$ and when $t = T_2$, $m = M_2$ ... (ii)

$\therefore$ From (ii), we get $\mu T_1 = \log M - \log M_1$

$\mu T_2 = \log M - \log M_2$ ... (iii)

Subtracting (iii) from (iv), we get, ... (iv)

$$\mu (T_2 - T_1) = \log M_1 - \log M_2 = \log (M_1/M_2)$$

Hence $$\mu = \frac{\log (M_1/M_2)}{T_2 - T_1}$$

Let the mass reduce to half its initial value in time T.

i.e. when $t = T$, $m = \frac{1}{2} M$

$\therefore$ From (ii), we get $\mu T = \log M - \log (M/2) = \log 2$

Thus, $$\boxed{T = \frac{1}{\mu} \log 2 = \frac{(T_2 - T_1) \log 2}{\log (M_1/M_2)}}$$

EXERCISE 2.2

1. Radium decomposes at the rate proportional to the amount present. If 5% of the original amount disappears in 50 years, how much will remain after 100 years ? **Ans.** 90.25%
2. If 30% of a radioactive substance disappeared in 10 days, how long will it take for 90% of it to disappear ? **(Dec. 2009, 2008) Ans.** 64.5 days
3. Radium decomposes at the rate proportional to the quantity of radium present. Suppose that it is found that in 25 years approximately 1.1% of certain quantity of radium has decomposed. Determine approximately how long will it take for one half of the original amount of radium to decompose. **Ans.** 1564.66 ≈ 1565 years.
4. Radium decomposes at the rate proportional to the amount present. If a fraction M of the original amount disappears in 1 year, how much will remain at the end of 21 years ? **Ans.** $(1 - 1/M)^{21}$ times the original amount.

2.6 NEWTON'S LAW OF COOLING

According to this law, the *temperature of a body changes at a rate which is proportional to the difference in temperature between that of the surrounding medium and that of the body itself.*

If θ_0 is the temperature of the surroundings and θ that of the body at any time t, then

$$\boxed{\frac{d\theta}{dt} = -k (\theta - \theta_0)} \quad \text{where } k \text{ is a constant.}$$

Illustrations on Newton's Law of Cooling :

Ex. 1 : *A metal ball is heated to a temperature of 100°C and at time t = 0 it is placed in water which is maintained at 40°C. If the temperature of the ball is reduced to 60°C in 4 minutes, find the time at which the temperature of the ball is 50°C.* **(May 2011, 2006, 2018)**

Sol. : Let the temperature of the ball be T°C at time t min. Then the differential equation is given by

$$\frac{dT}{dt} = -k (T - 40) \quad \text{or} \quad \frac{dT}{T - 40} = -k\, dT$$

Integration gives $-kt = \log (T - 40) + \log C$ ... (1)

At t = 0, T = 100. This gives $\log C = -\log 60$ ... (2)

and hence (2) becomes : $-kt = \log \frac{T - 40}{60}$ ... (3)

But T = 60 at t = 4. Substituting these values in (3), we obtain

$$-4k = \log \frac{1}{3} \text{ or } k = \frac{1}{4} \log 3$$

Hence, equation (3) gives :

$$-\frac{t}{4}\log 3 = \log\frac{T-40}{60}$$

When T = 50, we obtain $\boxed{t = \frac{4\log 6}{\log 3} = 6.5 \text{ minutes}}$

Ex. 2 : *A body originally at 80°C cools down to 60°C in 20 minutes, the temperature of the air being 40°C. What will be the temperature of the body after 40 minutes from the original ?* ***(May 2008, 2007, 2006; Dec. 2011, Nov. 2014)***

Sol. : If θ is the temperature of the body at any time t, then

$$\frac{d\theta}{dt} = -k(\theta - 40), \text{ where k is a constant.}$$

Integrating, $\int \frac{d\theta}{\theta - 40} = -k\int dt + \log C$, where C is a constant

or $\log(\theta - 40) = -kt + \log C$

i.e. $\theta - 40 = Ce^{-kt}$ ... (1)

When $t = 0$, $\theta = 80°$ and when $t = 20$, $\theta = 60°$

$\therefore$ $40 = C$, and $20 = Ce^{-20k}$

$\therefore$ $k = \frac{1}{20}\log 2$ ... (2)

Thus (1) becomes $\theta - 40 = 40\, e^{\left(-\frac{1}{20}\log 2\right)t}$

When t = 40 min., $\theta = 40 + 40\,e^{-2\log 2} = 40 + 40\,e^{\log(1/4)} = 40 + 40 \times \frac{1}{4}$

$\boxed{\theta = 50°C}$

Ex. 3 : *According to Newton's law of cooling, the rate at which a substance cools in moving air is proportional to the difference between the temperature of the substance and that of the air. If the temperature of the air is 300°C and the substance cools from 370°C to 340°C in 15 minutes, find when the temperature will be 310°C.*

Sol. : Let T be the temperature of the substance at the time t minutes.

Then, by Newton's law of cooling the differential equation is,

$$\frac{dT}{dt} = -k(T-300) \text{ or } \frac{dT}{T-300} = -k\,dt \quad ... (1)$$

Integrating between the limits t = 0, T = 370 and t = 15, T = 340,

$$\int_{370}^{340}\frac{dT}{T-300} = -k\int_0^{15} dt \Rightarrow [\log(T-300)]_{370}^{340} = -k[t]_0^{15}$$

or $\log 40 - \log 70 = -15k$

$\therefore$ $k = \frac{1}{15}\log\frac{7}{4}$ ... (2)

Integrating between the limits t = 0, T = 370 and t = t, T = 310

$$\int_{370}^{310}\frac{dT}{T-300} = -k\int_0^{t} dt \Rightarrow [\log(T-300)]_{370}^{310} = -k[t]_0^{t}$$

$$\log 10 - \log 70 = -kt = -\left(\frac{1}{15}\log\frac{7}{4}\right)t$$

$\therefore$ $\boxed{t = \frac{15(\log 7)}{\left(\log\frac{7}{4}\right)} = 52.2 \text{ minutes}}$

Ex. 4 : *Water at temperature 100° C cools in 10 minutes to 60° C in a room temperature of 20° C. Find when the temperature will be 30°C.*

Sol. : Let T be the temperature of the water at any time t minutes. Then by Newton's law of cooling, the differential equation is

$$\frac{dT}{dt} = -k(T-20)$$

or $$\frac{dT}{T-20} = -k\,dt \qquad \ldots (1)$$

Integrating (i) between the limits t = 0, T = 100°C and t = 10, T = 60°C

$$\int_{100}^{60} \frac{dT}{T-20} = -k\int_{0}^{10} dt$$

$\therefore$ $$[\log(T-20)]_{100}^{60} = -k\,[t]_0^{10}$$

or $$\log 40 - \log 80 = -10k$$

$\therefore$ $$k = \frac{1}{10}\log 2 \qquad \ldots (2)$$

Next, Integrating (1) between the limits t = 0, T = 100°C and t = t, T = 30°C.

$$\int_{100}^{30} \frac{dT}{T-20} = -k\int_{0}^{t} dT$$

$\therefore$ $$[\log(T-20)]_{100}^{30} = -k\,[t]_0^{t}$$

or $$\log 10 - \log 80 = -kt = -\left(\frac{1}{10}\log 2\right)t \qquad \left(\because k = \frac{1}{10}\log 2\right)$$

$\therefore$ $$t = \frac{\log 8}{\left(\frac{1}{10}\log 2\right)} = 10\,\frac{\log 8}{\log 2}$$

or $$\boxed{t = 30 \text{ minutes}}$$

Ex. 5 : *According to Newton's law of cooling, the rate at which a substance cools in moving air is proportional to the difference between the temperature of the substance and that of the air. If the temperature of the air is 30°C and the substance cools from 100°C to 70°C in 15 minutes, find when the temperature will be 40°C.* ***(Dec. 2008, 2006, 2005, 2016)***

Sol. : Let the unit of time be a minute and T be the temperature of the substance at any instant t. Then by Newton's law of cooling, we have

$$\frac{dT}{dt} = -k(T-30)$$

or $$\frac{dT}{T-30} = -k\,dt$$

Integrating, $$\log(T-30) = -kt + C \qquad \ldots (1)$$

Initially, when t = 0, T = 100

$\therefore$ from (1), $$C = \log 70$$

Substituting the value of C in (1), we have

$$\log(T-30) = -kt + \log 70$$

or $$kt = \log 70 - \log(T-30) \qquad \ldots (2)$$

Also, when t = 15, T = 70, gives

$$15k = \log 70 - \log 40 \qquad \ldots (3)$$

Dividing (2) by (3), we have $$\frac{t}{15} = \frac{\log 70 - \log(T-30)}{\log 70 - \log 40} \qquad \ldots (4)$$

Now, when T = 40, we have from (4),

$$\frac{t}{15} = \frac{\log 70 - \log 10}{\log 70 - \log 40} = \frac{\log_e 7}{\log_e 7/4} = \frac{\log_{10} 7}{\log_{10}(7/4)} = 3.48$$

$\therefore$ $$\boxed{t = 15 \times 3.48 = 52.20}$$

Hence the temperature will be 40°C after 52.2 minutes.

Ex. 6 : *If the temperature of the body drops from 100°C to 60°C in one minute when the temperature of the surrounding is 20°C, what will be the temperature of the body at the end of the second minute ?* ***(May 2005; Dec. 2006, 2005, 2013, 2017)***

Sol. : Let T be the temperature of the body at time t minutes. The differential equation is given by

$$\frac{dT}{dt} = -k(T-20) \quad \text{or} \quad \frac{dT}{T-20} = -k\,dt \qquad \dots (1)$$

Integrating, $\log(T-20) = -kt + C$

Initially at $t = 0$, $T = 100$, gives $C = \log 80$

$\therefore$ $kt = \log 80 - \log(T-20) \qquad \dots (2)$

Also at $t = 1$, $T = 60$, gives $k = \log 80 - \log 40$

$$\therefore \quad t = \frac{\log 80 - \log(T-20)}{\log 80 - \log 40}$$

For $t = 2$,

$$2 = \frac{\left(\log \frac{80}{T-20}\right)}{\log 2} \quad \text{or} \quad \log 2^2 = \log \frac{80}{T-20}$$

$$\therefore \quad 4 = \frac{80}{T-20} \quad \text{or} \quad T - 20 = 20$$

$$\therefore \quad \boxed{T = 40°C}$$

Alternatively, Integrating (1) between the limits t = 0, T = 100 and t = 1, T = 60, we obtain

$$\int_{100}^{60} \frac{dT}{T-20} = -k\int_0^1 dt \Rightarrow [\log(T-20)]_{100}^{60} = -k\,[t]_0^1$$

$$\therefore \quad k = \log 80 - \log 40 = \log 2$$

Next, integrate (1) between the limits t = 0, T = 100 and t = 2, T = T,

$$\int_{100}^{T} \frac{dT}{T-20} = -k\int_0^1 dt \Rightarrow [\log(T-20)]_{100}^{T} = -k\,[t]_0^2$$

$$\therefore \quad \log(T-20) - \log 80 = -2k \Rightarrow \log\left(\frac{T-20}{80}\right) = -2\log 2 \qquad (\because k = \log 2)$$

$$\therefore \quad \frac{T-20}{80} = \frac{1}{4}$$

$$\text{or} \quad \boxed{T = 40°C}$$

EXERCISE 2.3

1. If a thermometer is taken outdoors where the temperature is 0°C, from a room in which the temperature is 21°C and the reading drops to 10°C in 1 minute, how long after its removal will the reading be 5°C ?

   [Hint : $\frac{dT}{dt} = -k(T-0)$; $t = 0$, $T = 21$ and $t = 1$, $T = 10$; $k = \log_e 2.1 = \ln 2.1$] **Ans.** 1.93424 min = 1 min 56 sec.

2. Water at temperature 100°C cools in 10 minutes to 88°C in a room of temperature 25°C. Find the temperature of water after 20 minutes. **(Dec. 2008, May 2017)**

   [Hint : $\frac{d\theta}{dt} = -k(\theta - 25)$; $t = 0$, $\theta = 100$ and $t = 10$, $\theta = 88$; $k = \frac{1}{10}\log\frac{25}{21}$] **Ans.** when $t = 20$, $\theta = 77.9°C$

3. A body at temperature 100°C is placed in a room whose temperature is 20°C and cools to 60°C in 5 minutes. Find its temperature after a further interval of 3 minutes. **(Nov./Dec. 19, Dec. 2009, 2006, 18; May 2009, 15)**

   [Hint : $t = 0$, $T = 100$; $t = 5$, $T = 60$; $k = \frac{1}{5}\log 2$; find T when $t = 8$.] **Ans.** 46.4°C

4. When a thermometer is placed in a hot liquid bath at temperature T, the temperature θ indicated by the thermometer rises at the rate of $T - \theta$. For a bath at 95°C, the temperature reads 15°C at a certain instant (t = 0) and 35° at t = 10 second. What will be its temperature at t = 20 sec. ? **(Dec. 2007)**

   [Hint : θ = Temperature of thermometer, T = 95 = temperature of hot liquid bath; $t = 0$, $\theta = 15$ and $t = 10$, $\theta = 35$; $k = \frac{1}{10}\log\frac{4}{3}$.] **Ans.** 50°C

5. A copper ball is heated to a temperature of 100°C. Then at time t = 0 it is placed in water which is maintained at a temperature of 30°C. At the end of 3 minutes, the temperature of the ball is reduced to 70°C. Find the time at which the temperature of the ball drops to 31°C. [**Hint :** t = 0, T = 100 and t = 3, T = 70.]

Ans. $22.78 \approx 23$ min

6. Two friends A and B order coffee and receive cups of equal temperature at the same time. A adds a small amount of cool cream immediately but does not drink his coffee until 10 minutes later, B waits for 10 minutes and adds the same amount of cool cream and begins to drink. Assuming the Newton's law of cooling, decide who drinks the hotter coffee ?

Ans. B drinks hotter coffee.

2.7 SIMPLE ELECTRIC CIRCUITS

We shall consider circuits made up of

(i) three passive elements – resistance, inductance, capacitance and

(ii) an active element – voltage source which may be a battery or a generator.

1. **Table of Elements, Symbols and Units :**

Sr. No.	Element	Symbol	Unit
1.	Time	t	second
2.	Quantity of electricity (electric charge)	q	coulomb
3.	Current (= time rate flow of charge)	$i = \frac{dq}{dt}$	ampere (A)
4.	Resistance, R	$\xrightarrow{i}$ + R −	ohm (Ω)
5.	Inductance, L	$\xrightarrow{i}$ + L −	henry (H)
6.	Capacitance, C	$\xrightarrow{i}$ + − C	farad (F)
7.	Electromotive force or voltage (constant), E	$\xrightarrow{i}$ − + Battery, E = constant	volt
8.	Variable voltage generator	+ Generator, E = Variable voltage	volt

2. **Basic Relations :**

(i) $i = \frac{dq}{dt}$ or $q = \int i\, dt$ [$\because$ current is the rate of flow of electricity]

(ii) Voltage drop across resistance R = Ri (Ohm's law)

(iii) Voltage drop across inductance $L = L\frac{di}{dt}$.

(iv) Voltage drop across capacitance $C = \frac{q}{C}$.

3. **Kirchhoff's law :** The formulation of differential equations for an electrical circuit depends on the following two Kirchhoff's laws which are of cardinal importance :

(I) The algebraic sum of the **voltage** drops around any closed circuit is equal to the resultant electromotive force in the circuit.

(II) The algebraic sum of the **currents** flowing into (or from) any node is zero.

4. Differential Equations :

(i) Circuit involving L and R along with a Voltage Source (battery) E, all in Series : Consider a circuit containing resistance R and inductance L in series with a voltage source (battery) E.

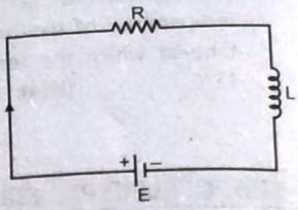

Fig. 2.4

Let i be the current flowing in the circuit at any time t. Then by Kirchhoff's first law, we have

Sum of voltage drops across R and L = E

i.e. $$Ri + L\frac{di}{dt} = E \qquad \ldots (1)$$

or $\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}$ which is a linear differential equation.

$$\text{I.F.} = e^{\int \frac{R}{L} dt} = e^{Rt/L}$$

and therefore its solution is

$$\therefore \quad i \cdot e^{Rt/L} = \int \frac{E}{L} e^{Rt/L} dt + c = \frac{E}{L} \cdot \frac{L}{R} \cdot e^{Rt/L} + C \qquad \ldots (2)$$

Here, $$i = \frac{E}{R} + Ce^{-Rt/L}$$

If initially there is no current in the circuit, i.e. $i = 0$, when $t = 0$, we have $C = -\frac{E}{R}$. Thus, (2) becomes

$$\boxed{i = \frac{E}{R}(1 - e^{-Rt/L})}.$$

As $t \to \infty$, $i = \frac{E}{R}$, which shows that i increases with t and attains the maximum value $\frac{E}{R}$.

(ii) Circuits involving R and C along with a Voltage Source (battery) E all in Series : Consider a circuit containing resistance R and capacitance C in series with a voltage source (battery) E.

Let i be the current flowing in the circuit at any time t. Then by Kirchhoff's first law, we have

Sum of voltage drops across R and C = E (e.m.f.)

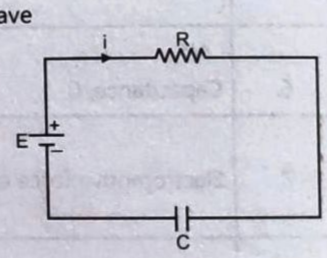

Fig. 2.5

i.e. $$Ri + \frac{q}{C} = E.$$

Since $i = \frac{dq}{dt}$, this equation in terms of q can be written as

$$R\frac{dq}{dt} + \frac{q}{C} = E \text{ or } \frac{dq}{dt} + \frac{q}{RC} = \frac{E}{R}$$

which is a linear differential equation.

