Math 1564 K: Midterm 2

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36/60

Please **read all instructions** carefully before beginning.

- Total there are 5 problems on ~~6~~ 5 pages (excluding this one).
- You have 75 minutes to complete this exam.
- No CALCULATORS.
- There are no aids of any kind (notes, text, computers, phones, etc.) allowed.
- you can assume that $\mathbb{R}^n$, $\mathcal{P}_n = \{$ polynomials of degree less or equal to $n\}$, and $M_{m\times n}$ with the usual addition and scalar multiplication on the respected spaces are vector spaces.
- All vector spaces are real and finite dimensional.
- Write your answers in the box when provided.
- Show all your work.
- Good luck!

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1. (4+4pts) Short Answer questions. Show your work to get full credits.

(a) Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{D} = \{\mathbf{d}_1, \mathbf{d}_2\}$ be bases for a vector space $\mathcal{V}$. Suppose

$$\mathbf{d}_1 = 2\mathbf{b}_1 + \mathbf{b}_2$$
$$\mathbf{d}_2 = \mathbf{b}_1 + \mathbf{b}_2$$

Find the change of coordinates matrix $[id]_{\mathcal{B}\to\mathcal{D}}$

$\boxed{\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}}$ X

$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

0

$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{matrix} 2b_1+b_2 \\ b_1+b_2 \end{matrix}$

(b) Let u and v be vectors in a real inner product space V. Show that if $||u-v||^2 = ||u||^2 + ||v||^2$, then $u \perp v$.

$||u-v||^2$

?? $\Rightarrow ||u||^2 + ||v||^2 - 2||u \cdot v||$ X

0

if $||u||^2 + ||v||^2 - 2||u \cdot v|| = ||u||^2 + ||v||^2$

$\Rightarrow -2||u \cdot v|| = 0$ this is not $||u-v||^2$.

$\Rightarrow u \cdot v = 0$

$\Rightarrow u \perp v$

0

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2. (3+3+3+3 pts)

Circle the correct answer for the following. You do not need to give a reasoning. No partial credits. For the **following questions**, let A be an $n \times n$ matrix; B and U be $m \times n$ matrices.

(a) $\det(-A) = -\det(A)$.

$\det(\lambda A) = \lambda^n \det(A)$

$\boxed{F}$

T F

$\det(-1 A) = -1^n \det(A)$

$n =$ even odd

(b) Let B be an $m \times n$ matrix. Then the least solution of $Bx = b$ is the vector $\widehat{x} = (B^T B)^{-1} B^T b$.

$QQ^T =$ ortho. proj.

$\boxed{T}$

T F

F

(c) If the columns of A are orthonormal in $\mathbb{R}^n$, then $\det(A) = 1$.

$\boxed{T}$

T F

F

(d) If a square matrix has orthonormal columns then it aslo has orthonormal rows.

$\boxed{F}$

T F

T

3

3. (15 pts)
Let $\mathcal{P}_2$ be all real polynomials of degree less or equal to 2 with an ordered basis $\mathcal{B} = \{1, x, x^2\}$. Define $T : \mathcal{P}_2 \to \mathcal{P}_2$ to be

$$T(p) = p(x-1)$$

(For example, $T(3 + x + x^2) = 3 + (x-1) + (x-1)^2 = 3 - x + x^2$.)
(i) Find $[T]_{\mathcal{B}\to\mathcal{B}}$ and $([T]_{\mathcal{B}\to\mathcal{B}})^{-1}$
(ii) Find $[T^{-1}]_{\mathcal{B}\to\mathcal{B}}$
(iii) Use (ii) to find $T^{-1}(x^2 + 2x)$. (Your answer should be a polynomial for (iii)).

(i) $\boxed{\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}}$ (ii) $\boxed{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}}$

$x-1 \to x$
$x+1$

(iii) $\boxed{x^2+4x+3}$

$$[T]_{\mathcal{B}\to\mathcal{B}} = \left[[T(b_1)]_{\mathcal{B}} \,\middle|\, [T(b_2)]_{\mathcal{B}} \cdots [T(b_n)]_{\mathcal{B}} \right]$$

$T(1) = 1$

$T(x) = x-1$

$T(x^2) = (x-1)^2 = x^2 - 2x + 1$

$$[T]_{\mathcal{B}\to\mathcal{B}} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left[\begin{matrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{matrix}\,\middle|\,\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right]$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$$

$$\downarrow$$

$$\left.\begin{matrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\,\right|\,\begin{matrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{matrix} \to \left.\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\,\right|\,\begin{matrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{matrix}$$

$T^{-1} : T(p) = p(x+1)$

$$\Rightarrow [T^{-1}]_{\mathcal{B}\to\mathcal{B}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$T(1) = 1$

$T(x) = x+1$

$T(x^2) = (x+1)^2 = x^2+2x+1$

$T^{-1}(x^2) = x^2+2x+1 \quad T^{-1}(2x) = 2(x+1) = 2x+2$

x^2+4x+3

15

4. (10 pts)

Let $A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$. Denote the range of A by $\mathcal{R}_A$.

