

# *Computer Graphics*

*by Ruen-Rone Lee*  
**ICL/ITRI**



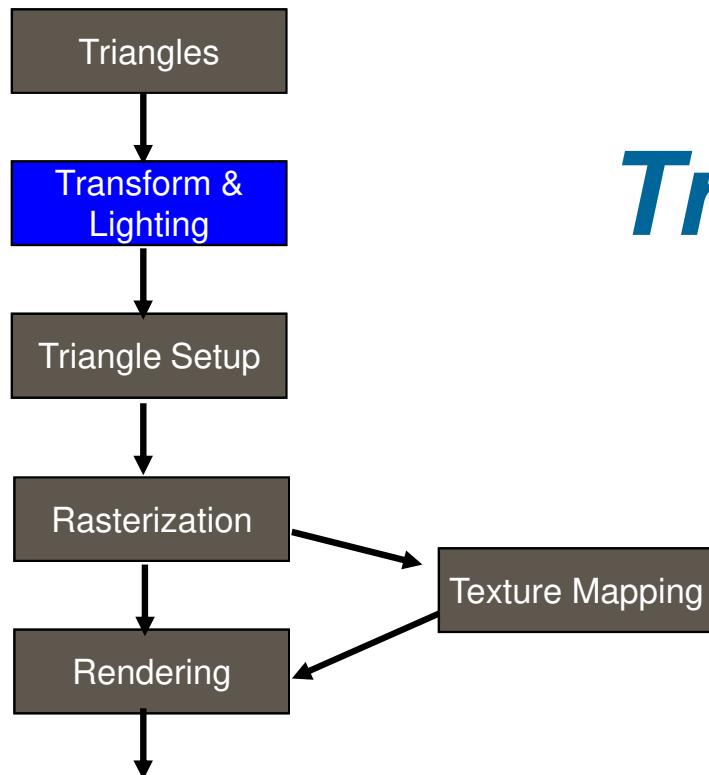
# *Wrap up from last class*

- ◆ **Primitives**
- ◆ **Rasterization – Generate Pixels that Constitute the Primitive**
  - Line Rasterization
  - Triangle Rasterization
- ◆ **Deriving Attributes (Colors, Depths, Texture Coordinates) of Pixels inside a Triangle**



## *Part I:*

# *Conventional 3D Graphics Pipeline*



## *Transformation*

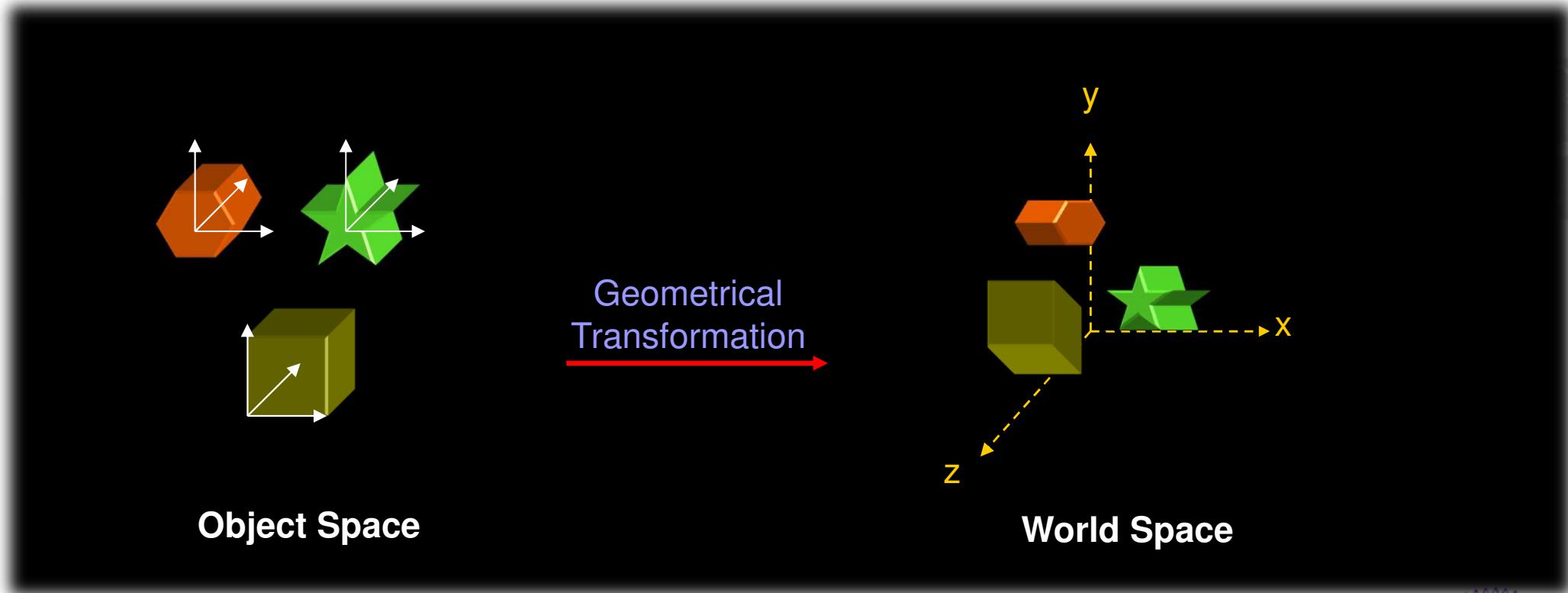
Conventional 3D Graphics Pipeline

*3D Viewing Process*



# 3D Viewing Process

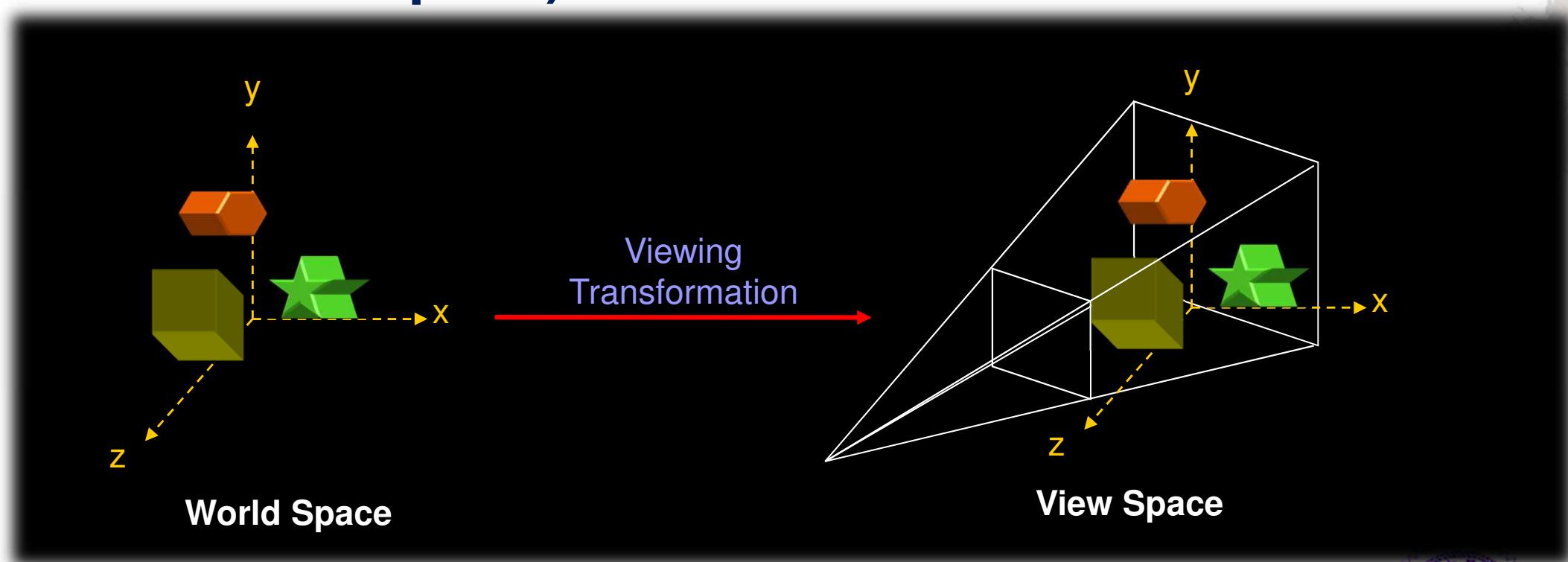
- ◆ Geometrical Transformation
  - From Object Space to World Space



# 3D Viewing Process

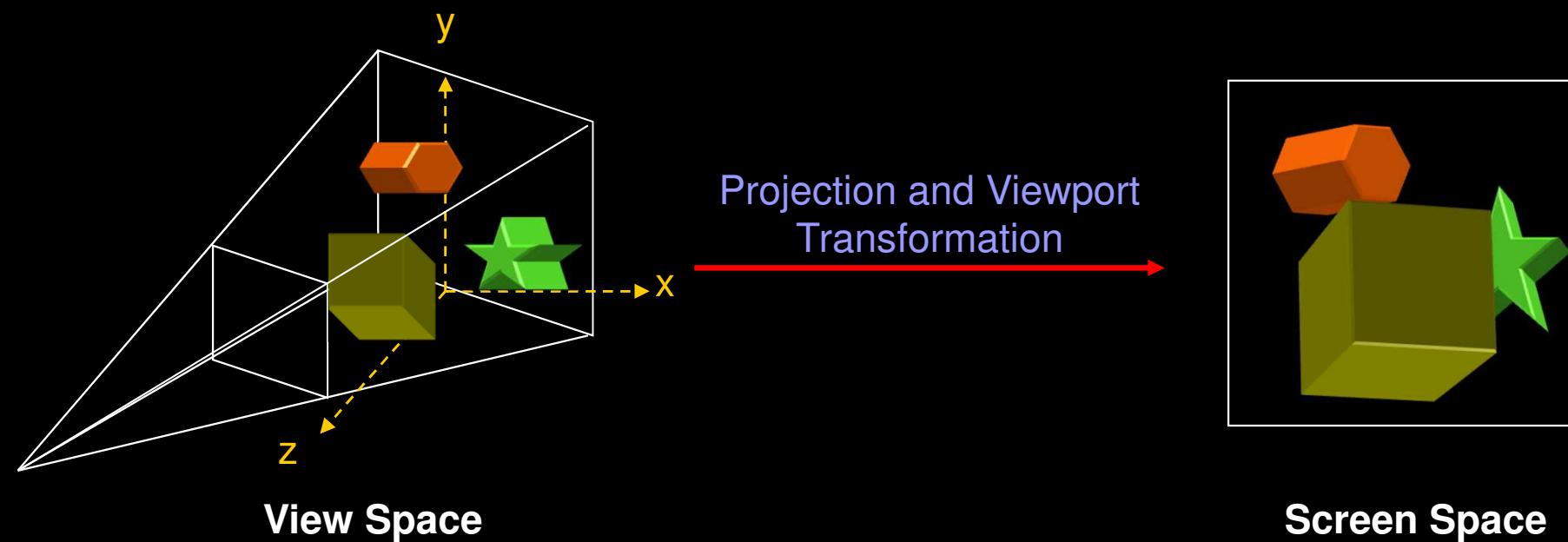
## ◆ Viewing Transformation

- From World Space to View Space (Eye Space or Camera Space)



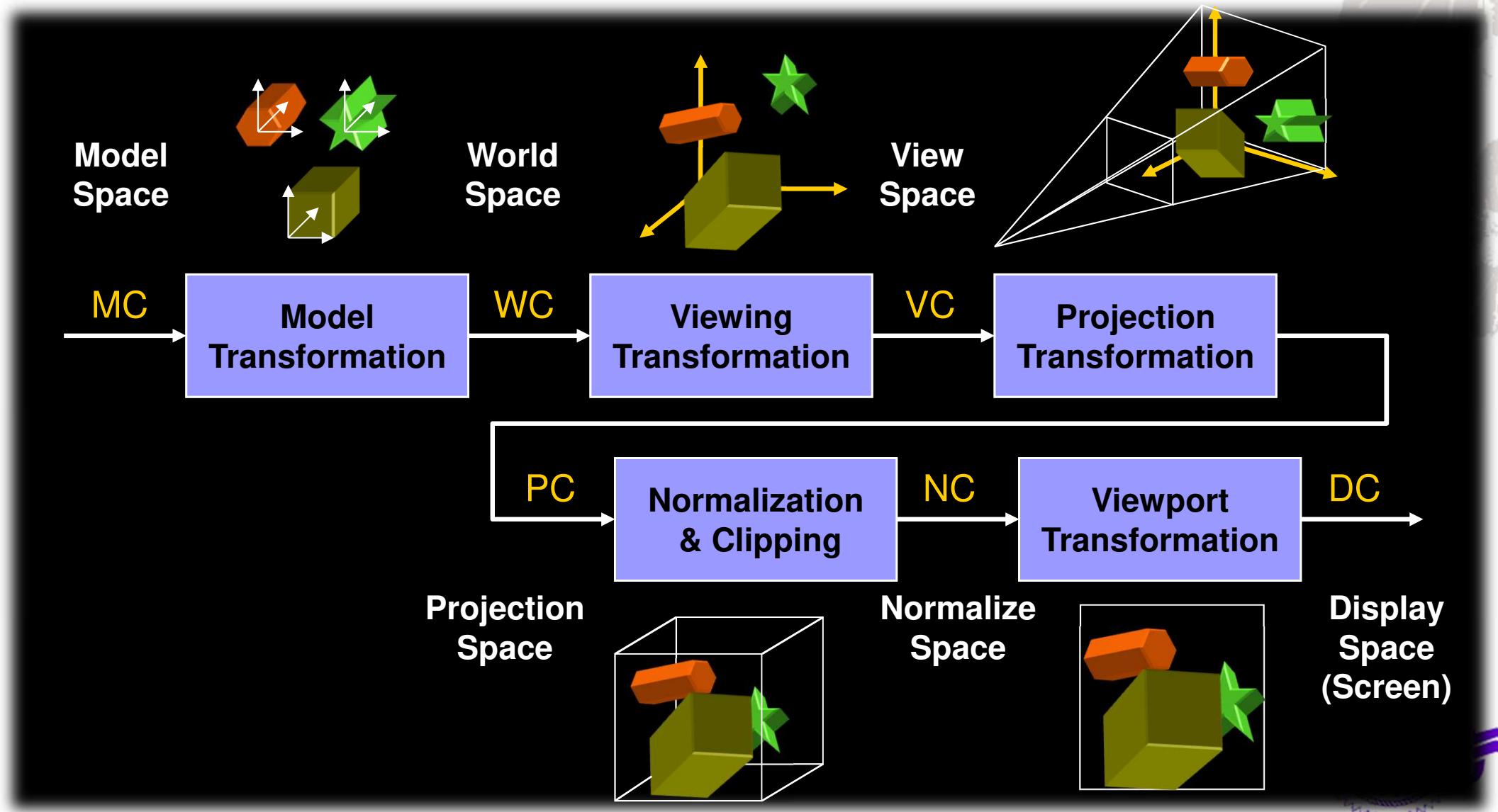
# 3D Viewing Process

- ◆ **Projection and Viewport Transformation**
  - From View Space (3D) to Screen Space (2D)



# 3D Viewing Process

## ◆ Coordinates Transformation



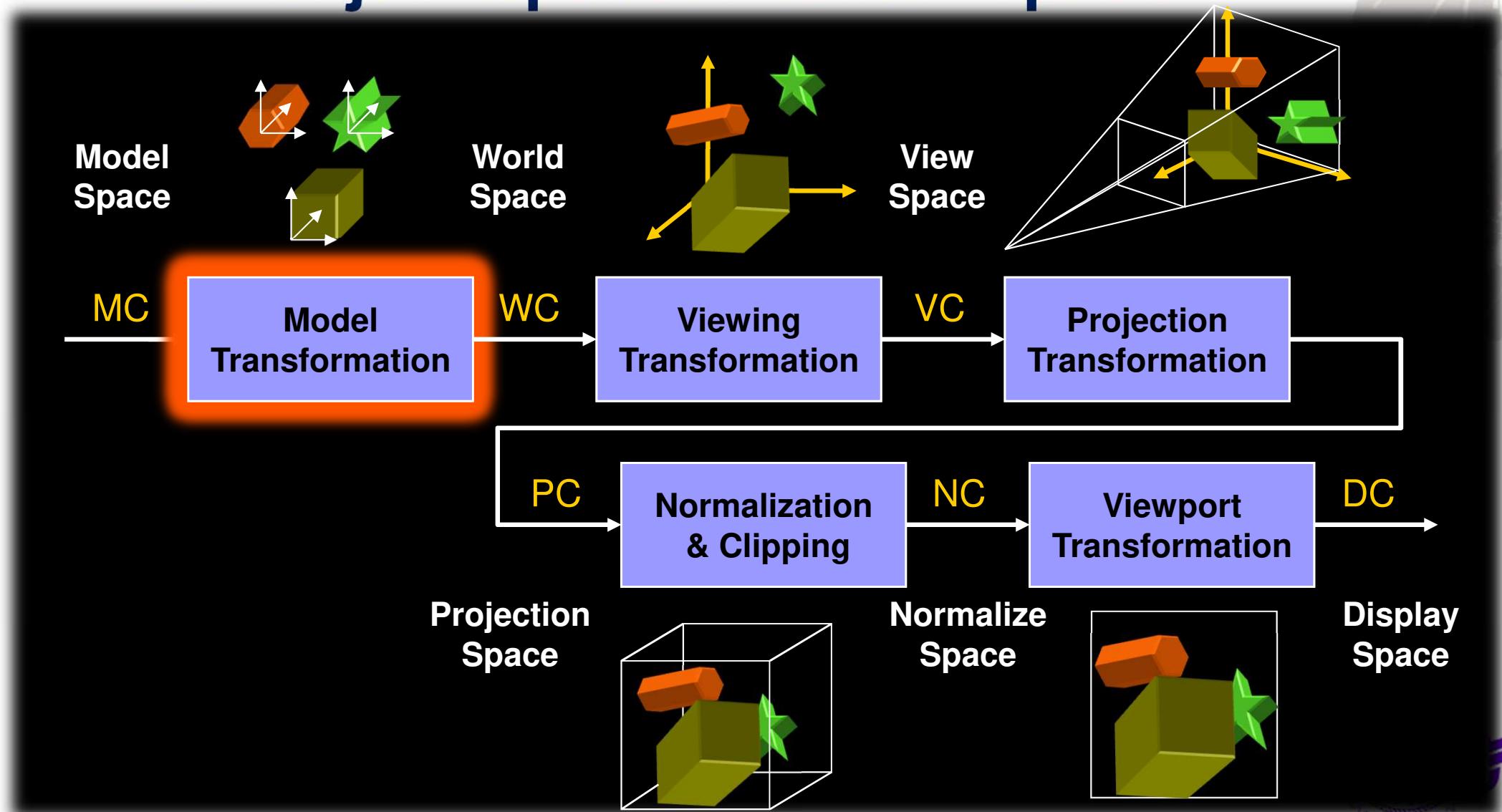
# *Geometrical Transformations*

*From Object Space to World Space*



# *Geometrical Transformation*

## ◆ From Object Space to World Space



# 2D Transformation

## ◆ Translation

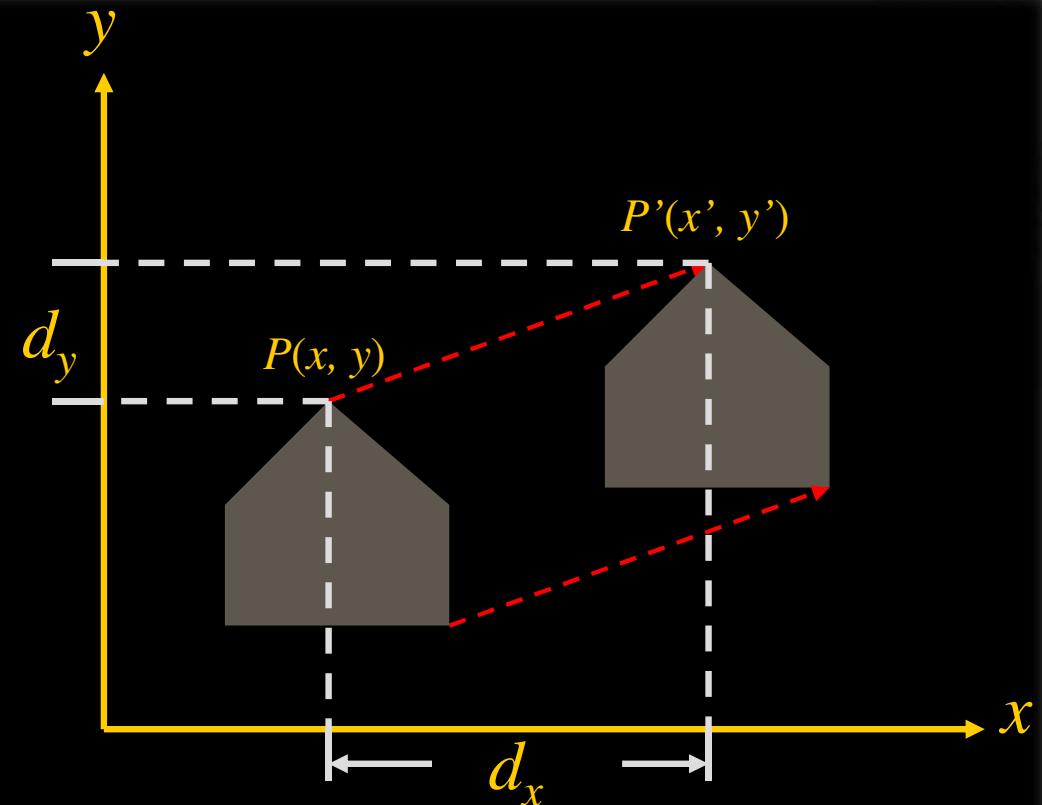
$P(x, y)$  move to  $P'(x', y')$

$$x' = x + d_x$$

$$y' = y + d_y$$

$$P = \begin{bmatrix} x \\ y \end{bmatrix}, P' = \begin{bmatrix} x' \\ y' \end{bmatrix}, T = \begin{bmatrix} d_x \\ d_y \end{bmatrix}$$

$$P' = P + T(d_x, d_y)$$



# 2D Transformation

## ◆ Scaling

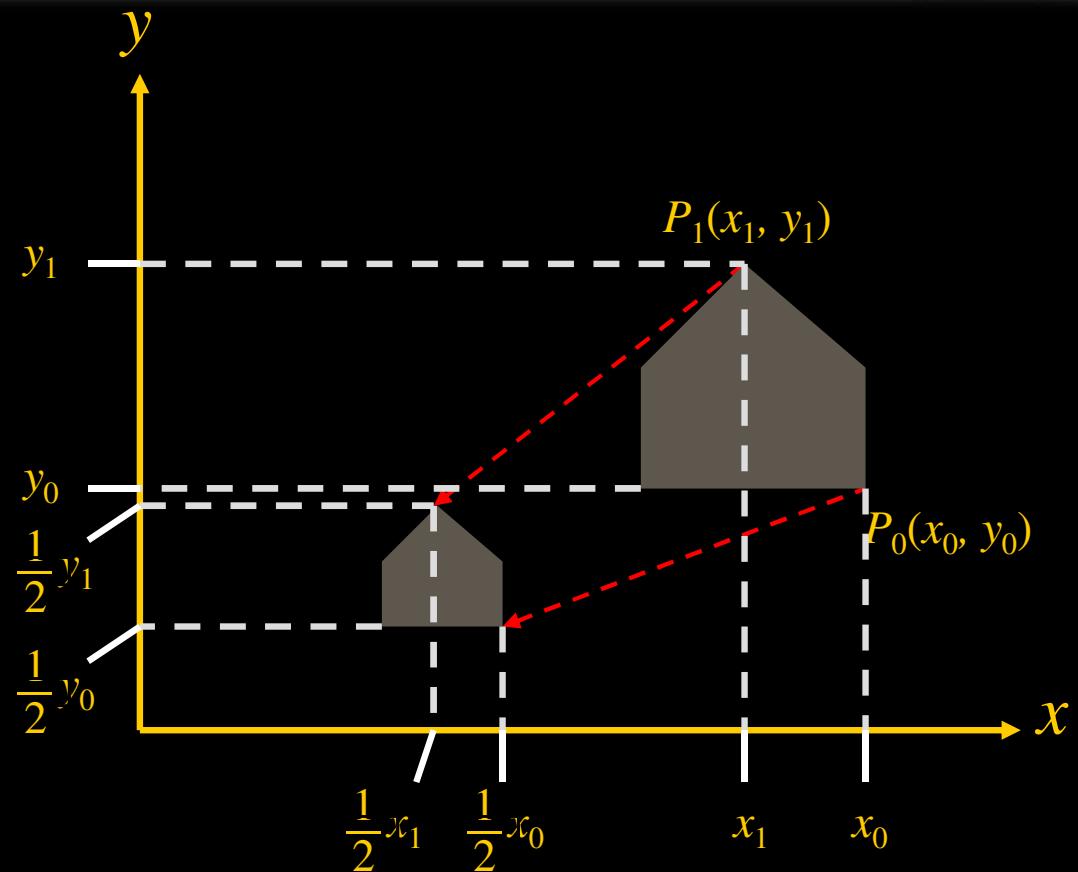
Scale by  $s_x$  along the  $x$  axis  
and by  $s_y$  along the  $y$  axis

$$x' = s_x \cdot x$$

$$y' = s_y \cdot y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

$$P' = S(s_x, s_y) \cdot P$$



# 2D Transformation

## ◆ Rotation

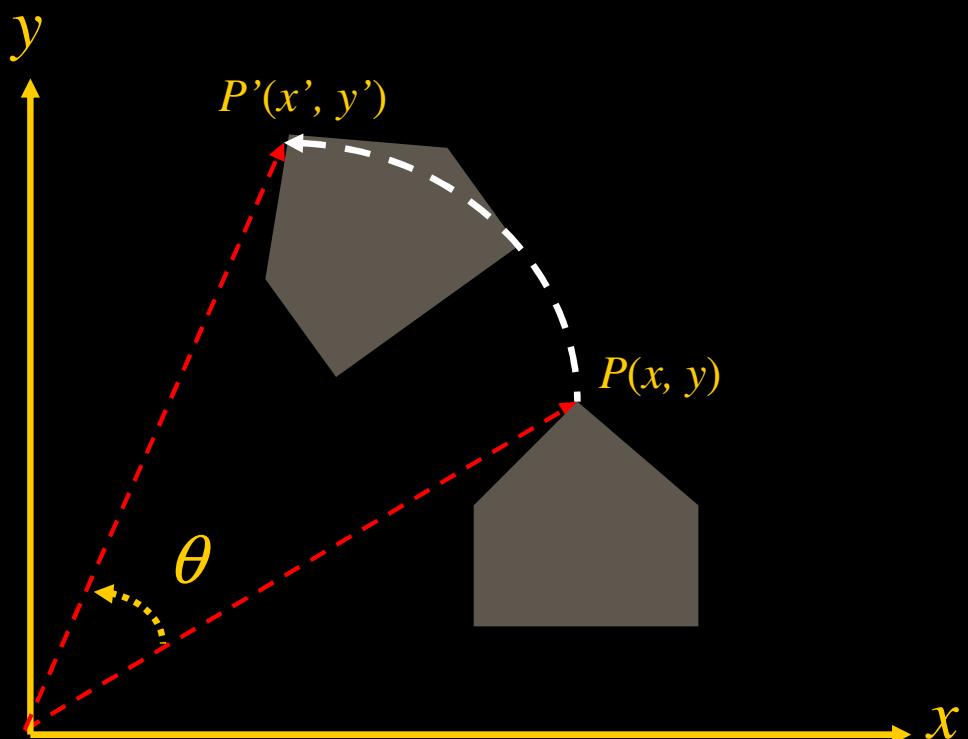
Rotate by an angle  $\theta$  with respect to the origin

$$x' = x \cdot \cos \theta - y \cdot \sin \theta$$

$$y' = x \cdot \sin \theta + y \cdot \cos \theta$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

$$P' = R(\theta) \cdot P$$



# 2D Transformation

## ◆ Derivation of the rotation equation

$$x = r \cdot \cos \phi$$

$$y = r \cdot \sin \phi$$

$$x' = r \cdot \cos(\phi + \theta)$$

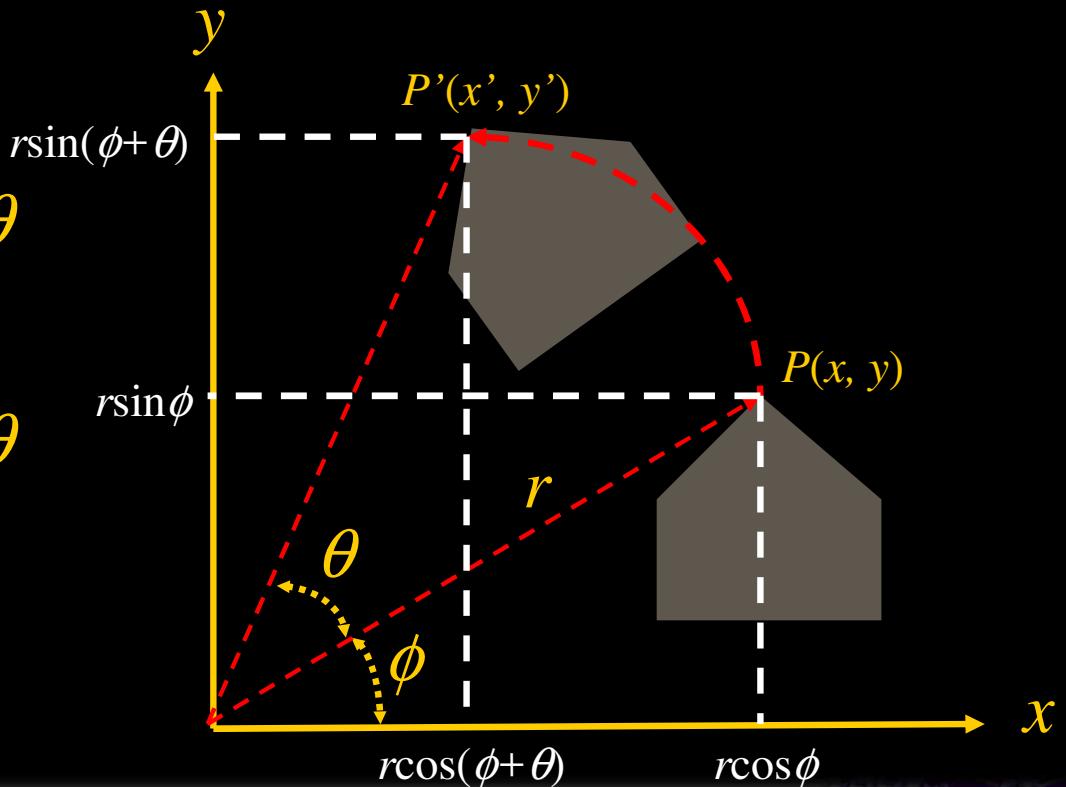
$$= r \cdot \cos \phi \cdot \cos \theta - r \cdot \sin \phi \cdot \sin \theta$$

$$y' = r \cdot \sin(\phi + \theta)$$

$$= r \cdot \cos \phi \cdot \sin \theta + r \cdot \sin \phi \cdot \cos \theta$$

$$x' = x \cdot \cos \theta - y \cdot \sin \theta$$

$$y' = x \cdot \sin \theta + y \cdot \cos \theta$$



# *Homogeneous Coordinates*

- ◆ Matrix representations for translation, scaling, and rotation

$$P' = T + P$$

Translation

$$P' = S \cdot P$$

Scaling

$$P' = R \cdot P$$

Rotation

- ◆ How to derive a consistent way to represent all the transformations?

