Homework 8

Problem 33

(a) Derive the operator identity

$$[\hat{p}, \hat{f}(\hat{x})] = -i\hbar \hat{f}'(\hat{x}),$$

where $\hat{f}(\hat{x})$ and $\hat{f}'(\hat{x})$ are operator-valued functions of $\hat{x}$ defined by their spectral decompositions

$$\hat{f}(\hat{x}) = \int dx\, |x\rangle f(x) \langle x|, \quad \hat{f}'(\hat{x}) = \int dx\, |x\rangle f'(x) \langle x|,$$

where $f'(x) = df(x)/dx$. *Hint*: see Eqs. (5.48) in the Lecture Notes for a similar derivation.

(b) Show that

$$[\hat{x}, \hat{f}(\hat{p})] = i\hbar \hat{f}'(\hat{p}),$$

where $\hat{f}(\hat{p})$ and $\hat{f}'(\hat{p})$ are functions of $\hat{p}$ defined similarly to $\hat{f}(\hat{x})$ and $\hat{f}'(\hat{x})$ of part (a).

Solution

(a) For any $|\psi\rangle$ we have

$$\left.\begin{aligned} \langle x|\hat{p}\hat{f}(\hat{x})|\psi\rangle &= -i\hbar \frac{d}{dx}\langle x|\hat{f}(\hat{x})|\psi\rangle = -i\hbar \frac{d}{dx}[f(x)\langle x|\psi\rangle] \\ \langle x|\hat{f}(\hat{x})\hat{p}|\psi\rangle &= f(x)\langle x|\hat{p}|\psi\rangle = -i\hbar f(x)\frac{d}{dx}\langle x|\psi\rangle \end{aligned}\right\} \implies \langle x|[\hat{p}, \hat{f}(\hat{x})]|\psi\rangle = -i\hbar f'(x)\langle x|\psi\rangle.$$

This gives

$$[\hat{p}, \hat{f}(\hat{x})]|\psi\rangle = \int dx\, |x\rangle\langle x|[\hat{p}, \hat{f}(\hat{x})]|\psi\rangle = -i\hbar \int dx\, |x\rangle f'(x)\langle x|\psi\rangle = -i\hbar \hat{f}'(\hat{x})|\psi\rangle.$$

Since $|\psi\rangle$ is arbitrary, this relation is equivalent to the operator identity $[\hat{p}, \hat{f}(\hat{x})] = -i\hbar \hat{f}'(\hat{x})$. ■

(b) Similarly,

$$\langle p|[\hat{x}, \hat{f}(\hat{p})]|\psi\rangle = i\hbar \left\{ \frac{d}{dp}[f(p)\langle p|\psi\rangle] - f(p)\frac{d}{dp}\langle p|\psi\rangle \right\} = i\hbar f'(p)\langle p|\psi\rangle,$$

so that

$$[\hat{x}, \hat{f}(\hat{p})]|\psi\rangle = i\hbar \int dp\, |p\rangle f'(p)\langle p|\psi\rangle = i\hbar \hat{f}'(\hat{p})|\psi\rangle,$$

for all $|\psi\rangle$, which implies the operator identity $[\hat{x}, \hat{f}(\hat{p})] = i\hbar \hat{f}'(\hat{p})$. ■

Problem 34

Use the relations established in Problem 33 to show that

$$\hat{H} = \frac{\hat{p}^2}{2m} + \text{const}$$

is the only translationally invariant (i.e., commuting with $\hat{p}$) Hamiltonian yielding the Heisenberg equation of motion (see, e.g., Eqs. (4.14) in the Lecture Notes) for the position operator

$$\frac{d}{dt}\hat{x}_t = \frac{\hat{p}_t}{m}.$$

Solution

Because $\hat{p}$ exhausts the complete set of commuting operators, any operator that commutes with $\hat{p}$ (such as any translationally invariant Hamiltonian) is a function of $\hat{p}$ and the dressed momentum operator is independent of time, $\hat{p}_t = \hat{p}$. Let

$$\hat{H} = \hat{H}(\hat{p}) = \int dp\, |p\rangle H(p) \langle p|.$$

The Heisenberg equation of motion for the position operator reads

$$\frac{d}{dt}\hat{x}_t = \frac{i}{\hbar}[\hat{H}(\hat{p}),\hat{x}]_t = -\frac{i}{\hbar}i\hbar\hat{H}'(\hat{p}) = \hat{H}'(\hat{p}) = \int dp\,|p\rangle H'(p)\langle p|.$$

