Question 1

Sunday, October 22, 2023 6:49 PM

1. Consider the following linear transformations $T, S : \mathbb{R}_3[x] \to \mathbb{R}_3[x]$ given by

$$Tp(x) = p'(x) \quad \text{and} \quad Sp(x) = p(x+1)$$

and consider the following bases of $\mathbb{R}^3[x]$:

$$\mathcal{E} = \{1, x, x^2, x^3\}$$

and

$$\mathcal{B} = \{1, 1+x, (1+x)^2, (1+x)^3\}$$

(a) Find $[T]_{\mathcal{E}\to\mathcal{B}}, [T]_{\mathcal{B}\to\mathcal{B}}, [T]_{\mathcal{B}\to\mathcal{E}}, [T]_{\mathcal{E}\to\mathcal{E}}$.

$$[T]_{e \to B} = \begin{pmatrix} [T(e_1)]_B & \cdots & [T(e_n)]_B \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 & 1 & -2 & 3 \\ 0 & 0 & 2 & -6 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$3x^2 + 6x + 3$

$$[T]_{B \to B} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$[T]_{B \to E} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$[T]_{E \to E} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(b) Find [S] ... [S] ...

(b) Find $[S]_{\mathcal{E}\to\mathcal{E}}$, $[S]_{\mathcal{B}\to\mathcal{B}}$.

$$[S]_{E\to E} = \left([S(e_1)]_e \cdots [S(e_n)]_e \right)$$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$[S]_{B\to B} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(c) Find $[T \circ S]_{\mathcal{E}\to\mathcal{E}}$.

$$T\circ S = T(S(p(x))) \qquad [T]_{B\to E}\,[S]_{E\to B}$$
$$\Rightarrow T(p(x+1))$$
$$\Rightarrow (p(x+1))\frac{d}{dx}$$

$$[T\circ S]_{E\to E} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(d) $[T \circ S]_{\mathcal{B}\to\mathcal{B}}$.

$$T\circ S = T(S(p(x))) \qquad [T]_{E\to B}\,[S]_{B\to E}$$
$$\Rightarrow T(p(x+1))$$

$$\Rightarrow \left(P(x+1)\right)\frac{d}{dx}$$

$$[T \circ S]_{B\to B} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(e) Use $[T]_{\mathcal{E}\to\mathcal{E}}$ to find a basis for the kernel and image of L.

$$[T]_{E\to E} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

→ Basis

$$\ker(T) = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$\text{image}(T) = \{ y \mid T(p(x)) = p'(x) = y \}$$

$$\text{im}(T) = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 3 \\ 0 \end{pmatrix} \right\}$$

Question 8

Tuesday, October 24, 2023 9:22 AM

8. Consider the subspace of $\mathbb{R}^4$

$$U = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} : a - b + c - d = 0 \text{ and } a + d = 0 \right\}$$

$-s - b + c - s = 0$

$b = c - 2s$

of $\mathbb{R}^4$.

(a) Find a basis for U.

(b) Find an orthonormal basis $\mathcal{C}$ for U.

(c) Let $x = (1,2,3,4)^T \in \mathbb{R}^4$. Find the orthogonal projection of x onto the space U : $\text{Proj}_U\, x$.

(d) Find an orthonormal basis of $\mathbb{R}^4$ that contains the vectors from $\mathcal{C}$ from (b).

(e) Find the matrix representation of the orthogonal projection of $\mathbb{R}^4$ onto the space U with respect to the basis that you obtained from (d).

(f) Find the matrix representation of the orthogonal projection of $\mathbb{R}^4$ onto the space U with respect to the standard basis of $\mathbb{R}^4$.