***Homogeneous coordinate representation***



# *Homogeneous Coordinates*

- ◆ A point  $(x, y)$  in 2D coordinates is represented by  $(x, y, W)$  in homogeneous coordinates
- ◆ Two homogeneous coordinates,  $(x, y, W)$  and  $(x', y', W')$ , represent the same point if and only if  $(x', y', W') = (tx, ty, tW)$  for some  $t \neq 0$
- ◆ Homogeneous coordinates to Cartesian Coordinates ( $W \neq 0$ )

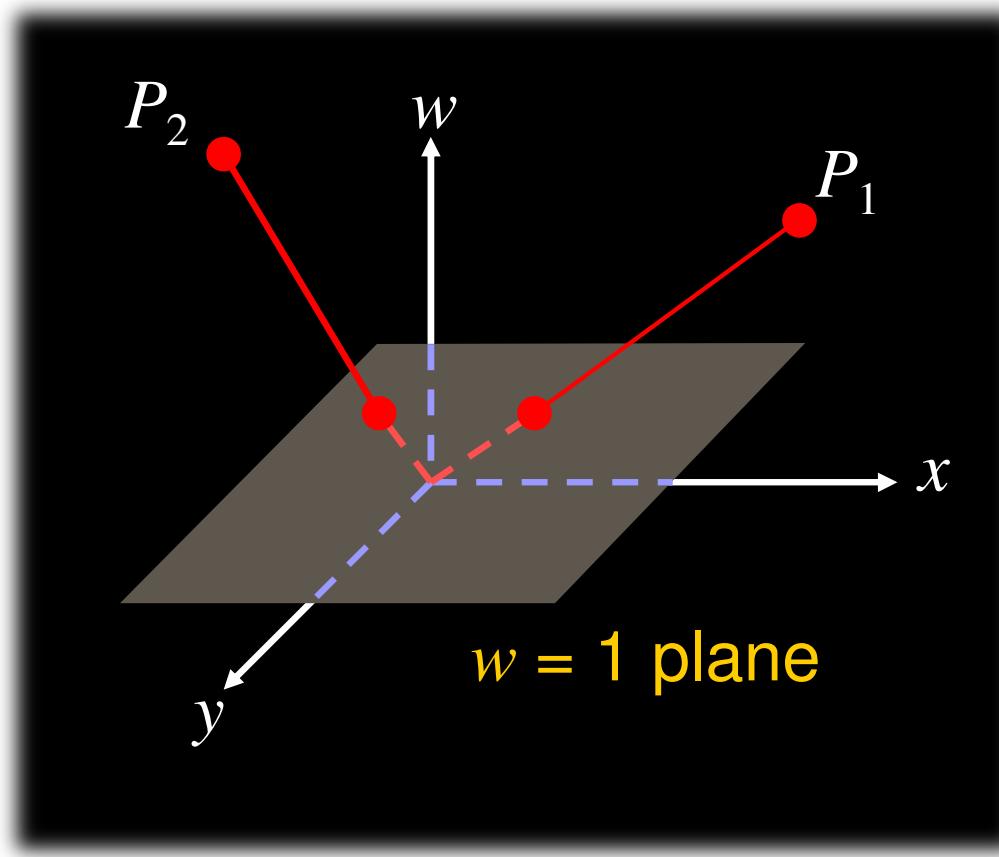
$$(x, y, W) = (\textcolor{red}{x/W}, \textcolor{red}{y/W}, 1)$$

$$(x', y', W') = (x'/W', y'/W', 1) = (\textcolor{red}{x/W}, \textcolor{red}{y/W}, 1)$$



# *Homogeneous Coordinates*

- ◆ All triples of the form  $(tx, ty, tW)$ , with  $t \neq 0$ , get a line in homogeneous coordinates



# *Translation in Homogeneous Rep.*

## ◆ 3x3 matrix form

$$\begin{aligned}x' &= x + d_x \\y' &= y + d_y\end{aligned}\quad \xrightarrow{\text{---}} \quad \begin{bmatrix}x' \\ y'\end{bmatrix} = \begin{bmatrix}d_x \\ d_y\end{bmatrix} + \begin{bmatrix}x \\ y\end{bmatrix} \quad \xrightarrow{\text{---}} \quad \begin{bmatrix}x' \\ y' \\ 1\end{bmatrix} = \begin{bmatrix}1 & 0 & d_x \\ 0 & 1 & d_y \\ 0 & 0 & 1\end{bmatrix} \cdot \begin{bmatrix}x \\ y \\ 1\end{bmatrix}$$

$$P' = T(d_x, d_y) \cdot P, \quad T(d_x, d_y) = \begin{bmatrix}1 & 0 & d_x \\ 0 & 1 & d_y \\ 0 & 0 & 1\end{bmatrix}$$



# Scaling in Homogeneous Rep.

## ◆ 3x3 matrix form

$$\begin{aligned}x' &= s_x \cdot x \\y' &= s_y \cdot y\end{aligned}\quad \text{-----} \rightarrow \begin{bmatrix}x' \\ y'\end{bmatrix} = \begin{bmatrix}s_x & 0 \\ 0 & s_y\end{bmatrix} \cdot \begin{bmatrix}x \\ y\end{bmatrix} \quad \text{-----} \rightarrow \begin{bmatrix}x' \\ y' \\ 1\end{bmatrix} = \begin{bmatrix}s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1\end{bmatrix} \cdot \begin{bmatrix}x \\ y \\ 1\end{bmatrix}$$

$$P' = S(s_x, s_y) \cdot P, \quad S(s_x, s_y) = \begin{bmatrix}s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1\end{bmatrix}$$



# *Rotation in Homogeneous Rep.*

## ◆ 3x3 matrix form

$$x' = x \cdot \cos \theta - y \cdot \sin \theta$$

$$y' = x \cdot \sin \theta + y \cdot \cos \theta$$

$$\begin{matrix} \xrightarrow{\text{---}} & \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} \end{matrix}$$

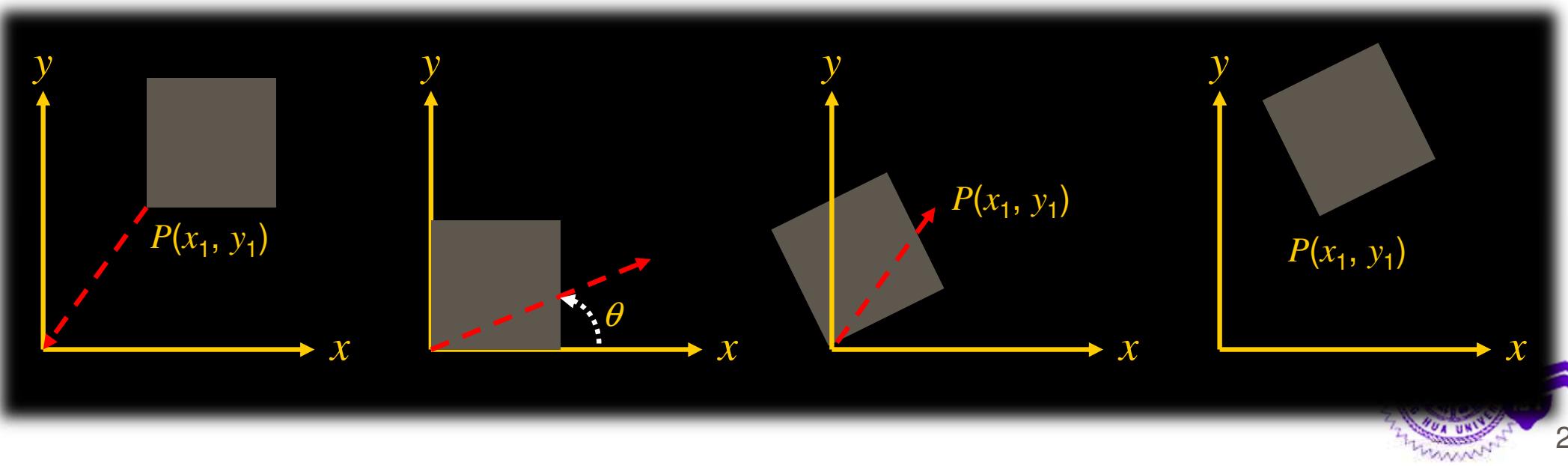
$$\begin{matrix} \xrightarrow{\text{---}} & \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \end{matrix}$$

$$P' = R(\theta) \cdot P, \quad R(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



# *Composition of 2D Transformations*

- ◆ Rotate an object about an arbitrary point  $P = (x_1, y_1)$ 
  - Translate from  $P$  to origin
  - Rotate by  $\theta$  degree
  - Translate back from origin to  $P$



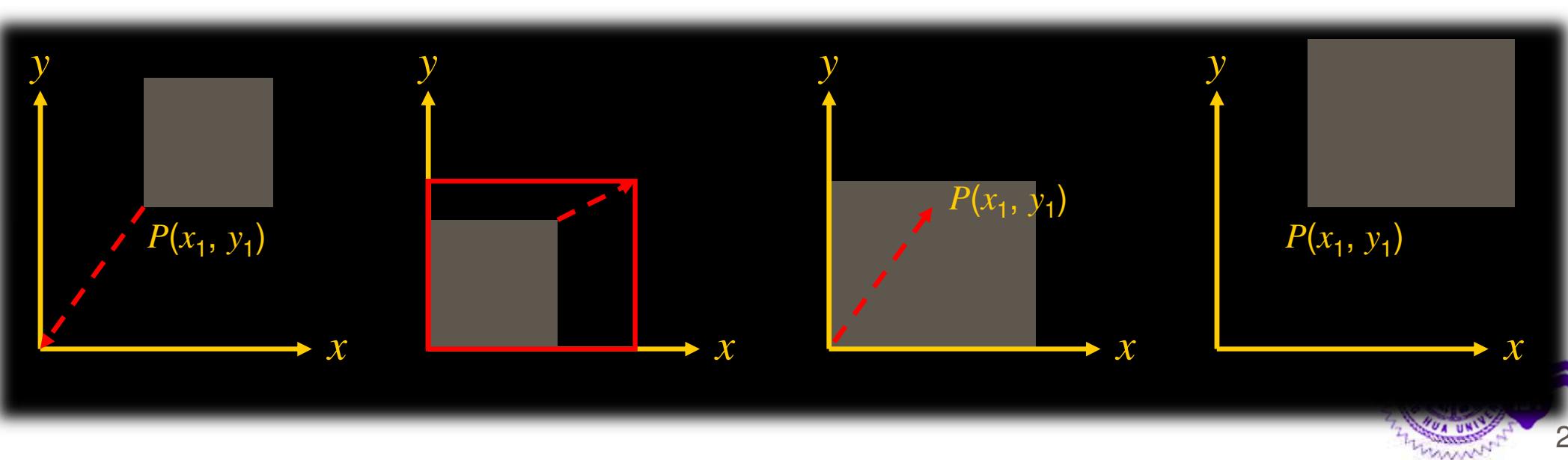
# *Composition of 2D Transformations*

- ◆ Rotate an object about an arbitrary point  $P = (x_1, y_1)$ 
  - Translate from  $P$  to origin,  $T(-x_1, -y_1)$
  - Rotate by  $\theta$  degree,  $R(\theta)$
  - Translate back from origin to  $P$ ,  $T(x_1, y_1)$

$$T(x_1, y_1) \cdot R(\theta) \cdot T(-x_1, -y_1) = \begin{bmatrix} 1 & 0 & x_1 \\ 0 & 1 & y_1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -x_1 \\ 0 & 1 & -y_1 \\ 0 & 0 & 1 \end{bmatrix}$$

# *Composition of 2D Transformations*

- ◆ Scale an object about an arbitrary point  $P = (x_1, y_1)$ 
  - Translate from  $P$  to origin
  - Scale by  $(s_x, s_y)$
  - Translate back from origin to  $P$



# *Composition of 2D Transformations*

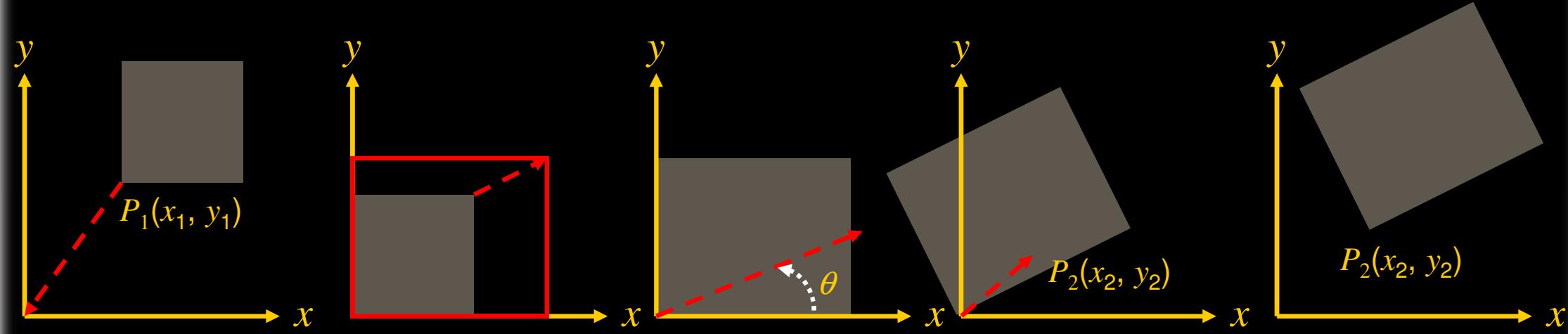
- ◆ Scale an object about an arbitrary point  $P = (x_1, y_1)$ 
  - Translate from  $P$  to origin,  $T(-x_1, -y_1)$
  - Scale by  $(s_x, s_y)$ ,  $S(s_x, s_y)$
  - Translate back from origin to  $P$ ,  $T(x_1, y_1)$

$$T(x_1, y_1) \cdot S(s_x, s_y) \cdot T(-x_1, -y_1) = \begin{bmatrix} 1 & 0 & x_1 \\ 0 & 1 & y_1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -x_1 \\ 0 & 1 & -y_1 \\ 0 & 0 & 1 \end{bmatrix}$$

# *Composition of 2D Transformations*

- ◆ Scale and rotate about  $P_1 = (x_1, y_1)$  and translate to point  $P_2 = (x_2, y_2)$

$$T(x_2, y_2) \cdot R(\theta) \cdot S(s_x, s_y) \cdot T(-x_1, -y_1)$$



# *Composition of 2D Transformations*

- ◆ Scale and rotate about  $P_1 = (x_1, y_1)$  and translate to point  $P_2 = (x_2, y_2)$ 
  - Translate from  $P_1$  to origin,  $T(-x_1, -y_1)$
  - Scale by  $(s_x, s_y)$ ,  $S(s_x, s_y)$
  - Rotate by  $\theta$  degree,  $R(\theta)$
  - Translate from origin to  $P_2$ ,  $T(x_2, y_2)$

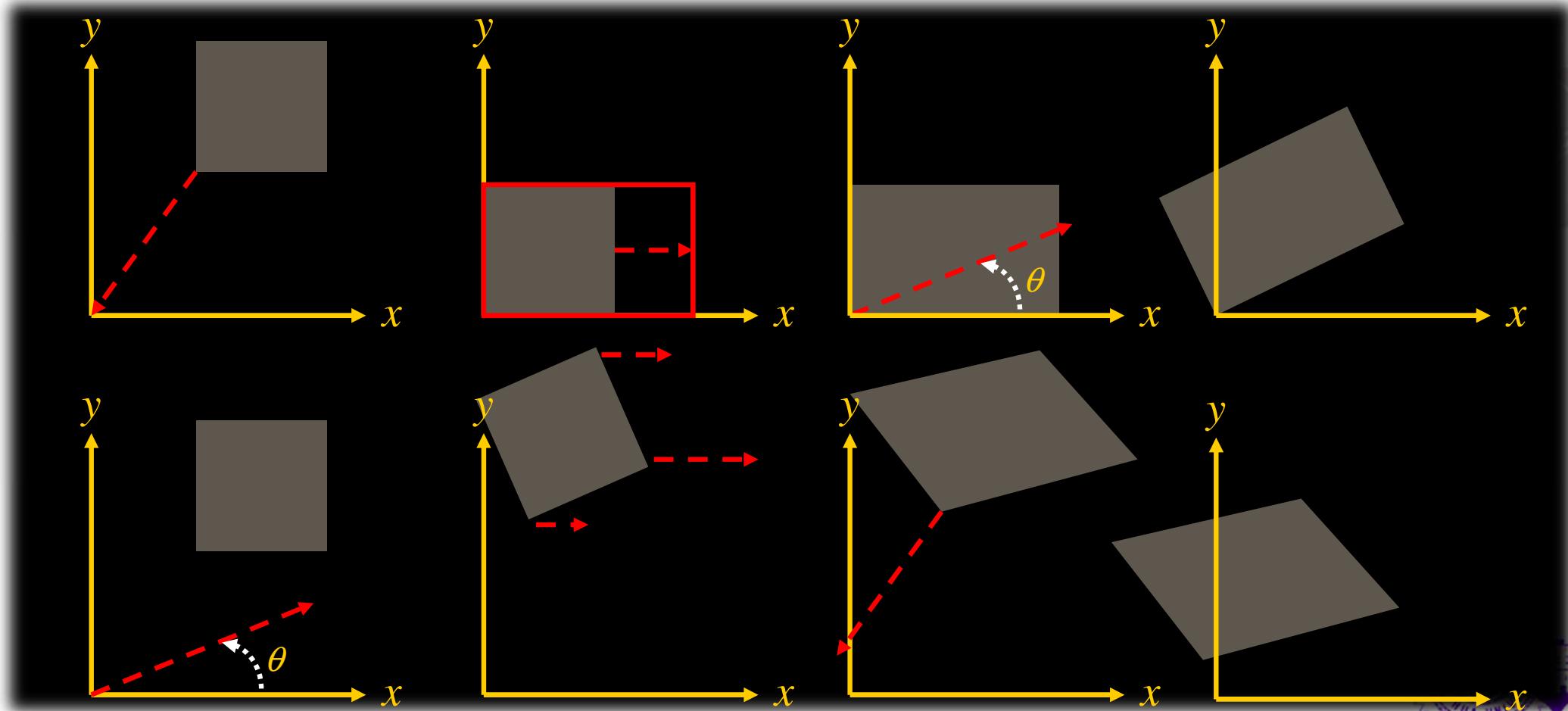
$$T(x_2, y_2) \cdot R(\theta) \cdot S(s_x, s_y) \cdot T(-x_1, -y_1)$$



# *Composition of 2D Transformations*

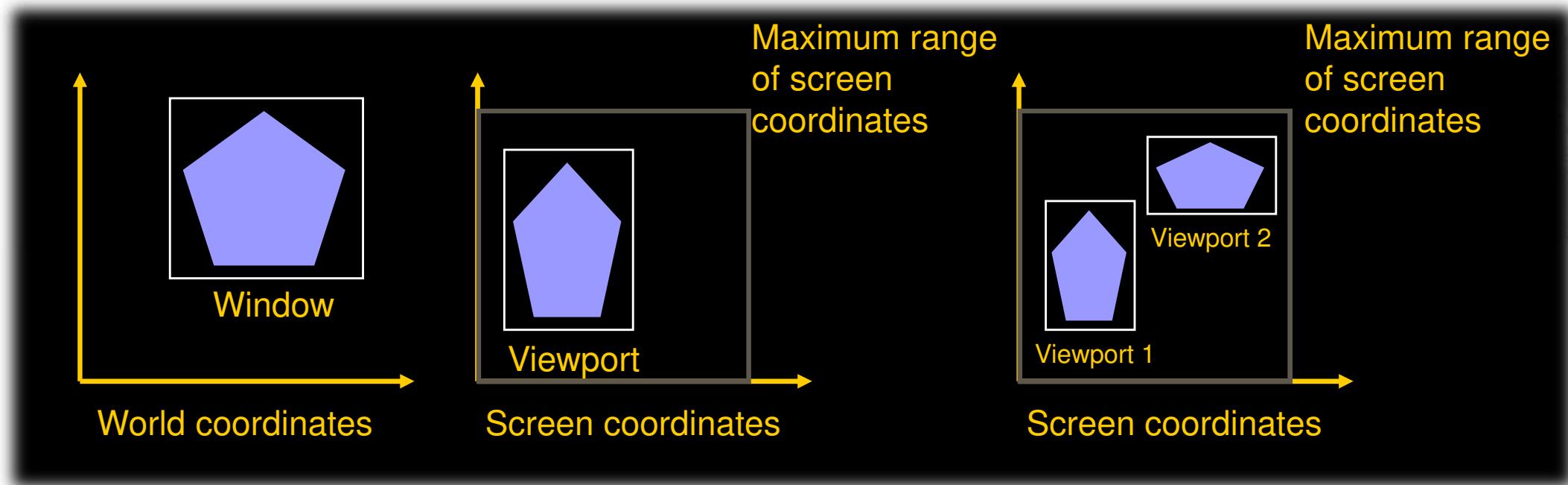
## ◆ Commutativity

- $R(\theta) \cdot S(s_x, s_y) \cdot T(d_x, d_y) \neq T(d_x, d_y) \cdot S(s_x, s_y) \cdot R(\theta)$



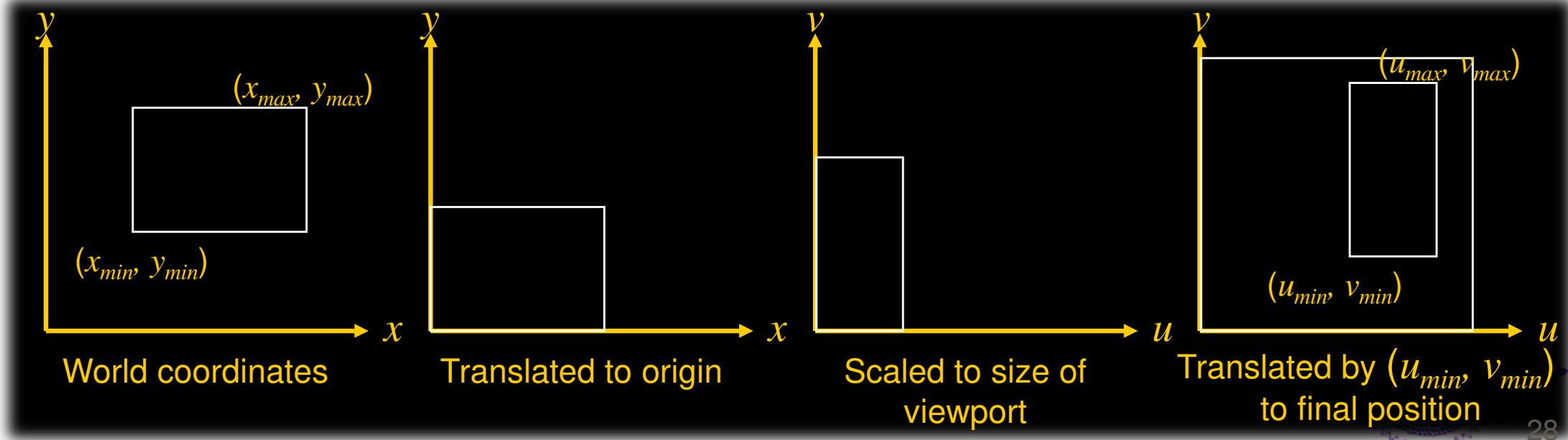
# *Window to Viewport Transformation*

## ◆ Application of 2D Transformation



# *Window to Viewport Transformation*

- ◆ Steps in transforming a world-coordinate window into a viewport
  - Window translated to origin
  - Window scaled to size of viewport
  - Translated by  $(u_{min}, v_{min})$  to final position



# *Window to Viewport Transformation*

- ◆ Composition matrix in transforming a world coordinate window into a viewport

$$M_{WV} = T(u_{\min}, v_{\min}) \cdot S\left(\frac{u_{\max} - u_{\min}}{x_{\max} - x_{\min}}, \frac{v_{\max} - v_{\min}}{y_{\max} - y_{\min}}\right) \cdot T(-x_{\min}, -y_{\min})$$



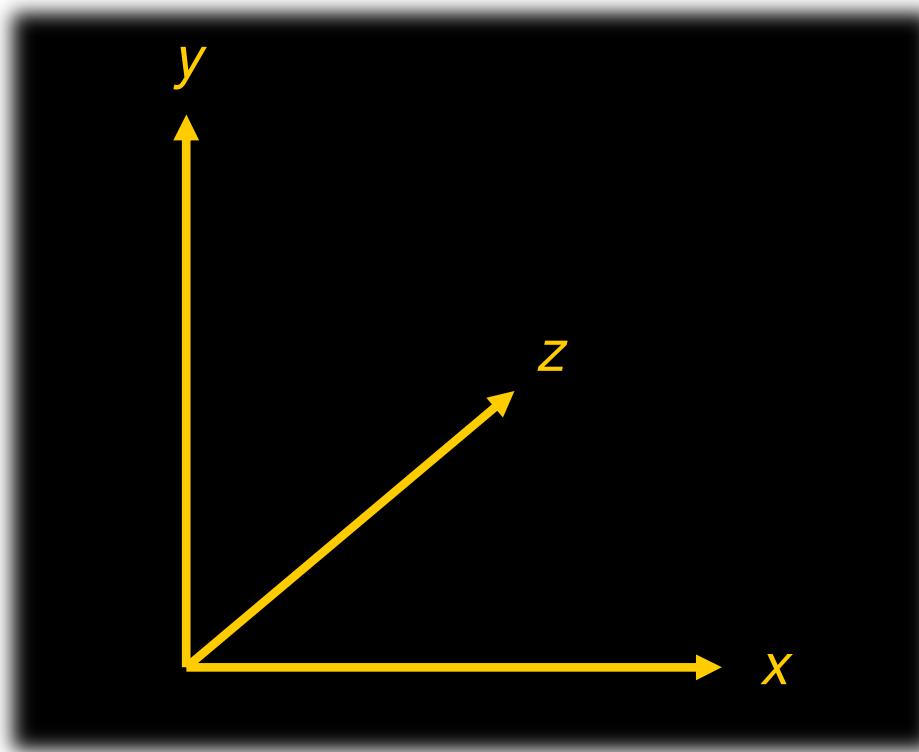
# *3D Geometrical Transformations*

*Similar to 2D Geometrical Transformations*



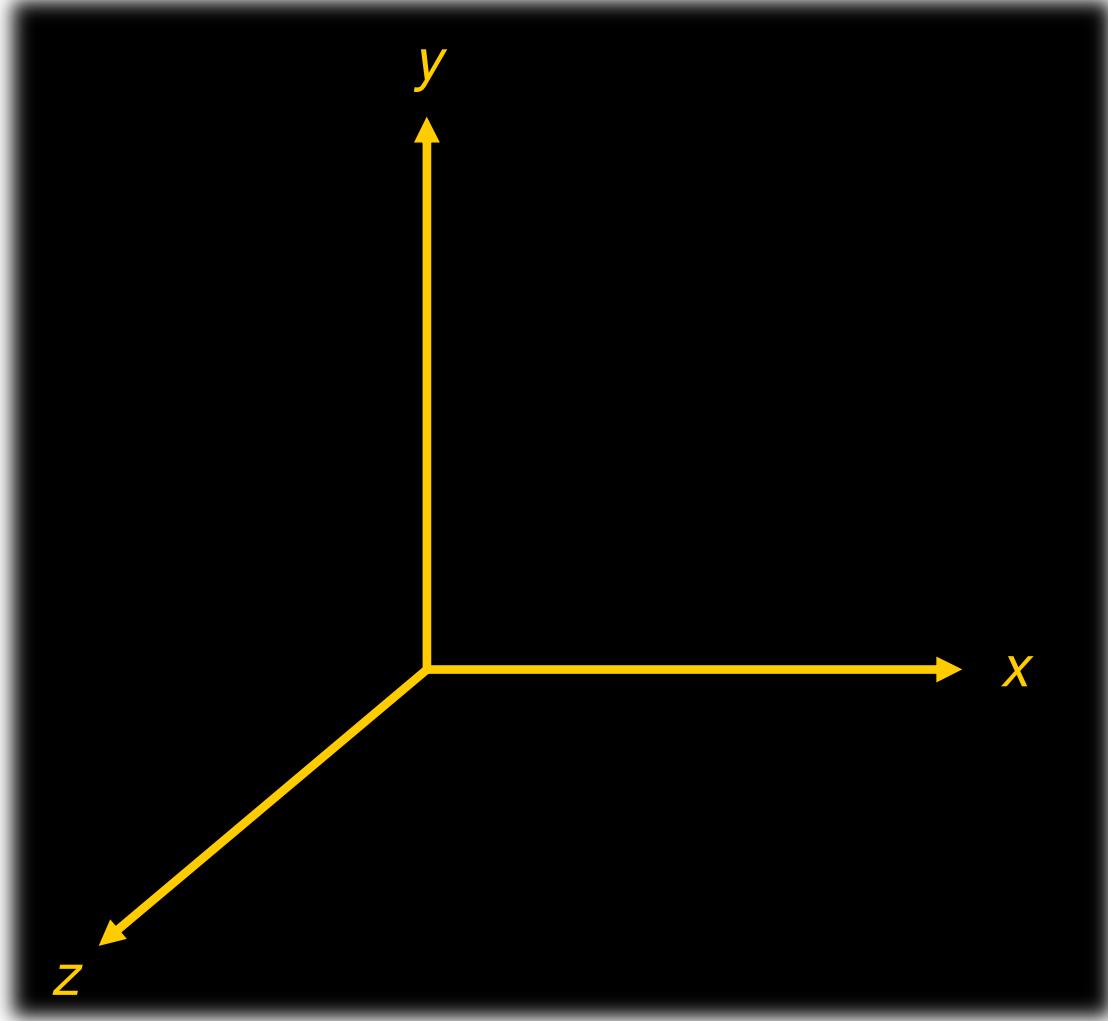
# *3D Coordinate Systems*

- ◆ **Left-hand Coordinate Systems**
  - Direct3D



# *3D Coordinate Systems*

- ◆ Right-hand Coordinate Systems
  - OpenGL



# 3D Transformation

## ◆ Translation

$P(x, y, z)$  move to  $P'(x', y', z')$

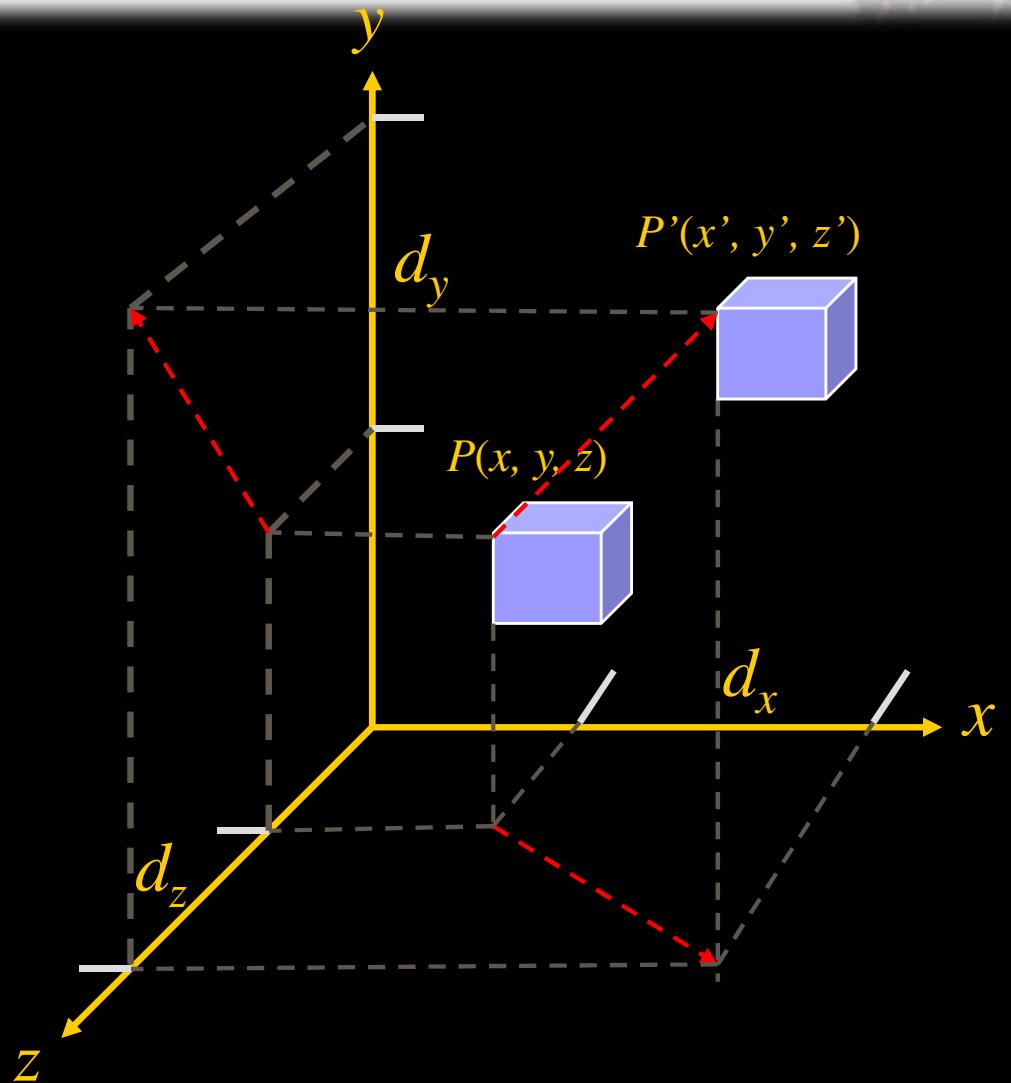
$$x' = x + d_x$$

$$y' = y + d_y$$

$$z' = z + d_z$$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix} + \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$P' = T(d_x, d_y, d_z) + P$$



# 3D Transformation

## ◆ Scaling

Scale by  $s_x$ ,  $s_y$ , and  $s_z$  with respect to the  $x$  axis,  $y$  axis, and  $z$  axis

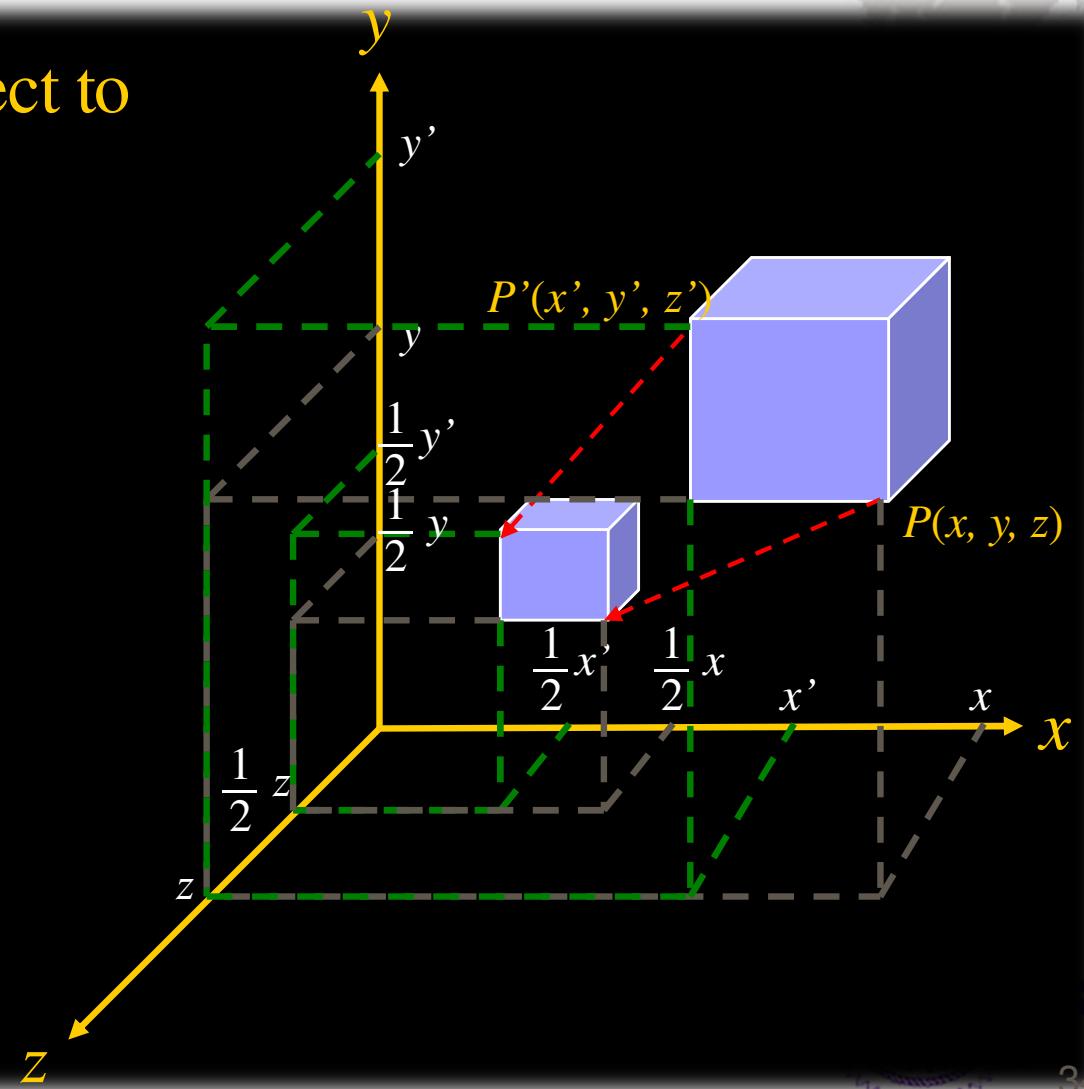
$$x' = s_x \cdot x$$

$$y' = s_y \cdot y$$

$$z' = s_z \cdot z$$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$P' = S(s_x, s_y, s_z) \cdot P$$



# 3D Transformation

## ◆ Rotation in $x$ -axis

Rotate an angle  $\theta$  in  $x$ -axis

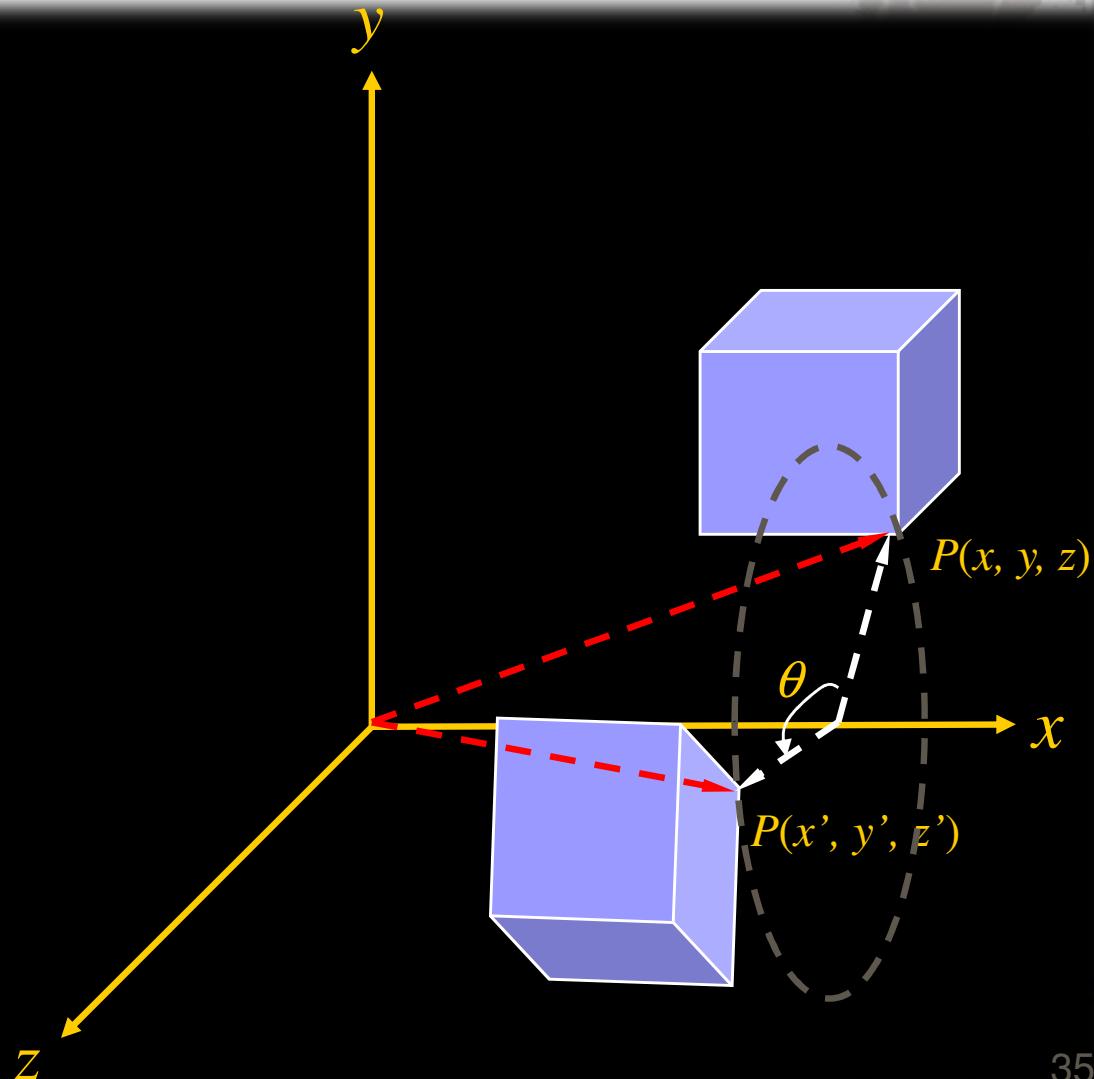
$$x' = x$$

$$y' = y \cdot \cos \theta - z \cdot \sin \theta$$

$$z' = y \cdot \sin \theta + z \cdot \cos \theta$$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$P' = R_x(\theta) \cdot P$$



