Homework 5

Problem 19

Matrix elements of spin 1/2 operator $\hat{S}_{\mathbf{n}}$ in $|\pm\mathbf{x}\rangle$ basis form 2×2 matrix

$$\hat{S}_{\mathbf{n}} = \begin{pmatrix} \langle +\mathbf{x}|\hat{S}_{\mathbf{n}}|+\mathbf{x}\rangle & \langle +\mathbf{x}|\hat{S}_{\mathbf{n}}|-\mathbf{x}\rangle \\ \langle -\mathbf{x}|\hat{S}_{\mathbf{n}}|+\mathbf{x}\rangle & \langle -\mathbf{x}|\hat{S}_{\mathbf{n}}|-\mathbf{x}\rangle \end{pmatrix}.$$

Find this matrix and verify that it yields the correct commutator $[\hat{S}_{\mathbf{y}}, \hat{S}_{\mathbf{z}}] = i\hbar\hat{S}_{\mathbf{x}}$.

Solution

The diagonal matrix elements of $\hat{S}_{\mathbf{n}}$ are the expectation values

$$\langle \pm\mathbf{x}|\hat{S}_{\mathbf{n}}|\pm\mathbf{x}\rangle = \langle S_{\mathbf{n}}\rangle_{\pm\mathbf{x}} = \pm(\hbar/2)\, n_{\mathbf{x}}.$$

Taking into account that in $|\pm\mathbf{z}\rangle$ basis

$$|\pm\mathbf{x}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm 1\end{pmatrix}, \quad \hat{S}_{\mathbf{n}} = \frac{\hbar}{2}\begin{pmatrix} n_{\mathbf{z}} & n_- \\ n_+ & -n_{\mathbf{z}}\end{pmatrix}, \quad n_\pm = n_{\mathbf{x}} \pm i n_{\mathbf{y}},$$

we get the off-diagonal matrix elements:

$$\langle \pm\mathbf{x}|\hat{S}_{\mathbf{n}}|\mp\mathbf{x}\rangle = \frac{\hbar}{4}(1, \pm 1)\begin{pmatrix} n_{\mathbf{z}} & n_- \\ n_+ & -n_{\mathbf{z}}\end{pmatrix}\begin{pmatrix} 1 \\ \mp 1\end{pmatrix} = \frac{\hbar}{4}\left[2n_{\mathbf{z}} \pm (n_+ - n_-)\right] = \frac{\hbar}{2}(n_{\mathbf{z}} \pm i n_{\mathbf{y}}).$$

The matrix we seek is then given by

$$\hat{S}_{\mathbf{n}} = \frac{\hbar}{2}\begin{pmatrix} n_{\mathbf{x}} & n_{\mathbf{z}} + i n_{\mathbf{y}} \\ n_{\mathbf{z}} - i n_{\mathbf{y}} & -n_{\mathbf{x}}\end{pmatrix}, \quad \text{in } |\pm\mathbf{x}\rangle \text{ basis.}$$

Note that $\mathrm{tr}\hat{S}_{\mathbf{n}} = 0$ and $\det\hat{S}_{\mathbf{n}} = -\hbar^2/4$, as it should be. As an ultimate test, we evaluate the commutator $[\hat{S}_{\mathbf{y}}, \hat{S}_{\mathbf{z}}]$:

$$[\hat{S}_{\mathbf{y}}, \hat{S}_{\mathbf{z}}] = \frac{\hbar^2}{4}\left[\begin{pmatrix} 0 & i \\ -i & 0\end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\right] = i\frac{\hbar^2}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix} = i\hbar\hat{S}_{\mathbf{x}}.$$

Problem 20

Because spin 1/2 operator $\hat{S}_{\mathbf{n}}$ has only two eigenvalues, any function of this operator can be written as the first-degree polynomial $\hat{f}(\hat{S}_{\mathbf{n}}) = c_0\hat{\mathbb{1}} + c_1\hat{S}_{\mathbf{n}}$. Find this polynomial for $\hat{f}(\hat{S}_{\mathbf{n}}) = e^{-i\theta\hat{S}_{\mathbf{n}}/\hbar}$, where θ is a real dimensionless number.

