Homework 6

Problem 23

(a) Operators $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$ are vectors, operator $\hat{C}$ is a scalar. Show that

$$[\hat{\mathbf{A}}*\hat{\mathbf{B}},\hat{C}] = \hat{\mathbf{A}}*[\hat{\mathbf{B}},\hat{C}] + [\hat{\mathbf{A}},\hat{C}]*\hat{\mathbf{B}},$$

where the symbol $*$ stands for either the dot product $\bullet$ or the cross product $\times$, so that $\hat{\mathbf{A}}*\hat{C}\hat{\mathbf{B}} = \hat{\mathbf{A}}\hat{C}*\hat{\mathbf{B}}$.

(b) Using the formula derived in part (a), vector identity $\boldsymbol{\alpha}\cdot(\boldsymbol{\beta}\times\boldsymbol{\gamma}) = (\boldsymbol{\alpha}\times\boldsymbol{\beta})\cdot\boldsymbol{\gamma}$ and the operator relation $[\hat{\mathbf{A}},\mathbf{a}\cdot\hat{\mathbf{J}}] = i\hbar(\mathbf{a}\times\hat{\mathbf{A}})$, show that the dot product of two vector operators $\hat{\mathbf{A}}\cdot\hat{\mathbf{B}}$ commutes with the angular momentum operator $\hat{\mathbf{J}}$.

Solution

(a) We have

$$\begin{aligned}[\hat{\mathbf{A}}*\hat{\mathbf{B}},\hat{C}] &= \hat{\mathbf{A}}*\hat{\mathbf{B}}\hat{C} - \hat{C}\hat{\mathbf{A}}*\hat{\mathbf{B}} = \hat{\mathbf{A}}*\hat{\mathbf{B}}\hat{C} - \hat{\mathbf{A}}*\hat{C}\hat{\mathbf{B}} + \hat{\mathbf{A}}\hat{C}*\hat{\mathbf{B}} - \hat{C}\hat{\mathbf{A}}*\hat{\mathbf{B}} \\ &= \hat{\mathbf{A}}*\big(\hat{\mathbf{B}}\hat{C} - \hat{C}\hat{\mathbf{B}}\big) + \big(\hat{\mathbf{A}}\hat{C} - \hat{C}\hat{\mathbf{A}}\big)*\hat{\mathbf{B}} = \hat{\mathbf{A}}*[\hat{\mathbf{B}},\hat{C}] + [\hat{\mathbf{A}},\hat{C}]*\hat{\mathbf{B}}. \quad \blacksquare\end{aligned}$$

(b) Let $\hat{C} = \mathbf{a}\cdot\hat{\mathbf{J}}$. Then

$$\begin{aligned}[\hat{\mathbf{A}}\cdot\hat{\mathbf{B}},\mathbf{a}\cdot\hat{\mathbf{J}}] &= \hat{\mathbf{A}}\cdot[\hat{\mathbf{B}},\mathbf{a}\cdot\hat{\mathbf{J}}] + [\hat{\mathbf{A}},\mathbf{a}\cdot\hat{\mathbf{J}}]\cdot\hat{\mathbf{B}} = i\hbar\big\{\hat{\mathbf{A}}\cdot(\mathbf{a}\times\hat{\mathbf{B}}) + (\mathbf{a}\times\hat{\mathbf{A}})\cdot\hat{\mathbf{B}}\big\} \\ &= i\hbar\big\{-\hat{\mathbf{A}}\cdot(\hat{\mathbf{B}}\times\mathbf{a}) + (\mathbf{a}\times\hat{\mathbf{A}})\cdot\hat{\mathbf{B}}\big\} = i\hbar\big\{-(\hat{\mathbf{A}}\times\hat{\mathbf{B}})\cdot\mathbf{a} + \mathbf{a}\cdot(\hat{\mathbf{A}}\times\hat{\mathbf{B}})\big\} = i\hbar\mathbf{a}\cdot\big(-\hat{\mathbf{A}}\times\hat{\mathbf{B}} + \hat{\mathbf{A}}\times\hat{\mathbf{B}}\big) = \hat{\mathbb{0}},\end{aligned}$$

or $\mathbf{a}\cdot[\hat{\mathbf{A}}\cdot\hat{\mathbf{B}},\hat{\mathbf{J}}] = \hat{\mathbb{0}}$. Because vector $\mathbf{a}$ here is arbitrary, this relation is equivalent to $[\hat{\mathbf{A}}\cdot\hat{\mathbf{B}},\hat{\mathbf{J}}] = \hat{\mathbb{0}}$. $\blacksquare$

