Lecture 9.2: Midterm II Review

2013/11/06

Distributions

You still need to know them:
Discrete:

- Bernoulli
- Binomial
- Geometric
- Discrete Uniform
- Hypergeometric
- Multinomial

Continuous

You still need to know them:
Discrete:

- Uniform(a, b)
- Gamma: Gamma function $\Gamma(k) = (k-1)\Gamma(k-1)$
- Exponential
- Beta
- Normal

Moment Generating Functions

- Definition: $M_X(t) = \mathbb{E}(\exp(tX))$
- $\mathbb{E}(X^k) = \frac{d^k}{dt^k} M_X(t)\Big|_{t=0}$
- Use MGF's to find $\mu = \mathbb{E}(X)$ and $\sigma^2 = \text{Var}(x)$.
  We did this for $Z \sim$Normal$(0, 1)$
- Say $Y = aX + b$, then $M_Y(t) = \exp(bt)M_X(at)$.
  We did this to show MGF of $X \sim \sigma Z + \mu \sim$ Normal(μ, σ^2)
- Uniqueness Theorem. Say you have X_1 and X_2. Then
  $F_1(x_1) = F_2(x_2)$ IFF $M_{X_1}(t) = M_{X_2}(t)$

Inequalities

- Markov's Inequality: For $u(x)$ a real-valued non-negative function
$$P(U(X) \geq c) \leq \frac{\mathbb{E}(U(X))}{c}$$
- Chebyshev's Inequality:
$$P(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}$$
- Jensen's Inequality: for $g()$ convex (look up definition of convex)
$$\mathbb{E}(g(X)) \geq g(\mathbb{E}(X))$$
So for $h()$ concave
$$\mathbb{E}(h(X)) \leq h(\mathbb{E}(X))$$

Inequalities

- Transformation of a RV: For $Y = g(X)$ a 1:1 transformation (every y in the range was mapped from a unique x in the domain)
$$f_Y(y) = f_X(x) \left| \frac{d}{dy} g^{-1}(y) \right|$$
Example in class was deriving the log-Normal distribution

Joint Distributions

- Joint PMF/PDF and its properties IFF
  1. $f(x) \geq 0$ for all vectors x
  2. Sums/integrates to 1
- Joint CDF and the IFF conditions
- Marginal PMF/PDF's

Independence

RV's $X_1, \ldots, X_k$ are said to be independent if $\forall a_i < b_i$

$$\begin{aligned} & \mathbb{P}(a_1 \leq X_1 \leq b_1, \ldots, a_k \leq X_k \leq b_k) \\ = \quad & \mathbb{P}(a_1 \leq X_1 \leq b_1) \ldots \mathbb{P}(a_k \leq X_k \leq b_k) \end{aligned}$$

Also, the RV's are independent IFF

$$F(x_1, \ldots, x_k) = \prod_{i=1}^{n} F_i(x_i) \quad f(x_1, \ldots, x_k) = \prod_{i=1}^{n} f_i(x_i)$$

This is different than

$$F(x_1, \ldots, x_k) = \prod_{i=1}^{n} F(x_i) \quad f(x_1, \ldots, x_k) = \prod_{i=1}^{n} f(x_i)$$

Independence

- Theorem that X and Y are independent IFF $\exists g(x), h(y)$ s.t. $f(x, y) = g(x)h(y)$
- If X and Y are independent, then $\mathbb{E}(g(X)h(Y)) = \mathbb{E}_X(g(X)) \times \mathbb{E}_Y(h(y))$
- If X and Y are independent $Z = X + Y$, $M_Z(t) = M_X(t)M_Y(t)$

Conditional PMF/PDF

- $f(x_2|x_1) = \frac{f(x_1,x_2)}{f_{X_1}(x_1)}$
- Poisson Example: $Y = X_1 + X_2$ where X_1 and X_2 are Poisson λ_1 and λ_2 respectively. Then
  - $X_1|Y = y$ is Binomial$\left(y, \frac{\lambda_1}{\lambda_1+\lambda_2}\right)$
  - $X_2|Y = y$ is Binomial$\left(y, \frac{\lambda_2}{\lambda_1+\lambda_2}\right)$
- $f(x_1, x_2) = f(x_2|x_1)f_{X_1}(x_1) = f(x_1|x_2)f_{X_2}(x_2)$
- If independent, then $f(x_2|x_1) = f_{X_2}(x_2)$

Other Facts

- $$\mathbb{E}(X_1 + X_2) = \mathbb{E}_{X_1}(X_1) + \mathbb{E}_{X_2}(X_2)$$

  always

- $$\text{Var}(X_1 + X_2) = \text{Var}(X_1) + \text{Var}(X_2)$$

  if independent.

- $\text{Cov}(X, Y) = \mathbb{E}((X - \mu_x)(Y - \mu_Y)) = \mathbb{E}(XY) - \mathbb{E}_X(X)\mathbb{E}_Y(Y)$ and its 3 properties:
  1. $\text{Cov}(aX, bY) = ab\text{Cov}(X, Y)$
  2. $\text{Cov}(X + a, Y + b) = \text{Cov}(X, Y)$
  3. $\text{Cov}(X, aX + b) = a\text{Var}(x)$