Solution

The projectors $\hat{\mathcal{P}}_\pm = |\pm\mathbf{n}\rangle\langle\pm\mathbf{n}|$ onto the two orthogonal eigenspaces of $\hat{S}_{\mathbf{n}}$ corresponding to eigenvalues $\pm\hbar/2$ satisfy

$$\hat{\mathcal{P}}_+ + \hat{\mathcal{P}}_- = \hat{\mathbb{1}}, \quad \hat{\mathcal{P}}_+ - \hat{\mathcal{P}}_- = 2\hat{S}_{\mathbf{n}}/\hbar.$$

The spectral decomposition of the operator $e^{-i\theta\hat{S}_{\mathbf{n}}/\hbar}$ then reads

$$e^{-i\theta\hat{S}_{\mathbf{n}}/\hbar} = e^{-i\theta/2}\hat{\mathcal{P}}_+ + e^{i\theta/2}\hat{\mathcal{P}}_- = \cos(\theta/2)(\hat{\mathcal{P}}_+ + \hat{\mathcal{P}}_-) - i\sin(\theta/2)(\hat{\mathcal{P}}_+ + \hat{\mathcal{P}}_-) = \cos(\theta/2)\hat{\mathbb{1}} - i\sin(\theta/2)(2\hat{S}_{\mathbf{n}}/\hbar).$$

The right-hand side of this relation is the polynomial we seek.

Alternatively, one can get this result be expanding $e^{-i\theta\hat{S}_{\mathbf{n}}/\hbar}$ in Taylor series and using the relation $\hat{S}_{\mathbf{n}}^2 = (\hbar/2)^2\hat{\mathbb{1}}$, which implies that $\hat{S}_{\mathbf{n}}^{2k} = (\hbar/2)^{2k}\hat{\mathbb{1}}$ for $k = 0, 1, 2, \dots$ and thus $\hat{S}_{\mathbf{n}}^{2k+1} = \hat{S}_{\mathbf{n}}^{2k}\hat{S}_{\mathbf{n}} = (\hbar/2)^{2k}\hat{S}_{\mathbf{n}} = (\hbar/2)^{2k+1}(2\hat{S}_{\mathbf{n}}/\hbar)$, so that

$$e^{-i\theta\hat{S}_{\mathbf{n}}/\hbar} = \sum_{k=0}^{\infty}\left[\frac{(-i\theta/2)^{2k}}{(2k)!}\hat{\mathbb{1}} + \frac{(-i\theta/2)^{2k+1}}{(2k+1)!}(2\hat{S}_{\mathbf{n}}/\hbar)\right] = \cos(\theta/2)\hat{\mathbb{1}} - i\sin(\theta/2)(2\hat{S}_{\mathbf{n}}/\hbar).$$

Problem 21

Show that for spin 1/2 operators $(\mathbf{a}\cdot\hat{\mathbf{S}})(\mathbf{b}\cdot\hat{\mathbf{S}}) = \alpha\hat{\mathbb{1}} + \boldsymbol{\beta}\cdot\hat{\mathbf{S}}$, where α and $\boldsymbol{\beta}$ are, respectively, a scalar and a vector that you need to find.

Solution

Because $(\mathbf{a}\cdot\hat{\mathbf{S}})(\mathbf{b}\cdot\hat{\mathbf{S}})$ is a geometric scalar bilinear in $\mathbf{a}$ and $\mathbf{b}$ and linear in $\hat{\mathbf{S}}$, it must have the form

$$(\mathbf{a}\cdot\hat{\mathbf{S}})(\mathbf{b}\cdot\hat{\mathbf{S}}) = \alpha\hat{\mathbb{1}} + \boldsymbol{\beta}\cdot\hat{\mathbf{S}} \ \text{ with } \ \alpha \propto \hbar^2(\mathbf{a}\cdot\mathbf{b}), \ \ \boldsymbol{\beta} \propto \hbar(\mathbf{a}\times\mathbf{b}),$$

where $\hbar$'s take care of the units. The missing numerical coefficients can be found by comparing this expression with, e.g., $\hat{S}_{\mathbf{z}}^2 = (\hbar/2)^2\hat{\mathbb{1}}$ and $\hat{S}_{\mathbf{x}}\hat{S}_{\mathbf{y}} = i(\hbar/2)\hat{S}_{\mathbf{z}}$. This gives

$$\alpha = \frac{\hbar^2}{4}(\mathbf{a}\cdot\mathbf{b}), \quad \boldsymbol{\beta} = \frac{i\hbar}{2}(\mathbf{a}\times\mathbf{b}).$$

