Question 1

Sunday, November 5, 2023 3:56 PM

1. Consider $\mathbb{R}_2[x]$ with inner product

$$< f, g >= \int_0^1 f(x)g(x)dx$$

. Apply Gram-Schmidt to $1, x, x^2$ to get an orthonormal basis for $\mathbb{R}_2[x]$.

$\beta = \{1, x, x^2\}$

$\boxed{q_1 = 1}$

$$q_2 = \frac{v_2 - \langle v_2, q_1 \rangle q_1}{\| \quad \cdots \quad \|}$$

$$\frac{x - \left[\int_0^1 x \cdot 1 \, dx\right] \cdot 1}{\| \quad \cdots \quad \|} = x - \frac{1}{2}x^2\Big]_0^1 = \frac{x - \frac{1}{2}}{\| \cdots \|}$$

$$\| \cdot \cdot \| = \langle x - \tfrac{1}{2}, x - \tfrac{1}{2} \rangle^{1/2}$$

$$(x - \tfrac{1}{2})(x - \tfrac{1}{2}) = \int_0^1 x^2 - x + \tfrac{1}{4} \, dx = \tfrac{1}{3}x^3 - \tfrac{1}{2}x^2 + \tfrac{1}{4}x\Big]_0^1 = \tfrac{4}{12} - \tfrac{6}{12} + \tfrac{3}{12} = \tfrac{1}{12}$$

$$\sqrt{1/12} = \frac{1}{\sqrt{12}} \Rightarrow \boxed{\sqrt{12}\,x - \frac{\sqrt{12}}{2}} = q_2$$

$$q_3 = \frac{v_3 - \langle v_3, q_1 \rangle q_1 - \langle v_3, q_2 \rangle q_2}{\| \quad \cdots \quad \|}$$

$$\frac{x^2 - \left[\int_0^1 x^2 dx\right] \cdot 1 - \left[\int_0^1 x^2(\sqrt{12}x - \tfrac{1}{2}\sqrt{12}) \, dx\right](\sqrt{12}x - \tfrac{1}{2}\sqrt{12})}{\| \quad \cdots \quad \|}$$

$$\left(\sqrt{12}\int_0^1 x^2(x - \tfrac{1}{2})\,dx\right)\sqrt{12}(x - \tfrac{1}{2}) \Rightarrow 12\left[\int x^3 - \tfrac{1}{2}x^2 \, dx\right](x - \tfrac{1}{2}) = 12\left(\tfrac{1}{4}x^4 - \tfrac{1}{6}x^3\right)\Big]_0^1 (x - \tfrac{1}{2})$$

$$\Rightarrow 12\left(\frac{1}{12}\right)\left(x-\frac{1}{2}\right) = x-\frac{1}{2}$$

$$\Rightarrow x^2 - \frac{1}{3} - \left(x - \frac{1}{2}\right) \Rightarrow \frac{x^2 - x + \frac{1}{6}}{\| \cdots \|}$$

$$\| \ldots \| = \langle x^2 - x + \tfrac{1}{6}, x^2 - x + \tfrac{1}{6} \rangle^{1/2} = \left[\int_0^1 \left(x^2 - x + \tfrac{1}{6}\right)^2 dx\right]^{1/2}$$

$$\Rightarrow \frac{1}{6\sqrt{5}}$$

$$e_3 = 6\sqrt{5}\left(x^2 - x + \frac{1}{6}\right) \Rightarrow \boxed{6\sqrt{5}x^2 - 6\sqrt{5}x + \sqrt{5}}$$