# 3D Transformation

## ◆ Rotation in y-axis

Rotate an angle  $\theta$  in  $y$ -axis

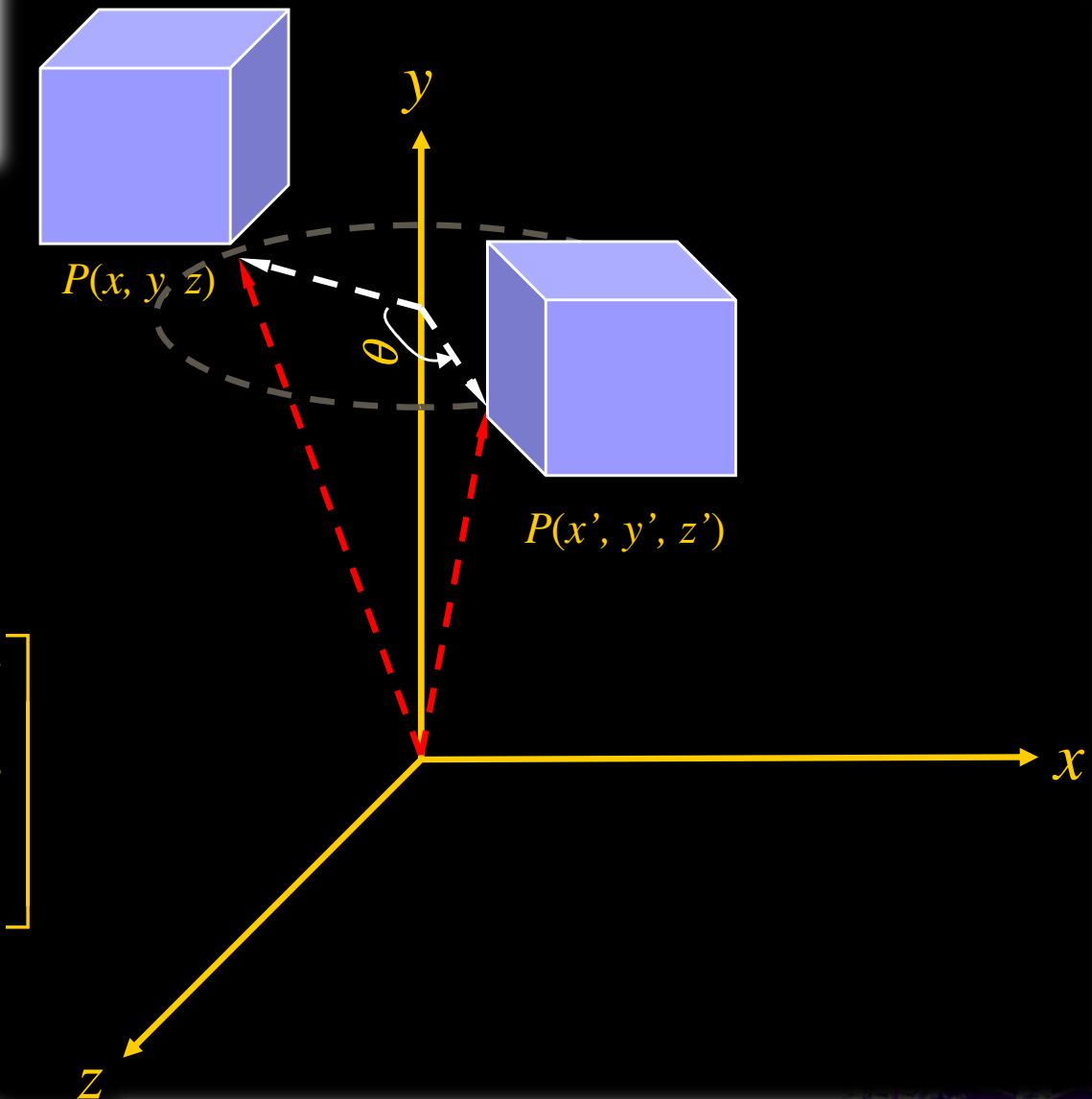
$$x' = x \cdot \cos \theta + z \cdot \sin \theta$$

$$y' = y$$

$$z' = -x \cdot \sin \theta + z \cdot \cos \theta$$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$P' = R_y(\theta) \cdot P$$



# 3D Transformation

## ◆ Rotation in $z$ -axis

Rotate an angle  $\theta$  in  $z$ -axis

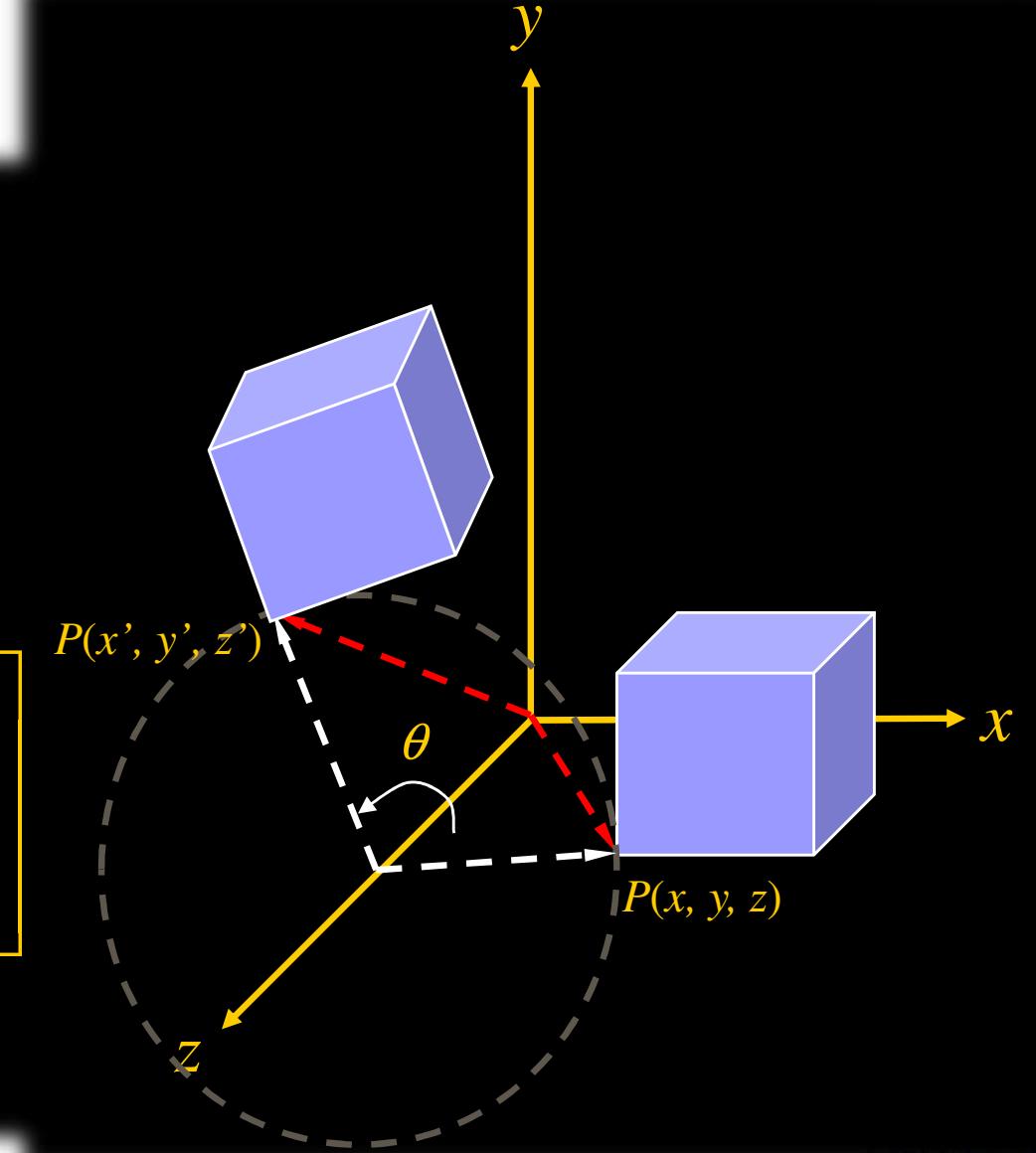
$$x' = x \cdot \cos \theta - y \cdot \sin \theta$$

$$y' = x \cdot \sin \theta + y \cdot \cos \theta$$

$$z' = z$$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$P' = R_z(\theta) \cdot P$$



# *Homogeneous Coordinates*

- ◆ A point  $(x, y, z)$  in 3D coordinates is represented by  $(x, y, z, W)$  in homogeneous coordinates
- ◆ Two homogeneous coordinates  $(x, y, z, W)$  and  $(x', y', z', W')$  represent the same point if and only if  $(x', y', z', W') = (tx, ty, tz, tW)$  for some  $t \neq 0$
- ◆ Cartesian Coordinates

$$(x, y, z, W) = (\textcolor{red}{x/W}, \textcolor{red}{y/W}, \textcolor{red}{z/W}, 1)$$

$$(x', y', z', W') = (x'/W', y'/W', z'/W', 1) = (\textcolor{red}{x/W}, \textcolor{red}{y/W}, \textcolor{red}{z/W}, 1)$$

# *Translation in Homogeneous Rep.*

## ◆ 4x4 matrix form

$$x' = x + d_x$$

$$y' = y + d_y \quad \text{---->}$$

$$z' = z + f_z$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} d_x \\ d_y \\ d_z \\ 1 \end{bmatrix} + \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\text{---->} \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$P' = T(d_x, d_y, d_z) \cdot P, \quad T(d_x, d_y, d_z) = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# Scaling in Homogeneous Rep.

## ◆ 4x4 matrix form

$$\begin{aligned}x' &= s_x \cdot x \\y' &= s_y \cdot y \\z' &= s_z \cdot z\end{aligned}\quad \text{-----} \rightarrow \begin{bmatrix}x' \\ y' \\ z'\end{bmatrix} = \begin{bmatrix}s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z\end{bmatrix} \cdot \begin{bmatrix}x \\ y \\ z\end{bmatrix}$$

$$\text{-----} \rightarrow \begin{bmatrix}x' \\ y' \\ z' \\ 1\end{bmatrix} = \begin{bmatrix}s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1\end{bmatrix} \cdot \begin{bmatrix}x \\ y \\ z \\ 1\end{bmatrix} \quad P' = S(s_x, s_y, s_z) \cdot P$$

# *Rotation in Homogeneous Rep.*

## ◆ 4x4 matrix form (Rotate in $x$ -axis)

$$x' = x$$

$$y' = y \cdot \cos \theta - z \cdot \sin \theta$$

$$z' = y \cdot \sin \theta + z \cdot \cos \theta$$

$$\dashrightarrow \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\dashrightarrow \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta & 0 \\ 0 & \sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$P' = R_x(\theta) \cdot P$$

# *Rotation in Homogeneous Rep.*

## ◆ 4x4 matrix form (Rotate in y-axis)

$$x' = x \cdot \cos \theta + z \cdot \sin \theta$$

$$y' = y$$

$$z' = -x \cdot \sin \theta + z \cdot \cos \theta$$

$$\dashrightarrow \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\dashrightarrow \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos \theta & 0 & \sin \theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin \theta & 0 & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$P' = R_y(\theta) \cdot P$$

# *Rotation in Homogeneous Rep.*

## ◆ 4x4 matrix form (Rotate in $z$ -axis)

$$\begin{aligned}x' &= x \cdot \cos \theta - y \cdot \sin \theta \\y' &= x \cdot \sin \theta + y \cdot \cos \theta \\z' &= z\end{aligned}\quad \dashrightarrow \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta & 0 & 0 \\ \sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\dashrightarrow \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta & 0 & 0 \\ \sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$P' = R_z(\theta) \cdot P$$

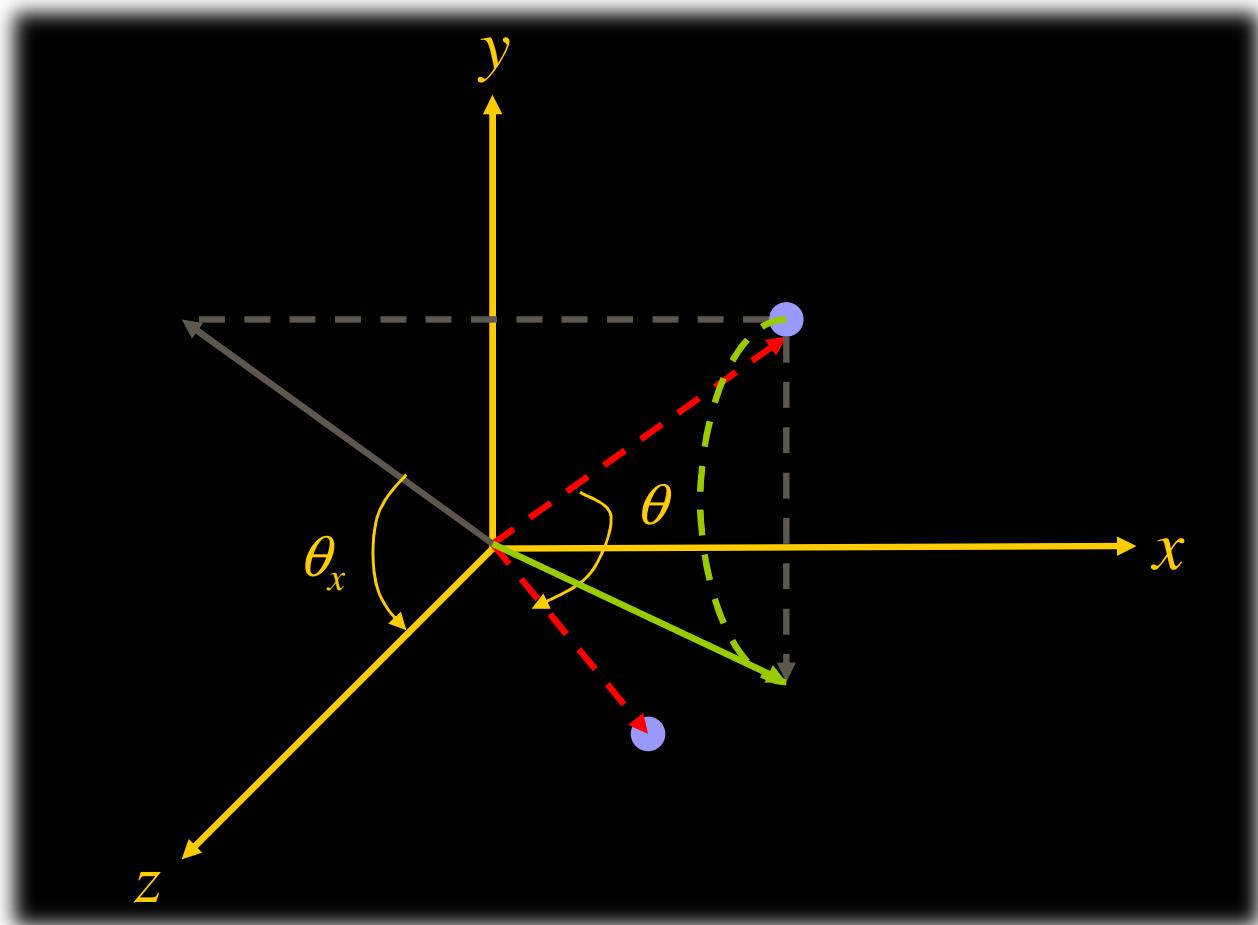
# *Composition of 3D Transformations*

*Rotation about the origin*  
*Rotation about an arbitrary point*  
*Rotation about an arbitrary axis*



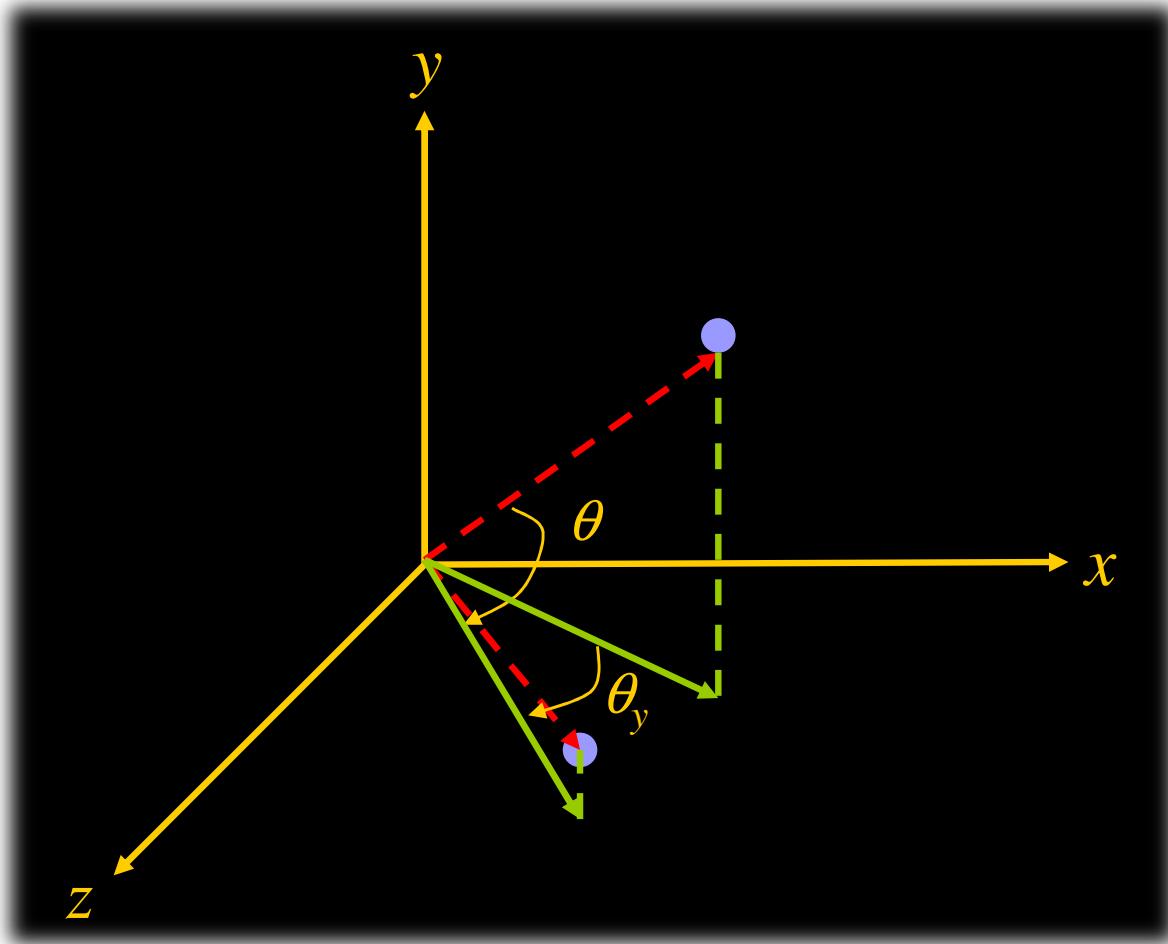
# *Composition of 3D Transformations*

- ◆ Rotation about the origin by  $\theta$  degree
  - Step 1: Rotate in  $x$ -axis by  $\theta_x$  degree



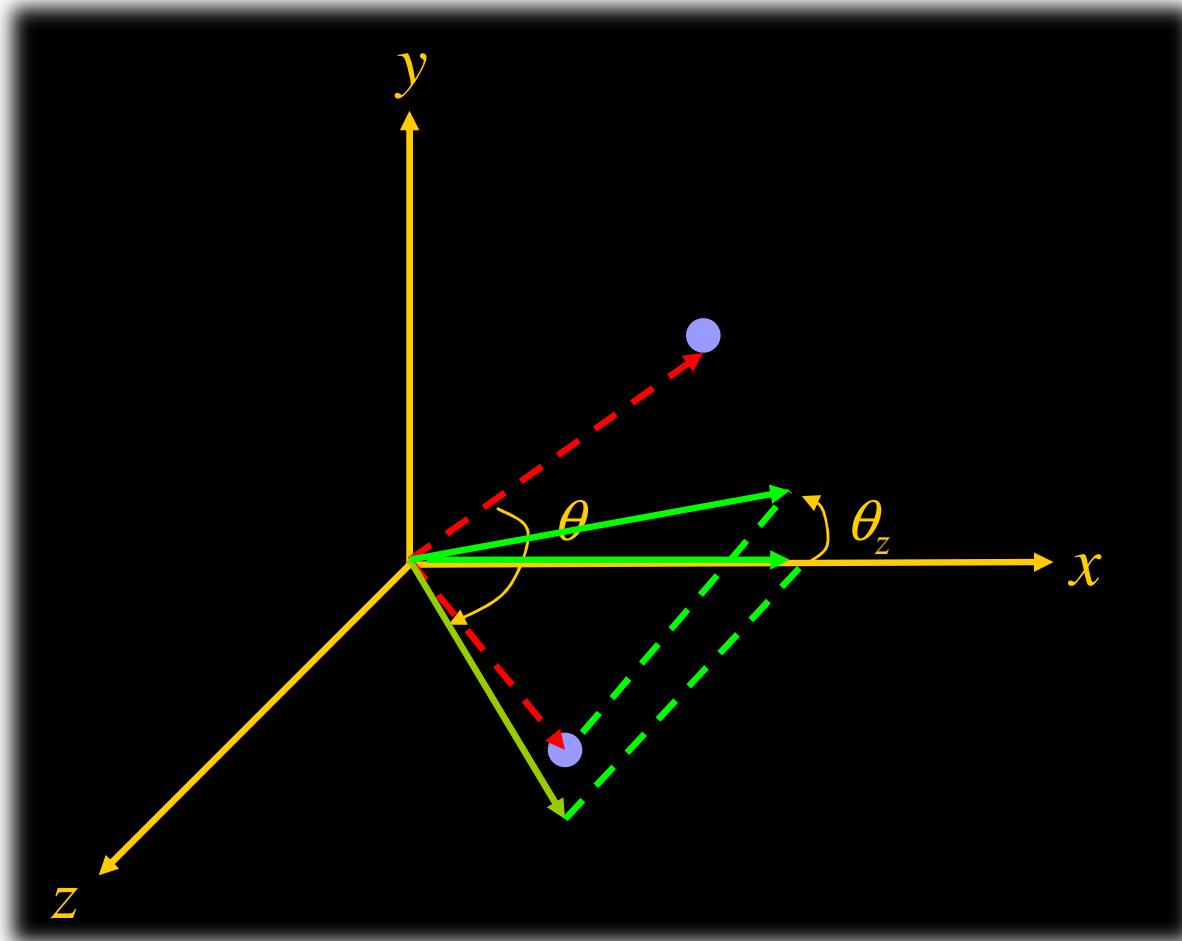
# *Composition of 3D Transformations*

- ◆ Rotation about the origin by  $\theta$  degree
  - Step 2: Rotate in  $y$ -axis by  $\theta_y$  degree



# *Composition of 3D Transformations*

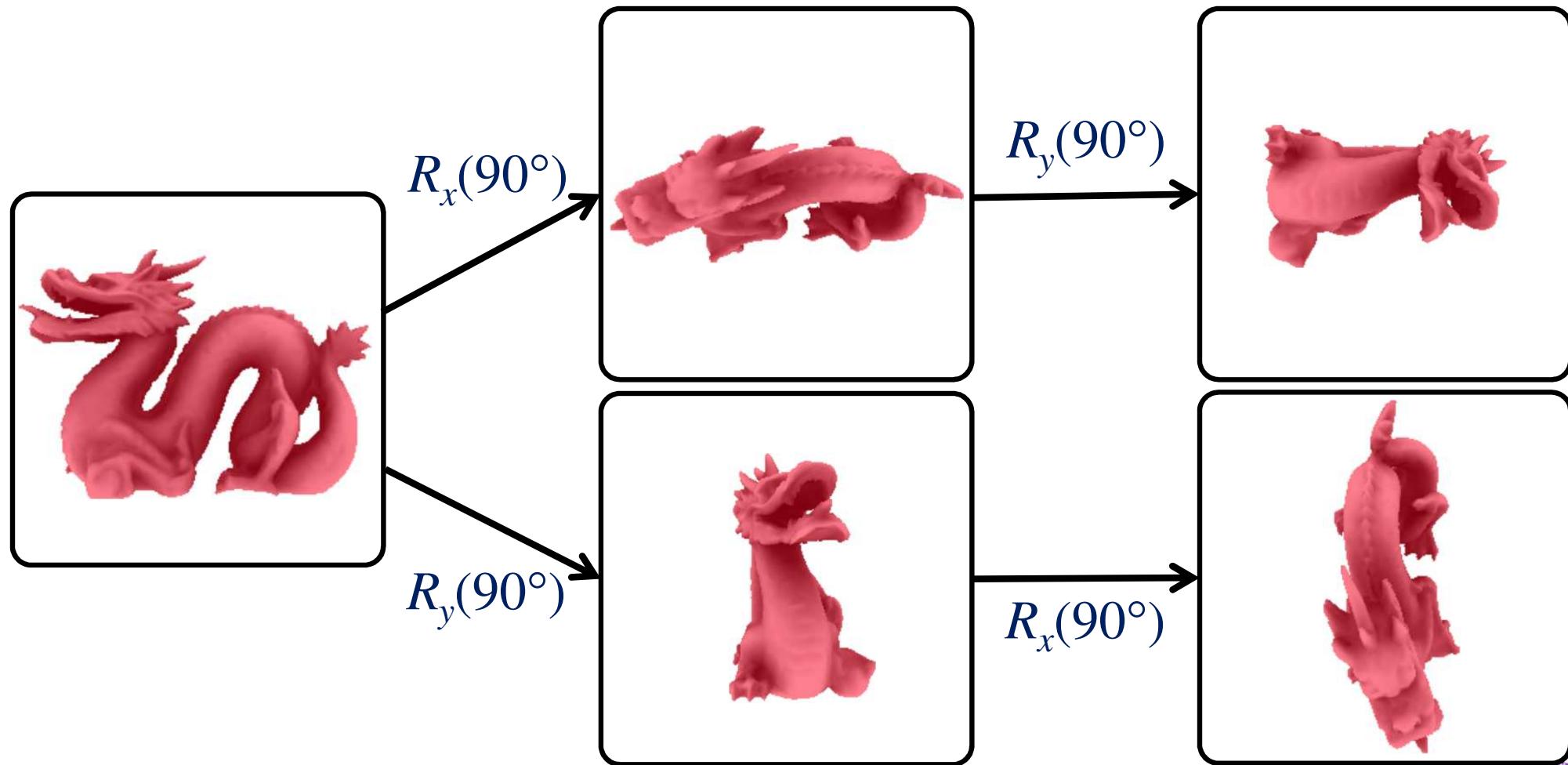
- ◆ Rotation about the origin by  $\theta$  degree
  - Step 3: Rotate in  $z$ -axis by  $\theta_z$  degree



# *Composition of 3D Transformations*

- ◆ Commutative property is not held for rotation

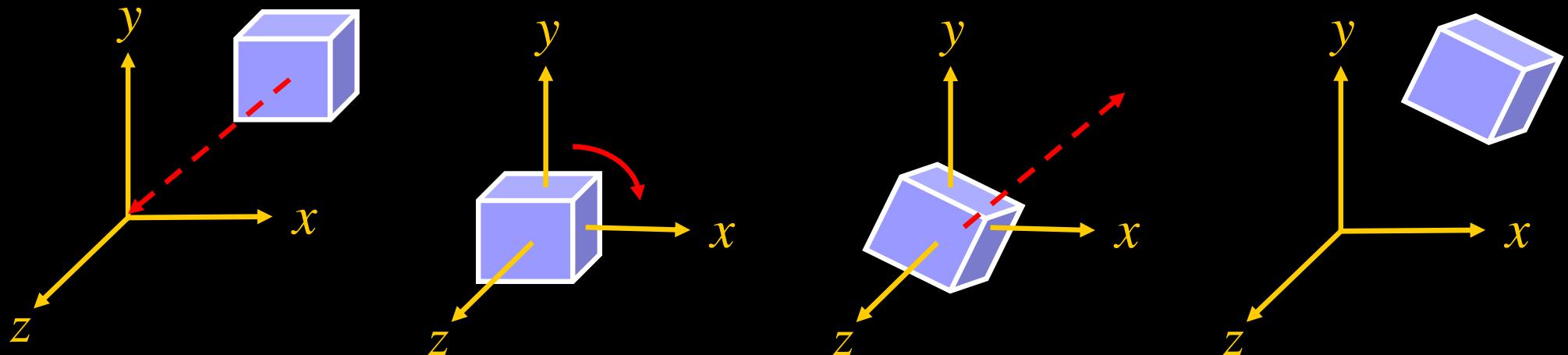
$$R_y(\theta_y) \cdot R_x(\theta_x) \neq R_x(\theta_x) \cdot R_y(\theta_y)$$



# *Composition of 3D Transformations*

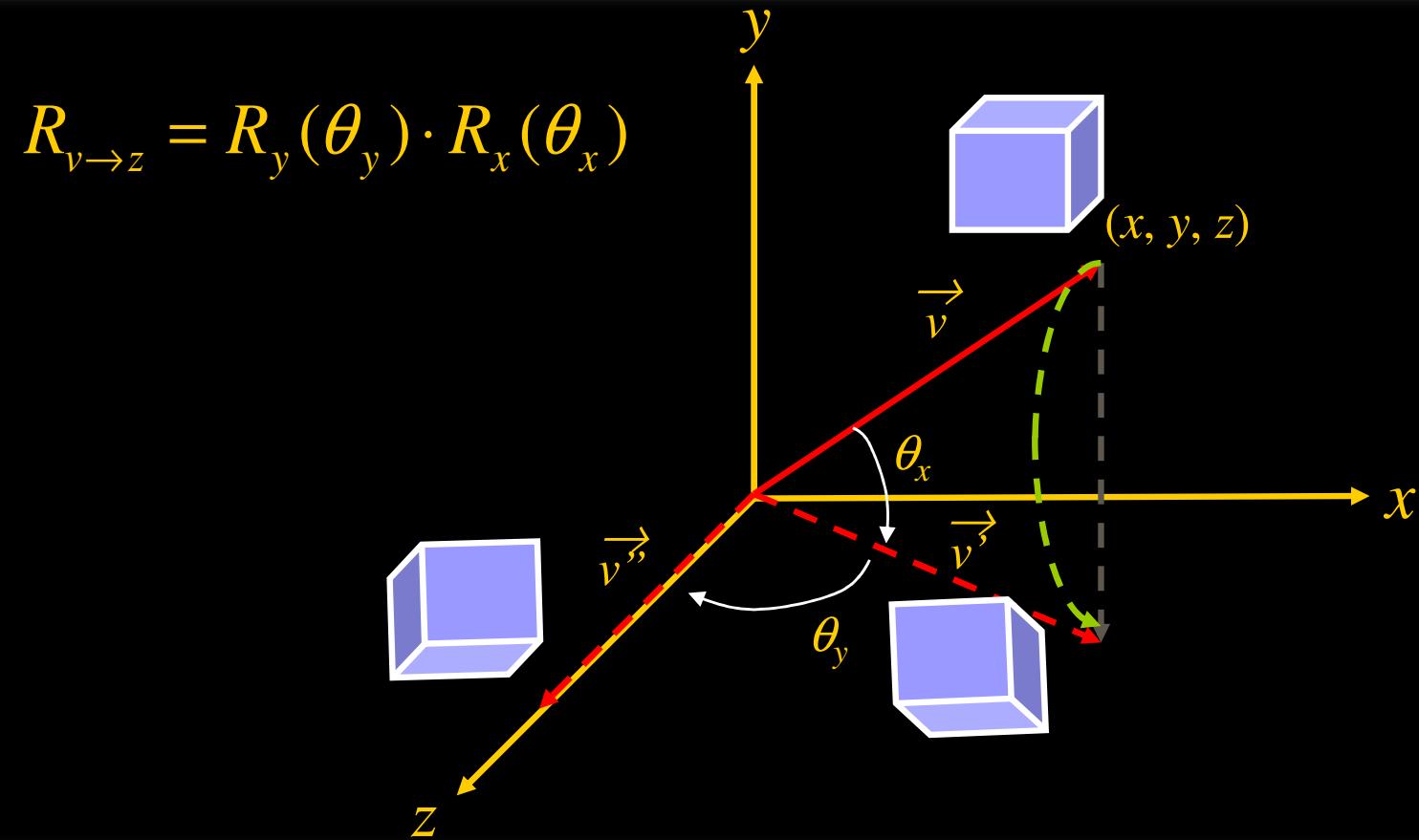
- ◆ Rotation about an arbitrary point  $P_0(x, y, z)$ 
  - Translate from  $P_0$  to origin
  - Rotate by  $\theta$  degree //  $R(\theta) = R_x(\theta_x) \cdot R_y(\theta_y) \cdot R_z(\theta_z)$
  - Translate from origin back to  $P_0$

$$P' = (T(x, y, z) \cdot R(\theta) \cdot T(-x, -y, -z)) \cdot P$$



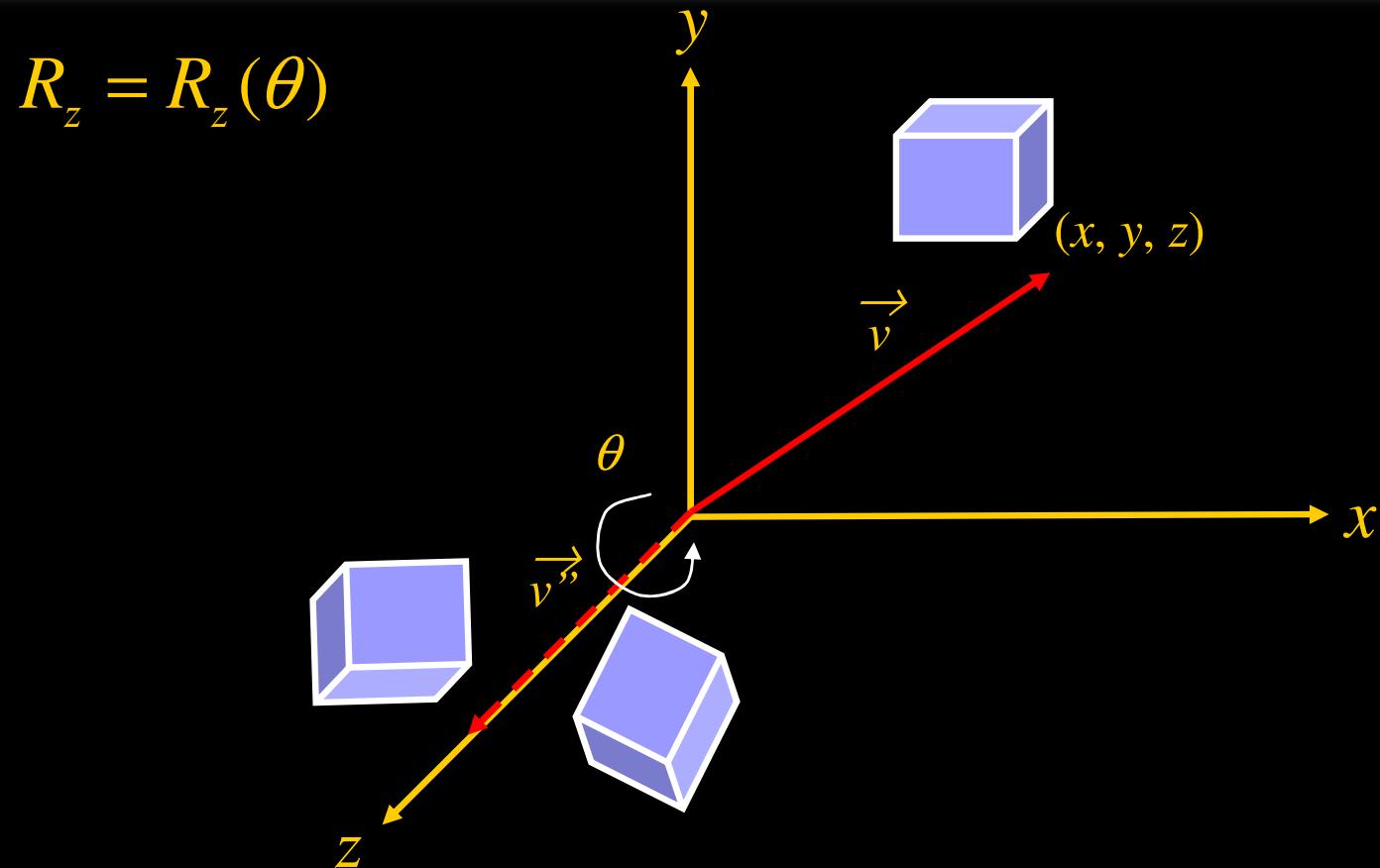
# *Composition of 3D Transformations*

- ◆ Rotation about an arbitrary vector  $\nu$ 
  - Step 1: Rotate the vector  $\nu$  to align with  $z$ -axis



