Question 1

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1. Consider $\mathbb{R}_2[x]$ with inner product

$$< f, g >= \int_0^1 f(x)g(x)dx$$

. Apply Gram-Schmidt to $1, x, x^2$ to get an orthonormal basis for $\mathbb{R}_2[x]$.

$\beta = \{1, x, x^2\}$

$\boxed{q_1 = 1}$

$$q_2 = \frac{v_2 - \langle v_2, q_1 \rangle q_1}{\| \ldots \|}$$

$$\frac{x - \left[\int_0^1 x \cdot 1 \, dx\right] \cdot 1}{\| \ldots \|} = x - \frac{1}{2}x^2\Big]_0^1 = \frac{x - \frac{1}{2}}{\| \ldots \|}$$

$$\| \ldots \| = \langle x - \tfrac{1}{2}, x - \tfrac{1}{2} \rangle^{1/2}$$

$$\left(x - \tfrac{1}{2}\right)\left(x - \tfrac{1}{2}\right) = \int_0^1 x^2 - x + \tfrac{1}{4} \, dx = \tfrac{1}{3}x^3 - \tfrac{1}{2}x^2 + \tfrac{1}{4}x\Big]_0^1 = \tfrac{4}{12} - \tfrac{6}{12} + \tfrac{3}{12} = \tfrac{1}{12}$$

$$\sqrt{1/12} = \frac{1}{\sqrt{12}} \Rightarrow \boxed{\sqrt{12}\,x - \frac{\sqrt{12}}{2}} = q_2$$

$$q_3 = \frac{v_3 - \langle v_3, q_1 \rangle q_1 - \langle v_3, q_2 \rangle q_2}{\| \ldots \|}$$

$$\frac{x^2 - \left[\int_0^1 x^2 dx\right] \cdot 1 - \left[\int_0^1 x^2 \left(\sqrt{12}\,x - \tfrac{1}{2}\sqrt{12}\right) dx\right] \left(\sqrt{12}\,x - \tfrac{1}{2}\sqrt{12}\right)}{\| \ldots \|}$$

$$\left(\sqrt{12} \int_0^1 x^2 \left(x - \tfrac{1}{2}\right) dx\right) \sqrt{12}\left(x - \tfrac{1}{2}\right) \Rightarrow 12\left[\int x^3 - \tfrac{1}{2}x^2 \, dx\right]\left(x - \tfrac{1}{2}\right) = 12\left(\tfrac{1}{4}x^4 - \tfrac{1}{6}x^3\right)\Big]_0^1 \left(x - \tfrac{1}{2}\right)$$

$\Rightarrow 12\left(\frac{1}{12}\right)\left(x-\frac{1}{2}\right) = x-\frac{1}{2}$

$\Rightarrow x^2-\frac{1}{3}-\left(x-\frac{1}{2}\right) \Rightarrow \dfrac{x^2-x+\frac{1}{6}}{\|\ldots\|}$

$\|\ldots\| = \langle x^2-x+\frac{1}{6}, x^2-x+\frac{1}{6}\rangle^{1/2} = \left[\int_0^1 (x^2-x+\frac{1}{6})^2\,dx\right]^{1/2}$

$\Rightarrow \frac{1}{6\sqrt{5}}$

$q_3 = 6\sqrt{5}\left(x^2-x+\frac{1}{6}\right) \Rightarrow \boxed{6\sqrt{5}x^2-6\sqrt{5}x+\sqrt{5}}$

Question 2

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2. Consider $C[-\pi,\pi]$, vector space of continuous functions defined on interval $[-\pi,\pi]$. Define inner product

$$< f, g >= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)dx$$

.

(a) Show that the set

$$F_n = \{\frac{1}{\sqrt{2}}, \sin(x), \cos(x), \sin(2x), \cos(2x), \ldots, \sin(nx), \cos(nx)\}$$

where $n \in \mathbb{N}$ is an orthonormal set.