Problem 24

(a) Use the formula derived in Problem 23(a), the Jacobi identity $\boldsymbol{\alpha}\times(\boldsymbol{\beta}\times\boldsymbol{\gamma}) + \boldsymbol{\beta}\times(\boldsymbol{\gamma}\times\boldsymbol{\alpha}) + \boldsymbol{\gamma}\times(\boldsymbol{\alpha}\times\boldsymbol{\beta}) = \mathbf{0}$, and the relation $[\hat{\mathbf{A}},\mathbf{a}\cdot\hat{\mathbf{J}}] = i\hbar(\mathbf{a}\times\hat{\mathbf{A}})$ to show that

$$[\hat{\mathbf{A}}\times\hat{\mathbf{B}},\mathbf{a}\cdot\hat{\mathbf{J}}] = i\hbar\,\mathbf{a}\times(\hat{\mathbf{A}}\times\hat{\mathbf{B}}).$$

(b) Using the formula derived in Problem 23(a) and the relations $[\hat{\mathbf{J}},\mathbf{a}\cdot\hat{\mathbf{A}}] = i\hbar(\mathbf{a}\times\hat{\mathbf{A}})$ and $\hat{\mathbf{A}}\times\hat{\mathbf{J}} + \hat{\mathbf{J}}\times\hat{\mathbf{A}} = 2i\hbar\hat{\mathbf{A}}$, show that

$$[\hat{\mathbf{J}}^2,\hat{\mathbf{A}}] = \alpha\hat{\mathbf{A}} + \beta\hat{\mathbf{A}}\times\hat{\mathbf{J}}$$

with the coefficients α and β that you need to find.

Solution

(a) We have

$$[\hat{\mathbf{A}}\times\hat{\mathbf{B}},\mathbf{a}\cdot\hat{\mathbf{J}}] = \hat{\mathbf{A}}\times[\hat{\mathbf{B}},\mathbf{a}\cdot\hat{\mathbf{J}}] + [\hat{\mathbf{A}},\mathbf{a}\cdot\hat{\mathbf{J}}]\times\hat{\mathbf{B}} = i\hbar\big\{\hat{\mathbf{A}}\times(\mathbf{a}\times\hat{\mathbf{B}}) + (\mathbf{a}\times\hat{\mathbf{A}})\times\hat{\mathbf{B}}\big\}.$$

Consider the Jacoby identity for vectors $\mathbf{a},\mathbf{A},\mathbf{B}$ (not yet operators!), $\mathbf{a}\times(\mathbf{A}\times\mathbf{B}) + \mathbf{A}\times(\mathbf{B}\times\mathbf{a}) + \mathbf{B}\times(\mathbf{a}\times\mathbf{A}) = \mathbf{0}$. It can be also written as $\mathbf{a}\times(\mathbf{A}\times\mathbf{B}) = \mathbf{A}\times(\mathbf{a}\times\mathbf{B}) + (\mathbf{a}\times\mathbf{A})\times\mathbf{B}$. Because the order of vectors $\mathbf{A}$ and $\mathbf{B}$ is maintained throughout this relation, they can be replaced with vector operators $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$, which yields the operator identity

$$\hat{\mathbf{A}}\times(\mathbf{a}\times\hat{\mathbf{B}}) + (\mathbf{a}\times\hat{\mathbf{A}})\times\hat{\mathbf{B}} = \mathbf{a}\times(\hat{\mathbf{A}}\times\hat{\mathbf{B}}).$$

With this identity taken into account, the above expression for the commutator $[\hat{\mathbf{A}}\times\hat{\mathbf{B}},\mathbf{a}\cdot\hat{\mathbf{J}}]$ turns to

$$[\hat{\mathbf{A}}\times\hat{\mathbf{B}},\mathbf{a}\cdot\hat{\mathbf{J}}] = i\hbar\,\mathbf{a}\times(\hat{\mathbf{A}}\times\hat{\mathbf{B}}). \quad \blacksquare$$

