Lecture 2.3: Law of Total Probability

2013/09/13

Independence

We say that two events A and B are independent if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \times \mathbb{P}(B)$$

Otherwise we say they are dependent.

Independence and Conditional Probability

Say events A and B are independent, then

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(A) \times \mathbb{P}(B)}{\mathbb{P}(B)} = \mathbb{P}(A)$$

i.e. B occurring does not affect the probability of A.

Mutual Independence

The k events $A_1, \ldots, A_k$ are said to be mutually independent if for every $j = 1, \ldots, k$ and every subset of distinct indices $i_1, \ldots, i_j$, we have

$$\mathbb{P}(A_{i_1} \cap \ldots \cap A_{i_j}) = \mathbb{P}(A_{i_1}) \times \ldots \times \mathbb{P}(A_{i_j})$$

Previously... Use of "{" and "}" on Slides 9 and 11

In the example on slide 10/14 where E denoted the event that the sum of two dice rolls is six and F event that the first die is four.

The $\mathbb{P}(\{4, 2\})$ was the probability of rolling 4 and then 2.

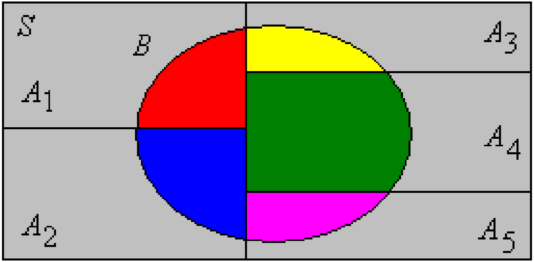
vs.

In the example with about pairwise independence not implying mutual independence $E = \{1, 2\}$ is the event of picking either 1 or 2. $F = \{1, 3\}$ is event of picking either 1 or 3.

Goals for Today

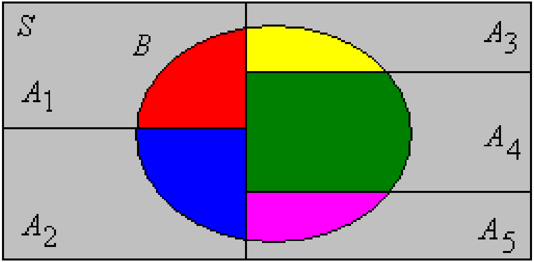
- Introduce the Law of Total Probability
- Examples

Law of Total Probability: Graphical Illustration



$$\begin{aligned}\mathbb{P}(B) &= \mathbb{P}(B \cap A_1) + \mathbb{P}(B \cap A_2) + \mathbb{P}(B \cap A_3) + \\ &\quad \mathbb{P}(B \cap A_4) + \mathbb{P}(B \cap A_5)\end{aligned}$$

Law of Total Probability: Graphical Illustration



$$\begin{aligned}\mathbb{P}(B) \quad = \quad & \mathbb{P}(B|A_1)\cdot\mathbb{P}(A_1)+\mathbb{P}(B|A_2)\cdot\mathbb{P}(A_2)+\mathbb{P}(B|A_3)\cdot\mathbb{P}(A_3)+\\ & \mathbb{P}(B|A_4)\cdot\mathbb{P}(A_4)+\mathbb{P}(B|A_5)\cdot\mathbb{P}(A_5)\end{aligned}$$

Partition (Letters Flipped from Slide with Graphics)

Law of Total Probability (Letters Flipped from Slide with Graphics)

Example of LOTP: Cards

Suppose we draw two cards from a well shuffled deck. One partition of the sample space is:

$$\begin{aligned} A_1 &= \{\text{The first card is an Ace}\} \\ A_1^c &= \{\text{The first card is not an Ace}\} \end{aligned}$$

It is often useful to look for partitions of the sample space that simplify a computation!

What is the probability the second card in the deck is an ace?

Example of LOTP: Cards

Condition on whether the first card is an ace or not:

Example of LOTP: Winning Games

A soccer team wins 60% of its games when it scores the first goal, and 10% of its games when the opposing team scores first. If the team scores the first goal about 30% of the time, what fraction of the games does it win? Let W be the event that the team wins, and SF be the event that it scores first. Then the LOTP says:

Example of LOTP: Monty Hall

Back to everyone's favorite game show! You've heard of this problem before, so you will switch. Assume WLOG the car is behind door 1. Let P_i be the event that the contestant chooses door i.

Example of LOTP: Monty Hall

You need to associated probabilities to all random events. But there is the one scenario where the contestant picks Door 1, and the host chooses at random which of the two doors that do not hide a car to open. So we apply the law of total probability on an already conditioned event:

Next Time:

- Bayes Theorem