# *Composition of 3D Transformations*

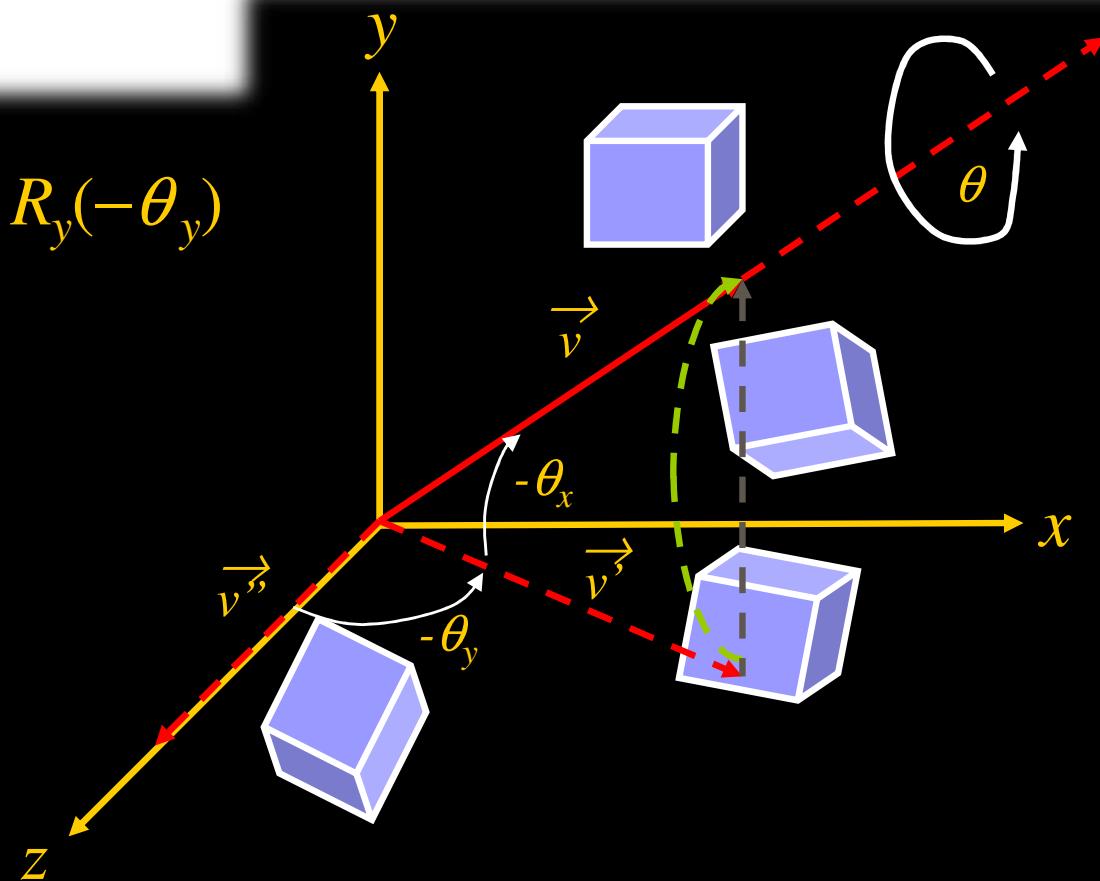
- ◆ Rotation about an arbitrary vector  $\nu$ 
  - Step 2: Rotate in  $z$  by  $\theta$  degree



# *Composition of 3D Transformations*

- ◆ Rotation about an arbitrary vector  $\nu$ 
  - Step 3: Rotate the  $z$ -axis back to align with vector  $\nu$

$$R_{z \rightarrow \nu} = R_x(-\theta_x) \cdot R_y(-\theta_y)$$



# *Composition of 3D Transformations*

- ◆ Rotation about an arbitrary vector  $\nu$ 
  - Step 1: Rotate the vector  $\nu$  to align with  $z$ -axis
  - Step 2: Rotate in  $z$  by  $\theta$  degree
  - Step 3: Rotate the  $z$ -axis back to align with vector  $\nu$

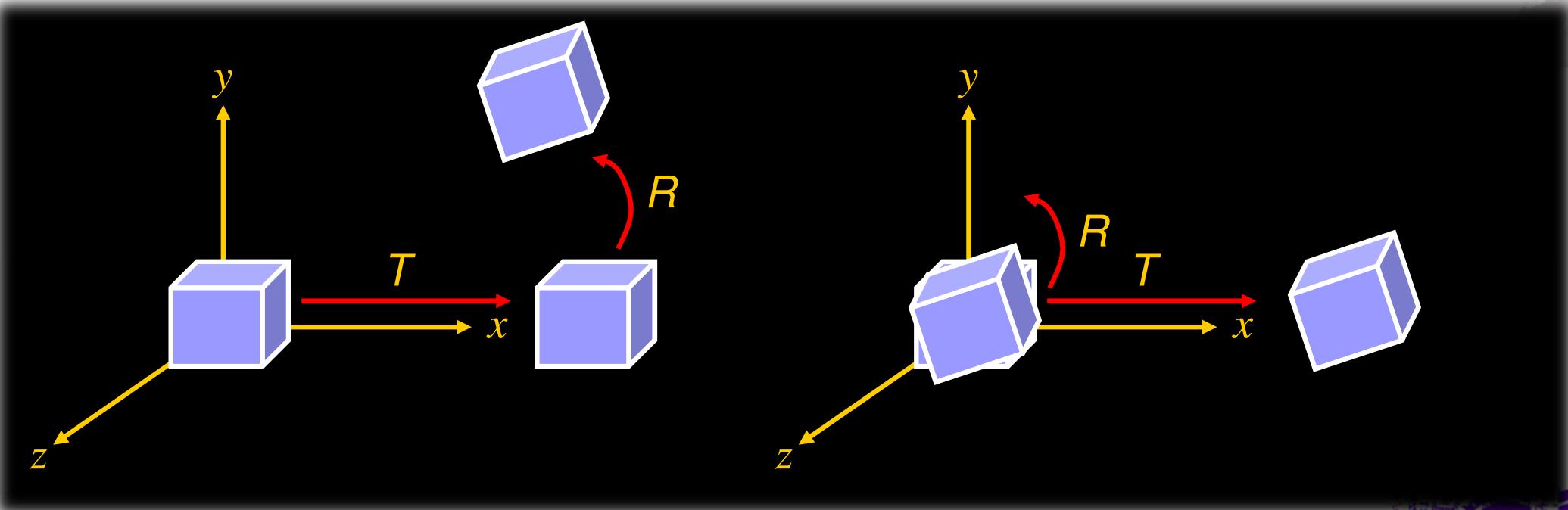
$$P' = \underbrace{(((R_x(-\theta_x) \cdot R_y(-\theta_y)) \cdot R_z(\theta)) \cdot (R_y(\theta_y) \cdot R_x(\theta_x)))}_{\text{Step 3}} \cdot P$$
$$\qquad\qquad\qquad \underbrace{\qquad\qquad\qquad}_{\text{Step 2}} \qquad\qquad\qquad \underbrace{\qquad\qquad\qquad}_{\text{Step 1}}$$



# *Composition of 3D Transformations*

## ◆ Commutative Property

- Commutative property is not held for matrix composition
- e.g.  $T \cdot R \neq R \cdot T$



# *Application Order of Transformation*

- ◆ Think of the *model* transformation in reverse order to get the right sequence of transformation in OpenGL
  - *An example of OpenGL matrix multiplication*

```
glMatrixMode(GL_MODELVIEW) ;  
glLoadIdentity() ;  
glMultMatrixf(N) ; /* apply transformation N */  
glMultMatrixf(M) ; /* apply transformation M */  
glMultMatrixf(L) ; /* apply transformation L */  
glBegin(GL_POINTS) ;  
    glVertex3f(v) ; /* draw transformed vertex v */  
glEnd() ;
```

The vertex transformation is  $I(N(M(Lv))) = (INML)v = (NML)v$

# *Application Order of Transformation*

- ◆ If you are not thinking in reverse order, than all the transformation matrices derived should be transposed
  - *For example , the Direct3D matrix representation is applied in this order*

The previous transformation example becomes

$$(((vL^T)M^T)N^T)I = v(L^TM^TN^T) = v(L^TM^TN^T)$$

If  $L, M, N, I$  is represented as transformation matrix in OpenGL

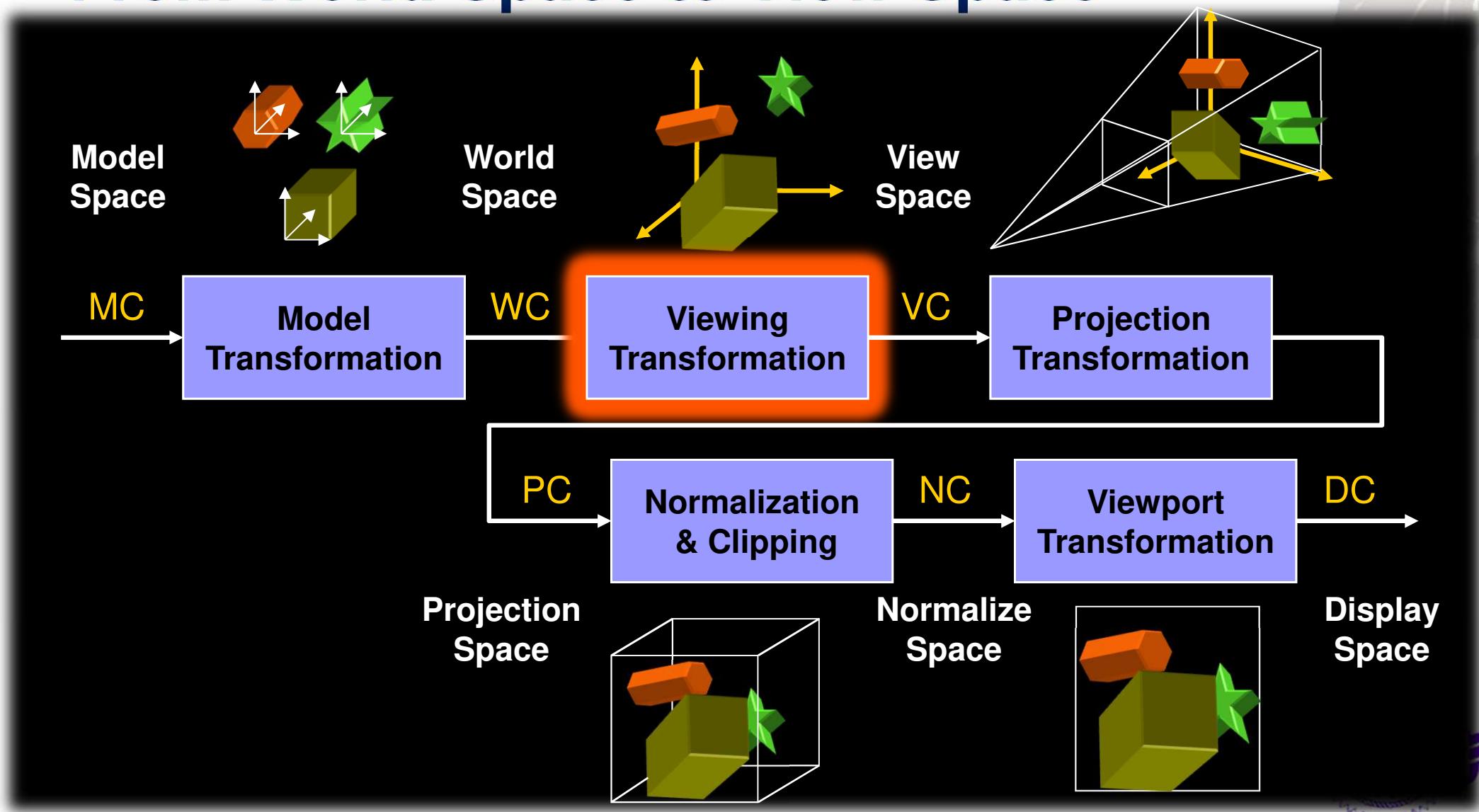
# *Viewing Transformation*

*From World Space to View Space*



# 3D Viewing Process

## ◆ From World Space to View Space



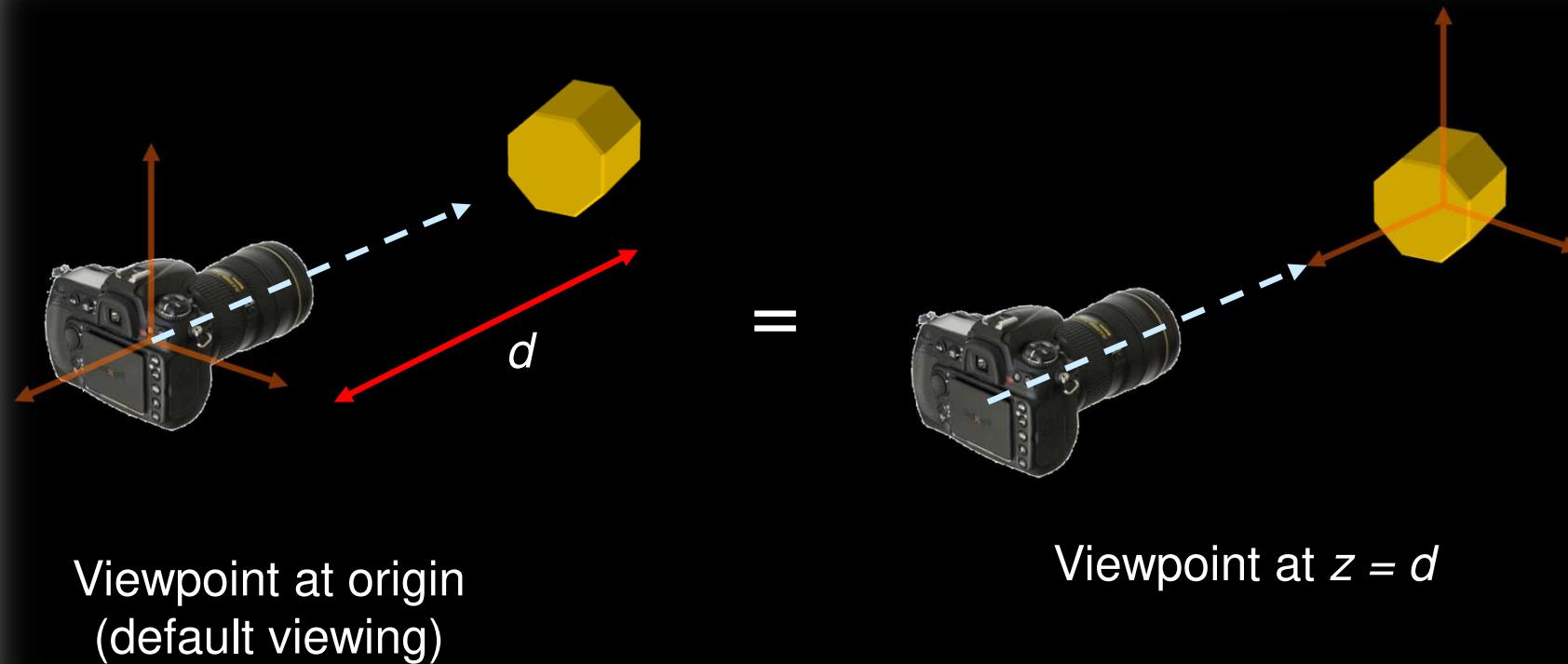
# *Viewing Transformation*

- ◆ It defines the position and orientation of the viewpoint
- ◆ It is composed of a sequence of translation and rotation transformations
- ◆ In OpenGL, the default viewpoint is located at the origin and looking toward the negative z-axis direction



# *Viewing Transformation*

## ◆ Example



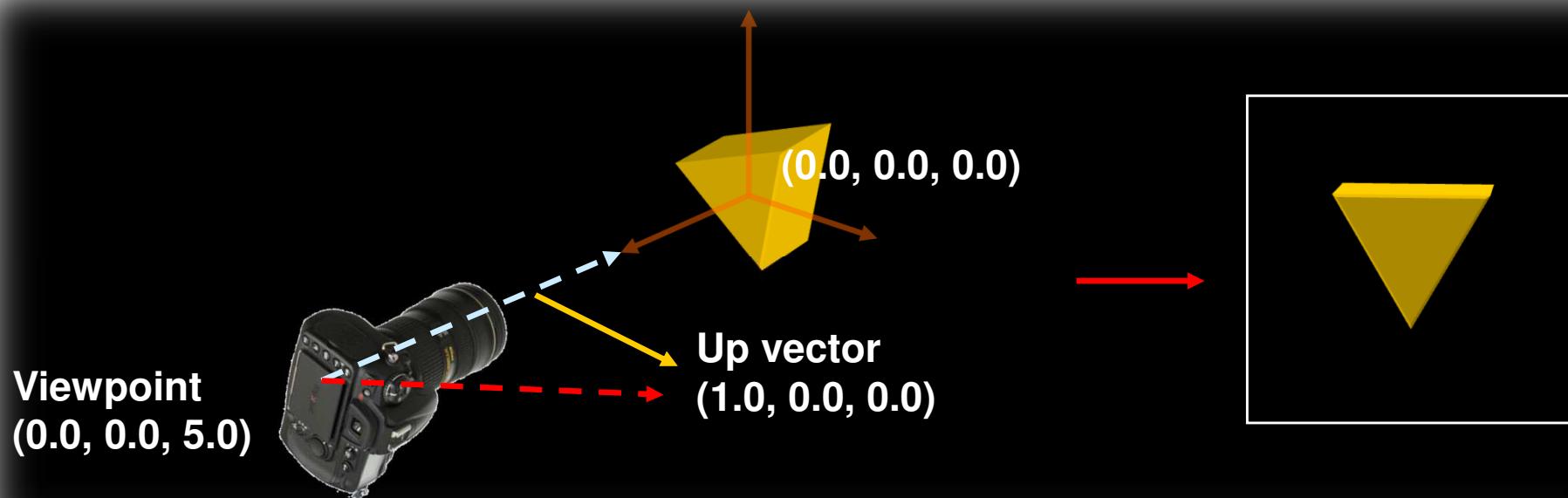
# *Specify the Viewing Parameters*

- ◆ You need to define the viewpoint, viewing direction, and the pose when you are looking through a specific direction
- ◆ `gluLookat(ex, ey, ez,  
              cx, cy, cz,  
              ux, uy, uz)` // eye coordinates  
                                  // view center coordinates  
                                  // up vector
  - Viewpoint is at (*ex, ey, ez*)
  - Viewing direction is from (*ex, ey, ez*) toward (*cx, cy, cz*)
  - Up direction is from (0, 0, 0) to (*ux, uy, uz*)



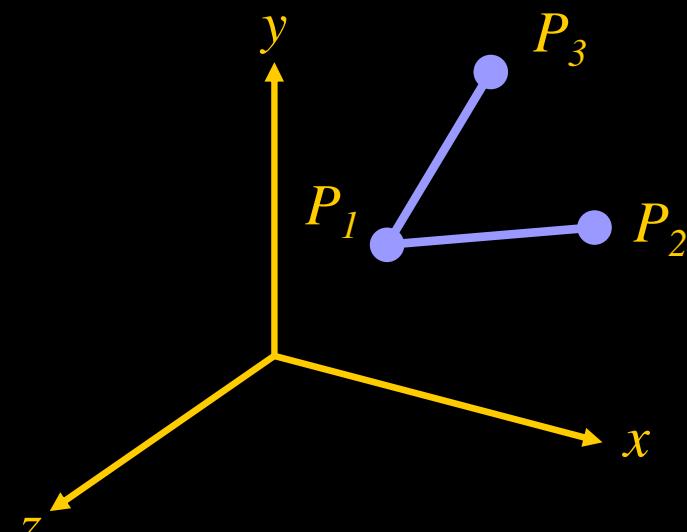
# *Specify the Viewing Parameters*

- ◆ **gluLookat(0.0, 0.0, 5.0, 0.0, 0.0, 0.0, 1.0, 0.0, 0.0)**
  - **Viewpoint is at (0.0, 0.0, 5.0)**
  - **Viewing direction is from (0.0, 0.0, 5.0) toward (0.0, 0.0, 0.0)**
  - **Up direction is (1.0, 0.0, 0.0)**

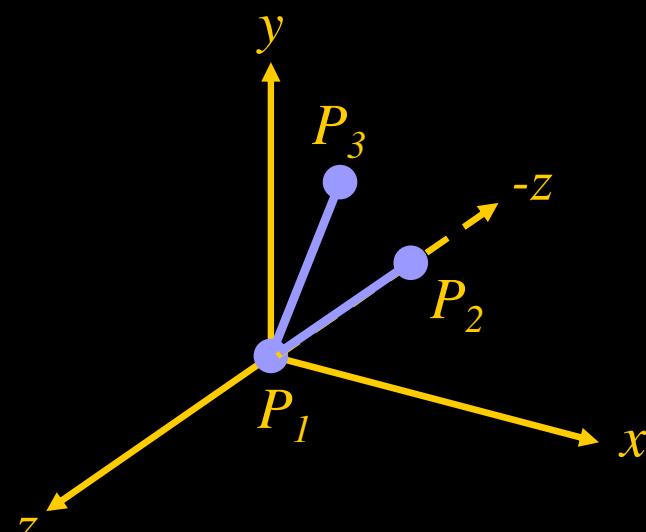


# *Concept of Viewing Transformation*

- ◆ Transforming  $P_1$ ,  $P_2$ , and  $P_3$  such that  $\overrightarrow{P_1P_2}$  aligns with the **negative z-axis** and  $\overrightarrow{P_1P_3}$  lies on the **yz-plane**



Initial position



Final position

# *Concept of Viewing Transformation*

- ◆ Transforming  $P_1$ ,  $P_2$ , and  $P_3$  from their initial position to their final position
  - Step 1: Translate  $P_1$  to the origin
  - Step 2: Rotate about the  $y$ -axis such that  $\overrightarrow{P_1P_2}$  lies on the  $yz$ -plane
  - Step 3: Rotate about the  $x$ -axis such that  $\overrightarrow{P_1P_2}$  align with the **negative**  $z$ -axis
  - Step 4: Rotate about the  $z$ -axis such that  $\overrightarrow{P_1P_3}$  lies on the  $yz$ -plane



# *Concept of Viewing Transformation*

- ◆ Transforming  $P_1$ ,  $P_2$ , and  $P_3$  from their initial position to their final position
  - Step 1: Translate  $P_1$  to the origin

$$P'_1 = T(-x_1, -y_1, -z_1) \cdot P_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$P'_2 = T(-x_1, -y_1, -z_1) \cdot P_2 = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \\ 1 \end{bmatrix}$$
$$P'_3 = T(-x_1, -y_1, -z_1) \cdot P_3 = \begin{bmatrix} x_3 - x_1 \\ y_3 - y_1 \\ z_3 - z_1 \\ 1 \end{bmatrix}$$



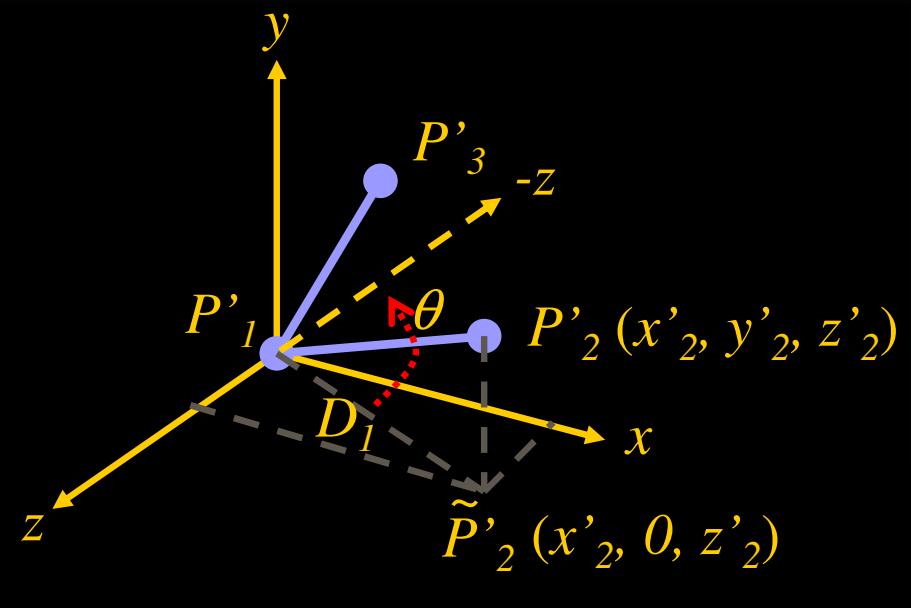
# Concept of Viewing Transformation

- ♦ Transforming  $P_1$ ,  $P_2$ , and  $P_3$  from their initial position to their final position
  - Step 2: Rotate about the  $y$ -axis such that  $\overrightarrow{P'_1 P'_2}$  lies in the  $(y, z)$  plane

$$P''_2 = R_y(\theta) \cdot P'_2 = \begin{bmatrix} 0 \\ y_2 - y_1 \\ D_1 \\ 1 \end{bmatrix}$$

where  $D_1 = |\overrightarrow{P'_1 \tilde{P}'_2}|$

Besides,  $P''_1 = R_y(\theta) \cdot P'_1$  and  
 $P''_3 = R_y(\theta) \cdot P'_3$

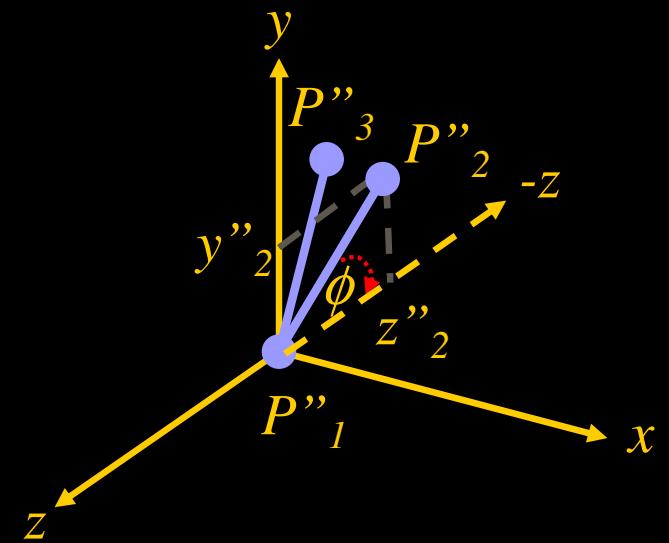


# Concept of Viewing Transformation

- ◆ Transforming  $P_1$ ,  $P_2$ , and  $P_3$  from their initial position to their final position
  - Step 3: Rotate about the x-axis such that  $\overrightarrow{P_1 P_2}$  lies on the z-axis

$$\begin{aligned}P''_2 &= R_x(-\phi) \cdot P''_2 \\&= R_x(-\phi) \cdot R_y(\theta) \cdot P'_2 \\&= R_x(-\phi) \cdot R_y(\theta) \cdot T \cdot P_2\end{aligned}$$

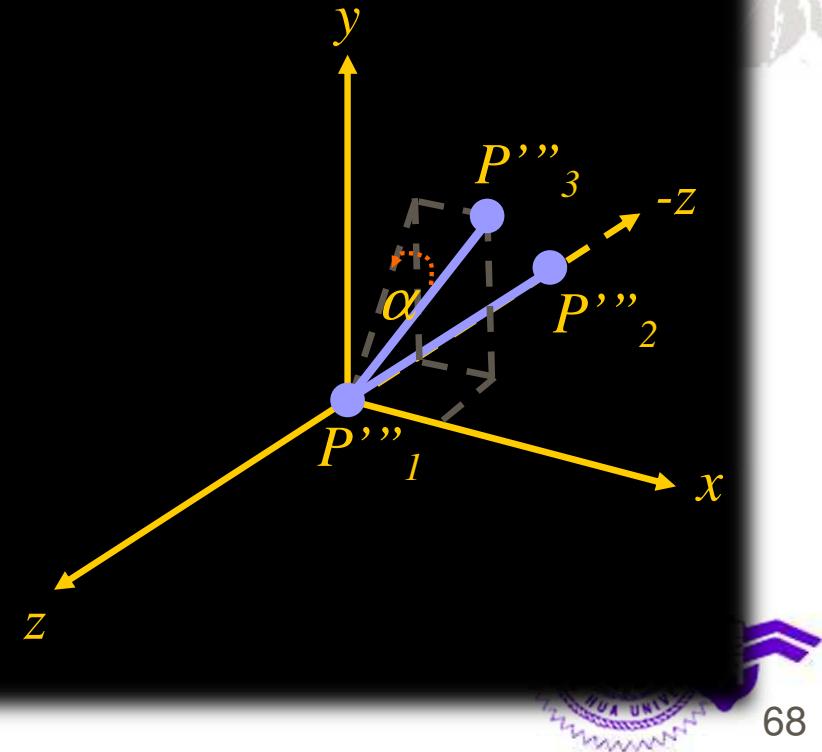
$$\begin{aligned}P''_1 &= R_x(-\phi) \cdot R_y(\theta) \cdot T \cdot P_1 \\P''_3 &= R_x(-\phi) \cdot R_y(\theta) \cdot T \cdot P_3\end{aligned}$$



# Concept of Viewing Transformation

- ◆ Transforming  $P_1$ ,  $P_2$ , and  $P_3$  from their initial position to their final position
  - Step 4: Rotate about the z-axis such that  $\overrightarrow{P_1 P_3}$  lies on the yz-plane

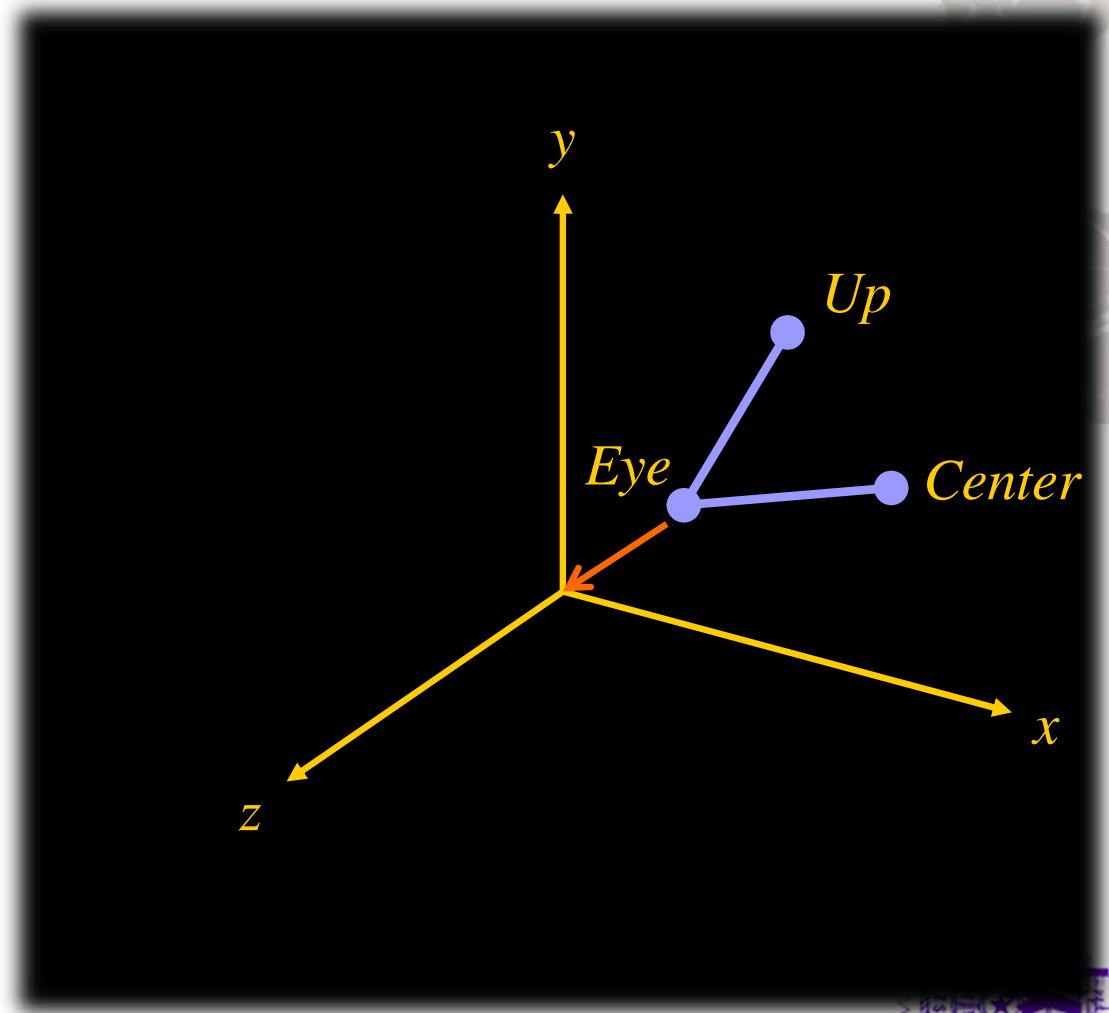
$$\begin{aligned}P'''_3 &= R_z(\alpha) \cdot P''_3 \\&= R_z(\alpha) \cdot R_x(-\phi) \cdot P''_3 \\&= R_z(\alpha) \cdot R_x(-\phi) \cdot R_y(\theta) \cdot P'_3 \\&= R_z(\alpha) \cdot R_x(-\phi) \cdot R_y(\theta) \cdot T \cdot P_3 \\P'''_1 &= R_z(\alpha) \cdot R_x(-\phi) \cdot R_y(\theta) \cdot T \cdot P_1 \\P'''_2 &= R_z(\alpha) \cdot R_x(-\phi) \cdot R_y(\theta) \cdot T \cdot P_2\end{aligned}$$



# Fast Viewing Matrix Derivation

- ◆ Translate eye position to the origin

$$T = \begin{bmatrix} 1 & 0 & 0 & -eye_x \\ 0 & 1 & 0 & -eye_y \\ 0 & 0 & 1 & -eye_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



# Fast Viewing Matrix Derivation

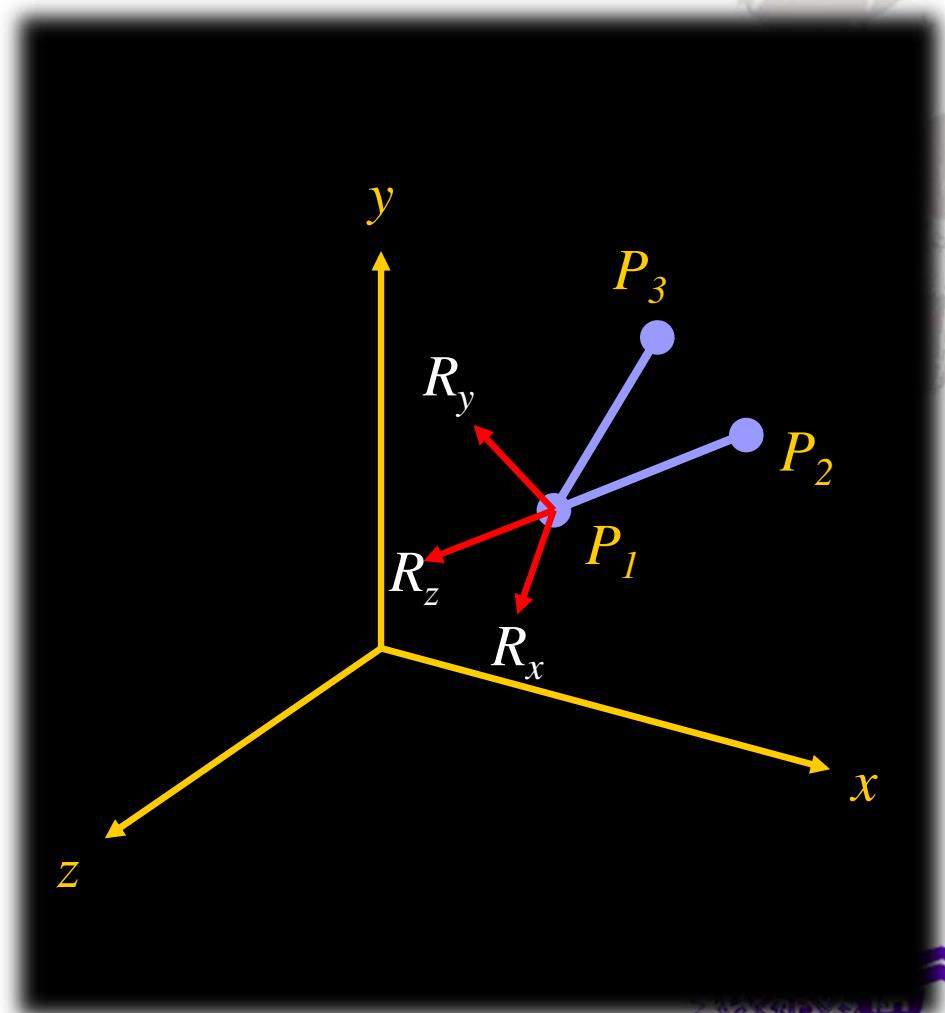
- ◆ Find the bases with respect to the new space

$$R = \begin{bmatrix} r_{1x} & r_{2x} & r_{3x} \\ r_{1y} & r_{2y} & r_{3y} \\ r_{1z} & r_{2z} & r_{3z} \end{bmatrix}$$

$$R_z = [r_{1z} \quad r_{2z} \quad r_{3z}]^T = \frac{-\overrightarrow{P_1 P_2}}{|\overrightarrow{P_1 P_2}|}$$

$$R_x = [r_{1x} \quad r_{2x} \quad r_{3x}]^T = \frac{\overrightarrow{P_1 P_2} \times \overrightarrow{P_1 P_3}}{|\overrightarrow{P_1 P_2} \times \overrightarrow{P_1 P_3}|}$$

$$R_y = [r_{1y} \quad r_{2y} \quad r_{3y}]^T = R_z \times R_x$$



# Fast Viewing Matrix Derivation

## ◆ Coordinates Conversion

$$x' \times (\mathbf{r}_{1x}, \mathbf{r}_{2x}, \mathbf{r}_{3x}) + y' \times (\mathbf{r}_{1y}, \mathbf{r}_{2y}, \mathbf{r}_{3y}) + z' \times (\mathbf{r}_{1z}, \mathbf{r}_{2z}, \mathbf{r}_{3z}) \\ = x \times (1, 0, 0) + y \times (0, 1, 0) + z \times (0, 0, 1)$$

$$\begin{matrix} & \text{Inverse matrix} \leftarrow \\ \downarrow & \end{matrix} \quad \begin{bmatrix} \mathbf{r}_{1x} & \mathbf{r}_{1y} & \mathbf{r}_{1z} \\ \mathbf{r}_{2x} & \mathbf{r}_{2y} & \mathbf{r}_{2z} \\ \mathbf{r}_{3x} & \mathbf{r}_{3y} & \mathbf{r}_{3z} \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\boxed{\begin{bmatrix} \mathbf{r}_{1x} & \mathbf{r}_{2x} & \mathbf{r}_{3x} \\ \mathbf{r}_{1y} & \mathbf{r}_{2y} & \mathbf{r}_{3y} \\ \mathbf{r}_{1z} & \mathbf{r}_{2z} & \mathbf{r}_{3z} \end{bmatrix}} \begin{bmatrix} \mathbf{r}_{1x} & \mathbf{r}_{1y} & \mathbf{r}_{1z} \\ \mathbf{r}_{2x} & \mathbf{r}_{2y} & \mathbf{r}_{2z} \\ \mathbf{r}_{3x} & \mathbf{r}_{3y} & \mathbf{r}_{3z} \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \mathbf{r}_{1x} & \mathbf{r}_{2x} & \mathbf{r}_{3x} \\ \mathbf{r}_{1y} & \mathbf{r}_{2y} & \mathbf{r}_{3y} \\ \mathbf{r}_{1z} & \mathbf{r}_{2z} & \mathbf{r}_{3z} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \mathbf{r}_{1x} & \mathbf{r}_{2x} & \mathbf{r}_{3x} \\ \mathbf{r}_{1y} & \mathbf{r}_{2y} & \mathbf{r}_{3y} \\ \mathbf{r}_{1z} & \mathbf{r}_{2z} & \mathbf{r}_{3z} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$