(b) Proceeding as in Problem 23(b), we write

$$\mathbf{a}\cdot[\hat{\mathbf{J}}^2,\hat{\mathbf{A}}] = [\hat{\mathbf{J}}\cdot\hat{\mathbf{J}},\mathbf{a}\cdot\hat{\mathbf{A}}] = \hat{\mathbf{J}}\cdot[\hat{\mathbf{J}},\mathbf{a}\cdot\hat{\mathbf{A}}] + [\hat{\mathbf{J}},\mathbf{a}\cdot\hat{\mathbf{A}}]\cdot\hat{\mathbf{J}} = i\hbar\big\{\hat{\mathbf{J}}\cdot(\mathbf{a}\times\hat{\mathbf{A}}) + (\mathbf{a}\times\hat{\mathbf{A}})\cdot\hat{\mathbf{J}}\big\}$$
$$= i\hbar\big\{-\hat{\mathbf{J}}\cdot(\hat{\mathbf{A}}\times\mathbf{a}) + (\mathbf{a}\times\hat{\mathbf{A}})\cdot\hat{\mathbf{J}}\big\} = i\hbar\big\{-(\hat{\mathbf{J}}\times\hat{\mathbf{A}})\cdot\mathbf{a} + \mathbf{a}\cdot(\hat{\mathbf{A}}\times\hat{\mathbf{J}})\big\} = i\hbar\,\mathbf{a}\cdot\big(-\hat{\mathbf{J}}\times\hat{\mathbf{A}} + \hat{\mathbf{A}}\times\hat{\mathbf{J}}\big).$$

Since $\mathbf{a}$ is arbitrary, this yields the operator identity

$$[\hat{\mathbf{J}}^2,\hat{\mathbf{A}}] = i\hbar\big(-\hat{\mathbf{J}}\times\hat{\mathbf{A}} + \hat{\mathbf{A}}\times\hat{\mathbf{J}}\big).$$

Substituting here $-\hat{\mathbf{J}}\times\hat{\mathbf{A}} = -2i\hbar\hat{\mathbf{A}} + \hat{\mathbf{A}}\times\hat{\mathbf{J}}$, we arrive at

$$[\hat{\mathbf{J}}^2,\hat{\mathbf{A}}] = 2\hbar^2\hat{\mathbf{A}} + 2i\hbar\,\hat{\mathbf{A}}\times\hat{\mathbf{J}}. \qquad \blacksquare$$

Problem 25

A unit vector $\mathbf{n}$ specified in the spherical polar coordinates by the angles θ and ϕ can be obtained by first rotating $\mathbf{z}$ by θ about $\mathbf{y}$, and then rotating the resulting vector by ϕ about $\mathbf{z}$. Verify that the rotated spin 1/2 state vectors

$$\hat{R}(\phi\mathbf{z})\hat{R}(\theta\mathbf{y})|\pm\mathbf{z}\rangle$$

coincide (up to phase factors) with the eigenvectors of $\hat{S}_{\mathbf{n}}$

$$|+\mathbf{n}\rangle = \cos(\theta/2)|+\mathbf{z}\rangle + e^{i\phi}\sin(\theta/2)|-\mathbf{z}\rangle, \quad |-\mathbf{n}\rangle = \sin(\theta/2)|+\mathbf{z}\rangle - e^{i\phi}\cos(\theta/2)|-\mathbf{z}\rangle.$$

Suggestion: write the rotation operators as first-degree polynomials of spin 1/2 operators [see Problem 20].

Solution

Taking into account the relations

$$\hat{R}(\alpha\mathbf{n}_0) = e^{-i\alpha\hat{S}_{\mathbf{n}_0}/\hbar} = \cos(\alpha/2)\hat{\mathbb{1}} - i\sin(\alpha/2)(2\hat{S}_{\mathbf{n}_0}/\hbar),$$
$$\hat{S}_{\mathbf{z}}|\pm\mathbf{z}\rangle = \pm(\hbar/2)|\pm\mathbf{z}\rangle \implies \hat{R}(\phi\mathbf{z})|\pm\mathbf{z}\rangle = e^{-i\phi\hat{S}_{\mathbf{z}}/\hbar}|\pm\mathbf{z}\rangle = e^{\mp i\phi/2}|\pm\mathbf{z}\rangle,$$
$$\hat{S}_{\mathbf{y}}|\pm\mathbf{z}\rangle = \frac{1}{2i}\big(\hat{S}_+ - \hat{S}_-\big)|\pm\mathbf{z}\rangle = \pm i(\hbar/2)|\mp\mathbf{z}\rangle,$$

