Question 5

Sunday, December 3, 2023 5:25 PM

5. Consider matrix

$$A = \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}$$

(a) Determine an SVD of A.

$$A = \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix} \qquad A^TA = \begin{pmatrix} 2 & 0 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 6 & 13 \end{pmatrix}$$

$$(4-\lambda)(13-\lambda) - 36 = p(\lambda) \qquad \lambda = 1,\ 16$$

$$E_1 = \text{null}\begin{pmatrix} 3 & 6 \\ 6 & 12 \end{pmatrix} = \text{span}\left\{\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right\} \approx \text{span}\left\{\frac{1}{\sqrt5}\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right\}$$

$$E_{16} = \text{null}\begin{pmatrix} -12 & 6 \\ 6 & -3 \end{pmatrix} = \text{span}\left\{\begin{pmatrix} 1/2 \\ 1 \end{pmatrix}\right\} \approx \text{span}\left\{\frac{2}{\sqrt5}\begin{pmatrix} 1/2 \\ 1 \end{pmatrix}\right\}$$

$$\sigma_1 = 4 \qquad \sigma_2 = 1$$

$$\Sigma = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$$

$$u_i = \frac{Av_i}{\sigma} \qquad u_1 = \frac{\begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1/\sqrt5 \\ 2/\sqrt5 \end{pmatrix}}{4} = \frac{1}{4}\begin{pmatrix} 8/\sqrt5 \\ 4/\sqrt5 \end{pmatrix} = \begin{pmatrix} 2/\sqrt5 \\ 1/\sqrt5 \end{pmatrix}$$

$$u_2 = \frac{\begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} -2/\sqrt5 \\ 1/\sqrt5 \end{pmatrix}}{1} = \begin{pmatrix} -1/\sqrt5 \\ 2/\sqrt5 \end{pmatrix}$$

$$A = \begin{pmatrix} 2/\sqrt5 & -1/\sqrt5 \\ 1/\sqrt5 & 2/\sqrt5 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \sqrt5/5 & -2/\sqrt5 \\ 2/\sqrt5 & 1/\sqrt5 \end{pmatrix}^T$$

(b) Write A in the form of $\sum_{i=1}^{r} \sigma_i u_i v_i^T$, a sum of several rank-1 matrices.

$$A = \sum_{i=1}^{2} \sigma_i u_i v_i^T \Rightarrow 4\begin{pmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}\begin{pmatrix} \sqrt{5}/5 \\ 2/\sqrt{5} \end{pmatrix} + 1\begin{pmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}\begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}$$

(c) Notice that A is invertible. Determine an SVD of A^{-1}. Do you need to start from scratch?

$$A^{-1} = (U\Sigma V^T)^{-1} = V^{T^{-1}} \Sigma^{-1} U^{-1} \Rightarrow V \Sigma^{-1} U^T$$

$$\begin{pmatrix} \sqrt{5}/5 & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}\begin{pmatrix} 1/4 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix}^T = A^{-1}$$

(d) Determine an SVD of A^T. Do you need to start from scratch?

$$A^T = (U\Sigma V^T)^T = V\Sigma^T U^T =$$

$$\begin{pmatrix} \sqrt{5}/5 & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix}^T$$