(g) Use the answer from (f) to calculate $\text{Proj}_U\, x$

a) $a = -s$
$b = t - 2s$
$c = t$
$d = s$

$$\left\langle \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \\ 0 \\ 1 \end{pmatrix} \right\rangle$$

b) GS Process

$$q_1 = \frac{1}{\|q_1\|} q_1 = \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$$

$$q_2 = \frac{v_2 - \langle v_2, q_1 \rangle q_1}{\| - - - - - \|}$$

$\langle v_2, q_1 \rangle$

$$(-1,-2,0,1)\begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} = 0 + \frac{-2}{\sqrt{2}} + 0 + 0$$

$$= -\frac{2}{\sqrt{2}}\begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ -2/2 \\ -2/2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ -1 \\ 0 \end{pmatrix}$$

$$V_2 - \begin{pmatrix} 0 \\ -1 \\ -1 \\ 0 \end{pmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \frac{\begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}}{\|\cdot\cdot\|}$$

$$= \begin{pmatrix} -1/2 \\ -1/2 \\ 1/2 \\ 1/2 \end{pmatrix}$$

c) (c) Let $x = (1,2,3,4)^T \in \mathbb{R}^4$. Find the orthogonal projection of x onto the space U : $\mathrm{Proj}_U\, x$.

$\langle q_1, q_2 \rangle$ ON set Span $\bar{U}$

$$\mathrm{proj}_{\bar{U}} x = \sum_{i=1}^{2} \mathrm{proj}_{q_i} x$$

$$\frac{\langle x, q_1 \rangle}{\langle q_1, q_1 \rangle} q_1 + \frac{\langle x, q_2 \rangle}{\langle q_2, q_2 \rangle} q_2$$

$$\begin{pmatrix} 0 \\ 5/2 \\ 5/2 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix} = \begin{bmatrix} -1 \\ 3/2 \\ 7/2 \end{bmatrix}$$

$\left[\ 1\ \right]$

(d) Find an orthonormal basis of $\mathbb{R}^4$ that contains the vectors from $\mathcal{C}$ from (b).

$$C = \left\langle \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}, \begin{pmatrix} -1/2 \\ -1/2 \\ 1/2 \\ 1/2 \end{pmatrix} \right\rangle$$

let vectors $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ be an extension

Then $q_3 = \dfrac{v_3 - \langle v_3, q_1 \rangle q_1 - \langle v_3, q_2 \rangle q_2}{\| \quad - - - \quad \|}$

$$v_3 - 0 - \left(-\frac{1}{2}\begin{pmatrix} -1/2 \\ -1/2 \\ 1/2 \\ 1/2 \end{pmatrix}\right) \Rightarrow v_3 + \frac{1}{2}\begin{pmatrix} -1/2 \\ -1/2 \\ 1/2 \\ 1/2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{matrix} -1/4 \\ -1/4 \\ 1/4 \\ 1/4 \end{matrix}$$

$$\Rightarrow \begin{bmatrix} 3/4 \\ -1/4 \\ 1/4 \\ 1/4 \end{bmatrix} \frac{1}{\sqrt{12/16}} \Rightarrow \frac{2}{\sqrt{3}} \begin{bmatrix} 3/4 \\ -1/4 \\ 1/4 \\ 1/4 \end{bmatrix} \Rightarrow \begin{bmatrix} 3/2\sqrt{3} \\ -1/2\sqrt{3} \\ 1/2\sqrt{3} \\ 1/2\sqrt{3} \end{bmatrix} \quad \frac{3}{2\sqrt{3}} \Rightarrow \frac{3\sqrt{3}}{6} \quad \boxed{\frac{\sqrt{3}}{2}}$$

$$\Rightarrow \begin{bmatrix} \sqrt{3}/2 \\ -\sqrt{3}/6 \\ \sqrt{3}/6 \\ \sqrt{3}/6 \end{bmatrix}$$

$q_4 = \dfrac{v_4 - \langle v_4, q_1 \rangle q_1 - \langle v_4, q_2 \rangle q_2 - \langle v_4, q_3 \rangle q_3}{\| \quad - - - \quad \|}$

$$\frac{v_4 - 0 - \left(\frac{1}{2}\begin{pmatrix} -1/2 \\ -1/2 \\ 1/2 \\ 1/2 \end{pmatrix}\right) - \left(\frac{\sqrt{3}}{6}\begin{pmatrix} \sqrt{3}/2 \\ -\sqrt{3}/6 \\ \sqrt{3}/6 \\ \sqrt{3}/6 \end{pmatrix}\right)}{\| \quad - - - - - - \quad \|}$$

