$\cancel{P_+ = |+n\rangle\langle +n| \quad \begin{pmatrix} |+z\rangle \\ |-z\rangle \end{pmatrix} (\langle +z|, \langle -z|) \quad \langle +z|+z\rangle}$

Q19 $S_n = n \cdot \hat{S}$ in $|\pm x\rangle$ Basis

$$S_n = \begin{pmatrix} \langle +x|S_n|+x\rangle & \langle +x|S_n|-x\rangle \\ \langle -x|S_n|+x\rangle & \langle -x|S_n|-x\rangle \end{pmatrix} \qquad n_x^2 + n_y^2 + n_z^2 = 1$$

$$\Rightarrow \frac{\hbar}{2}\begin{pmatrix} n_x & n_- \\ n_+ & -n_x \end{pmatrix} = \hat{S}_n \qquad n_\pm = n_y \pm i n_z$$

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad S_z = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$[S_y, S_z] = \frac{\hbar^2}{4}(S_y S_z - S_z S_y) \Rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix} = \frac{\hbar^2}{2}\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$$

$$i\hbar\left(\frac{\hbar}{2}\right)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{\hbar^2}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

Q20 $f(\hat{S}_n) = c_0 \mathbb{1} + c_1 \hat{S}_n$ let $f(\hat{S}_n) = e^{-i\theta \frac{S_n}{\hbar}}$

$\cancel{f(\hat{S}_n) = f(0)\mathbb{1} + f'(0)\hat{S}_n \text{ where } f(0) = e^{-i\theta\frac{0}{\hbar}} = \mathbb{1}}$

$\cancel{f'(S_n) = -\frac{i\theta}{\hbar} e^{-i\theta S_n/\hbar} \quad S_n = 0}$

$\cancel{f'(0) = -\frac{i\theta}{\hbar}\mathbb{1}}$

$\cancel{f(\hat{S}_n) = \mathbb{1} - \frac{i\theta}{\hbar}\mathbb{1} S_n}$

$\cancel{\Rightarrow f(\hat{S}_n) = \mathbb{1} - \frac{i\theta}{\hbar} S_n}$

Spin 1/2 operator $\hat{S}_n$ has 2 eigenvalues $\therefore$ 2 projectors

$\mathbb{1} = \sum_a P_a \quad \therefore \quad P_+ + P_- = \mathbb{1}$ ← Sys. of eq. to yield

$S_n = \sum_a a P_a \quad \therefore \quad S_n = \frac{\hbar}{2} P_+ + \frac{-\hbar}{2} P_- \Rightarrow P_+ - P_- = \frac{2}{\hbar} S_n$

$P_\pm = \frac{1}{2}\left[\mathbb{1} \pm \frac{2}{\hbar} S_n\right] \qquad f(S_n) = \sum_a f(a) P_a \quad \therefore \quad f(S_n) = f(\tfrac{\hbar}{2})P_+ + f(-\hbar/2)P_-$

$f(\hbar/2) = e^{i\theta(\hbar/2)/\hbar} \Rightarrow e^{i\theta/2} \qquad f(-\hbar/2) = e^{-i\theta/2}$

$$e^{i\theta/2}\left(\frac{1}{2}\right)\left(\mathbb{1} + \frac{2}{\hbar} S_n\right) + e^{-i\theta/2}\left(\frac{1}{2}\right)\left(\mathbb{1} - \frac{2}{\hbar} S_n\right)$$

$$\frac{1}{2}\left(e^{i\theta/2}\mathbb{1} + e^{i\theta/2}\frac{2}{\hbar} S_n + e^{-i\theta/2}\mathbb{1} - e^{-i\theta/2}\frac{2}{\hbar} S_n\right)$$

$$\underbrace{\frac{1}{2}\left(e^{i\theta/2} + e^{-i\theta/2}\right)}\mathbb{1} + \frac{1}{2}\left(e^{i\theta/2} - e^{-i\theta/2}\right)\frac{2}{\hbar} S_n$$

$$\underline{f(S_n) = \cos\frac{\theta}{2}\mathbb{1} + i\sin\frac{\theta}{2}\left(\frac{2}{\hbar}\right) S_n}$$

$Q_2)$ $(a\cdot S)(b\cdot S) = \alpha \mathbb{1} + \beta\cdot S$ $\quad \alpha \in \mathbb{R}$, $\beta \in \mathbb{C}^n$

