Simply-Typed Lambda Calculus with Extensions

CS231

1 Language of Booleans and Integers

1.1 Syntax

$$
\begin{array}{lcl}
\mathtt{t} & ::= & \mathtt{true} \mid \mathtt{false} \mid \mathtt{if\ t\ then\ t\ else\ t} \\
 & \mid & \mathtt{n} \mid \mathtt{t\ +\ t} \mid \mathtt{t\ >\ t} \\
\mathtt{n} & ::= & \text{integer constant} \\
\mathtt{v} & ::= & \mathtt{true} \mid \mathtt{false} \mid \mathtt{n} \\
\mathtt{T} & ::= & \mathtt{Bool} \mid \mathtt{Int}
\end{array}
$$

1.2 Small-Step Operational Semantics

$$\frac{}{\mathtt{if\ true\ then\ t_2\ else\ t_3} \longrightarrow \mathtt{t_2}} \quad \text{(E-IfTrue)}$$

$$\frac{}{\mathtt{if\ false\ then\ t_2\ else\ t_3} \longrightarrow \mathtt{t_3}} \quad \text{(E-IfFalse)}$$

$$\frac{\mathtt{t_1} \longrightarrow \mathtt{t_1'}}{\mathtt{if\ t_1\ then\ t_2\ else\ t_3} \longrightarrow \mathtt{if\ t_1'\ then\ t_2\ else\ t_3}} \quad \text{(E-If)}$$

$$\frac{\mathtt{t_1} \longrightarrow \mathtt{t_1'}}{\mathtt{t_1\ +\ t_2} \longrightarrow \mathtt{t_1'\ +\ t_2}} \quad \text{(E-Plus1)}$$

$$\frac{\mathtt{t_2} \longrightarrow \mathtt{t_2'}}{\mathtt{v_1\ +\ t_2} \longrightarrow \mathtt{v_1\ +\ t_2'}} \quad \text{(E-Plus2)}$$

$$\frac{\mathrm{n} = \mathrm{n}_1\ [\![+]\!]\ \mathrm{n}_2}{\mathtt{n_1\ +\ n_2} \longrightarrow \mathtt{n}} \quad \text{(E-PlusRed)}$$

$$\frac{\mathtt{t_1} \longrightarrow \mathtt{t_1'}}{\mathtt{t_1\ >\ t_2} \longrightarrow \mathtt{t_1'\ >\ t_2}} \quad \text{(E-GT1)}$$

$$\frac{\mathtt{t_2} \longrightarrow \mathtt{t_2'}}{\mathtt{v_1\ >\ t_2} \longrightarrow \mathtt{v_1\ >\ t_2'}} \quad \text{(E-GT2)}$$

$$\frac{\mathrm{v} = \mathrm{n}_1\ [\![>]\!]\ \mathrm{n}_2}{\mathtt{n_1\ >\ n_2} \longrightarrow \mathtt{v}} \quad \text{(E-GTRed)}$$

1.3 Static Type System

$$\frac{}{\mathtt{true : Bool}} \quad \text{(T-True)} \qquad\qquad \frac{}{\mathtt{false : Bool}} \quad \text{(T-False)}$$

$$\frac{\texttt{t}_1 : \texttt{Bool} \quad \texttt{t}_2 : \texttt{T} \quad \texttt{t}_3 : \texttt{T}}{\texttt{if } \texttt{t}_1 \texttt{ then } \texttt{t}_2 \texttt{ else } \texttt{t}_3 : \texttt{T}} \quad \text{(T-IF)}$$

$$\frac{}{\texttt{n} : \texttt{Int}} \quad \text{(T-NUM)}$$

$$\frac{\texttt{t}_1 : \texttt{Int} \quad \texttt{t}_2 : \texttt{Int}}{\texttt{t}_1 \texttt{ + } \texttt{t}_2 : \texttt{Int}} \quad \text{(T-PLUS)}$$

$$\frac{\texttt{t}_1 : \texttt{Int} \quad \texttt{t}_2 : \texttt{Int}}{\texttt{t}_1 \texttt{ > } \texttt{t}_2 : \texttt{Bool}} \quad \text{(T-GT)}$$

2 Simply-Typed Lambda Calculus

2.1 Syntax

$$\begin{array}{lll} \texttt{t} & ::= & \texttt{x} \mid \texttt{function x:T -> t} \mid \texttt{t t} \\ \texttt{v} & ::= & \texttt{function x:T -> t} \\ \texttt{T} & ::= & \texttt{T}_1 \rightarrow \texttt{T}_2 \end{array}$$

