**Math 391 Probability: Midterm Exam II Solutions**
**Friday, November 8, 2013**

Name: ______________________________________________

Please read the following instructions carefully:

1. You must show your work/thinking to receive full credit.
2. Write your answers on the exam. If you need more room, you can write on the back of the exam pages. If this still isn't enough room, you can attach pages to the exam.

1. The joint density function of $(X, Y)$ is

$$f(x,y) = \begin{cases} x+y & 0<x<1,\ 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Are $X$ and $Y$ independent? Why or why not?

**Solution:** No, because $f(x,y) \neq f_X(x)f_Y(y)$.

$$\begin{aligned} f_X(x) &= \int_0^1 f(x,y)dy = \int_0^1 (x+y)dy = xy + \frac{y^2}{2}\Big|_{y=0}^{y=1} = x + \frac{1}{2} \\ f_Y(y) &= \int_0^1 f(x,y)dx = y + \frac{1}{2} \\ f_X(x)f_Y(y) &= \left(x+\frac{1}{2}\right)\left(y+\frac{1}{2}\right) \neq x+y \end{aligned}$$

(b) Find the density function of $X$ (including the support)

**Solution:** From a)

$$f_X(x) = x + \frac{1}{2}$$

for $0 < x < 1$

(c) Find $F_X(0.75|Y=y)$

**Solution:** $y$ is now a given value, not a variable.

$$\begin{aligned} F_X(0.75|Y=y) &= \int_{-\infty}^{0.75} f(x|y)dx = \int_{-\infty}^{0.75} \frac{f(x,y)}{f_Y(y)}dx \\ &= \int_0^{0.75} \frac{x+y}{y+\frac{1}{2}}dx = \frac{1}{y+\frac{1}{2}}\left(\frac{x^2}{2}+yx\right)\Big|_{x=0}^{x=0.75} = \frac{1}{y+\frac{1}{2}}\left(\frac{0.75^2}{2}+0.75y\right) \end{aligned}$$

(d) Find the conditional density function $f(x|y)$ (including the support)

**Solution:** From c)

$$f(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{x+y}{y+\frac{1}{2}}$$

for $0 < x < 1$

2. Say $\mathbb{P}\left(-\sqrt{5} < X < \sqrt{5}\right) = 0.3$ and $\mathbb{E}(X) = 0$. Find a lower bound for $\mathrm{Var}(X)$.

**Solution:**

$$
\begin{array}{rcl}
 & & \mathbb{P}\left(-\sqrt{5} < X < \sqrt{5}\right) = \mathbb{P}(X^2 < 5) = 0.3 \\
 & \Rightarrow & \mathbb{P}(X^2 \geq 5) = 0.7 \\
\text{but} & & \mathbb{P}(X^2 \geq 5) \leq \dfrac{\mathbb{E}(X^2)}{5} \text{ by Markov's Inequality since } X^2 \text{ is non-negative and } 5 > 0 \\
\text{so} & & 5 \times \mathbb{P}(X^2 \geq 5) \leq \mathbb{E}(X^2) \\
 & \Rightarrow & \mathbb{E}(X^2) \geq 5 \times 0.7 = 3.5 \\
\text{and} & & \mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \mathbb{E}(X^2) - 0 \geq 3.5
\end{array}
$$

3. Say we have a $\Gamma(\text{shape} = k, \text{scale} = \theta)$ random variable with $k > 0$ and $\theta > 0$:

$$f(x) = \frac{1}{\theta^k \Gamma(k)} x^{k-1} \exp(-x/\theta)$$

(a) Show in whatever manner you wish that $\text{Var}(X) = k\theta^2$.

**Solution:** Easiest way is using MGF's $M_X(t) = (1 - \theta t)^{-k}$

- $\mathbb{E}[X] = \frac{d}{dt} M_X(t)\big|_{t=0}$

$$\frac{d}{dt} M_X(t)\big|_{t=0} = -k(1-\theta t)^{-k-1}(-\theta)\big|_{t=0} = k\theta(1-\theta t)^{-k-1}\big|_{t=0} = k\theta$$