$1 \quad -2 - 3 = -4$

$1 \quad -5 \quad = -4$

$= \quad \begin{pmatrix} -1/4 \\ \ldots \end{pmatrix} \quad \begin{pmatrix} 1/4 \\ \ldots \end{pmatrix}$

$$v_4 - \begin{pmatrix} -3/12 \\ 3/12 \\ 3/12 \end{pmatrix} - \begin{pmatrix} -4/12 \\ 4/12 \\ 4/12 \end{pmatrix}$$

$$\overline{\|\ - - -\ \|}$$

$$\Rightarrow \begin{pmatrix} 0 \\ 1/3 \\ -1/3 \\ -1/3 \end{pmatrix} \frac{3}{\sqrt{3}} = \begin{bmatrix} 0 \\ 3/3\sqrt{3} \\ -3/3\sqrt{3} \\ -3/3\sqrt{3} \end{bmatrix}$$

(e) Find the matrix representation of the orthogonal projection of $\mathbb{R}^4$ onto the space U with respect to the basis that you obtained from (d).

$$C = \left\langle \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}, \begin{pmatrix} -1/2 \\ -1/2 \\ 1/2 \\ 1/2 \end{pmatrix}, \begin{bmatrix} \sqrt{3}/2 \\ -\sqrt{3}/6 \\ \sqrt{3}/6 \\ \sqrt{3}/6 \end{bmatrix}, \begin{bmatrix} 0 \\ 3/3\sqrt{3} \\ -3/3\sqrt{3} \\ -3/3\sqrt{3} \end{bmatrix} \right\rangle$$

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 3 & 7 \\ 0 & 0 & 2 & 8 \end{bmatrix}$$

(f) Find the matrix representation of the orthogonal projection of $\mathbb{R}^4$ onto the space U with respect to the standard basis of $\mathbb{R}^4$.

UU^T $\quad (-1/2$

$$\left[\mathrm{Proj}_U(\cdot)\right]_{E\to E} = U U \qquad C = \left\langle \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}, \begin{pmatrix} -1/2 \\ 1/2 \\ 1/2 \end{pmatrix} \right\rangle$$

$$\begin{pmatrix} 0 \\ \sqrt{2}/2 \\ \sqrt{2}/2 \\ 0 \end{pmatrix}\left(0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right) + \begin{pmatrix} -1/2 \\ -1/2 \\ 1/2 \\ 1/2 \end{pmatrix}\left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

$$= \begin{pmatrix} 1/4 & 1/4 & -1/4 & -1/4 \\ 1/4 & 3/4 & 1/4 & -1/4 \\ -1/4 & 1/4 & 3/4 & 1/4 \\ -1/4 & -1/4 & 1/4 & 1/4 \end{pmatrix}$$

respect to the standard basis of $\mathbb{R}^4$.

(g) Use the answer from (f) to calculate $\mathrm{Proj}_U\, x$

$$\begin{pmatrix} 1/4 & 1/4 & -1/4 & -1/4 \\ 1/4 & 3/4 & 1/4 & -1/4 \\ -1/4 & 1/4 & 3/4 & 1/4 \\ -1/4 & -1/4 & 1/4 & 1/4 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \begin{bmatrix} -1 \\ 3/2 \\ 7/2 \\ 1 \end{bmatrix}$$

Question 7

Monday, October 23, 2023 10:53 PM

7. Let $M \in M_n(\mathbb{R})$. Characterize M such that $< \cdot, \cdot >_M : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by $< x, y >_M = (Mx)^T(My)$ is an inner product. Justify your answer.

if $\langle \cdot, \cdot \rangle_M : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$

$\langle x, y \rangle \longmapsto (Mx)^T(My)$

Then it holds for positive Definitness

$(Mx)^T(Mx) > 0$

$x^T M^T M x > 0$ and thus

if $Mx = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \Rightarrow a_1^2 + \dots + a_n^2 = 0$ iff $Mx = 0$

in order for $M\vec{x} = \vec{0}$, x must contain trivial solution $(\vec{x} = \vec{0})$. for $M(x) = \vec{0}$ to hold for $x = \vec{0}$, Then $\exists A = M^{-1}$ where

$AMx = A\vec{0} \Rightarrow x = \vec{0}$

Thus M is <u>invertible</u>.

