11/30 Lecture

Thursday, November 30, 2023 1:55 PM

Singular-Value Decomposition

$A \in M_{m \times n}(\mathbb{R})$

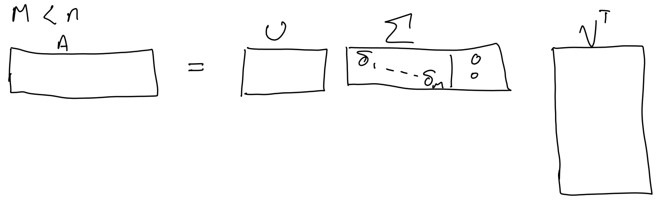
$A = U \Sigma V^T$ where U + V are orthogonal matrices

$U \in M_m(\mathbb{R})$

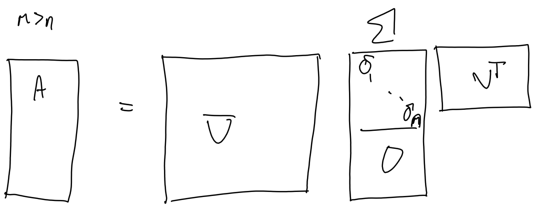
$\Sigma \in M_{mn}(\mathbb{R})$

$V \in M_n(\mathbb{R})$

① $m < n$



② $m > n$



convenient to write $AV = U\Sigma$

$\therefore AV_i = U_i \delta_i$ for $1 \le i \le r$

$U_i = \frac{1}{\delta_i} AV_i$

ex1

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

① form A^TA

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

② orthogonally Diagonalize

$A^TA = QDQ^{-1}$ $\lambda_1 = 3$ $\lambda_2 = 1$ $\lambda_3 = 0$

$\delta_1 = \sqrt{3}$ $\delta_2 = 1$ $\delta_3 = 0$

$E_{\lambda_1} = \text{Span}\left\{\frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}\right\} \leftarrow v_1$

$E_{\lambda_2} = \text{Span}\left\{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\right\} \leftarrow v_2$

$E_{\lambda_3} = \text{Span}\left\{\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\right\} \leftarrow v_3$

③ Images of V_i under A

$$U_1 = \frac{AV_1}{\delta_1} = \frac{\frac{1}{\sqrt{6}}\begin{pmatrix} 3 \\ 3 \end{pmatrix}}{\sqrt{3}}$$

$$U_2 = \frac{\frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}}{1}$$

$U_3 = NA$ since $\delta_3 = 0$

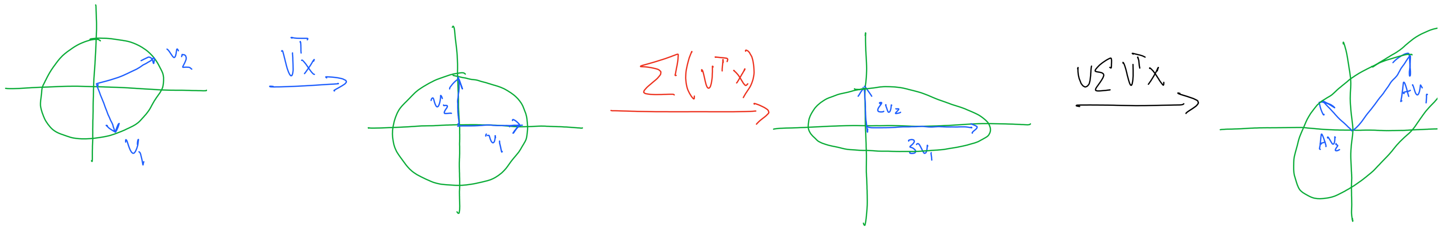
(4) $U\Sigma' V'$

$2\times 2 \quad 2\times 3 \quad 3\times 3$

$$A = U\Sigma V^T = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix}$$

ex)

$A = \begin{pmatrix} 2 & -1 \\ 2 & 2 \end{pmatrix}$ SVD $A = \begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{array}{c} \begin{pmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix} \end{array}\begin{array}{l} v_1 \\ v_2 \end{array}$



Prop let $Rank(A) = r$ then singular values of A

$\sigma_1, \dots, \sigma_r$ are non zero but $\sigma_{r+1}, \dots, \sigma_n$ are zero

Proof

$\{Av_1, \dots, Av_r\}$ span the $Col(A)$

The Rank of A = $dim(Col(A)) = r$

- Relation of SVD to the 4 Fundamental Spaces

- from studio $\underbrace{(Av_1 \dots Av_r)}$ span $Col(A)$ → $Av_1 = \sigma_1 u_1$