Here $$\text{I.F.} = e^{\int \frac{1}{RC} dt} = e^{\frac{t}{RC}}$$

and therefore its solution is

$$q\,e^{\frac{t}{RC}} = \int \frac{E}{R} e^{\frac{t}{RC}} dt + B = \frac{E}{R}\left(RC\, e^{\frac{t}{RC}}\right) + B$$

$$q = EC + B\,e^{-\frac{t}{RC}}$$

Assuming $q = q_0$ when $t = 0$

$$q_0 = EC + B \text{ giving } B = q_0 - EC$$

$$\therefore \quad q = EC + (q_0 - EC)\, e^{-t/RC}$$

or $$q = EC\,(1 - e^{-t/RC}) + q_0 e^{-t/RC}$$

$$\therefore \quad i = \frac{dq}{dt} = EC\frac{1}{RC} e^{-t/RC} - \frac{q_0}{RC} e^{-t/RC}$$

or $$\boxed{i = \left(\frac{E}{R} - \frac{q_0}{RC}\right) e^{-t/RC}}$$

(iii) Circuit involving L and C both in Series, after Removing Source Applied e.m.f. : Consider a circuit containing inductance L and capacitance C in series without applied e.m.f.

Let i be the current flowing in the circuit at any time t. Then by Kirchhoff's first law, we have sum of voltage drops across L and C = 0

$$L\frac{di}{dt} + \frac{q}{C} = 0$$

i.e.

$$\frac{di}{dt} = -\frac{q}{LC}$$

$$\frac{di}{dq}\frac{dq}{dt} = -\frac{q}{LC}$$

$\therefore$

$$i\frac{di}{dq} = -\frac{q}{LC}$$

$$\int i\,di = -\int \frac{q}{LC}\,dq + A \Rightarrow \frac{i^2}{2} = -\frac{q^2}{2LC} + A$$

$$i^2 = -\frac{q^2}{LC} + B$$

Assuming $i = 0,\quad q = q_0$ when $t = 0 \quad \therefore \quad B = \frac{q_0^2}{LC}$

$$i^2 = \frac{\left(q_0^2 - q^2\right)}{LC}$$

$$i = \pm\frac{\sqrt{\left(q_0^2 - q^2\right)}}{\sqrt{LC}}$$

Since q decreases as t increases, $i = \frac{dq}{dt} = -\frac{1}{\sqrt{LC}}\sqrt{q_0^2 - q^2}$

$$-\frac{dq}{\sqrt{q_0^2 - q^2}} = \frac{dt}{\sqrt{LC}}$$

Integrating $\qquad \cos^{-1}\left(\frac{q}{q_0}\right) = \frac{t}{\sqrt{LC}} + C$

Assuming $q = q_0$, when $t = 0 \;\therefore\; C = 0$

$\therefore$

$$\frac{t}{\sqrt{LC}} = \cos^{-1}\left(\frac{q}{q_0}\right)$$

$$\boxed{q = q_0\cos\left(\frac{t}{\sqrt{LC}}\right)}$$

USEFUL FORMULAE

1.	$\int e^{at}\sin bt\,dt = \frac{e^{at}}{a^2 + b^2}(a\sin bt - b\cos bt)$
2.	$\int e^{at}\cos bt\,dt = \frac{e^{at}}{a^2 + b^2}(a\cos bt + b\sin bt)$
3.	$\int e^{at}\sin bt\,dt = \frac{e^{at}}{\sqrt{a^2 + b^2}}\sin(bt - \phi)$, where $\phi = \tan^{-1}\left(\frac{b}{a}\right)$
4.	$\int e^{at}\cos bt\,dt = \frac{e^{at}}{\sqrt{a^2 + b^2}}\cos(bt - \phi)$, where $\phi = \tan^{-1}\left(\frac{b}{a}\right)$

Illustrations on Electrical Circuits :

Ex. 1 : *A resistance of 100 ohms, an inductance of 0.5 henry are connected in series with a battery of 20 volts. Find the current in a circuit as a function of t.* **(Nov./Dec. 2019, May 2005, 2014, 2017, 2018)**

Sol. : By Kirchhoff's law, we have $L\frac{dI}{dt} + RI = E$. (Read i = I)

$$\frac{dI}{dt} + \frac{R}{L}I = \frac{E}{L} \text{ which is linear. Here } P = \frac{R}{L}, Q = \frac{E}{L}.$$

$$\text{I.F.} = e^{\int \frac{R}{L}dt} = e^{\frac{Rt}{L}}$$

G.S. is
$$I e^{\frac{Rt}{L}} = \int \frac{E}{L}\cdot e^{\frac{Rt}{L}} dt + A = \frac{E}{R}e^{\frac{Rt}{L}} + A$$

∴ But at t = 0, I = 0 $0 = \frac{E}{R} + A$ giving $A = -\frac{E}{R}$

∴ G.S. becomes
$$I e^{\frac{Rt}{L}} = \frac{E}{R}\left(-1 + e^{\frac{Rt}{L}}\right) \text{ or } I = \frac{E}{R}\left(1 - e^{-\frac{Rt}{L}}\right)$$

Given : R = 100 ohms, L = 0.5 henry, E = 20 volts

∴
$$\boxed{I = \frac{20}{100}\left(1 - e^{-\frac{100t}{0.5}}\right) = \frac{1}{5}(1 - e^{-200t})}$$

Ex. 2 : *A voltage $E e^{-at}$ is applied at t = 0 to a circuit containing inductance L and resistance R. Show that the current at any time t is $\frac{E}{R - aL}\left(e^{-at} - e^{-\frac{Rt}{L}}\right)$.* **(May 11, 09, 07, Dec. 16)**

Sol. : Let i be the current in the circuit at any time t. Also a voltage Ee^{-at} is applied at t = 0 to a circuit containing inductance L and resistance R. Then by Kirchhoff's law, we have,

$$L\frac{di}{dt} + Ri = Ee^{-at} \Rightarrow \frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}e^{-at} \qquad \text{... (1)}$$

which is a linear differential equation.

∴
$$\text{I.F.} = e^{\frac{Rt}{L}}$$

Solution of (1) is given by

$$i.e^{\frac{R}{L}t} = \frac{E}{L}\int e^{-at}.e^{\frac{R}{L}t} dt + B = \frac{E}{L}\int e^{\left(\frac{R}{L} - a\right)t} dt + B = \frac{E}{L}.\frac{e^{\left(\frac{R}{L} - a\right)t}}{(R/L - a)} + B$$

$$i\cdot e^{\frac{R}{L}t} = \frac{E}{R - aL}\cdot e^{\left(\frac{R}{L} - a\right)t} + B$$

Given $i = 0, t = 0, B = -\frac{E}{R - aL}$

∴
$$\boxed{i = \frac{E}{R - aL}\left(e^{-at} - e^{-\frac{R}{L}t}\right)}$$

Ex. 3 : *In a circuit containing inductance L, resistance R and voltage E, the current I is given by : $E = RI + L\frac{dI}{dt}$. Given L = 640 H, R = 250 Ω and E = 500 volts. I being zero when t = 0. Find the time that elapses, before it reaches 90% of its maximum value.* **(May 2008, 2006, 2004, 2015)**

Sol. : Given that
$$E = RI + L\frac{dI}{dt} \Rightarrow \frac{dI}{dt} + \frac{R}{L}I = \frac{E}{L} \text{ which is a linear differential equation.}$$

$$\text{I.F.} = e^{\frac{R}{L}\int dt} = e^{\frac{Rt}{L}}$$

G.S.

$$I e^{\frac{Rt}{L}} = \frac{E}{L} \int e^{\frac{Rt}{L}} dt + A = \frac{E}{L} \frac{e^{\frac{Rt}{L}}}{R/L} + A = \frac{E}{R} e^{\frac{Rt}{L}} + A$$

$$I = \frac{E}{R} + A e^{\frac{-Rt}{L}}$$

I being zero when $t = 0 \therefore \quad 0 = \frac{E}{R} + A e^0 \quad \therefore A = -\frac{E}{R}$

$$I = \frac{E}{R} - \frac{E}{R} e^{-\frac{Rt}{L}}, \quad I = \frac{E}{R}\left(1 - e^{-\frac{Rt}{L}}\right) \qquad \ldots (1)$$

Maximum value of I be I_{max} which is obtained when $t \to \infty$

$$I_{max} = \frac{E}{R}(1 - e^{-\infty}) = \frac{E}{R}(1 - 0) = \frac{E}{R}$$

$$I_{max} = \frac{90}{100}\frac{E}{R} \quad \text{(90\% of max.)}$$

Putting in (1) if $t = t_1$ for $90\% \ I_{max}$.

$$\frac{9}{10}\frac{E}{R} = \frac{E}{R}\left(1 - e^{-\frac{Rt_1}{L}}\right)$$

$$\frac{9}{10} = 1 - e^{-\frac{Rt_1}{L}}$$

$$e^{-\frac{Rt_1}{L}} = 1 - \frac{9}{10} = \frac{1}{10}$$

$$-\frac{Rt_1}{L} = -\log_e 10$$

$$t_1 = \frac{L}{R}\log_e 10$$

Given $L = 640$, $R = 250$ $\therefore$ $\quad t_1 = \frac{640}{250}\log_e 10$

$\therefore$ $\boxed{t_1 = \frac{64}{25}\log_e 10 = 5.89 \text{ sec}}$ which is the required time.

***Ex. 4 :** Show that the differential equation for the current 'i' in an electrical circuit containing an inductance L and a resistance R in series and acted on by an electromotive force E sin ωt satisfies the equation $L\frac{di}{dt} + Ri = E \sin \omega t$.* **(Dec. 2011, May 2019)**

Find the value of the current at an time t, if initially there is no current in the circuit.

Sol. : Given $\quad Ri + L\frac{di}{dt} = E \sin \omega t$ or $\frac{di}{dt} + \frac{R}{L} i = \frac{E}{L} \sin \omega t$ which is a linear differential equation.

Here $\quad \text{I.F.} = e^{\int \frac{R}{L} dt} = e^{\frac{Rt}{L}}$

The general solution is $\quad i\,(\text{I.F.}) = \int \frac{E}{L} \sin \omega t \cdot (\text{I.F.})\, dt + c$

i.e.

$$i\, e^{\frac{Rt}{L}} = \frac{E}{L} \int e^{\frac{Rt}{L}} \sin \omega t \, dt + c = \frac{E}{L} \frac{e^{\frac{Rt}{L}}}{\sqrt{[(R/L)^2 + \omega^2]}} \sin\left(\omega t - \tan^{-1}\frac{L\omega}{R}\right) + c$$

or

$$i = \frac{E}{\sqrt{(R^2 + \omega^2 L^2)}} \sin(\omega t - \phi) + c e^{-Rt/L} \qquad \ldots (1)$$

where, $\tan \phi = L\omega/R$

Initially when $t = 0,\ i = 0\ \therefore\quad 0 = \frac{E \sin(-\phi)}{\sqrt{(R^2 + \omega^2 L^2)}} + c$ i.e. $c = \frac{E \sin \phi}{\sqrt{(R^2 + \omega^2 L^2)}}$

Thus, (1) takes the form $\quad i = \frac{E \sin(\omega t - \phi)}{\sqrt{(R^2 + \omega^2 L^2)}} + \frac{E \sin \phi}{\sqrt{(R^2 + \omega^2 L^2)}} \cdot e^{-Rt/L}$

$$\boxed{i = \frac{E}{\sqrt{(R^2 + \omega^2 L^2)}} [\sin(\omega t - \phi) + \sin \phi \cdot e^{-Rt/L}]}$$ which gives the current at any time t.

Ex. 5 : *The equation of an L-R circuit is given by* $L\frac{dI}{dt} + RI = 10 \sin t$. *If* $I = 0$, *at* $t = 0$, *express I as a function of t.* ***(Dec. 2008, 2005)***

Sol. : Given $\quad L\frac{dI}{dt} + RI = 10 \sin t$

$\therefore \quad \frac{dI}{dt} + \frac{R}{L} I = \frac{10}{L} \sin t$ is a linear differential equation

Here $\quad$ I.F. $= e^{\frac{Rt}{L}}$

G.S. is $\quad I \cdot e^{Rt/L} = \frac{10}{L} \int e^{Rt/L} \sin t \, dt + B = \frac{10}{L} \frac{e^{Rt/L}}{\sqrt{\frac{R^2}{L^2} + 1}} \sin(t - \phi) + B$ where $\tan \phi = \frac{L}{R}$

$$I = \frac{10}{\sqrt{R^2 + L^2}} \cdot \sin(t - \phi) + B\, e^{-\frac{Rt}{L}}$$

When $t = 0,\ \ I = 0,\ \therefore B = \frac{10}{\sqrt{R^2 + L^2}} \sin \phi$

$$\boxed{I = \frac{10}{\sqrt{R^2 + L^2}} \left[\sin(t - \phi) + \sin \phi\, e^{-\frac{Rt}{L}}\right]}$$

Ex. 6 : *A constant electromotive force E volts is applied to a circuit containing a constant resistance R ohms in series and a constant inductance L henries. If the initial current is zero, show that the current builds up to half its theoretical maximum in (L log 2)/R seconds.*

(Dec. 2011, 2009, 2008, 2006, 2017; May 2013)

Sol. : Let i be the current in the circuit at any time t. By Kirchhoff's law, we have

$$L\frac{di}{dt} + Ri = E \quad \text{or} \quad \frac{di}{dt} + \frac{R}{L} i = \frac{E}{L} \qquad \text{... (1)}$$

which is a linear differential equation.

Here $\quad$ I.F. $= e^{\int \frac{R}{L} dt} = e^{\frac{Rt}{L}}$

The general solution of equation (1) is

$$i\,(\text{I.F.}) = \int \frac{E}{L} (\text{I.F.})\, dt + c$$

or $\quad i \cdot e^{\frac{Rt}{L}} = \int \frac{E}{L} \cdot e^{\frac{Rt}{L}} dt + c = \frac{E}{L} \cdot \frac{L}{R} e^{\frac{Rt}{L}} + c$

or $\quad i = \frac{E}{R} + c\, e^{-\frac{Rt}{L}} \qquad \text{... (2)}$

Initially, when $t = 0,\ \ i = 0$ so that $c = -\frac{E}{R}$

Thus, (2) becomes, $\quad i = \frac{E}{R}\left(1 - e^{-\frac{Rt}{L}}\right) \qquad \text{... (3)}$

This equation gives the current in the circuit at any time t. Clearly, i increases with t and attains the maximum value $\frac{E}{R}$.

Let the current in the circuit be half its theoretical maximum after a time T seconds.

Then, $$\frac{1}{2}\cdot\frac{E}{R} = \frac{E}{R}\left(1 - e^{-\frac{RT}{L}}\right)$$

or $$e^{-\frac{RT}{L}} = \frac{1}{2}$$

or $$-\frac{RT}{L} = \log\frac{1}{2} = -\log 2$$

∴ $$\boxed{T = (L\log 2)/R}$$

Ex. 7 : *An electrical circuit contains an inductance of 5 henries and a resistance of 12 ohms in series with an e.m.f. 120 sin (20t) volts. Find the current at t = 0.01, if it is zero when t = 0.* **(Dec. 2006)**

Sol. : $$\frac{di}{dt} + \frac{R}{L}i = \frac{120\sin(20t)}{L}$$ which is a linear differential equation.

$$\text{I.F.} = e^{\frac{Rt}{L}}$$ and its G.S. is

$$i\cdot e^{\frac{Rt}{L}} = \frac{120}{L}\int e^{\frac{Rt}{L}}\sin(20t)\,dt + A$$

$$= \frac{120}{L}\,\frac{e^{\frac{Rt}{L}}}{\sqrt{R^2/L^2 + 400}}\sin(20t - \phi) + A \quad \text{where } \tan\phi = \frac{20L}{R}$$

$$i = \frac{120}{\sqrt{R^2 + 400L^2}}\sin(20t - \phi) + A e^{-\frac{Rt}{L}}$$

To find the current i at t = 0.01, given L = 5 H, R = 12 Ω and

$$\phi = \tan^{-1}\left[\frac{20(5)}{12}\right] = 1.451367401$$

we get $$\boxed{i = 0.023729634 \text{ ampere.}}$$

Ex. 8 : *The equation of electromotive force in terms of current i for an electrical circuit having resistance R and a condenser of capacity C in series, is* $E = Ri + \int \frac{i}{C}\,dt$. *Find the current i at any time t, when* $E = E_0 \sin \omega t$.

Sol. : The given equation can be written as $Ri + \int \frac{i}{C}\,dt = E_0 \sin \omega t$.

Differentiating both sides w.r.t. t, we have

$$R\frac{di}{dt} + \frac{i}{C} = \omega E_0 \cos\omega t \quad \text{or} \quad \frac{di}{dt} + \frac{i}{RC} = \frac{\omega E_0}{R}\cos\omega t \qquad \ldots (1)$$

which is linear differential equation.

$$\text{I.F.} = e^{\int \frac{1}{RC}dt} = e^{\frac{t}{RC}}$$

Here the general solution of equation (1) is

$$i.\,e^{\frac{t}{RC}} = \int \frac{\omega E_0}{R}\cos\omega t\cdot e^{\frac{t}{RC}}\,dt + k = \frac{\omega E_0}{R}\cdot\frac{e^{\frac{t}{RC}}}{\sqrt{\left(\frac{1}{RC}\right)^2 + \omega^2}}\cos\left(\omega t - \tan^{-1}\frac{\omega}{\frac{1}{RC}}\right) + k$$

$$= \frac{\omega C E_0}{\sqrt{1 + R^2C^2\omega^2}}\,e^{\frac{t}{RC}}\cos(\omega t - \phi) + k \quad \text{where } \tan\phi = RC\omega$$

or $$\boxed{i = \frac{\omega C E_0}{\sqrt{1 + R^2C^2\omega^2}}\cos(\omega t - \phi) + k\,e^{-\frac{t}{RC}}}$$ which gives the current at any time t.

