# Question 5

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5. Here are some facts about matrices that will come in handy in the future.

(a) A square matrix $A \in M_n(\mathbb{R})$ is called **diagonal** if all of its entries that are not on the main diagonal are equal zero, that is, $A$ is diagonal if $(A)_{ij} = 0$ for all $i \neq j$. Here is an example of a diagonal matrix:

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Prove that if $A, B \in M_n(\mathbb{R})$ are both diagonal then both $A + B$ and $AB$ are diagonal as well.

Consider A and B arbitrary matrices in where they are Diagonal. Prove that A+B and AB are Diagonal via induction.

Base case: $n=1$.

Consider $A, B \in M_1(\mathbb{R})$. Then A & B are Diagonal if $A = (\alpha)$, $B = (\psi)$ s.t. $\alpha, \psi \in \mathbb{R}$

Then $A+B = (\alpha + \psi) \in M_1(\mathbb{R})$ still Diagonal

and $AB = (\alpha\psi) \in M_1(\mathbb{R})$ still Diagonal

Induction hypothesis: $n = k$

Consider $A, B \in M_k(\mathbb{R})$ s.t. A & B are Diagonal

$$A = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & a_{kk} \end{pmatrix} \quad B = \begin{pmatrix} b_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & b_{kk} \end{pmatrix}$$

Assume that for $A, B \in M_k(\mathbb{R})$ the following holds

A+B is Diagonal and AB is Diagonal

Induction Step: $n = k+1$

$A, B \in M_{k+1}(\mathbb{R})$ $A = \begin{pmatrix} a_{11} & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \vdots & & a_{kk} & 0 \\ 0 & \cdots & \cdots & 0 \\ 0 & \cdots & 0 & a_{k+1\,k+1} \end{pmatrix}$

$B = \begin{pmatrix} b_{11} & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \vdots & & b_{kk} & 0 \\ 0 & \cdots & \cdots & 0 \\ 0 & \cdots & 0 & b_{k+1\,k+1} \end{pmatrix}$

Since $A+B$ for $A, B \in M_k(\mathbb{R})$ is Diagonal

$\Rightarrow A+B \begin{pmatrix} a_{11}+b_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{kk}+b_{kk} \end{pmatrix}$ Diagonal

$\Rightarrow A+B$ for $A, B \in M_{k+1}(\mathbb{R})$

$$A+B = \begin{pmatrix} a_{11}+b_{11} & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & a_{kk}+b_{kk} & 0 \\ 0 & \cdots & 0 & a_{k+1\,k+1}+b_{k+1\,k+1} \end{pmatrix}$$

IOW: $\{a_{i,k+1}\}_{i=1}^{k} = \{0\}$

and $\{a_{k+1,i}\}_{i=1}^{k} = \{0\}$

likewise for B. Thus A+B for the Rows/columns $k+1$ of $A, B$ is still zero making A+B Diagonal for $A, B \in M_{k+1}(\mathbb{R})$

for AB where $A, B \in M_{k+1}(\mathbb{R})$

$$AB = \begin{pmatrix} a_{11}b_{11} & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & a_{kk}b_{kk} & 0 \\ 0 & \cdots & 0 & a_{k+1,k+1}b_{k+1,k+1} \end{pmatrix}$$

Still Diagonal

Still zero

Since
$$\{a_{i,k+1}\}_{i=1}^{k} = \{0\}$$
$$\{a_{k+1,i}\}_{i=1}^{k} = \{0\}$$

Likewise for B, the k+1 Rows and columns are still zero making AB Diagonal for $A, B \in M_{k+1}(\mathbb{R})$

(b) For a square matrix $A \in M_n(\mathbb{R})$ the **trace** of $A$, denoted $tr(A)$, is the sum of all of its entries on the main diagonal, that is $tr(A) = \sum_{i=1}^{n}(A)_{ii}$. Here is an example of a trace computation:

$$tr\begin{pmatrix} 3 & 5 & 7 \\ 0 & 2 & 4 \\ 8 & 1 & -1 \end{pmatrix} = 3 + 2 + (-1) = 4.$$

i. For $A, B \in M_n(\mathbb{R})$ prove that $tr(AB) = tr(BA)$.

ii. Show that $tr(\cdot) : M_n(\mathbb{R}) \to \mathbb{R}$ is a linear map.

i) The trace for $C = AB$ is Defined as

$$tr(C) = \sum_{i}^{n} c_{ii} \Rightarrow \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}b_{ji}$$

we can swap the sums
$$\Rightarrow \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij}b_{ji}$$

and

Because a and b are $\in \mathbb{R}$, they are commutive $\Rightarrow \sum_{j=1}^{n}\sum_{i=1}^{n} b_{ji}\, a_{ij} \Rightarrow trace(BA)$

since square matrices

ii. Show that $tr(\cdot) : M_n(\mathbb{R}) \to \mathbb{R}$ is a linear map.