Alternatively, this result can be obtained by a brute force calculation. Indeed, taking into account that

$$\hat{S}_{\mathbf{x}}\hat{S}_{\mathbf{y}} = -\hat{S}_{\mathbf{y}}\hat{S}_{\mathbf{x}} = i(\hbar/2)\hat{S}_{\mathbf{z}}, \quad \hat{S}_{\mathbf{y}}\hat{S}_{\mathbf{z}} = -\hat{S}_{\mathbf{z}}\hat{S}_{\mathbf{y}} = i(\hbar/2)\hat{S}_{\mathbf{x}}, \quad \hat{S}_{\mathbf{z}}\hat{S}_{\mathbf{x}} = -\hat{S}_{\mathbf{z}}\hat{S}_{\mathbf{x}} = i(\hbar/2)\hat{S}_{\mathbf{y}}, \quad \hat{S}_{\mathbf{x}}^2 = \hat{S}_{\mathbf{y}}^2 = \hat{S}_{\mathbf{z}}^2 = (\hbar/2)^2\hat{\mathbb{1}},$$

we obtain

$$\begin{aligned}(\mathbf{a}\cdot\hat{\mathbf{S}})(\mathbf{b}\cdot\hat{\mathbf{S}}) &= (a_{\mathbf{x}}\hat{S}_{\mathbf{x}} + a_{\mathbf{y}}\hat{S}_{\mathbf{y}} + a_{\mathbf{z}}\hat{S}_{\mathbf{z}})(b_{\mathbf{x}}\hat{S}_{\mathbf{x}} + b_{\mathbf{y}}\hat{S}_{\mathbf{y}} + b_{\mathbf{z}}\hat{S}_{\mathbf{z}}) \\ &= (\hbar/2)^2(a_{\mathbf{x}}b_{\mathbf{x}} + a_{\mathbf{y}}b_{\mathbf{y}} + a_{\mathbf{z}}b_{\mathbf{z}})\hat{\mathbb{1}} + i(\hbar/2)\big\{(a_{\mathbf{y}}b_{\mathbf{z}} - a_{\mathbf{z}}b_{\mathbf{y}})\hat{S}_{\mathbf{x}} + (a_{\mathbf{z}}b_{\mathbf{x}} - a_{\mathbf{x}}b_{\mathbf{z}})\hat{S}_{\mathbf{y}} + (a_{\mathbf{x}}b_{\mathbf{y}} - a_{\mathbf{y}}b_{\mathbf{x}})\hat{S}_{\mathbf{z}}\big\} \\ &= (\hbar/2)^2(\mathbf{a}\cdot\mathbf{b})\hat{\mathbb{1}} + i(\hbar/2)(\mathbf{a}\times\mathbf{b})\cdot\hat{\mathbf{S}}.\end{aligned}$$

This relation holds only for spin 1/2 operators. Remarkably, it remains valid even if vectors $\mathbf{a}$ and $\mathbf{b}$ are replaced with vector operators $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$ that commute with $\hat{\mathbf{S}}$ but not necessarily with each other.

Problem 22

(a) Evaluate $\Delta S_{\mathbf{n}_1}$, $\Delta S_{\mathbf{n}_2}$, and $\big\langle[\hat{S}_{\mathbf{n}_1}, \hat{S}_{\mathbf{n}_2}]\big\rangle \equiv \langle\mathbf{n}|[\hat{S}_{\mathbf{n}_1}, \hat{S}_{\mathbf{n}_2}]|\mathbf{n}\rangle$ for spin 1/2 in the state $|\mathbf{n}\rangle$.
Feel free to use the relations

$$\langle\mathbf{a}\cdot\mathbf{S}\rangle_{\mathbf{n}} = \langle\mathbf{n}|\mathbf{a}\cdot\hat{\mathbf{S}}|\mathbf{n}\rangle = \frac{\hbar}{2}(\mathbf{a}\cdot\mathbf{n}), \quad \Delta_{\mathbf{n}}(\mathbf{a}\cdot\mathbf{S}) = \frac{\hbar}{2}|\mathbf{a}\times\mathbf{n}|, \quad \big[\mathbf{a}\cdot\hat{\mathbf{S}}, \mathbf{b}\cdot\hat{\mathbf{S}}\big] = i\hbar(\mathbf{a}\times\mathbf{b})\cdot\hat{\mathbf{S}}.$$

(b) Verify that the expressions obtained in part (a) obey the uncertainty relation

$$\Delta S_{\mathbf{n}_1}\Delta S_{\mathbf{n}_2} \geq \frac{1}{2}\Big|\big\langle[\hat{S}_{\mathbf{n}_1}, \hat{S}_{\mathbf{n}_2}]\big\rangle\Big|.$$

Suggestion: As you will discover, part (b) reduces to demonstrating a certain relation between unit dimensionless vectors $\mathbf{n}$, $\mathbf{n}_1$, and $\mathbf{n}_2$. This can be done, for example, by switching to the spherical polar coordinates with $\mathbf{z} = \mathbf{n}$.