# Fast Viewing Matrix Derivation

- ◆ Final Viewing Matrix is defined as

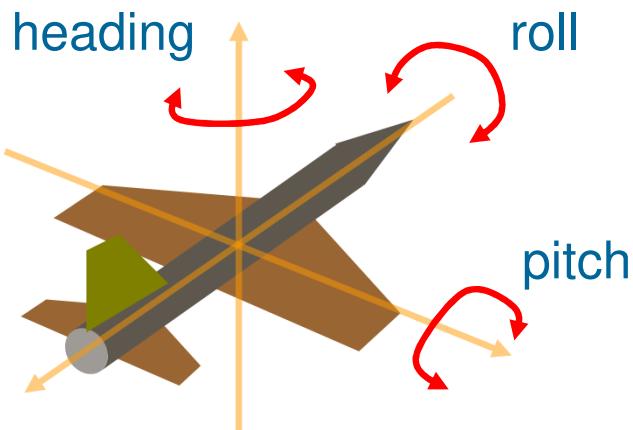
$$M_{view} = R \bullet T = \begin{bmatrix} r_{1x} & r_{2x} & r_{3x} & 0 \\ r_{1y} & r_{2y} & r_{3y} & 0 \\ r_{1z} & r_{2z} & r_{3z} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -eye_x \\ 0 & 1 & 0 & -eye_y \\ 0 & 0 & 1 & -eye_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



# Custom Viewing

## ◆ Pilot view

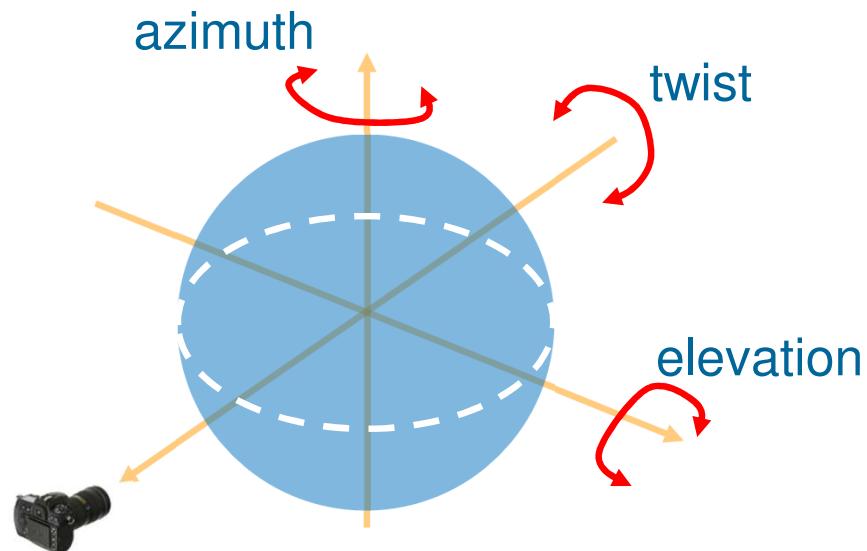
```
void pilotView(GLdouble planex, GLdouble planey,  
              GLdouble planez, GLdouble roll,  
              GLdouble pitch, GLdouble heading)  
{  
    glRotated(roll, 0.0, 0.0, 1.0);  
    glRotated(pitch, 1.0, 0.0, 0.0);  
    glRotated(heading, 0.0, 1.0, 0.0); // yaw  
    glTranslated(-planex, -planey, -planez);  
}
```



# Custom Viewing

## ◆ Polar view

```
void polarView(GLdouble distance, GLdouble twist,  
               GLdouble elevation, GLdouble azimuth)  
{  
    glTranslate(0.0, 0.0, -distance);  
    glRotated(twist, 0.0, 0.0, 1.0);  
    glRotated(elevation, 1.0, 0.0, 0.0);  
    glRotated(azimuth, 0.0, 1.0, 0.0);  
}
```



# *Transformation in Reverse Order*

- ◆ Each vertex is doing modeling transformation first and then viewing transformation

$$\begin{aligned} VMv &= V(Mv) = V(M_n M_{n-1} \dots M_2 M_1 M_0 v) \\ &= V(M_n(M_{n-1}(\dots(M_2(M_1(M_0 v))))\dots))) \end{aligned}$$



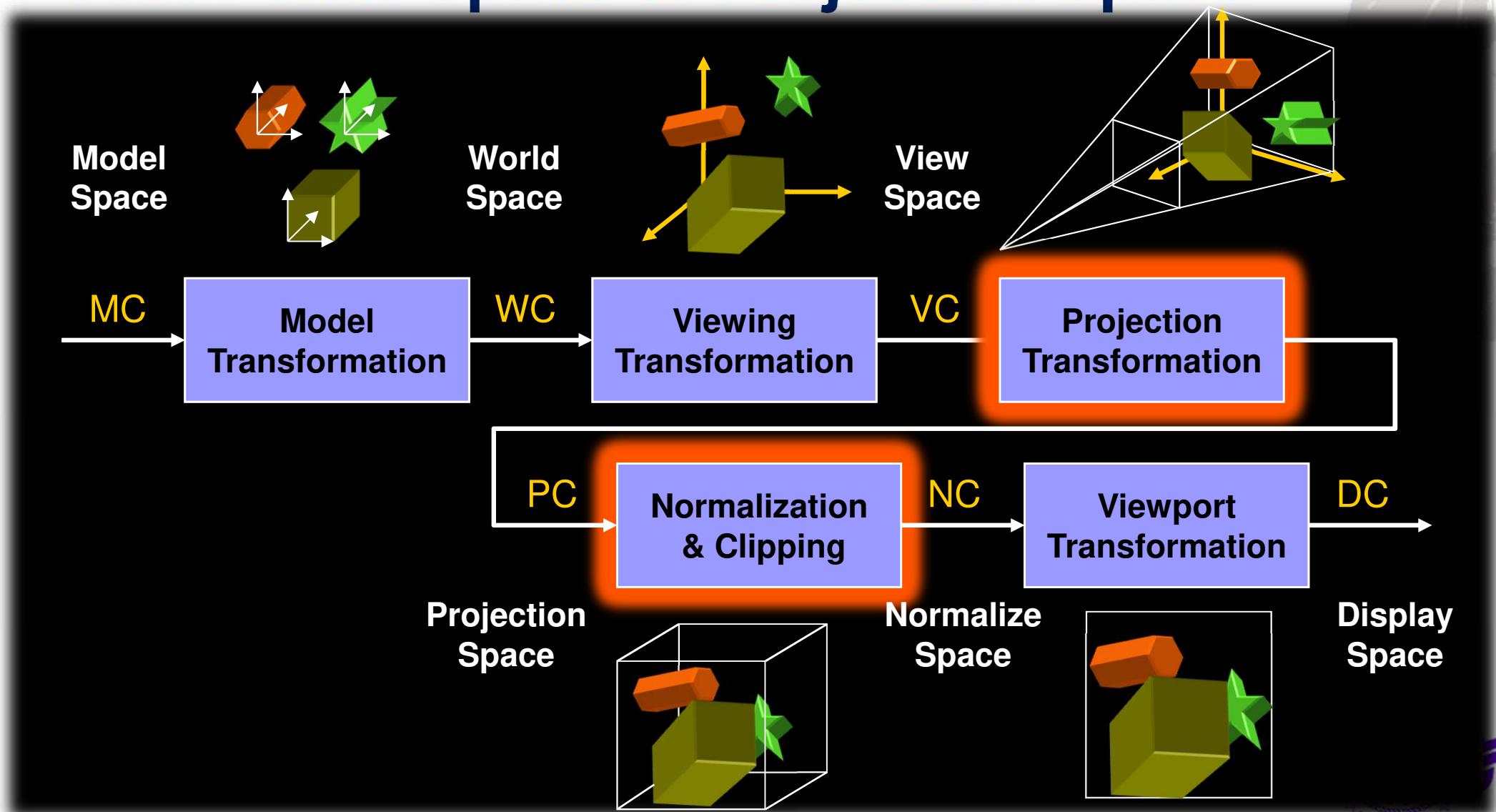
# *Projection*

*From View Space to Projection Space*



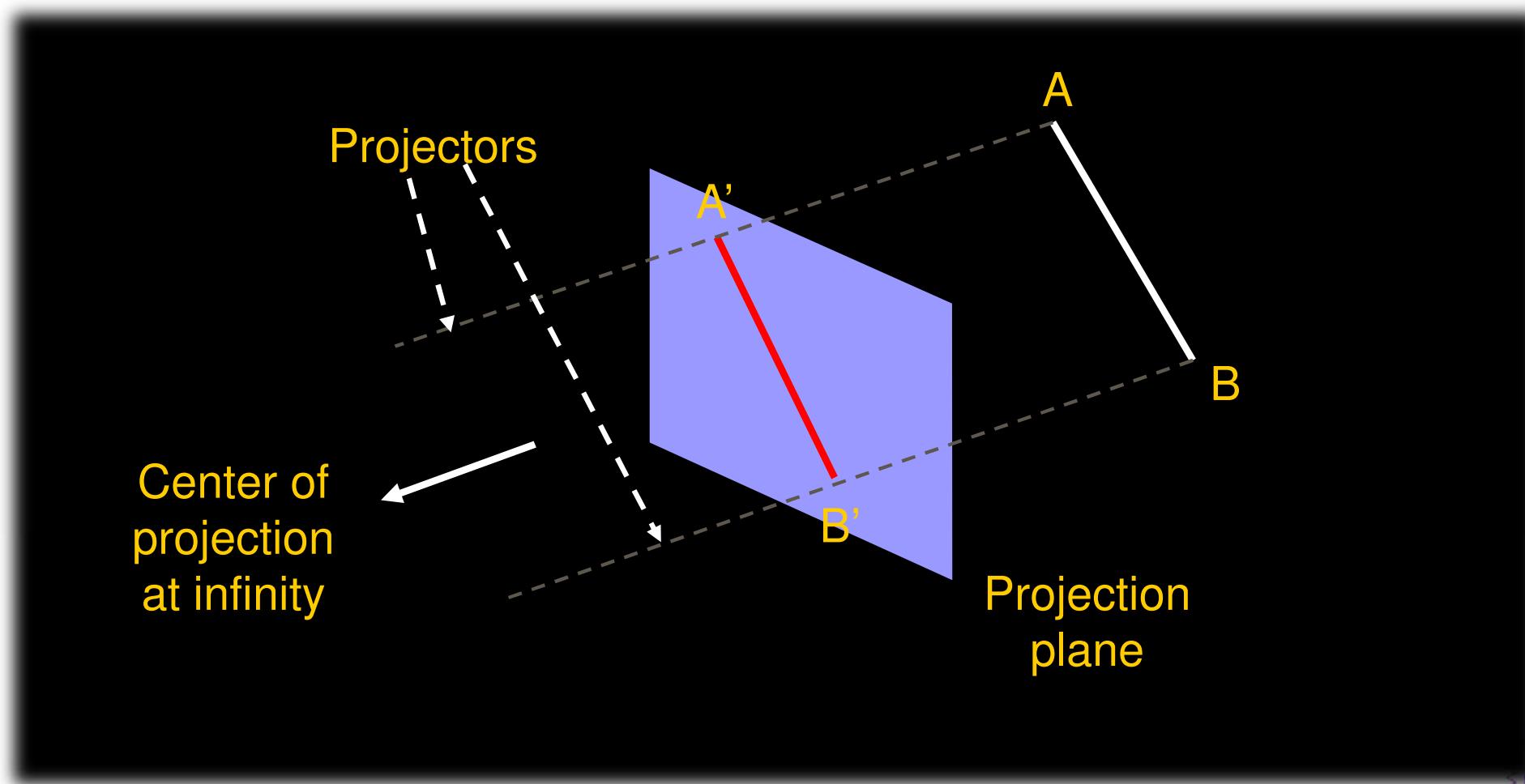
# 3D Viewing Process

## ◆ From View Space to Projection Space



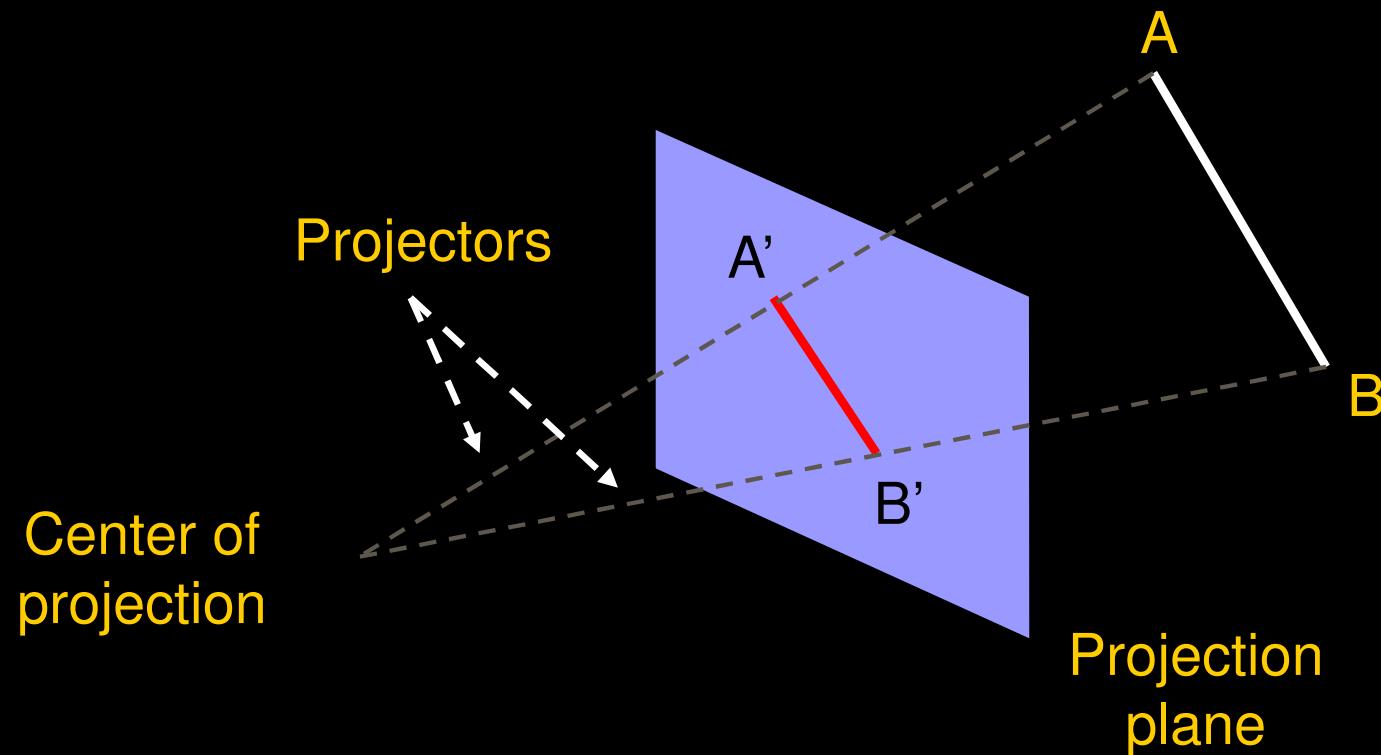
# Projections

## ◆ Parallel projection



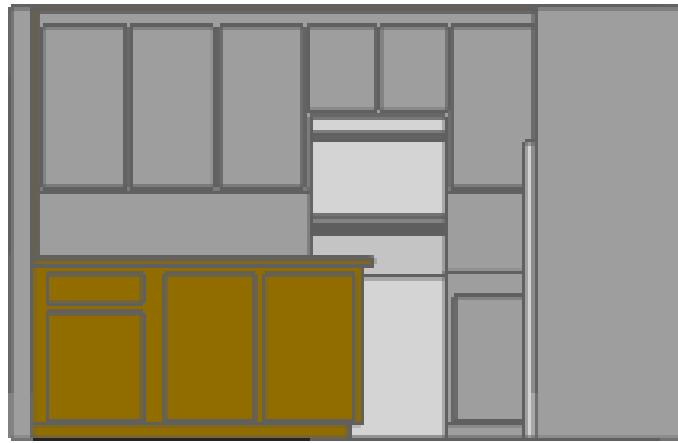
# Projections

## ◆ Perspective projection

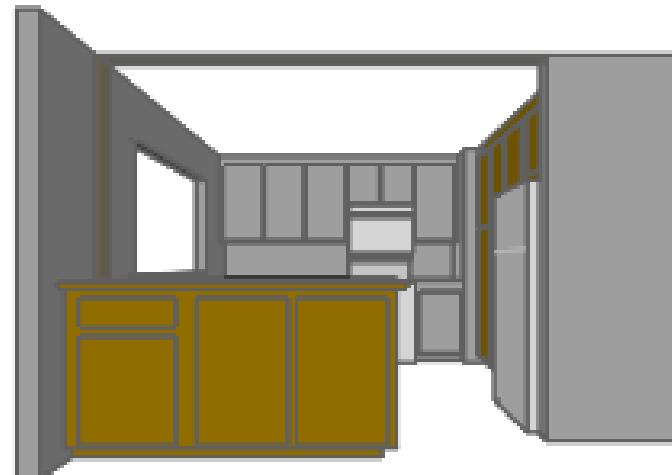


# *Projections*

- ◆ Parallel projection vs. perspective projection



Parallel projection



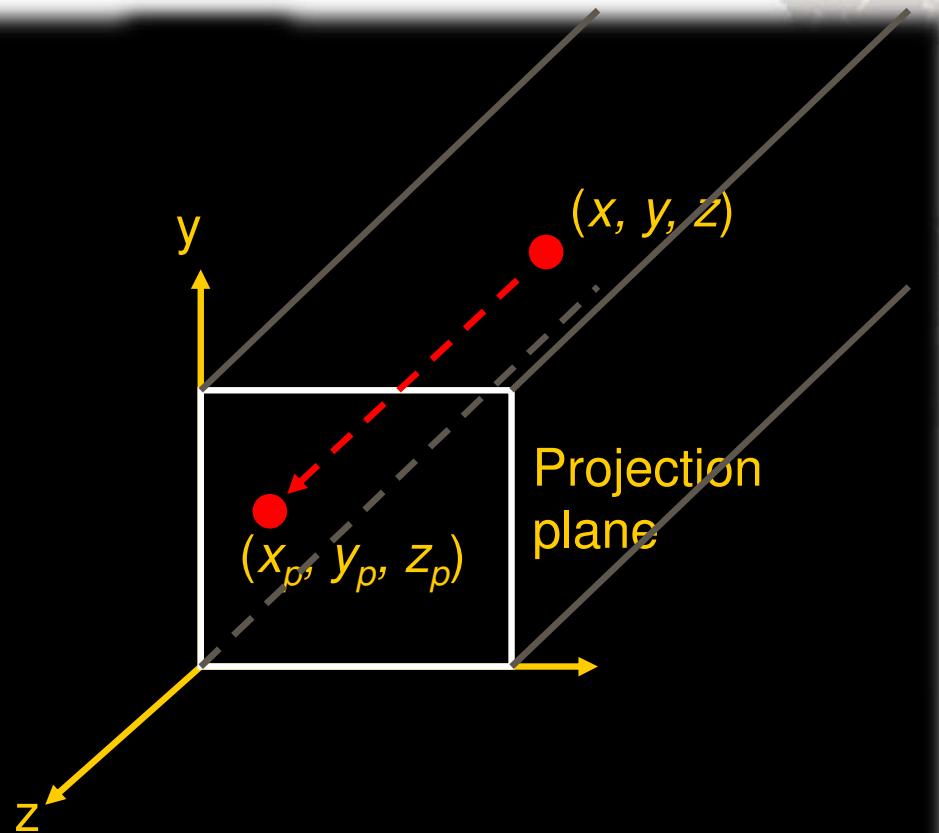
Perspective projection

# Parallel Projection

- ◆ Projection onto a projection plane at  $z = 0$

$$x_p = x, y_p = y, z_p = 0$$

$$\begin{bmatrix} x_p \\ y_p \\ z_p \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \\ 1 \end{bmatrix}$$
$$= M_{ort} \cdot P$$



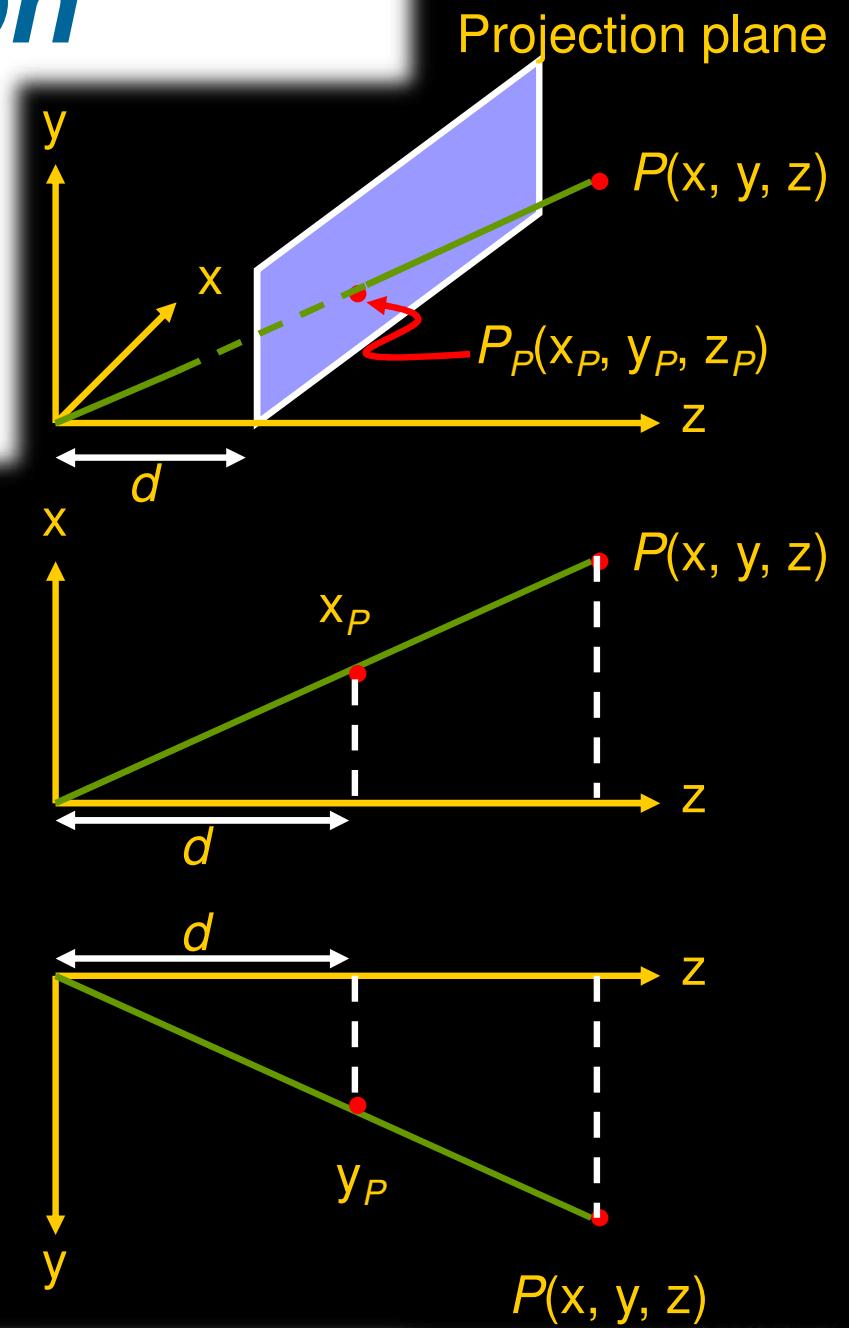
# Perspective Projection

- ◆ Projection onto a plane at  $z = d$

$$\frac{x_p}{d} = \frac{x}{z}, \frac{y_p}{d} = \frac{y}{z}$$

$$x_p = \frac{x}{z/d}, y_p = \frac{y}{z/d}$$

$$M_{per} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/d & 0 \end{bmatrix}$$



# Perspective Projection

$$\begin{bmatrix} X \\ Y \\ Z \\ W \end{bmatrix} = M_{per} \cdot P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/d & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$[X \quad Y \quad Z \quad W]^T = \left[ x \quad y \quad z \quad \frac{z}{d} \right]^T$$

From homogeneous coordinates to Cartesian coordinate

$$\left( \frac{X}{W}, \frac{Y}{W}, \frac{Z}{W} \right) = (x_p, y_p, z_p) = \left( \frac{x}{z/d}, \frac{y}{z/d}, d \right)$$

# Perspective Projection

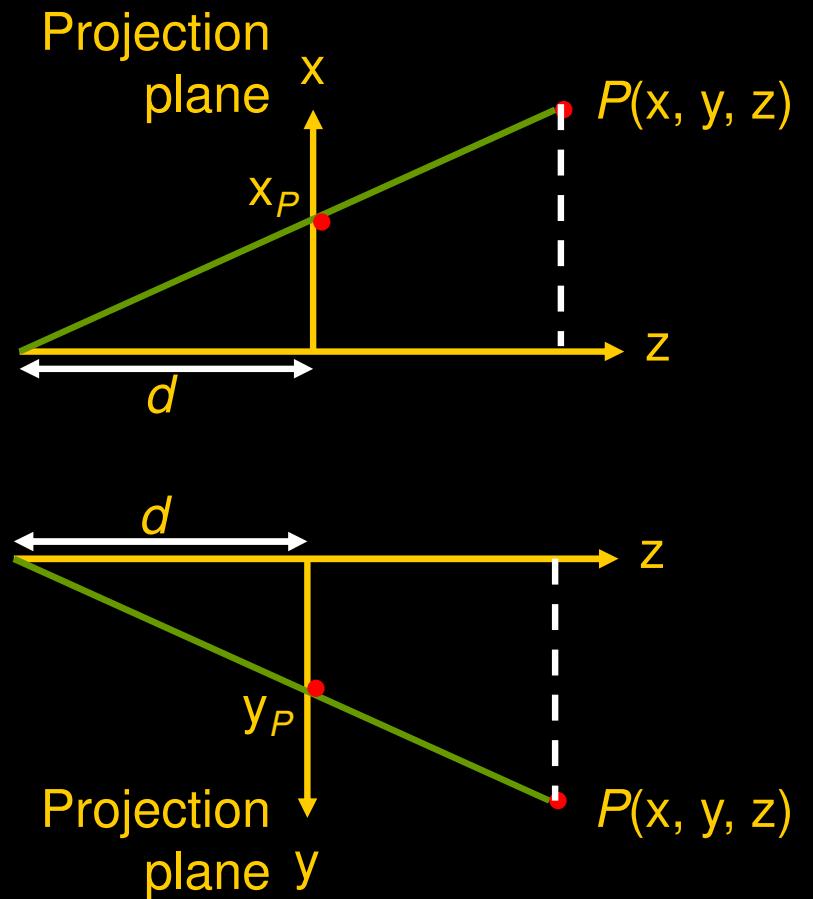
## ◆ Alternative Perspective Projection

$$\frac{x_p}{d} = \frac{x}{z+d}, \frac{y_p}{d} = \frac{y}{z+d}$$

$$x_p = \frac{d \cdot x}{z+d} = \frac{x}{(z/d)+1}$$

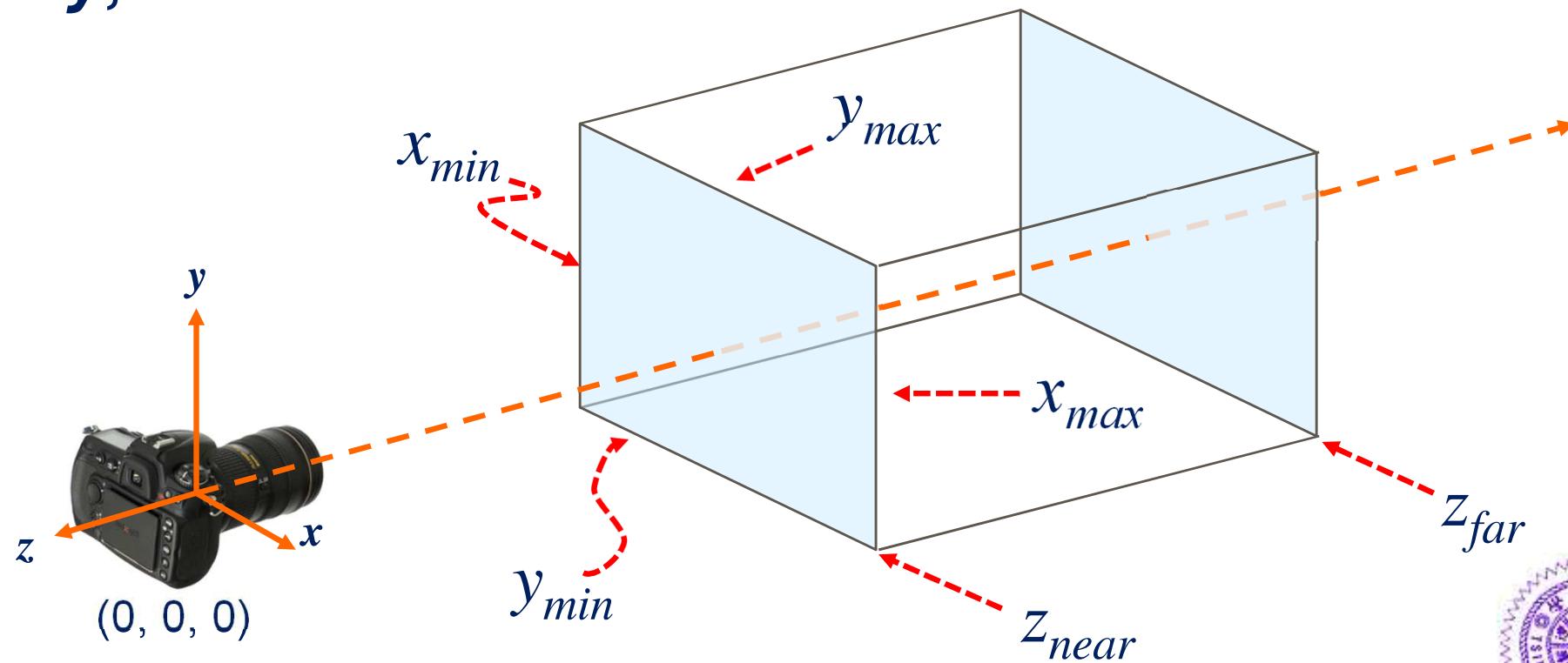
$$y_p = \frac{d \cdot y}{z+d} = \frac{y}{(z/d)+1}$$

$$M'_{per} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1/d & 1 \end{bmatrix}$$



# View Volume

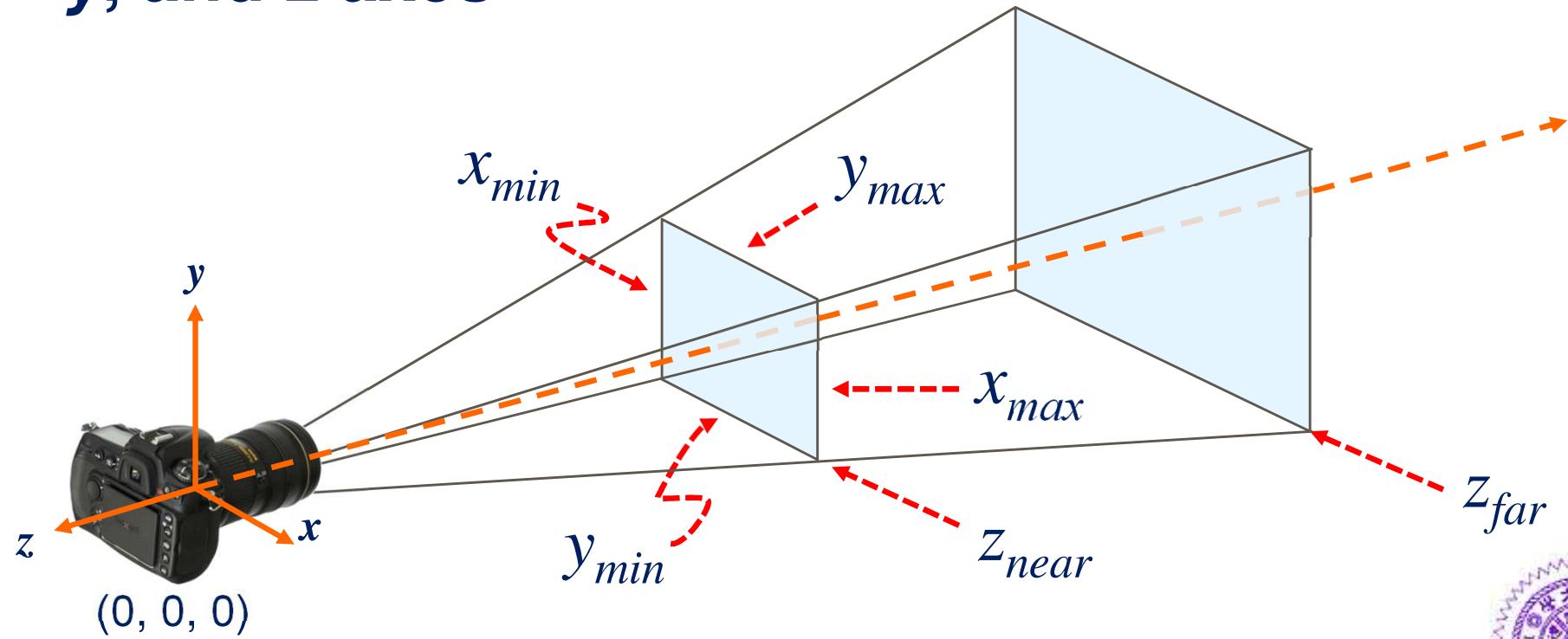
- ◆ View Volume in Orthographic Projection
  - Assume the eye is at the origin, then specify  $x_{max}$ ,  $x_{min}$ ,  $y_{max}$ ,  $y_{min}$ ,  $z_{near}$ ,  $z_{far}$  as the boundaries of x, y, and z axes



