Question 1

Thursday, October 12, 2023 1:59 PM

1. Consider bijective map $f : \mathcal{D} \to \mathcal{C}$. In class, we have stated that there exists map $g : \mathcal{C} \to \mathcal{D}$ such that $f \circ g = \mathrm{id}_{\mathcal{D}}$ and $g \circ f = \mathrm{id}_{\mathcal{D}}$. g in this case is called the inverse of f, denoted by f^{-1}. In the context of linear maps, we say linear map $T \in \mathcal{L}(V, W)$ is **invertible** if there exists linear map $S \in \mathcal{L}(W, V)$ such that $T \circ S = \mathrm{id}_W$ and $S \circ T = \mathrm{id}_V$. In this case, we denote S by T^{-1}.

   (a) Let $T \in \mathcal{L}(V, W)$. Show that its inverse T^{-1}, if it exists, is unique.

Consider Set $V = \{v_1, \dots, v_n\}$

and $W = \{w_1, \dots, w_n\}$ Basis for $\bar{V}$ + $\bar{W}$ resp.

Then $T \in \mathcal{L}(V, W)$ is such that $T(\vec{V}) = \vec{W}$

That is $T\left(\sum_i^n c_i v_i\right) = \sum_i^n c_i w_i$.

Assume G and h are inverses of T

Such that $G, h \in \mathcal{L}(\bar{W}, \bar{V})$

as such $T \circ G = \mathrm{id}_W$ and $T \circ h = \mathrm{id}_W$.

$G \circ T = Id_V$ and $h \circ T = Id_V$

$h = h \circ Id_W$

$= h \circ (T \circ G)$

$(h \circ T) \circ G$

$Id_V \circ G$

$= G$ $\Rightarrow$ $\underline{h = G}$ inverse is $\underline{\text{unique}}$

(b) Show that $T \in \mathcal{L}(V, W)$ is invertible iff it is bijective, i.e., an isomorphism.

① if T Bijective $\Rightarrow$ T invertible

Since T is Bijective, $\ker(T) = \{\vec{0}\}$

and since T Bijective, Then for every $v_i \in \{v_1, \dots, v_n\}$

Basis for $\bar{V}$ $T(v_i) = w_i$, $w_i \in \{w_1, \dots, w_n\}$ Basis for $\bar{W}$.

likewise $\dim(\mathrm{Im}(T)) = \dim(\bar{W})$ and $\exists$ a one-to one map Between $\bar{V}$ and $\bar{W}$. T is linear

$T(\alpha x + \lambda y) = \alpha T(x) + \lambda T(y)$

assume $\exists\ T^{-1}$ s.t. $T(T^{-1}(x))\ Id_W$.

and say $T(x) = x'$, $T(y) = y'$

Then $T(\alpha T^{-1}(x') + \lambda T^{-1}(y')) = \alpha x' + \lambda y'$

$\Rightarrow T^{-1}$ linear and T is invertible

② if T invertible $\Rightarrow$ T bijective?

Since T invertible, $\exists\ T^{-1} = S$

Thus $S \circ T = Id_V$ implies injectivity where every element in $\bar{V}$ is mapped to one in $\bar{W}$. and $T \circ S = Id_W$ meaning T surjective since any and all elements in $\bar{W}$ are mapped to some $\bar{V}$. Thus T Bijective.

(c) Give two examples of linear maps that are not invertible for two different reasons.

① $f : \mathbb{R} \to 0$

$x \mapsto 0$

This is surjective
But not injective

$\therefore$ Not invertible

② $T : \mathbb{R}^3 \to \mathbb{R}^2$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} x \\ y \end{bmatrix}$ Not invertible

Question 2

Saturday, October 14, 2023 5:37 PM

2. In class, we have shown that if two finite-dimensional vector spaces are of the same dimension, then they are isomorphic. Show the converse is also true.