Question 6

Monday, October 23, 2023 7:09 PM

6. In class we mentioned that $< A, B >_1 = tr(AB^T)$ defines an inner product on on $M_{m\times n}(\mathbb{R})$ and in studio covered that $< A, B >_2 = tr(A^T B)$ is an inner product. Is there a typo in terms of where the transpose operation is on?

$$\langle A, B\rangle_1 : M_{m\times n}(\mathbb{R}) \to \mathbb{R}$$

$$A, B \longmapsto tr(AB^T)$$

$$(AB)_{m,n} = \sum_{j=1}^{n} A_{mj} B_{jn}$$

① $\langle x, y\rangle = \langle y, x\rangle \; \forall x, y \in \bar{V}$

② $\langle x, ky\rangle = k\langle x, y\rangle \; \forall x, y \in \bar{V} \; \forall k \in \mathbb{R}$

③ $\langle x, y+z\rangle = \langle x, y\rangle + \langle x, z\rangle \; \forall x, y, z \in \bar{V}$

④ $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0$ iff $x = 0$

$$\underset{A}{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}} \underset{B}{\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

$$(AB^T)_{i,j} = \sum_{k=1}^{n} A_{ik} B_{jk}$$

$$tr(\underset{m\times n}{A}\underset{n\times m}{B^T}) = \sum_{p=1}^{m}\sum_{k=1}^{n} A_{pk} B_{pk}$$

$$tr(\underset{n\times m}{A^T}\underset{m\times n}{B}) = \sum_{k=1}^{n}\sum_{p=1}^{m} A_{kp} B_{kp}$$

Consider the Definitions of $tr(AB^T)$ and $tr(A^TB)$

The ips $\langle A, B\rangle_1 : M_{m\times n}(\mathbb{R}) \to \mathbb{R}$

$$A, B \longmapsto tr(AB^T)$$

is an inner product space since

① $\langle A, B\rangle = tr(AB^T) = \sum_{p=1}^{m}\sum_{k=1}^{n} A_{pk} B_{pk}$

$BA^T \qquad = \sum_{p=1}^{m}\sum_{k=1}^{n} B_{pk} A_{pk}$

$\Rightarrow tr(BA^T) = \langle B, A\rangle$

$\forall A, B \in M_{n\times n}(\mathbb{R})$

② $\langle A, \alpha B\rangle = tr(A(\alpha B)^T)$

$= \alpha\, tr(AB^T)$

$= \alpha\langle A, B\rangle \quad \forall \alpha \in \mathbb{R}$

③ $\langle A, B+C\rangle = tr(A(B+C)^T)$

$\Rightarrow \sum_{p=1}^{m}\sum_{k=1}^{n} A_{pk}(B_{pk} + C_{pk})$

$\Rightarrow \sum_{p=1}^{m}\sum_{k=1}^{n} A_{pk} B_{pk} + A_{pk} C_{pk}$

$\Rightarrow \sum_{p=1}^{m}\sum_{k=1}^{n} A_{pk} B_{pk} + \sum_{p=1}^{m}\sum_{k=1}^{n} A_{pk} C_{pk}$

$\Rightarrow tr(AB^T) + tr(AC^T)$

$\Rightarrow \langle A, B\rangle + \langle A, C\rangle$ ✓

④ $\langle A, A\rangle > 0 \wedge \langle A, A\rangle = 0$ iff $A = 0$

$\langle A, A\rangle = tr(AA^T) = \sum_{p=1}^{m}\sum_{k=1}^{n} A_{pk}A_{pk} = 0$ when $A @ pk = 0$

and $p \in \{1, \dots, m\}, k \in \{1, \dots, n\}$

every element in A is zero ✓

Similarly $\langle A, B\rangle_2 : M_{m \times n}(\mathbb{R}) \longrightarrow \mathbb{R}$

$A, B \longmapsto tr(A^T B)$

is an inner product space as it satisfies

① Symmetry ② Homogeneity ③ Distributivity ④ Pos. Def.

