Question 3

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3. Let $A, B \in M_n(\mathbb{R})$ and $\lambda \in \mathbb{R}$. Prove or disprove the following claims:

(a) $|A + B| = |A| + |B|$

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ False

$\det(A) = 1$ $\det(B) = +1$

$\det(A+B) = 0$ since $A+B = [0]$, singular

$\boxed{0 \neq 2}$

(b) $|\lambda A| = \lambda |A|$

False

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ Consider $\lambda = 7$

$|\lambda A| = 49$ $\lambda |A| = 7$

$49 \neq 7$

(c) $|\lambda A| = \lambda^n |A|$

TRUE

consider the expansion of λA via columns

$$\lambda A = \lambda\begin{bmatrix} a_1, \dots, a_n \end{bmatrix} \Rightarrow \begin{bmatrix} \lambda a_1, \dots, \lambda a_n \end{bmatrix}$$

Taking det of $\nearrow$ can factor out each Scalar λ from the cols $\Rightarrow \det(\lambda a_1, \dots, \lambda a_n)$

$$= \lambda \det(a_1, \dots, \lambda a_n) \Rightarrow (\lambda \cdot \lambda \cdot \dots \cdot \lambda) \det(a_1, \dots, a_n)$$

$$\Rightarrow \lambda^n(\det(a_1, \dots, a_n)) \Rightarrow \underline{\lambda^n |A|}$$

(d) If A is anti-symmetric (that is, $A^T = -A$) and n is odd then A is not invertible.

any anti-symmetric matrix is such that the elements down the main diagonal are zero.

$$\begin{bmatrix} 0 & a_2 & \cdots & a_n \\ a_2 & 0 & & \vdots \\ \vdots & & 0 \quad 0 & a_{n-1} \\ a_n & \cdots & a_{n-1} & 0 \end{bmatrix} \qquad A_{ij} = -A_{ji}$$

if you Reduce A to echelon form s.t it is upper triangular, you can take the total product $\prod_{i,j}^{n} A_{ij}$ of the main Diagonal.

Note: Since this A has an odd # of columns,

The $n-1$ column middle col/Row is

the $\frac{n-1}{2}+1$ column, middle col/row is a row that remains with a zero in its $\frac{n-1}{2}+1$ entry which falls on the main diagonal in echelon form. Thus $\prod_{i,j}^{n}$ will include a 0 in the term where $i,j = \frac{n-1}{2}+1$ making $\det = 0 \Rightarrow A$ is singular.

(e) If A is anti-symmetric (that is, $A^T = -A$) and n is even then A is not invertible.

consider $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $n=2$

$\det(A) = 1 \neq 0 \Rightarrow A$ is invertible

False

(f) If $AB = 0$ then $|A^2| + |B^2| = 0$.

if $AB = [0] \Rightarrow (AB)_{ij} = \sum_{k}^{n} A_{ik} B_{kj} = 0$
$\forall i,j \in \{1, \dots, n\}$

$(A^2)_{ij} = \sum_{k}^{n} A_{ik} A_{kj}$ $(B^2)_{ij} = \sum_{k}^{n} B_{ik} B_{kj}$

$A = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \end{bmatrix}$

$$\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\det(A^2) = \underline{1}$ $\qquad \underline{AB = 0}$

$\det(B^2) = 0 \qquad 0 \neq 1 \Rightarrow$ false

(g) If $|A+B| = |A|$ then B is the zero matrix.

$A = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$A+B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad \det(A+B) = 0$

$\det(A) = 0$ $\qquad \underline{\text{False}}$