Lecture 2.2: Independence

2013/09/11

Previously... Conditional Probability

The conditional probability of an event A, given the event B, is defined by

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

provided $\mathbb{P}(B) > 0$.

This is read as "the probability of A given B."

Previously... Multiplication Rule (of Probability)

Another way to consider conditional probability is via the multiplication rule:

For any events A and B

$$\mathbb{P}(A \cap B) = \mathbb{P}(A|B) \times \mathbb{P}(B) = \mathbb{P}(B|A) \times \mathbb{P}(A)$$

For any n events $E_1, \ldots, E_n$

$$\begin{aligned}
& \mathbb{P}(E_1 \cap E_2 \cap \ldots \cap E_n) \\
= \;& \mathbb{P}(E_2 \cap \ldots \cap E_n | E_1) \times \mathbb{P}(E_1) \\
= \;& \mathbb{P}(E_3 \cap \ldots \cap E_n | E_1, E_2) \times \mathbb{P}(E_2|E_1) \times \mathbb{P}(E_1) \\
= \;& \ldots \\
= \;& \mathbb{P}(E_n|E_1, \ldots, E_{n-1}) \times \ldots \times \mathbb{P}(E_2|E_1) \times \mathbb{P}(E_1)
\end{aligned}$$

Previously... Monty Hall Problem

Assume w/o loss of generality that the car is behind Door 1:

$$
\begin{aligned}
\mathbb{P}(\text{Open 2} \cap \text{Picked 1}) &= \mathbb{P}(\text{Open 2} \mid \text{Picked 1}) \times \mathbb{P}(\text{Picked 1}) = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6} \\
\mathbb{P}(\text{Open 3} \cap \text{Picked 1}) &= \mathbb{P}(\text{Open 3} \mid \text{Picked 1}) \times \mathbb{P}(\text{Picked 1}) = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{P}(\text{Open 1} \cap \text{Picked 2}) &= \mathbb{P}(\text{Open 1} \mid \text{Picked 2}) \times \mathbb{P}(\text{Picked 2}) = 0 \\
\mathbb{P}(\text{Open 3} \cap \text{Picked 2}) &= \mathbb{P}(\text{Open 3} \mid \text{Picked 2}) \times \mathbb{P}(\text{Picked 2}) = \frac{1}{3}
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{P}(\text{Open 1} \cap \text{Picked 3}) &= \mathbb{P}(\text{Open 1} \mid \text{Picked 3}) \times \mathbb{P}(\text{Picked 3}) = 0 \\
\mathbb{P}(\text{Open 2} \cap \text{Picked 3}) &= \mathbb{P}(\text{Open 2} \mid \text{Picked 3}) \times \mathbb{P}(\text{Picked 3}) = \frac{1}{3}
\end{aligned}
$$

We'll look at another (perhaps more intuitive) solution once we've seen the law of total probability.

Goals for Today

- Example of multiplication rule
- Define independence
- Examples

Multiplication Rule Example

An ordinary deck of 52 playing cards is randomly divided into 4 piles of 13 cards each. Compute the probability that each pile has exactly 1 ace.

Define the following events $E_i, i = 1, 2, 3, 4$ as follows:

- $E_1 = \{$The ace of spades (WLOG) is in any of the four piles$\}$
- $E_2 = \{$The ace of spades and hearts are in different piles$\}$
- ...
- ...

Independence

Independence and Conditional Probability

Example

Suppose we toss two fair dice. Let E denote the event that the sum of the dice is six and F denote the event that the first die equals four. Show these events are not independent:

Mutual Independence

Example

Pairwise independent events that are not mutually independent:

Example

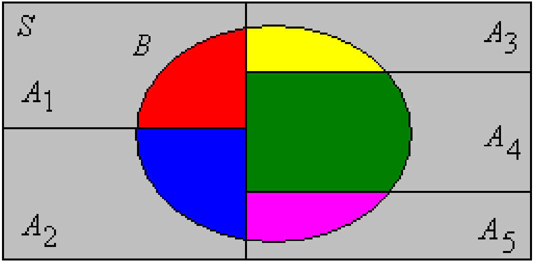
Three brands of IPA's (X, Y, Z) are to be ranked according to taste by a judge. Let

1. Event A: Brand X is preferred over Y
2. Event B: Brand X is ranked best
3. Event C: Brand X is ranked second best.
4. Event D: Brand X is ranked third best

If the judge is actually a fraud and ranks the beers randomly, is event A independent of B, C, and D?

Example

Next Time: The Law of Total Probability



$$\mathbb{P}(B) = \mathbb{P}(B|A_1)\cdot\mathbb{P}(A_1)+\mathbb{P}(B|A_2)\cdot\mathbb{P}(A_2)+\mathbb{P}(B|A_3)\cdot\mathbb{P}(A_3)+ \mathbb{P}(B|A_4)\cdot\mathbb{P}(A_4)+\mathbb{P}(B|A_5)\cdot\mathbb{P}(A_5)$$