Question 1

Sunday, December 3, 2023 1:54 PM

1. Prove or disprove the following claims.
   (a) The representation of quadratic form $q(\vec{x}) = \vec{x}^T A\vec{x}$, where A is a symmetric matrix, is unique.

$x_1^2 + 2x_1x_2 + x_2^2 \quad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

$Q(\vec{x}) = \vec{x}^T A\vec{x}$

$Q(\vec{x}) = \vec{x}^T B\vec{x}$

$\Rightarrow \vec{x}^T A\vec{x} = \vec{x}^T B\vec{x}$

$\Rightarrow \vec{x}^T QDQ^T\vec{x} = \vec{x}^T PHP^T\vec{x}$

$\Rightarrow [x]_Q^T D[x]_Q = [x]_P^T H [x]_P$

D & H Share Same Eigenvalues along main Diagonal $\Rightarrow A = B$ if They are Real & Symmetric, They are one in the other

$\Rightarrow$ Unique $\Rightarrow$ Statement TRUE

   (b) Consider $q(\vec{x}) = \vec{x}^T A\vec{x}$ where $A \in M_2(\mathbb{R})$. $q(\vec{x})$ is not a quadratic form if A is not symmetric.

Counter-example

$A = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}$

$Q(\vec{x}) = \vec{x}^T A\vec{x}$

$= [x_1 \; x_2]\begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$= \begin{matrix} x_1 + 3x_2 \\ 0x_1 + 2x_2 \end{matrix} = \underline{x_1^2 + 3x_2x_1 + 2x_2^2}$

False

(c) If $A \succ 0$ then $A^7 \succ 0$.

- if A pos. def. $\Rightarrow$ A^7 pos. def

TRUE

A is symmetric $\Rightarrow$ $A = QDQ^{-1}$ for Q orthogonal
& D diagonal

$D = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}$ where $\lambda_i > 0$ e-vals $\forall i \in \{1, \dots, n\}$

$A^7 = QD^7Q^{-1} \Rightarrow D = \begin{bmatrix} \lambda_1^7 & & \\ & \ddots & \\ & & \lambda_n^7 \end{bmatrix} \Rightarrow$ all $\lambda_i^7 > 0$

$\Rightarrow$ $A \succ 0$ since still symmetric & all evals > 0

TRUE

(d) If $A \prec 0$, then $A^4 \prec 0$.

- Negative Definite

$A = QDQ^{-1}$ $D = \begin{bmatrix} \lambda_1 & & 0 \\ 0 & \ddots & \lambda_n \end{bmatrix}$ $\lambda_i < 0$ $\forall i$

But any $\alpha \in \mathbb{R}$ s.t. $\alpha < 0$

$\Rightarrow$ $\alpha^4 > 0$

$\Rightarrow$ $\lambda_i^4 > 0$ $\forall i$

$\Rightarrow$ $A^4 = QD^4Q^{-1}$ $D^4 = \begin{bmatrix} \lambda_1^4 & 0 \end{bmatrix}$

$$\ldots A^4 = Q D^4 Q^T, \; D^4 = \begin{bmatrix} \ddots & & \\ & \ddots & \\ 0 & & \lambda_n^4 \end{bmatrix}$$

Where The eigenvalues are all > 0

$\Rightarrow$ A^4 is P.D. $\Rightarrow$ Statement false

(e) If $A \succ 0$ and $B \prec 0$ then $A - B \succ 0$.

Consider $\underset{(PD)}{A} = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}$ $\underset{(ND)}{B} = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}$

$Q(x) = \vec{x}^T (A-B) \vec{x} \Rightarrow \vec{x}^T A \vec{x} - \vec{x}^T B \vec{x} \Rightarrow$ This can be said as $Q(\vec{x}) = g(\vec{x}) + h(\vec{x})$

Since $g(\vec{x})$ is > 0 $\forall \vec{x} \neq 0$ and $h(\vec{x})$ is < 0 $\forall \vec{x} \neq 0$ $\Rightarrow$ $Q(x) > 0$ $\forall x \neq 0$ $\Rightarrow$ $A-B$ is PD $\forall \vec{x} \neq 0$. Statement TRUE

(f) Every matrix has a singular value decomposition.