2.2 Substitution

$[\texttt{x} \mapsto \texttt{v}]\texttt{x} = \texttt{v}$
$[\texttt{x} \mapsto \texttt{v}]\texttt{x}' = \texttt{x}'$, where $\texttt{x} \neq \texttt{x}'$
$[\texttt{x} \mapsto \texttt{v}]\texttt{function x:T -> t}_0 = \texttt{function x:T -> t}_0$
$[\texttt{x} \mapsto \texttt{v}]\texttt{function x}_0\texttt{:T -> t}_0 = \texttt{function x}_0\texttt{:T -> } [\texttt{x} \mapsto \texttt{v}]\texttt{t}_0$, where $\texttt{x} \neq \texttt{x}_0$
$[\texttt{x} \mapsto \texttt{v}]\texttt{t}_1\ \texttt{t}_2 = [\texttt{x} \mapsto \texttt{v}]\texttt{t}_1\ [\texttt{x} \mapsto \texttt{v}]\texttt{t}_2$

2.3 Small-Step Operational Semantics

$$\frac{}{\texttt{((function x:T -> t) v)} \longrightarrow [\texttt{x} \mapsto \texttt{v}]\texttt{t}} \quad \text{(E-APPBETA)}$$

$$\frac{\texttt{t}_1 \longrightarrow \texttt{t}_1'}{\texttt{t}_1\ \texttt{t}_2 \longrightarrow \texttt{t}_1'\ \texttt{t}_2} \quad \text{(E-APP1)} \qquad \frac{\texttt{t}_2 \longrightarrow \texttt{t}_2'}{\texttt{v}_1\ \texttt{t}_2 \longrightarrow \texttt{v}_1\ \texttt{t}_2'} \quad \text{(E-APP2)}$$

2.4 Static Type System

Γ is a finite function from variable names to types.

$$\frac{\Gamma(\texttt{x}) = \texttt{T}}{\Gamma \vdash \texttt{x} : \texttt{T}} \quad \text{(T-VAR)}$$

$$\frac{\Gamma, \texttt{x:T}_1 \vdash \texttt{t} : \texttt{T}_2}{\Gamma \vdash \texttt{function x:T}_1 \texttt{ -> t} : \texttt{T}_1 \rightarrow \texttt{T}_2} \quad \text{(T-FUN)}$$

$$\frac{\Gamma \vdash \texttt{t}_1 : \texttt{T}_2 \rightarrow \texttt{T} \quad \Gamma \vdash \texttt{t}_2 : \texttt{T}_2}{\Gamma \vdash \texttt{t}_1\ \texttt{t}_2 : \texttt{T}} \quad \text{(T-APP)}$$

3 Extensions

We augment our language with a unit value, pairs, tagged unions, `let`, `letrec`.

3.1 Syntax

```
t  ::=  () | (t,t) | fst t | snd t
     |  left t | right t | (match t with left x -> t | right x -> t)
     |  let x=t in t | letrec x=v in t
v  ::=  () | (v,v) | left v | right v
T  ::=  True | False | T ∧ T | T ∨ T
```

3.2 Small-Step Operational Semantics

$$\frac{\texttt{t}_1 \longrightarrow \texttt{t}_1'}{(\texttt{t}_1,\texttt{t}_2) \longrightarrow (\texttt{t}_1',\texttt{t}_2)} \quad \text{(E-PAIR1)}$$

$$\frac{\texttt{t}_2 \longrightarrow \texttt{t}_2'}{(\texttt{v}_1,\texttt{t}_2) \longrightarrow (\texttt{v}_1,\texttt{t}_2')} \quad \text{(E-PAIR2)}$$

$$\frac{\texttt{t} \longrightarrow \texttt{t}'}{\texttt{fst t} \longrightarrow \texttt{fst t}'} \quad \text{(E-FST)}$$

$$\frac{}{\texttt{fst}\ (\texttt{v}_1,\texttt{v}_2) \longrightarrow \texttt{v}_1} \quad \text{(E-FSTRED)}$$

$$\frac{\texttt{t} \longrightarrow \texttt{t}'}{\texttt{snd t} \longrightarrow \texttt{snd t}'} \quad \text{(E-SND)}$$

$$\frac{}{\texttt{snd}\ (\texttt{v}_1,\texttt{v}_2) \longrightarrow \texttt{v}_2} \quad \text{(E-SNDRED)}$$

$$\frac{\texttt{t} \longrightarrow \texttt{t}'}{\texttt{left t} \longrightarrow \texttt{left t}'} \quad \text{(E-LEFT)}$$

$$\frac{\texttt{t} \longrightarrow \texttt{t}'}{\texttt{right t} \longrightarrow \texttt{right t}'} \quad \text{(E-RIGHT)}$$