$\therefore$ <u>NOT A TYPO</u>

Question 5

Monday, October 23, 2023 6:22 PM

5. True or false. Remember to justify your answer.
   (a) There exists a non-zero upper-triangular matrix $A \in M_2(\mathbb{R})$ such that A^2 is the zero matrix.
   (b) Let $A \in M_n(\mathbb{R})$. If $AB = BA$ for every $B \in M_n(\mathbb{R})$ then $A = \lambda I_n$ for some $\lambda \in \mathbb{R}$.
   (c) Let $A, B \in M_n(\mathbb{R})$. Then AB is invertible if and only if both A and B are invertible.
   (d) Let $A \in M_n(\mathbb{R})$. A is NOT invertible if and only if there exists $B \in M_n(\mathbb{R})$ such that $AB = 0$.
   (e) Let $A, B \in M_n(\mathbb{R})$. If both A and B are invertible then $AB = BA$.
   (f) Let $A \in M_n(\mathbb{R})$. If A is invertible then $A + I$ is also invertible.
   (g) If $A^2 - I$ is invertible then $A - I$ is invertible.

a) $\begin{bmatrix} x & y \\ 0 & z \end{bmatrix}\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} = \begin{bmatrix} x^2 & xy+yz \\ 0 & z^2 \end{bmatrix}$

$x^2 = 0$ $\quad x = 0$

$xy + yz = 0$ $\quad z = 0 \quad y \in \mathbb{R}$ ✓

$z^2 = 0$

$\begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$ ✓

TRUE

(b) Let $A \in M_n(\mathbb{R})$. If $AB = BA$ for every $B \in M_n(\mathbb{R})$ then $A = \lambda I_n$ for some $\lambda \in \mathbb{R}$.

let $B = E_{i,j}$ The matrix That is 1 @ i,j and zero everywhere else.

Then $AE_{ij} = \begin{pmatrix} | & & | & & | \\ 0 & \cdots & a_{\cdot,i} & \cdots & 0 \\ | & & | & & | \end{pmatrix}$

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^{1,2} = \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix}$

$E_{ij}A = \begin{pmatrix} \text{---} & 0 & \text{---} \\ \text{---} & a_{j,\cdot} & \text{---} \\ \text{---} & 0 & \text{---} \end{pmatrix}$

if $AE_{ij} = E_{ij}A$, then $a_{,i} = a_{j,}$

$\therefore$ True for Diagonal matrix

$E_{i,i}A = AE_{ii}$ $\therefore = \lambda I_n$

(c) Let $A, B \in M_n(\mathbb{R})$. Then AB is invertible if and only if both A and B are invertible.

$AB(AB)^{-1} = I_n$ $\quad (AB)^{-1}AB$ $\quad \Rightarrow$

$A\underbrace{BB^{-1}}_{I_n}A^{-1} = I_n$ $\quad B^{-1}\underbrace{A^{-1}A}_{I_n}B = I_n$

$\Rightarrow \underbrace{AI_n}A^{-1}$ $\quad B^{-1}B = in$

$\Rightarrow AA^{-1} = I_n$ $\therefore$ True

if A and B inv $\Leftarrow$

Then AB is invertible

$\underbrace{AA^{-1}}_{I_n} = I_n$

$\Rightarrow \underbrace{AB}\underbrace{B^{-1}A^{-1}} = I_n$

$(AB)(AB)^{-1} \rightarrow$ AB is invertible

(d) Let $A \in M_n(\mathbb{R})$. A is NOT invertible if and only if there exists $B \in M_n(\mathbb{R})$ such that $AB = 0$.

False

Consider $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $B = [0]$
zero matrix

$AB = 0$ and A is invertible.

(e) Let $A, B \in M_n(\mathbb{R})$. If both A and B are invertible then $AB = BA$.

false $\quad A = \begin{bmatrix} 2 & 7 \\ 2 & 8 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$

inv $\quad\quad$ inv

$$AB = \begin{bmatrix} -3 & 23 \\ -4 & 26 \end{bmatrix} \neq \begin{bmatrix} 6 & 27 \\ 4 & 17 \end{bmatrix} = BA$$

(f) Let $A \in M_n(\mathbb{R})$. If A is invertible then $A + I$ is also invertible.

let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $A + I_n = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

which is NOT invertible - false

(g) If $A^2 - I$ is invertible then $A - I$ is invertible.