***Ex. 9 :** Find the current i in the circuit having resistance R and condenser of capacity C in series with emf E sin ωt.*

(May 2006; Dec. 2010, 2007)

Sol. : We have $Ri + \frac{q}{C} = E \sin \omega t \Rightarrow R\frac{dq}{dt} + \frac{q}{C} = E \sin \omega t.$

or $\frac{dq}{dt} + \frac{q}{RC} = \frac{E}{R} \sin \omega t$ which is a linear differential equation

Here $\text{I.F.} = e^{\frac{t}{RC}}$

G.S. is $q \cdot e^{\frac{t}{RC}} = \int \frac{E}{R} \sin \omega t \; e^{\frac{t}{RC}} \; dt + A$

$$= \frac{E}{R} \cdot \frac{e^{\frac{t}{RC}}}{\sqrt{1/R^2C^2 + \omega^2}} \sin(\omega t - \phi) + A, \quad \text{where } \tan\phi = RC\omega$$

$$= EC \frac{e^{\frac{t}{RC}}}{\sqrt{1 + R^2C^2\omega^2}} \sin(\omega t - \phi) + A$$

$\therefore$ $$q = \frac{EC}{\sqrt{1 + R^2C^2\omega^2}} \sin(\omega t - \phi) + A\, e^{-\frac{t}{RC}}$$

$$\boxed{i = \frac{dq}{dt} = \frac{EC\omega}{\sqrt{1 + R^2C^2\omega^2}} \cos(\omega t - \phi) - \frac{A}{RC}\, e^{-\frac{t}{RC}}}$$

***Ex. 10 :** A circuit consists of resistance 'R' ohms and a condenser of 'C' farads connected to a constant e.m.f. E. If $\frac{q}{C}$ is the voltage of the condenser at time t after closing the circuit, show that the voltage at time t is $E\left(1 - e^{-\frac{t}{CR}}\right)$.* **(May 2007, 2005; Dec. 2009, 2013)**

Sol. : The differential equation for the circuit is $Ri + \frac{q}{C} = E \Rightarrow i + \frac{q}{RC} = \frac{E}{R}$

$$\frac{dq}{dt} + \left(\frac{1}{RC}\right) q = \frac{E}{R} \text{ which is a linear differential equation}$$

Here $\text{I.F.} = e^{\int \frac{1}{RC} dt} = e^{t/RC}$

G.S. is $q \cdot e^{t/RC} = \int \frac{E}{R} \cdot e^{t/RC}\, dt + B \Rightarrow q\, e^{t/RC} = \frac{Ee^{t/RC}}{R/RC} + B$

or $\frac{q}{C} e^{t/RC} = E\, e^{t/RC} + B_1$

At $t = 0$, $q = 0$, $\therefore$ $0 = E + B_1 \quad \therefore \quad B_1 = -E$

$\therefore$ $$\frac{q}{C} e^{\frac{t}{RC}} = E\, e^{\frac{t}{RC}} - E$$

$$\boxed{\frac{q}{C} = E\left(1 - e^{-\frac{t}{RC}}\right)}$$

***Ex. 11 :** The charge 'Q' on the plate of a condenser of capacity 'C' charged through a resistance 'R' by a steady voltage 'V' satisfies the differential equation $R\frac{dQ}{dt} + \frac{Q}{C} = V$. If $Q = 0$ at $t = 0$, show that $Q = CV[1 - e^{-t/RC}]$. Find the current flowing into the plate.*

(Dec. 2005; May 2008, 2004)

Sol. : The given equation is $R\frac{dQ}{dt} + \frac{Q}{C} = V$ i.e. $\frac{dQ}{dt} + \frac{1}{RC} Q = \frac{V}{R}$ which is a linear differential equation.

$$\text{I.F.} = e^{\int \frac{1}{RC} dt} = e^{t/RC}$$

G.S. is

$$Q\, e^{t/RC} = \int \frac{V}{R}\, e^{t/RC} \cdot dt + c_1 = \frac{V}{R} \cdot RC\, e^{t/RC} + c_1$$

$$Q = CV + c_1 e^{-t/RC}$$

for $t = 0, Q = 0 \therefore c_1 = -CV$

$$0 = CV + c_1$$

$\therefore$

$$Q = CV - CV\, e^{-t/RC}$$

$\therefore$

$$Q = CV\left[1 - e^{-\frac{t}{RC}}\right]$$

and

$$i = \frac{dQ}{dt} = CV\left[\frac{1}{RC} e^{-\frac{t}{RC}}\right]$$

$$\boxed{i = \frac{V}{R} e^{-\frac{t}{RC}}}$$

Ex. 12 : *An voltage $200\, e^{-5t}$ is applied to a circuit containing resistance $R = 20$ ohms and condenser of capacity $C = 0.01$ farads in series. Find the change and current at any time, assuming that $t = 0$, $q = 0$.*

Sol. : The differential equation for the R-C circuit is

$$Ri + \frac{q}{C} = E \quad \text{or} \quad R\frac{dq}{dt} + \frac{q}{C} = E \ \text{(in terms of q)}$$

or

$$\frac{dq}{dt} + \frac{q}{RC} = \frac{E}{R} \qquad \ldots (1)$$

Given : $R = 20$, $C = 0.01$ and $E = 200\, e^{-5t}$.

$\therefore$

$$\frac{dq}{dt} + \frac{q}{20 \times 0.01} = \frac{200\, e^{-5t}}{20} \quad \text{or} \quad \frac{dq}{dt} + 5q = 10\, e^{-5t} \qquad \ldots (2)$$

which is a linear differential equation.

$$\text{I.F} = e^{\int 5dt} = e^{5t}$$

G.S. is

$$q\, e^{5t} = \int (10\, e^{-5t})\, e^{5t}\, dt + A = 10t + A \qquad \ldots (3)$$

Given : $t = 0, q = 0 \ \therefore A = 0$

From G.S. (3),

$$q\, e^{5t} = 10t$$

or

$$q = 10t\, e^{-5t}$$

and

$$i = \frac{dq}{dt} = 10\,(e^{-5t} - 5t\, e^{-5t})$$

or

$$\boxed{i = 10\,(1 - 5t)\, e^{-5t}}$$

EXERCISE 2.5

1. In a circuit of resistance R, self inductance L, the current i is given by $L\frac{di}{dt} + Ri = E \cos pt$, where E, p are constants. Find the current at time t.

   Ans. $i = \frac{E}{L^2p^2 + R^2}\,[pL \sin pt + R \cos pt] + ce^{-Rt/L}$

2. When a switch is closed in a circuit containing a battery E, a resistance R and an inductance L, the current i builds up at rate given by $L\frac{di}{dt} + Ri = E$. Find i as a function of t. How long will it be, before the current has reached one half its maximum value, if $E = 6$ volts, $R = 100$ ohms and $L = 0.1$ henry ?

   [**Hint :** $i = \frac{E}{R}\left(1 - e^{-\frac{R}{L}t}\right)$, $i_{max} = \frac{E}{R}$ and $t = \frac{L}{R}\log$]

   Ans. 0.0006931 sec.

3. Solve the equation, $L\frac{di}{dt} + Ri = 200 \cos (300\, t)$, where $R = 100$ ohms, $L = 0.05$ henry and find i, given that $i = 0$ when $t = 0$. What value does i approach after a long time ? **Ans.** $i = \frac{40}{409}\,(20 \cos 300\, t + 3 \sin 300\, t)$

   $-\frac{800}{409}\, e^{-200t}$ and $\frac{40}{\sqrt{409}}$

4. Find the current at any time $t > 0$ in a circuit having in series a constant electromotive force 40 volt, a resistor 10 ohms and an inductor 0.2 henry, given that initial current is zero. Also find the current when $E(t) = 150 \cos 200\, t$. **Ans.** $i(t) = 4\,(1 - e^{-50\,t})$;

$$i(t) = \frac{3}{170}(50 \cos 200\,t + 200 \sin 200\,t) - \frac{15}{17} e^{-50t}$$

5. A capacitor $C = 0.01$ Farad in series with a resistor $R = 20$ ohms is charged from a battery $E = 10$ volts. Assuming that initially the capacitor is completely uncharged, determine the charge $Q(t)$, voltage $v(t)$ on the capacitor and current $i(t)$ in the circuit.

(Nov. 2015, May 2016)

Ans. $Q(t) = 0.1\,(1 - e^{-5t})$, $v(t) = \dfrac{Q}{C} = 10\,(1 - e^{-5t})$,

$$i(t) = \frac{dQ}{dt} = 0.5\, e^{-5t}$$

2.8 RECTILINEAR MOTION

Rectilinear motion is a motion of a body along a straight line. Let a body of mass m start moving from a fixed point O along a straight line OX under the action of a force F. After any time t, let it be moving at P where OP = x, then

(i) its velocity $(v) = \dfrac{dx}{dt}$

(ii) its acceleration $(a) = \dfrac{dv}{dt}$ or $\dfrac{d}{dt}\left(\dfrac{dx}{dt}\right) = \dfrac{d^2x}{dt^2}$

Also,
$$a = \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = \frac{dv}{dx} \cdot v$$

$\therefore$
$$a = v\frac{dv}{dx}$$

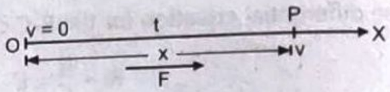

Fig. 2.6

$\therefore$ acceleration $= a = \dfrac{dv}{dt}$ or $v\dfrac{dv}{dx}$ or $\dfrac{d^2x}{dt^2}$

Newton's second law of motion states that $F = \dfrac{d}{dt}(mv)$.

If m is constant then
$$F = m\frac{dv}{dt} = ma$$

$\therefore$ $F = m\dfrac{dv}{dt}$ or $mv\dfrac{dv}{dx}$ or $m\dfrac{d^2x}{dt^2}$, where F is the effective force.

D'Alembert's Principle : Algebraic sum of the forces acting on a body along a given direction is equal to the product of mass × acceleration in that direction.

i.e. $\boxed{\text{Net force = Mass × Acceleration}}$

Note : The forces usually are : (i) vertically downward, (ii) tension in elastic string or spring, (iii) reactions or stresses at points in contact with other bodies, (iv) forces of attraction, (v) forces of resistance due to wind and friction etc.

Due precaution must be taken about the direction.

Force (Net) means *algebraic sum* of the forces acting along that direction i.e. direction of motion.

Illustrations on Rectilinear Motion :

Ex. 1 : *A body of mass m, falling from rest is subjected to the force of gravity and an air-resistance proportional to the square of the velocity 'kv^2'. If it falls through a distance x and possesses a velocity v at that instant, prove that* $\dfrac{2\,kx}{m} = \log\left(\dfrac{a^2}{a^2 - v^2}\right)$ *where* $mg = ka^2$.

(May 2006, 2005, 2004; Dec. 2011, 2016, 2017)

Sol. : The forces acting on the body are :

(i) the weight acting downwards $= mg$

(ii) the air resistance acting upwards $= -kv^2$

$\therefore$ Net force on the body $F = mg - kv^2$

Equation of motion is,

$$m\,v\frac{dv}{dx} = mg - kv^2$$

$$m\,v\frac{dv}{dx} = k\,(a^2 - v^2) \quad \text{(Given that } mg = ka^2\text{)}$$

Integrating,
$$\int \frac{v\,dv}{a^2 - v^2} = \int \frac{k}{m}\,dx + c$$

$$-\frac{1}{2}\log(a^2 - v^2) = \frac{kx}{m} + c$$

$\because$ $x = 0,\ v = 0 \ \therefore\ c = -\frac{1}{2}\log a^2$

$\therefore$
$$\frac{1}{2}[\log a^2 - \log(a^2 - v^2)] = \frac{kx}{m}$$

$$\boxed{\frac{2kx}{m} = \log\left(\frac{a^2}{a^2 - v^2}\right)}$$ which is the desired relation.

Fig. 2.7

***Ex. 2 :** A body starts moving from rest is opposed by a force per unit mass of value 'cx' and resistance per unit mass of value 'bv^2', where x and v are the displacement and velocity of the body at that instant. Show that the velocity of the body is given by*

$$v^2 = \frac{c}{2b^2}(1 - e^{-2bx}) - \frac{cx}{b}$$

(Dec. 2009, 2005, Nov. 2015, May 2017)

Sol. : By Newton's second law of motion, the equation of motion of the body is

$$v\frac{dv}{dx} = -cx - bv^2$$

or
$$v\frac{dv}{dx} + bv^2 = -cx \qquad \text{... (1)}$$

This is Bernoulli's linear differential equation, put $v^2 = z$ and $2v\frac{dv}{dx} = \frac{dz}{dx}$, equation (1) becomes

$$\frac{1}{2}\frac{dz}{dx} + bz = -cx$$

or
$$\frac{dz}{dx} + 2bz = -2cx \qquad \text{... (2)}$$

which is linear equation.

The general solution of (2) is

$$\text{I.F.} = e^{\int 2b\,dx} = e^{2bx}$$

$$z \cdot e^{2bx} = \int -2cx \cdot e^{2bx}\,dx + c_1$$

$$= -2c\int x\,e^{2bx}\,dx + c_1 \quad \text{[Integrating by parts]}$$

$$= -2c\left[x \cdot \frac{e^{2bx}}{2b} - \int 1 \cdot \frac{e^{2bx}}{2b}\,dx\right] + c_1$$

$$= -\frac{cx}{b}e^{2bx} + \frac{c}{2b^2}e^{2bx} + c_1$$

or
$$v^2 \cdot e^{2bx} = -\frac{cx}{b}e^{2bx} + \frac{c}{2b^2}e^{2bx} + c_1$$

or
$$v^2 = -\frac{cx}{b} + \frac{c}{2b^2} + c_1 e^{-2bx} \qquad \text{... (3)}$$

Initially, when $x = 0,\ v = 0 \ \therefore\ \frac{c}{2b^2} + c_1 = 0$ or $c_1 = -\frac{c}{2b^2}$

Substituting the value of c_1 in (3), we have

$$v^2 = -\frac{cx}{b} + \frac{c}{2b^2} - \frac{c}{2b^2}e^{-2bx}$$

or
$$\boxed{v^2 = \frac{c}{2b^2}(1 - e^{-2bx}) - \frac{cx}{b}}$$

Ex. 3 : Velocity of escape from the earth. *Determine the least velocity with which a particle must be projected vertically upwards so that it does not return to the earth. Assume that it is acted upon by the gravitational attraction of the earth only.* **(May 2008)**

Sol. : Let r be the variable distance of the particle from the earth's centre. By Newton's law of gravitation, the acceleration a of the particle is proportional to $\frac{1}{r^2}$ $\left(\text{i.e. } a \propto \frac{1}{r^2}\right)$.

$$\therefore \quad a = v\frac{dv}{dr} = -\frac{k}{r^2} \qquad \dots (1)$$

where v is the velocity of the particle when its distance from the earth's centre is r, the acceleration is negative because v is decreasing.

On the surface of the earth, $r = R$, the radius of the earth and $a = -g$, the acceleration of gravity at the surface.

i.e. $$-g = -\frac{k}{R^2} \quad \text{or} \quad k = gR^2$$

$\therefore$ From (1), $$v\frac{dv}{dr} = -\frac{gR^2}{r^2}$$

or $$v\,dv = -\frac{gR^2}{r^2}\,dr$$

Integrating, $$\int v\,dv = -gR^2\int\frac{dr}{r^2} + c$$

or $$\frac{v^2}{2} = \frac{gR^2}{r} + c$$

or $$v^2 = \frac{2gR^2}{r} + 2c \qquad \dots (2)$$

Let v_0 be the initial velocity of projection from the surface of the earth. Then,

$$v = v_0 \text{ when } r = R$$

$\therefore$ From (2), $$v_0^2 = 2gR + 2c \dots (3)$$

Substituting (3) from (2) (to eliminate c),

$$v^2 - v_0^2 = \frac{2gR^2}{r} - 2gR$$

or $$v^2 = \frac{2gR^2}{r} + (v_0^2 - 2gR)$$

The particle will never return to earth if its velocity v during ascent remains positive. For, if v vanishes, the particle comes to rest and then descends, so that v becomes negative.

Now, as the particle rises upwards, $\frac{2gR^2}{r}$ goes on decreasing. The velocity will remain positive if and only if $v_0^2 - 2gR \ge 0$, i.e. if $v_0 \ge \sqrt{2gR}$

$\therefore$ The least velocity of projection is $\boxed{v_0 = \sqrt{2gR}}$

A particle projected with this velocity will never return to the earth, i.e. will escape from the earth. This velocity is called the velocity of escape from the earth.

Ex. 4 : *A body of mass m falls from rest under gravity in a fluid whose resistance to motion at any instant is mK times its velocity, where K is a constant. Find the terminal velocity of the body and also the time taken to acquire one-half of its limiting speed.* **(Dec. 07, 18)**

Sol. : Let v be the velocity of the body at any time t. Then the equation of motion of the body is given by,

$$m\frac{dv}{dt} = mg - m\,Kv$$

or $$\frac{dv}{dt} = g - Kv \qquad \dots (1)$$

Separating the variables in (1) and then integrating it, we obtain

$$t = \frac{-1}{K}\log(g - Kv) + C \qquad \dots (2)$$

Since the body is falling from rest, we have $v = 0$ when $t = 0$. This gives $C = \frac{1}{K} \log g$.

and therefore (2) becomes, $t = \frac{1}{K} \log \frac{g}{g - Kv} = \frac{-1}{K} \log \left(1 - \frac{K}{g} v\right)$ ... (3)

We know that the terminal velocity is attained when the weight of the body exactly balances the resistance to motion. If V is the terminal velocity attained by the body, then

$$mg = m\,KV$$

or

$$V = g/K \quad \text{... (4)}$$

Substituting (4) in (3), we obtain

$$t = -\frac{V}{g} \log \left(1 - \frac{v}{V}\right) \quad \text{... (5)}$$

Let $t = t_1$ when $v = \frac{1}{2} V$. Then (5) gives

$$\boxed{t_1 = -\frac{V}{g} \log \left(\frac{1}{2}\right) = \frac{V}{g} \log 2}$$

Hence, the body attains one-half of its limiting speed at time $\frac{V}{g} \log 2$ seconds, where V is the terminal velocity of the body.