$$\frac{\texttt{t} \longrightarrow \texttt{t}'}{\texttt{match t with left x}_1\ \texttt{->}\ \texttt{t}_1\ \texttt{|}\ \texttt{right x}_2\ \texttt{->}\ \texttt{t}_2 \longrightarrow \texttt{match t}'\ \texttt{with left x}_1\ \texttt{->}\ \texttt{t}_1\ \texttt{|}\ \texttt{right x}_2\ \texttt{->}\ \texttt{t}_2} \quad \text{(E-MATCH)}$$

$$\frac{}{\texttt{match left v with left x}_1\ \texttt{->}\ \texttt{t}_1\ \texttt{|}\ \texttt{right x}_2\ \texttt{->}\ \texttt{t}_2 \longrightarrow [\texttt{x}_1 \mapsto \texttt{v}]\texttt{t}_1} \quad \text{(E-MATCHLEFT)}$$

$$\frac{}{\texttt{match right v with left x}_1\ \texttt{->}\ \texttt{t}_1\ \texttt{|}\ \texttt{right x}_2\ \texttt{->}\ \texttt{t}_2 \longrightarrow [\texttt{x}_2 \mapsto \texttt{v}]\texttt{t}_2} \quad \text{(E-MATCHRIGHT)}$$

$$\frac{\texttt{t}_1 \longrightarrow \texttt{t}_1'}{\texttt{let x=t}_1\ \texttt{in t}_2 \longrightarrow \texttt{let x=t}_1'\ \texttt{in t}_2} \quad \text{(E-LET)}$$

$$\frac{}{\texttt{let x=v in t} \longrightarrow [\texttt{x} \mapsto \texttt{v}]\texttt{t}} \quad \text{(E-LETRED)}$$

$$\frac{}{\texttt{letrec x=v in t} \longrightarrow \texttt{let x=}[\texttt{x} \mapsto \texttt{letrec x=v in x}]\texttt{v in t}} \quad \text{(E-LETREC)}$$

3.3 Static Type System

$$\frac{}{\Gamma \vdash \texttt{()} : \texttt{True}} \quad \text{(T-TRUE)}$$

$$\frac{\Gamma \vdash \texttt{t} : \texttt{False}}{\Gamma \vdash \texttt{t} : \texttt{T}} \quad \text{(T-FALSE)}$$

$$\frac{\Gamma \vdash \texttt{t}_1 : \texttt{T}_1 \qquad \Gamma \vdash \texttt{t}_2 : \texttt{T}_2}{\Gamma \vdash (\texttt{t}_1,\texttt{t}_2) : \texttt{T}_1 \wedge \texttt{T}_2} \quad \text{(T-PAIR)}$$

$$\frac{\Gamma \vdash \texttt{t} : \texttt{T}_1 \wedge \texttt{T}_2}{\Gamma \vdash \texttt{fst t} : \texttt{T}_1} \quad \text{(T-FST)}$$

$$\frac{\Gamma \vdash \texttt{t} : \texttt{T}_1 \wedge \texttt{T}_2}{\Gamma \vdash \texttt{snd t} : \texttt{T}_2} \quad \text{(T-SND)}$$

$$\frac{\Gamma \vdash \texttt{t} : \texttt{T}_1}{\Gamma \vdash \texttt{left t} : \texttt{T}_1 \vee \texttt{T}_2} \quad \text{(T-LEFT)}$$

$$\frac{\Gamma \vdash \texttt{t} : \texttt{T}_2}{\Gamma \vdash \texttt{right t} : \texttt{T}_1 \vee \texttt{T}_2} \quad \text{(T-RIGHT)}$$

$$\frac{\Gamma \vdash \texttt{t} : \texttt{T}_1 \vee \texttt{T}_2 \qquad \Gamma, \texttt{x}_1{:}\texttt{T}_1 \vdash \texttt{t}_1 : \texttt{T} \qquad \Gamma, \texttt{x}_2{:}\texttt{T}_2 \vdash \texttt{t}_2 : \texttt{T}}{\Gamma \vdash \texttt{match t with left x}_1 \texttt{ -> t}_1 \texttt{ | right x}_2 \texttt{ -> t}_2 : \texttt{T}} \quad \text{(T-MATCH)}$$

$$\frac{\Gamma \vdash \texttt{t}_1 : \texttt{T}_1 \qquad \Gamma, \texttt{x}{:}\texttt{T}_1 \vdash \texttt{t}_2 : \texttt{T}}{\Gamma \vdash \texttt{let x=t}_1 \texttt{ in t}_2 : \texttt{T}} \quad \text{(T-LET)}$$

$$\frac{\Gamma, \texttt{x}{:}\texttt{T}_1 \vdash \texttt{v}_1 : \texttt{T}_1 \qquad \Gamma, \texttt{x}{:}\texttt{T}_1 \vdash \texttt{t}_2 : \texttt{T}}{\Gamma \vdash \texttt{letrec x=v}_1 \texttt{ in t}_2 : \texttt{T}} \quad \text{(T-LETREC)}$$