$A \in M_{m \times n}(\mathbb{R})$. $A^T A$ is Symmetric $\Rightarrow$

$A^T A = Q D Q^T$ for Q orth + D Diagonal

and $A^T A$ is PSD meaning all eigenvalues are ≥ 0. if $\text{Rank}(A^T A) = r \Rightarrow r = \text{Rank}(A)$ (By Rank-nullity). For each $\lambda_i = \sigma_i^2$, $\exists$ a unit eigenvector for the eigenspace s.t $E_{\sigma_i^2} = \text{Span}\left\{ \frac{q}{\|q\|} \right\}$

Thus $\{q_1, \ldots, q_r\}$ Col vectors of Q form

$\{q_1, \dots, q_n\}$, col vectors of Q ...

an orthonormal Basis where $\{q_1, \dots q_r\}$ are e-vectors of all $\sigma_i \neq 0$ and $q_{r+1}, \dots, q_n$ are e-vectors of all $\sigma_i = 0$.

Since $A^T A q_i = \sigma_i^2 q_i \Rightarrow$ (q_i e-vector of $A^T A$)

we can Consider $AA^T = U\Sigma V^T V \Sigma^T U^T \Rightarrow U(\Sigma \Sigma^T)U^T$

$\Rightarrow$ where for e-vector u_i of AA^T, $AA^T u_i = \sigma_i^2 u_i$

$$u_i = \frac{1}{\sigma_i} A q_i \Rightarrow U_R = A Q_R \Sigma_R^{-1}$$

$\Rightarrow AQ_R = U_R \Sigma_R$ But for $1, \dots, R$ where $q_{i \to R}$ is for $\sigma_i \neq 0$. for $R \to n$, this ON Basis is extended to include it

s.t. $AQ = U\Sigma \Rightarrow A = U\Sigma Q^{-1} \Rightarrow \boxed{A = U\Sigma V^T}$

SVD

TRUE

(g) Similar matrices must have the same singular values.

false. if $A = P^{-1}BP$ where $A \cong B$, then A + B share the same eigenvalues But singular values are derived from AA^T or A^TA/

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix}$$

$B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} A \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ But $\delta_A = 3 \,\&\, 2$

$\delta_B = 6 \,\&\, 4$

$\delta_B \neq \delta_A$ False

(h) If A and B are real symmetric matrices such that $A^3 = B^3$, then A must be equal to B.

$A = QDQ^{-1}$ & $B = PXP^{-1}$ & $A^3 = B^3$

$\Rightarrow QD^3Q^{-1} = PX^3P^{-1}$

$A^3 = QD^3Q^T = B^3 = PX^3P^T \Rightarrow$ cube Rooting each element in D^3 & X^3 Reveals that The eigenvalues are the same $\Rightarrow$ Regardless of Q & P, $A = B$ Since it is Real & Symmetric $\Rightarrow$ Statement TRUE

Question 2

Sunday, December 3, 2023 4:12 PM

2. Orthogonally diagonalize the following matrices

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$$

$\det(A - \lambda I) = P(\lambda)$

$$\begin{matrix} -\lambda & 1 & 1 & -\lambda & 1 \\ 1 & -\lambda & 1 & 1 & -\lambda \\ 1 & 1 & -\lambda & 1 & 1 \\ - & - & - + & + & + \end{matrix}$$

$-\lambda^3 + 1 + 1 + \lambda + \lambda + \lambda$

$-\lambda^3 + 3\lambda + 2 = P(\lambda)$

$\lambda(-\lambda^2 + 3) + 2 = P(\lambda)$

$\lambda = -1, 2$

$E_{\lambda=-1} = \text{nul}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \text{Span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$

$E_{\lambda=2} = \text{nul}\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} = \text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$

$$A = \underbrace{\begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{2} & 0 & 1/\sqrt{3} \\ 0 & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{2} & 0 & 1/\sqrt{3} \\ 0 & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}^{-1}}_{Q^{-1}}$$

