Homework 4

Problem 14

$\hat{A} = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}$ is a linear operator in a two-dimensional Hilbert space.

(a) Verify that $\hat{A} \neq \hat{A}^\dagger$ and $[\hat{A}, \hat{A}^\dagger] \neq \hat{0}$, i.e., that $\hat{A}$ is neither Hermitian nor unitary.

(b) Solve the eigenvalue problem $\hat{A}|\phi\rangle = a|\phi\rangle$, $|\phi\rangle \neq |\mathrm{null}\rangle$. Do vectors $|\phi\rangle$ satisfying these equations span the space?

Solution

(a) We have

$$\hat{A}^\dagger = \begin{pmatrix} 0 & 0 \\ -i & 0 \end{pmatrix} \neq \hat{A}, \quad [\hat{A}, \hat{A}^\dagger] = \left[\begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -i & 0 \end{pmatrix} \right] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \neq \hat{0}. \quad \blacksquare$$

(b) Substituting $|\phi\rangle = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ into $\hat{A}|\phi\rangle = a|\phi\rangle$, we obtain

$$\begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = a \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \quad \Longrightarrow \quad \begin{cases} i\phi_2 = a\phi_1, \\ 0 = a\phi_2. \end{cases}$$

These equations have no non-trivial solutions if $a \neq 0$. Indeed, for $a \neq 0$ the second equation gives $\phi_2 = 0$, then the first equation shows that $\phi_1 = 0$ as well. On the contrary, for $a = 0$ the first equation gives $\phi_2 = 0$, whereas ϕ_1 can be arbitrary. Thus, $\hat{A}$ has only one eigenvalue and only one linearly independent eigenvector:

$$a = 0, \quad |\phi\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

To span a two-dimensional space, one needs two linearly independent vectors, one is not enough.

Problem 15

$\hat{A}$ is a linear operator (not necessarily Hermitian) acting in a two-dimensional Hilbert space.
Express $\det \hat{A}$ via $\operatorname{tr} \hat{A}$ and $\operatorname{tr} \hat{A}^2$.

Solution

For $\hat{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ we have

$$\operatorname{tr} \hat{A} = A_{11} + A_{22}, \quad \operatorname{tr} \hat{A}^2 = A_{11}^2 + A_{22}^2 + 2A_{12}A_{21}, \quad \det \hat{A} = A_{11}A_{22} - A_{12}A_{21}.$$

This gives

$$(\operatorname{tr} \hat{A})^2 = A_{11}^2 + A_{22}^2 + 2A_{11}A_{22} = \operatorname{tr} \hat{A}^2 + 2(A_{11}A_{22} - A_{12}A_{21}) = \operatorname{tr} \hat{A}^2 + 2 \det \hat{A},$$

so that

$$\det \hat{A} = \frac{1}{2} \left[(\operatorname{tr} \hat{A})^2 - \operatorname{tr} \hat{A}^2 \right].$$

Since the the traces in the right-hand side of the relation are independent of the choice of the basis [see Problem 13(a)], so is the left-hand side, i.e., $\det \hat{A}$. This proves such independence for a two-dimensional Hilbert space. The same is true for any finite-dimensional Hilbert space.

If $\hat{A}$ were Hermitian, we would be able to work in the basis of its eigenvectors. If $a_\pm$ are the two eigenvalues of $\hat{A}$ (not necessarily different), then

$$\operatorname{tr} \hat{A} = a_+ + a_-, \quad \operatorname{tr} \hat{A}^2 = a_+^2 + a_-^2, \quad \det \hat{A} = a_+ a_-,$$

so that

$$(\operatorname{tr} \hat{A})^2 = (a_+ + a_-)^2 = a_+^2 + a_-^2 + 2a_+a_- = \operatorname{tr} \hat{A}^2 + 2 \det \hat{A} \quad \Longrightarrow \quad \det \hat{A} = \frac{1}{2} \left[(\operatorname{tr} \hat{A})^2 - \operatorname{tr} \hat{A}^2 \right].$$

Problem 16

(a) Observable A corresponds to the Hermitian operator

$$\hat{A} = \begin{pmatrix} 1 & e^{-i\pi/3} \\ e^{i\pi/3} & -1 \end{pmatrix}.$$

Find all possible outcomes of measurement of A.

(b) Find the state vector $|\psi\rangle$ representing the state for which a measurement of A is certain to yield the largest of the possible outcomes found in part (a).

(c) Observable B corresponds to the operator

$$\hat{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Evaluate the expectation value of B in the state found in part (b).