① $tr(\cdot)$ is closed under Addition

let $A, B \in M_n(\mathbb{R})$

$Tr(A+B) \Rightarrow Tr(C)$ where $(C)_{ij} = (A)_{ij} + (B)_{ij}$

$\Rightarrow \sum_{i=1}^{n} C_{ii} \Rightarrow \sum_{i=1}^{n}(a_{ii}+b_{ii}) \Rightarrow \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii}$

$\Rightarrow Tr(A) + tr(B)$

② $Tr(\cdot)$ is closed under Scalar multiplication

let $A \in M_n(\mathbb{R})$ and $\alpha \in \mathbb{R}$

Then $Tr(\alpha A) \Rightarrow \sum_{i=1}^{n} \alpha a_{ii} \Rightarrow \alpha \sum_{i=1}^{n} a_{ii} \Rightarrow \alpha Tr(A)$

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(c) For a square matrix $A \in M_n(\mathbb{R})$ the **transposed** of $A$, denoted $A^T$, is the matrix obtained by turning each row of $A$ into a column by order, that is $(A^T)_{i,j} = (A)_{j,i}$. Here is an example of a transposed computation:

$$\begin{pmatrix} 3 & 5 & 7 \\ 0 & 2 & 4 \\ 8 & 1 & -1 \end{pmatrix}^T = \begin{pmatrix} 3 & 0 & 8 \\ 5 & 2 & 1 \\ 7 & 4 & -1 \end{pmatrix}.$$

For $A, B \in M_n(\mathbb{R})$ prove that $(AB)^T = B^T A^T$.

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Since $AB = \sum_{k=1}^{1} A_{ik} B_{kj}$ for $AB \in i,j$

The $(AB)^T$ flips all Rows to columns

Thus $(AB)^T_{i,j} = \sum_{k=1}^{n} A_{jk} B_{ki}$

$$\begin{array}{cc} A & B \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \begin{pmatrix} e & f \\ g & h \end{pmatrix} \end{array}$$

$$AB = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$

$$(AB)^T = \begin{pmatrix} ae+bg & ce+dg \\ af+bh & cf+dh \end{pmatrix}$$

$B^T A^T$ can for arbitrary $i,j$ be written as

$$(B^T A^T)_{i,j} = \sum_{k=1}^{n} B^T_{ik} A^T_{kj}$$

$$\Rightarrow \sum_{k=1}^{n} B_{ki} A_{jk} \Rightarrow \sum_{k=1}^{n} A_{jk} B_{ki} = (AB)^T_{ij}$$

(d) Let $A \in M_{m\times n}(\mathbb{R})$, $B \in M_{n\times k}(\mathbb{R})$, and $C \in M_{k\times \ell}(\mathbb{R})$. Prove that

$7\times 6$ $\quad$ $6\times 4$

$\underset{n\times l}{BC}$ $\quad$ $7\times 4$

$$A(BC) = (AB)C.$$

$A(BC) = A\left(\sum_{q=1}^{k} B_{iq} C_{qj}\right)$ for any arbitrary $i \le n$, $j \le l$

$\Rightarrow \sum_{x=1}^{n} A_{fx} \left(\sum_{q=1}^{k} B_{iq} C_{qj}\right)$ $\quad$ $f \le M$

$\Rightarrow \sum_{x=1}^{n} \sum_{q=1}^{k} A_{fx} B_{iq} C_{qj}$ But since

$1 \le i \le n$

$n \quad k$

$$\Rightarrow \sum_{i=1}^{n} \sum_{q=1}^{k} A_{fi} B_{iq} C_{qj}$$ x can be subbed for i

$$\Rightarrow \left(\sum_{i=1}^{n} a_{fi} B_{iq}\right) \sum_{q=1}^{k} C_{qj}$$

$$\Rightarrow (AB)C$$