Question 7

Sunday, November 19, 2023 9:35 PM

7. Prove or disprove the following claims.

(a) If $A \in M_3(\mathbb{R})$ has rows equal to v $2v$ $3v$ for some $v \in \mathbb{R}^3$ and A has a nonzero eigenvalue then A is diagonalizable.

TRUE

$$\begin{matrix} a-\lambda & b & c & a-\lambda & b \\ 2a & 2b-\lambda & 2c & 2a & 2b-\lambda \\ 3a & 3b & 3c-\lambda & 3a & 3b \end{matrix}$$

$(a-\lambda)(2b-\lambda)(3c-\lambda) + \cancel{6abc} + \cancel{6abc} - \cancel{6abc} + 3ac\lambda - \cancel{6abc} + \cancel{6bc\lambda} - \cancel{6abc} + 2ab\lambda$

$\cancel{6abc} - \cancel{3ac\lambda} - \cancel{6bc\lambda} + 2b\lambda^2 + a\lambda^2 + 3c\lambda^2 - \lambda^3$

$\lambda^2(2b + a + 3c - \lambda)$ $\quad \lambda = 0$ and $2b + a + 3c \neq 0$

for $\lambda = 0$, $\dim(E_0) = 2$ $\quad GM = AM = 2$

for $\lambda = a + 2b + 3c \neq 0$ $\quad \dim E = 1$ $\quad AM = GM = 1$

$\forall \lambda_i \quad GM_{\lambda_i} = AM_{\lambda_i} \Rightarrow$ Diagonizable

(b) If $A \in M_4(\mathbb{R})$ has characteristic polynomial $q_A(x) = x^2(x+5)(x+6)$ and

$$\begin{pmatrix} 0 \\ -1 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 7 \\ 5 \\ 3 \\ 4 \end{pmatrix} \in null(A)$$

then A is diagonalizable.

TRUE

Consider each E-Val of A Based on $q_A(\lambda)$

for $\lambda = 0$, The span of $E_0 = $ nullity A ; $\dim(E_0) = 2 = AM$

Additionally eigenvalues -5 and -6 each have eigenvectors with eigenspaces of span 1.

$\Rightarrow \forall \lambda_i$ The $GM(\lambda_i) = AM(\lambda_i) \Rightarrow A$ Digonizable

(c) Let $A \in M_n(\mathbb{R})$. If 0 is an eigenvalue of A then its geometric multiplicity is equal to $n - rank A$.

By Rank – nullity

$\dim\{Rank(A)\} + \dim\{Null(A)\} = n \quad \therefore \dim\{null(A)\} = n - \dim(Rank(A))$

and Geometric multiplicity for $\lambda = 0$ is

The nullity of matrix since for any

main diagonal elements $A - \lambda I_n \Rightarrow A - \vec{0} \Rightarrow A$

$GM(\lambda = 0) = \dim(null(A))$

TRUE

(d) There exists $A \in M_5(\mathbb{R})$ which is diagonalizable and satisfies $rank A = 1$ and $tr A = 0$.

if $Rank(A) = 1 < 5 \Rightarrow \det(A) = 0 \Rightarrow 0$ is eigenvector

Note By Rank-nullity $\dim(null A) = 4 \Rightarrow \dim\{E_{\lambda=0}\} = 4$

if $\exists$ Diagonal D of $A = PDP^{-1}$ for $P \in GL_5(\mathbb{R})$

Then $tr(A) = tr(D) = 0$

But if D is Digonal of A of the form $A = PDP^{-1} \Rightarrow D$ has the eigenvalues of A down its main digonal. Since Rank $A = 1$, $\exists$ 1 nonzero eigenvalue as the 5^{th} element on main diagonal in D.

This contradicts $tr(A) = 0$ since all 4 other e-vals $= 0$

and $\lambda_5 \neq 0$. $\Longrightarrow$ Statement false.

(e) If $A \in M_n(\mathbb{R})$ is diagonalizable and 2 is the only eigenvalue of A then $A = 2I$.

if $\lambda = 2$ for $A \in M_n(\mathbb{R}) \Rightarrow$ The AM of $\lambda = 2$ is going to be n since if Diagonizable $\Rightarrow \sum AM = n \Rightarrow AM(2) = n$. But Diagonizable also implies $AM = GM \; \forall \lambda_i \Rightarrow GM$ of $\lambda = 2$ is also n. $\Rightarrow \dim\{\text{null}\{A - \lambda I_n\}\} = n \Rightarrow$ every column is a free variable $\Rightarrow A - \lambda I_n = [0] \Rightarrow \underline{A = 2I_n}$

Statement TRUE

(f) If $A, B \in M_n(\mathbb{R})$ have the same eigenvalues and A is diagonalizable then so is B.

False

consider $A =$ zero matrix

A has 0 as its eigenvalues

Consider also $B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

This matrix only has eigenvalues of 0

B is Not Diagonizable.