Remark : When resistance becomes equal to the weight, the acceleration becomes zero and particle continues to fall with a constant velocity, called the *limiting or terminal velocity.*

Ex. 5 : *A particle moves in a straight line under the action of an attraction varying inversely as the $\frac{3}{2}$th power of the distance. Show that the velocity acquired by falling from an infinite distance to a distance 'a' from the centre is equal to the velocity which would be acquired in moving from rest at a distance 'a' to a distance $\frac{a}{4}$.*

Sol. : The equation of motion is $v \frac{dv}{dx} = -\lambda x^{-3/2}$, where λ is a constant and taking unit mass.

This gives $v\,dv = -\lambda x^{-3/2}\,dx$

If v_1 is the velocity acquired in moving from rest at infinity to a distance a, we have

$$\int_0^{v_1} v\,dv = -\lambda \int_{\infty}^{a} x^{-3/2}\,dx \quad \text{or} \quad \frac{v_1^2}{2} = \frac{2\lambda}{\sqrt{a}} \quad \text{... (1)}$$

If v_2 is the velocity acquired in moving from rest at a distance 'a' to a distance a/4, we have,

$$\int_0^{v_2} v\,dv = -\lambda \int_a^{a/4} x^{-3/2}\,dx$$

or

$$\frac{v_2^2}{2} = \lambda \cdot 2\left[\frac{1}{\sqrt{x}}\right]_a^{a/4} = \frac{2\lambda}{\sqrt{a}} \quad \text{...(2)}$$

From (1) and (2), $\boxed{v_1 = v_2.}$

Ex. 6 : *A particle of mass m is projected vertically upward under gravity, the resistance of the air being mk times the velocity. Show that the greatest height attained by the particle is $\frac{V^2}{g}$ [$\lambda - \log (1 + \lambda)$], where V is the greatest velocity which the above mass will attain when it falls freely and λV is the initial velocity.*

Sol. : Let v be the velocity of the particle at time t. The forces acting on the particle are :

(i) its weight mg acting vertically downwards,

(ii) the resistance mkv of the air acting vertically downwards.

Accelerating force on the particle $= -mg - mkv$

$\therefore$ By Newton's second law, the equation of motion of the particle is,

$$mv\frac{dv}{dx} = -mg - mkv$$

or $$v\frac{dv}{dx} = -g - kv \quad ...(1)$$

When the particle falls freely (under gravity), equation (1) becomes (changing g to –g)

$$v\frac{dv}{dx} = g - kv \quad ...(2)$$

When the particle attains the greatest velocity V, its acceleration is zero.

$\therefore$ From (2), $$0 = g - kV \quad \text{or} \quad k = \frac{g}{V}$$

Putting this value of k in (1), we have,

$$v\frac{dv}{dx} = -g - \frac{g}{V}v = -\frac{g}{V}(V + v)$$

or $$\frac{v}{V+v}dv = -\frac{g}{V}dx$$

Integrating, $$\int \frac{v}{V+v}dv = -\frac{g}{V}\int dx + c$$

or $$\int\left(1 - \frac{V}{V+v}\right)dv = -\frac{g}{V}x + c$$

or $$v - V\log(V+v) = -\frac{g}{V}x + c \quad ...(3)$$

Initially, when $x = 0$, $v = \lambda V$ $\therefore$ From (3), we have, $\lambda V - V\log(V + \lambda V) = c$

or $$c = V[\lambda - \log V(1+\lambda)]$$

Substituting the value of c in (3),

$$v - V\log(V+v) = -\frac{g}{V}x + V[\lambda - \log V(1+\lambda)] \quad ...(4)$$

Let h be the greatest height attained by the particle. Then $x = h$ when $v = 0$

$\therefore$ From (4), we have, $$-V\log V = -\frac{g}{V}h + V[\lambda - \log V(1+\lambda)]$$

or $$\frac{g}{V}h = V\lambda - V[\log V(1+\lambda) - \log V] = V\lambda - V\log\frac{V(1+\lambda)}{V}$$

or $$\boxed{h = \frac{V^2}{g}[\lambda - \log(1+\lambda)]}$$

Ex. 7 : *A paratrooper and his parachute weigh 50 kg. At the instant parachute opens, he is travelling vertically downward at the speed of 20 m/s. If the air resistance varies directly as the instantaneous velocity and it is 20 newtons when the velocity is 10 m/s, find the limiting velocity, the position and the velocity of the paratrooper at any time t.*

Sol. : Let v m/s be the velocity of the paratrooper t seconds after the parachute opens. The forces acting on the paratrooper are :

(i) the weight 50 kg acting vertically downwards,

(ii) the air resistance kv acting vertically upwards.

Accelerating force = (50 – kv) N

By Newton's second law, the accelerating force is,

$$m\frac{dv}{dt} = \frac{W}{g}\frac{dv}{dt} = \frac{50}{g}\frac{dv}{dt}$$

$\therefore$ The equation of motion is $$\frac{50}{g}\frac{dv}{dt} = 50 - kv \quad ...(1)$$

When $v = 10$ m/s, the air resistance $kv = 20$ N

$\therefore$ $$k \cdot 10 = 20 \Rightarrow k = 2$$

Substituting the value of k in (1), we have,

$$\frac{50}{g}\frac{dv}{dt} = 50 - 2v \quad \text{or} \quad \frac{dv}{25 - v} = \frac{g}{25}\,dt$$

Integrating, $$-\log(25 - v) = \frac{gt}{25} + c \qquad \text{...(2)}$$

When t = 0, v = 20 ∴ c = − log 5

Substituting the value of c in (2), we have,

$$-\log(25 - v) = \frac{gt}{25} - \log 5$$

or $$\log\frac{25 - v}{5} = -\frac{gt}{25} \quad \text{or} \quad \frac{25 - v}{5} = e^{-gt/25}$$

or $$v = 5(5 - e^{-gt/25}) \qquad \text{...(3)}$$

which gives the velocity of the paratrooper at any time t.

The limiting velocity is the velocity when $t \to \infty$.

∴ From (3), the limiting velocity = 25 m/s.

Now, from (3), we have, $$\frac{dx}{dt} = 5(5 - e^{-gt/25})$$

or $$dx = 5(5 - e^{-gt/25})\,dt$$

Integrating, $$x = 5\left(5t + \frac{25}{g}e^{-gt/25}\right) + C' \qquad \text{...(4)}$$

Initially, when t = 0, x = 0 ∴ $C' = -\frac{125}{g}$

Substituting the value of C' in (4), the position of the paratrooper at any time t is given by,

$$\boxed{x = 25t - \frac{125}{g}(1 - e^{-gt/25})}$$

Ex. 8 : *Assuming that the resistance to movement of a ship through water in the form of $a^2 + b^2v^2$, where v is the velocity and a and b are constants, write down the differential equation for retardation of the ship moving with engine stopped. Prove that the time in which the speed falls to one half its original value u is given by, $\frac{W}{abg}\tan^{-1}\frac{abu}{2a^2 + b^2u^2}$, where W is the weight of the ship.*

(May 2011, Dec. 2007, May 2016)

Sol. : $$m\frac{dv}{dt} = -(a^2 + b^2v^2) \quad \text{but} \quad m = \frac{W}{g}$$

∴ $$\frac{W}{g}\frac{dv}{dt} = -(a^2 + b^2v^2) \quad \text{or} \quad \frac{W}{g}\left(\frac{dv}{a^2 + b^2v^2}\right) = -dt$$

or $$\frac{W}{gb^2}\int_u^{u/2}\frac{dv}{\frac{a^2}{b^2} + v^2} = -\int_0^t dt$$

∴ $$\frac{W}{gb^2}\left[\frac{b}{a}\tan^{-1}\frac{bv}{a}\right]_u^{u/2} = -t$$

or $$t = \frac{W}{abg}\left[\tan^{-1}\frac{bv}{a}\right]_{u/2}^{u} = \frac{W}{abg}\left[\tan^{-1}\frac{bu}{a} - \tan^{-1}\frac{bu}{2a}\right]$$

$$= \frac{W}{abg}\tan^{-1}\frac{\left(\frac{bu}{a} - \frac{bu}{2a}\right)}{\left(1 + \frac{b^2u^2}{2a^2}\right)} = \frac{W}{abg}\tan^{-1}\left[\frac{bu/2a}{(2a^2 + b^2u^2)/2a^2}\right]$$

∴ $$\boxed{t = \frac{W}{abg}\tan^{-1}\left(\frac{abu}{2a^2 + b^2u^2}\right)}$$

Ex. 9 : *The distance x descended by a parachuter satisfies the differential equation* $v\frac{dv}{dx} = g\left(1-\frac{v^2}{k^2}\right)$ *where v is velocity, k, g constants. If v = 0 and x = 0 at time t = 0, show that* $x = \frac{k^2}{g}\log\cosh\left(\frac{gt}{k}\right)$. **(May 20011, 2006, 2005; Dec. 2013, 2009)**

Sol. : Since,

$$v\frac{dv}{dx} = \frac{dv}{dt}$$

$$\therefore \quad \frac{dv}{dt} = g\left(\frac{k^2-v^2}{k^2}\right)$$

$$\int \frac{dv}{k^2-v^2} = \int \frac{g}{k^2}\,dt + A$$

$$\therefore \quad \frac{1}{k}\tanh^{-1}\frac{v}{k} = \frac{g}{k^2}t + A$$

Using v = 0, at t = 0 we get, A = 0

$$\therefore \quad \tanh^{-1}\left(\frac{v}{k}\right) = \frac{g}{k}t$$

$$\text{or} \quad v = k\tanh\left(\frac{g}{k}t\right)$$

i.e.

$$\frac{dx}{dt} = k\tanh\left(\frac{g}{k}t\right)$$

$$\therefore \quad \int dx = k\int \tanh\left(\frac{g}{k}t\right)dt$$

i.e.

$$x = \frac{k}{(g/k)}\left[\log\cosh\frac{gt}{k}\right] + B$$

$$\text{or} \quad x = \frac{k^2}{g}\log\cosh\left(\frac{gt}{k}\right) + B$$

Also x = 0 when t = 0. $\therefore B = 0$

Hence,

$$\boxed{x = \frac{k^2}{g}\log\cosh\left(\frac{gt}{k}\right)}$$

Ex. 10 : *A particle is moving in a straight line with an acceleration* $k\left[x+\frac{a^4}{x^3}\right]$ *directed towards origin. If it starts from rest at a distance a from the origin, prove that it will arrive at origin at the end of time* $\frac{\pi}{4\sqrt{k}}$. **(May 2004, 2007, 2008, 2013; Dec. 2006)**

Sol. : Equation of motion is

$$v\frac{dv}{dx} = -k\left(x+\frac{a^4}{x^3}\right)$$

$$\int v\,dv = -k\int\left(x+\frac{a^4}{x^3}\right)dx$$

or

$$\frac{v^2}{2} = -k\left[\frac{x^2}{2}-\frac{a^4}{2x^2}\right] + C$$

When x = a, v = 0, $\therefore C = 0$

$$\therefore \quad v^2 = k\left(\frac{a^4-x^4}{x^2}\right)$$

Since acceleration is directed towards origin,

$$\therefore \quad v = -\sqrt{k}\sqrt{\frac{a^4-x^4}{x^2}} \quad \text{or} \quad \frac{dx}{dt} = \frac{-\sqrt{k}\sqrt{a^4-x^4}}{x}$$

$$\int \frac{x\,dx}{\sqrt{a^4-x^4}} = -\sqrt{k}\int dt,$$

Put $x^2 = u \quad \therefore \quad x\,dx = \frac{du}{2}$

$$\frac{1}{2}\int \frac{du}{\sqrt{a^4-u^2}} = -\sqrt{k}\cdot t + C$$

$$\therefore \quad \sin^{-1}\left(\frac{u}{a^2}\right) = -2\sqrt{k}\cdot t + C_1$$

or

$$\sin^{-1}\left(\frac{x^2}{a^2}\right) = -2\sqrt{k}\cdot t + C_1$$

When $t = 0,\ x = a\ \therefore\ C_1 = \frac{\pi}{2}$

$$\therefore \quad \sin^{-1}\left(\frac{x^2}{a^2}\right) = \frac{\pi}{2} - 2\sqrt{k}\cdot t$$

At $x = 0$,

$$0 = \frac{\pi}{2} - 2\sqrt{k}\cdot t$$

$$\therefore \quad \boxed{t = \frac{\pi}{4\sqrt{k}}}$$

***Ex. 11 :** The distance 'x' descended by a person falling by means of a parachute satisfies the differential equation* $\left(\frac{dx}{dt}\right)^2 = k^2\,[1 - e^{-\frac{2gx}{k^2}}]$ *where 'k' and 'g' are constants and $x = 0$ when $t = 0$. Show that* $x = \frac{k^2}{g}\log\cosh\left(\frac{gt}{k}\right)$. **(May 2007)**

Sol. : The equation is,

$$\left(\frac{dx}{dt}\right)^2 = k^2\left[1 - e^{-\frac{2gx}{k^2}}\right]$$

$$\therefore \quad \frac{dx}{dt} = k\sqrt{1 - e^{-\frac{2gx}{k^2}}}$$

$$\frac{dx}{\sqrt{1 - e^{-\frac{2gx}{k^2}}}} = k\,dt \quad \text{or} \quad \int \frac{e^{\frac{gx}{k^2}}}{\sqrt{e^{\frac{2gx}{k^2}} - 1}}\,dx = k\int dt$$

$$\frac{k^2}{g}\int \frac{du}{\sqrt{u^2 - 1}} = kt + C \qquad \left(\because e^{\frac{gx}{k^2}} = u\right)$$

$$\therefore \quad \frac{k^2}{g}\cosh^{-1}u = kt + C \qquad \left(\because \int \frac{du}{\sqrt{u^2-1}} = \cosh^{-1}u\right)$$

When $t = 0,\ x = 0$. But $e^{\frac{gx}{k^2}} = u\ \therefore$ Put $x = 0,\ \therefore\ e^0 = u \Rightarrow u = 1$. Hence, when $t = 0,\ u = 1$.

$$\frac{k^2}{g}\cosh^{-1}1 = 0 + C$$

We note here that $\cosh(0) = 1, \quad \cosh^{-1}(1) = 0 \qquad \therefore \quad C = 0.$

$$\Rightarrow \quad \frac{k^2}{g}\cosh^{-1}u = kt \Rightarrow \cosh^{-1}u = \frac{gt}{k}$$

$$\therefore \quad u = \cosh\left(\frac{gt}{k}\right) \Rightarrow e^{\frac{gx}{k^2}} = \cosh\left(\frac{gt}{k}\right) \quad \text{or} \quad \frac{gx}{k^2} = \log\cosh\left(\frac{gt}{k}\right)$$

$$\therefore \quad \boxed{x = \frac{k^2}{g}\log\cosh\left(\frac{gt}{k}\right)}$$

***Ex. 12 :** A body of mass 'm' falls from rest under the influence of gravity and a retarding force due to air resistance proportional to square of the velocity. Find the velocity and distance described as a function of time. Hence, show that the velocity of the body approaches the limiting value.* **(May 2009, Dec. 2004)**

Sol. : The forces acting on the body are :

(i) the weight mg acting downwards.

(ii) Retarding force due to air resistance is mkv^2 (mk is the constant of proportionality).

$\therefore$ Net force by virtue of which body falls $= mg - mkv^2 = m\,(g - kv^2)$

By D'Alembert's principle, $\quad m\frac{dv}{dt} = m\,(g - kv^2) \quad$ or $\quad \frac{dv}{dt} = k\left(\frac{g}{k} - v^2\right)$

Take $\quad \frac{g}{k} = a^2 \Rightarrow$

We have $$\frac{dv}{dt} = \frac{g}{a^2}(a^2 - v^2)$$

$$\therefore \quad \int \frac{dv}{a^2 - v^2} = \frac{g}{a^2} \int dt + C$$

$$\frac{1}{a} \tanh^{-1} \frac{v}{a} = \frac{gt}{a^2} + C$$

When $t = 0$, $v = 0$. $\therefore$ $C = 0$

$$\tanh^{-1} \frac{v}{a} = \frac{gt}{a} \qquad \boxed{v = a \tanh\left(\frac{gt}{a}\right)} \qquad \text{...(1)}$$

But $$v = \frac{ds}{dt}$$

$$\therefore \quad \frac{dS}{dt} = a \tanh\left(\frac{gt}{a}\right)$$

$$dS = a \int \tanh\left(\frac{gt}{a}\right) dt + A \quad \therefore \; S = \frac{a^2}{g} \log \cosh\left(\frac{gt}{a}\right) + A$$

When $t = 0$, $S = 0$, $\therefore$ $A = 0$

$$S = \frac{a^2}{g} \log \cosh\left(\frac{gt}{a}\right) \qquad \text{...(2)}$$

Equations (1) and (2) describe velocity and distance as a function of time.

