Question 4

Sunday, December 3, 2023 4:35 PM

4. Let $A \in M_n(\mathbb{R})$ be a symmetric matrix, with eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p$. Show that

$A = QDQ^{-1} = \sum \lambda q_i q_i^T$

$$\max_{\|\vec{x}\|\neq 0} \frac{\vec{x}^T A \vec{x}}{\vec{x}^T \vec{x}} = \lambda_1$$

$$\min_{\|\vec{x}\|\neq 0} \frac{\vec{x}^T A \vec{x}}{\vec{x}^T \vec{x}} = \lambda_p$$

Comment at what vectors x the max and min values are attained.

$Q(X) = \vec{x}^T A\vec{x}$, Since A Symmetric $\Rightarrow A = QDQ^{-1}$

$$\Rightarrow \max \frac{x^T (QDQ^{-1})\vec{x}}{x^T x} = \lambda_1$$

we say $Q^{-1}\vec{x} \Rightarrow Q^T\vec{x} = [x]_Q$ & likewise for $\vec{x}^T Q = [x]_Q^{-1}$

$$\Rightarrow \max \frac{[x]_Q^{-1} D [x]_Q}{x^T x} = \frac{\sum_{i=1}^{n} \lambda_i [x]_{Qi}^2}{x^T x} = \lambda_1$$

$$\Rightarrow \frac{\sum_{i=1}^{n} \lambda_i x_i^2}{\sum_{i=1}^{n} x_i^2}$$ is maximized @ $\lambda = 1$

we know $\lambda_1 \geq \lambda_i \geq \lambda_n$

$\Rightarrow x_i^2 \lambda_1 \geq x_i^2 \lambda_i \geq x_i^2 \lambda_n$

$\Rightarrow \sum^{n} x_i^2 \lambda_1 \geq \sum_{i=1}^{n} x_i^2 \lambda_i \geq \sum_{i=1}^{n} x_i^2 \lambda_n$

$i=1$ $\quad$ $i=1$

→ Divide out for indices i

$$\lambda_1 \geq \frac{\sum_{i=1}^{n} x_i^2 \lambda_i}{\sum_i x_i} \geq \lambda_n$$

max when $x = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ Since λ_1 is at A_{11}

and min when $x = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$

in other words the first vector in U when A is orthogonally diagonalized is max & last vector in U is minimized