$\{u_1, \dots, u_r\}$ span $Col(A)$

- $\{v_{r+1} \dots v\}$ span $null(A)$

$A = U\Sigma' V^T$

→ Apply transpose

$A^T = V^T \Sigma^T U^T \Rightarrow A^T U = V\Sigma^T$

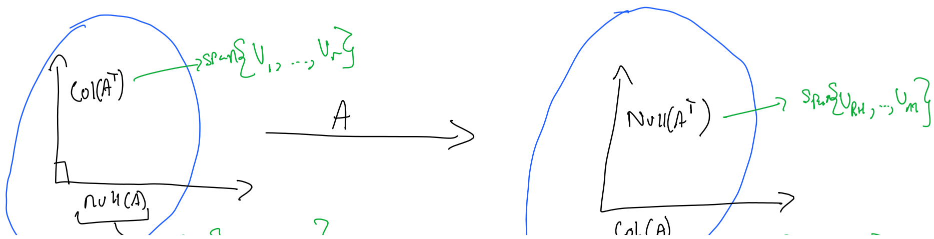
$\Rightarrow A^T u_i = \sigma_i v_i$

$Av_i = \sigma_i u_i$

$\{\underbrace{A^T u_1, \dots, A^T u_r}\}$ spans $Col(A^T)$ → $A^T u_1 = \sigma_1 v_1$

$\Rightarrow \{v_1, \dots, v_r\}$ span $Col(A)$

$\{u_{r+1}, \dots u_n\}$ spans $Null(A)$



$$\boxed{\begin{aligned} Av_i &= \sigma_i u_i \\ A^T u_i &= \sigma_i v_i \end{aligned} \qquad 1 \le i \le r}$$

Best low Rank approximation

$$A = \begin{pmatrix} | & & | \\ u_1 & \dots & u_m \\ | & & | \end{pmatrix} \begin{pmatrix} \sigma_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \sigma_r \end{pmatrix} \begin{pmatrix} — v_1^T — \\ \vdots \\ — v_n^T — \end{pmatrix}$$

$$\Rightarrow \sum_{i=1}^{r} \sigma_i \vec{u}_i \vec{v}_i^T$$

$A \approx \sigma_1 u_1 v_1^T$ — Rank 1 appx

$A \approx \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T \longrightarrow$ Rank 2 appx

$\vdots$

$A = \sum_{i=1}^{r} \sigma_i u_i v_i^T$

Least Square Solution + SVD

$$A = U \Sigma V^T$$

Prop $\quad \vec{x}^* = \frac{b \cdot u_1}{\sigma_1} v_1 + \dots + \frac{b \cdot u_r}{\sigma_r} v_r$

is a least square solution to $A\vec{x} = \vec{b}$

Proof
NT $\quad Ax^*$ is proj of $\vec{b}$ onto $Col(A)$

$$A\vec{x}^* = \text{Proj}_{col(A)} \vec{b}$$

Consider $\quad A\vec{x}^* = A \sum_{i=1}^{r} \frac{b \cdot u_i}{\sigma_i} v_i$

$$\Rightarrow \sum_{i=1}^{r} \frac{b \cdot u_i}{\sigma_i} A v_i$$

$$\Rightarrow \sum_{i=1}^{r} \frac{b \cdot u_i}{\sigma_i} \sigma_i u_i$$

$$\Rightarrow \sum_{i=1}^{r} (\vec{b} \cdot \vec{u}_i) \vec{u}_i \Rightarrow \sum_{i=1}^{r} \frac{b \cdot u_i}{\| u_i \|} u_i$$

$\Rightarrow \text{Proj}_{Col(A)} b$ Since $\{v_1, \dots, v_r\}$ spans $Col(A)$

Back to Digonal matrix

Let $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

$D^2 = \begin{pmatrix} \lambda_1^2 & & 0 \\ & \ddots & \\ 0 & & \lambda_n^2 \end{pmatrix}$

$$D^2 + 3D + 2I_n = \begin{pmatrix} \lambda_1^2 + 3\lambda + 2 & & 0 \\ & \ddots & \\ 0 & & \lambda_n^2 + 3\lambda_n + 2 \end{pmatrix}$$

$p(x)$ is a polynomial of finite degree

$$P(D) = \begin{pmatrix} P(\lambda) & & 0 \\ & \ddots & \\ 0 & & P(\lambda_n) \end{pmatrix}$$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$$

$$e^D = 1 + D + \frac{D^2}{2} + \frac{D^3}{D!} + \dots + \frac{D^n}{n!}$$

$$\Rightarrow \begin{pmatrix} 1 + \lambda + \frac{1}{2}\lambda^2 + \dots & & 0 \\ & \ddots & \\ 0 & & 1 + \lambda_n + \frac{1}{2}\lambda_n^2 + \dots \end{pmatrix}$$

Let $A = XDX^{-1}$

$$e^A = X e^D X^{-1}$$

$y = y(t)$, $t \geq 0$

Solve IVT $\frac{dy}{dt} = ky$, $y(0) = y_0$

Solution is $y = y_0 e^{kt}$