we obtain

$$\hat{R}(\phi\mathbf{z})\hat{R}(\theta\mathbf{y})|\pm\mathbf{z}\rangle = \hat{R}(\phi\mathbf{z})\big\{\cos(\theta/2)\hat{\mathbb{1}} - i\sin(\vartheta/2)(2\hat{S}_{\mathbf{y}}/\hbar)\big\}|\pm\mathbf{z}\rangle$$
$$= \hat{R}(\phi\mathbf{z})\big\{\cos(\theta/2)|\pm\mathbf{z}\rangle \pm \sin(\theta/2)|\mp\mathbf{z}\rangle\big\} = e^{\mp i\phi/2}\cos(\theta/2)|\pm\mathbf{z}\rangle \pm e^{\pm i\phi/2}\sin(\theta/2)|\mp\mathbf{z}\rangle.$$

Comparison with the above expressions for $|\pm\mathbf{n}\rangle$ shows that

$$\hat{R}(\phi\mathbf{z})\hat{R}(\theta\mathbf{y})|\pm\mathbf{z}\rangle = \pm e^{-i\phi/2}|\pm\mathbf{n}\rangle. \qquad \blacksquare$$

Problem 26

Using the relations

$$\hat{\mathbf{J}}^2 = \frac{1}{2}\big(\hat{J}_+\hat{J}_- + \hat{J}_-\hat{J}_+\big) + \hat{J}_{\mathbf{z}}^2, \quad \hat{J}_{\mathbf{n}} = \mathbf{n}\cdot\hat{\mathbf{J}} = \frac{1}{2}\big(n_+\hat{J}_- + n_-\hat{J}_+\big) + n_{\mathbf{z}}\hat{J}_{\mathbf{z}},$$
$$\hat{\mathbf{J}}^2|j,m;\mathbf{z}\rangle = \hbar^2 j(j+1)|j,m;\mathbf{z}\rangle, \quad \hat{J}_{\mathbf{z}}|j,m;\mathbf{z}\rangle = \hbar m|j,m;\mathbf{z}\rangle, \quad \hat{J}_{\pm}|j,m;\mathbf{z}\rangle = \hbar\sqrt{j(j+1) - m(m\pm 1)}\,|j,m\pm 1;\mathbf{z}\rangle,$$

compute the expectation values $\langle J_{\mathbf{n}}\rangle_{m,\mathbf{z}} \equiv \langle j,m;\mathbf{z}|\hat{J}_{\mathbf{n}}|j,m;\mathbf{z}\rangle$ and $\langle J_{\mathbf{n}}^2\rangle_{m,\mathbf{z}} \equiv \langle j,m;\mathbf{z}|\hat{J}_{\mathbf{n}}^2|j,m;\mathbf{z}\rangle$.

Solution

It is obvious that only operators containing products of equal number of the raising and lowering operators $\hat{J}_\pm$ contribute to the expectation values we wish to calculate. (Note that this would not be so for the matrix elements $\langle j,m;\mathbf{z}|f(\hat{J}_\mathbf{n})|j,m';\mathbf{z}\rangle$ with $m \neq m'$.) Denoting operators making no contribution to the result by the ellipsis $(\cdots)$ and taking into account that

$$\hat{J}_+\hat{J}_- + \hat{J}_-\hat{J}_+ = 2\big(\hat{\mathbf{J}}^2 - \hat{J}_\mathbf{z}^2\big), \quad n_+n_- = n_\mathbf{x}^2 + n_\mathbf{y}^2 = 1 - n_\mathbf{z}^2,$$

we obtain

$$\begin{aligned}
\langle J_\mathbf{n}\rangle_{m,\mathbf{z}} &= \langle j,m;\mathbf{z}|\{n_\mathbf{z}\hat{J}_\mathbf{z} + \ldots\}|j,m;\mathbf{z}\rangle = \hbar m n_\mathbf{z}, \\
\langle J_\mathbf{n}^2\rangle_{m,\mathbf{z}} &= \langle j,m;\mathbf{z}|\Big\{n_\mathbf{z}^2\hat{J}_\mathbf{z}^2 + \frac{1}{4}n_+n_-\big(\hat{J}_+\hat{J}_- + \hat{J}_-\hat{J}_+\big) + \ldots\Big\}|j,m;\mathbf{z}\rangle \\
&= \langle j,m;\mathbf{z}|\Big\{n_\mathbf{z}^2\hat{J}_\mathbf{z}^2 + \frac{1}{2}\big(1-n_\mathbf{z}^2\big)\big(\hat{\mathbf{J}}^2 - \hat{J}_\mathbf{z}^2\big) + \ldots\Big\}|j,m;\mathbf{z}\rangle \\
&= \langle J_\mathbf{n}\rangle_{m,\mathbf{z}}^2 + \frac{\hbar^2}{2}\big[j(j+1) - m^2\big]\big(1-n_\mathbf{z}^2\big).
\end{aligned}$$

