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MATHEMATICS

How does FFT work?

Asked 7 years, 2 months ago Modified 1 year, 5 months ago Viewed 9k times

17

For five years I tried to understand how Fourier transform works. Read a lot of articles, but nobody could explain it in simple terms. Two weeks ago I stumbled upon the video about a 100 years old machine that calculates Fourier series mechanically: https://www.youtube.com/watch?v=NAsM30MAHLg - I watched it and suddenly it became very clear! It turns out to be very simple thing!

Now I want to understand how FFT works. And again nobody can explain it in simple terms.

Can anyone explain it to a non mathematician? Thanks.

algorithms fourier-analysis fourier-transform recursive-algorithms fast-fourier-transform

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edited May 26, 2022 at 9:17
Sandipan Dey
2,101 10 11

asked Sep 5, 2016 at 8:32
shal
291 2 6

3 It would make this question much more clear too, if you elaborate on your understanding of an ordinary Fourier transform. Then the other users would now what terms to use to explain FFT to you – Yuriy S Sep 5, 2016 at 8:38

7 Many people can explain it in simple terms, but you have to give a starting point. As it is it is *unclear what you are asking*? Remember that asking for a guided walkthru of a chapter in a book will be closed as *too broad*. – Jyrki Lahtonen Sep 5, 2016 at 8:45

1 It's well explained in Algorithms by Dasgupta, Papadimitriou and Vazirani. – xavierm02 Sep 5, 2016 at 8:51

1 I found this video the other day, maybe it helps getting a bit more intuition: youtube.com/watch?v=FjmwwDHT98c – PlasmaHH Sep 6, 2016 at 13:27

1 I gave a quick (one paragraph) but surprisingly complete explanation of the discrete Fourier transform here: math.stackexchange.com/a/582626/40119. The discrete Fourier transform simply changes basis to a special basis, the "discrete Fourier basis", which is a basis of eigenvectors for the cyclic shift operator S on $\mathbb{C}^n$. You can easily compute these eigenvectors yourself right now. Because any shift-invariant linear operator commutes with S, a simultaneous diagonalization theorem tells us that any shift-invariant operator is diagonalized by the discrete Fourier basis. – littleO Sep 12, 2016 at 11:42

4 Answers

Sorted by: Highest score (default)

13

As said by @user289075, the FFT is just an efficient computational procedure and does not explain the theory behind the discrete Fourier transform.

The key idea is that to compute the transform of a signal n samples long, you can compute the transforms of the two halves of length $n/2$ independently and recombine them to get the global transform. This principle is applied recursively, i.e. to compute the transform on $n/2$ points you combine two transforms on $n/4$ points and so on until n cannot be divided anymore.

By means of complexity analysis, one can show that the number of arithmetical operations is reduced from n^2 (for a naive implementation) to $n \log n$, a significant saving.

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edited Sep 5, 2016 at 9:14

answered Sep 5, 2016 at 9:01
user65203

8

The FFT is just an algorithm for computing the discrete Fourier transform (DFT). It turns out that the DFT matrix is highly symmetric (due to the symmetry and periodicity properties of e^{ix}). The FFT is just a matrix factorisation of the DFT into a series of sparse matrices. You can multiply the sparse matrices and obtain the DFT in fewer operations than you could by simply multiplying the DFT matrix.

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answered Sep 5, 2016 at 8:46

Matthew Hampsey
595 2 6 17

Yes, I realize that this is just an optimization of DFT. An indicator of me grasping DFT was the fact that I could write the entire algorithm myself from scratch and everything worked correctly. Now I would like to understand FFT the same way, so that I could write it myself and it would work correctly. Is there any detailed explanation of the symmetry of DFT matrix, with illustrations? – shal Sep 5, 2016 at 9:29

This PDF looks like it works through the factorisation in detail for n = 4
robotics.itee.uq.edu.au/~elec3004/ebooks/... – Matthew Hampsey Sep 5, 2016 at 11:03

8

I feel that all the theoretical elements are present in the others' responses, but not the pragmatic reason why FFT reduces the complexity by a significant factor (and hence the popularity of this algorithm among other things).