# View Volume

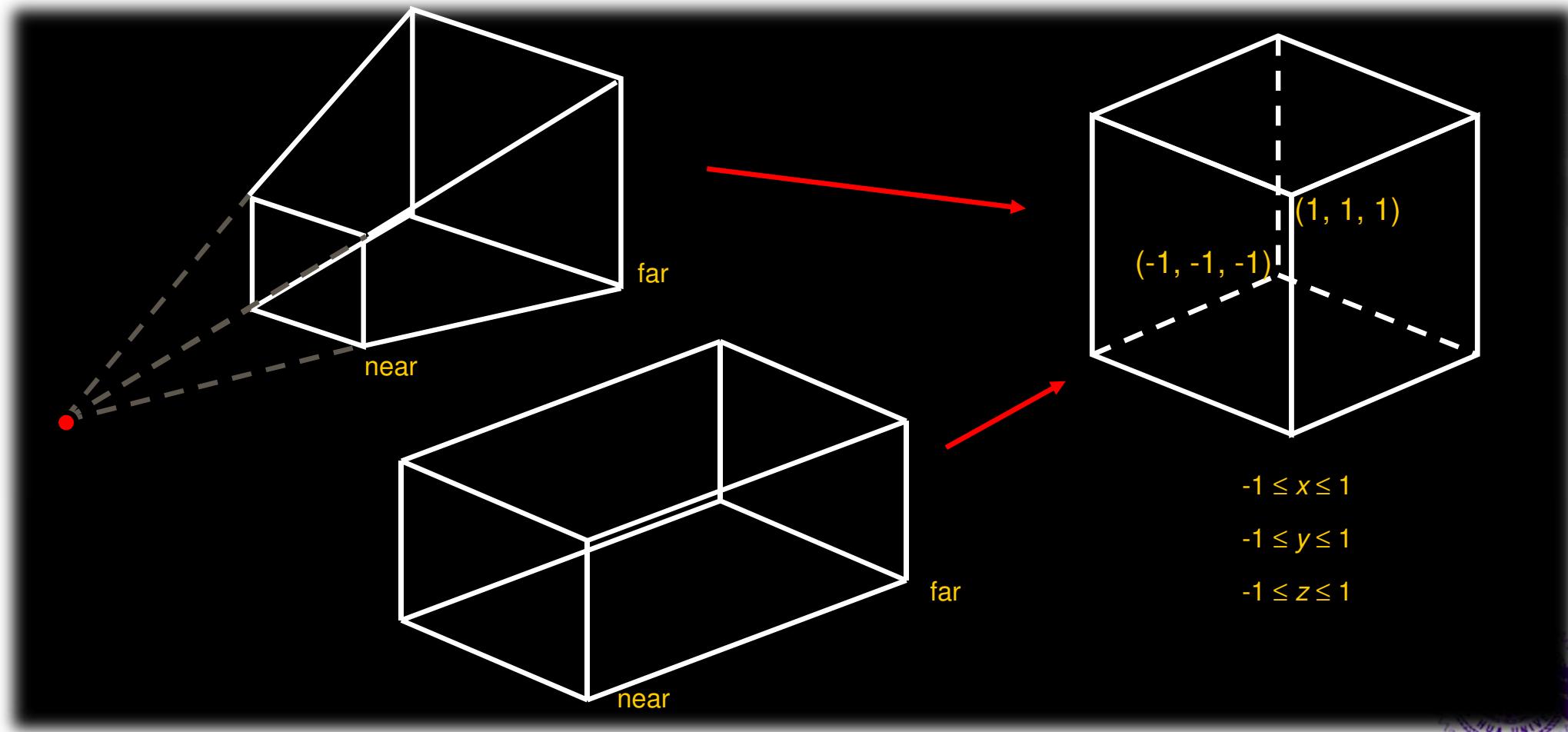
## ◆ View Volume in Perspective Projection

- Assume the eye is at the origin, then specify  $x_{max}$ ,  $x_{min}$ ,  $y_{max}$ ,  $y_{min}$ ,  $z_{near}$ ,  $z_{far}$  as the boundaries of x, y, and z axes



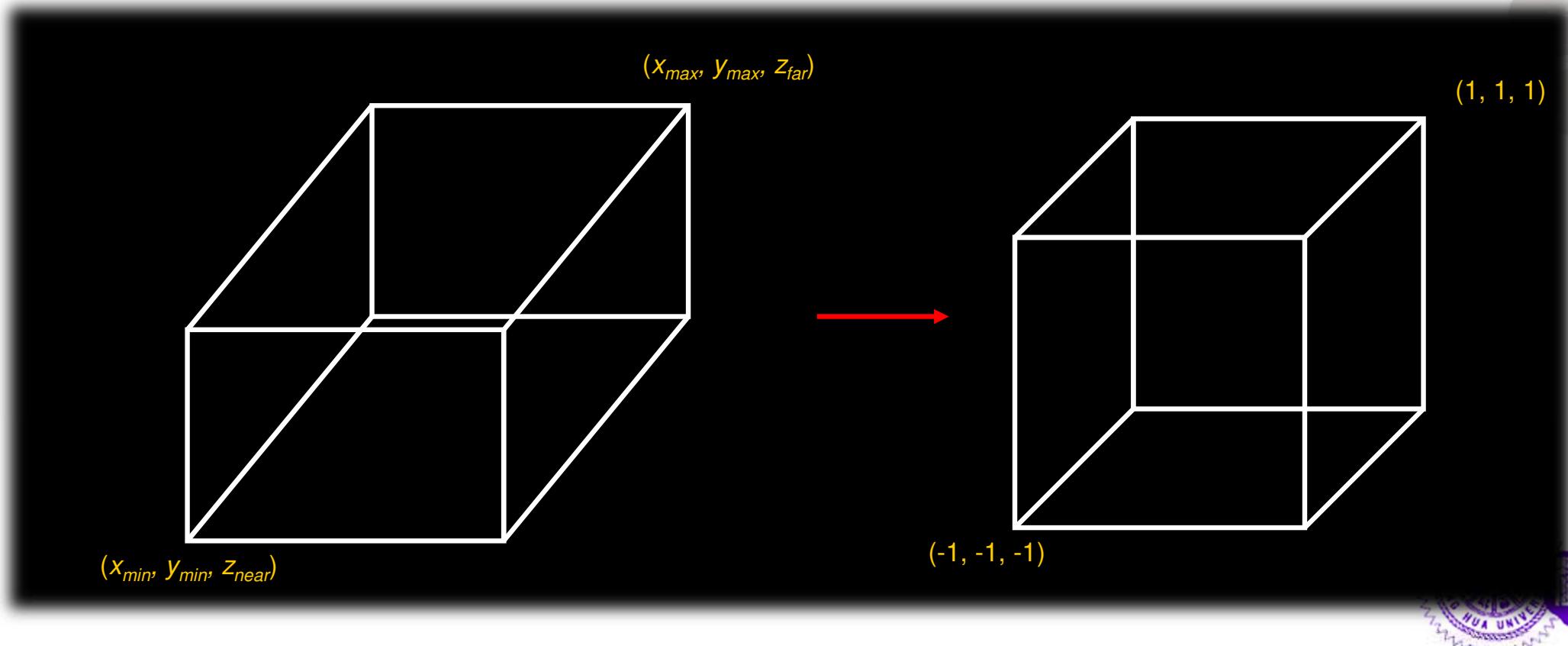
# *View Volume Normalization*

- ◆ Make the view volume clipping easy under normalized clipping volume



# Orthogonal Normalization

- ◆ Composition of orthogonal normalization
  - Translate the view volume center to origin
  - Scale to have each side of length 2



# Orthogonal Normalization

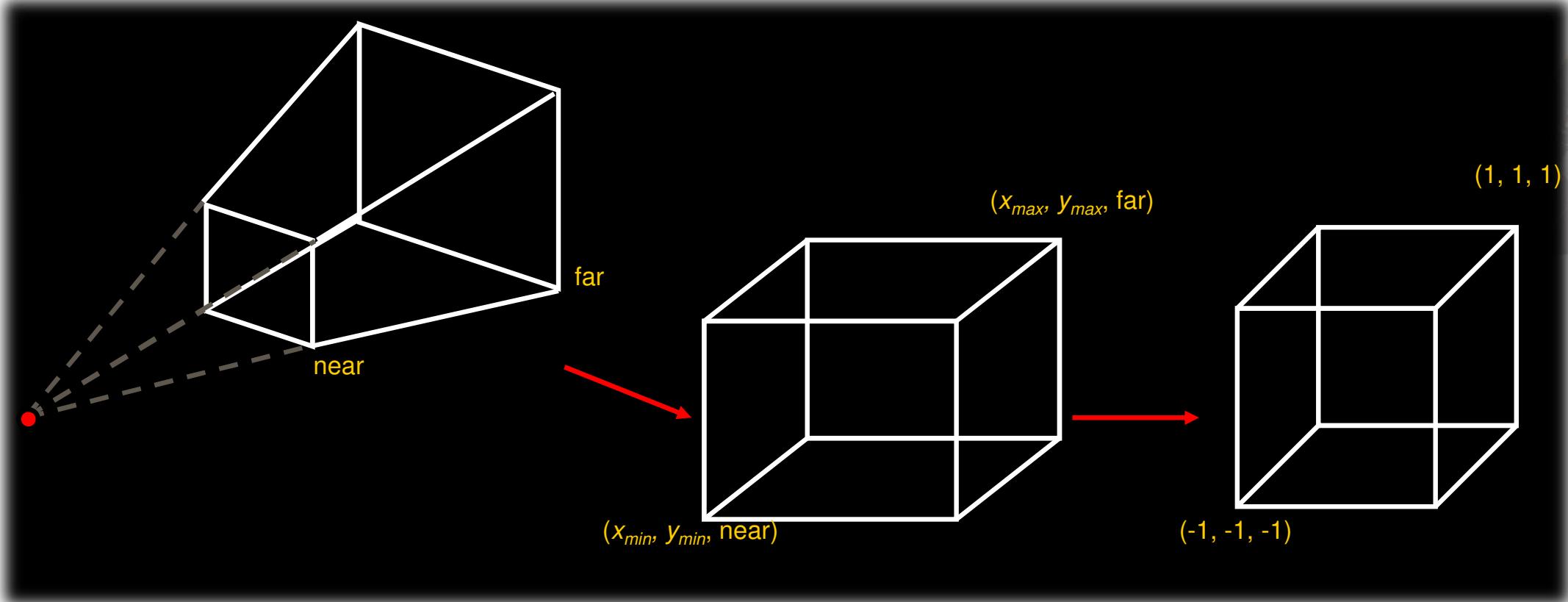
## ◆ Composition of orthogonal normalization

$$M_{orthonorm} = S\left(\frac{2}{x_{\max} - x_{\min}}, \frac{2}{y_{\max} - y_{\min}}, \frac{2}{z_{\text{far}} - z_{\text{near}}}\right) \\ \cdot T\left(-\frac{x_{\max} + x_{\min}}{2}, -\frac{y_{\max} + y_{\min}}{2}, -\frac{z_{\text{far}} + z_{\text{near}}}{2}\right)$$

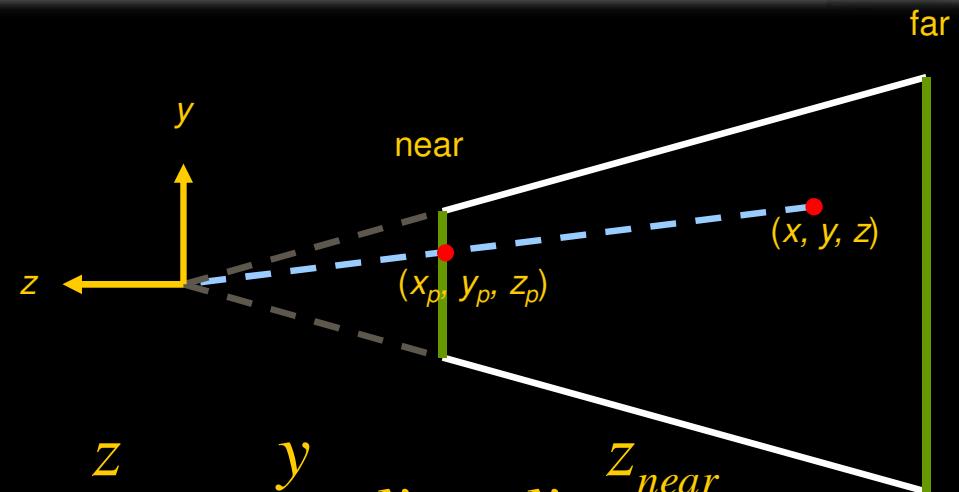
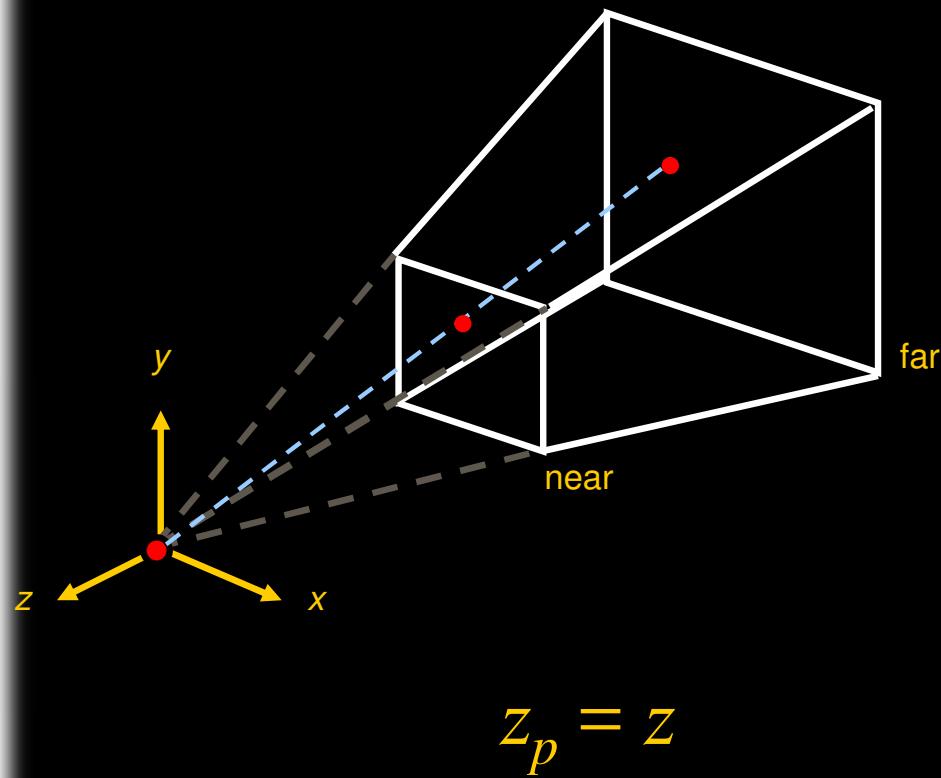
$$= \begin{bmatrix} \frac{2}{x_{\max} - x_{\min}} & 0 & 0 & -\frac{x_{\max} + x_{\min}}{x_{\max} - x_{\min}} \\ 0 & \frac{2}{y_{\max} - y_{\min}} & 0 & -\frac{y_{\max} + y_{\min}}{y_{\max} - y_{\min}} \\ 0 & 0 & \frac{2}{z_{\text{far}} - z_{\text{near}}} & -\frac{z_{\text{far}} + z_{\text{near}}}{z_{\text{far}} - z_{\text{near}}} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# Perspective Normalization

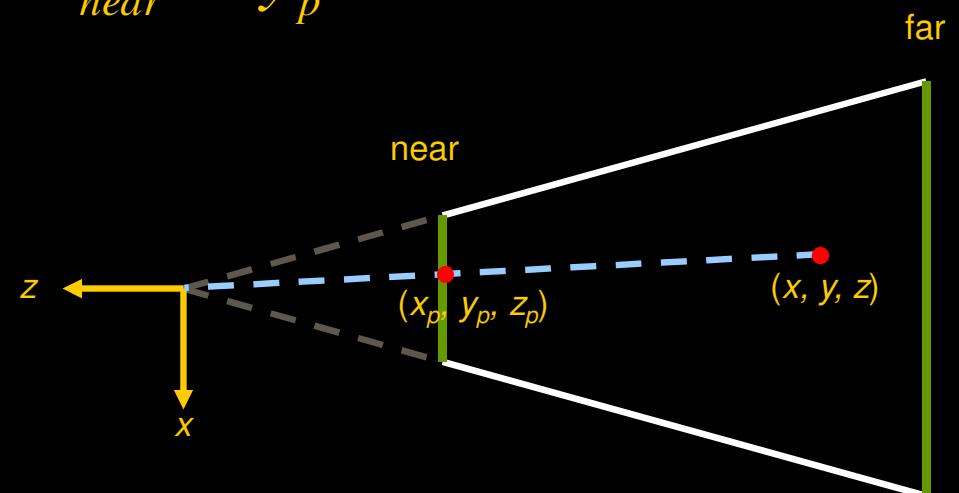
## ◆ Composition of perspective normalization



# Perspective Normalization

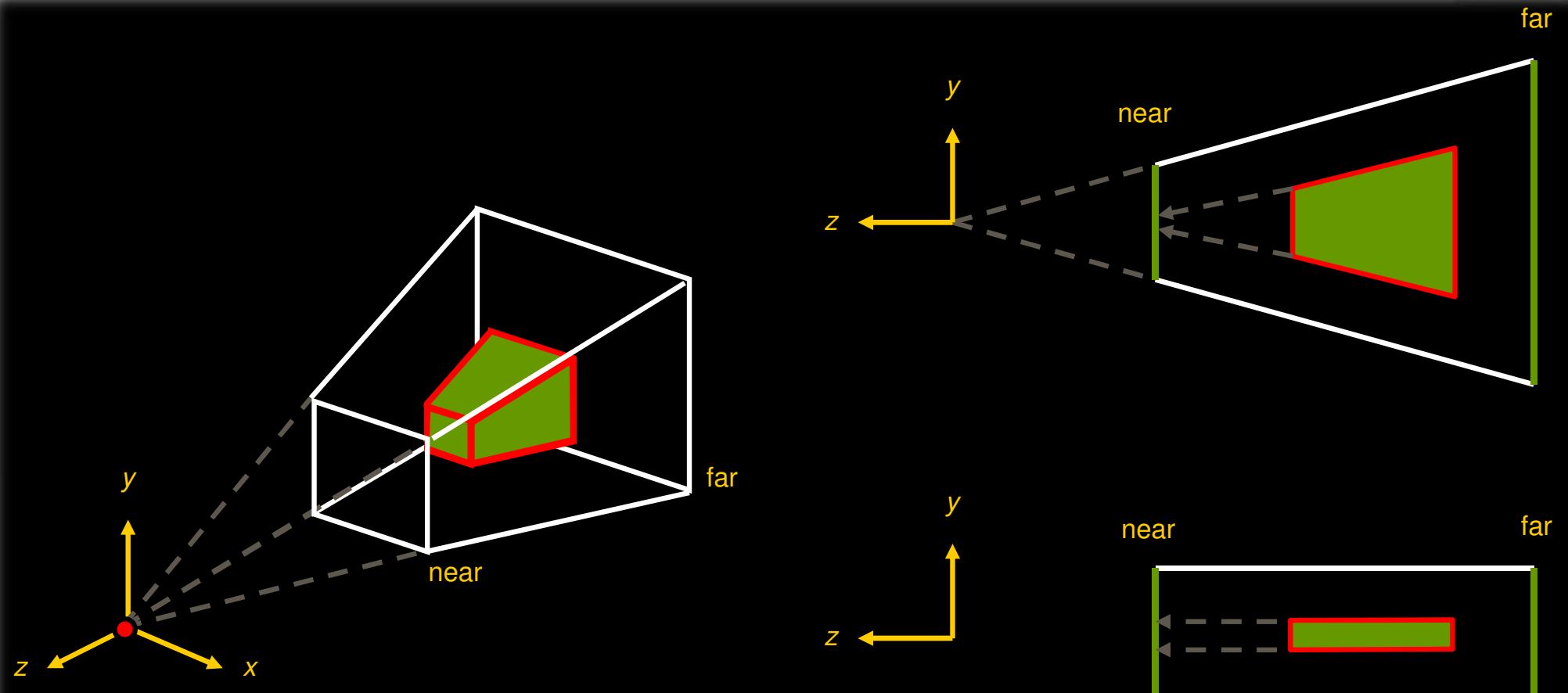


$$\frac{z}{z_{near}} = \frac{y}{y_p}, y_p = y \cdot \frac{z_{near}}{z}$$



$$\frac{z}{z_{near}} = \frac{x}{x_p}, x_p = x \cdot \frac{z_{near}}{z}$$

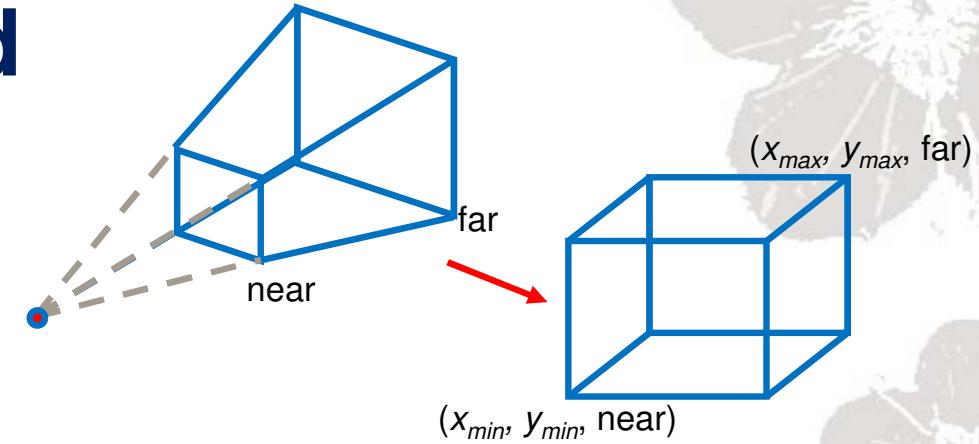
# Perspective Normalization



# Perspective Normalization

## ◆ From Frustum to Cuboid

$$\begin{bmatrix} x' \\ y' \\ z' \\ w' \end{bmatrix} = \begin{bmatrix} z_{near} & 0 & 0 & 0 \\ 0 & z_{near} & 0 & 0 \\ 0 & 0 & S_z & T_z \\ 0 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



$$x_p = \frac{x'}{w'} = x \cdot \frac{z_{near}}{z}$$

$$y_p = \frac{y'}{w'} = y \cdot \frac{z_{near}}{z}$$

$$z_p = \frac{z'}{w'} = \frac{S_z \cdot z + T_z}{z}$$

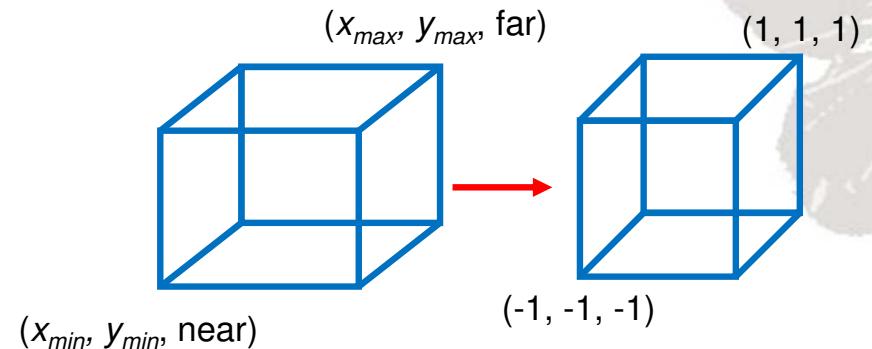
$$\begin{cases} z_p = z_{near}, \text{ if } z = z_{near} \\ z_p = z_{far}, \text{ if } z = z_{far} \end{cases}$$

$$\begin{cases} \frac{S_z \cdot z_{near} + T_z}{z_{near}} = z_{near} \\ \frac{S_z \cdot z_{far} + T_z}{z_{far}} = z_{far} \end{cases}$$

$$\begin{cases} S_z = z_{far} + z_{near} \\ T_z = -z_{far} \cdot z_{near} \end{cases}$$

# Perspective Normalization

- ◆ From Cuboid to Cube
  - Similar to orthogonal normalization



$$M_{orthonorm} = S\left(\frac{2}{x_{\max} - x_{\min}}, \frac{2}{y_{\max} - y_{\min}}, \frac{2}{z_{\max} - z_{\min}}\right)$$

$$\cdot T\left(-\frac{x_{\max} + x_{\min}}{2}, -\frac{y_{\max} + y_{\min}}{2}, -\frac{z_{\max} + z_{\min}}{2}\right)$$

# Perspective Normalization

From cuboid to cube

From frustum to cuboid

$$M_{persnorm} = \begin{bmatrix} \frac{2}{x_{\max} - x_{\min}} & 0 & 0 & -\frac{x_{\max} + x_{\min}}{x_{\max} - x_{\min}} \\ 0 & \frac{2}{y_{\max} - y_{\min}} & 0 & -\frac{y_{\max} + y_{\min}}{y_{\max} - y_{\min}} \\ 0 & 0 & \frac{2}{z_{\max} - z_{\min}} & -\frac{z_{\max} + z_{\min}}{z_{\max} - z_{\min}} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

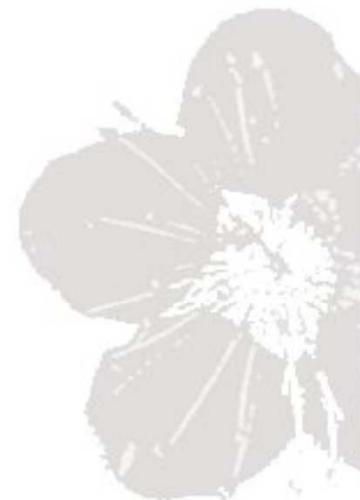
$$\cdot \begin{bmatrix} z_{near} & 0 & 0 & 0 \\ 0 & z_{near} & 0 & 0 \\ 0 & 0 & z_{far} + z_{near} & -z_{far}z_{near} \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2 \cdot z_{near}}{x_{\max} - x_{\min}} & 0 & -\frac{x_{\max} + x_{\min}}{x_{\max} - x_{\min}} & 0 \\ 0 & \frac{2 \cdot z_{near}}{y_{\max} - y_{\min}} & -\frac{y_{\max} + y_{\min}}{y_{\max} - y_{\min}} & 0 \\ 0 & 0 & \frac{z_{far} + z_{near}}{z_{\max} - z_{\min}} & \frac{-2z_{far}z_{near}}{z_{\max} - z_{\min}} \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Final perspective normalization matrix



# *Clipping*

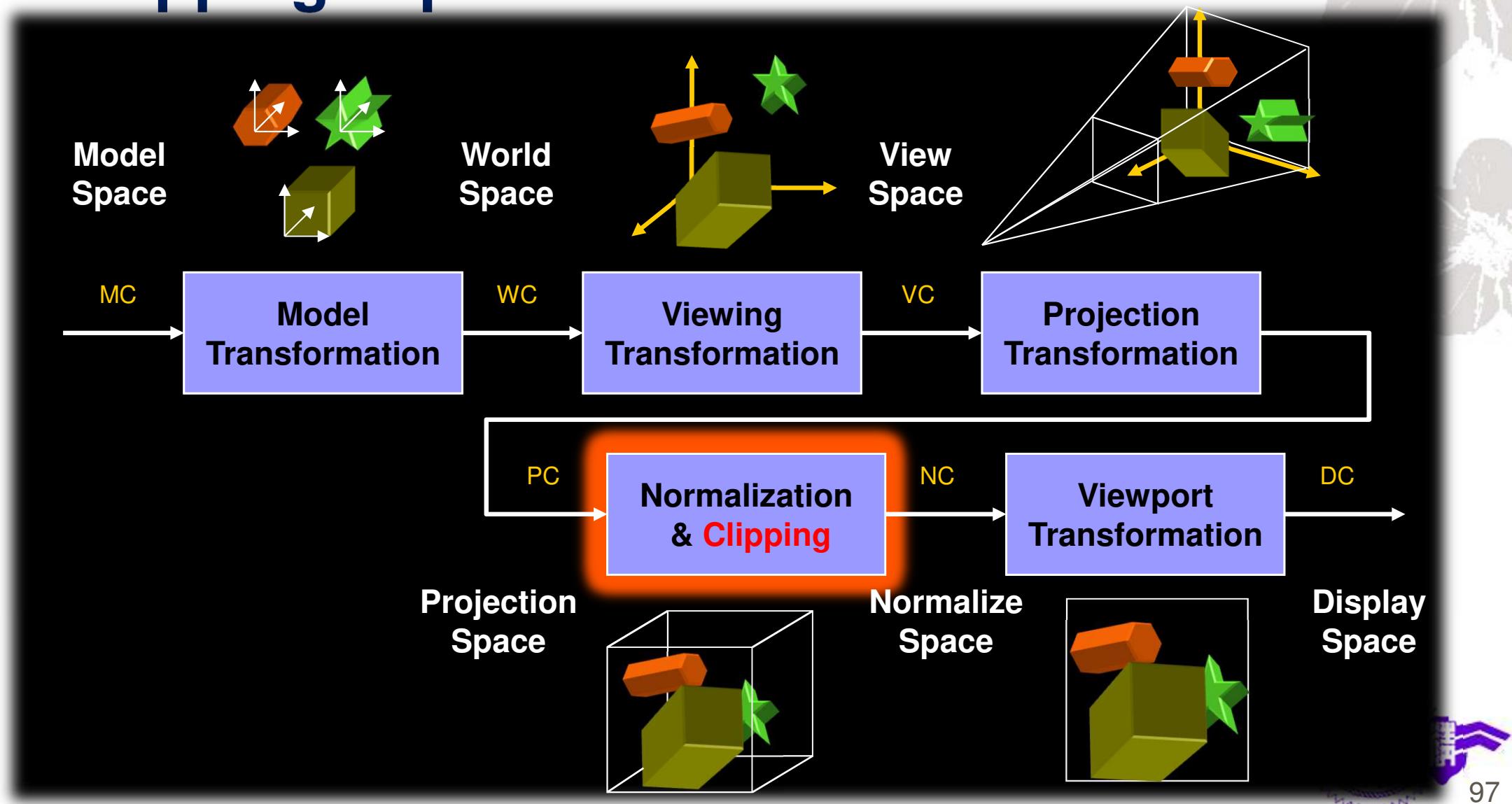


*2D Clipping  
View Volume Clipping*



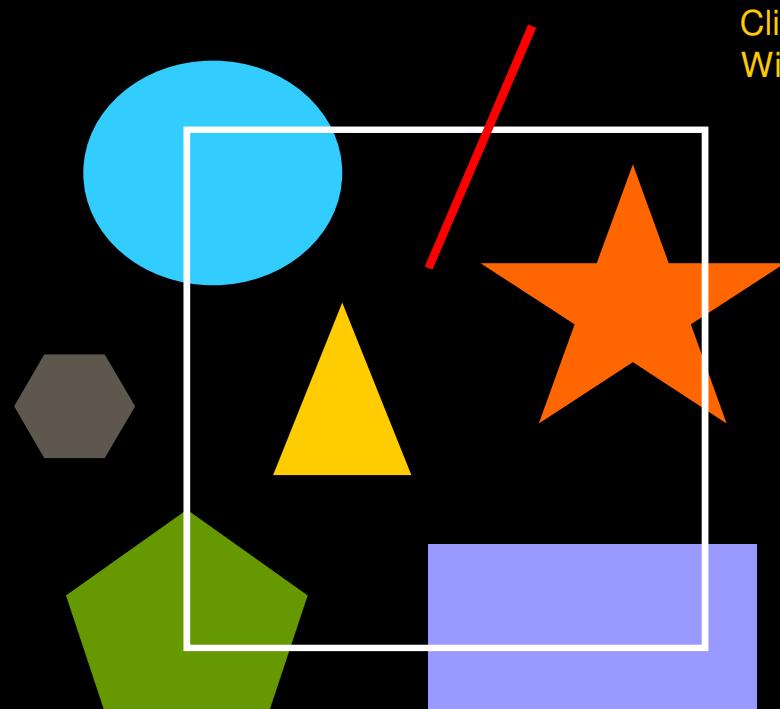
# *Review of 3D Viewing Process*

- ◆ Clipping is performed after normalization



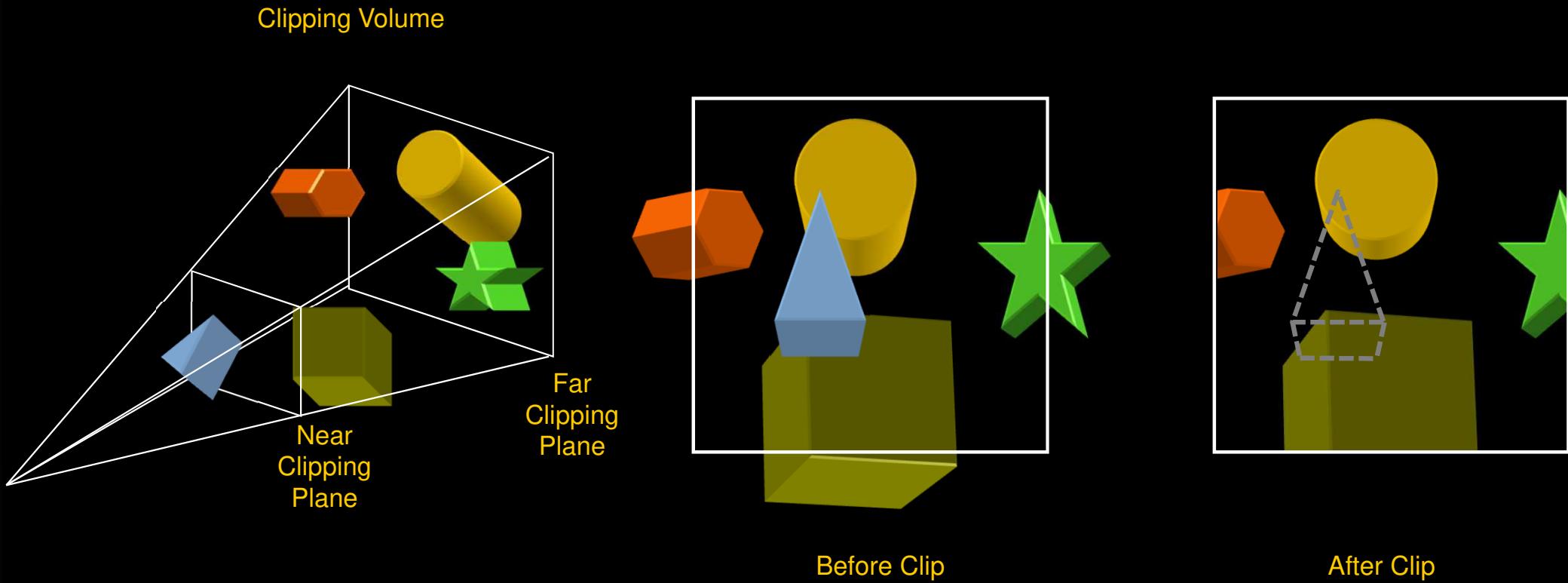
# 2D Clipping

## ◆ Clipping against 2D clipping window



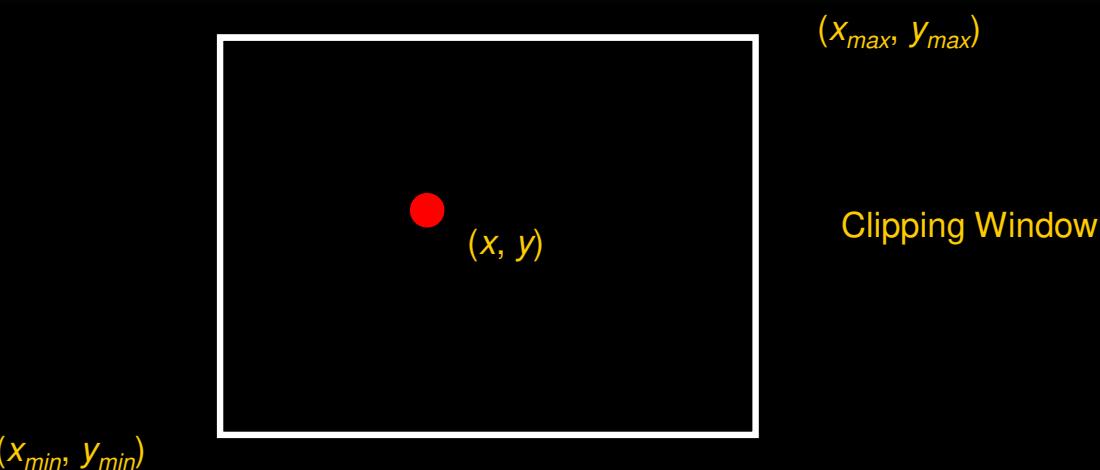
# 3D Clipping

## ◆ Clipping against 3D clipping volume



# 2D Clipping

- ◆ Define a clipping window with  $(x_{min}, y_{min})$  and  $(x_{max}, y_{max})$  as the lower-left and top-right coordinates, respectively.
- ◆  $x_{min} \leq x \leq x_{max}$  and  $y_{min} \leq y \leq y_{max}$  must be satisfied for a point  $(x, y)$  to be inside the clipping window