(i) Find an orthonormal basis of $\mathcal{R}_A$. (Note that the third column of A is the sum of the first two columns of A.)

(ii) Find $proj_{\mathcal{R}_A} b$.

(i) $\boxed{\left\{ \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} \right\}}$ ✓

(ii) $\boxed{\begin{pmatrix} 11/6 \\ 7/6 \\ 2/6 \end{pmatrix}}$ ✓

$w_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$q_1 = \dfrac{w_1}{\|w_1\|}$

$q_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$w_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} - proj_{q_1} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \cdot q_1$

$= \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 0$

$proj_{q_1} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 0$

$w_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

$q_2 = \dfrac{w_2}{\|w_2\|}$

$q_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

$w_3 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} - proj_{q_1} \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} - proj_{q_2} \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \quad \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$

$w_3 = \begin{matrix} 2 \\ 0 \\ 1 \end{matrix} - \sqrt{2} q_1 - \sqrt{3} q_2$

$\begin{matrix} 2 \\ 0 \\ 1 \end{matrix} - \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$proj_{\mathcal{R}_A} b = proj_{q_1} b + proj_{q_2} b$

$= \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \cdot q_1 \quad \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$ ✓

$\frac{3}{\sqrt{2}} q_1 + \frac{1}{\sqrt{3}} q_2$

$= \begin{pmatrix} \frac{3}{2} \\ \frac{3}{2} \\ 0 \end{pmatrix} + \begin{pmatrix} 1/3 \\ -1/3 \\ 1/3 \end{pmatrix} = \begin{pmatrix} 9/6 + 2/6 \\ 9/6 - 2/6 \\ 0 + 2/6 \end{pmatrix} = \begin{pmatrix} 11/6 \\ 7/6 \\ 2/6 \end{pmatrix}$

10

5. (15 pts)

Let $v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, and $L = span\{v_1, v_2\}$.

(i) Find a unit vector $w \in \mathbb{R}^3$ such that $w \perp L$.

(ii) Find the volume of the parallelepiped formed by the vectors v_1, v_2, w.

(iii) Find the area of the triangle formed by the points $(0,0,0), (1,1,0), (1,0,1)$ in $\mathbb{R}^3$.

(i) $\boxed{\begin{pmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}}$ (ii) $\boxed{\cancel{-1/3} \quad \frac{3}{\sqrt{3}}}$ (iii) $\boxed{0}$

5 $\quad$ 3 $\quad$ 0 $\quad$ 0

let $w = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \in \mathbb{R}^3$

$w \cdot v_1 = 0 \qquad w \cdot v_2 = 0$

$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 0 \qquad \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 0$

$w_1 + w_2 = 0 \qquad w_1 + w_3 = 0$

$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right)$

$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right)$

$w_3 = w_2$

$w_3 = -w_1 \qquad s\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

$w = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

$\|w\| = \begin{pmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$

volume $= \det \begin{pmatrix} 1 & 1 & -1/\sqrt{3} \\ 1 & 0 & 1/\sqrt{3} \\ 0 & 1 & 1/\sqrt{3} \end{pmatrix} \xrightarrow[R_3 \to R_2]{R_2 \to R_1} \begin{pmatrix} 1 & 0 & 1/\sqrt{3} \\ 0 & 1 & 1/\sqrt{3} \\ 1 & 1 & -1/\sqrt{3} \end{pmatrix} \xrightarrow{R_3 - R_1 - R_2} \begin{pmatrix} 1 & 0 & 1/\sqrt{3} \\ 0 & 1 & 1/\sqrt{3} \\ 0 & 0 & -3/\sqrt{3} \end{pmatrix}$

$= \det \begin{pmatrix} 1 & 0 & 1/\sqrt{3} \\ 0 & 1 & 1/\sqrt{3} \\ 0 & 0 & -1/3 \end{pmatrix} = -1/3$

$\det \begin{vmatrix} 1 & 1 & -1/\sqrt{3} \\ 1 & 0 & 1/\sqrt{3} \\ 0 & 1 & 1/\sqrt{3} \end{vmatrix} = 1 \begin{vmatrix} 0 & 1/\sqrt{3} \\ 1 & 1/\sqrt{3} \end{vmatrix} - 1 \begin{vmatrix} 1 & -1/\sqrt{3} \\ 1 & 1/\sqrt{3} \end{vmatrix} + 0$

$1\left(0 - 1/\sqrt{3}\right) - 1\left(1/\sqrt{3} - (-1/\sqrt{3})\right)$

$1(-1/\sqrt{3}) - 1(2/\sqrt{3})$

$-1/\sqrt{3} - 2/\sqrt{3} = \left|-3/\sqrt{3}\right| = -1/3$ $\quad$ 8

$\det \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 0$

lin. dep.