Applying the DFT amounts to multiplying a matrix to a vector (your original, sampled, signal, for example, each entry of your vector being one sample). That matrix is the Fourier matrix. Say you have $2n$ samples, the matrix to consider is then (up to a constant factor)

$$(F_{2n})_{jk} = \exp(i\pi \times jk/n)$$

where $i = \sqrt{-1}$. These are roots of unity.

Now the magic happens because of a so-called **butterfly** property of these matrices, you can decompose them in a product of matrices: each of which is cheap to apply to a vector (cf. below)

For example, take F_4, it is given (up to a factor) by:

$$F_4 \propto \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{pmatrix} \propto \left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & i \\ \hline 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -i \end{array}\right) \left(\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 1 & i^2 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & i^2 \end{array}\right) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Let's look at the first two matrices: their structure is very typical, (you could show by recursion how this generalizes) but more about this later. What matters for us now is that there are **only 2 non-zero entries per line**. This means that when considering the application of one of those matrices to a vector, the number of operations to do is a modest constant times n. The number of

matrices in the decomposition, on the other hand, grows like $\log n$. So overall the complexity scales like $n \times \log n$ where an unstructured, basic, matrix-vector product would scale like n^2.

(To clarify this point: starting from the right, you apply one matrix to one vector, this costs $O(n)$, then you apply a matrix to the resulting vector, again $O(n)$, since there are $O(\log n)$ matrices, the overall complexity is $O(n \log n)$ where O signifies (roughly) "of the order of", for more about big-O notation, check out the wikipedia page (1))

Finally, the last matrix in the decomposition is a *bit-reversal* matrix. Not going in the details it can be computed extremely efficiently. (But even naively, the cost is obviously not more than linear in n).

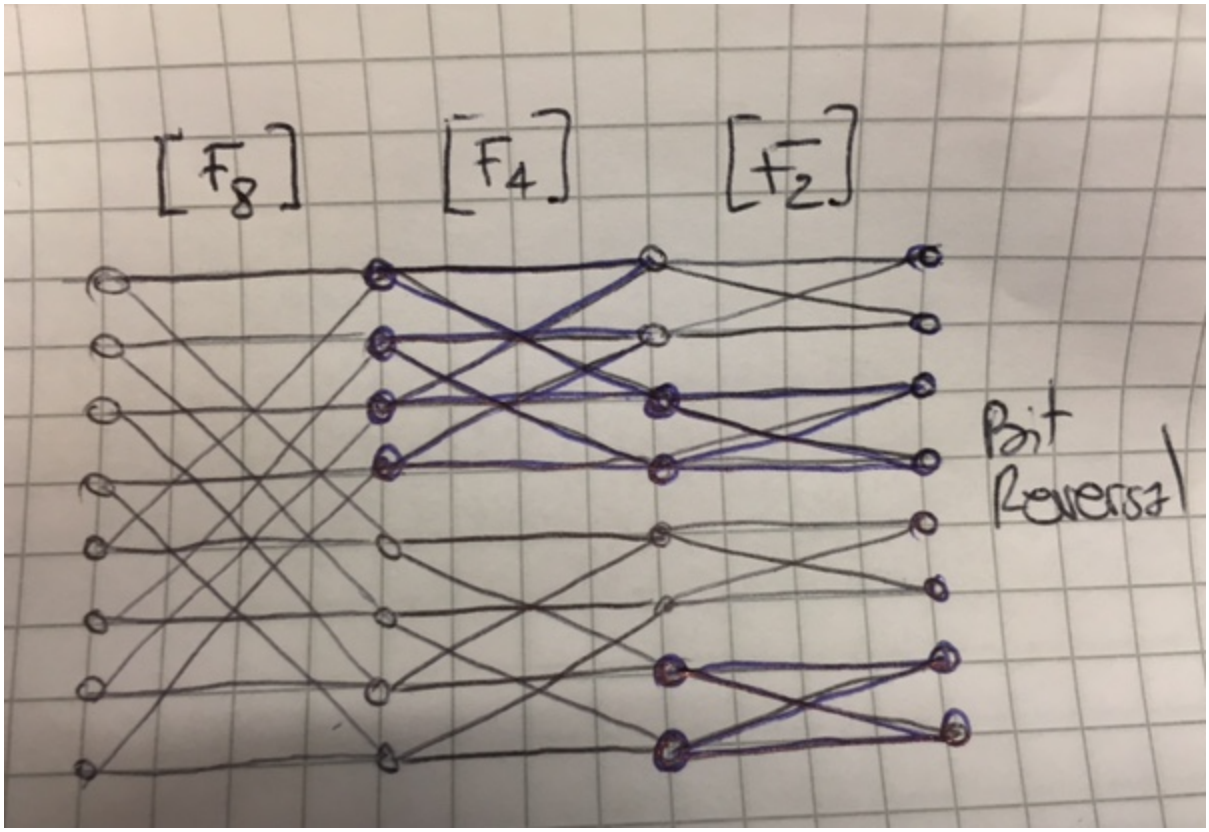
Provided I haven't lost you here, let's generalize a bit: the matrix F_{2n} has the following form (this is not hard to check)

