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Tuesday, November 28, 2023 17:40

1. Determine the singular values of

   a. $\begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix}$

   b. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{pmatrix}$

2. Consider $A = \begin{pmatrix} 4 & -2 \\ 2 & -1 \\ 0 & 0 \end{pmatrix}$. We are going to form A's SVD $A = U\Sigma V^T$ from scratch. Note $A \in M_{3\times 2}(\mathbb{R})$, $U \in M_3(\mathbb{R})$, $\Sigma \in M_{3\times 2}(\mathbb{R})$, $V \in M_{2\times 2}(\mathbb{R})$, where U, V are orthogonal matrices.

   a. Orthogonally diagonalize A^TA such that $A^TA = QDQ^T$.

   b. Note that columns of Qin (a) form an orthonormal basis for $\mathbb{R}^2$. Matrix V is set to be equal to Q. Call Q's column vectors v_i's, aka eigenvectors of A^TA.

   c. Use your result in (a) to determine A's singular values $\sigma_1 \geq \sigma_2$. Form Σ by populating its diagonal entries with A's singular values. Note that since $\Sigma \in M_{3\times 2}(\mathbb{R})$, $\Sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{pmatrix}$

   d. U has three columns to be populated. If Av_i is not zero, the i-th column of U is exactly $\dfrac{1}{\sigma_i} Av_i$. Note that $Av_1 \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, $Av_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ in our example. This means U's first column is taken care of. What about the second and third column? We know U is orthogonal. We need to come up an o.n. set of two vectors such that they are orthogonal to $\dfrac{1}{\sigma_1} Av_1$. Can you come up with 2 that fit the requirement? With these 3 clumns, we have formed U.

   e. We have obtained SVD of A as $A = U\Sigma V^T$. Notice that we can write it as $AV = U\Sigma$.

3. Can you determine the SVD for matrix 1.a by inspection?

4. With the notation from question 2, consider $V = \begin{pmatrix} | & & | \\ v_1 & \dots & v_n \\ | & & | \end{pmatrix}$. Let $Av_1, \dots, Av_r$ be the non-zero vectors in the collection $\{Av_1, \dots, Av_n\}$. Show that $Av_1, \dots, Av_r$ spans Col(A). Show also that $v_{r+1}, \dots, v_n$ spans Nul(A).

I described and stated the result of SVD but was one step sho
presenting the factorization result. Part 2 is a concrete examp
factorization with numbers. I will show more examples on Th

I have talked about the definition of singular values of A whic
positive square roots of eigenvalues of A^TA. We use the co
that $\lambda_1 \geq \dots \geq \lambda_p$ and $\sigma_1 \geq \dots \geq \sigma_p$

There are different definitions of SVD. We use the one ir
U,V are square orthogonal matrices, Σ is of the same si
Please stress this part since they seemed confused in cl

Q

$= \begin{pmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}$

D

$= \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

$(Av_1\ Av_2) =$

$\begin{pmatrix} -2\sqrt{5} & 0 \\ -\sqrt{5} & 0 \\ 0 & 0 \end{pmatrix}$

Note Av_2 is zero vector.

One U is $\begin{pmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ -\frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

U=V=

$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $\Sigma = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$