Lecture 2.1: Conditional Probability

2013/09/09

Previously...

- We went over some examples of combinations/permutations
- Used the definition of probability to:
  - Derive the hypergeometric distribution
  - Solve the Birthday Problem
- Introduced the Monty Hall Problem.

Previously... Hypergeometric Distribution

- N total number of balls in the urn
- n number of balls drawn from the urn without replacement
- K number of "success" balls in the urn (in our case, black)
- X is the (random) number of "success" balls drawn

What is the probability that we draw k successes?

$$\mathbb{P}(X = k) = \frac{\binom{K}{k} \times \binom{N-K}{n-k}}{\binom{N}{n}}$$

Note the difference between K and k.

Previously... Birthday Problem

We showed the solution to the birthday problem. Let

$$
\begin{aligned}
E &= \text{At least two people have the same birthday} \\
E^c &= \text{No two people have the same birthday}
\end{aligned}
$$

we have

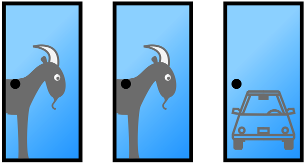
$$P(E) = 1 - P(E^c) \;\; = \;\; 1 - \frac{P_n^{365}}{365^n}$$

For $n = 2$:

$$
\begin{aligned}
P(E^c) &= \frac{\text{Both do not have the same birthday}}{\#\text{ of possible birthday pairs}} \\
&= \frac{365 \times 364}{365 \times 365} = \frac{P_n^{365}}{365^2}
\end{aligned}
$$

Monty Hall Problem

You are presented with 3 closed doors. Behind two of the doors are goats, and behind the third is a car. You pick a door, which remains closed. The host, who knows behind which of the doors lies the car, then opens one of other two doors to reveal a goat. He asks "Do you want to switch doors or not?"



What do you do? We will demonstrate the most effective strategy via conditional probability

Goals for today

1. Define conditional probability
2. Introduce the multiplication rule
3. Demonstrate the solution to the Monty Hall Problem

Conditional Probability

Often times we want to evaluate the effect of one event occurring on a subsequent event:

- What is the probability of sampling an $A\clubsuit$ after having sampled a $10\heartsuit$?
- What is the probability of flipping a heads given that you flipped a heads just before hand?
- Sampling from an urn without replacement

Example Demonstrating You Already Know Cond. Prob.

Let's suppose I take a random sample of 100 Reed students to study their smoking habits.

	Smoker	Not Smoker	Total
Male	19	41	60
Female	12	28	40
Total	31	69	100

- What is the probability of a randomly selected male smoking?
- What is the probability that a randomly selected smoker is female?

Conditional Probability

Back to Example

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Conditional Probability Satisfies Kolmogorov Axiom 3

If A_1 and A_2 are mutually exclusive, then

Multiplication Rule (of Probability)

Another way to consider conditional probability is via the multiplication rule:

Often times trying to evaluate the probability of $A \cap B$ is difficult, but evaluating them as a sequence:

1. What is the probability of event A occurring? THEN
2. What is the probability of event B occurring given that event A occurred?

Example

Let

- E_1 be the event that you pick an ace of spades on your first card
- E_2 be the event that you pick a spade on your second card

$$\mathbb{P}(E_1 \cap E_2) = \mathbb{P}(E_2 | E_1) \cdot \mathbb{P}(E_1) = \frac{12}{51} \times \frac{1}{52}$$

Multiplication Rule Extended to n Events

For any n events $E_1, \ldots, E_n$

Monty Hall Problem

Let's solve this with what we've learned today:

Some notes:

1. Assume w/o loss of generality that the car is behind Door 1
2. Choices of doors by both the host and the contestant are random.
3. Think in terms of events $O_y \cap P_x$
   - O_y: Host opens door y
   - P_x: contestant picks door x

   and how "switching" doors related to these events to winning vs losing.

Monty Hall Problem

Assume w/o loss of generality that the car is behind Door 1:

Monty Hall Problem

What trips people up is they think the probability of the car being behind the door you selected goes up from $\frac{1}{3}$ to $\frac{1}{2}$.

Another way of thinking about the problem is the strategy of switching works only if you didn't pick the car originally:

- Probability of $\frac{2}{3}$ that you didn't pick the car initially.
- Probability of $\frac{1}{3}$ that you did.

Next Time:

- Independence of events