Solution

(a) Possible measurement outcomes are eigenvalues $a_\pm$ of $\hat{A}$. Since $\operatorname{tr}\hat{A} = a_+ + a_- = 0$, the eigenvalues can be written as $a_\pm = \pm a$. Substitution into $\det\hat{A} = a_+a_- = -2$ then gives $a = \sqrt{2}$, hence

$$a_\pm = \pm\sqrt{2}.$$

(b) Plugging $|\psi\rangle \propto \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ into $\hat{A}|\psi\rangle = a_+|\psi\rangle$, we obtain

$$\psi_1 + e^{-i\pi/3}\psi_2 = \sqrt{2}\,\psi_1 \quad\Longrightarrow\quad \psi_2 = e^{i\pi/3}(\sqrt{2}-1)\psi_1 \quad\Longrightarrow\quad |\psi\rangle = c\begin{pmatrix} e^{-i\pi/3} \\ \sqrt{2}-1 \end{pmatrix}.$$

The normalization condition $\|\psi\|^2 = 1$ yields $|c|^2\big[1 + (\sqrt{2}-1)^2\big] = 2\sqrt{2}(\sqrt{2}-1)|c|^2 = 1$, hence

$$|\psi\rangle = \frac{1}{\big[2\sqrt{2}(\sqrt{2}-1)\big]^{1/2}}\begin{pmatrix} e^{-i\pi/3} \\ \sqrt{2}-1 \end{pmatrix} \quad \text{(up to a phase factor).}$$

(c) The expectation value of B in the state represented by the state vector $|\psi\rangle$ found in part (b) is given by

$$\langle B\rangle_\psi = \langle\psi|\hat{B}|\psi\rangle = \frac{1}{2\sqrt{2}(\sqrt{2}-1)}\big(e^{i\pi/3}, \sqrt{2}-1\big)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} e^{-i\pi/3} \\ \sqrt{2}-1 \end{pmatrix} = \frac{e^{i\pi/3} + e^{-i\pi/3}}{2\sqrt{2}} = \frac{\cos(\pi/3)}{\sqrt{2}} = \frac{1}{2\sqrt{2}}.$$

Problem 17

(a) $\hat{A}$ is Hermitian operator on a two-dimensional Hilbert space.
Express eigenvalues of $\hat{A}$ via $\operatorname{tr}\hat{A}$ and $\operatorname{tr}\hat{A}^2$.

(b) $|\varphi\rangle$ and $|\psi\rangle$ are normalized vectors with an inner product $\langle\varphi|\psi\rangle = \alpha$, where α is a complex number.
Using the relation derived in part (a) and the identity $\operatorname{tr}\big(|\Phi\rangle\langle\Psi|\big) = \langle\Psi|\Phi\rangle$ [see Problem 13(c)], find eigenvalues of the operator

$$\hat{A} = |\varphi\rangle\langle\psi| + |\psi\rangle\langle\varphi|.$$

(c) Verify that the eigenvalues found in part (b) are real numbers, as they should be.

Solution

(a) Eigenvalues $a_\pm$ of $\hat{A}$ (we do not assume that $a_+ \neq a_-$) obey the equations

$$a_+ + a_- = \operatorname{tr}\hat{A}, \quad a_+^2 + a_-^2 = \operatorname{tr}\hat{A}^2.$$

The first of these equations is satisfied by

$$a_\pm = \frac{1}{2}\operatorname{tr}\hat{A} \pm a.$$

Substituting $a_\pm$ in this form into the second equation, we find

$$a^2 = \frac{1}{2}\operatorname{tr}\hat{A}^2 - \left(\frac{1}{2}\operatorname{tr}\hat{A}\right)^2,$$

so that

$$a_\pm = \frac{1}{2}\operatorname{tr}\hat{A} \pm \sqrt{\frac{1}{2}\operatorname{tr}\hat{A}^2 - \left(\frac{1}{2}\operatorname{tr}\hat{A}\right)^2}.$$

Note that since the eigenvalues must be real, this relation implies that $\left(\operatorname{tr}\hat{A}\right)^2 \leq 2\operatorname{tr}\hat{A}^2$ for all Hermitian operators on two-dimensional Hilbert spaces.

(a) We have

$$\begin{aligned}
\operatorname{tr}\hat{A} &= \operatorname{tr}\big\{|\varphi\rangle\langle\psi| + |\psi\rangle\langle\varphi|\big\} = \langle\psi|\varphi\rangle + \langle\varphi|\psi\rangle = \alpha + \alpha^* = 2\operatorname{Re}\alpha,\\
\operatorname{tr}\hat{A}^2 &= \operatorname{tr}\Big\{|\varphi\rangle\langle\psi|\psi\rangle\langle\varphi| + |\psi\rangle\langle\varphi|\varphi\rangle\langle\psi| + |\varphi\rangle\langle\psi|\varphi\rangle\langle\psi| + |\psi\rangle\langle\varphi|\psi\rangle\langle\varphi|\Big\} = 2\langle\varphi|\varphi\rangle\langle\psi|\psi\rangle + \langle\varphi|\psi\rangle^2 + \langle\psi|\varphi\rangle^2\\
&= 2 + \alpha^2 + \alpha^{*2} = 2\big[1 + (\operatorname{Re}\alpha)^2 - (\operatorname{Im}\alpha)^2\big].
\end{aligned}$$

