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Friday, November 10, 2023 09:29

This is a quick overview for essentials of con
class. This will be useful for later material.

1. Define $\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R},\ i^2 = -1\}$. Recall that if $z = a + bi \in \mathbb{C}$. Re(z), the real part, is a whereas Im(z), the imaginary part, is b. The conjugate of z, denoted by $\overline{z}$, is given by $a - bi$.
   Modulus of z, denoted by $|z|$, is given by $\sqrt{a^2 + b^2}$, which gives the "length" of z.

   Determine the real and imaginary parts of $\frac{i-4}{2i-3}$ and its modulus.

2. If a complex number is of unit length, we can represent it as $\cos\theta + i\sin\theta$.

   a. Draw a sketch and interpret what θ means for a complex number on the complex plane. θ is usu. called the argument of this complex number.

   Please draw ample sketches to help them un

   b. We can use trig identities to show that for two complex number $\cos\theta_1 + i\sin\theta_1$ and $\cos\theta_2 + i\sin\theta_2$, their product is given by $\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)$. With a sketch, interpret this result.

   c. It can be shown that $e^{i\theta} = \cos\theta + i\sin\theta$. Use this to justify part b.

   d. With the identity $e^{i\theta} = \cos\theta + i\sin\theta$, show that

   $$(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$$

   This is called DeMoivre's formula. Interpret this result geometrically. What happens when we multiply a complex number with a complex number of modulus 1?

   e. With the help of DeMoivre's, explain $1, -1, i, -i$ are roots to $z^4 = 1$ on $\mathbb{C}$. How many roots do you suspect $z^3 = 1$ has? What are they? Can you draw a sketch on the complex plane to help explain this?

3. For $z = a + bi \in \mathbb{C}$, we can write z as $z = re^{i\theta}$, where $e^{i\theta} = \cos\theta + i\sin\theta$. This is called the polar form of a complex number. Here $r = |z|$ and $\theta = \arctan\frac{b}{a}$. Note that as a complex number $e^{i\theta}$ is of modulus 1, aka, unit length.

   a. Write 1+i in its polar form. With the help its polar form, determine $(1+i)^6$. What are its real part and imaginiary part? What are its modulus and argument?

   b. Determine the polar form of 8, 6i, and
   $$\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right)^7$$

   c. Explain what happens when we multiply two complex numbers together in terms of the product's modulus and argument.

   $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ as matrix representation of a

   d. Show that if $z = re^{i\theta}$, then the n-th root of z is given by

   $$r^{\frac{1}{n}}\left(\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right),\ k = 0, 1, \ldots,\ n-1$$

   Find all cubic roots of -8.

   If you have time left, you can also start talk

4. Consider the map $T : \mathbb{C} \to \mathbb{C}$, defined as $z \mapsto (1 + \sqrt{3}i)\, z$. Interpret this map geometrically. How do we represent this map as a matrix -vector multiplication if this is a linear trasnformation from $\mathbb{R}^2 \to \mathbb{R}^2$.

5. On the complex plane, draw a sketch that relates the following complex numbers
   $z,\ \overline{z},\ iz, -z$

6. Consider vector space $\mathbb{C}^2$. Determine the null space of $\begin{pmatrix} 1-2i & 5 \\ 1 & 1+2i \end{pmatrix}$.