F_n is an orthonormal set

① $\langle q_i, q_j \rangle = 0$ for $i \neq j$

$$\langle \frac{1}{\sqrt{2}}, \sin(x) \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{\sqrt{2}}(\sin(x))dx = \frac{1}{\pi\sqrt{2}}\int_{-\pi}^{\pi} \sin(x)dx = \frac{1}{\pi\sqrt{2}} \cos(x)\Big]_{-\pi}^{\pi} = 0$$

② $\langle q_i, q_j \rangle = 1$ for $i = j$

$$\langle \sin(x), \sin(x) \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin^2 x\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1-\cos(2x)}{2} dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} 1 - \cos(2x)\, dx \Rightarrow \frac{1}{2\pi}\left(\int_{-\pi}^{\pi} 1 - \int_{-\pi}^{\pi} \cos(2x)dx\right)$$

$$\frac{1}{2\pi}\left(2\pi - \frac{1}{2}\sin(2x)\Big]_{-\pi}^{\pi}\right) = \frac{1}{2\pi}\left(2\pi - \frac{1}{2}(\sin(2\pi) - \sin(-2\pi))\right)$$

$$\Rightarrow \frac{1}{2\pi}\left(2\pi - \frac{1}{2}(0-0)\right) = \frac{1}{2\pi}(2\pi) = \boxed{1}$$

(b) Determine the orthogonal projection of function $f(x) = x$ onto the space spanned by F_n. This is usually called the n-th order Fourier approximation of function $f(x)$. If we represent this projection as

$$a_0\frac{1}{\sqrt{2}} + b_1 \sin(x) + c_1 \cos(x) +, \ldots, + b_n \sin(nx) + c_n \cos(nx)$$

then $a_0, b_1, c_1, \ldots, b_n, c_n$ are called the Fourier coefficients of function $f(x)$.

The Projection of $f(x)=x$ onto F_n is

$$\text{proj}_{F_n} f(x) = \sum_{i=1}^{n} \text{proj}_{q_i} f(x) \Rightarrow \sum_{i=1}^{n} \langle f(x), q_i \rangle q_i$$

for $q_i = \frac{1}{\sqrt{2}}$, $\langle x, \frac{1}{\sqrt{2}} \rangle \frac{1}{\sqrt{2}} = \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\pi}\right)\left(\frac{1}{\sqrt{2}}\right)\int_{-\pi}^{\pi} x\,dx = \frac{1}{2\pi}\left(\frac{1}{2}x^2\right)\Big]_{-\pi}^{\pi}$

$$\Rightarrow \frac{1}{2\pi}\left(\frac{\pi^2}{2} - \frac{\pi^2}{2}\right) = 0$$

for $i=2$ $\quad \langle x, \sin(x) \rangle \sin x = \frac{\sin(x)}{\pi}\int_{-\pi}^{\pi} x\sin(x)\,dx = \frac{2\pi\sin(x)}{\pi}$

$$= \frac{2}{1}\sin(x)$$

for $i=3$ $\quad \langle x, \cos(x) \rangle \cos(x) = \frac{\cos(x)}{\pi}\int_{-\pi}^{\pi} x\cos(x)\,dx = \boxed{0}$

for $i=4$ $\quad \langle x, \sin(2x) \rangle \sin 2x = \frac{\sin(2x)}{\pi}\int_{-\pi}^{\pi} x\sin(2x)\,dx = -\frac{2}{2}\sin(2x)$

for $i=5$ $\quad \langle x, \cos(2x) \rangle \cos(2x) = 0$

for $i=6$ $\langle x, \sin(3x) \rangle \sin(3x) = \frac{2}{3}\sin(3x)$

for $i=8$ $\langle x, \sin(4x) \rangle \sin(4x) = -\frac{2}{4}\sin(4x)$

for $i=10$ $\langle x, \sin(5x) \rangle \sin(5x) = \frac{2}{5}\sin(5x)$

$$\sum_{i=1}^{n} \frac{2\sin(ix)}{c_i}$$

where $c \in \{1, -2, 3, -4, \ldots, n\}$

Question 3

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3. Consider $v \in \mathbb{R}^n$ and subspace $U \subseteq \mathbb{R}^n$. We know that we can write v as a sum of $v_1 \in U$ and $v_2 \in U^\perp$. Show that this decomposition is unique.

If we say $V = V_1 + V_2$, and $V_1 \in \bar{U}$, Then $V_1 = \text{proj}_{\bar{U}} \vec{V}$

$\Rightarrow V_1 = P\vec{V}$ for $P = [\text{Proj}_{\bar{U}}(\cdot)]$

assume $\exists$ another Decompisition of $V = \tilde{V}_1 + \tilde{V}_2$

$\vec{V} = PV + V_2$. But $\tilde{V}_1$ is also $\text{proj}_{\bar{U}} \vec{V} \Rightarrow PV$

$\vec{V} = PV + \tilde{V}_2$ Thus $V_1 = \tilde{V}_1$

and By $\cancel{PV} + V_2 = \cancel{PV} + \tilde{V}_2 \therefore V_2 = \tilde{V}_2$

Question 4

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4. Four data points in $\mathbb{R}^3$ with coordinates are given as follows.

$$(-1, 2, 9), (0, 1, 1), (2, 0, 0), (1, 2, -1)$$

.

