Question 5

Sunday, November 19, 2023 8:42 PM

5. Each of the following you are given a linear map. Determine whether it is diagonalizable.

(a) $T : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ given by

$$TA = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} A$$

$P_T(\lambda) = (1-\lambda)(4-\lambda) - 4 \Rightarrow 4 - 5\lambda + \lambda^2 - 4 \Rightarrow \lambda^2 - 5\lambda$; $\lambda = 0$ & 5

for $\lambda = 0$, $Am = 1$, $Gm = 1$

$E_{\lambda=0} = \text{null}\begin{Bmatrix} 1 & 2 \\ 2 & 4 \end{Bmatrix} = \text{span}\left\{\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right\}$

for $\lambda = 5$, $Am = 1$, $Gm = 1$

$E_{\lambda=5} = \text{null}\begin{Bmatrix} -4 & 2 \\ 2 & -1 \end{Bmatrix} = \text{span}\left\{\begin{pmatrix} 1 \\ -2 \end{pmatrix}\right\}$

$\therefore \ \forall \lambda_i \quad Am_{\lambda_i} = Gm_{\lambda_i} \Rightarrow$ Diagonizible

(b) $T : \mathbb{R}_2[x] \to \mathbb{R}_2[x]$ given by $Tp(x) = x(p(x+1) - p(x))$

$\mathcal{B} = \langle 1, x, x^2 \rangle$

$P(x) = 1 \longrightarrow x(P(x+1) - P(x)) \Rightarrow x(0) = 0$

$P(x) = x \longrightarrow x(x+1-x) = x$

$P(x) = x^2 \longrightarrow x((x+1)^2 - x^2) = 2x^2 + x$

$x^2 + 2x + 1 - x^2$

$[T]_B = \Big[[Tb_1]_B \ [Tb_2]_B \ [Tb_3]_B \Big] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

$P_T(\lambda) = -\lambda \begin{vmatrix} 1-\lambda & 1 \\ \end{vmatrix} = -\lambda[(1-\lambda)(2-\lambda) - 0] \Rightarrow (-\lambda)(1-\lambda)(2-\lambda) = P_T(\lambda)$

$$P_{T_B}(\lambda) = -\lambda \begin{vmatrix} \cdots \\ 0 & 2-\lambda \end{vmatrix} = -\lambda[(1-\lambda)(2-\lambda)-0] \Rightarrow (\lambda)(1-\lambda)(2-\lambda) = P_{T_B}(\lambda)$$

$\lambda = 0, 1, 2$

for $\lambda=0$ $E_{\lambda=0} = \text{null}\left\{\begin{matrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{matrix}\right\} = \text{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}\right\}$ $AM=GM=1$

for $\lambda=1$ $E_{\lambda=1} = \text{null}\left\{\begin{matrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{matrix}\right\} = \text{span}\left\{\begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$ $AM=GM=1$

for $\lambda=2$ $E_{\lambda=2} = \text{null}\left\{\begin{matrix} -2 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{matrix}\right\} = \text{span}\left\{\begin{pmatrix}0\\1\\1\end{pmatrix}\right\}$ $AM=GM=1$

✓ **Diagonizable**

(c) Let V be a vector space and $B = (v_1, v_2, v_3)$ a basis for V. Here we consider the linear transformation $T : V \to V$ which satisfies $Tv_1 = 5v_1$, $Tv_2 = v_2 + 2v_3$ and $Tv_3 = 2v_2 + v_3$.

$$[T]_B = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

$$P_{T_B}(\lambda) = (5-\lambda)[(1-\lambda)(1-\lambda)-4] \Rightarrow (5-\lambda)(1-2\lambda+\lambda^2-4) \Rightarrow$$
$$(5-\lambda)(\lambda-3)(\lambda+1) = P(\lambda)$$

for $\lambda=5$

$E_5 = \text{null}\left\{\begin{matrix} 0 & 0 & 0 \\ 0 & -4 & 2 \\ 0 & 2 & -4 \end{matrix}\right\} = \text{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}\right\}$ $AM=GM=1$

for $\lambda=3$

$E_3 = \text{null}\left\{\begin{matrix} 2 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 2 & -2 \end{matrix}\right\} = \text{span}\left\{\begin{pmatrix}0\\1\\1\end{pmatrix}\right\}$ $AM=GM=1$

for $\lambda=-1$

$E_{-1} = \text{null}\left\{\begin{matrix} 6 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{matrix}\right\} = \text{span}\left\{\begin{pmatrix}0\\1\\-1\end{pmatrix}\right\}$ $AM=GM=1$

Diagonizable since $\forall \lambda_i$ $AM_i = GM_i$ ✓