Problem Set 0

0-1. (a) $\{1\}$

(b) 8

(c) 4

0-2. (a) $3/2$

(b)

1 : 1, 2, 3, 4, 5, 6 2 : 2, 4, 6, 8, 10, 12

3 : 3, 6, 9, 12, 15, 18 4 : 4, 8, 12, 16, 20, 24

5 : 5, 10, 15, 20, 25, 30 6 : 6, 12, 18, 24, 30, 36

$$\left(\sum_{i=1}^{6} i\right)\cdot\left(\sum_{i=1}^{6} i\right)\cdot 1/36 = \left(\frac{7\cdot 6}{2}\right)^2 \cdot 1/36 = 49/4$$
$$= 12.25$$

(c) X and Y are independent each other

So, by linearity of expectation,

$$E[X+Y] = E[X] + E[Y] = 1.5 + 12.25$$
$$= 13.75$$

0-3. (a) True (b) False (c) True

0-4. $P(n) ::=$ "For any integer $n \geq 1$, $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$".

Base case ($n=1$): $\sum_{i=1}^{1} i^3 = 1 = \left(\frac{1\cdot 2}{2}\right)^2 = 1$. So base case holds.

Inductive step: Assume $P(n)$ such that $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$ is true.

$$\sum_{i=1}^{n+1} i^3 = \left(\sum_{i=1}^{n} i^3\right) + (n+1)^3 = \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3$$

$$= \frac{n^2(n+1)^2}{4} + \frac{4(n+1)^3}{4} = \frac{(n+1)^2(n^2+4n+4)}{4} = \left(\frac{(n+1)(n+2)}{2}\right)^2$$

So $P(n+1)$ holds. $\square$

0-5.

Pf. by induction

$P(n) ::=$ "For any connected undirected graph $G=(V,E)$ where $|V|=n$, if $|E|=|V|-1$, then G is acyclic.

Base case ($n=1$): Only one node, no edge. It's acyclic itself.

Inductive step: Assume $P(n)$.

Let's build a tree with $n+1$ nodes and n edges. We will remove a node from the graph to make a graph with n nodes. If we get rid of a vertex whose degree is more than 1, then the graph would not be a connected graph, so we should pick a node degree 1. Then we get a graph with n vertices, $n-1$ edges, which by assumption ($P(n)$) is acyclic. When we reattach the removed node and edge, it would not incur a cycle, as the node is a leaf of tree. So $P(n+1)$ holds.