$(A - I)(A + I)$

$A^2 - IA + IA - I^2$

$= A^2 - I$

$$\left[(A - I)(A + I)\right]\left[(A - I)(A + I)\right]^{-1} = I_n$$

$$(A - I)\underbrace{(A + I)(A + I)^{-1}}_{I_n}(A - I)^{-1}$$

$(A - I) I_n (A - I)^{-1} \quad I_n$

$(A-I)\, I_n (A-I)^{-1} = I_n$

$(A-I)(A-I)^{-1} = I_n$

Thus $A-I$ invertible if A^2-I inv

Question 4

Sunday, October 22, 2023 10:51 PM

4. Consider curve defined by $49x^2 - 30\sqrt{3}xy + 19y^2 = 64$ on $\mathbb{R}^2$.

(a) Show that with respect to basis

$$\mathcal{B} = < \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}, \begin{pmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix} >$$

the curve is an ellipse.

$E = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ $\quad x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Consider $[id]_{B \to e} = \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix}$ Change of Basis mtrx

$$\begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1/2\, x + \sqrt{3}/2\, y \\ -\frac{\sqrt{3}}{2}x + \frac{1}{2}y \end{bmatrix} = \begin{bmatrix} U \\ S \end{bmatrix}$$

$U = \frac{1}{2}x + \frac{\sqrt{3}}{2}y$

$x = \left(U - \frac{\sqrt{3}}{2}y\right)(2)$

$x = 2U - \sqrt{3}y$

$x = 2U - \sqrt{3}\left(\frac{1}{2}S + \frac{\sqrt{3}}{2}U\right)$

$x = 2U - \frac{\sqrt{3}}{2}S - \frac{3}{2}U$

$\boxed{x = \frac{1}{2}U - \frac{\sqrt{3}}{2}S}$

$S = -\frac{\sqrt{3}}{2}x + \frac{1}{2}y$

$S = \frac{-\sqrt{3}}{2}\left(2U - \sqrt{3}y\right) + \frac{1}{2}y$

$S = -\sqrt{3}U + \frac{3}{2}y + \frac{1}{2}y$

$S = -\sqrt{3}U + \cancel{\frac{7}{2}}y + 2y$

$\boxed{y = \frac{1}{2}S + \frac{\sqrt{3}}{2}U}$

$$49\left(\frac{1}{2}u-\frac{\sqrt{3}}{2}s\right)\left(\frac{1}{2}u-\frac{\sqrt{3}}{2}s\right)-30\sqrt{3}\left(\frac{1}{2}u-\frac{\sqrt{3}}{2}s\right)\left(\frac{\sqrt{3}}{2}u+\frac{1}{2}s\right)+19\left(\frac{\sqrt{3}}{2}u+\frac{1}{2}s\right)\left(\frac{\sqrt{3}}{2}u+\frac{1}{2}s\right)=64$$

$$49\left(\frac{1}{4}u^2-\frac{\sqrt{3}}{2}su+\frac{3}{4}s^2\right)-30\sqrt{3}\left(\frac{\sqrt{3}}{4}u^2-\frac{1}{2}su-\frac{\sqrt{3}}{4}s^2\right)+19\left(\frac{3}{4}u^2+\frac{\sqrt{3}}{2}su+\frac{1}{4}s^2\right)=64$$

u^2

$$\frac{49}{4}u^2-\frac{90}{4}u^2+\frac{57}{4}u^2=4u^2$$

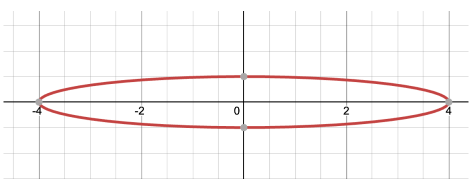
su

$$-\frac{49\sqrt{3}}{2}su+\frac{30\sqrt{3}}{2}su+\frac{19\sqrt{3}}{2}su=0$$

s^2

$$\frac{147}{4}s+\frac{90}{4}s^2+\frac{19}{4}s^2=256s^2$$

$$4u^2+256s^2=64$$



ellipse

(b) Show that with respect to basis

$$\mathcal{C} = \left\langle \begin{pmatrix} 2 \\ 2\sqrt{3} \end{pmatrix}, \begin{pmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix} \right\rangle$$

the curve is the unit circle.