if $Q \Rightarrow P$

Converse: if $P \Rightarrow Q$

if 2 V.S. have same dimension, then isomorphic

if 2 V.S. are isomorphic, then same dimension

Proof|

Consider $\bar{V}$ with basis $\{v_1, \dots, v_n\}$ and

$\bar{\omega}$ ...... $\{w_1, \dots, w_n\}$

if $\bar{V}$ and $\bar{\omega}$ are isomorphic, $\exists$ a

linear map $T \in \mathcal{L}(\bar{V}, \bar{\omega})$, an isomorphism

s.t. $T(v_i) = w_i \quad \forall i \; \{1, \dots, N\}$ dim of(V).

as such, if T is an isomorphism for $\bar{V} \to \bar{\omega}$

Then T is bijective meaning there is a

one-to-one relation between $\bar{V}$ and $\bar{\omega}$.

as such since $T(v_i) = \omega_i \quad \forall i \in \{1, \dots, n\}$, Then

$\dim(\bar{V}) = \dim(\bar{\omega})$.

T is a bijective map, thus $\ker(T) = \{\vec{0}\}$

and by Rank-nullity

$$\dim(\ker(T)) + \dim(\operatorname{im}(T)) = \dim V$$

$$0 + \dim(W) = \underline{\dim(V)}$$

Question 3

Saturday, October 14, 2023 5:50 PM

3. Let $A \in M_{m\times n}(\mathbb{R})$ and consider the linear map T_A associated with A defined as $T_A : \mathbb{R}^n \to \mathbb{R}^m$ where $T_A(x) = Ax$. Show that

(a) T_A is injective if there is a pivot in every col of rref(A).

If There is a pivot in every col of RREF(A), Then There are n Pivots. $\Rightarrow$ The Col(A) creates a subspace with dimension n. That means any vector x in The Domain $\mathbb{R}^n$, $x \in \mathbb{R}^n$ is sent via T to a Unique vector $y \in \mathbb{R}^m$. essentially y is a linear Combination of The columns of A, specified By A. Since There is a Pivot in every column of A (Ref), Then the nullspace of A, i.e. The $\ker(T_A) = \{\vec{0}\}$. Thus, This implies T_A is injective.

(b) T_A is surjective if there is a pivot in every row of rref(A).

If There is a pivot in every Row of RREF(A), Then There are m Pivots. This implies that any augmented matrix representing The equation $A\vec{x} = \vec{b}$ has only one solution.

This implies that for any vector $\vec{b} \in \mathbb{R}^m$, there is at least one $\vec{x}$ that exists in $\mathbb{R}^n$.

as such the image of $T = \{y \mid y = Ax, x \in \mathbb{R}^n\}$ and $\dim(\mathrm{im}\,T) = m$, # pivots. Thus

T_A is surjective.

Question 4

Saturday, October 14, 2023 6:19 PM

4. Consider the following matrices:

$$\overset{2\times 2}{} A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}, \quad B = \overset{2\times 3}{\begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & 1 \end{pmatrix}}, \quad F = \overset{1\times 3}{\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}}$$

$$C = \underset{3\times 2}{\begin{pmatrix} 1 & 2 \\ 3 & 1 \\ 2 & -1 \end{pmatrix}}, \quad D = \underset{3\times 3}{\begin{pmatrix} 1 & 0 & 3 \\ 1 & 1 & -2 \\ 2 & -1 & 1 \end{pmatrix}}, \quad E = \overset{3\times 1}{\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}}$$

Compute the following expressions if defined: AB, BA, D^2, B^2, DC, CB, BC, FE, EF, CE, EC.

$$AB = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 2 & 5 \\ 0 & -1 & 5 \end{pmatrix}$$

$$BA = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} = \text{und}$$

$$D^2 = \begin{pmatrix} 1 & 0 & 3 \\ 1 & 1 & -2 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 3 \\ 1 & 1 & -2 \\ 2 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 7 & -3 & 6 \\ -2 & 3 & -1 \\ 3 & -2 & 9 \end{pmatrix}$$

$$B^2 = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & 1 \end{pmatrix} = \text{und}$$

$$DC = \underset{3\times 3}{\begin{pmatrix} 1 & 0 & 3 \\ 1 & 1 & -2 \\ 2 & -1 & 1 \end{pmatrix}} \begin{pmatrix} 1 & 2 \\ 3 & 1 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 7 & -1 \\ 0 & 5 \\ 1 & 2 \end{pmatrix}$$

$$CB = \begin{pmatrix} 1 & 2 \\ 3 & 1 \\ 2 & -1 \end{pmatrix} \overset{3\times 2}{\begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & 1 \end{pmatrix}} = \begin{pmatrix} 5 & 2 & 5 \\ 5 & 1 & 10 \\ 0 & -1 & 5 \end{pmatrix}$$