$B = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$

$$(3-\lambda)(3-\lambda)-4=P(\lambda)$$

$$9-6\lambda+\lambda^2-4=P(\lambda)$$

$$\lambda^2-6\lambda+5 \Rightarrow (\lambda-5)(\lambda-1) \quad \lambda=5 \ \& \ 1$$

$$E_5 = \text{null}\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} = \text{Span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$$

$$E_1 = \text{null}\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = \text{Span}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$$

$$B = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}^{-1}$$

Question 3

Sunday, December 3, 2023 4:27 PM

3. Determine all matrices C such that $C^2 = B$ in problem 2.

$$B = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}^{-1}$$

if $C^2 = B \Rightarrow CC = B \Rightarrow CC = QDQ^{-1}$

$$\Rightarrow C = QD^{1/2}Q^{-1}$$

$\Rightarrow$

$$C = Q\begin{bmatrix} \pm\sqrt{5} & 0 \\ 0 & \pm 1 \end{bmatrix} Q^{-1} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{5} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}^{-1} \quad (1)$$

$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} -\sqrt{5} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}^{-1} \quad (2)$$

$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{5} & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}^{-1} \quad (3)$$

$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} -\sqrt{5} & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}^{-1} \quad (4)$$

Question 4

Sunday, December 3, 2023 4:35 PM

4. Let $A \in M_n(\mathbb{R})$ be a symmetric matrix, with eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p$. Show that

$A = QDQ^{-1} = \sum \lambda q_i q_i^T$

$$\max_{\|\vec{x}\|\neq 0} \frac{\vec{x}^T A \vec{x}}{\vec{x}^T \vec{x}} = \lambda_1$$

$$\min_{\|\vec{x}\|\neq 0} \frac{\vec{x}^T A \vec{x}}{\vec{x}^T \vec{x}} = \lambda_p$$

Comment at what vectors x the max and min values are attained.

$Q(x) = \vec{x}^T A \vec{x}$, Since A Symmetric $\Rightarrow A = QDQ^{-1}$

$$\Rightarrow \max \frac{x^T (QDQ^{-1}) \vec{x}}{x^T x} = \lambda_1$$

we say $Q^{-1}\vec{x} \Rightarrow Q^T \vec{x} = [x]_Q$ & likewise for $\vec{x}^T Q = [x]_Q^{-1}$

$$\Rightarrow \max \frac{[x]_Q^{-1} D [x]_Q}{x^T x} = \frac{\sum_{i=1}^{n} \lambda_i [x]_{Qi}^2}{x^T x} = \lambda_1$$

$$\Rightarrow \frac{\sum_{i=1}^{n} \lambda_i x_i^2}{\sum_{i=1}^{n} x_i^2}$$ is maximized @ $\lambda = 1$

we know $\lambda_1 \geq \lambda_i \geq \lambda_n$

$\Rightarrow x_i^2 \lambda_1 \geq x_i^2 \lambda_i \geq x_i^2 \lambda_n$

$\Rightarrow \sum^n x_i^2 \lambda_1 \geq \sum_{i=1}^n x_i^2 \lambda_i \geq \sum_{i=1}^n x_i^2 \lambda_n$

$i=1$

$\rightarrow$ Divide out for indices i

$$\lambda_1 \geq \frac{\sum_{i=1}^{n} x_i^2 \lambda_i}{\sum_i x_i} \geq \lambda_p$$

max when $x = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ Since λ_1 is at A_{11}

and min when $x = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$

in other words the first vector in U when A is orthogonally Diagonized is max & last vector in U is minimized

Question 5

Sunday, December 3, 2023 5:25 PM

5. Consider matrix

$$A = \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}$$

(a) Determine an SVD of A.