$$F_{2n} \propto \begin{pmatrix} I_n & A_n \\ I_n & -A_n \end{pmatrix} \begin{pmatrix} F_n & 0 \\ 0 & F_n \end{pmatrix} B_{2n}$$

where B_{2n} is a bit-reversal matrix of appropriate dimensions, F_n is the DFT matrix of order n, and hence this matrix can itself be expressed as a product etc.. I_n is the identity matrix of size $n \times n$ and A_n is a diagonal matrix with the powers of i from 0 to $n-1$.

By the way you can now observe that it's desirable to have the size of your original sample-vector be a power of 2. If it is not the case, it is *cheaper* to pad your vector with zeros (this is actually called *zero padding*)

The last thing to clarify is the **butterfly** idea (see also [2]). For this, nothing better than a small drawing (sorry I didn't want to do this in Tikz, if there's a hero out there...). If you represent the vectors after each stage by dots and you connect the dots from the original vector to the vector after the matrix has been applied, you get this nice diagram (ignoring the bit reversal)



I've tried to make clear on the diagram that the first level corresponds to 4 applications of F_2, the next level to 2 applications of F_4 etc. Just to make it clearer, starting from the right it means that the first matrix multiplication after the bit-reversal will

- for the first entry, need to look at the first and the second entry of the original vector (corresponding to a non-zero entries at positions (1,1) and (1,2) in that level's matrix)
- for the next entry, need to look at the first and the second entry (this makes the first block in the "F2" column (corresponding to non-zero entries at positions (2,1) and (2,2))

etc.. Further, observe that, at each level, we can consider *blocks of entries* independently. Again this leads to further optimization that can make the FFT very very efficient.

Ps: I've ignored all multiplication constants since that's an O(1) computation, it's easy to check what they are and it will depend on conventions you're using.

1. https://en.wikipedia.org/wiki/Big_O_notation
2. https://en.wikipedia.org/wiki/Butterfly_diagram

see also: https://en.wikipedia.org/wiki/Cooley%E2%80%93Tukey_FFT_algorithm#Pseudocode

where some pseudocode is available illustrating how you would code the FFT recursively.

Another good resource to see how to code the FFT from scratch:
https://jakevdp.github.io/blog/2013/08/28/understanding-the-fft/

Share Cite Follow edited Sep 12, 2016 at 10:53 answered Sep 8, 2016 at 21:21

tibL
2,110 13 20

Thank you man, for your effort, but I still don't get it. I guess I need a private patient tutor for that, but not now. What I did get is: * what butterfly effect is * that DFT can be calculated via matrices. * regarding Big O notation - I knew it before. Well, it's very disappointing, to know that I am unable to understand something in this world... – shal Sep 12, 2016 at 7:51

but then it looks like you've understood everything that you asked for? maybe clarify what still confuses you? – tibL Sep 12, 2016 at 9:41