- $\mathbb{E}[X^2] = \frac{d^2}{dt^2} M_X(t)\big|_{t=0}$

$$\begin{aligned} \frac{d^2}{dt^2} M_X(t)\big|_{t=0} &= \frac{d}{dt} k\theta(1-\theta t)^{-k-1}\big|_{t=0} = k\theta(-k-1)(1-\theta t)^{-k-2}(-\theta)\big|_{t=0} \\ &= k\theta^2(k+1) \end{aligned}$$

- So

$$\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = k\theta^2(k+1) - k^2\theta^2 = k^2\theta^2 + k\theta^2 - k^2\theta^2 = k\theta^2$$

(b) Now, say we have $X_1, \ldots, X_n$ independent Gamma random variables where

$$X_i \sim \Gamma(\text{shape} = k_i, \text{scale} = \theta)$$

random variables. What is the distribution of $Y = X_1 + \ldots + X_n$?

**Solution:** Since the $n$ Gamma RV's are independent, we know

$$M_Y(t) = \prod_{i=1}^n M_{X_i}(t) = \prod_{i=1}^n (1-\theta t)^{-k_i} = (1-\theta t)^{-\sum_{i=1}^n k_i}$$

which itself is the MGF of a Gamma $(\text{shape} = \sum_{i=1}^n k_i, \text{scale} = \theta)$. So by the uniqueness of MGF's theorem, we know that $Y$ is

$$\text{Gamma}\left(\text{shape} = \sum_{i=1}^n k_i, \text{scale} = \theta\right)$$

(c) How does your answer change if the $X_1, \ldots, X_n$ are not only independent, but also *identically distributed*? i.e. that

$$X_i \sim \Gamma(\text{shape} = k, \text{scale} = \theta)$$

**Solution:** In this case

$$\sum_{i=1}^n k = nk$$

So

$$Y \sim \text{Gamma}\left(\text{shape} = nk, \text{scale} = \theta\right)$$

4. Show that the MGF of a Normal($\mu, \sigma^2$) random variable is $\exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right)$

**Solution:** The easiest/quickest way to go about this is to first find the MGF of $Z \sim \text{Normal}(0,1)$, and then set $X = \sigma Z + \mu$ and apply the linear transformation MGF theorem.

$$\begin{aligned}
\mathbb{E}[\exp(Xt)] &= \int_{-\infty}^{\infty} \exp(xt)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}x^2\right)dx = \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}x^2 + xt\right)dx \\
&= \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}x^2 + \frac{2xt}{2} - \frac{t^2}{2} + \frac{t^2}{2}\right)dx \\
&= \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left(x^2 - 2xt + t^2\right) + \frac{t^2}{2}\right)dx \\
&= \exp\left(\frac{t^2}{2}\right)\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}(x-t)^2\right)dx = \exp\left(\frac{t^2}{2}\right)\times 1
\end{aligned}$$

since what was in the integral was the pdf of a Normal($t, 1$) random variable. For $Y = aX + b$, $M_Y(t) = \exp(bt)M_X(at)$. So since $X = \sigma Z + \mu$ we have

$$M_X(t) = \exp(\mu t)\exp\left(\frac{1}{2}(t\sigma)^2\right) = \exp\left(\mu t + \frac{1}{2}t^2\sigma^2\right)$$

5. Let $X$ be an Exponential($\lambda$) random variable i.e.

$$f(x) = \lambda \exp(-\lambda x) \text{ for } 0 < x < \infty$$

Let $Y = \exp(-\lambda X)$. Find the distribution of $Y$ including its support. Identify this distribution.

**Solution:**

$$\begin{aligned}
y &= g(x) = \exp(-\lambda x) \\
x &= g^{-1}(y) = -\frac{\log y}{\lambda} \\
\frac{d}{dy} g^{-1}(y) &= -\frac{d}{dy}\frac{\log y}{\lambda} = -\frac{1}{\lambda y} \\
f_Y(y) &= f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| = \lambda \exp\left(-\lambda\left(-\frac{\log y}{\lambda}\right)\right)\frac{1}{y\lambda} \\
&= \lambda \exp(\log(y)) \frac{1}{y\lambda} = \frac{y}{y} = 1
\end{aligned}$$

The support of $Y$ is

$$\left(\exp(-\lambda \times 0), \lim_{x\to\infty} \exp(-\lambda \times x)\right) = (1, 0) = (0, 1)$$

This is the Uniform$(0, 1)$ random variable. This is in fact in inverse of HW6, Question 2.