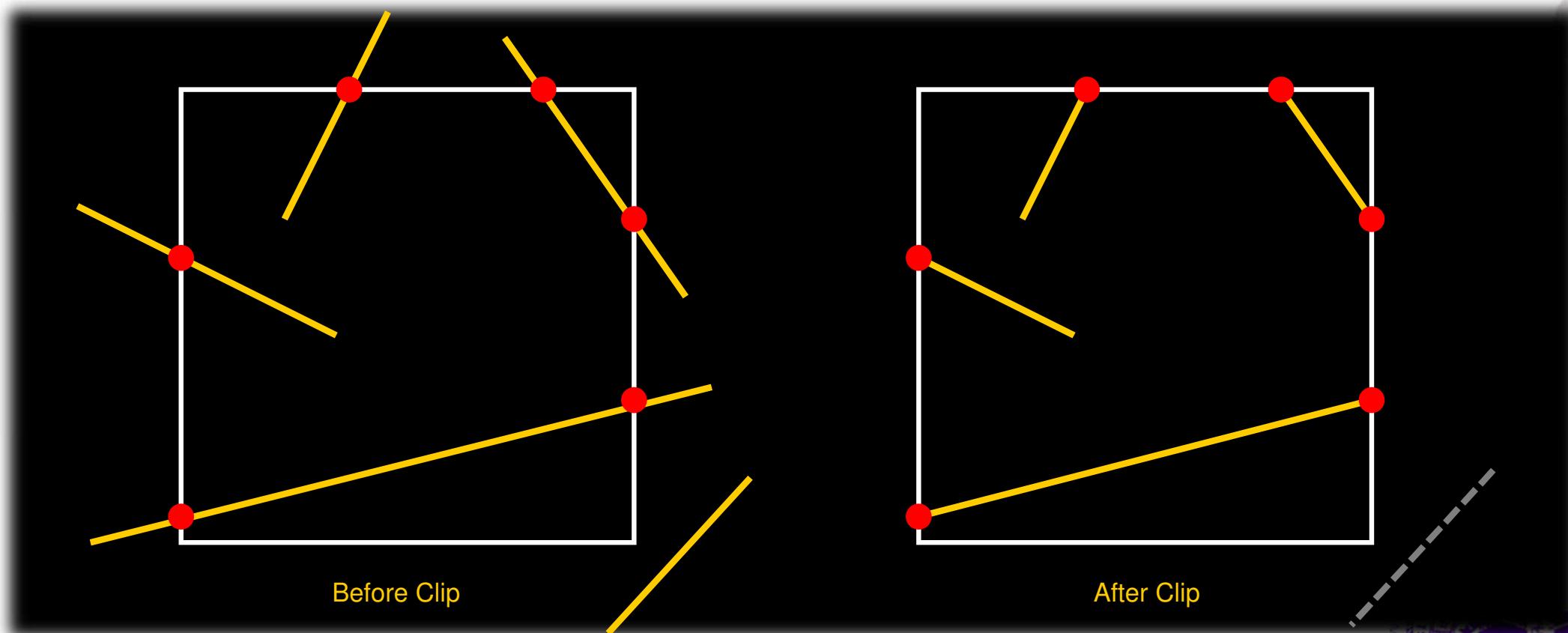


# *2D Clipping*

- ◆ Clipping line segments
- ◆ Clipping polygons
- ◆ Other primitives can be converted into line segments or polygons before clipping

# *Clipping 2D Line Segments*

- ◆ Compute intersections with each side of clipping window



# Clipping 2D Line Segments

- ◆ For each line segment,  $y = mx + b$ , compute the intersection with each side of clipping window

$$\begin{cases} y = mx + b \\ x = x_{\min} \end{cases}$$

$$y_{\min} \leq y \leq y_{\max}$$

$$\begin{cases} y = mx + b \\ x = x_{\max} \end{cases}$$

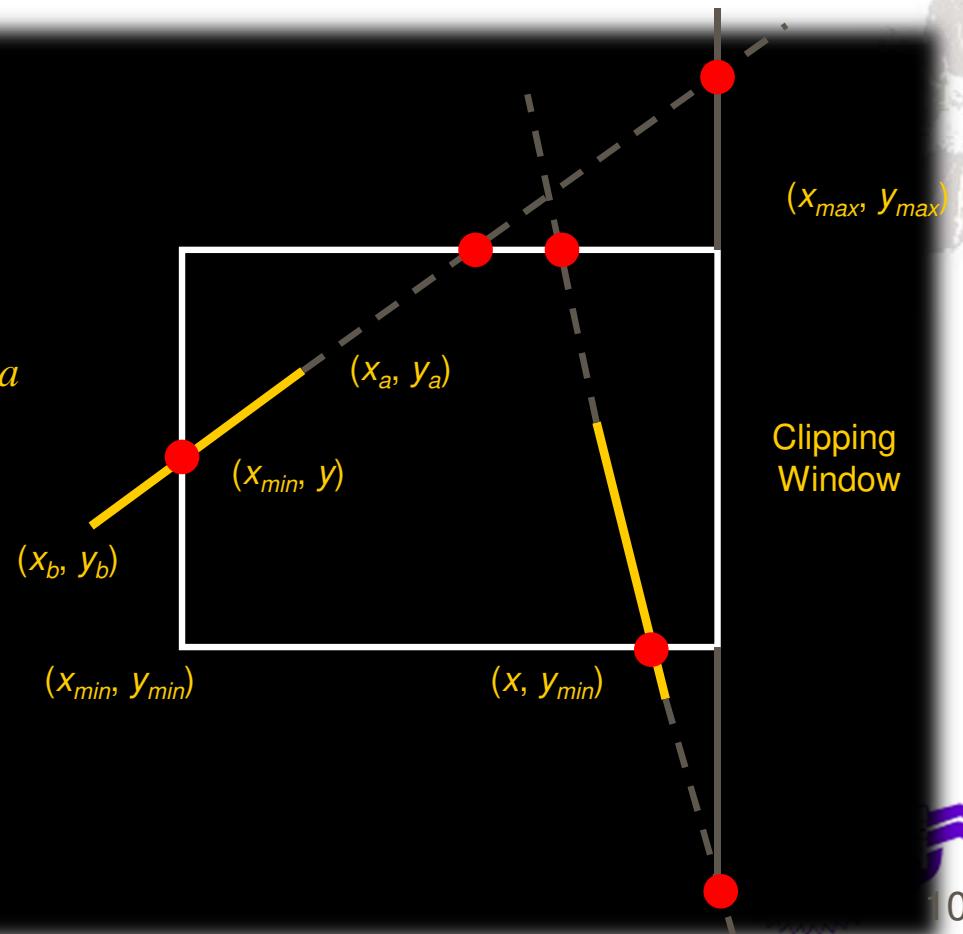
$$y_b \leq y \leq y_a \text{ if } y_b \leq y_a$$

$$\begin{cases} y = mx + b \\ y = y_{\min} \end{cases}$$

$$x_{\min} \leq x \leq x_{\max}$$

$$\begin{cases} y = mx + b \\ y = y_{\max} \end{cases}$$

$$x_b \leq x \leq x_a \text{ if } x_b \leq x_a$$



# Clipping 2D Line Segments

- ◆ Compute intersection point using parametric equations
  - Example for intersection with  $y = y_{\min}$

$$\begin{cases} x = x_a + t_0(x_b - x_a) \\ y = y_a + t_0(y_b - y_a) \end{cases} \quad \begin{cases} x = x_{\min} + t_1(x_{\max} - x_{\min}) \\ y = y_{\min} \end{cases}$$

$$\begin{cases} x = x_a + t_0(x_b - x_a) = x_{\min} + t_1(x_{\max} - x_{\min}) \\ y = y_a + t_0(y_b - y_a) = y_{\min} \end{cases}$$

Solving  $t_0$  and  $t_1$ , assume  $t_0 = a$  and  $t_1 = b$ ,  
if  $0 \leq a \leq 1$  and  $0 \leq b \leq 1$ , then

$x = x_{\min} + b(x_{\max} - x_{\min})$ ,  $y = y_{\min}$  is the intersection point

# *Clipping 2D Line Segments*

- ◆ Intersection derivation costs a lot of computations, such as division and multiplication.
- ◆ Reduce the number of intersection calculation can greatly speed up the clipping process.
- ◆ Do the intersection calculation only when it is necessary

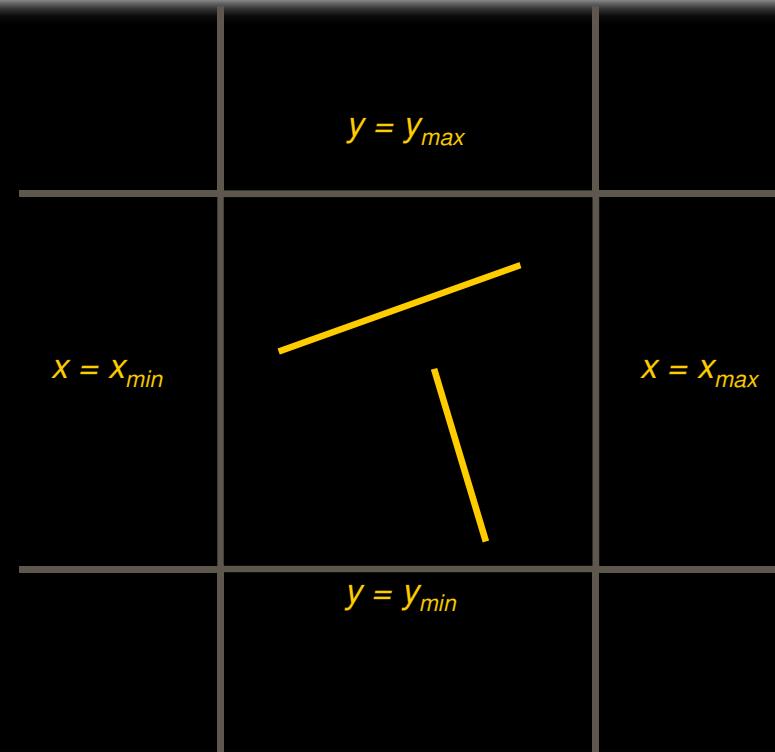
# *Clipping 2D Line Segments*

- ◆ **Cohen-Sutherland line clipping algorithm**
  - Step 1: check for trivially accepted
  - Step 2: Region check for trivially rejected
  - Step 3: Subdivide the line segment, repeat Step 1 and Step 2 until what remains is complete inside the clip rectangle or can be trivially rejected



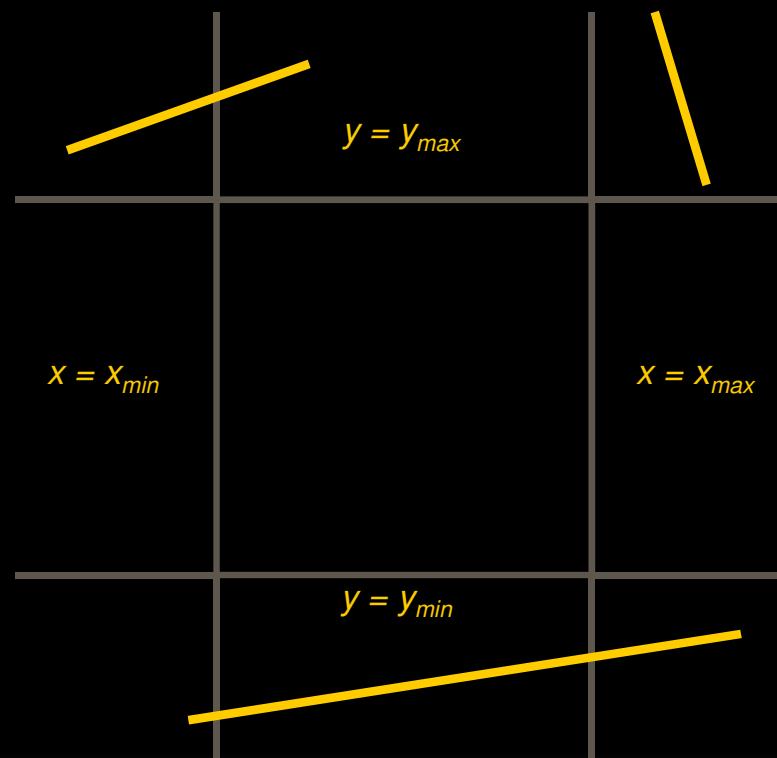
# Cohen-Sutherland Algorithm

- ◆ Case for trivially accepted
  - Both endpoints of a line segments lie inside the clipping rectangle



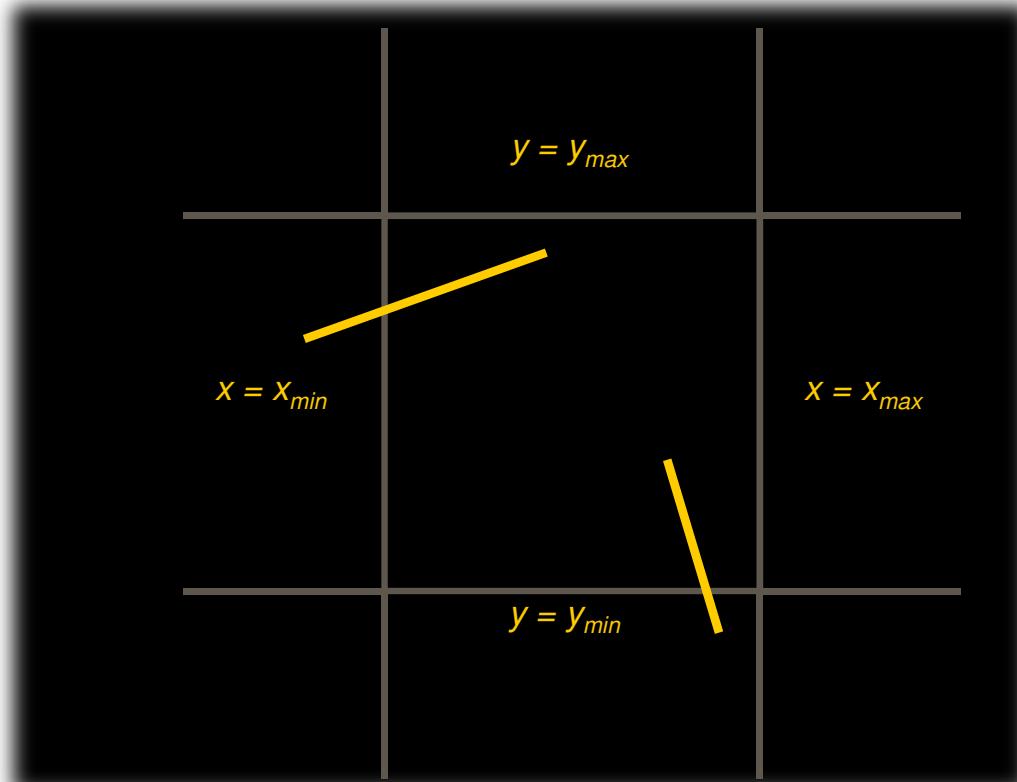
# Cohen-Sutherland Algorithm

- ◆ Case for trivially rejected
  - Both endpoints of a line segments lie outside one of the clipping boundary



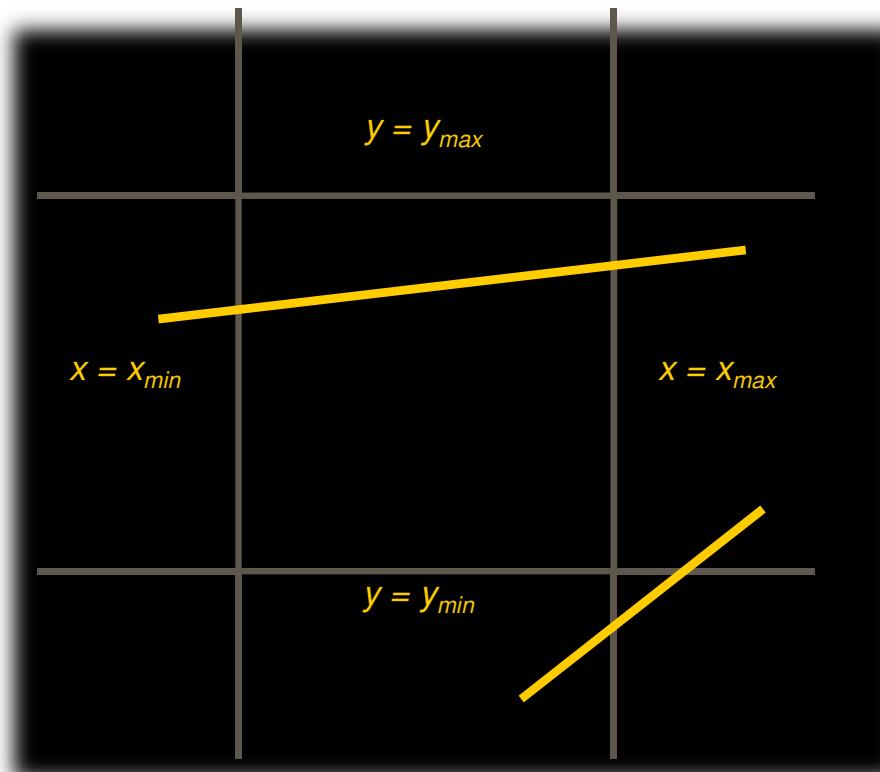
# Cohen-Sutherland Algorithm

- ◆ Cases that require intersection calculation
  - One endpoint of a line segments lies inside the clipping rectangle and the other endpoint is outside



# Cohen-Sutherland Algorithm

- ◆ Cases that require intersection calculation
  - Both endpoints of a line segments lie outside the clipping rectangle and the endpoints are not on the same side of a clipping boundary



# Cohen-Sutherland Algorithm

## ◆ Outcodes

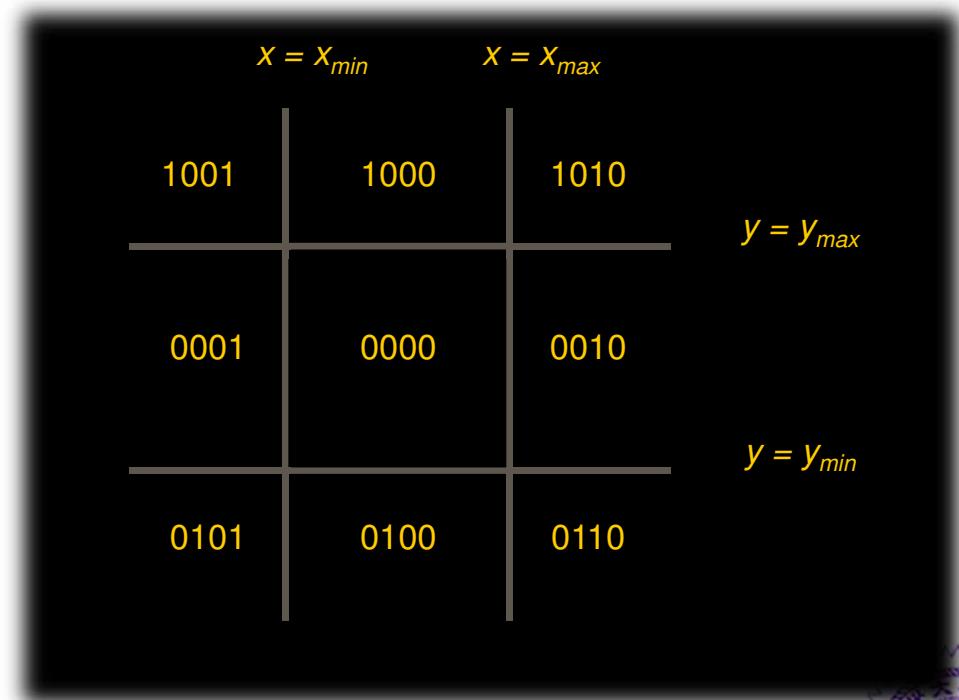
- For each endpoint  $(x, y)$  of a line segment, an outcode  $(b_0, b_1, b_2, b_3)$  is defined as

$b_0 = 1$ , if  $y > y_{max}$

$b_1 = 1$ , if  $y < y_{min}$

$b_2 = 1$ , if  $x > x_{max}$

$b_3 = 1$ , if  $x < x_{min}$



# Cohen-Sutherland Algorithm

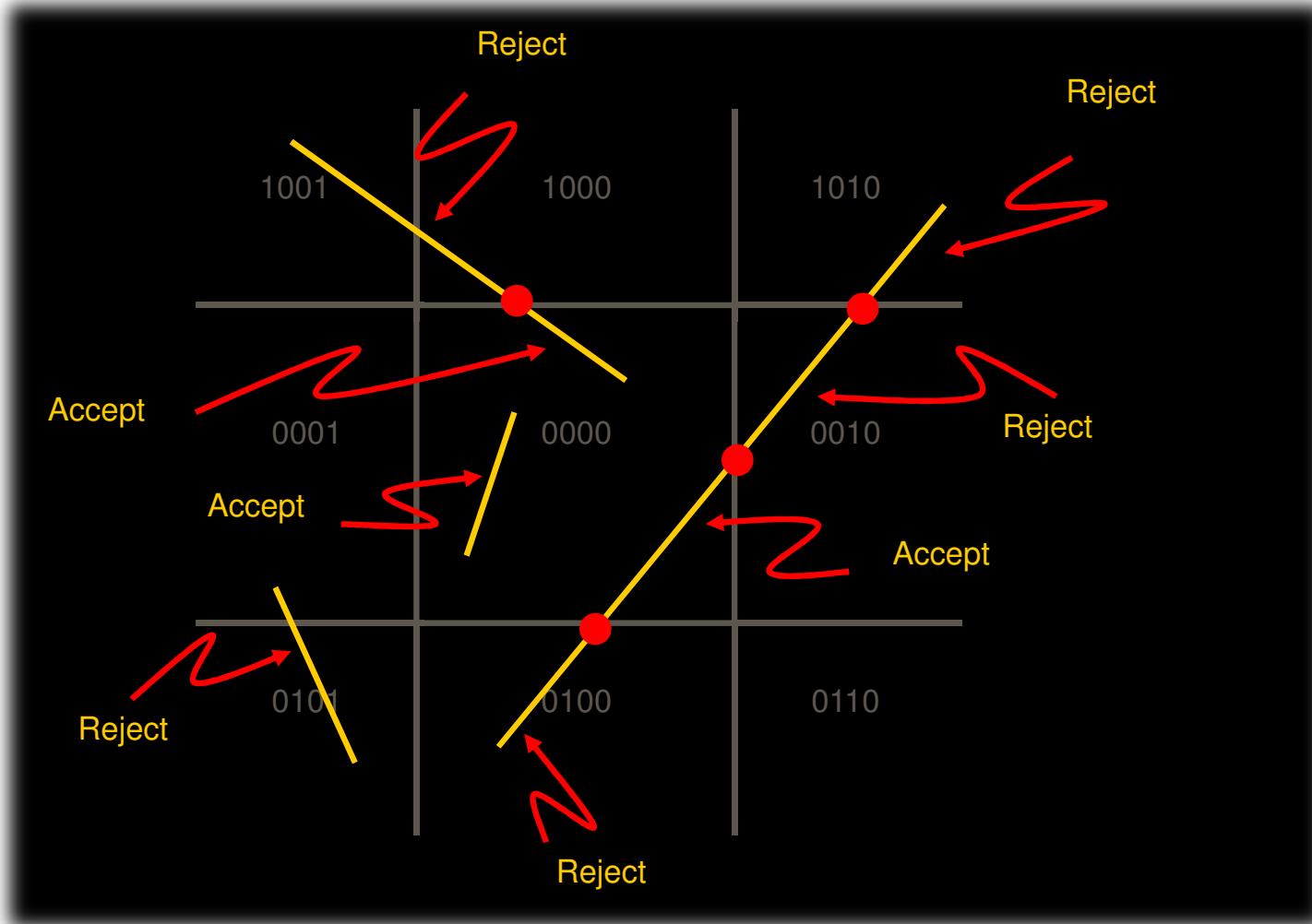
- ◆ Assume  $(b_0, b_1, b_2, b_3)_A$  and  $(b_0, b_1, b_2, b_3)_B$  are the outcodes of endpoints (A, B) of a line segment, respectively
  - If  $((b_0, b_1, b_2, b_3)_A) \& ((b_0, b_1, b_2, b_3)_B) \neq 0000$  implies trivially rejected
  - if  $(b_0, b_1, b_2, b_3)_A = (b_0, b_1, b_2, b_3)_B = 0000$  implies trivially accepted

1001	1000	1010
0001	0000	0010
0101	0100	0110



# Cohen-Sutherland Algorithm

## ◆ Examples



# *Cohen-Sutherland Algorithm*

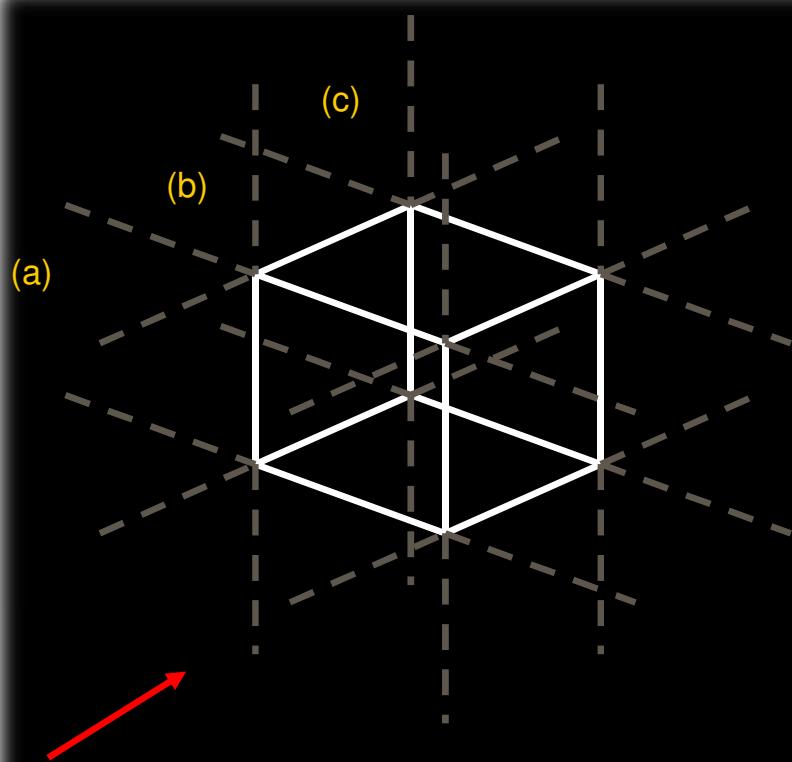
## ◆ Advantages

- Most line segments can be easily accepted or rejected without any intersection calculation
- Less intersection calculations are required in compare to brute force approach



# Clipping in 3D

- ◆ Extension of Cohen-Sutherland algorithm
  - 6-bit outcode



101001	101000	101010
100001	100000	100010
100101	100100	100110
(a)		
001001	001000	001010
000001	000000	000010
000101	000100	000110
(b)		

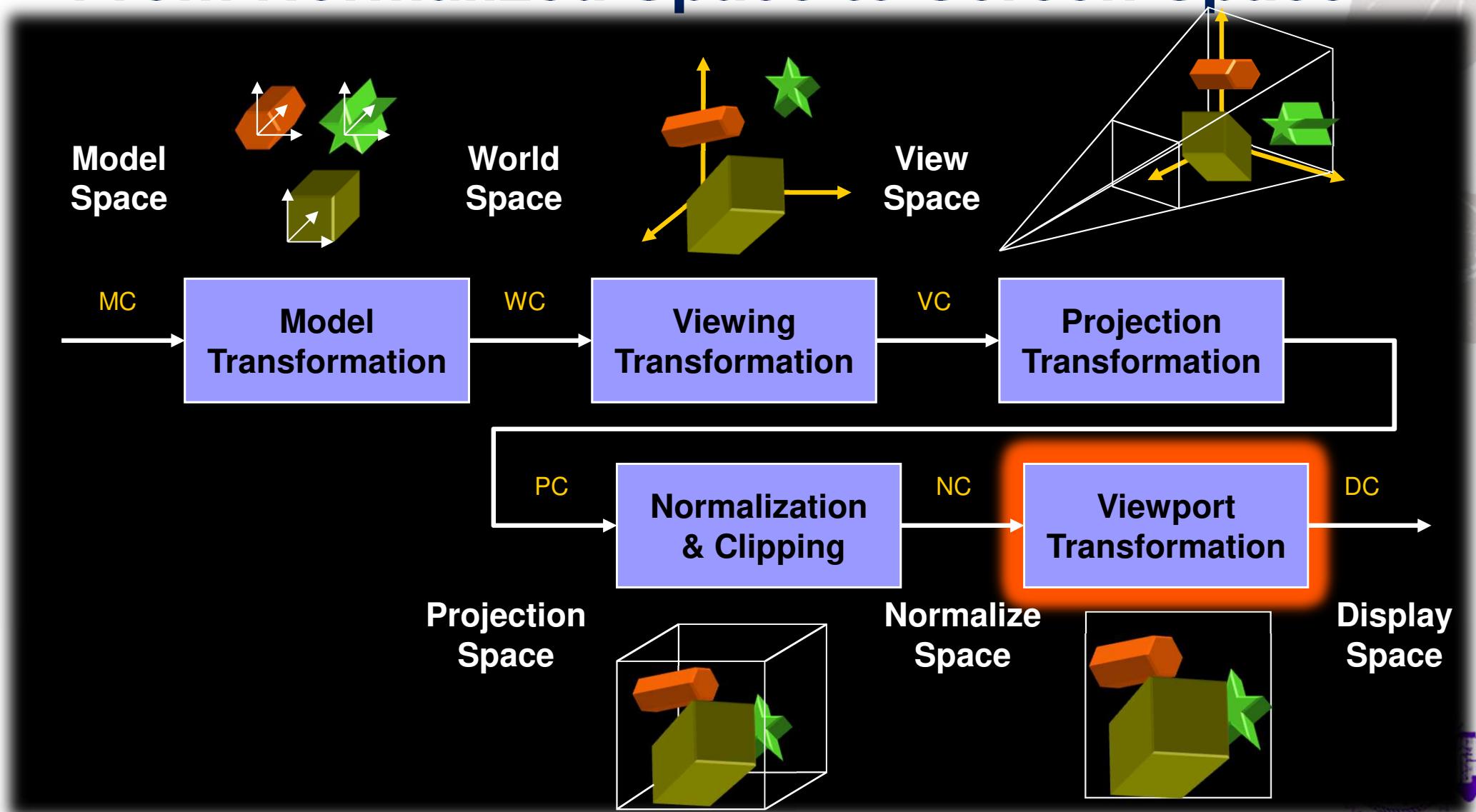
011001	011000	011010
010001	010000	010010
010101	010100	010110
(c)		

# *Viewport Transformation*



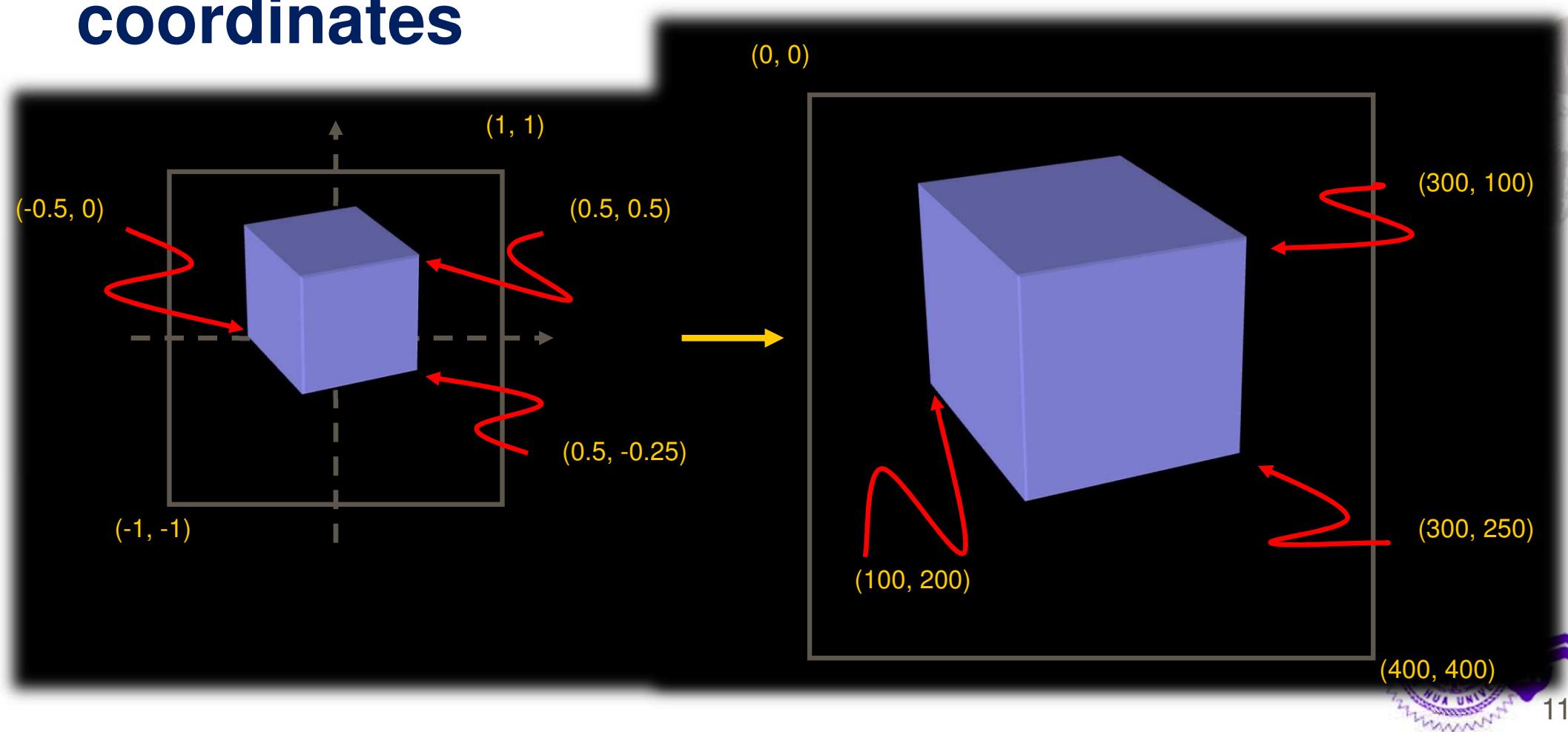
# *Review of 3D Viewing Process*

## ◆ From Normalized Space to Screen Space



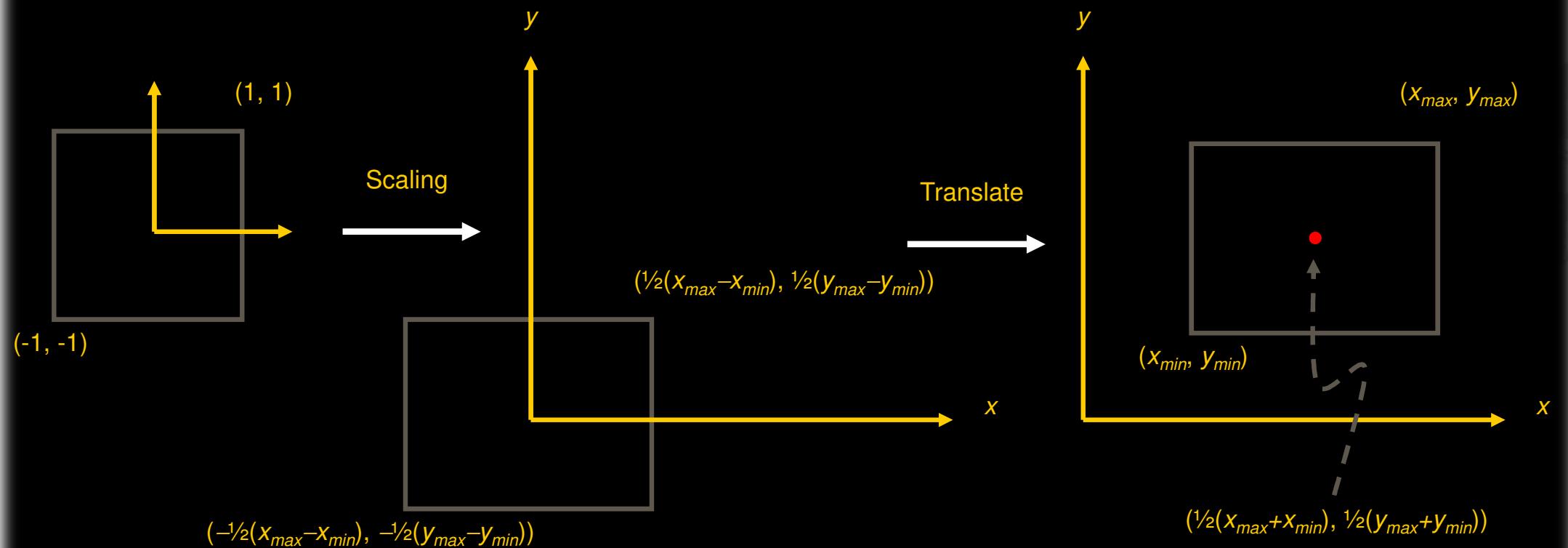
# Viewport Transformation

- ◆ Transform from normalized clipping coordinates to 2D display device coordinates