1 What confuses me is that I can't write the working algorithm it from scratch. Seems like I kind of understand it, but not enough to write it myself. As if I see all the pieces, but can't combine them together (for example, I can write slow DFT myself from the moment I saw that peculiar machine I mentioned in my original post) Anyway, thank you, your rant made it much clearer! I don't need it for job or anything anyway, I am driven by a pure curiosity :) – shal Sep 12, 2016 at 10:25

1 maybe have a look at this: jakevdp.github.io/blog/2013/08/28/understanding-fft ? – tibL Sep 12, 2016 at 10:52

This one is also excellent: pythonnumericalmethods.berkeley.edu/notebooks/... – Sandipan Dey May 26, 2022 at 0:04

0

As per the `python` implementation here of the 1D Cooley-Tukey FFT algorithm (with the assumption that the length of the time-domain input signal x is a power of 2, i.e., $N = 2^m$ for some $m \in \square$ otherwise we have to zero-pad the signal etc.) the following code can be used:

```
def FFT(x, inverse=False):
    N = len(x)
    if N == 1:
        return x
    else:
        X_even = FFT(x[::2], inverse) # even subproblems
        X_odd = FFT(x[1::2], inverse) # odd subproblems
        factor = np.exp(-2j*(-1 if inverse else 1)*np.pi*np.arange(N)/ N)
        X = np.concatenate(\
            [X_even+factor[:N//2]*X_odd,
             X_even+factor[N//2:]*X_odd])
        return X
```

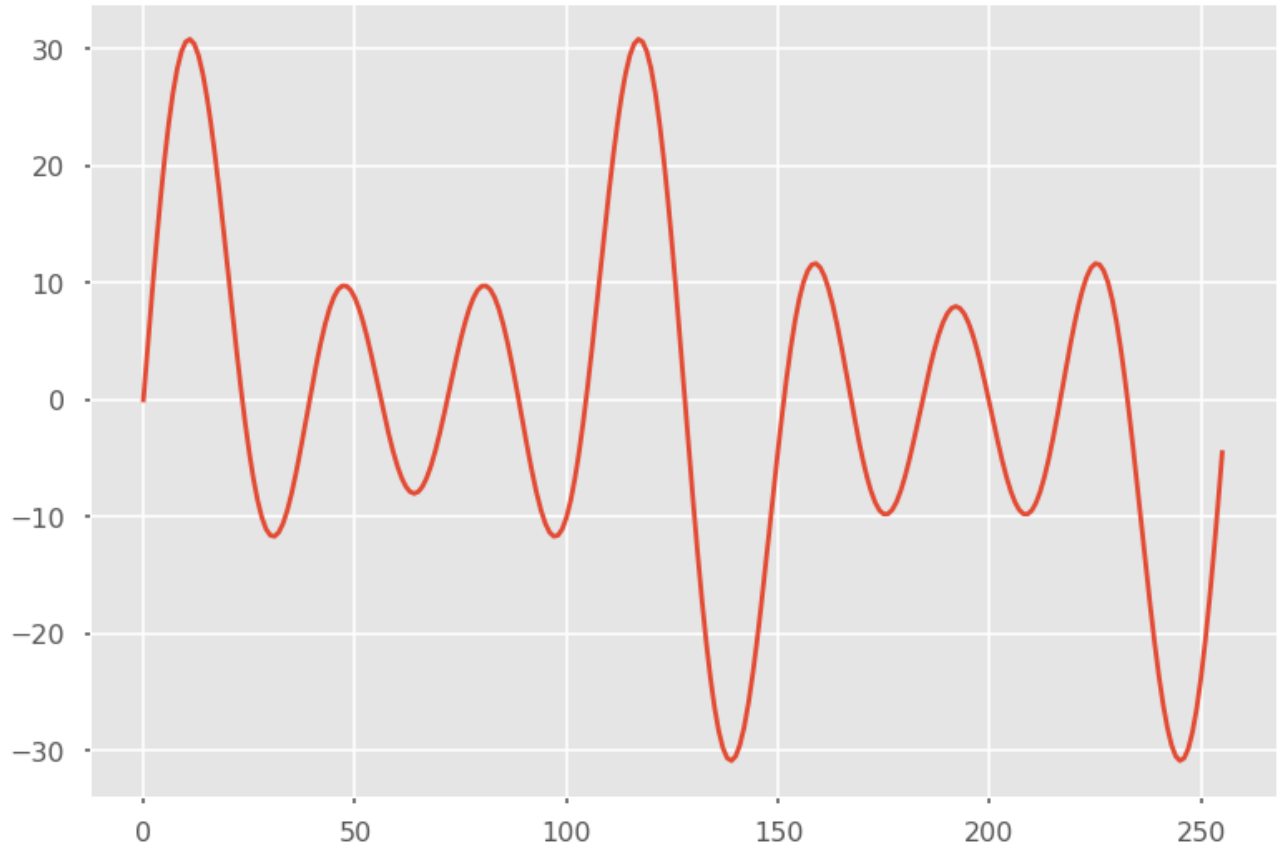
The above divide-and-conquer algorithm has complexity $O(N \log N)$, since computation of F_n uses $F_{\frac{n}{2}}$, $\forall n > 1$, it takes $\log N$ stages, each with linear time, which gives a considerable speedup over the naive DFT implementation with complexity $O(N^2)$, by exploiting the bit-reverse structures of the matrix M (shown below) consisting of N^{th} roots of unity in the complex plane:

$$M = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & w_N & w_N^2 & \dots & w_N^{N-1} \\ 1 & w_N^2 & w_N^4 & \dots & w_N^{2(N-1)} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & w_N^{N-1} & w_N^{2(N-1)} & \dots & w_N^{(N-1)^2} \end{bmatrix}, \text{ where } w_N = e^{-2\pi i/N}$$

We have the recurrence relation $T(N) = 2T(N/2) + O(N)$, which has the bound $T(N) = \Theta(N \log N)$, by the Master theorem.

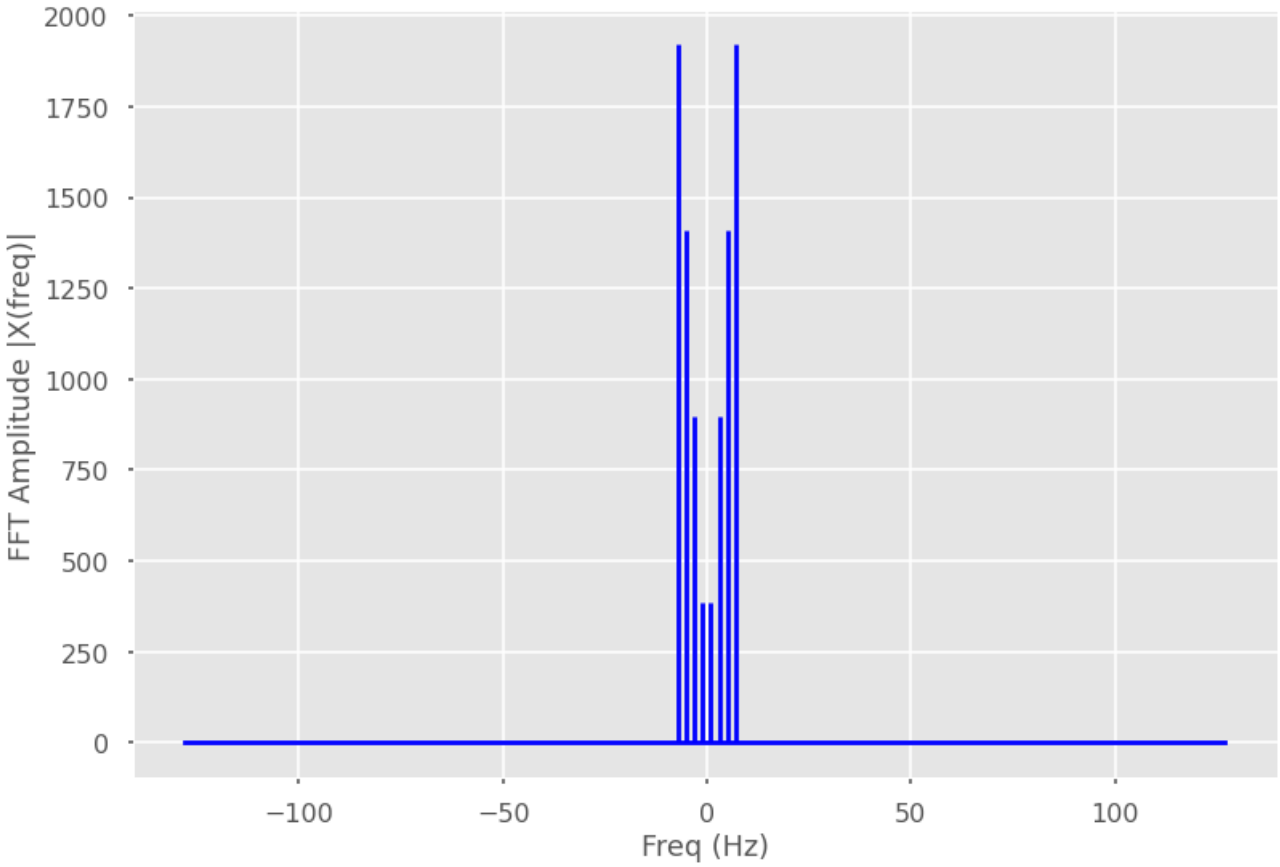
With the following signal in the time domain as shown below and applying the FFT on the input signal x we obtain the frequency domain output signal X:

```
import numpy as np
import matplotlib.pylab as plt

# sampling rate
sr = 256
# sampling interval
ts = 1.0/sr
t = np.arange(0,1,ts)

# time domain signal
x = np.zeros_like(t)
for f in range(1,9,2):
    x += (2*f+1)*np.sin(2*np.pi*f*t)
plt.plot(x)
plt.show()
```