To find the limiting value, we use result (1).

$$v = a \tanh\left(\frac{gt}{a}\right) \qquad \text{or} \qquad v = a\left(\frac{e^{gt/a} - e^{-gt/a}}{e^{gt/a} + e^{-gt/a}}\right)$$

$$v = a\left(\frac{1 - e^{-2gt/a}}{1 + e^{-2gt/a}}\right)$$

As $t \to \infty, v \to v_\infty$ $\therefore$ $$v_\infty = a\left(\frac{1-0}{1+0}\right) = a$$

But $a^2 = \frac{g}{k}$ $\therefore$ $$a = \sqrt{\frac{g}{k}}$$

$$\therefore \quad \boxed{\text{Limiting value of } v = v_\infty = a = \sqrt{\frac{g}{k}}}$$

Ex. 13 : *A particle of mass m is projected upward with velocity V. Assuming the air resistance k times its velocity, write the equation of motion and show that it will reach maximum height in time* $\frac{m}{k} \log\left(1 + \frac{kV}{mg}\right)$ *and distance travelled at any time t is* **(Dec. 2008)**
$\left(\frac{mV}{k} + \frac{m^2 g}{k^2}\right)\left(1 - e^{-\frac{kt}{m}}\right) - \frac{gmt}{k}$.

Sol. : Net force acting on the body $= -mg - kv$. By D'Alembert's principle,

$$m\frac{dv}{dt} = -mg - kv \qquad \text{or} \qquad \frac{dv}{dt} = -g - \frac{k}{m}v$$

$$\therefore \quad \int \frac{dv}{g + \frac{k}{m}v} = -\int dt + C \qquad \text{or} \qquad \frac{m}{k}\log\left(g + \frac{kv}{m}\right) = -t + C$$

When $t = 0$, $v = V$ $$\therefore \; C = \frac{m}{k}\log\left(g + \frac{kV}{m}\right)$$

$$\therefore \quad t = \frac{m}{k}\log\left(\frac{g + \frac{kV}{m}}{g + \frac{kv}{m}}\right) \qquad \text{...(1)}$$

It will attain maximum height when $v = 0$, $t = t_1$

$$\therefore \qquad t_1 = \frac{m}{k} \log\left(\frac{g + \frac{kV}{m}}{g}\right) \qquad \text{or} \qquad t_1 = \frac{m}{k} \log\left(1 + \frac{kV}{mg}\right)$$

which is the required time.

From (1),

$$\frac{k}{m} t = \log\left(\frac{g + \frac{kV}{m}}{g + \frac{kv}{m}}\right) \qquad \text{or} \qquad e^{\frac{kt}{m}} = \frac{g + \frac{kV}{m}}{g + \frac{kv}{m}}$$

$$\therefore \qquad g + \frac{k}{m} v = \left(g + \frac{kV}{m}\right) e^{-\frac{kt}{m}}$$

$$\therefore \qquad v = -\frac{gm}{k} + \left(V + \frac{mg}{k}\right) e^{-\frac{kt}{m}}$$

But

$$v = \frac{ds}{dt} = -\frac{gm}{k} + \left(V + \frac{mg}{k}\right) e^{-\frac{kt}{m}}$$

Integrating,

$$s = -\frac{gm}{k} \int dt + \left(V + \frac{mg}{k}\right) \int e^{-\frac{kt}{m}} dt + A$$

$$s = -\frac{gm}{k} t - \frac{m}{k} \left(V + \frac{mg}{k}\right) e^{-\frac{kt}{m}} + A$$

When $t = 0$, $s = 0$, $A = \frac{m}{k}\left(V + \frac{mg}{k}\right)$

$$\boxed{s = -\frac{gm}{k} t + \frac{m}{k}\left(V + \frac{mg}{k}\right)\left(1 - e^{-\frac{kt}{m}}\right)}$$

Ex. 14 : *A particle of unit mass is projected vertically upward with velocity u. Assuming that the air resistance is k times the instantaneous velocity of the particle, show that the particle will return to point of projection with velocity V given by,* **(Dec. 2006, 2005)**

$$V + u = \frac{g}{k} \log\left(\frac{g + ku}{g - kV}\right)$$

Sol. : The equation of motion is,

$$v \frac{dv}{dx} = -kv - g \qquad \text{or} \qquad \frac{v\,dv}{kv + g} = -dx$$

$$\therefore \qquad \int \frac{kv\,dv}{kv + g} = -k \int dx + C \qquad \text{or} \qquad \int \left(1 - \frac{g}{kv + g}\right) dv = -kx + C$$

$$\therefore \qquad v - \frac{g}{k} \log(kv + g) = -kx + C$$

When $x = 0$, $v = u$, $\therefore\ C = u - \frac{g}{k} \log(ku + g)$

$$v - \frac{g}{k} \log(kv + g) + \frac{g}{k} \log(ku + g) = -kx + u$$

$$v + \frac{g}{k} \log\left(\frac{ku + g}{kv + g}\right) = -kx + u$$

Let $x = x_1$ be maximum height attained. At this instant, $v = 0$.

$$\frac{g}{k} \log\left(\frac{ku + g}{g}\right) = -kx_1 + u \qquad \text{...(1)}$$

From this instant, body will start moving down. Algebraic sum of net forces $= -kv + g$

$\therefore$ The equation of motion is,

$$v \frac{dv}{dx} = -kv + g \qquad \text{or} \qquad \frac{v\,dv}{g - kv} = dx$$

$$\text{or} \qquad \frac{-kv\,dv}{g - kv} = -k\,dx \qquad \text{or} \qquad \int \left(1 - \frac{g}{g - kv}\right) dv = -k \int dx$$

or $$v + \frac{g}{k}\log(g - kv) = -kx + C_1$$

When $x = 0,\ v = 0 \therefore C_1 = \frac{g}{k}\log g$

$$\therefore \quad v + \frac{g}{k}\log(g - kv) - \frac{g}{k}\log g = -kx$$

or $$v + \frac{g}{k}\log\left(\frac{g - kv}{g}\right) = -kx$$

Body when reaches the point of projection, $x = x_1,\ v = V$

$$\therefore \quad V + \frac{g}{k}\log\left(\frac{g - kV}{g}\right) = -kx_1 \qquad \text{...(2)}$$

By using equation (1),

$$V + \frac{g}{k}\log\left(\frac{g - kV}{g}\right) = \frac{g}{k}\log\left(\frac{ku + g}{g}\right) - u$$

$$\therefore \quad \boxed{V + u = \frac{g}{k}\log\left(\frac{g + ku}{g - kV}\right)}$$

Ex. 15 : *A particle of unit mass moves in a horizontal straight line OA with an acceleration $\frac{k}{r^3}$ at a distance r and directed towards 0. If initially the particle was at rest at a distance a from 0, show that it will be at a distance $\frac{a}{2}$ from 0 at the end of time $\frac{a^2}{2}\sqrt{\frac{3}{k}}$.*

(May 2009, Dec. 2004)

Sol. : Equation of motion is,

$$v\frac{dv}{dr} = -\frac{k}{r^3}$$

$$\therefore \quad \int v\,dv = -\int \frac{k}{r^3}\,dr + C$$

$$\therefore \quad \frac{v^2}{2} = \frac{k}{2r^2} + C$$

When $r = a,\ v = 0 \therefore C = \frac{-k}{2a^2}$

$$\therefore \quad \frac{v^2}{2} = \frac{k}{2r^2} - \frac{k}{2a^2}$$

$$v^2 = k\left(\frac{a^2 - r^2}{a^2 r^2}\right) \Rightarrow v = \frac{\sqrt{k}}{a}\frac{\sqrt{a^2 - r^2}}{r}$$

But $$v = \frac{dr}{dt} = \frac{\sqrt{k}}{a}\frac{\sqrt{a^2 - r^2}}{r}$$

$$\int \frac{r}{\sqrt{a^2 - r^2}}\,dr = \frac{\sqrt{k}}{a}\int dt + C_1$$

$$\therefore \quad -\sqrt{a^2 - r^2} = \frac{\sqrt{k}}{a}t + C_1$$

Also $r = a,\ t = 0 \Rightarrow C_1 = 0$

$$-\sqrt{a^2 - r^2} = \frac{\sqrt{k}}{a}t \quad \text{or} \quad a^2 - r^2 = \frac{k}{a^2}t^2$$

When $t = \frac{a^2}{2}\sqrt{\frac{3}{k}}$

$$a^2 - r^2 = \frac{k}{a^2}\frac{a^4}{4}\left(\frac{3}{k}\right) = \frac{3a^2}{4}$$

$$r^2 = a^2 - \frac{3a^2}{4}$$

$$\therefore \quad r^2 = \frac{a^2}{4} \Rightarrow \boxed{r = \frac{a}{2}}$$

Ex. 16 : *A particle is projected vertically upwards with velocity V_1 and the resistance of the air produces a retardation Kv^2, where v is the velocity. Find the velocity V_2 with which the particle will return to the point of projection.*

Sol. : During the upward motion, let x be the distance of the particle at time t.

Taking the resistance to be mKv^2, the equation of motion may be written as :

$$m\,v\,\frac{dv}{dx} = -\,mg - mKv^2$$

or

$$v\,\frac{dv}{dx} = -\,g - Kv^2 \quad \text{... (1)}$$

Separating the variables in (1) and then integrating it, we obtain

$$\int dx = -\int \frac{v\,dv}{g + Kv^2} \qquad \text{... (2)}$$

Suppose that the greatest height attained is H. At this time the velocity is zero. Initially, i.e. at x = 0, we have $v = V_1$. Hence equation (2) is written as :

$$\int_0^H dx = -\int_{V_1}^{0} \frac{v\,dv}{g + Kv^2} = \int_0^{V_1} \frac{v\,dv}{g + Kv^2}$$

i.e.

$$H = \frac{1}{2K}\log\left(1 + \frac{K}{g}V_1^2\right) \qquad \text{... (3)}$$

During the downward motion, let y be the distance fallen in time t seconds after starting from the highest point. The equation of motion is now,

$$m\,v\,\frac{dv}{dy} = mg - mKv^2$$

or

$$v\,\frac{dv}{dy} = g - Kv^2 \quad \text{... (4)}$$

Separating the variables, we find

$$dy = \frac{v\,dv}{g - Kv^2} \quad \text{... (5)}$$

Now, y varies from y = 0 to y = H and v varies from v = 0 to v = V_2. Hence, we obtain from equation (5),

$$\int_0^H dy = \int_0^{V_2} \frac{v\,dv}{g - Kv^2} = -\frac{1}{2K}\left[\log\left(g - Kv^2\right)\right]_0^{V_2}$$

i.e.

$$H = -\frac{1}{2K}\log\left(1 - \frac{K}{g}V_2^2\right) \qquad \text{... (6)}$$

Equating (3) and (6), we obtain

$$\frac{1}{1 - \frac{K}{g}V_2^2} = 1 + \frac{K}{g}V_1^2$$

i.e.

$$V_2^2 = \frac{V_1^2}{1 + \frac{K}{g}V_1^2}$$

We therefore have

$$\boxed{\frac{1}{V_2^2} = \frac{1 + \frac{K}{g}V_1^2}{V_1^2} = \frac{1}{V_1^2} + \frac{K}{g}}$$

which gives the required velocity V_2.

Ex. 17 : *The resistance to the motion of a car of mass m varies as the square of its speed (i.e. kv^2) and the effective horse-power exerted at the road wheels is constant and equal to P. Show that the distance in which the car can accelerate from speed v_0 to v_1 is given by*

$$\frac{m}{3k} \log \frac{P - k v_0^3}{P - k v_1^3}$$

Sol. : If F is the pull exerted by the engine, then it is given that $F v = \text{power} = P$ i.e. $F = \frac{P}{v}$.

∴ The equation of motion of the car is

$$mv \frac{dv}{dx} = F - kv^2 = \frac{P}{v} - kv^2 \quad \text{or} \quad \frac{dx}{m} = \frac{v^2\, dv}{P - kv^3}$$

Hence, on integrating, the required distance x is given by,

$$\frac{x}{m} = \int_{v_0}^{v_1} \frac{v^2\, dv}{P - kv^3} = -\frac{1}{3k} [\log (P - kv^3)]_{v_0}^{v_1} = -\frac{1}{3k} [\log (P - kv_1^3) - \log (P - kv_0^3)]$$

or

$$x = \frac{m}{3k} \log \frac{P - kv_0^3}{P - kv_1^3}$$

Ex. 18 : *The acceleration of a moving particle being proportional to the cube of its velocity and negative, show that the distance passed over in time t is given by,*

$$S = \frac{\left(\sqrt{2 k v_0^2 t + 1} - 1\right)}{kv_0}.$$

the initial velocity being v_0, and the distance being measured from the position of the particle at time $t = 0$.

Sol. : The equation of motion is, $m \frac{dv}{dt} = -\, mk\, v^3$ where mk is the constant of proportionality.

$$\frac{dv}{dt} = -\,k\, v^3 \qquad \text{or} \qquad \int \frac{dv}{v^3} = -k \int dt + C$$

$$-\frac{1}{2v^2} = -\; kt + C$$

But $v = v_0$ when $t = 0$, ∴ $C = \frac{-1}{2\, v_0^2}$

$$-\frac{1}{2v^2} = -\, kt - \frac{1}{2v_0^2}$$

$$\frac{1}{v^2} = 2kt + \frac{1}{v_0^2} = \frac{2kt\, v_0^2 + 1}{v_0^2}$$

$$v = \frac{v_0}{\sqrt{2kt\, v_0^2 + 1}}$$

But $v = \frac{ds}{dt}$;

$$\int ds = v_0 \int \frac{dt}{\sqrt{2kt\, v_0^2 + 1}} + d$$

$$S = \frac{v_0}{2kv_0^2} \int \frac{2kv_0^2\, dt}{\sqrt{2kt\, v_0^2 + 1}} + d \quad \text{(Note this step)}$$

$$S = \frac{1}{2kv_0} \left(2 \sqrt{2kt\, v_0^2 + 1}\right) + d = \frac{\sqrt{2kt\, v_0^2 + 1}}{k\, v_0} + d$$

Since $S = 0$, when $t = 0$, ∴ $d = -\frac{1}{kv_0}$ ∴ $\boxed{S = \frac{1}{kv_0}\left(\sqrt{1 + 2kt\, v_0^2} - 1\right)}$

Ex. 19 : *A ship of mass 45,000 Mg starts from rest under the force of a constant propeller thrust of 9,00,000 N. (a) Find its velocity as a function of time t given that the resistance in newtons is 1,50,000 v with v = velocity measured in ms^{-1}, (b) Find the terminal velocity (i.e. v when $t \to \infty$) in kilometers per hour.*

Sol. : Since mass (kg) × acceleration (ms^{-2}) = net force (N)

then,

$$45 \times 10^6 \frac{dv}{dt} = 9{,}00{,}000 - 15 \times 10^4\, v$$

or $$\frac{dv}{dt} + \frac{v}{300} = \frac{1}{50} \qquad \text{... (1)}$$

$\therefore$ $$\text{I.F.} = e^{\frac{t}{300}}$$

G.S. is $$ve^{\frac{t}{300}} = \frac{1}{50}\int e^{\frac{t}{300}}\, dt + C = 6e^{\frac{t}{300}} + C$$

(a) When $t = 0$, $v = 0$, $C = -6$ and $\boxed{v = 6\left(1 - e^{-\frac{t}{300}}\right)}$

(b) As $t \to \infty$, $v \to 6$, the terminal velocity is $\boxed{v = 6 \text{ ms}^{-1} = 21.6 \text{ km per hour}}$. This may also be obtained from (1) since, as v approaches a limiting value, $\frac{dv}{dt} \to 0$. Then $v = 6$ as before.

Ex. 20 : *A boat is being towed at the rate 20 km per hour. At the instant (t = 0) that the towing line is cast off, a man in the boat begins to row in the direction of motion exerting a force of 9 N. If the combined mass of the man and boat is 225 kg and the resistance (N) is equal to 26.25 v, where v is measured in ms^{-1}, find the speed of the boat after 1/2 minute.*

Sol. : Since mass (kg) × acceleration (ms^{-2}) = net force (N)

then, $$225\frac{dv}{dt} = 90 - 26.25\, v \quad \text{or} \quad \frac{dv}{dt} + \frac{7}{60}v = \frac{2}{5}$$

Integrating, $$ve^{7t/60} = \frac{2}{5}\int e^{7t/60}\, dt + C = \frac{120}{35}e^{7t/60} + C$$

When $t = 0$, $$v = \frac{20000}{3600} = \frac{50}{9},\ C = \frac{134}{63}$$

and $$\boxed{v = \frac{24}{7} + \frac{134}{63}e^{-7t/60}}$$

When $t = 30$, $$\boxed{v = \frac{24}{7} + \frac{134}{63}e^{-3.5} = 3.5 \text{ ms}^{-1}}$$

EXERCISE 2.4

1. A vehicle starts from rest and its acceleration is given by $k\left(1 - \frac{t}{T}\right)$, where k is a constant and T is time taken to attain highest speed. Find the highest speed and distance travelled till the speed is attained.

   Ans. : $\frac{kT}{2}, \frac{kT^2}{3}$

2. A chain is coiled up near the edge of a smooth table and it just starts to fall over the edge. When a length x has fallen, its velocity v is given by, $xv\frac{dv}{dx} + v^2 = gx$. Show that if $v = 0$ at $x = 0$, then $v^2 = \frac{2}{3}gx$. **Hint :** Put $v^2 = t$

3. If $m\frac{dv}{dt} = X$ and $I\frac{dw}{dt} = aX$, where m, a, I are constants and X, v, w are functions of t, show that v_0, w_0 are the initial values of v, w respectively. $v = v_0 + \frac{I}{am}(w - w_0)$

   Hint : Eliminate X.

4. A bullet is fired into a sand tank, its retardation is proportional to the square root of its velocity is $(k\sqrt{v})$. How long will it take to come to rest if it enters the sand tank with an initial velocity v_0. **(May 2014)**

   Hint : $\frac{dv}{dt} = -k\sqrt{v}$; $t = 0$, $v = v_0$. **Ans.** $t = \frac{2}{k}\sqrt{v_0}$

2.9 SIMPLE HARMONIC MOTION

If a particle moves on a straight line, so that the force acting on it is always directed towards a fixed point on the line and proportional to its distance from the point, the particle is said to move in Simple Harmonic Motion.

