Lecture 4.1: Discrete Expectation + Variance

2013/09/23

Previously... Discrete Random Variables

Definition

If the set of all possible values of a random variable X is a countable set, $x_1, x_2, \ldots$, then X is called a discrete random variable. The function

$$f(x) = \mathbb{P}(X = x) \text{ for } x = x_1, x_2, \ldots$$

that assigns the probability to each possible value x will be called the probability mass function (PMF). Note: called a discrete probability density function in text, while some other texts use $p(x)$

Previously... Properties

Theorem

A function $f(x)$ is a discrete pdf IFF it satisfies both of the following properties for at most a countably infinite set of real $x_1, x_2, \ldots$

1. $f(x_i) \geq 0$ *for all* x_i*: non-negative prob*
2. $\sum_{x_i} f(x_i) = 1$*: sums to one*

Previously... Bernoulli Random Variable

Suppose we have a trial/experiment whose outcome can be classified as "success" or "failure." Let $X = 1$ if the outcome is a success and $X = 0$ if the outcome is a failure, then

$$f(x) = \begin{cases} 1 - p & \text{if } x = 0 \\ p & \text{if } x = 1 \end{cases}$$

where $0 \leq p \leq 1$ is the probability of a success.

Previously... Binomial Random Variable

Say we have n independent trials of a Bernoulli random variable each with probability of success p. Then the probability that we have x successes (I used k last class) is

$$f(x) = \mathbb{P}(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$$

We say that X is a Binomial(n, p) distribution.

Example: $n = 10$ die rolls and define getting a 1 as a success i.e. $p = \frac{1}{6}$. Let X be the (random) number of 1's. Then $X \sim$ Binomial$(10, \frac{1}{6})$

Previously... Geometric Random Variable

Once again say we have success/failure Bernoulli trials with probability p of success. What is the probability that it takes x independent trials until we get a success?

$$f(x) = \mathbb{P}(X = x) \quad = \quad (1-p)^{x-1}p$$

We say $X \sim$ Geometric(p) distribution.

Goals for Today

- Think intuitively build up the notion of expected value and variance
- Define them mathematically
- Derive the expected value of a Binomial(n, p) RV
- Go over some properties of expectation and variance

Intuitively Thinking: Expected Value

Easy example: Coin flips. Say we flip a fair coin $n = 10$ times with probability $p = \frac{1}{2}$ of heads.

How many heads do you expect to get?

$n \times p = 10 \times \frac{1}{2} = 5$

Intuitively Thinking: Expected Value

Slightly more complicated example: Say you have a discrete random variable X with PMF:

x	2	3	4	10	11
$f(x) = \mathbb{P}(X = x)$	$\frac{15}{100}$	$\frac{25}{100}$	$\frac{10}{100}$	$\frac{30}{100}$	$\frac{20}{100}$

E.g. We observe $X = 3$ with prob .25

Is the value we expect to observe:

$$\frac{2+3+4+10+11}{5} \quad = \quad 6\ ?$$

Intuitively Thinking: Expected Value

No, each of the x's have different probability of occurring.

For each x, we assign weight $f(x) = \mathbb{P}(X = x)$.
i.e. for all x, we have $x \cdot f(x)$:

$$2 \times \frac{15}{100} + 3 \times \frac{25}{100} + 4 \times \frac{10}{100} + 10 \times \frac{30}{100} + 11 \times \frac{20}{100}$$

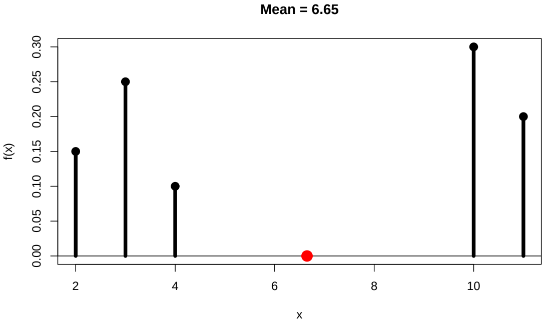
and not $$2 \times \frac{1}{5} + 3 \times \frac{1}{5} + 4 \times \frac{1}{5} + 10 \times \frac{1}{5} + 11 \times \frac{1}{5}$$

Expected Value

The expected value is a weighted average of all possible values x.
This can be thought of as a measure of center.