We can obtain the centered frequency spectrum with FFT as follows:

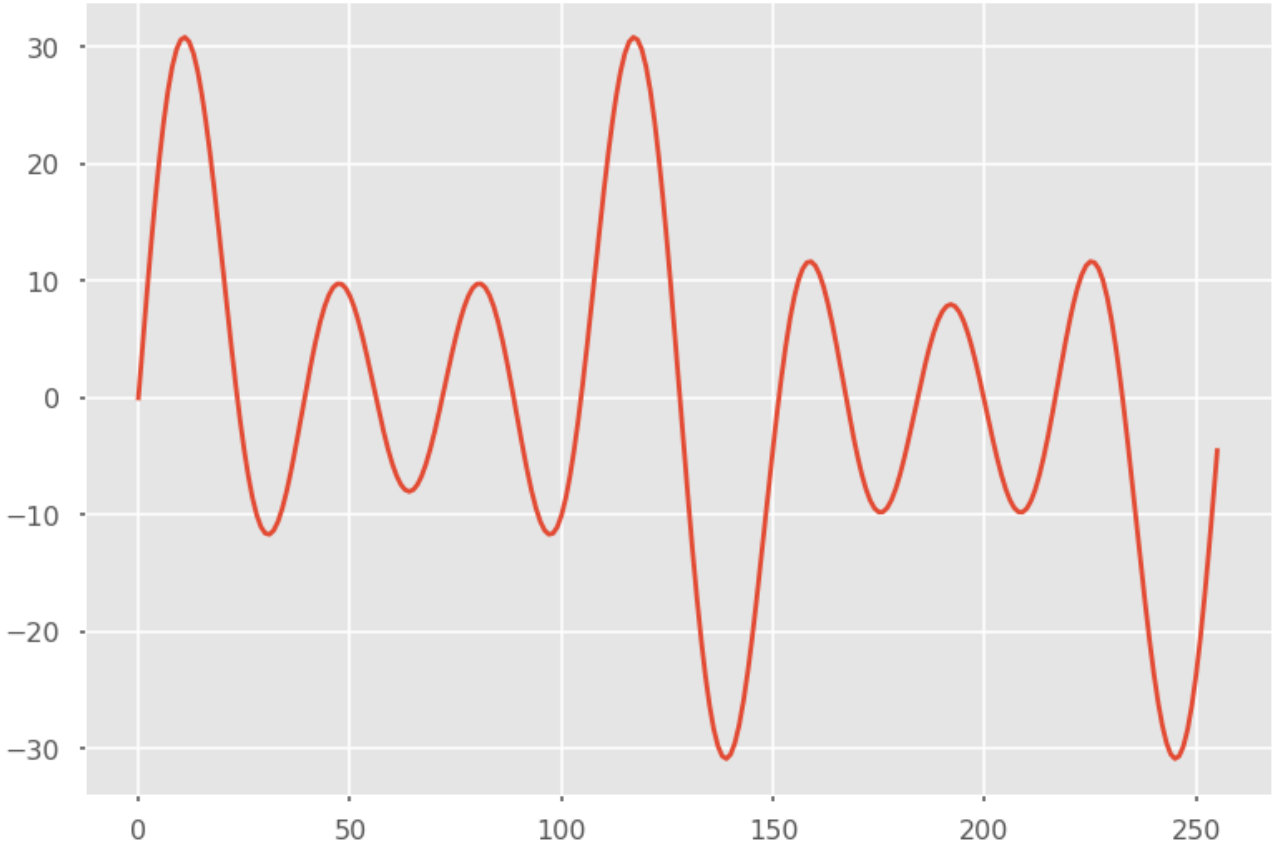
```
X = FFT(x)

# plot frequency domain signal by centering (fftshift)
N = len(X)
n = np.arange(N)
T = N/sr
freq = n/T

plt.stem(freq-N//2, np.roll(abs(X),N//2), 'b', \
         markerfmt=" ", basefmt="-b")
plt.xlabel('Freq (Hz)')
plt.ylabel('FFT Amplitude |X(freq)|')
```



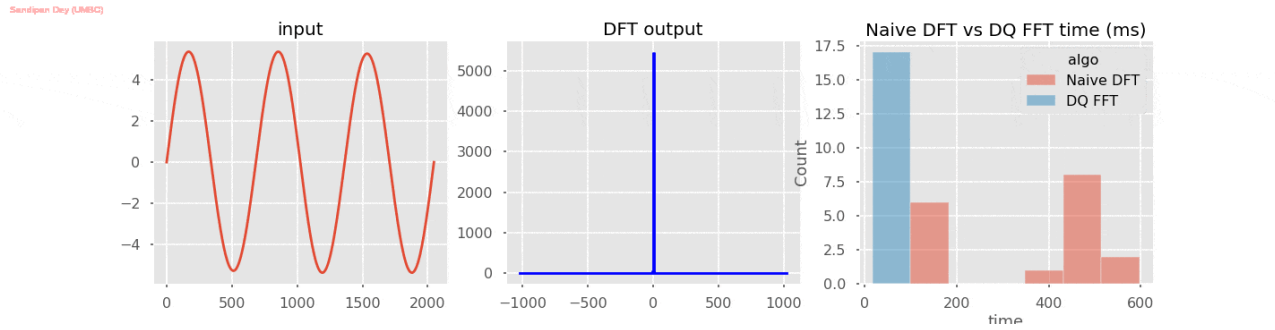
As expected, since we used 4 frequency values, namely $1, 3, 5, 7$ to construct the input signal in the time domain, the frequency domain output also contains the same frequencies with the corresponding proportional amplitudes.

Finally reconstruct the signal in time domain with inverse FFT:

```
x = FFT(X, inverse=True) / N
plt.plot(np.real(x))
```



The following animation compares the distribution of time taken by naive DFT vs. the above divide-conquer FFT algorithms when the input signals are of length 2^{10} or 2^{11} :



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edited May 27, 2022 at 0:46

answered May 26, 2022 at 0:34

Sandipan Dey

2,101 10 11