MATH-1564-K1,K2,K3 Linear Algebra with Abstract Vector Spaces

Homework 4

1. In each of the following you are given a statement, which may be true or false. Determine whether the statement is correct and show how you reached this conclusion.
   (a) $\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \in \mathrm{span}\{\begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix}\}$
   (b) $2 + 3x + 2x^2 - x^3 \in \mathrm{span}\{1 - x^3, 2 + x + x^2, 3 - x\}$
   (c) $\mathrm{span}\{\begin{pmatrix} 5 & -2 \\ -5 & -3 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 4 & -1 \end{pmatrix}\} \subseteq \mathrm{span}\{\begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix}\}$
   (d) $\mathrm{span}\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\} = \mathrm{span}\{\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}\}$
   (e) $\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix}\}$ spans $\mathbb{R}^2$.
   (f) $\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}\}$ spans $\mathbb{R}^2$.
   (g) $\{1 - x + x^2, x - x^2 + x^3, 1 + x^2 - x^3, x^3\}$ spans $\mathbb{R}_3[x]$.

2. In each of the following you are given a vector space (you do not need to prove that this is indeed a vector space). Find a spanning set for each of these vector spaces.
   (a) $\{\begin{pmatrix} a+b+c \\ a-2b \\ 3a-2c \\ 4c-b \end{pmatrix} : a, b, c \in \mathbb{R}\}$
   (b) $\{A \in M_2(\mathbb{R}) : A\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\}$
   (c) $\{p(x) \in \mathbb{R}_3[x] : p'(1) = 0\}$
   (d) $\{p(x) \in \mathbb{R}_n[x] : p(1) = p(-1)\}$

3. In each of the following you are given a set, determine whether it is linearly independent or linearly dependent, show how you reach your conclusion.
   (a) $\{\begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix}\}$
   (b) $\{\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}\}$
   (c) $\{1 - x^3, 2 + x + x^2, 3 - x, 1 + x + x^2 + x^3\}$
   (d) $\{f(x) = \sin^2 x, g(x) = \cos^2(x), h(x) = 1\}$ (Note that $h(x)$ is the constant function which is equal to 1 for every x).

4. Let V be a vector space and w_1, w_2, w_3 in V be such that $\{w_1, w_2, w_3\}$ is linearly independent. Prove or disprove the following claims.
   (a) The set $\{w_1 + w_2 + w_3, w_2 + w_3, w_3\}$ is linearly independent.

(b) The set $\{w_1 + 2w_2 + w_3, w_2 + w_3, w_1 + w_2\}$ is linearly independent.

5. Let V be a vector space and let $S \subset V$ and $T \subset V$ be two finite subsets of V. Prove or disprove the following claims.

   (a) If $S \subset T$ and S is linearly independent then T is linearly independent.

   (b) If $S \subset T$ and T is linearly independent then S is linearly independent.

   (c) If S and T are linearly independent then $S \cap T$ is either empty or linearly independent (Remark: sometimes people consider an empty set to be linearly independent).

   (d) If S and T are linearly independent then $S \cup T$ is linearly independent.

   (e) If $W = \text{span}S$ and $U = \text{span}T$ then $W + U = \text{span}(S \cup T)$.

6. The following claims are either **true or false**. Determine which case is it for each claim and prove your answer.

   (a) Let V be a vector space which satisfies dim$V$=3. Then there exist, a subspace W of V and a subspace U of W (that is, $U \subset W \subset V$) such that dim$U$=1 and dim$W$=2.

   (b) Let V be a vector space which satisfies dim$V$=3 and let W be a **non–trivial** subspace of V and U be a **non–trivial** subspace of W (that is, $U \subset W \subset V$) then dim$U$=1 and dim$W$=2.

   (c) Let V be a vector space which satisfies dim$V$=3 and let $v_1, v_2, v_3 \in V$ be such that $\{v_1, v_2\}$ are linearly independent, $\{v_2, v_3\}$ are linearly independent, and $\{v_3, v_1\}$ are linearly independent. Then $\{v_1, v_2, v_3\}$ is a basis for V.

   (d) Let V be a vector space and $v_1, ..., v_n \in V$ then: $\{v_1, ..., v_n\}$ is linearly independent iff $\dim\big(\text{span}\{v_1, ..., v_n\}\big) = n$.

   (e) Let V be a vector space and let $V_1, V_2, V_3 \subset V$ be such that $V_1 + V_2 = V_1 + V_3$ and dimV_2 =dimV_3 then $V_2 = V_3$. (The sum of two subspaces was defined in previous HW's).