Question 7

Sunday, November 5, 2023 6:46 PM

7. Consider $A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} = (v_1 \,|v_2|\, v_3)$ with v_1, v_2, v_3 are the columns of A.

(a) Use Gram-Schmidt process to construct an orthonormal set $\{q_1, q_2, q_3\}$ such that for $j = 1, 2, 3$,

$$\operatorname{span}\{q_1, \ldots, q_j\} = \operatorname{span}\{v_1, \ldots, v_j\}.$$

$$\left\{ \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} -1/2 \\ -1/2 \\ -1/2 \\ -1/2 \end{bmatrix} \right\}$$

(b) Use the answer from (i), find r_{ij}, for $1 \le i \le j \le 3$ such that

$$v_1 = r_{11}q_1, \quad v_2 = r_{12}q_1 + r_{22}q_2, \quad v_3 = r_{13}q_1 + r_{23}q_2 + r_{33}q_3.$$

$$\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = R_{11} \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix} \qquad R_{11} = 2$$

$$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = R_{12} \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix} + R_{22} \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} \qquad \begin{matrix} R_{12} = 1 \\ R_{22} = 1 \end{matrix}$$

$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = R_{13} \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix} + R_{23} \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} + R_{33} \begin{bmatrix} -1/2 \\ -1/2 \\ -1/2 \\ -1/2 \end{bmatrix}$$

$$R_{13} = 0 \qquad R_{23} = 1 \qquad R_{33} = 1$$

(c) Denote $Q = (q_1 \,|q_2|\, q_3)$ and $R = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{pmatrix}$. Show that indeed $A = QR$ and

$\mathrm{Col}(A) = \mathrm{Col}(Q)$.

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} = \begin{bmatrix} q_1, & q_2, & q_3 \end{bmatrix} \begin{bmatrix} R \end{bmatrix}$$

$$\begin{pmatrix} 1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & -1/2 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}}_{A}$$ ✓

$\mathrm{Col}(A) = \mathrm{Col}(Q)$

① $\mathrm{Col}(A) \subseteq \mathrm{Col}(Q)$

every column vector can be written as a l.c of col vectors q_1, q_2, q_3. Conf Part B.

② $\mathrm{Col}(Q) \subseteq \mathrm{Col}(A)$

$$q_1 = \frac{1}{2} a_1$$

$$q_2 = a_2 - \frac{1}{2} a_1$$

$$q_3 = a_3 - \left(a_2 - \frac{1}{2} a_1\right)$$

$\therefore \mathrm{Col}(Q) = \mathrm{Col}(A)$

(d) Show that $Q^T Q = I_3$ and $QQ^T = q_1 q_1^T + q_2 q_2^T + q_3 q_3^T$. Therefore QQ^T is the orthogonal projection onto $\mathrm{Col}(Q) = \mathrm{Col}(A)$.

$$\underset{3 \times 4}{\begin{pmatrix} 1/2 & -1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \end{pmatrix}} \underset{4 \times 3}{\begin{pmatrix} 1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & -1/2 \end{pmatrix}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ ✓

$$\underset{4\times 3}{\begin{pmatrix} 1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & -1/2 \\ -1/2 & 1/2 & -1/2 \end{pmatrix}} \underset{3\times 4}{\begin{pmatrix} 1/2 & -1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \end{pmatrix}} = \begin{pmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{pmatrix} \begin{pmatrix} 1/2 & -1/2 & 1/2 & -1/2 \end{pmatrix}$$

$$+ \begin{pmatrix} 1/2 \\ 1/2 \\ -1/2 \\ 1/2 \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 & 1/2 & 1/2 \end{pmatrix}$$

$$+ \begin{pmatrix} 1/2 \\ -1/2 \\ -1/2 \\ -1/2 \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 & -1/2 & -1/2 \end{pmatrix}$$

$$= \begin{pmatrix} 3/4 & 1/4 & 1/4 & -1/4 \\ 1/4 & 3/4 & -1/4 & 1/4 \\ 1/4 & -1/4 & 3/4 & 1/4 \\ -1/4 & 1/4 & 1/4 & 3/4 \end{pmatrix}$$