Solution

(a) We have

$$\Delta S_{\mathbf{n}_1} = \Delta(\mathbf{n}_1\cdot\mathbf{S}) = \frac{\hbar}{2}|\mathbf{n}_1\times\mathbf{n}|, \quad \Delta S_{\mathbf{n}_2} = \Delta(\mathbf{n}_2\cdot\mathbf{S}) = \frac{\hbar}{2}|\mathbf{n}_2\times\mathbf{n}|,$$

$$\big\langle[\hat{S}_{\mathbf{n}_1}, \hat{S}_{\mathbf{n}_2}]\big\rangle = i\hbar\langle\mathbf{n}|(\mathbf{n}_1\times\mathbf{n}_2)\cdot\hat{\mathbf{S}}|\mathbf{n}\rangle = \frac{i\hbar^2}{2}\,\mathbf{n}\cdot(\mathbf{n}_1\times\mathbf{n}_2).$$

(b) Substituting $\Delta S_{\mathbf{n}_1}$, $\Delta S_{\mathbf{n}_2}$, and $\big\langle[\hat{S}_{\mathbf{n}_1}, \hat{S}_{\mathbf{n}_2}]\big\rangle$ found in part (a) into the uncertainty relation, we obtain

$$|\mathbf{n}\times\mathbf{n}_1||\mathbf{n}\times\mathbf{n}_2| \geq |\mathbf{n}\cdot(\mathbf{n}_1\times\mathbf{n}_2)|.$$

We wish to show that this inequality holds for all unit dimensionless vectors $\mathbf{n}$, $\mathbf{n}_1$, and $\mathbf{n}_2$, as it must. Following the suggestion, we switch to the spherical polar coordinates with $\mathbf{z}=\mathbf{n}$. In these coordinates, vectors $\mathbf{n}_{1,2}$ are specified by the pairs of angles $(\theta_{1,2}, \phi_{1,2})$. Taking into account that $0 \leq \theta_{1,2} \leq \pi$ and thus $\sin\theta_{1,2} \geq 0$, we find

$$|\mathbf{n}\times\mathbf{n}_{1,2}| = \sin\theta_{1,2},$$

$$|\mathbf{n}\cdot(\mathbf{n}_1\times\mathbf{n}_2)| = |n_{1x}n_{2y} - n_{1y}n_{2x}| = \sin\theta_1\sin\theta_2\big|\cos\phi_1\sin\phi_2 - \sin\phi_1\cos\phi_2\big| = \sin\theta_1\sin\theta_2\big|\sin(\phi_1-\phi_2)\big|.$$

The above inequality then assumes the form

$$\sin\theta_1\sin\theta_2 \geq \sin\theta_1\sin\theta_2\big|\sin(\phi_1-\phi_2)\big| \;\; - \;\text{obviously true!} \quad \blacksquare$$

Alternatively, the above inequality can be derived by purely geometric means. We have

$$(\mathbf{n}\times\mathbf{n}_1)\times(\mathbf{n}\times\mathbf{n}_2) = \underbrace{\big((\mathbf{n}\times\mathbf{n}_1)\cdot\mathbf{n}_2\big)}_{\mathbf{n}\cdot(\mathbf{n}_1\times\mathbf{n}_2)}\mathbf{n} - \underbrace{\big((\mathbf{n}\times\mathbf{n}_1)\cdot\mathbf{n}\big)}_{0}\mathbf{n}_2 = \big(\mathbf{n}\cdot(\mathbf{n}_1\times\mathbf{n}_2)\big)\mathbf{n},$$

where we used vector identities $\boldsymbol{\alpha}\times(\boldsymbol{\beta}\times\boldsymbol{\gamma}) = (\boldsymbol{\alpha}\cdot\boldsymbol{\gamma})\boldsymbol{\beta} - (\boldsymbol{\alpha}\cdot\boldsymbol{\beta})\boldsymbol{\gamma}$ and $\boldsymbol{\alpha}\cdot(\boldsymbol{\beta}\times\boldsymbol{\gamma}) = (\boldsymbol{\alpha}\times\boldsymbol{\beta})\cdot\boldsymbol{\gamma}$. This yields the relation

$$\big|\mathbf{n}\cdot(\mathbf{n}_1\times\mathbf{n}_2)\big| = \big|(\mathbf{n}\times\mathbf{n}_1)\times(\mathbf{n}\times\mathbf{n}_2)\big|.$$

The inequality $|\mathbf{n}\times\mathbf{n}_1||\mathbf{n}\times\mathbf{n}_2| \geq |\mathbf{n}\cdot(\mathbf{n}_1\times\mathbf{n}_2)|$ (see above) can now be written as

$$|\mathbf{N}_1||\mathbf{N}_2| \geq |\mathbf{N}_1\times\mathbf{N}_2|, \quad \mathbf{N}_{1,2} = \mathbf{n}\times\mathbf{n}_{1,2},$$

which is obviously valid for all $\mathbf{N}_{1,2}$.