$3\times 2 \qquad 2\times 3$

$$BC = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 1 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 7 & -1 \\ 7 & 4 \end{pmatrix}$$

$$Fe = \underset{1\times 3}{\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}} \underset{3\times 1}{\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}} = 2$$

$$Ef = \underset{3\times 1}{\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}} \underset{1\times 3}{\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ -1 & -2 & -3 \end{pmatrix}$$

$$CE: \underset{3\times 2}{\begin{pmatrix} 1 & 2 \\ 3 & 1 \\ 2 & -1 \end{pmatrix}} \underset{3\times 1}{\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}} = \text{und}$$

$$EC \quad \underset{3\times 1}{\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}} \underset{3\times 2}{\begin{pmatrix} 1 & 2 \\ 3 & 1 \\ 2 & -1 \end{pmatrix}} = \text{und}$$

Question 5

Sunday, October 15, 2023 4:13 PM

5. Here are some facts about matrices that will come in handy in the future.

(a) A square matrix $A \in M_n(\mathbb{R})$ is called **diagonal** if all of its entries that are not on the main diagonal are equal zero, that is, A is diagonal if $(A)_{ij} = 0$ for all $i \neq j$. Here is an example of a diagonal matrix:

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Prove that if $A, B \in M_n(\mathbb{R})$ are both diagonal then both $A + B$ and AB are diagonal as well.

Consider A and B arbitrary matrices in where They are Diagonal. Prove That A+B and AB are Diagonal via induction.

Base case: $n=1$.

Consider $A, B \in M_1(\mathbb{R})$. Then A + B are Diagonal if $A = (\alpha)$, $B = (\psi)$ s.t. $\alpha, \psi \in \mathbb{R}$

Then $A+B = (\alpha+\psi) \in M_1(\mathbb{R})$ still Diagonal

and $AB = (\alpha\psi) \in M_1(\mathbb{R})$ still Diagonal

Induction hypothesis: $n = k$

Consider $A, B \in M_k(\mathbb{R})$ s.t. A+B are Diagonal

$$A = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & a_{kk} \end{pmatrix} \quad B = \begin{pmatrix} b_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & b_{kk} \end{pmatrix}$$

Assume That for $A, B \in M_k(\mathbb{R})$ The following holds

$A+B$ is Diagonal and AB is Diagonal

Induction Step: $n = k+1$

$A, B \in M_{k+1}(\mathbb{R})$ $A = \begin{pmatrix} a_{11} & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \vdots & & a_{kk} & 0 \\ 0 & \cdots & \cdots & \vdots \\ 0 & \cdots & 0 & a_{k+1\,k+1} \end{pmatrix}$

$B = \begin{pmatrix} b_{11} & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \vdots & & b_{kk} & 0 \\ 0 & \cdots & \cdots & \vdots \\ 0 & \cdots & 0 & b_{k+1\,k+1} \end{pmatrix}$

Since $A+B$ for $A, B \in M_k(\mathbb{R})$ is Diagonal

$\Rightarrow A+B$ $\begin{pmatrix} a_{11}+b_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{kk}+b_{kk} \end{pmatrix}$ Diagonal

$\Rightarrow A+B$ for $A, B \in M_{k+1}(\mathbb{R})$

$A+B = \begin{pmatrix} a_{11}+b_{11} & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & a_{kk}+b_{kk} & 0 \\ 0 & \cdots & 0 & a_{k+1\,k+1}+b_{k+1\,k+1} \end{pmatrix}$

