Lecture 3.2: Introduction to Random Variables

2013/09/17

Previously... Bayes Theorem (Simplified Form)

Let A and B be events such that $\mathbb{P}(A) > 0$, then Bayes theorem states:

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A|B) \cdot \mathbb{P}(B)}{\mathbb{P}(A)}$$

Previously... Bayes Theorem (Expanded Form)

Now say we have a collection of events $B_1, \ldots, B_k$ that form a partition of the sample space S, then for each $j = 1, \ldots, k$

$$\begin{aligned}\mathbb{P}(B_j|A) &= \frac{\mathbb{P}(A|B_j) \cdot \mathbb{P}(B_j)}{\mathbb{P}(A)} \\ &= \frac{\mathbb{P}(A|B_j) \cdot \mathbb{P}(B_j)}{\sum_{i=1}^{k} \mathbb{P}(A|B_i) \cdot \mathbb{P}(B_i)}\end{aligned}$$

i.e. Apply the law of total probability to the denominator

Previously: Screening for a Disease

For two tests with the exact same sensitivity and specificity:

$\mathbb{P}(D)$	$\mathbb{P}(D \mid P)$	$\mathbb{P}(D^c \mid P)$
0.01	0.5	0.5
0.1	0.917	0.083

The error rate $\mathbb{P}(D^c|P)$ dropped from 0.5 to 0.083! The change is due to the prevalence of the disease $\mathbb{P}(D)$.

Moral of the Story: In many cases, how well your diagnostic test performs depends on the prevalence of the disease.

Previously... Bayesian Statistics

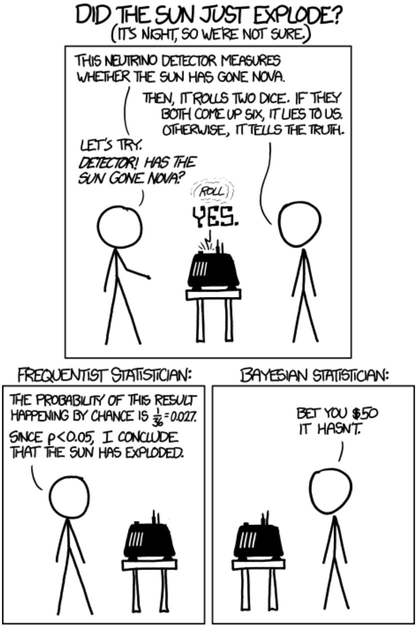
To express our belief about θ from a Bayesian perspective, we have:

1. A prior distribution $\mathbb{P}(\theta)$. It reflects our prior belief about θ.
2. The likelihood function $\mathbb{P}(X|\theta)$. This is the mechanism that generates the data X *given* θ.
3. A posterior distribution $\mathbb{P}(\theta|X)$. We update our belief about θ after observing data X.

$$\mathbb{P}(\theta|X) = \frac{\mathbb{P}(X|\theta) \cdot \mathbb{P}(\theta)}{\mathbb{P}(X)}$$

Frequentists do not have a notion of 1. and 3.

Bayes Theorem XKCD



Bayes Theorem XKCD

Let

- N be the event that the sun going nova (explode)
  i.e. the parameter of interest
- X be the event the detector gives the answer "Yes."
  i.e. the data given N

Then by Bayes Theorem:

$$\mathbb{P}(N|X) = \frac{\mathbb{P}(X|N)\mathbb{P}(N)}{\mathbb{P}(X)} = \frac{\mathbb{P}(X|N)\mathbb{P}(N)}{\mathbb{P}(X|N)\mathbb{P}(N) + \mathbb{P}(X|N^c)\mathbb{P}(N^c)}$$

Bayes Theorem XKCD

We set

$$
\begin{aligned}
\mathbb{P}(X|N^c) &= \frac{1}{36} \quad \text{i.e. probability detector lies to you is } \frac{1}{36} \\
\mathbb{P}(X|N) &= \frac{35}{36} \quad \text{i.e. probability detector doesn't lie to you is } \frac{35}{36} \\
\mathbb{P}(N) &= 0.000001 \quad \text{our prior belief the sun didn't go nova}
\end{aligned}
$$

So then

$$
\begin{aligned}
\mathbb{P}(N|X) &= \frac{\mathbb{P}(X|N)\mathbb{P}(N)}{\mathbb{P}(X)} = \frac{\mathbb{P}(X|N)\mathbb{P}(N)}{\mathbb{P}(X|N)\mathbb{P}(N) + \mathbb{P}(X|N^c)\mathbb{P}(N^c)} \\
&= \frac{\frac{35}{36} \times 0.000001}{\frac{35}{36} \times 0.000001 + \frac{1}{36} \times 0.999999} \approx \frac{0}{0 + \frac{1}{36}} = 0
\end{aligned}
$$

Random Variable

Random Variable

Random Variable Illustration

Distribution Function

Properties of Distribution Function

Example

Plot of Random Variable

Lemmas

Example

Next Time

- Discrete random variables