Rewrite LHS in $|\pm x\rangle$ Basis

$$\begin{bmatrix} \langle x|S_a|x\rangle & \langle x|S_a|-x\rangle \\ \langle -x|S_a|x\rangle & \langle -x|S_a|-x\rangle \end{bmatrix}\begin{bmatrix} \langle x|S_b|x\rangle & \langle x|S_b|-x\rangle \\ \langle -x|S_b|+x\rangle & \langle -x|S_b|-x\rangle \end{bmatrix} = \frac{\hbar^2}{4}\begin{bmatrix} a_x & a_- \\ a_+ & -a_x \end{bmatrix}\begin{bmatrix} b_x & b_- \\ b_+ & -b_x \end{bmatrix}$$

$$= \frac{\hbar^2}{4}\begin{bmatrix} a_x b_x + a_- b_+ & a_x b_- - a_- b_x \\ a_+ b_x - a_x b_+ & a_+ b_- + a_x b_x \end{bmatrix} \Rightarrow \frac{\hbar^2}{4}\left(\begin{bmatrix} a_x b_x & 0 \\ 0 & a_x b_x \end{bmatrix} + \begin{bmatrix} a_- b_+ & a_x b_- - a_- b_x \\ a_+ b_x - a_x b_+ & a_+ b_- \end{bmatrix}\right)$$

$$\Rightarrow \frac{\hbar^2}{4} a_x b_x \mathbb{1} + \frac{\hbar^2}{4}\begin{bmatrix} a_- b_+ & a_x b_- - a_- b_x \\ a_+ b_x - a_x b_+ & a_+ b_- \end{bmatrix}$$

$$\begin{bmatrix} \lambda_x & \lambda_- \\ \lambda_+ & -\lambda_x \end{bmatrix} = \begin{bmatrix} \langle x|S_\lambda|x\rangle & \langle x|S_\lambda|-x\rangle \\ \langle -x|S_\lambda|x\rangle & \langle -x|S_\lambda|-x\rangle \end{bmatrix} = \lambda\cdot\hat{S}$$

$$\lambda_x = a_- b_+ = (a_y - i a_z)(b_y + i b_z) \Rightarrow a_y b_y - i a_z b_y + i a_y b_z - i^2 a_z b_z$$
$$\Rightarrow a_y b_y + i(a_y b_z - a_z b_y) + a_z b_z$$

$$-\lambda_x = a_y b_y + a_z b_z - i(a_y b_z - a_z b_y)$$
$$\lambda_- = (a_x b_- - a_- b_x) = a_x b_z - a_z b_x + i(a_y b_x - a_x b_y)$$
$$\lambda_+ = (a_+ b_x - a_x b_+) = -(a_x b_z - a_z b_x) + i(b_x a_y - a_x b_y)$$

$\lambda_x = -(-\lambda_x) \cancel{=0}$ thus $\alpha = \frac{\hbar^2}{4}(a_x b_x)$ $\quad \beta = \frac{\hbar^2}{4}\begin{bmatrix} i(a_z b_y - a_y b_z) \\ 0 \\ . \end{bmatrix}$

Q22 $\Delta S_{n_1} = \frac{\hbar}{2}|n_1 \times n|$ $\quad \Delta S_{n_2} = \frac{\hbar}{2}|n_2 \times n|$

$\langle [S_{n_1}, S_{n_2}] \rangle = \langle i\hbar (n_1 \times n_2) \cdot S \rangle \Rightarrow i\hbar \langle \underbrace{(n_1 \times n_2)}_{\text{"c"}} \cdot \hat{S} \rangle = i\hbar \left(\frac{\hbar}{2}(n_1 \times n_2) \cdot n\right)$

$\Rightarrow i \frac{\hbar^2}{2}(n_1 \times n_2) \cdot n$

b) $\frac{\hbar^2}{4}|n_1 \times n||n_2 \times n| \geq i\frac{\hbar^2}{4}|(n_1 \times n_2) \cdot n|$

$\Rightarrow ||n_1 \times n|| ||n_2 \times n|| \geq ||(n_1 \times n_2) \cdot n|$

$n_1 \times n = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ n_{11} & n_{12} & n_{13} \\ n_x & n_y & n_z \end{bmatrix} = \sqrt{(n_{12}n_z - n_{13}n_y)\hat{i} - (n_{11}n_z - n_{13}n_x)\hat{j} + (n_{11}n_y - n_{12}n_x)\hat{k}}$

$n_2 \times n = \sqrt{(n_{22}n_z - n_{23}n_y)\hat{i} - (n_{21}n_z - n_{23}n_x)\hat{j} + (n_{21}n_y - n_{22}n_x)\hat{k}}$

Multiplying these together produces product of

$\sqrt{(\ldots\hat{i} - \ldots\hat{j} + \ldots\hat{k})(\ldots\hat{i} - \ldots\hat{j} + \ldots\hat{k})} \geq [(n_{12}n_{23} - n_{13}n_{22})\hat{i} - (n_{11}n_{23} - n_{13}n_{21})\hat{j} +$
$(n_{11}n_{22} - n_{12}n_{21})\hat{k}] \cdot (n_x\hat{i} + n_y\hat{j} + n_z\hat{k})$

and as shown through cross product expansion + subsequent conversion to spherical coordinates