Equating the right-hand side of this equation to $\hat{p}/m = \int dp\,|p\rangle(p/m)\langle p|$, we obtain

$$\frac{d}{dp}H(p) = \frac{p}{m} \quad\Longrightarrow\quad H(p) = \frac{p^2}{2m} + \text{const} \quad\Longrightarrow\quad \hat{H} = \frac{\hat{p}^2}{2m} + \text{const}. \qquad \blacksquare$$

Problem 35

Evaluate $\langle p^2\rangle_\psi$ for

$$\textbf{(a)}\quad \psi(x) \propto e^{-|x/a|}, \qquad \textbf{(b)}\quad \psi(x) \propto \big(1-|x/a|\big)\theta\big(1-|x/a|\big).$$

(Don't forget the normalization coefficients.)

Solution

(a) We have

$$\frac{d\psi}{dx} = -\frac{1}{a}\frac{d|x|}{dx}\psi(x) = -\frac{1}{a}\,\text{sign}(x)\psi(x) \quad\Longrightarrow\quad \left|\frac{d\psi}{dx}\right|^2 = \frac{1}{a^2}|\psi(x)|^2$$

and

$$\langle p^2\rangle = \hbar^2\int dx\left|\frac{d\psi}{dx}\right|^2 = \frac{\hbar^2}{a^2}\int dx\,|\psi(x)|^2 = \frac{\hbar^2}{a^2}.$$

(Notice that we were able to get the answer without explicitly evaluating the normalization coefficient.)

(b) In this case, we need to find the normalization coefficient c in $\psi(x) = c\big(1-|x/a|\big)\theta\big(1-|x/a|\big)$. We have

$$1 = \int dx\,|\psi(x)|^2 = |c|^2\int_{-a}^{a}dx\,\big(1-|x/a|\big)^2 = \frac{2a}{3}|c|^2 \quad\Longrightarrow\quad |c|^2 = \frac{3}{2a}.$$

Taking into account that

$$\left|\frac{d\psi}{dx}\right| = \frac{|c|}{a}\theta\big(1-|x/a|\big),$$

we obtain

$$\langle p^2\rangle = \hbar^2\int dp\left|\frac{d\psi}{dx}\right|^2 = \frac{\hbar^2|c|^2}{a^2}\int_{-a}^{a}dx = \frac{2\hbar^2|c|^2}{a} = \frac{3\hbar^2}{a^2}.$$

Problem 36

States $\psi_\pm$ correspond to the position-space wave functions

$$\psi_\pm(x) = \langle x|\psi_\pm\rangle = \frac{1}{\sqrt{2}}\big[\varphi_1(x) \pm i\varphi_2(x)\big],$$

where $\varphi_1(x)$ and $\varphi_2(x)$ are real functions that satisfy

$$\int_{-\infty}^{\infty}dx\,\varphi_1^2(x) = \int_{-\infty}^{\infty}dx\,\varphi_2^2(x) = 1, \quad \int_{-\infty}^{\infty}dx\,\varphi_1(x)\varphi_2(x) = 0, \quad \varphi_1(x) = \varphi_1(-x), \quad \varphi_2(x) = \varphi_2(-x).$$

(a) Show that state vectors $|\psi_+\rangle$ and $|\psi_-\rangle$ are orthogonal.

(b) Show that states $\psi_\pm$ are characterized by the same probability densities in the position and momentum spaces,

$$|\psi_+(x)|^2 = |\psi_-(x)|^2, \quad |\psi_+(p)|^2 = |\psi_-(p)|^2.$$

(This example shows that the position- and the momentum-space probability densities do not specify the state.)

Solution

(a) Taking into account the properties of $\varphi_{1,2}(x)$, we obtain

$$\langle\psi_+|\psi_-\rangle = \int dx\,\psi_+^*(x)\psi_-(x) = \int dx\,\psi_-^2(x) = \frac{1}{2}\int dx\left[\varphi_1^2(x) - \varphi_2^2(x) - 2i\,\varphi_1(x)\varphi_2(x)\right] = 0. \qquad \blacksquare$$

(b) Because $\psi_+(x) = \psi_-^*(x)$, the two states $\psi_\pm$ have the same position-space probability densities:

$$|\psi_+(x)|^2 = |\psi_-(x)|^2 = \psi_+(x)\psi_-(x) = \frac{1}{2}\left[\varphi_1^2(x) + \varphi_2^2(x)\right]. \qquad \blacksquare$$