Expected Value

You can also think of the mean as the center of mass or balance point (marked with red point):



Expected Value of a Binomial(n, p)

If $X \sim$ Binomial(n, p) i.e. X is distributed Binomial with n trials and probability of success p, then

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

We also know that for PMF's:

$$\sum_x f(x) = 1$$

so in our case

$$\sum_{x=0}^{n} f(x) = 1$$

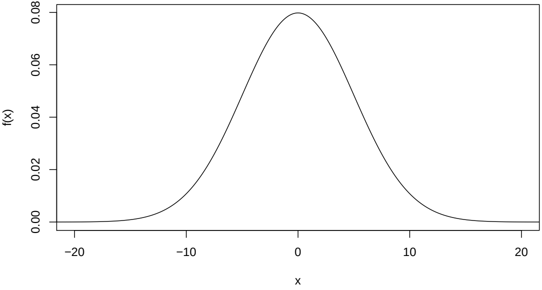
Expected Value of a Binomial(n, p)

Expected Value of a Binomial(n, p)

Linearity of Expectations

Intuitively Thinking: Measures of Spread

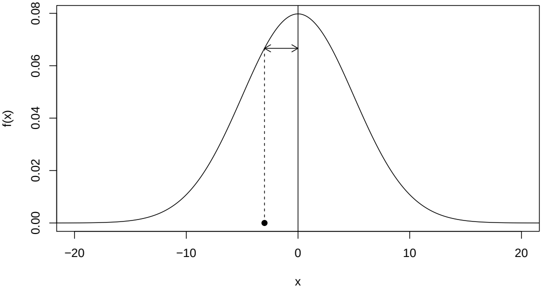
Consider the following (continuous) distribution with $\mu = 0$. Let's build a measure of expected "spread".



Let's define "spread" as the absolute deviation from μ: $|x - \mu|$.
i.e. $+$'ve & $-$'ve deviations of the same magnitude are treated the same.

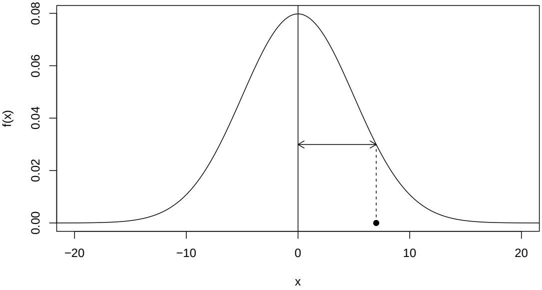
Intuitively Thinking: Measures of Spread

When $x = -3.0$, the abs. deviation from μ is $|-3.0 - \mu| = 3.0$.
Note $f(x) = P(X = x) = 0.066$.



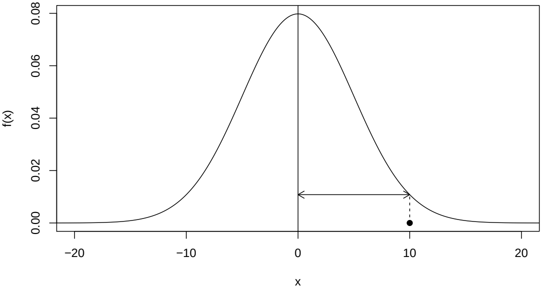
Intuitively Thinking: Measures of Spread

When $x = 7.0$, the abs. deviation from μ is $|7.0 - \mu| = 7.0$.
Note $f(x) = P(X = x) = 0.030$.



Intuitively Thinking: Measures of Spread

When $x = 10.0$, the abs. deviation from μ is $|10.0 - \mu| = 10.0$.
Note $f(x) = P(X = x) = 0.011$.



Intuitively Thinking: Measures of Spread

x	Abs Deviation $\lvert x-\mu\rvert$	Weight $f(x)=P(X=x)$
-3.0	$\lvert -3.0-0\rvert = 3.0$	0.066
7.0	$\lvert 7.0-0\rvert = 7.0$	0.030
10.0	$\lvert 10.0-0\rvert = 10.0$	0.011

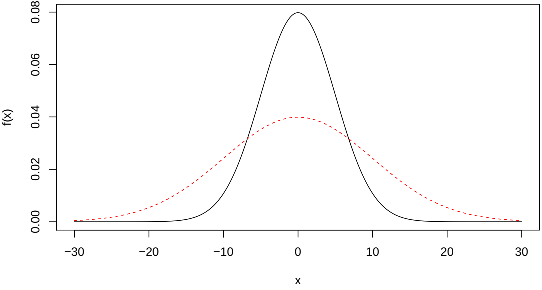
So say we do this for all x and take a weighted average of the $|x-\mu|$ where the weights are $f(x)$.

Voilà: Our notion of expected spread.

Variance

Variance

Both the following curves have the same mean, but the dashed red curve has bigger variance than the the black one.



Standard Deviation

Alternate Expression of Variance

Alternate Expression of Variance

Next Time

- More on discrete random variables