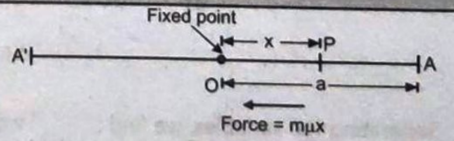

Fig. 2.8

(I) Let O be the fixed point and P be the position of particle at any time t. Let OP = x. Force acting on particle is $m\mu x$, where m is mass of the particle and μ a constant considered positive.

Equation of motion is

$$m\frac{d^2x}{dt^2} = -m\mu x \quad \text{or} \quad v\frac{dv}{dx} = -\mu x$$

$$v\,dv = -\mu x\,dx$$

Integrating, we get

$$v^2 = -\mu x^2 + A$$

Assuming that particle starts from point A (OA = a) and its initial velocity is zero.

When $x = a$, $v = 0$, $\therefore$ $A = \mu a^2$ $\therefore$ $v^2 = \mu(a^2 - x^2)$

or

$$v = \frac{dx}{dt} = -\sqrt{\mu}\sqrt{(a^2 - x^2)}$$

–ve sign is attached because x decreases as t increases.

$$\therefore \quad -\frac{dx}{\sqrt{a^2 - x^2}} = \sqrt{\mu}\,dt$$

Integrating,

$$\cos^{-1}\frac{x}{a} = \sqrt{\mu}\,t + B \qquad \left(\because \int -\frac{dx}{\sqrt{a^2 - x^2}} = \cos^{-1}\frac{x}{a}\right)$$

When $t = 0$, $x = a$ $\therefore$ $B = 0$

we get,

$$x = a\cos\left(\sqrt{\mu}\cdot t\right)$$

Particle will reach O in time t_1, given by

$$0 = a\cos\left(\sqrt{\mu}\cdot t_1\right)$$

$$\therefore \quad \boxed{t_1 = \frac{\pi}{2\sqrt{\mu}}}$$

Its velocity at that time will be $\sqrt{\mu}$ a.

As soon as it will cross O, the direction of force will change, however, particle will move with velocity $\sqrt{\mu}$ a and ultimately come to rest at A'. OA = OA' and time taken by the particle to travel from O and A' will be $\frac{\pi}{2\sqrt{\mu}}$.

Due to attraction, particle will start moving towards O. It is a to-fro motion i.e. oscillatory motion. Due to this, it is called Simple Harmonic Motion. Period of oscillation is $\frac{2\pi}{\sqrt{\mu}}$.

(II) Hooke's Law : Suppose that an elastic string of negligible weight hangs vertically with its upper end fixed and with a mass of M kg attached to its lower end. Let l metres be the natural length of the string and let T be its tension when it is stretched to a length x metres. Then the extension of the string beyond its natural length is given by the ratio $(x - l)/l$. The tension and the extension are related by Hooke's law :

The tension of an elastic string or spring is proportional to the extension of the string beyond its natural length. We thus have

$$T = \frac{\lambda}{l}(x - l)$$

where λ is a constant called the *modulus of elasticity* of the string. The value of λ depends on the material of which the string is composed. Clearly, $\lambda = T$ when $x = 2l$, i.e., λ is that force which would stretch the string to twice its natural length.

To determine motion, let v metres/sec. be the velocity of mass M when the extension is x metres. Then the tension is λx. The acceleration of the mass is $\frac{dv}{dt}$ or $v\frac{dv}{dx}$ so that the equation of motion may be written as,

$$Mv\frac{dv}{dx} = Mg - \lambda x$$

Separating the variables, we find :

$$v\,dv = \left(g - \frac{\lambda}{M}x\right)dx$$

Integrating both sides, we obtain :

$$\frac{v^2}{2} = gx - \frac{\lambda}{M}\frac{x^2}{2} + C \quad \text{... (1)}$$

To calculate C, we assume that $v = V_0$ when $x = 0$. This gives $C = V_0^2/2$. Hence, equation (1) becomes :

$$\frac{v^2}{2} = gx - \frac{\lambda}{M}\frac{x^2}{2} + \frac{V_0^2}{2}$$

or

$$\boxed{v^2 = 2gx - \frac{\lambda}{M}x^2 + V_0^2}$$

which gives the velocity of the mass when the extension is x metres.

(III) Motion of a Particle of Mass m Suspended by an Elastic String (or Extensible Spring) : Let OA be an elastic string of natural length 'l' with fixed end 'O'. Let a mass 'm' be attached to 'A' so that the string elongates and the mass 'm' reaches a position of equilibrium at B. Consider the equilibrium of the mass 'm' at 'B'. It is acted on by force 'mg' downwards due to gravity and tension 'T' upwards due to extension. In equilibrium,

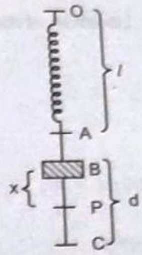

Fig. 2.9

$$mg = T = \frac{\lambda AB}{l} \quad \text{(by Hooke's law)}$$

If

$$AB = c \quad \text{then} \quad c = \frac{mgl}{\lambda} \qquad \dots (1)$$

Now, let the particle be pulled down upto 'C' and then let go. Let BC = d (say). The mass will go up due to tension in the 'spring'. Let the mass be at 'P' between B and C and BP = x (say).

The force acting on the mass is mg – T downwards.

∴ Equation of motion is,

$$m \cdot \frac{d^2x}{dt^2} = mg - T = mg - \frac{\lambda \times \text{Extension}}{\text{Natural length}}$$

$$= mg - \frac{\lambda\,(AB + x)}{l} = mg - \frac{\lambda}{l}\left(\frac{mgl}{\lambda} + x\right) = -\frac{\lambda x}{l}$$

$$\therefore \quad \boxed{\frac{d^2x}{dt^2} = -\frac{\lambda}{lm}\,x} \qquad \dots (2)$$

This equation is same as that of S.H.M. except $\mu = \frac{\lambda}{lm}$.

Hence the particle of mass m will execute S.H.M. of period $\frac{2\pi}{\sqrt{\frac{\lambda}{lm}}}$. The expression $\frac{mgl}{\lambda}$ is called static extension of the spring and denoted by c. Under this case, period of oscillation $2\pi\sqrt{\frac{lm}{\lambda}} = 2\pi\sqrt{\frac{c}{g}}$

$$\therefore \quad \text{Period} = 2\pi\sqrt{\frac{\text{Static extension}}{g}}$$

Importance of c is obvious if period of oscillation is compared with that of simple pendulum of length l i.e. period of oscillation of mass is that of simple pendulum of length $\frac{lmg}{\lambda}$, i.e. static extension of spring.

Illustrations on Simple Harmonic Motion :

Ex. 1 : *A spring of negligible weight hangs vertically. A mass m is attached to the other end. If the mass is moving with velocity V_0 when the spring is unstretched, find the velocity v as a function of the stretch x (Take λ as Young's modulus of the spring).*

Sol. : If x is the increase in length of the spring when velocity of the mass m is v; then the equation of motion is

$$mv\frac{dv}{dx} = mg - \lambda x$$

$$\int mv\,dv = \int (mg - \lambda x)\,dx$$

$$m\frac{v^2}{2} = mgx - \lambda\frac{x^2}{2} + c$$

$$x = 0, \; v = v_0 \quad \therefore \quad \frac{mv_0^2}{2} = c$$

$$\therefore \quad \boxed{mv^2 = 2mgx - \lambda x^2 + mv_0^2}$$

Ex. 2 : *Two identical loads are suspended from the end of a spring. Find the motion imparted to one load if the other breaks loose in case the increase in length of spring under action of one load at rest is a.*

Sol. :

When both loads are attached, we have,

$$2mg = \lambda \cdot \frac{2a}{l}$$

where λ is Young's modulus and l is natural length of spring.

$$\lambda = \frac{mgl}{a}$$

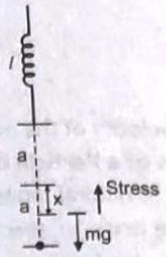

Fig. 2.10

Equation of motion is

$$mv\frac{dv}{dx} = mg - \lambda\frac{(x+a)}{l}$$

$$\therefore \quad \frac{1}{2}m\frac{d}{dx}(v^2) = mg - \frac{mg}{a}\cdot(x+a) = -\frac{mgx}{a}$$

$$\therefore \quad \frac{d}{dx}(v^2) = -\frac{2g}{a}x$$

$$\therefore \quad v^2 = -\frac{g}{a}x^2 + A$$

When $x = a,\ v = 0 \ \therefore\ A = ga$

$$\therefore \quad v^2 = \frac{g}{a}(a^2 - x^2)$$

$$\therefore \quad v = \frac{dx}{dt} = \pm\sqrt{\frac{g}{a}}\sqrt{a^2 - x^2}$$

As t increases, x decreases.

Hence
$$\frac{dx}{dt} = -\sqrt{\frac{g}{a}}\sqrt{a^2 - x^2} \quad \text{or} \quad \frac{-dx}{\sqrt{a^2 - x^2}} = \sqrt{\frac{g}{a}}\,dt$$

$$\cos^{-1}\frac{x}{a} = \sqrt{\frac{g}{a}}\,t + B$$

When $t = 0,\ x = a,\ \therefore\ B = 0$

$$\therefore \quad \boxed{x = a\cos\left(\sqrt{\frac{g}{a}}\cdot t\right)}$$

Ex. 3 : *A particle of mass m is attached to one end of a light elastic string of natural length a and modulus* $\frac{mg}{k}$. *The other end of the string is fixed to a point O and the particle is allowed to fall from rest at O. Obtain velocity of the particle and show that the highest magnitude is* $\sqrt{ag(2+k)}$.

(Dec. 2007)

Sol. : Here OA is the natural length of the string. If P is a subsequent position of the particle such that AP = x (Refer Fig. 2.11) then the forces acting on the mass m are

(i) the weight mg acting downwards (ii) the tension T in the string acting upwards.

By D-Alembert's principle, the equation of motion is $mv\frac{dv}{dx} = mg - T$, where $T = \frac{\lambda \cdot x}{l}$.

Given λ = modulus of elasticity $= \frac{mg}{k}$ $\cdot$ x = extended length = x, l = natural length = a

$$\therefore \quad T = \frac{mg}{k}\cdot x\frac{1}{a} = \frac{mg}{ak}x$$

$\therefore$ Equation of motion is

$$mv\frac{dv}{dx} = mg - \frac{mg}{ak}x \Rightarrow v\frac{dv}{dx} = g - \frac{g}{ak}x = \frac{g}{ak}(ak - x)$$

$$\int v\,dv = \frac{g}{ak}\int(ak - x)\,dx + C \Rightarrow \frac{v^2}{2} = \frac{g}{ak}\frac{(ak - x)^2}{-2} + C \quad \text{... (1)}$$

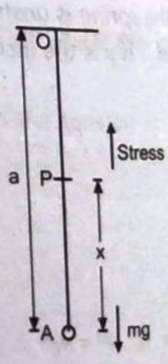

Fig. 2.11

[Since particle falls from rest with initial velocity u, then equation of motion is

$$mv\frac{dv}{dx} = mg$$

$$\int v\,dv = g\int dx \quad \text{or} \quad \frac{v^2}{2} = gx + C_1$$

$$v^2 = 2gx + C_2$$

When $x = 0,\ v = u \quad \therefore C_2 = u^2$

$$v^2 = 2gx + u^2$$

$$x = 0 \Rightarrow \text{ it is a free fall through height a}$$

$$\therefore \boxed{v^2 = 2ag + u^2}$$

When $x = 0,\ u = 0 \Rightarrow v^2 = 2ag$

$$\therefore \boxed{v = \sqrt{2ag}}$$

Equation (1) becomes, $\dfrac{2ag}{2} = \dfrac{g}{ak}\dfrac{(ak-0)^2}{-2} + C$

$$\therefore C = ag + \frac{agk}{2} = \frac{ag}{2}(2+k)$$

From equation (1), $\dfrac{v^2}{2} = \dfrac{g}{ak}\dfrac{(ak-x)^2}{-2} + \dfrac{ag}{2}(2+k)$

$$v^2 = -\frac{g}{ak}(ak-x)^2 + ag(2+k)$$

v is maximum when $ak - x = 0 \Rightarrow x = ak$

$$v_{max}^2 = ag(2+k)$$

$$\therefore \boxed{v_{max} = \sqrt{ag(2+k)}}$$

***Ex. 4 :** An elastic string without weight of natural length l and modulus of elasticity being weight of n-grams, is suspended by one end and a mass m is attached to the other, show that the time of oscillations is $2\pi\sqrt{\dfrac{ml}{ng}}$.*

Sol. : Let the weight be at a distance x below the position of equilibrium then equation of motion will be,

$$m\frac{d^2x}{dt^2} = mg - \frac{\lambda(x+S_0)}{l} = mg - \frac{\lambda S_0}{l} - \frac{\lambda x}{l}$$

$$= mg - mg - \frac{\lambda x}{l} \qquad \left(\because mg = \frac{\lambda S_0}{l}\right)$$

$$\frac{d^2x}{dt^2} = \frac{-\lambda x}{lm}$$

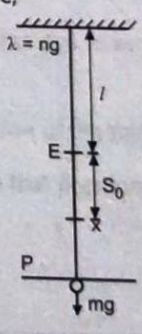

Fig. 2.12

This equation is same as that of S.H.M. except

$$\mu = \frac{\lambda}{lm}$$

Hence the particle of mass m will execute S.H.M. of period $= \dfrac{2\pi}{\sqrt{\lambda/lm}}$.

$$\boxed{\text{Period} = 2\pi\cdot\sqrt{\frac{lm}{\lambda}} = 2\pi\sqrt{\frac{lm}{ng}}}$$

Ex. 5 :** A point executing simple harmonic motion has velocities v_1 and v_2 and accelerations a_1 and a_2 in two positions respectively. Show that the distance between the two positions is* $\left|\dfrac{v_1^2 - v_2^2}{a_1 + a_2}\right|$ ***(May 2006, 2005, 2004)

Sol. :

$$\frac{d^2x_1}{dt^2} = -n^2 x_1 \qquad \therefore \quad a_1 = -n^2 x_1 \qquad \text{... (1)}$$

$$\frac{d^2x_2}{dt^2} = -n^2 x_2 \qquad \therefore \quad a_2 = -n^2 x_2 \qquad \text{... (2)}$$

We have

$$v\frac{dv}{dx} = -n^2 x$$

$$v_1\frac{dv}{dx_1} = -n^2 x_1 \qquad\qquad v_2\frac{dv}{dx_2} = -n^2 x_2$$

$$\int v_1\, dv = -n^2 \int x_1\, dx_1 \qquad\qquad \int v_2\, dv = -n^2 \int x_2\, dx_2$$

$$\frac{v_1^2}{2} = -\frac{n^2 x_1^2}{2} + A \qquad\qquad \frac{v_2^2}{2} = -\frac{n^2 x_2^2}{2} + B$$

When $x_1 = a$, $v_1 = 0$ $\qquad$ When $x_2 = a$, $v_2 = 0$

$$A = \frac{n^2 a^2}{2} \qquad\qquad B = \frac{n^2 a^2}{2}$$

$\therefore$

$$v_1^2 = -n^2 x_1^2 + n^2 a^2 \qquad\qquad v_2^2 = -n^2 x_2^2 + n^2 a^2$$

$$v_1^2 = n^2 (a^2 - x_1^2) \qquad\qquad v_2^2 = n^2 (a^2 - x_2^2)$$

$$v_1 = n\sqrt{a^2 - x_1^2} \qquad\qquad v_2 = n\sqrt{a^2 - x_2^2}$$

$\therefore$

a is the amplitude of S.H.M.

$$v_1^2 - v_2^2 = n^2 \left(x_2^2 - x_1^2\right) = n^2 \left(\frac{a_2^2}{n^4} - \frac{a_1^2}{n^4}\right) \qquad \text{by using (1) and (2)}$$

$$v_1^2 - v_2^2 = \frac{\left(a_2^2 - a_1^2\right)}{n^2}$$

$$n^2 = \frac{\left(a_2^2 - a_1^2\right)}{\left(v_1^2 - v_2^2\right)}$$

Also,

$$x_1 - x_2 = \frac{a_2 - a_1}{n^2} = \frac{(a_2 - a_1)\left(v_1^2 - v_2^2\right)}{\left(a_2^2 - a_1^2\right)}$$

$\therefore$

$$\boxed{x_1 - x_2 = \frac{v_1^2 - v_2^2}{a_1 + a_2}}$$

***Ex. 6 :** A mass of 2 kg is hung on a light spiral spring and produces a static deflection of 1/4 m. A mass of 2 kg is suddenly added to the original mass. Show that the maximum elongation produced is 0.75 m.*

Sol. : Let k be the spring constant. Then in the equilibrium position for the 2 kg mass, we have

$$2g = k\left(\frac{1}{4}\right) \quad \text{or} \quad k = 8g \qquad \text{... (1)}$$

Let x m be the extension of the spring when a mass of 2 kg is suddenly added to the original mass. Then $\left(x + \frac{1}{4}\right)$ m is the total elongation produced by the 4 kg mass and the equation of motion of the 4 kg mass is given by

$$4 \cdot v \cdot \frac{dv}{dx} = 4g - k\left(x + \frac{1}{4}\right) \qquad \text{...(2)}$$

where v m/sec is its velocity at any time t secs. Substituting (1) in (2), we obtain

$$4 \cdot v \cdot \frac{dv}{dx} = 4g - 8g\left(x + \frac{1}{4}\right) = 2g - 8gx$$

or

$$v \cdot \frac{dv}{dx} = \frac{g}{2} - 2gx$$

Separating the variables and integrating, we obtain $\frac{v^2}{2} = \frac{g}{2}x - 2g \cdot \frac{x^2}{2} + C.$ ... (3)

From the initial conditions, we have x = 0 and v = 0 at t = 0. This gives C = 0.

Hence, equation (3) becomes : $v^2 = gx - 2gx^2$ ... (4)

The elongation will be maximum when v = 0. Equation (4) then gives :

x = 0 or x = 1/2.