$A = \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}$ $A^TA = \begin{pmatrix} 2 & 0 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 6 & 13 \end{pmatrix}$

$(4-\lambda)(13-\lambda) - 36 = p(\lambda)$ $\lambda = 1, 16$

$E_1 = \text{null}\begin{pmatrix} 3 & 6 \\ 6 & 12 \end{pmatrix} = \text{span}\left\{\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right\} \simeq \text{span}\left\{\frac{1}{\sqrt{5}}\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right\}$

$E_{16} = \text{null}\begin{pmatrix} -12 & 6 \\ 6 & -3 \end{pmatrix} = \text{span}\left\{\begin{pmatrix} 1/2 \\ 1 \end{pmatrix}\right\} \simeq \text{span}\left\{\frac{2}{\sqrt{5}}\begin{pmatrix} 1/2 \\ 1 \end{pmatrix}\right\}$

$\sigma_1 = 4$ $\sigma_2 = 1$

$$\Sigma = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$$

$u_i = \dfrac{Av_i}{\sigma}$ $u_1 = \dfrac{\begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}}{4} = \frac{1}{4}\begin{pmatrix} 8/\sqrt{5} \\ 4/\sqrt{5} \end{pmatrix} = \begin{pmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}$

$u_2 = \dfrac{\begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}}{1} = \begin{pmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$

$$A = \begin{pmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \sqrt{5}/5 & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}^T$$

(b) Write A in the form of $\sum_{i=1}^{r} \sigma_i u_i v_i^T$, a sum of several rank-1 matrices.

$$A = \sum_{i=1}^{2} \sigma_i u_i v_i^T \Rightarrow 4\begin{pmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}\begin{pmatrix} \sqrt{5}/5 \\ 2/\sqrt{5} \end{pmatrix} + 1\begin{pmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}\begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}$$

(c) Notice that A is invertible. Determine an SVD of A^{-1}. Do you need to start from scratch?

$$A^{-1} = (U\Sigma V^T)^{-1} = V^{T^{-1}} \Sigma^{-1} U^{-1} \Rightarrow V \Sigma^{-1} U^T$$

$$\begin{pmatrix} \sqrt{5}/5 & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}\begin{pmatrix} 1/4 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix}^T = A^{-1}$$

(d) Determine an SVD of A^T. Do you need to start from scratch?

$$A^T = (U\Sigma V^T)^T = V \Sigma^T U^T =$$

$$\begin{pmatrix} \sqrt{5}/5 & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix}^T$$

Question 6

Monday, December 4, 2023   11:10 AM

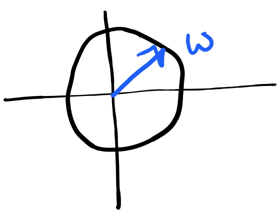
6. What can one say about a matrix's eigenvalues and its singular values? Consider the following. Note that here σ_1 is the largest singualr value and σ_n the smallest of matrix A.

   (a) Let $A \in M_2(\mathbb{R})$. Let $\boldsymbol{w} \in \mathbb{R}^2$ be a unit vector. Show that

$$\sigma_2 \leq \|A\boldsymbol{w}\| \leq \sigma_1$$

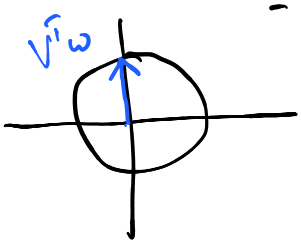
   by tracing what happens to $\boldsymbol{w}$ under the three matrices in A's SVD.

- if we say $A = U\Sigma V^T$ w is transformed By A as such

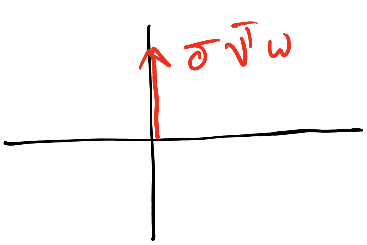


for some $\vec{w}$ of unit length, it exists on the unit circle.

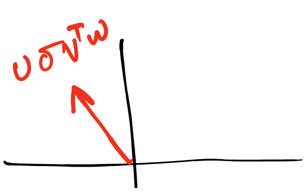
V^T $\downarrow$



- Under V^T, w gets transformed into a different Basis



Applying Σ to $V^T x$ scales it By singular values



- Applying U to $\Sigma V^T x$ Rotates it again to the Left-singular

Vectors of A

The Norm of This new Vector Can Be analyzed as follows:

- each component is stretched By σ_i

So since $\vec{\omega}$ is of length 1, the stretching & contraction of $\vec{\omega}$ is Bounded By σ_i for Both the largest & Smallest singular values.