Consider $\begin{bmatrix} x \\ y \end{bmatrix} = \alpha \begin{bmatrix} 2 \\ 2\sqrt{3} \end{bmatrix} + \beta \begin{bmatrix} -\sqrt{3}/2 \\ 1/2 \end{bmatrix}$

$$\begin{cases} x = 2\alpha - \frac{\sqrt{3}}{2}\beta \\ y = 2\sqrt{3}\alpha + \frac{1}{2}\beta \end{cases}$$

$$49\left(2\alpha-\frac{\sqrt{3}}{2}\beta\right)^2-30\sqrt{3}\left(2\alpha-\frac{\sqrt{3}}{2}\beta\right)\left(2\sqrt{3}\alpha+\frac{1}{2}\beta\right)+19\left(2\sqrt{3}\alpha+\frac{1}{2}\beta\right)^2=64$$

when simplifying, we get

$$64\alpha^2 + 64\beta^2 = 64$$

$$\Rightarrow \alpha^2 + \beta^2 = 1 \quad \pi \text{ unit } \underline{\text{circle}}$$

Question 3

Sunday, October 22, 2023 10:27 PM

3. Consider the transformations S and T and the bases B and E from Q1. Find the following matrices of transition from basis to basis:

$$[id]_{\mathcal{B}\to\mathcal{E}}, [id]_{\mathcal{E}\to\mathcal{B}}$$

Check that the formula of transition from basis to basis holds in the following cases:

$$[T]_{\mathcal{B}} = [id]_{\mathcal{E}\to\mathcal{B}}[T]_{\mathcal{E}}[id]_{\mathcal{B}\to\mathcal{E}}$$
$$[S]_{\mathcal{E}} = [id]_{\mathcal{B}\to\mathcal{E}}[S]_{\mathcal{B}}[id]_{\mathcal{E}\to\mathcal{B}}$$

$$[id]_{B\to E} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$x^3 + 3x^2 + 3x + 1$
$-3x^2 - 6x - 3$
$+ 3x + 3$
-1

$$[id]_{e\to B} = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[T]_B = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} [T]_E \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \checkmark$$

$$[S]_E = E\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} [S]_B \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \checkmark$$

Question 2

Sunday, October 22, 2023 7:20 PM

2. Consider the following ordered bases of $\mathbb{R}^3$:

$$\mathcal{B} =< \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} >$$

$$\mathcal{C} =< \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} >$$

$$\mathcal{E} =< \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} >$$

Find the following matrices of transition from basis to basis:

$$[id]_{\mathcal{B}\to\mathcal{E}}, [id]_{\mathcal{C}\to\mathcal{E}}, [id]_{\mathcal{E}\to\mathcal{B}}, [id]_{\mathcal{E}\to\mathcal{C}}, [id]_{\mathcal{B}\to\mathcal{C}}, [id]_{\mathcal{C}\to\mathcal{E}}.$$

$$[id]_{B\to E} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad [id]_{C\to E} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ -1 & 0 & 3 \end{bmatrix}$$

$$[id]_{E\to B} = \begin{bmatrix} 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \end{bmatrix}$$

$$[id]_{E\to C} = \begin{bmatrix} -3/8 & 3/4 & 1/8 \\ 3/4 & -1/2 & -1/4 \\ -1/8 & 1/4 & 3/8 \end{bmatrix}$$

$$[id]_{B\to C} = \begin{bmatrix} 3/8 & 7/8 & -1/4 \\ 1/4 & -3/4 & 1/2 \\ 1/8 & 5/8 & 1/4 \end{bmatrix}$$

$$[\mathrm{id}]_{C \to e} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ -1 & 0 & 3 \end{bmatrix}$$