Hence $\boxed{\text{the maximum elongation produced} = 0.5 + 0.25 = 0.75 \text{ m.}}$

Ex. 7 : *An elastic spring of natural length l is fixed at a point A. To the lower end is attached a particle of mass m so that the spring stretches to a length 2l. If the particle is dropped from A, show that it descends a distance $l\left(2 + \sqrt{3}\right)$ before coming to rest.*

(Dec. 2006, 2005)

Sol. : The tension in the spring is given by $T = \frac{\lambda(2l - l)}{l} = \lambda.$

Since this balances the weight of the particle, we have $\lambda = mg$ ... (1)

At time t, let x be the extension of the spring beyond its natural length. Then the equation of motion of the particle is given by

$$m \cdot v \cdot \frac{dv}{dx} = mg - T = mg - \lambda\frac{x}{l} = mg - mg \cdot \frac{x}{l} = mg\left(1 - \frac{x}{l}\right)$$

or

$$v\,dv = g\left(1 - \frac{x}{l}\right)dx \qquad \text{... (2)}$$

Integrating both sides of (2), we find $\frac{v^2}{2} = gx - \frac{gx^2}{2l} + C$ ... (3)

Since the particle is dropped from A, we have $v = \sqrt{2gl}$ when x = 0. Substituting these values in (3), we find $C = lg$. Hence, equation (iii) becomes : $\frac{v^2}{2} = gx - \frac{g}{2l}x^2 + gl$ ... (4)

When v = 0, x is given by $x^2 - 2lx - 2l^2 = 0$

from which we obtain $x = l + l\sqrt{3}$. Hence, the distance by which the particle descends before coming to rest is given by

$$\boxed{x + l = 2l + l\sqrt{3} = l\left(2 + \sqrt{3}\right)}$$

Ex. 8 : *In the case of a stretched elastic spring which has one end fixed and a particle of mass m attached at the other end, the equation of motion is $m\frac{d^2x}{dt^2} = -\frac{mg}{e}(x - l)$, where l is the natural length of the string and e is the elongation due to weight mg. Find x and v under the condition that at t = 0, $x = x_0$ and v = 0.*

Sol. : Let O be the fixed end of the string and A the end to which particle is attached. If P is the position of the particle during the motion such that OP = x, then the equation of motion is $m\frac{d^2x}{dt^2} = -T$, where T is the tension in the string in that position.

If λ is the modulus of elasticity of the string, we have, as given

$$mg = \lambda\frac{e}{l} \quad \text{or} \quad \lambda = \frac{mg\,l}{e}$$

Now,

$$T = \lambda\frac{AP}{l} = \frac{\lambda}{l}(x - l) \quad \therefore \quad T = \frac{mg}{e}(x - l)$$

$\therefore$ The equation of motion is $m\frac{d^2x}{dt^2} = -\frac{mg}{e}(x - l)$

or

$$v\frac{dv}{dx} = -\frac{g}{e}(x - l)$$

$x = x_0$ for $v = 0$ $\therefore$ $\int_0^v v\,dv = -\frac{g}{e}\int_{x_0}^{x}(x - l)\,dx$

$$\frac{v^2}{2} = -\frac{g}{e}\left[\frac{x^2}{2} - lx\right]_{x_0}^{x}$$

$$\frac{v^2}{2} = -\frac{g}{e}\left[\frac{x^2}{2} - lx - \frac{x_0^2}{2} + lx_0\right]$$

$$v^2 = \frac{g}{e}\left[x_0^2 - 2lx_0 + l^2 - l^2 + 2lx - x^2\right] = \frac{g}{e}\left[(l - x_0)^2 - (l - x)^2\right]$$

$$v = \frac{dx}{dt} = \sqrt{\frac{g}{e}}\left[(l - x_0)^2 - (l - x)^2\right]^{1/2}$$

$$\int \frac{dx}{\sqrt{(l - x_0)^2 - (l - x)^2}} = \sqrt{\frac{g}{e}}\int dt$$

$$-\sin^{-1}\left(\frac{l - x}{l - x_0}\right) = \sqrt{\frac{g}{e}}\, t + C$$

At $t = 0,\ x = x_0 \ \therefore\ C = -\frac{\pi}{2}$

$\therefore$

$$\frac{\pi}{2} - \sin^{-1}\left(\frac{l - x}{l - x_0}\right) = \sqrt{\frac{g}{e}}\, t$$

$$\cos^{-1}\left(\frac{l - x}{l - x_0}\right) = \sqrt{\frac{g}{e}}\, t$$

$$\frac{l - x}{l - x_0} = \cos\sqrt{\frac{g}{e}}\, t$$

$$\boxed{x = l + (x_0 - l)\cos\left(\sqrt{\frac{g}{e}}\, t\right)}$$

and

$$\boxed{v = \frac{dx}{dt} = -(x_0 - l)\sqrt{\frac{g}{e}}\sin\left(\sqrt{\frac{g}{e}}\, t\right)}$$

EXERCISE 2.6

1. A particle is oscillating in a straight line about a centre of force O, towards which when at a distance x the force is $mn^2 x$ and a is the amplitude of the oscillation. When at a distance $\frac{a\sqrt{3}}{2}$ from O, the particle receives a blow in the direction of motion which generates a velocity na. If this velocity be away from O, show that the new amplitude is $a\sqrt{3}$.

   [**Hint :** $v\frac{dv}{dx} = -n^2 x$ when $x = \frac{a\sqrt{3}}{2}$, $v = \frac{3na}{2}$.

   For new amplitude, put $v = 0$. $x = a\sqrt{3}$.]

2. A particle executes S.H.M. When it is 2 cm from mid path, its velocity is 10 cm/sec. and when it is 6 cm from centre of its path, its velocity is 2 cm/sec. Find its period and its greatest acceleration.

   (Dec. 2004; May 2008, 2007) Ans. $\frac{2\pi}{\sqrt{3}}$, $\sqrt{336}$ cm/sec²

3. A particle of mass m is suspended from one end of spring whose other end is attached to a fixed point. If the extension in the length due to the mass of the particle is e, find the period of oscillation.

   (Dec. 2008) Ans. $2\pi\sqrt{e/g}$

4. A mass hangs from a fixed point by means of a tight elastic spring which obeys Hooke's law. The mass being given a small vertical displacement. If n is the number of oscillations per second in the ensuring S.H.M. and l is the length of the spring when the system is in equilibrium, show that the natural length of the spring is $l - \frac{g}{4\pi^2 n^2}$.

2.10 HEAT FLOW

The fundamental principles involved in the problems of heat conduction are :

(i) Heat flows from a higher temperature to the lower temperature.

(ii) The quantity of heat in a body is proportional to its mass and temperature.

(iii) **Fourier's Law of Heat Conduction :** The rate of heat flow across an area is proportional to the area and to the rate of change of temperature with respect to its distance normal to the area.

If q (cal/sec.) be the quantity of heat that flows across a slab of area A (cm^2) and thickness δx in one second, where the difference of temperature at the faces is δT, then by (iii) above

$$q = \text{Thermal conductivity} \times \text{Area} \times \text{Temperature gradient}$$

$$\boxed{q = -kA\frac{dT}{dx}}$$

where k is a constant depending upon the material of the body and is called the thermal conductivity.

Negative sign is attached because T decreases as x increases.

Ex. 1 : *Obtain a formula for the steady-state heat loss per unit time from a unit length of pipe of radius r_0 carrying steam at temperature T_0 if the pipe is covered with insulation of thickness w, the outer surface of which remains at the constant temperature T_1. What is the temperature distribution through the insulation; i.e., what is the temperature in the insulation as a function of the radius ?*

Sol. : Since the problem tells us that steady-state conditions have been reached, it follows that the heat loss per unit time from a unit length of the pipe is a constant independent of time, say Q.

Let us now consider a typical cross-section of the pipe and insulation, as suggested in Fig. 2.14.

Let T denote the temperature in the insulation at the radius r, it follows that dT/dr is the temperature gradient (or temperature change per unit length) in the direction perpendicular to the cylindrical area of radius r. Hence, by Fourier's law, we have for the amount of heat Q flowing through this general area per unit time,

$$Q = \text{Thermal conductivity} \times \text{Area} \times \text{Temperature gradient}$$

$$Q = -k(1 \times 2\pi r)\frac{dT}{dr}$$

$$dT = \frac{-Q}{2\pi k}\frac{dr}{r} \Rightarrow T = \frac{-Q}{2\pi k}\log r + c$$

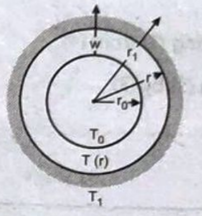

Fig. 2.13 : A typical cross-section of an insulated pipe

To determine the constant c, we use the fact that $T = T_0$ when $r = r_0$, from which

$$T_0 = \frac{-Q}{2\pi k}\log r_0 + c \quad \text{or} \quad c = T_0 + \frac{Q}{2\pi k}\log r_0$$

Substituting the value of c,

$$T = T_0 - \frac{Q}{2\pi k}(\log r - \log r_0) \qquad \text{... (1)}$$

Furthermore, $T = T_1$ when $r = r_0 + w = r_1$.

Hence

$$T_1 = T_0 - \frac{Q}{2\pi k}(\log r_1 - \log r_0)$$

$$\therefore \quad Q = \frac{(T_0 - T_1)\,2\pi k}{\log r_1 - \log r_0} \qquad \text{... (2)}$$

$$\therefore \quad \frac{Q}{2\pi k} = \frac{T_0 - T_1}{\log r_1 - \log r_0}$$

From (1), we get

$$T = T_0 - \left(\frac{T_0 - T_1}{\log r_1 - \log r_0}\right)(\log r - \log r_0)$$

$$T = T_0 - (T_0 - T_1)\frac{\log (r/r_0)}{\log (r_1/r_0)}$$

i.e.

$$\boxed{\frac{T_0 - T}{T_0 - T_1} = \frac{\log (r/r_0)}{\log (r_1/r_0)}}$$

Ex. 2 : *A pipe 20 cm in diameter contains steam at 150°C and is protected with a covering 5 cm thick for which k = 0.0025. If the temperature of the outer surface of the covering is 40°C, find the temperature half-way through the covering under steady-state conditions.*

(May 2011, 2009, 2005, 2018; Dec. 2009, May 2014)

Sol. : Let q cal/sec be the constant quantity of heat flowing out radially through a surface of the pipe having radius x cm and length 1 cm. Then the area of the lateral (belt) surface = $2\pi x$.

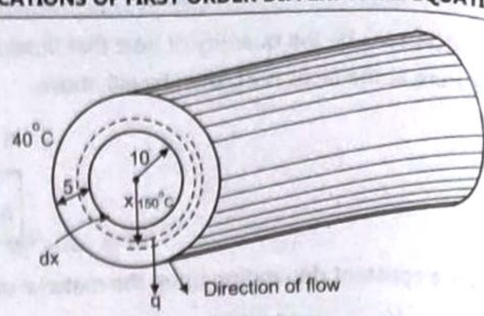

Fig. 2.14

Hence by Fourier's law, $$q = -k \cdot 2\pi x \cdot \frac{dT}{dx}$$

or $$dT = -\frac{q}{2\pi k} \cdot \frac{dx}{x}$$

Integrating, we have $$T = -\frac{q}{2\pi k} \log_e x + c$$

Since, $T = 150$, when $x = 10$

$\therefore$ $$150 = -\frac{q}{2\pi k} \log_e 10 + c \quad \text{... (1)}$$

Again since $T = 40$, when $x = 15$

$$40 = -\frac{q}{2\pi k} \log_e 15 + c \quad \text{... (2)}$$

Subtracting (2) from (1), $$110 = \frac{q}{2\pi k} \log_e 1.5 \text{... (3)}$$

Let $T = t$, when $x = 12.5$

$\therefore$ $$t = -\frac{q}{2\pi k} \log_e 12.5 + c \quad \text{... (4)}$$

Subtracting (1) from (4), $$t - 150 = -\frac{q}{2\pi k} \log_e 1.25 \quad \text{... (5)}$$

Dividing (5) by (3),

$$\frac{t-150}{110} = -\frac{\log_e 1.25}{\log_e 1.5},$$

hence $\boxed{t = 89.5°C}$

Ex. 3 : *A long hollow pipe has an inner diameter of 10 cm and outer diameter of 20 cm. The inner surface is kept at 200°C and the outer surface at 50°C. The thermal conductivity is 0.12. How much heat is lost per minute from a portion of the pipe 20 metres long ? Find the temperature at a distance x = 7.5 cm from the centre of the pipe.* **(Nov./Dec. 2019, Dec. 2011, 06, 05; May 07)**

Sol. : Here the isothermal surfaces are cylinders, the axis of each one of them is the axis of the pipe. Consider one such cylinder of radius x cm and length 1 cm. The surface area of this cylinder is $A = 2\pi x$ sq. cm. Let Q cal/sec be the quantity of heat flowing across this surface, then

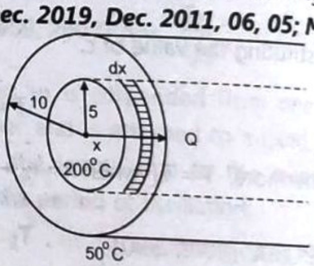

Fig. 2.15

$$Q = -kA\frac{dT}{dx} = -k \cdot 2\pi x\frac{dT}{dx}$$

or $$dT = -\frac{Q}{2\pi k} \cdot \frac{dx}{x}$$

Integrating, we have $$T = -\frac{Q}{2\pi k} \log_e x + c$$

Since $T = 200$, when $x = 5$ ... (1)

$\therefore$ $$200 = -\frac{Q}{2\pi k} \log_e 5 + c$$

Also $T = 50$, when $x = 10$ ... (2)

$\therefore$ $$50 = -\frac{Q}{2\pi k} \log_e 10 + c$$

Subtracting (3) from (2), we have ... (3)

$$150 = \frac{Q}{2\pi k} (\log_e 10 - \log_e 5)$$

or $$150 = \frac{Q}{2\pi k} \log_e 2 \quad \text{... (4)}$$

$\therefore$ $$Q = \frac{2\pi k \times 150}{\log_e 2} = \frac{300\pi \times 0.12}{\log_e 2} = 163 \text{ cal/sec}$$

Hence the heat lost per minute through 20 metre length of the pipe = 60 × 2000 Q = 120000 × 163 = 1956000 cal

Now, let

$$T = t, \quad \text{when} \quad x = 7.5$$

From (1),

$$t = -\frac{Q}{2\pi k} \log_e 7.5 + c \qquad \text{... (5)}$$

Subtracting (2) from (5), we have

$$t - 200 = -\frac{Q}{2\pi k} (\log_e 7.5 - \log_e 5)$$

or

$$t - 200 = -\frac{Q}{2\pi k} \log_e 1.5 \qquad \text{... (6)}$$

Dividing (6) by (4), we have

$$\frac{t - 200}{150} = -\frac{\log_e 1.5}{\log_e 2}$$

or

$$t = 200 - 150 \times 0.58 = 113$$

∴ When x = 7.5 cm, $\boxed{T = 113°C}$

Ex. 4 : *A steam pipe 20 cm in diameter is protected with a covering 6 cm thick for which the coefficient of thermal conductivity is k = 0.0003 cal/cm deg. sec. in steady state. Find the heat lost per hour through a meter length of the pipe, if the surface of the pipe is at 200°C and the outer surface of the covering is at 30°C.* ***(May 2006, 2005, 2015)***

Sol. : We have

$$q = -kA \frac{dt}{dx}$$

∴

$$q = -2\pi x \cdot k \cdot \frac{dt}{dx}$$

$$q \cdot \frac{dx}{x} = -2\pi k \, dt$$

$$\frac{q}{2\pi k} \int_{10}^{16} \frac{dx}{x} = -\int_{200}^{30} dt$$

$$\frac{q}{2\pi k} \log\left(\frac{16}{10}\right) = 200 - 30$$

$$q = \frac{170\,(2\pi k)}{\log(1.6)} \text{ cal/sec.}$$

$$\text{Required heat loss} = \frac{340 \times (3.14) \times 0.0003}{\log 1.6} \times 100 \times 60 \times 60$$

$$\boxed{\text{Heat loss} = 245443.3861 \text{ cal}}$$

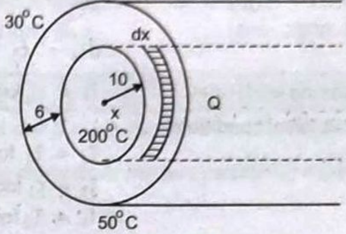

Fig. 2.16

Ex. 5 : *The inner and outer surfaces of a spherical shell are maintained at T_0 and T_1 temperatures respectively. If the inner and outer radii of the shell are r_0 and r_1 respectively and thermal conductivity of the shell is k, find the amount of heat lost from the shell per unit time. Find also the temperature distribution through the shell.* ***(Dec. 2007, Nov. 2015)***

Sol. : Let x be the thickness of the spherical shell. Then heat flowing through the section is

$$q = -k\,(4\pi x^2) \frac{dT}{dx}$$

$$q \frac{dx}{x^2} = -4\pi k \, dT$$

$$q \int_{r_0}^{r_1} \frac{1}{x^2} \, dx = -4\pi k \int_{T_0}^{T_1} dT$$

$$q\left(\frac{1}{r_0} - \frac{1}{r_1}\right) = -4\pi k\,(T_1 - T_0)$$

$$q = \frac{4\pi k\,(T_0 - T_1)\, r_0 r_1}{r_1 - r_0} \qquad \text{... (1)}$$

$$dT = -\frac{q}{4\pi k} \frac{dx}{x^2} \quad T = \frac{q}{4\pi k} \frac{1}{x} + c$$

When $T = T_0$, $x = r_0$,

$$c = T_0 - \frac{q}{4\pi k} \frac{1}{r_0}$$

$$T = \frac{q}{4\pi k}\frac{1}{x} + T_0 - \frac{q}{4\pi k}\frac{1}{r_0}$$

by using (1)

$$= \frac{(T_0 - T_1)\, r_0 r_1}{(r_1 - r_0)\, x} - \frac{(T_0 - T_1)\, r_0 r_1}{(r_1 - r_0)\, r_0} + T_0$$

$\Rightarrow$

$$\boxed{T = \frac{1}{r_1 - r_0}\left[\frac{(T_0 - T_1)\, r_0 r_1}{x} + T_1 r_1 - T_0 r_0\right]}$$

Ex. 6 : *One-dimensional steady-state heat conduction for a hollow cylinder with constant thermal conductivity k in the region $a \le r \le b$, the temperature T_r at a distance r, $(a \le r \le b)$ is given by $\frac{d}{dr}\left[r\frac{dT_r}{dr}\right] = 0$ with $T_r = T_1$, when $r = a$, and $T_r = T_2$ when $r = b$. Use this to determine steady-state temperature distribution T_r in the cylinder in terms of r.* **(May 2008)**

Sol. : Initial conditions are

$$T_r = T_1 \quad \text{for} \quad r = a$$

$$T_r = T_2 \quad \text{for} \quad r = b$$

We have $\quad \frac{d}{dr}\left[r\frac{dT_r}{dr}\right] = 0$

Integrating $r\frac{dT_r}{dr} = c_1$, where, c_1 is a constant.