Next, since $\varphi_{1,2}(x)$ are even functions of x, their Fourier transforms are real,

$$\varphi_{1,2}(p) = \int \frac{dx}{\sqrt{2\pi\hbar}}\, e^{-ipx/\hbar}\varphi_{1,2}(x) = \int \frac{dx}{\sqrt{2\pi\hbar}}\cos(px/\hbar)\,\varphi_{1,2}(x) = \varphi_{1,2}^*(p),$$

so that $\psi_+(p) = \psi_-^*(p) = \frac{1}{\sqrt{2}}\left[\varphi_1(p) + i\varphi_2(p)\right]$. Thereore, just as in the position space,

$$|\psi_+(p)|^2 = |\psi_-(p)|^2 = \psi_+(p)\psi_-(p) = \frac{1}{2}\left[\varphi_1^2(p) + \varphi_2^2(p)\right]. \qquad \blacksquare$$

Problem 37

(a) $|\psi\rangle$ is an eigenvector of Hermitian operator $\hat{H}$. Show that $\langle\psi|[\hat{A},\hat{H}]|\psi\rangle = 0$ for all linear operators $\hat{A}$.

(b) A one-dimensional motion of a particle of mass m in the potential $V(x)$ is described by the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \hat{V}(\hat{x}), \quad \hat{V}(\hat{x}) = \int dx\,|x\rangle V(x)\langle x|.$$

By substituting $\hat{A} = \hat{x}\hat{p}$ into the identity proved in part (a), show that

$$\langle\psi|\hat{p}^2/m|\psi\rangle = \alpha\langle\psi|\hat{x}\hat{V}'(\hat{x})|\psi\rangle,$$

where $|\psi\rangle$ is a bound-state eigenvector of $\hat{H}$, $\hat{V}'(\hat{x}) = \int dx\,|x\rangle V'(x)\langle x|$ with $V'(x) = dV(x)/dx$, and α is a numerical coefficient that you need to find. You may find useful the relations derived in Problem 33.

(c) Using the relation established in part (b), evaluate the momentum uncertainty Δp for bound states in the logarithmic potential well $V(x) = \varepsilon\ln|x/a|$.

Solution

(a) Since the eigenvalue E in $\hat{H}|\psi\rangle = E|\psi\rangle$ is real, we have

$$\langle\psi|[\hat{A},\hat{H}]|\psi\rangle = \langle\psi|\hat{A}\hat{H}|\psi\rangle - \langle\psi|\hat{H}\hat{A}|\psi\rangle = \langle\psi|\hat{A}\hat{H}|\psi\rangle - \langle\psi|\hat{A}^\dagger\hat{H}|\psi\rangle^* = E\langle\psi|\hat{A}|\psi\rangle - \left[E\langle\psi|\hat{A}^\dagger|\psi\rangle\right]^* = 0. \qquad \blacksquare$$

(b) Using the relations found in Problem 33, we obtain

$$[\hat{x}\hat{p},\hat{H}] = \frac{1}{2m}[\hat{x}\hat{p},\hat{p}^2] + [\hat{x}\hat{p},\hat{V}(\hat{x})] = \frac{1}{2m}[\hat{x},\hat{p}^2]\hat{p} + \hat{x}[\hat{p},\hat{V}(\hat{x})] = i\hbar\left\{\hat{p}^2/m - \hat{x}\hat{V}'(\hat{x})\right\}.$$

Substitution of this commutator into the identity derived in part (a) yields the ***quantum virial theorem***

$$\langle\psi|\hat{p}^2/m|\psi\rangle = \langle\psi|\hat{x}\hat{V}'(\hat{x})|\psi\rangle. \qquad \blacksquare$$

(c) For the logarithmic potential given above $V'(x) = \varepsilon/x$, so that $\hat{x}\hat{V}'(\hat{x}) = \int dx\,|x\rangle xV'(x)\langle x| = \varepsilon\hat{\mathbb{1}}$. The virial theorem of part (b) then yields

$$\langle p^2\rangle_\psi = \langle\psi|\hat{p}^2|\psi\rangle = m\langle\psi|\hat{x}\hat{V}'(\hat{x})|\psi\rangle = m\varepsilon\langle\psi|\hat{\mathbb{1}}|\psi\rangle = m\varepsilon.$$

Since for bound states $\langle p\rangle_\psi = 0$, the momentum uncertainty is

$$\Delta p = \left[\langle p^2\rangle_\psi\right]^{1/2} = \sqrt{m\varepsilon}.$$

Thus, all bound states in a logarithmic potential have the same momentum uncertainty and the same kinetic energy.