(b) Show part (a) algebraically.

$\|A\omega\| \Rightarrow \|U\Sigma V^T \omega\|$

$\|V^T\omega\| = \sqrt{\langle V^T\omega, V^T\omega\rangle}$, in $\mathbb{R}^n$ $\langle \cdot,\cdot \rangle$ is $x^T x$

$\Rightarrow \sqrt{(V^T\omega)^T(V^T\omega)} \Rightarrow \sqrt{\omega^T V V^T \omega} = \sqrt{\omega^T \omega} \Rightarrow$

$\sqrt{\langle \omega,\omega\rangle} \Rightarrow \|\omega\|$ $\quad \sigma_1 \left[\omega \right.$

$\therefore$

$\Rightarrow \|U\Sigma V^T\omega\| \Rightarrow \|U\Sigma\omega\| \Rightarrow \|\Sigma\omega\|$ Since U is orthogonal too

$\Rightarrow$ since $\vec{\omega}$ is of unit length, we essentially scale w's components By σ_1 & σ_2 Respectively. As a Result the maximum length That $\begin{bmatrix} \sigma_1 \omega_1 \\ \sigma_2 \omega_2 \end{bmatrix}$ can Be is σ_1 & likewise for its lower Bound which is

σ_2

(c) Let $A \in M_{m\times n}(\mathbb{R})$. Show that

$$\sigma_n\|v\| \leq \|Av\| \leq \sigma_1\|v\|$$

for any $v \in \mathbb{R}^n$.

for any $Av \Rightarrow \| U\Sigma V^T v \|$. Since V^T is orthogonal, it is norm-preserving indicating $\|V^T v\| = \|v\|$ and scaling $V^T v$ by each singular value Σ would be analogous to scaling the vector itself by Σ. One can consider the bounds of this scaling where it scales maximally by σ_1, the largest singular value and scales minimally by σ_n, the smallest....

$$\Rightarrow \quad \sigma_n \|v\| \leq \| U\Sigma V^T v \| \leq \sigma_1 \|v\|$$

$$\Rightarrow \quad \sigma_n \|v\| \leq \|Av\| \leq \sigma_1 \|v\|$$

(d) Let λ be a real eigenvalue of matrix $A \in M_n(\mathbb{R})$. Show that

$$\sigma_n \leq |\lambda| \leq \sigma_1$$

.

$Av = \lambda v$ for eigenvector $\vec{v}$ in $\mathbb{R}^n$

Taking norm of both sides $\Rightarrow \|Av\| = |\lambda| \|\vec{v}\|$

$\Rightarrow |\lambda| = \dfrac{\|Av\|}{\|v\|}$. By proof 6 c)

$\sigma_n \|v\| \leq \|Av\| \leq \sigma_1 \|v\| \Rightarrow \sigma_n \leq \dfrac{\|Av\|}{\|v\|} \leq \sigma_1$

$$\Rightarrow \sigma_n \leq |\lambda| \leq \sigma_1$$

(e) Consider the matrix A in Q5. Determine $\min_{\|x\|=1} \|Ax\|$. Comment at what vectors x the min value is attained.

$$A = \begin{pmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{5}/5 & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}^T$$

mapping vector $\begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$ will minimize $\|Av\|$ to 1

(f) Consider the matrix A in Q5. Determine $\max_{\|x\|=1} \|Ax\|$. Comment at what vectors x the max value is attained.

$\|Av\|$ maximized to $\sigma_1 = 4$

since $Av_i = \sigma_i u_i$

$\|Av_i\| = \sigma_i \|u_i\|$ @ $\sigma_i = 4$

$i = 1$

v_1 Thus vector is $\begin{bmatrix} \sqrt{5}/5 \\ 2/\sqrt{5} \end{bmatrix}$