Test 1

Problem t1

Is it possible to prepare a pure state of spin 1/2 particles for which

$$\text{(a) } |\langle S_{\mathbf{x}}\rangle| > \Delta S_{\mathbf{y}}\,? \qquad \text{(b) } |\langle S_{\mathbf{x}}\rangle| = \Delta S_{\mathbf{y}}\,?$$

If you answer *yes*, provide an example of the Bloch vector specifying the state. If you answer *no*, explain why not.

Solution

Consider a pure state of spin 1/2 specified by the Bloch vector $\mathbf{n}$. For this state

$$\langle S_{\mathbf{x}}\rangle^2 = (\hbar/2)^2(\mathbf{x}\cdot\mathbf{n})^2 = (\hbar/2)^2 n_{\mathbf{x}}^2,$$

$$(\Delta S_{\mathbf{y}})^2 = (\hbar/2)^2|\mathbf{y}\times\mathbf{n}|^2 = (\hbar/2)^2\big[1-(\mathbf{y}\cdot\mathbf{n})^2\big] = (\hbar/2)^2\big(1-n_{\mathbf{y}}^2\big)$$

[see, e.g., Homework Problem 1], so that

$$(\Delta S_{\mathbf{y}})^2 - \langle S_{\mathbf{x}}\rangle^2 = (\hbar/2)^2\big(1-n_{\mathbf{x}}^2-n_{\mathbf{y}}^2\big) = (\hbar/2)^2 n_{\mathbf{z}}^2 \geq 0.$$

The answers to the posed questions are now obvious:

(a) **No!** (b) **Yes!** Any state with $\mathbf{n}$ that lies in (x,y)-plane (i.e., such that $n_{\mathbf{z}}=0$) has this property.

Problem t2

Find the angle α $(0\leq\alpha\leq\pi)$ between the Bloch vectors corresponding to the spin 1/2 state vectors

$$|\psi_1\rangle = \cos(\pi/8)\,|{+\mathbf{n}}\rangle + e^{i\pi/4}\sin(\pi/8)\,|{-\mathbf{n}}\rangle, \quad |\psi_2\rangle = e^{i\pi/4}\cos(\pi/8)\,|{+\mathbf{n}}\rangle + \sin(\pi/8)\,|{-\mathbf{n}}\rangle.$$

You may find useful the trigonometric identities $1+\cos(2\theta)=2\cos^2\theta$, $1-\cos(2\theta)=2\sin^2\theta$.

Solution

The inner product of the two state vectors is

$$\langle\psi_1|\psi_2\rangle = e^{i\pi/4}\cos^2(\pi/8) + e^{-i\pi/4}\sin^2(\pi/8).$$

This gives

$$\big|\langle\psi_1|\psi_2\rangle\big|^2 = \cos^4(\pi/8)+\sin^4(\pi/8) = \frac{1}{4}\Big\{\big[1+\cos(\pi/4)\big]^2+\big[1-\cos(\pi/4)\big]^2\Big\} = \frac{1}{2}\big[1+\cos^2(\pi/4)\big] = \frac{3}{4}$$

and

$$\cos\alpha = 2\big|\langle\psi_1|\psi_2\rangle\big|^2 - 1 = 1/2 \quad\Longrightarrow\quad \theta = \pi/3.$$