Substituting these expressions into the relation derived in part (a), we obtain

$$a_\pm = \operatorname{Re}\alpha \pm \sqrt{1 - (\operatorname{Im}\alpha)^2}.$$

(b) The first term in the right-hand side of the above expression for $a_\pm$ is real. As for the second term, we have

$$(\operatorname{Im}\alpha)^2 \leq |\alpha|^2 = |\langle\varphi|\psi\rangle|^2 \leq \langle\varphi|\varphi\rangle\langle\psi|\psi\rangle = 1,$$

where we used the Schwartz inequality. This gives $1 - (\operatorname{Im}\alpha)^2 \geq 0$, hence $\sqrt{1 - (\operatorname{Im}\alpha)^2}$ is indeed real.

Problem 18

In matrix notations, spin 1/2 operator $\hat{S}_{\mathbf{n}} = \mathbf{n}\cdot\hat{\mathbf{S}}$ in $|\pm\mathbf{z}\rangle$ basis is given by

$$\hat{S}_{\mathbf{n}} = \begin{pmatrix} \langle+\mathbf{z}|\hat{S}_{\mathbf{n}}|+\mathbf{z}\rangle & \langle+\mathbf{z}|\hat{S}_{\mathbf{n}}|-\mathbf{z}\rangle \\ \langle-\mathbf{z}|\hat{S}_{\mathbf{n}}|+\mathbf{z}\rangle & \langle-\mathbf{z}|\hat{S}_{\mathbf{n}}|-\mathbf{z}\rangle \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} n_{\mathbf{z}} & n_- \\ n_+ & -n_{\mathbf{z}} \end{pmatrix}, \quad n_\pm = n_{\mathbf{x}} \pm i n_{\mathbf{y}}.$$

Solve the eigenvalue problem for $\hat{S}_{\mathbf{n}}$ and verify that the eigenvectors you found coincide (up to phase factors) with

$$|\mathbf{n}\rangle = \cos(\theta/2)|+\mathbf{z}\rangle + e^{i\phi}\sin(\theta/2)|-\mathbf{z}\rangle, \quad |-\mathbf{n}\rangle = \sin(\theta/2)|+\mathbf{z}\rangle - e^{i\phi}\cos(\theta/2)|-\mathbf{z}\rangle.$$

Solution

Let $s_\pm$ be the two eigenvalues of $\hat{S}_{\mathbf{n}}$. Taking into account that $\hat{S}_{\mathbf{n}}^2 = (\hbar/2)^2\hat{\mathbb{1}}$, we obtain

$$\left.\begin{aligned} s_+ + s_- &= \operatorname{tr}\hat{S}_{\mathbf{n}} = 0,\\ s_+^2 + s_-^2 &= \operatorname{tr}\hat{S}_{\mathbf{n}}^2 = 2(\hbar/2)^2 \end{aligned}\right\} \implies s_\pm = \pm\hbar/2.$$

The eigenvectors $|\pm\mathbf{n}\rangle$ corresponding to these eigenvalues are normalized solutions of $\hat{S}_{\mathbf{n}}|\pm\mathbf{n}\rangle = \pm(\hbar/2)|\pm\mathbf{n}\rangle$. Substituting here

$$|\pm\mathbf{n}\rangle \propto |+\mathbf{z}\rangle + \alpha_\pm|-\mathbf{z}\rangle,$$

we get

$$\begin{pmatrix} n_{\mathbf{z}} & n_- \\ n_+ & -n_{\mathbf{z}} \end{pmatrix} \begin{pmatrix} 1 \\ \alpha_\pm \end{pmatrix} = \pm \begin{pmatrix} 1 \\ \alpha_\pm \end{pmatrix} \quad \Longrightarrow \quad \alpha_\pm = \pm \frac{1 \mp n_{\mathbf{z}}}{n_-} = \pm \frac{n_+}{1 \pm n_{\mathbf{z}}} \quad \Longrightarrow \quad |\alpha_\pm|^2 = \frac{1 \mp n_{\mathbf{z}}}{1 \pm n_{\mathbf{z}}}.$$

Properly normalized eigenvectors then are

$$|\pm\mathbf{n}\rangle = \frac{1}{\sqrt{1+|\alpha_\pm|^2}} \big(|+\mathbf{z}\rangle + \alpha_\pm |-\mathbf{z}\rangle \big) = \frac{1}{\sqrt{2}} \left\{ \sqrt{1 \pm n_{\mathbf{z}}}\, |+\mathbf{z}\rangle \pm \frac{n_+}{\sqrt{1 \pm n_{\mathbf{z}}}} |-\mathbf{z}\rangle \right\}.$$

Plugging here $n_{\mathbf{z}} = \cos\theta$ and $n_+ = n_{\mathbf{x}} + i n_{\mathbf{y}} = e^{i\phi} \sin\theta$ with $0 \le \theta \le \pi$, we recover the quoted expressions.