IOW: $\{a_{i,k+1}\}_{i=1}^{k} = \{0\}$

and $\{a_{k+1,i}\}_{i=1}^{k} = \{0\}$

likewise for B. Thus $A+B$ for the

Rows/columns $k+1$ of A, B is still

zero making $A+B$ Diagonal for $A, B \in M_{k+1}(\mathbb{R})$

for AB where $A, B \in M_{k+1}(\mathbb{R})$

$$AB = \begin{pmatrix} a_{11}b_{11} & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & a_{kk}b_{kk} & 0 \\ 0 & \cdots & 0 & a_{k+1,k+1}b_{k+1,k+1} \end{pmatrix}$$

still Diagonal

Still zero

Since

$$\{a_{i,k+1}\}_{i=1}^{k} = \{0\}$$

$$\{a_{k+1,i}\}_{i=1}^{k} = \{0\}$$

likewise for B, the k+1 Rows and columns are still zero making AB Diagonal for $A, B \in M_{k+1}(\mathbb{R})$

(b) For a square matrix $A \in M_n(\mathbb{R})$ the **trace** of A, denoted $tr(A)$, is the sum of all of its entries on the main diagonal, that is $tr(A) = \sum_{i=1}^{n} (A)_{ii}$. Here is an example of a trace computation:

$$tr \begin{pmatrix} 3 & 5 & 7 \\ 0 & 2 & 4 \\ 8 & 1 & -1 \end{pmatrix} = 3 + 2 + (-1) = 4.$$

i. For $A, B \in M_n(\mathbb{R})$ prove that $tr(AB) = tr(BA)$.

ii. Show that $tr(\cdot) : M_n(\mathbb{R}) \to \mathbb{R}$ is a linear map.

i) The trace for $C = AB$ is Defined as

$$tr(C) = \sum_{i}^{n} c_{ii} \Rightarrow \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} b_{ji}$$

we can swap the sums

$$\Rightarrow \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij} b_{ji}$$

Beca... a and b are $\in \mathbb{R}$, they are commutive $\Rightarrow \sum_{j=1}^{n} \sum_{i=1}^{n} b_{ji}\, a_{ij} \Rightarrow$ trace(BA)

Since square matrices

ii. Show that $tr(\cdot) : M_n(\mathbb{R}) \to \mathbb{R}$ is a linear map.

① tr(·) is closed under Addition

let $A, B \in M_n(\mathbb{R})$

$Tr(A+B) \Rightarrow Tr(C)$ where $(C)_{ij} = (A)_{ij} + (B)_{ij}$

$\Rightarrow \sum_{i=1}^{n} C_{ii} \Rightarrow \sum_{i=1}^{n} (a_{ii} + b_{ii}) \Rightarrow \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii}$

$\Rightarrow Tr(A) + tr(B)$

② Tr(·) is closed under Scalar multiplication

let $A \in M_n(\mathbb{R})$ and $\alpha \in \mathbb{R}$

Then $Tr(\alpha A) \Rightarrow \sum_{i=1}^{n} \alpha a_{ii} \Rightarrow \alpha \sum_{i=1}^{n} a_{ii} \Rightarrow \alpha Tr(A)$

(c) For a square matrix $A \in M_n(\mathbb{R})$ the **transposed** of A, denoted A^T, is the matrix obtained by turning each row of A into a column by order, that is $(A^T)_{i,j} = (A)_{j,i}$. Here is an example of a transposed computation:

$$\begin{pmatrix} 3 & 5 & 7 \\ 0 & 2 & 4 \\ 8 & 1 & -1 \end{pmatrix}^T = \begin{pmatrix} 3 & 0 & 8 \\ 5 & 2 & 1 \\ 7 & 4 & -1 \end{pmatrix}.$$

For $A, B \in M_n(\mathbb{R})$ prove that $(AB)^T = B^T A^T$.

M

Since $AB = \sum_{k=1}^{1} A_{ik} B_{kj}$ for $AB \in i, j$

The $(AB)^T$ flips all Rows to Columns

Thus $(AB)^T_{i,j} = \sum_{k=1}^{n} A_{jk} B_{ki}$

$$A \qquad B$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

$$AB = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$

$$(AB)^T = \begin{pmatrix} ae+bg & ce+dg \\ af+bh & cf+dh \end{pmatrix}$$

$B^T A^T$ can for arbitrary i, j be written as

$$(B^T A^T)_{i,j} = \sum_{k=1}^{n} B^T_{ik} A^T_{kj}$$

$$\Rightarrow \sum_{k=1}^{n} B_{ki} A_{jk} \Rightarrow \sum_{k=1}^{n} A_{jk} B_{ki} = (AB)^T_{ij}$$