# Viewport Transformation

## ◆ Composition of viewport transformation



# Viewport Transformation

- ◆ Matrix representation of viewport transformation

$$M_{viewport} = T\left(\frac{x_{\max} + x_{\min}}{2}, \frac{y_{\max} + y_{\min}}{2}, 0\right) \cdot S\left(\frac{x_{\max} - x_{\min}}{2}, \frac{y_{\max} - y_{\min}}{2}, 1\right)$$
$$= \begin{bmatrix} \frac{x_{\max} - x_{\min}}{2} & 0 & 0 & \frac{x_{\max} + x_{\min}}{2} \\ 0 & \frac{y_{\max} - y_{\min}}{2} & 0 & \frac{y_{\max} + y_{\min}}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# Viewport Transformation

## ◆ Depth Range Adjustment

- E.g., set the depth range within [0, 1]

$$M_{viewport} = T\left(\frac{x_{max} + x_{min}}{2}, \frac{y_{max} + y_{min}}{2}, 0.5\right) \cdot S\left(\frac{x_{max} - x_{min}}{2}, \frac{y_{max} - y_{min}}{2}, \frac{1}{2}\right)$$

Translate the range from [-0.5, 0.5] to [0, 1]      Scale the range from [-1, 1] to [-0.5, 0.5]

# *OpenGL Transformations*

*glOrtho*

*glFrustum*

*gluPerspective*

*gluLookAt*



# *OpenGL Conventions*

- ◆ In order to derive the similar equations for OpenGL `glOrtho`, `glFrustum`, and `gluPerspective`, the following terms should be aware of
  - $x_{max} = Right$ ;  $x_{min} = Left$
  - $y_{max} = Top$ ;  $y_{min} = Bottom$
  - $z_{near} = -Near$ ;  $z_{far} = -Far$
- ◆ It is because the definition of Near and Far in the APIs are the distances to the viewpoint instead of the Z coordinates

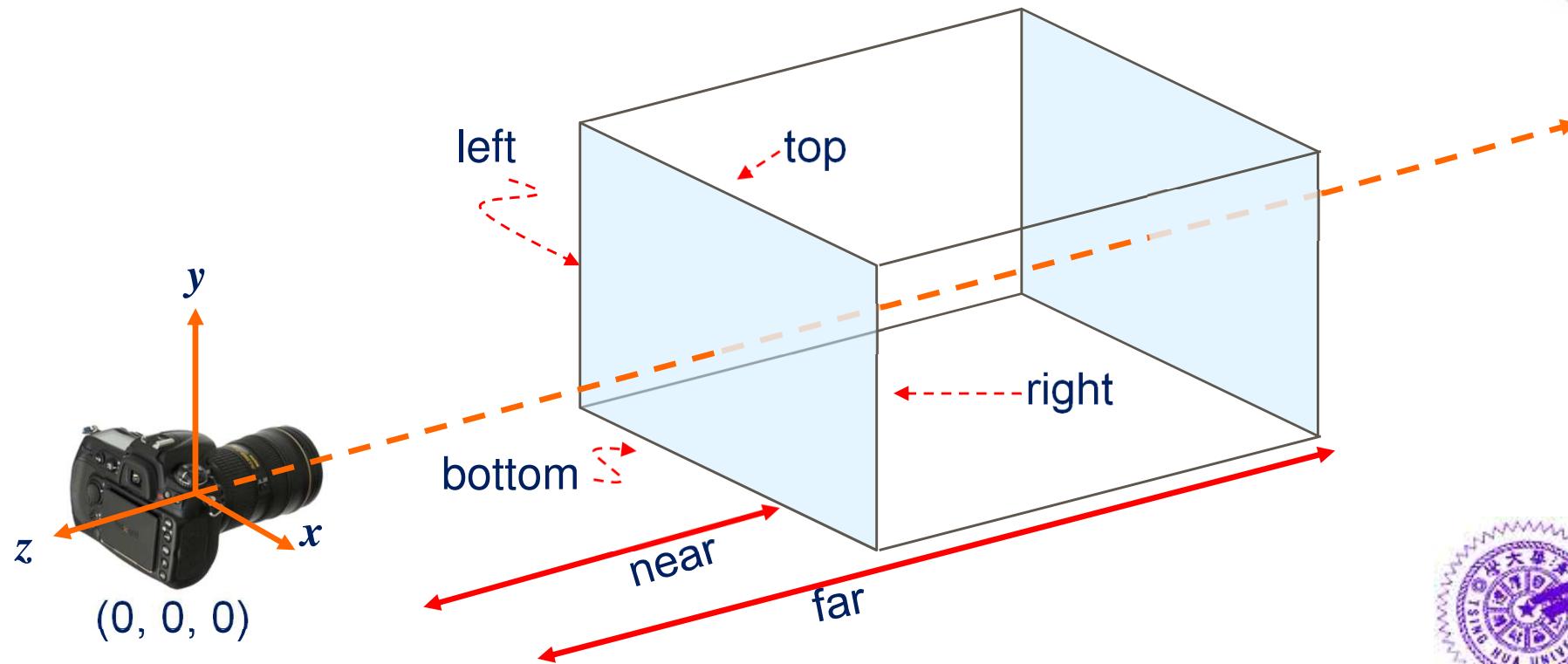


# glOrtho

## ◆ OpenGL Orthographic Projection

### ■ `glOrtho(left, right, bottom, top, near, far)`

- ▶ *left, right, bottom, top*: coordinates
- ▶ *near, far*: distances to viewpoint.



# gOrtho

- ◆ OpenGL Orthographic Transformation Matrix
  - Orthographic (parallel) projection and orthographic normalization

$$\begin{pmatrix} \frac{2}{Right-Left} & 0 & 0 & t_x \\ 0 & \frac{2}{Top-Bottom} & 0 & t_y \\ 0 & 0 & \frac{-2}{Far-Near} & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

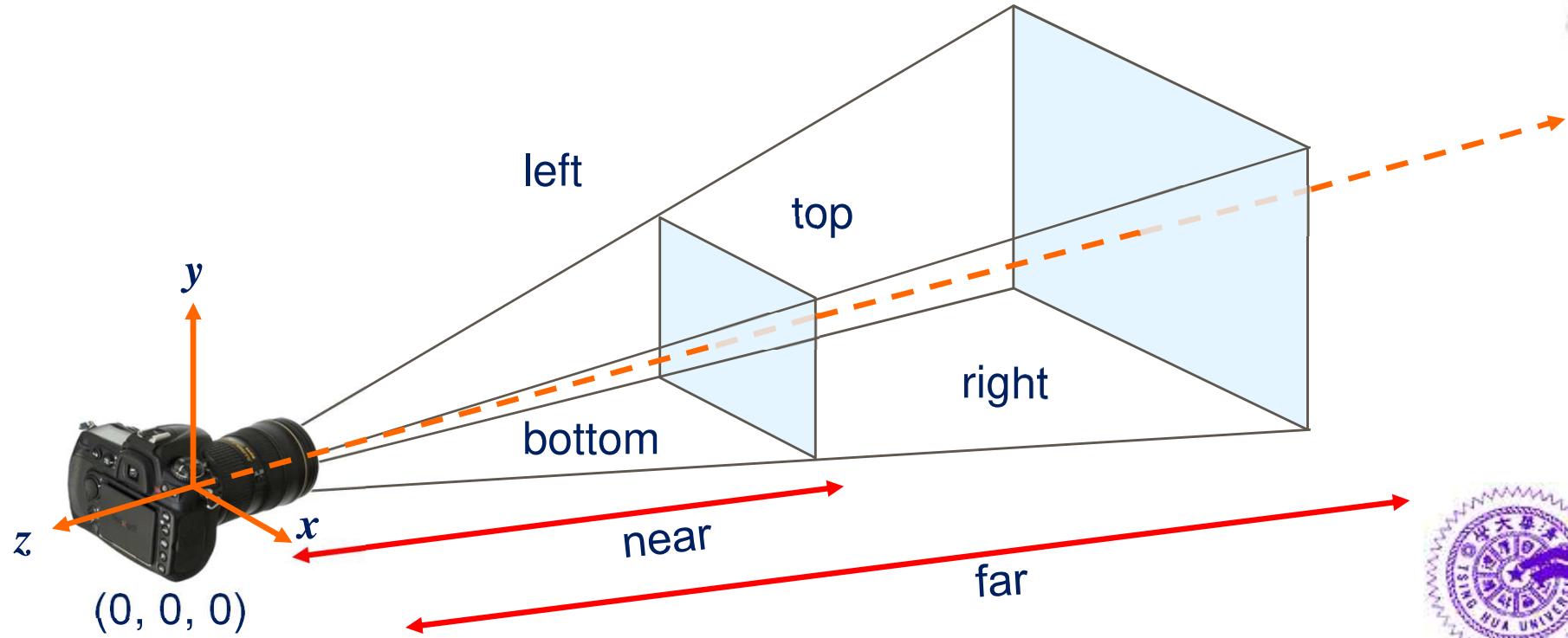
$$t_x = -\frac{Right+Left}{Right-Left}$$
$$t_y = -\frac{Top+Bottom}{Top-Bottom}$$
$$t_z = -\frac{Far+Near}{Far-Near}$$



# glFrustum

## ◆ OpenGL Perspective Projection

- **glFrustum(left, right, bottom, top, near, far)**
  - ▶ *left, right, bottom, top*: coordinates
  - ▶ *near, far*: distances to viewpoint



# glFrustum

- ◆ OpenGL Perspective Transformation Matrix
  - Perspective projection and perspective normalization

$$\begin{pmatrix} \frac{2 \text{ Near}}{\text{Right} - \text{Left}} & 0 & A & 0 \\ 0 & \frac{2 \text{ Near}}{\text{Top} - \text{Bottom}} & B & 0 \\ 0 & 0 & C & D \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$A = \frac{\text{Right} + \text{Left}}{\text{Right} - \text{Left}}$$

$$B = \frac{\text{Top} + \text{Bottom}}{\text{Top} - \text{Bottom}}$$

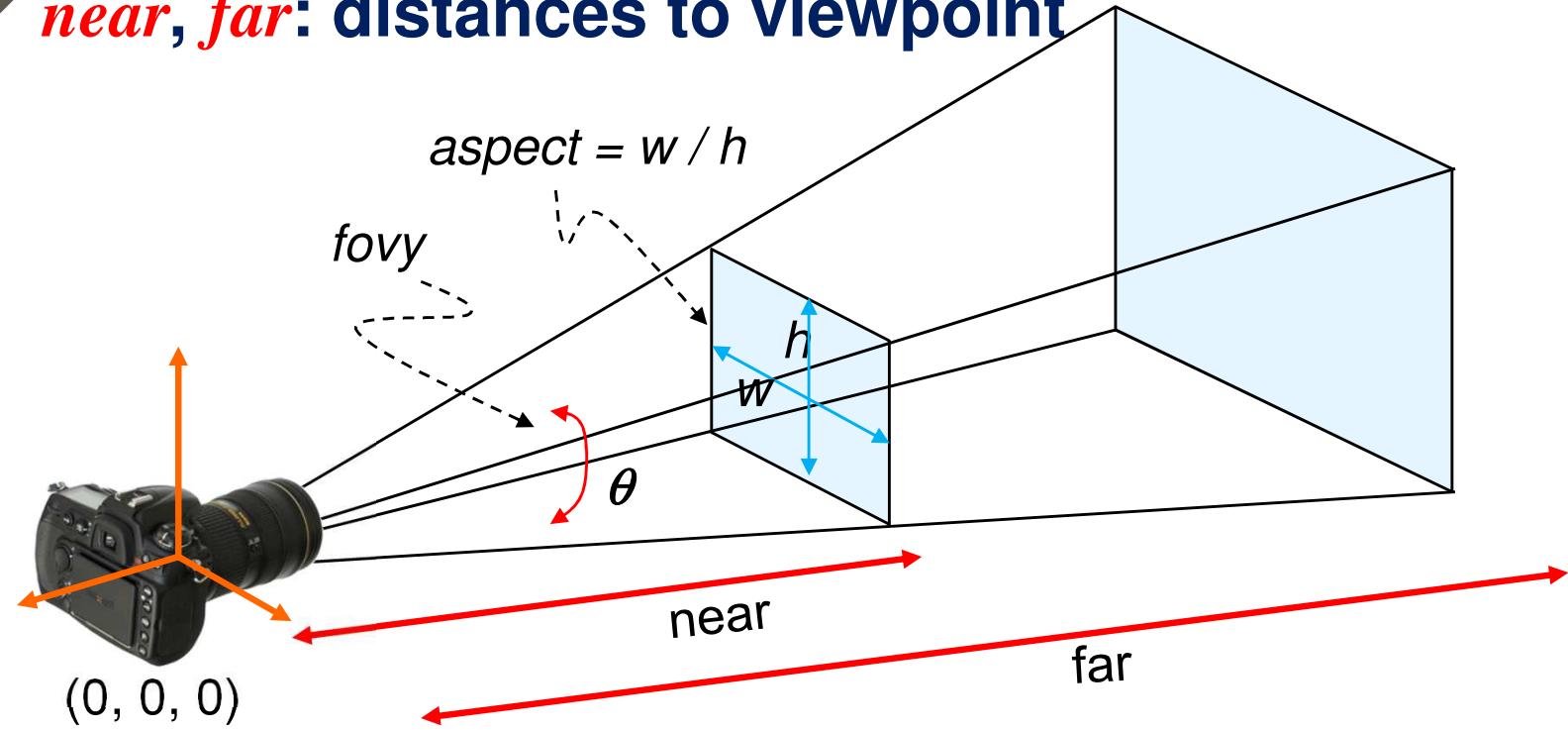
$$C = -\frac{\text{Far} + \text{Near}}{\text{Far} - \text{Near}}$$

$$D = -\frac{2\text{Far} \text{ Near}}{\text{Far} - \text{Near}}$$



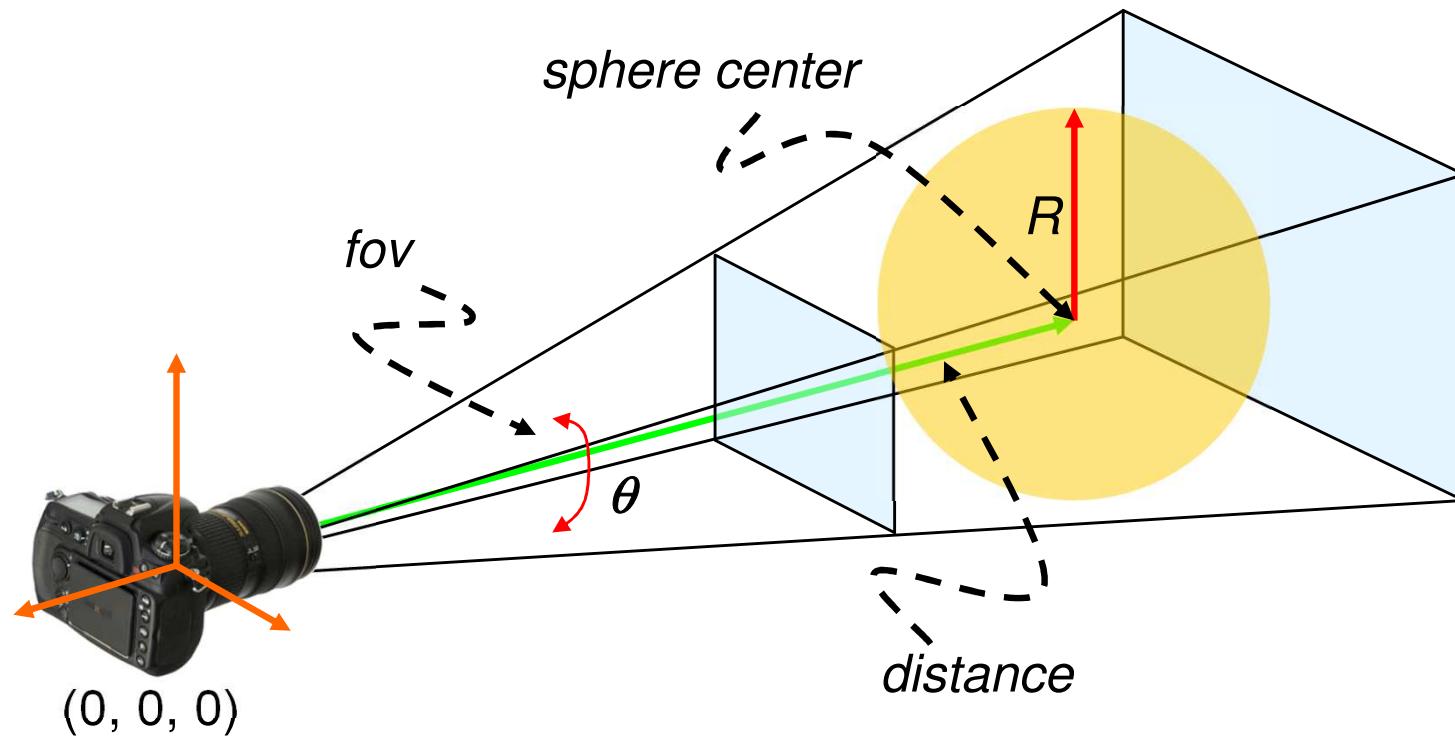
# *gluPerspective*

- ◆ OpenGL Perspective Projection
  - **gluPerspective(*fovy*, *aspect*, *near*, *far*)**
    - ▶ *fovy*: field of view angle
    - ▶ *aspect*: aspect ratio of width to height
    - ▶ *near*, *far*: distances to viewpoint



# *gluPerspective*

- ◆ How to derive a good field of view angle
  - Determine the bounding sphere of the scene
  - $fov = 2 \times \tan^{-1}(R/distance)$ 
    - ▶  $R$ : sphere radius;  $distance$ : camera to sphere center



# *gluPerspective*

- ◆ OpenGL Perspective Transformation Matrix
  - Perspective projection and perspective normalization

$$\begin{vmatrix} \frac{f}{\text{aspect}} & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & \frac{\text{far} + \text{near}}{\text{near} - \text{far}} & \frac{2 \cdot \text{far} \cdot \text{near}}{\text{near} - \text{far}} \\ 0 & 0 & -1 & 0 \end{vmatrix}$$

$$f = \cot \left( \frac{\text{fovy}}{2} \right)$$



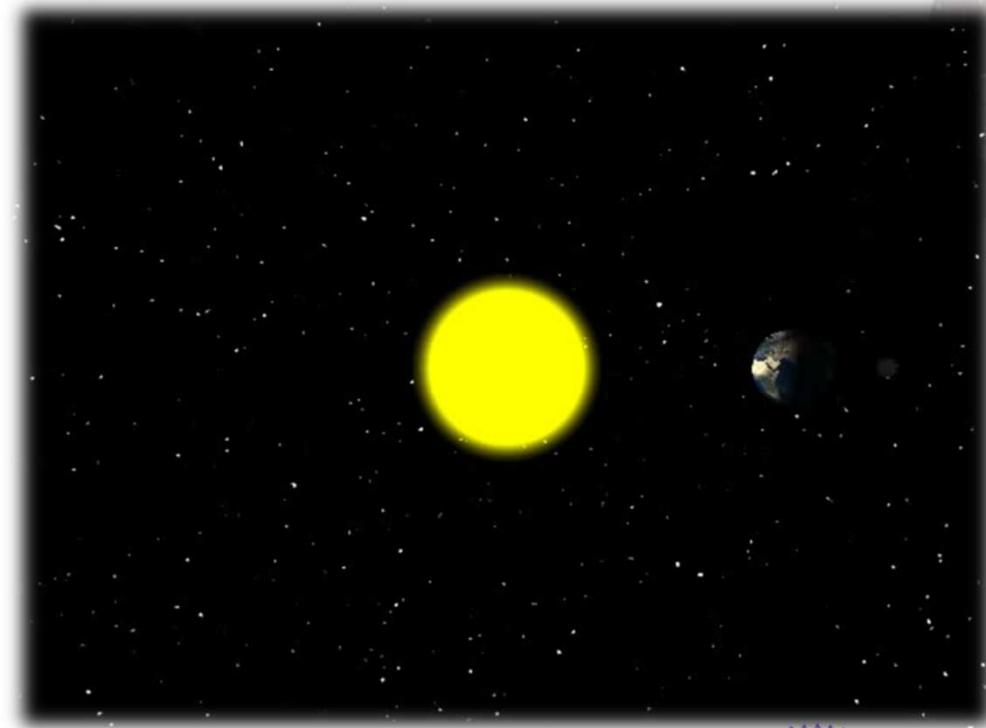
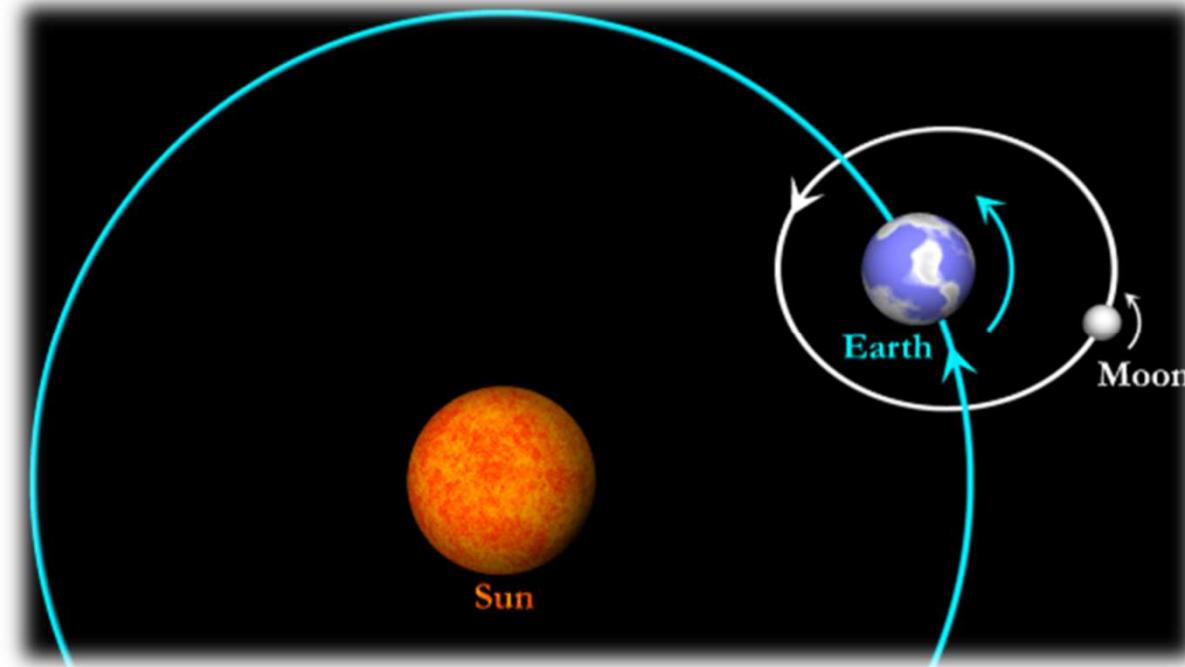
# *Managing Transformation Matrices*

***Matrix Stack***



# *Why need to manage the matrices?*

- ◆ Consider a simplified solar system with sun, earth, and moon



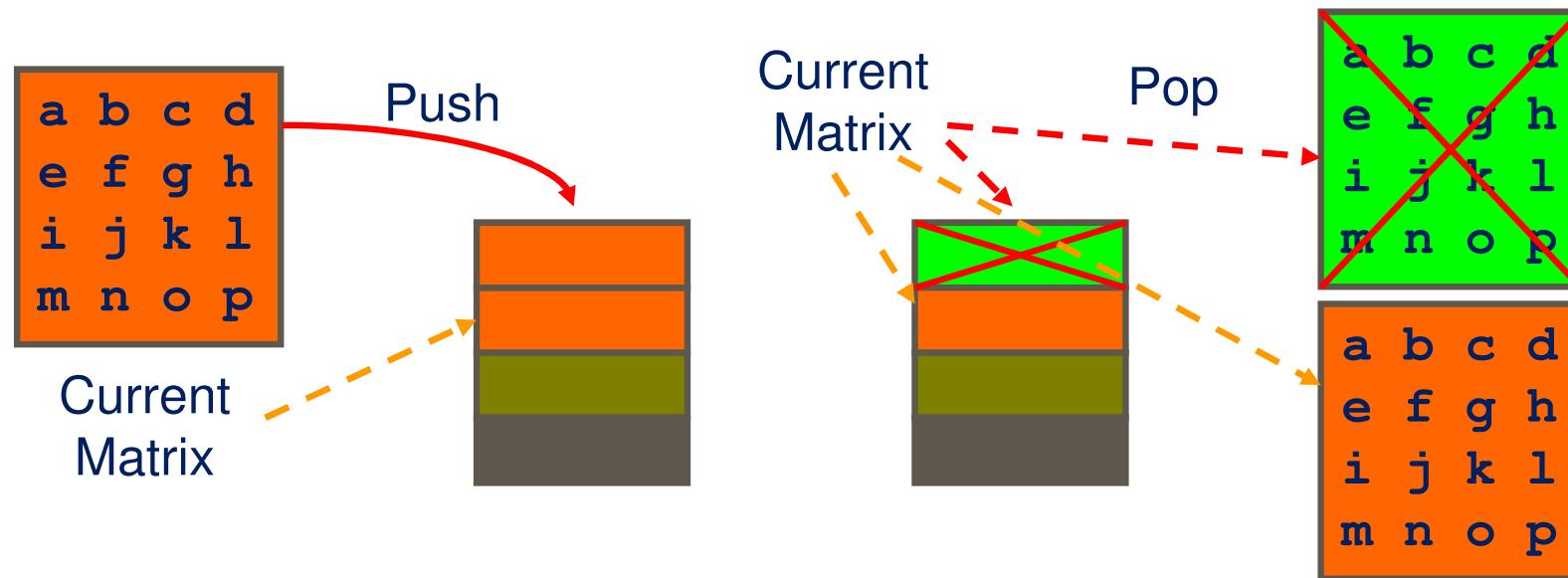
# *Matrix Stack*

- ◆ Matrix stack is useful to remember some transformation matrix at certain point and restore it back when necessary
- ◆ OpenGL used to manage the matrix stack for user but is now **deprecated**
- ◆ You can implemented a similar mechanism by your own or use other way to manage your matrices



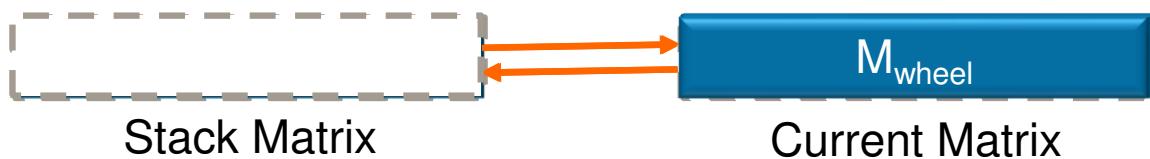
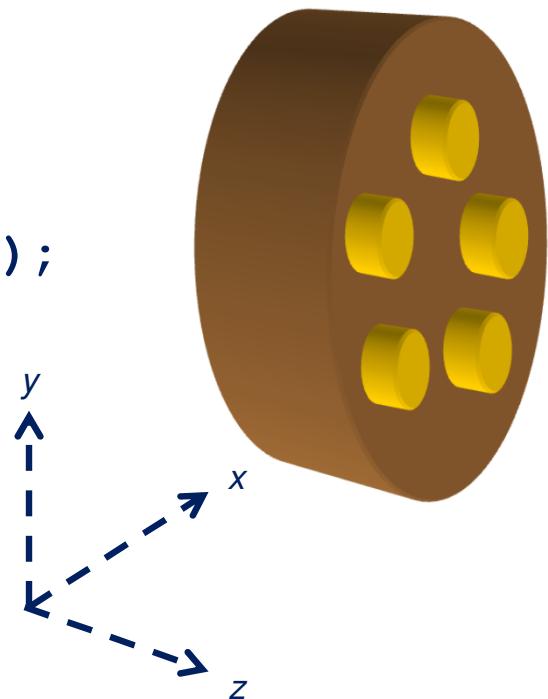
# Matrix Stack in OpenGL

- ◆ Top of the matrix stack is the current matrix
- ◆ Any transformation is deal with the current matrix, i.e. the top of matrix stack
- ◆ Use *PushMatrix* or *PopMatrix* to push into or pop out of the matrix stack



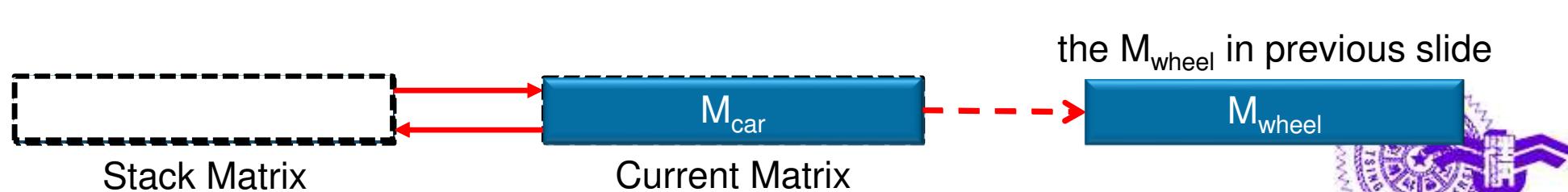
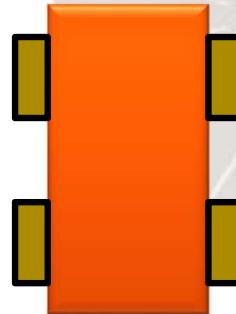
# *Example of Using Matrix Stack*

```
draw_wheel_and_bolts()
{
    int i;
    draw_wheel();
    for(i=0;i<5;i++) {
        glPushMatrix();
        glRotatef(72.0*i, 0.0, 0.0, 1.0);
        glTranslatef(3.0, 0.0, 0.0);
        draw_bolt();
        glPopMatrix();
    }
}
```



# *Example of Using Matrix Stack*

```
draw_body_and_wheel_and_bolts()
{
    → draw_car_body();
    → glPushMatrix();
    →     glTranslatef(40, 0, 30); /*move to first wheel position*/
    →     draw_wheel_and_bolts();
    → glPopMatrix();
    → glPushMatrix();
    →     glTranslatef(40, 0, -30); /*move to 2nd wheel position*/
    →     draw_wheel_and_bolts();
    → glPopMatrix();
    → ...
    →         /*draw last two wheels similarly*/
}
```

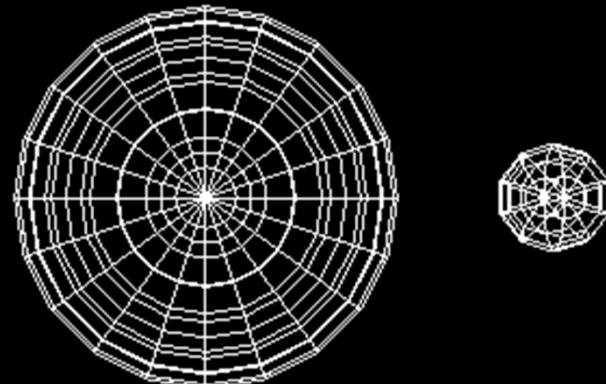


# *Composition of 3D Transformations*

## ◆ Example I: Building a Solar System

```
void display(void)
{
    glClear(GL_COLOR_BUFFER_BIT);
    glColor3f(1.0, 1.0, 1.0);
    glPushMatrix();
    glutWireSphere(1.0, 20, 16); /* draw sun */
    glRotatef((GLfloat) year, 0.0, 1.0, 0.0);
    glTranslatef(2.0, 0.0, 0.0);
    glRotatef((GLfloat) day, 0.0, 1.0, 0.0);
    glutWireSphere(0.2, 10, 8); /* draw smaller planet */
    glPopMatrix();
    glutSwapBuffers();
}
```

Restore the sun position



Store the sun position

Eg.  $360^\circ \times (\text{date}/365)$

Eg.  $360^\circ \times (\text{hour}/24)$

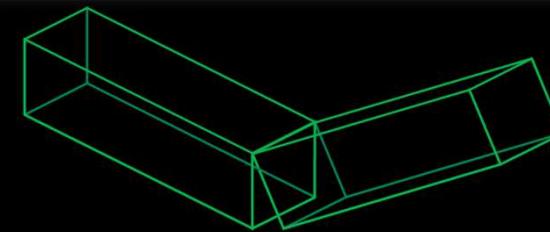
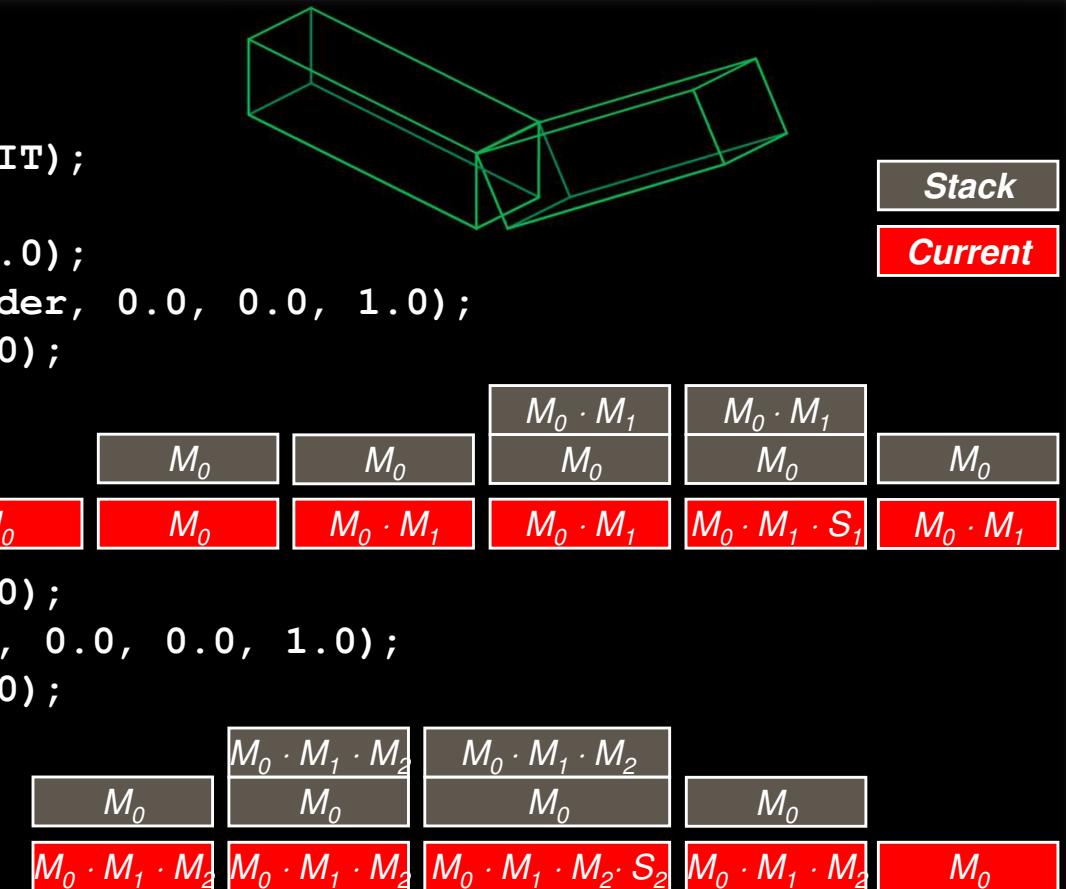
# Composition of 3D Transformations

## ◆ Example II: Articulated Robot Arm

```

void display(void)
{
    glClear(GL_COLOR_BUFFER_BIT);
    M0 → glPushMatrix();
    M1 → { glTranslatef(-1.0, 0.0, 0.0);
              glRotatef((GLfloat) shoulder, 0.0, 0.0, 1.0);
              glTranslatef(1.0, 0.0, 0.0);
    }
    M0 · M1 → glPushMatrix();
    M0 · M1 · S1 → glScalef(2.0, 0.4, 0.4);
                           glutWireCube(1.0);
    M0 · M1 → glPopMatrix();
    M2 → { glTranslatef(1.0, 0.0, 0.0);
              glRotatef((GLfloat) elbow, 0.0, 0.0, 1.0);
              glTranslatef(1.0, 0.0, 0.0);
    }
    M0 · M1 · M2 → glPushMatrix();
    M0 · M1 · M2 · S2 → glScalef(2.0, 0.4, 0.4);
                           glutWireCube(1.0);
    M0 · M1 · M2 → glPopMatrix();
    M0 → glPopMatrix();
    glutSwapBuffers();
}

```



# Q&A

