MATH-1564-K1,K2,K3 Linear Algebra with Abstract Vector Spaces

Homework 8

1. Consider $\mathbb{R}_2[x]$ with inner product

$$< f, g >= \int_0^1 f(x)g(x)dx$$

. Apply Gram-Schmidt to $1, x, x^2$ to get an orthonormal basis for $\mathbb{R}_2[x]$.

2. Consider $C[-\pi, \pi]$, vector space of continuous functions defined on interval $[-\pi, \pi]$. Define inner product

$$< f, g >= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x)dx$$

.

   (a) Show that the set

$$F_n = \{\frac{1}{\sqrt{2}}, \sin(x), \cos(x), \sin(2x), \cos(2x), \ldots, \sin(nx), \cos(nx)\}$$

   where $n \in \mathbb{N}$ is an orthonormal set.

   (b) Determine the orthogonal projection of function $f(x) = x$ onto the space spanned by F_n. This is usually called the n-th order Fourier approximation of function $f(x)$. If we represent this projection as

$$a_0 \frac{1}{\sqrt{2}} + b_1 \sin(x) + c_1 \cos(x) +, \ldots, +b_n \sin(nx) + c_n \cos(nx)$$

   then $a_0, b_1, c_1, \ldots, b_n, c_n$ are called the Fourier coefficients of function $f(x)$.

3. Consider $v \in \mathbb{R}^n$ and subspace $U \subseteq \mathbb{R}^n$. We know that we can write v as a sum of $v_1 \in U$ and $v_2 \in U^{\perp}$. Show that this decomposition is unique.

4. Four data points in $\mathbb{R}^3$ with coordinates are given as follows.

$$(-1, 2, 9), (0, 1, 1), (2, 0, 0), (1, 2, -1)$$

.

Determine coefficients c_1, c_2 such that the plane $z = c_1 x + c_2 y$ best fits the data.

5. Let $P \in \mathcal{L}(V)$ be an orthogonal projection map in inner product space V that projects vectors into subspace U. Show from first principle that $< x, Py >=< Px, y >=< Px, Py >$ for all $x, y, z \in V$.

6. True or False.

   (a) If A, B are symmetric matrices, then so are their product AB.

   (b) If A admits a QR factorization, i.e., $A = QR$, then $R = Q^T A$.

   (c) If $A \in M_{m \times n}(\mathbb{R})$, then $\text{rank}(A) = \text{rank}(A^T A)$.

   (d) Least square solution x^* to system $Ax = b$ is chosen so that Ax^* is as close as possible to b.

(e) If the cols of A are linearly independent, then the least square solution to system $Ax = b$ is unique.

(f) If $b \in \text{ Col}(A)$, then the least square solution x^* to system $Ax = b$ satisfies $Ax^* = b$.

(g) If $AA^T = A^T A$ for a square matrix, then A must be orthogonal.

(h) Let $A \in M_3(\mathbb{R})$ that represents an orthogonal projection with respect to standard basis in $\mathbb{R}^3$. There exists an orthogonal matrix $Q \in M_3(\mathbb{R})$ sucht that $Q^T AQ$ is diagonal.

7. Consider $A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} = (v_1 \,|v_2|\, v_3)$ with v_1, v_2, v_3 are the columns of A.

(a) Use Gram-Schmidt process to construct an orthonormal set $\{q_1, q_2, q_3\}$ such that for $j = 1, 2, 3$,

$$\text{span}\,\{q_1, \ldots, q_j\} = \text{span}\,\{v_1, \ldots, v_j\}\,.$$

(b) Use the answer from (i), find r_{ij}, for $1 \le i \le j \le 3$ such that

$$v_1 = r_{11} q_1, \quad v_2 = r_{12} q_1 + r_{22} q_2, \quad v_3 = r_{13} q_1 + r_{23} q_2 + r_{33} q_3.$$

(c) Denote $Q = (q_1 \,|q_2|\, q_3)$ and $R = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{pmatrix}$. Show that indeed $A = QR$ and $\text{Col}(A) = \text{Col}(Q)$.

(d) Show that $Q^T Q = I_3$ and $QQ^T = q_1 q_1^T + q_2 q_2^T + q_3 q_3^T$. Therefore QQ^T is the orthogonal projection onto $\text{Col}(Q) = \text{Col}(A)$.