(d) Let $A \in M_{m\times n}(\mathbb{R})$, $B \in M_{n\times k}(\mathbb{R})$, and $C \in M_{k\times \ell}(\mathbb{R})$. Prove that

7×6 $\qquad$ 6×4

$\underset{n\times l}{BC}$ $\qquad$ 7×4

$$A(BC) = (AB)C.$$

$A(BC) = A\left(\sum_{q=1}^{k} B_{iq} C_{qj}\right)$ for any arbitrary $i \leq n$, $j \leq l$

$$\Rightarrow \sum_{x=1}^{n} A_{fx} \left(\sum_{q=1}^{k} B_{iq} C_{qj}\right)$$

$f \leq M$

$$\Rightarrow \sum_{x=1}^{n} \sum_{q=1}^{k} A_{fx} B_{iq} C_{qj}$$

But since

$1 \leq i \leq n$

$$\Rightarrow \sum_{i=1}^{n} \sum_{q=1} A_{fi} B_{iq} C_{qj}$$ x can be subbed for i

$$\Rightarrow \left(\sum_{i=1}^{n} a_{fi} B_{iq}\right) \sum_{q=1}^{k} C_{qj}$$

$$\Rightarrow (AB)C$$

Question 6

Sunday, October 15, 2023 5:53 PM

6. Consider each of the following there is a claim, which might be **true or false**. If the claim is true then prove it, and if it is false then provide a counterexample. (For counter examples you may choose any n you wish, but if you want to prove a claim then you should prove it for all possible n's).
   (a) If $A \in M_n(\mathbb{R})$ satisfies $A^2 = 0$ then $A = 0$. (Here 0 is the zero matrix).
   (b) If $A, B \in M_n(\mathbb{R})$ are such that $AB = BA$ then $AB^2 = B^2A$.
   (c) Let $A, B, C \in M_n(\mathbb{R})$. If $AB = CB$ then $A = C$.
   (d) Let $A \in M_n(\mathbb{R})$, then $(A + I)^2 = A^2 + 2A + I$.
   (e) Let $A, B \in M_n(\mathbb{R})$, then $(A + B)^2 = A^2 + 2AB + B^2$.

a) if $A \in M_n(\mathbb{R})$, then

$$A^2 = \sum_{k=1}^{n} A_{ik}A_{kj} \quad \text{for arbitrary } i, j \leq n$$

if $A^2 = \sum_{k=1}^{n} A_{ik}A_{kj} = 0$, then $A = [0]$ false

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+bc & ab+bd \\ ca+dc & cb+d^2 \end{bmatrix}$$

$a^2 + bc = 0$
$ab + bd = 0$
$ca + dc = 0$
$cb + d^2 = 0$

$a = 1$
$1 + bc = 0$
$b + bd = 0$
$c + dc = 0$
$cb + d^2 = 0$

$bc = -1$
$b(d+1) = 0$
$c(d+1) = 0$
$cb + d^2 = 0$

$d = -1$

$bc = -1$
$cb + 1 = 0$

$cb = -1$

$c = 1$
$b = -1$

$\begin{bmatrix} 1 & -1 \end{bmatrix}$ False

$\begin{bmatrix} 1 & -1 \end{bmatrix}$ false

(b) If $A, B \in M_n(\mathbb{R})$ are such that $AB = BA$ then $AB^2 = B^2A$.

b) If $AB = BA$, Then $AB^2 = B^2A$

<u>Proof)</u> $AB^2 \Rightarrow ABB = BBA$

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj} = \sum_{k=1}^{n} B_{ik}A_{kj} = (BA)_{ij}$$

let $C = AB$. Then C also $= BA$

$$ABB = BBA \Rightarrow CB = BC \Rightarrow (CB)_{ij} = (BC)_{ij}$$

$$\Rightarrow \sum_{k=1}^{n} C_{ik}B_{kj} = \sum_{k=1}^{n} B_{ik}C_{kj}$$

But Since $C = AB, BA$

$$\Rightarrow \sum_{k=1}^{n}\left(\sum_{k=1}^{n} A_{ik}B_{kj}\right)B_{kj} = \sum_{k=1}^{n} B_{ik}\left(\sum_{k=1}^{n} A_{ik}B_{kj}\right)$$