$\therefore \quad dT_r = c_1 \frac{dr}{r}$

Integrating both sides, $\quad T_r = c_1 \log r + c_2$

We put initial conditions,

(1) $\quad T_r = T_1$ for $r = a$

$$T_1 = c_1 \log a + c_2 \quad \text{... (1)}$$

(2) $\quad T_r = T_2$ for $r = b$

$$T_2 = c_1 \log b + c_2 \quad \text{... (2)}$$

(1) – (2) gives $\quad T_1 - T_2 = c_1 \log\frac{a}{b}$

$$\therefore \quad c_1 = \frac{T_1 - T_2}{\log\frac{a}{b}}$$

and

$$c_2 = T_1 - \frac{T_1 - T_2}{\log\frac{a}{b}}\log a = T_1 - \frac{(T_1 - T_2)\log a}{\log a - \log b}$$

i.e.

$$c_2 = \frac{T_1(\log a - \log b) - (T_1 - T_2)\log a}{\log a - \log b} = -\frac{T_1 \log b + T_2 \log a}{\log a - \log b} = -\frac{T_2 \log a - T_1 \log b}{\log (a/b)}$$

$$\therefore \quad T_r = \left(\frac{T_1 - T_2}{\log (a/b)}\right)\log r + \frac{T_2 \log a - T_1 \log b}{\log (a/b)}$$

$$\therefore \quad \boxed{\left[\log\frac{a}{b}\right] T_r = (T_1 - T_2)\log r + T_2 \log a - T_1 \log b}$$

Ex. 7 : *For steady heat flow through the wall of a spherical shell of inner and outer radii r_1 and r_2 respectively, the temperature T at a distance r from the centre of the sphere is given by $r\frac{d^2T}{dr^2} + 2\frac{dT}{dr} = 0$. Integrate for T by substituting $\frac{dT}{dr} = y$ if u_1 and u_2 are temperatures at inner and outer surfaces. Find T in terms of r.* **(Dec. 2010)**

Sol. :

$$r\frac{d^2T}{dr^2} + 2\frac{dT}{dr} = 0$$

Put $\quad \frac{dT}{dr} = y$

$$\therefore \quad r\frac{dy}{dr} + 2y = 0 \qquad \text{or} \qquad \frac{dy}{y} + 2\frac{dr}{r} = 0$$

Integrating

$$\log y + \log r^2 = \log c \qquad \therefore \qquad y = \frac{c}{r^2}$$

$$\frac{dT}{dr} = \frac{c}{r^2} \qquad \therefore \qquad T = -\frac{c}{r} + d$$

Using;

$T = u_1, \quad r = r_1, \quad T = u_2, \quad r = r_2$

$$u_1 = -\frac{c}{r_1} + d, \quad u_2 = -\frac{c}{r_2} + d$$

$$u_1 - u_2 = c\left(\frac{1}{r_2} - \frac{1}{r_1}\right) = \left(\frac{r_1 - r_2}{r_1 r_2}\right) c$$

$$c = \frac{r_1 r_2}{r_1 - r_2}(u_1 - u_2)$$

$$d = u_1 + \frac{c}{r_1} = u_1 + \frac{1}{r_1}\frac{r_1 r_2}{r_1 - r_2}(u_1 - u_2) = \frac{u_2 r_2 - u_1 r_1}{r_2 - r_1}$$

$$\boxed{T = \frac{1}{r_2 - r_1}\left[(u_2 r_2 - u_1 r_1) - \frac{r_1 r_2 (u_2 - u_1)}{r}\right]}$$

EXERCISE 2.8

1. A pipe 10 cm in diameter contains steam at 100°C. It is covered with asbestos, 5 cm thick, for which k = 0.0006 and the outside surface is at 30°C. Find the amount of heat lost per hour from a meter long pipe. **(Dec. 2008, May 2013)**

Ans. 14,0000 cal/hr.

2.11 MISCELLANEOUS EXAMPLES

Ex. 1 : *In a chemical reaction in which two substances A and B initially of amount a and b respectively are concerned, the velocity of transformation $\frac{dx}{dt}$ at any time t is known to be equal to the product (a – x) (b – x) of the amounts of the two substances then remaining untransformed. Find t in terms of x if a = 0.7, b = 0.5 and x = 0.3 when t = 300 seconds.*

Sol. : We have

$$\frac{dx}{dt} = (a - x)(b - x) = \left(\frac{7}{10} - x\right)\left(\frac{1}{2} - x\right)$$

$$\frac{dx}{\left(\frac{7}{10} - x\right)\left(\frac{1}{2} - x\right)} = dt \quad \text{or} \quad 5\left[\frac{dx}{\frac{1}{2} - x} - \frac{dx}{\frac{7}{10} - x}\right] = dt$$

Integrating,

$$5\left[\log\left(\frac{7}{10} - x\right) - \log\left(\frac{1}{2} - x\right)\right] = t + c$$

But $x = \frac{3}{10}$ when t = 300.

$$5\left(\log\frac{2}{5} - \log\frac{1}{5}\right) = 300 + c \quad \Rightarrow c = 5\log 2 - 300$$

$\therefore$

$$t = 5\left[\log\left(\frac{7}{10} - x\right) - \log\left(\frac{1}{2} - x\right)\right] - 5\log 2 + 300$$

$\therefore$

$$\boxed{t = 5\left[\log\left(\frac{7}{10} - x\right) - \log\left(\frac{1}{2} - x\right) - \log 2\right] + 300}$$

Ex. 2 : *When investigating the stress in the material of a thick cylinder subjected to internal pressure, the following relations are found to exist; $p + r\frac{dp}{dr} = q$ and $p + q = 2a$, where p and q are the radial stress and the circumferential stress respectively and r is the radius. Express p as a function of r.*

Sol. : We have to eliminate q between the given equations.

$\because \quad q = 2a - p \qquad \therefore \qquad p + r\frac{dp}{dr} = 2a - p$

i.e. $\quad r\frac{dp}{dr} + 2p - 2a = 0 \qquad \text{or} \qquad \int\frac{dp}{p - a} + 2\int\frac{dr}{r} = 0$

$\therefore \quad \log(p - a) + 2\log r = \log c \Rightarrow (p - a) r^2 = c.$

$$\boxed{p = \frac{c}{r^2} + a}$$

Ex. 3 : *For a thick cylinder under internal pressure, if 'p' is the compressive stress and 'f' the tensile stress at a distance 'r' from the axis of the cylinder, the differential equation is* $r\frac{dp}{dr} + p + f = 0$. *Assuming* $f + ap = b$, $p = 0$. *When* $r = r_2$ *and* $p = p_1$, *when* $r = r_1$, *show that* $\left(\frac{r_1}{r_2}\right)^{a-1} = \left(\frac{1-a}{b}\right) p_1 + 1.$

Sol. : $r\frac{dp}{dr} + p + f = 0$ and $f = b - ap$.

$$r\frac{dp}{dr} + (p - ap + b) = 0$$

$$\frac{dp}{b-(a-1)p} + \frac{dr}{r} = 0 \quad \text{(by using V.S. form)}$$

But $p = 0$ when $r = r_2$; $p = p_1$ when $r = r_1$ and multiplying by $(a-1)$ throughout, we have,

$$(a-1)\int_{r_2}^{r_1} \frac{dr}{r} = \int_0^{p_1} \frac{-(a-1)\,dp}{b-(a-1)p}$$

$$\log\left(\frac{r_1}{r_2}\right)^{a-1} = \log\left(\frac{b-(a-1)p_1}{b}\right)$$

$$\left(\frac{r_1}{r_2}\right)^{a-1} = 1 - \left(\frac{a-1}{b}\right) p_1 = 1 + \left(\frac{1-a}{b}\right) p_1$$

Ex. 4 : *An equation in the theory of stability of an aeroplane is* $\frac{dv}{dt} = g\cos\alpha - kv$, *v being the velocity, g and k are constants. It is observed that at time t = 0, velocity is also zero. Solve completely.*

Sol. : We have $\quad \frac{dv}{dt} = g\cos\alpha - kv$ at $t = 0$, $v = 0$

We have $\quad \frac{dv}{dt} + kv = g\cos\alpha$ is a linear equation

$$\text{I.F.} = e^{kt}$$

G.S. is $\quad v\,e^{kt} = g\cos\alpha \int e^{kt}\,dt + C$

$$v\,e^{kt} = \frac{g}{k}\cos\alpha\, e^{kt} + C$$

$\therefore \quad v = \frac{g}{k}\cos\alpha + Ce^{-kt} \quad$ ... (1)

At $t = 0$, $v = 0$, $0 = \frac{g}{k}\cos\alpha + C \quad \therefore\ C = -\frac{g}{k}\cos\alpha$

$\therefore$ (1) becomes $\quad v = \frac{g}{k}\cos\alpha - \frac{g}{k}\cos\alpha\, e^{-kt}$

$$\boxed{v = \frac{g}{k}\cos\alpha\,(1 - e^{-kt})}$$

Ex. 5 : *The amount x of a substance present in a certain chemical reaction at time t is given by* $\frac{dx}{dt} + \frac{x}{10} = 2 - 1.5\,e^{-\frac{t}{10}}$. *If at t = 0, x = 0.5, find x at t = 10.* **(Dec. 2009)**

Sol. : $\quad \frac{dx}{dt} + \frac{x}{10} = 2 - 1.5\,e^{-\frac{t}{10}}$ linear equation

$$\text{I.F.} = e^{t/10}$$

G.S. is $\quad x\,e^{t/10} = \int e^{t/10}\left[2 - 1.5\,e^{-\frac{t}{10}}\right] dt + c$

$\therefore \quad x\,e^{t/10} = 2 \times 10 \times e^{t/10} - \frac{3}{2}t + c$

$$x = 20 - \frac{3}{2}t\,e^{-\frac{t}{10}} + c\,e^{-t/10}, \text{ at } t = 0,\ x = 0.5$$

$$\frac{1}{2} = 20 + c \therefore c = -\frac{39}{2}$$

$$x = 20 - \frac{3}{2} t e^{-\frac{t}{10}} - \frac{39}{2} e^{-\frac{t}{10}}$$

Now at $t = 10$,

$$x = 20 - 15 e^{-1} - \frac{39}{2} e^{-1}$$

$$\boxed{x = 20 - \frac{69}{2e}}$$

Ex. 6 : *A man invests Rs. 5000 at the rate of 6 percent per annum, interest being compounded continuously. When will the sum double itself? (Take $\log_e 2 = 0.693$)*

Sol. : Let x be the amount after t years. Then

$$\frac{dx}{dt} = \frac{6}{100} x \qquad \text{or} \qquad \int \frac{dx}{x} = \frac{6}{100} \int dt$$

$$\log x = \frac{6t}{100} + c$$

But when $t = 0$, $x = 5000 \therefore c = \log 5000$

$$\log x - \log 5000 = \frac{6t}{100}$$

$$\therefore \frac{6}{100} t = \log\left(\frac{x}{5000}\right)$$

To find t when x = 10,000

$$\frac{6}{100} t = \log\left(\frac{10{,}000}{5{,}000}\right) = \log 2$$

$$\boxed{t = \frac{100}{6} (0.693) = 11.55 \text{ years}}$$

Ex. 7 : Motion of a boat across a stream : *A boat is rowed with a velocity u directly across a stream of width a. If the velocity of the current is directly proportional to the product of the distances from the two banks, find the path of the boat and the distance downstream to the point where it lands.*

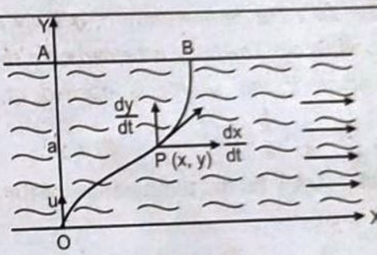

Fig. 2.17

Sol. : Taking the origin at the point from where the boat starts, let the axes be chosen as in Fig. 2.17. At any time t after its start from O, let the boat be at P (x, y), so that

$$\frac{dx}{dt} = \text{velocity of the current} = ky(a - y)$$

$$\frac{dy}{dt} = \text{velocity with which the boat is being rowed} = u$$

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{u}{ky(a-y)} \qquad \text{... (1)}$$

This gives the direction of the resultant velocity of the boat which is also the direction of the tangent to the path of the boat.

Now (1) is of variables separable form and we can write it as

$$y(a - y)\, dy = \frac{u}{k} dx$$

Integrating, we get $\frac{ay^2}{2} - \frac{y^3}{3} = \frac{u}{k} x + c$

Since $y = 0$ when $x = 0$, $\therefore c = 0$

Hence the equation of the path of the boat is

$$\boxed{x = \frac{k}{6u} y^2 (3a - 2y)}$$

Putting $y = a$, we get the distance AB, downstream where the boat lands $= ka^3/6u$.

Ex. 8 : *If the population of a country doubles in 50 years, in how many years will it treble under the assumption that the rate of increase is proportional to the number of inhabitants ?*

Sol. : Let y denote the population at time t years and y_0 the population at time t = 0. Then

$$\frac{dy}{dt} = ky \qquad \text{... (1)}$$

or $$\frac{dy}{y} = k\,dt, \text{ where k is the proportionality factor.}$$

Integrating, we have $\log y = kt + \log C$ or $y = Cekt$. ... (2)

At time $t = 0$, $y = y_0$ and from (2), $C = y_0$. Thus, $y = y_0 e^{kt}$ ... (3)

At $t = 50$, $y = 2y_0$. From (3), $2y_0 = y_0 e^{(50 k)}$ or $e^{(50 k)} = 2$.

When $y = 3y_0$, (3) gives $3 = e^{kt}$.

Then $$3^{50} = e^{(50 k)t} = e^{(50 k) t} = 2^t$$

$$50 \log (3) = t \log (2)$$

$\therefore$ $\boxed{t = 79 \text{ years}}$

Ex. 9 : *In a certain culture of bacteria, the rate of increase is proportional to the number present. If it is found that the number doubles in 4 hours, how many may be expected at the end of 12 hours ?*

Sol. : Let x denote the number of bacteria at time t hours. Then

$$\frac{dx}{dt} = kx \quad \text{or} \quad \frac{dx}{x} = k\,dt. \qquad \text{...(1)}$$

Integrating (1), we have $\log x = kt + \log C$ or $x = C e^{kt}$ ... (2)

Assuming that $x = x_0$ at time $t = 0$, $C = x_0$ and $x = x_0 e^{kt}$.

At time $t = 4$, $x = 2x_0$. Then $2x_0 = x_0 e^{4k}$ and $e^{4k} = 2$.

When t = 12, $\boxed{x = x_0 e^{12k} = x_0 (e^{4k})^3 = x_0 (2)^3 = 8x_0}$, that is, there are 8 times the original number.

Ex. 10 : *The temperature of a body decreases at a rate $k\theta$, where $\theta°$ is the amount of temperature of the body hotter than the surrounding air. The body is heated by a source which makes the body's temperature increase at a rate "at" where 't' is the time and 'a' is a constant. If this source is applied at t = 0, and the body is then at the temperature of the surrounding air, show that* $\theta = \frac{a}{k}\left(t - \frac{1}{k} + \frac{1}{k}e^{-kt}\right)$.

Sol. : Let T be the temperature of the body at any time t, and T_1 be the constant temperature of the surrounding; then $\theta = T - T_1$

or $$\frac{d\theta}{dt} = \frac{dT}{dt}. \qquad \text{... (1)}$$

Now due to cooling alone, the temperature of the body decreases at the rate $\frac{dT}{dt}$

which equals $-k\theta$ $\therefore$ $$\frac{dT}{dt} = -k\theta \qquad \text{... (2)}$$

And due to the source of heat, the body's temperature increases at the rate $\frac{dT}{dt} = at$ ... (3)

$\therefore$ The rate at which the body's temperature changes due to both these effects is given by $(-k\theta + at)$.

$\therefore$ $$\frac{dT}{dt} = -k\theta + at \quad \text{or} \quad \frac{d\theta}{dt} = -k\theta + at \qquad [\because \text{by (1)}]$$

$$\frac{d\theta}{dt} + k\theta = at \quad \therefore \text{ I.F.} = e^{kt}$$

and G.S. is $$\theta \cdot e^{kt} = a\int t e^{kt}\,dt + C = \frac{a}{k} e^{kt}\left(t - \frac{1}{k}\right) + C$$

When $t = 0$, $\theta = 0$ $\therefore$ $C = \frac{a}{k^2}$

$\therefore$ $$\boxed{\theta = \frac{a}{k}\left(t - \frac{1}{k} + \frac{1}{k}e^{-k/t}\right)}$$