Notice that because $m^2 \leq j^2 < j(j+1)$, this gives a finite uncertainty $\Delta J_\mathbf{n} = \big[\langle J_\mathbf{n}^2\rangle - \langle J_\mathbf{n}\rangle^2\big]^{1/2} > 0$ for all $\mathbf{n} \neq \pm\mathbf{z}$.

Problem 27

Vectors $|m;\mathbf{n}\rangle$ with $m = 0, \pm 1$ are simultaneous eigenvectors of spin 1 operators $\hat{\mathbf{S}}^2$ and $\hat{S}_\mathbf{n}$:

$$\hat{\mathbf{S}}^2|m;\mathbf{n}\rangle = 2\hbar^2|m;\mathbf{n}\rangle, \quad \hat{S}_\mathbf{n}|m;\mathbf{n}\rangle = \hbar m|m;\mathbf{n}\rangle, \quad \langle m;\mathbf{n}|m';\mathbf{n}\rangle = \delta_{m,m'}.$$

Write **(a)** the projectors $\hat{\mathcal{P}}_{m,\mathbf{n}} = |m;\mathbf{n}\rangle\langle m;\mathbf{n}|$ and **(b)** the rotation operator $\hat{R}(\theta\mathbf{n}) = e^{-i\theta\hat{S}_\mathbf{n}/\hbar}$ as second-degree polynomials of $\hat{S}_\mathbf{n}$.

Solution

(a) Solving the equations

$$\hat{\mathbb{1}} = \hat{\mathcal{P}}_{+1,\mathbf{n}} + \hat{\mathcal{P}}_{-1,\mathbf{n}} + \hat{\mathcal{P}}_{0,\mathbf{n}}, \quad \hat{S}_\mathbf{n}/\hbar = \hat{\mathcal{P}}_{+1,\mathbf{n}} - \hat{\mathcal{P}}_{-1,\mathbf{n}}, \quad (\hat{S}_\mathbf{n}/\hbar)^2 = \hat{\mathcal{P}}_{+1,\mathbf{n}} + \hat{\mathcal{P}}_{-1,\mathbf{n}},$$

we obtain

$$\hat{\mathcal{P}}_{m,\mathbf{n}} = \begin{cases} \dfrac{1}{2}\big[m(\hat{S}_\mathbf{n}/\hbar) + (\hat{S}_\mathbf{n}/\hbar)^2\big], & m \neq 0, \\ \hat{\mathbb{1}} - (\hat{S}_\mathbf{n}/\hbar)^2, & m = 0. \end{cases}$$

(b) The spectral decomposition of the rotation operator reads

$$\hat{R}(\theta\mathbf{n}) = e^{-i\theta\hat{S}_\mathbf{n}/\hbar} = \sum_m e^{-im\theta}\hat{\mathcal{P}}_{m,\mathbf{n}}.$$

Substituting here the projectors found in part (a), we get

$$\begin{aligned}
\hat{R}(\theta\mathbf{n}) &= e^{-i\theta}\frac{1}{2}\big[\hat{S}_\mathbf{n}/\hbar + (\hat{S}_\mathbf{n}/\hbar)^2\big] + e^{i\theta}\frac{1}{2}\big[-\hat{S}_\mathbf{n}/\hbar + (\hat{S}_\mathbf{n}/\hbar)^2\big] + \big[\hat{\mathbb{1}} - (\hat{S}_\mathbf{n}/\hbar)^2\big] \\
&= \hat{\mathbb{1}} - \frac{1}{2}\big(e^{i\theta} - e^{-i\theta}\big)\hat{S}_\mathbf{n}/\hbar + \frac{1}{2}\big(e^{i\theta} + e^{-i\theta} - 2\big)(\hat{S}_\mathbf{n}/\hbar)^2 \\
&= \hat{\mathbb{1}} - i\sin\theta(\hat{S}_\mathbf{n}/\hbar) - (1-\cos\theta)(\hat{S}_\mathbf{n}/\hbar)^2.
\end{aligned}$$