MATH-1564-K1,K2,K3 Linear Algebra with Abstract Vector Spaces

Homework 9

1. Compute the determinants of the following matrices:

$$\begin{pmatrix} 2 & 6 & 16 \\ -3 & -6 & 18 \\ 5 & 12 & 35 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ -1 & 5 & 2 \\ 3 & 2 & 0 \end{pmatrix} \quad \begin{pmatrix} 4 & 0 & 1 \\ -2 & 2 & -1 \\ 0 & 4 & -3 \end{pmatrix}, \begin{pmatrix} 4 & -4 & 2 & 1 \\ 1 & 2 & 0 & 3 \\ 2 & 0 & 3 & 4 \\ 0 & -1 & 2 & 1 \end{pmatrix}$$

2. (a) Let $a, b, c \in \mathbb{R}$. Prove that

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (c-a)(c-b)(b-a)$$

(b) Find the values of a for which the following set is a basis for $\mathbb{R}^3$:

$$\left\{ \begin{pmatrix} a-1 \\ -3 \\ -6 \end{pmatrix}, \begin{pmatrix} 3 \\ a+5 \\ 6 \end{pmatrix}, \begin{pmatrix} -3 \\ -3 \\ a-4 \end{pmatrix} \right\}$$

(c) Assume that,

$$\begin{vmatrix} a & x & l \\ b & y & m \\ c & z & n \end{vmatrix} = 2$$

Find:

$$\begin{vmatrix} 2a+3x & 2b+3y & 2c+3z \\ l+x & m+y & n+z \\ 7l & 7m & 7n \end{vmatrix}$$

3. Let $A, B \in M_n(\mathbb{R})$ and $\lambda \in \mathbb{R}$. Prove or disprove the following claims:

   (a) $|A+B| = |A| + |B|$

   (b) $|\lambda A| = \lambda |A|$

   (c) $|\lambda A| = \lambda^n |A|$

   (d) If A is anti-symmetric (that is, $A^T = -A$) and n is odd then A is not invertible.

   (e) If A is anti-symmetric (that is, $A^T = -A$) and n is even then A is not invertible.

   (f) If $AB = 0$ then $\left|A^2\right| + \left|B^2\right| = 0$.

   (g) If $|A+B| = |A|$ then B is the zero matrix.

4. (a) Compute the determinant of the following $n \times n$ matrix:

$$\begin{pmatrix} 4 & 1 & 1 & \dots & 1 \\ 1 & 4 & 1 & \dots & 1 \\ 1 & 1 & 4 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 4 \end{pmatrix}$$

(b) For the matrix,

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

Prove that $|A| = 1 + (-1)^{(n+1)}$. (Note the 1 on the left lowest corner).

5. For the following exercises $i = \sqrt{-1}$. And the field we are working with is $\mathbb{C}$.
   (a) Determine all complex numbers z such that $z^2 = i$.
   (b) Write $3 + 4i$ in its polar form.
   (c) Determine call $z \in \mathbb{C}$ such that $z^4 = 1$.
   (d) If $z \in \mathbb{C}$ is non-zero, what is z^{-1} in polar form? Compare $z, \bar{z}, z^{-1}$ in a sketch.
6. Let $f(x)$ be a polynomial with real coefficients. Show that if $z \in \mathbb{C}$ is a root of $f(x)$ then so is its complex conjugate $\bar{z}$.
7. Let $A \in M_n(\mathbb{R})$ that is invertible with eigen-pair (λ, v).
   (a) Is v an eigenvector of A^5? What's its corresponding eigenvalue? Generalize.
   (b) Is v an eigenvector of A^{-1}? What's its corresponding eigenvalue?
   (c) Is v an eigenvector of $A^2 + 3A + 6I_n$? What's its corresponding eigenvalue?