# 10/23

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Unless specified otherwise, the inner product on $\mathbb{R}^n$ is the default dot product.

1. Let vectors $u = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, v = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ in $\mathbb{R}^4$.
   a. Normalize $u$ to $\widehat{u}$.
   b. Orthogonally decompose $v$ to two parts, one part along the direction of $\widehat{u}$ and the other on $\widehat{u}^{\perp}$.

2. Consider the following set of vectors in $\mathbb{R}^4$. Use Gram-Schmidt process to produce an orthonormal set.
   $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \\ -1 \end{pmatrix} \right\}$

3. Consider plane spanned by vectors $u = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, v = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ in $\mathbb{R}^4$. Let $w = \begin{pmatrix} 0 \\ 2 \\ 1 \\ -1 \end{pmatrix}$ .
   a. Determine $Proj_u(w)$ and $Proj_v\,(w)$
   b. Let $U$ = span$\{u, v\}$ . Do you think $Proj_u(w)$ + $Proj_v\,(w)$ is going to give us the orthogonal projection of w onto space $U$? How do you verify this?

4. Let $u$ be a unit vector in $\mathbb{R}^n$. Consider $Proj_u\,(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ where $v \mapsto Proj_u\,(v)$.
   a. Before working out the math, can you guess the rank of this linear map?
   b. Determine the matrix representation of this map with respect to the standard basis.
   c. What is rank of the matrix you found?

5. In class, we define an orthonormal basis to be an orthonormal set that spans the underlying space.
   a. Note that we did not mention independence. The reason is because an orthonormal set is linearly independent. Show this result.
   b. An **orthogonal set** of vectors consists of vectors that are pairwise orthogonal. With this definition, is an orthogonal set always linearly independent?

6. True or False
   a. If $u \perp v,\ u \perp w$, then $u$ is orthogonal to any linear combination of $v$ and $w$.
   b. If $Q \in M_n\,(\mathbb{R})$ and the columns of $Q$ form an orthonormal set then $Q^T Q = I_n$.
   c. If $Q \in M_n\,(\mathbb{R})$ and the columns of $Q$ form an orthonormal set, then map $T_Q : \mathbb{R}^n \to \mathbb{R}^n$ defined by [Equation] preserves the norm. That is $||x|| = \ ||Qx||$.
   d. If $v \in Nul(A)$, then $v$ is orthogonal to rows of $A$.