11/27

Saturday, November 25, 2023 22:03

1. In class we defined definiteness of real symmetric matrices by considering its associated quadratic form. We can also define definiteness of a quadratic form in the similar fashion. That is, quadratic form $Q(\vec{x})$ is positive definite if $Q(\vec{x}) > 0$ for all non-zero $\vec{x}$ and is positive semi-definite if $Q(\vec{x}) \geq 0$ for all $\vec{x}$. Similarly, negative definite, negative semidifinite can also be defined. If $Q(\vec{x})$ fails to be any of these, it is then called indefinite. For each of the following: a. Determine the defniteness of the following quadratic forms. Note that you can either examine the quadratic form directly or consider the symmetric matrix associated with it. B.. Write it in terms of $\vec{x}^T A \vec{x}$ where A is a real symmetric matrix.
   a. $Q(x,y) = -x^2 - 2xy - y^2$
   b. $Q(x,y) = 3x^2 - 8xy + 3y^2$
2. Let $A \in M_{m\times n}(\mathbb{R})$.
   a. Show that $A^T A$ is symmetric positive semi-definite.
   b. Show that $A^T A$ is symmetric positive definite if A's columns are linear independent.
3. True or False
   a. If A is invertible and orthogonally diagonalizable, then so is A^{-1}.
   b. If A is symmetric , then rank(A) = rank(A^2)
   c. The sum of two positive definite matrices is also positive definite.
4. Consider linear map $T: \mathbb{R}^n \to \mathbb{R}^m$. Show that there exists an orthonormal basis $< v_1, \ldots, v_n >$ in $\mathbb{R}^n$ such that $Tv_1, \ldots, Tv_m$ form an orthogonal set in $\mathbb{R}^m$. Note that it is possible some of the images under T are zero vectors.
5. Last time we orthogonally diagonalized matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. Use this result to orthogonally diagnalize the following
   a. $\begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix}$
   b. $\begin{pmatrix} 9 & 1 & 1 \\ 1 & 9 & 1 \\ 1 & 1 & 9 \end{pmatrix}$
   c. $\begin{pmatrix} 0 & 5 & 5 \\ 5 & 0 & 5 \\ 5 & 5 & 0 \end{pmatrix}$

Many students skipped last lecture, where I covered spectral theorem anc forms. Please also review these results and definitions when you see fit.

1. If A is real symmetric, then A = QDQ^T for some orthogonal matrix (matrix D.
2. Q(x) and its corresponding x^TAx
3. Q(x) and x^Tax are positive definite iff all A's eigenvalues are positiv A's eigenvalues are non-negative.
4. Any quadratic form can be written as $Q(\vec{x}) = \lambda_1 c_1^2 + \ldots, \lambda_n c_n^2$ $\lambda_1, \ldots, \lambda_n$ are eigenvalues of A and $c_1, \ldots, c_n$ is the coordin eigenbasis of A.

Please do 1.a directly and also consider its symmetric matrix A's eigen

Please do this one in studio since I will need it for SVD on Tuesday.

This is also getting them ready for SVD. The basis is just the eigenbasis of

$A = QDQ^T$ or $D = Q^T AQ$. We have 3A, A+8I and 5A-I