Since $AB = BA$ @ ij, Then $B_{kj} = B_{ik}$

$$\Rightarrow \sum_{k=1}^{n}\left(\sum_{k=1}^{n} A_{ik}B_{kj}\right)B_{kj} = \sum_{k=1}^{n}\left(\sum_{k=1}^{n} A_{ik}B_{kj}\right)B_{kj}$$

$$AB^2 = B^2A$$

(c) Let $A, B, C \in M_n(\mathbb{R})$. If $AB = CB$ then $A = C$.

false if B is the zero matrix, $AB = 0$
$CB = 0$
$\forall A, B \in M_n(\mathbb{R})$

ex)

$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $CB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $A \neq C$

(d) Let $A \in M_n(\mathbb{R})$, then $(A + I)^2 = A^2 + 2A + I$.

Consider the expansion of $(A + I_n)^2$

$(A + I_n)(A + I_n) \Rightarrow A(A + I_n) + I_n(A + I_n)$

$\Rightarrow A^2 + A I_n + I_n A + (I_n)^2$

$A^2 + 2 A I_n + I_n$

$\Rightarrow A^2 + 2A + I_n$ ✓ claim is true

(e) Let $A, B \in M_n(\mathbb{R})$, then $(A + B)^2 = A^2 + 2AB + B^2$.

like above, lets expand $(A+B)^2 \Rightarrow (A+B)(A+B)$

$A(A+B) + B(A+B) \Rightarrow \underbrace{A^2 + AB + BA + B^2}$

Note: $AB \neq BA$ thus $\neq$ $A^2 + 2AB + B^2$ ✓

counterexample

$$\overset{A}{\begin{bmatrix}2&3\\4&5\end{bmatrix}}\overset{B}{\begin{bmatrix}1&0\\1&0\end{bmatrix}} \quad (A+B)^2 = \begin{bmatrix}3&3\\5&5\end{bmatrix}^2 = \begin{bmatrix}24&24\\40&40\end{bmatrix}$$

$$\begin{bmatrix}2&3\\4&5\end{bmatrix}\begin{bmatrix}2&3\\4&5\end{bmatrix} = \begin{bmatrix}16&21\\28&37\end{bmatrix}$$

$$\begin{bmatrix}2&3\\4&5\end{bmatrix}\begin{bmatrix}1&0\\1&0\end{bmatrix} = 2\begin{bmatrix}5&0\\9&0\end{bmatrix} = \begin{bmatrix}10&0\\18&0\end{bmatrix}$$

$$\begin{bmatrix}1&0\\1&0\end{bmatrix}^2 = \begin{bmatrix}1&0\\1&0\end{bmatrix}$$

$$\begin{matrix}1&0&10\\1&0&10\end{matrix} \quad RHS = \begin{bmatrix}27&21\\47&37\end{bmatrix} \neq \begin{bmatrix}24&24\\40&40\end{bmatrix}$$

Question 7

Sunday, October 15, 2023 10:09 PM

7. The following claims are either **true or false**. Determine which case is it for each claim and prove your answer.
   (a) For any two matrices $A \in M_{m\times n}(\mathbb{R})$ and $B \in M_{n\times k}(\mathbb{R})$ we have $\text{rank}(AB) = \text{rank}(A) \cdot \text{rank}(B)$.
   (b) If A is a square matrix then its column space is equal to its null space.
   (c) If $A \in M_{m\times n}(\mathbb{R})$ is such that the vectors in its rows are linearly independent then the vectors in its columns are also linearly independent.
   (d) If $A \in M_n(\mathbb{R})$ is such that the vectors in its rows are linearly independent then the vectors in its columns are also linearly independent

a) false.

let $A = \begin{bmatrix} 2 & 3 \\ 7 & 1 \end{bmatrix}$ RREF(A) $= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ RANK(A) = 2

$B = \begin{bmatrix} 4 & 12 \\ 8 & 1 \end{bmatrix}$ RREF (B) $= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ RANK(B) = 2

$AB = \begin{bmatrix} 8+24 & 24+3 \\ 28+8 & 84+1 \end{bmatrix} = \begin{bmatrix} 32 & 27 \\ 36 & 85 \end{bmatrix}$ RREF(AB) $= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ RANK(AB) = 2

RANK(AB) $= 2 \neq 4 =$ Rank(A) $\cdot$ Rank(B)

b) If A is a square matrix then its column space is equal to its null space.

Consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, Then Col(A) = Span$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

But Nul(A) $= \{\vec{0}\} \neq$ Span$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$, Thus false

c)

(c) If $A \in M_{m\times n}(\mathbb{R})$ is such that the vectors in its rows are linearly independent then the vectors in its columns are also linearly independent.

$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$ vectors $\{[1,2,1],[1,2,3]\}$

are linearly independent

But $\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}2\\2\end{bmatrix},\begin{bmatrix}1\\3\end{bmatrix}\right\}$ are LD

false

D)

(d) If $A \in M_n(\mathbb{R})$ is such that the vectors in its rows are linearly independent then the vectors in its columns are also linearly independent

True! If A is such a matrix with lin. ind. Rows

Then the Rank of A $Rank(A) = n$.

$\Rightarrow \dim(Rows(A)) = n$, n linearly independent Rows

Note: $Rank(A)$ implies it is equal to Both the

$\dim(Row(A))$ and $\dim(Col(A))$.

$\Rightarrow$ $n = \dim(col(A))$. Meaning n independent Columns

Thus all columns in A are linearly independent

Question 8

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8. For any two matrices $A \in M_{m\times n}(\mathbb{R})$ and $B \in M_{n\times k}(\mathbb{R})$ which satisfy $AB = 0$ prove that $\text{rank}(B) + \text{rank}(A) \le n$.

if $AB=0$, Then $(AB)_{ij} = \sum_{p=1}^{n} A_{ip}B_{pj} = 0$ arbitrary $i \le m$, $j \le k$

This implies that $B \in$ null(A), AKA $\ker(A)$

By Rank-nullity Theorem

$$\text{col}(B) \subseteq \text{null}(A)$$

$$\underbrace{\dim(\text{col}(B))}_{\text{Rank } B} \le \dim(\text{null}(A))$$

$$\text{Rank } B + \text{Rank } A \le \underbrace{\dim(\text{null}(A)) + \text{Rank}(A)}_{= n}$$

$\Rightarrow$ Rank B + Rank $A \le n$

Question 9

Sunday, October 15, 2023 10:43 PM

9. The following claims are either **true or false**. Determine which case is it for each claim and prove your answer.
   (a) For any two $m \times n$ matrices A and B we have $\text{rank}(A+B) = \text{rank}(A) + \text{rank}(B)$.
   (b) For any two $m \times n$ matrices A and B we have $\text{rank}(A+B) \leq \text{rank}(A) + \text{rank}(B)$.

3x5 3x5

a)
false

let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

$A+B = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ Rank $(A+B) = 2$

Rank (A) + Rank $(B) = 4 \neq 2$

B) let A and B Be Represented as follows

$A = \begin{pmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{pmatrix}$ $B = \begin{pmatrix} | & & | \\ b_1 & \cdots & b_n \\ | & & | \end{pmatrix}$ $A+B = \begin{pmatrix} | & & | \\ a_1+b_1 & \cdots & a_n+b_n \\ | & & | \end{pmatrix}$

where A is Spanned By $\{a_1, \ldots, a_n\}$ where B is Spanned By $\{b_1, \ldots, b_n\}$ and A+B is Spanned By $\{a_1, b_1, \ldots, a_n, b_n\}$

but every Column vector in A+B Can Be Rep. as a lin. Comb. of $\{a_1, \ldots, a_n\}$ and/or $\{b_1, \ldots, b_n\}$

Thus The Cols of A+B is Spanned By at

most 2n Basis vectors (Basis of A = n + Basis of B = n)

Thus Rank(A + B) $\leq$ Rank(A) + Rank(B)