Determine coefficients c_1, c_2 such that the plane $z = c_1 x + c_2 y$ best fits the data.

$$\begin{array}{cc|c} c_1 & c_2 & z \\ \end{array}$$
$$\left[\begin{array}{cc|c} -1 & 2 & 9 \\ 0 & 1 & 1 \\ 2 & 0 & 0 \\ 1 & 2 & -1 \end{array}\right]$$

has No solutions. We must take x^*, least square solution

$Ax = z$ is inconsistent

we must find $\text{proj}_{\text{Col}(A)} \vec{z}$

$$z = \begin{bmatrix} 9 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

where $\text{col}(A) = \text{Span}\left\{ \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 2 \end{bmatrix} \right\}$

$$= \sum_{i=1}^{2} \frac{\langle z, q_i \rangle}{\langle q_i, q_i \rangle} q_i \Rightarrow$$

$$\text{Proj}_{q_1} z = \frac{-9 + 0 + 0 + -1}{1 + 0 + 4 + 1} = \frac{-10}{6} = -\frac{5}{3} \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

$$\text{proj}_{q_2} z = \frac{18 + 1 + 0 + -2}{4 + 1 + 0 + 4} = \frac{17}{9} = \frac{17}{9} \begin{bmatrix} 2 \\ 1 \\ 0 \\ 2 \end{bmatrix}$$

$$x^* = \begin{bmatrix} -5/3 \\ 17/9 \end{bmatrix}$$

$c_1 = -5/3 \qquad c_2 = \frac{17}{9}$

Question 5

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5. Let $P \in \mathcal{L}(V)$ be an orthogonal projection map in inner product space V that projects vectors into subspace U. Show from first principle that $< x, Py >=< Px, y >=< Px, Py >$ for all $x, y, z \in V$.

if $\langle x, Py \rangle = \langle Px, y \rangle$

if we consider the spectral decompisition of vector any vector $\vec{v} \in \bar{V}$, then $\vec{v} = \vec{v}_U + \vec{v}_{U^\perp}$ $v_U \in U$ and $v_{U^\perp} \in U^\perp$. Then $x = x_U + x_{U^\perp}$

$\langle x_U + x_{U^\perp}, y_U \rangle = \langle x_U, y_U + y_{U^\perp} \rangle$

$\langle x_U, y_U \rangle + \langle x_{U^\perp}, y_U \rangle = \langle x_U, y_U \rangle + \langle x_U, y_{U^\perp} \rangle$

But since any vector in $U^\perp$ and U are made of Basis vectors orthogonal to one another, any vector $v \in U^\perp$ and $v' \in U$ are orthogonal and their inner product is zero.

$\langle x_U, y_U \rangle + \cancel{\langle x_{U^\perp}, y_U \rangle} = \langle x_U, y_U \rangle + \cancel{\langle x_U, y_{U^\perp} \rangle}$

$\langle x, Py \rangle = \langle x_U, y_U \rangle \Rightarrow \langle x, Py \rangle = \langle Px, Py \rangle$

Question 6

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6. True or False.

(a) If A, B are symmetric matrices, then so are their product AB.

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 7 \\ 5 & 8 \end{bmatrix}$$ false

(b) If A admits a QR factorization, i.e., $A = QR$, then $R = Q^T A$.

$A = QR \longrightarrow$ left multiply Q^T in

$Q^T A = Q^T Q R$ Note, Q has ON. columns

That span $col(A)$. Thus $Q^T Q = I_n$

$Q^T A = R$ <u>TRUE</u>

(c) If $A \in M_{m\times n}(\mathbb{R})$, then $\text{rank}(A) = \text{rank}(A^T A)$.

By Rank Nullity $\quad Rank(A) + \dim(null\,A) = n$

$Rank(A^T A) + \dim(null\,A^T A) = n \rightarrow$ since $A^T A$ is $n \times n$

consider $Ax = 0$. left multiply A^T

$= A^T A x = A^T 0 = 0$

$\Rightarrow x \in null(A^T A)$ and $null(A)$

Thus $\dim(null\,A) = \dim[null(A^T A)] \therefore$ $Rank\,A = Rank\,A^T A$

(d) Least square solution x^* to system $Ax = b$ is chosen so that Ax^* is as close as possible to b.

TRue.

$\|b - Ax^*\| \leq \|b - \vec{v}\| \quad \forall \vec{v} \in col(A)$

(e) If the cols of A are linearly independent, then the least square solution to system $Ax = b$ is unique.

TRue, if A is linearly independent and x^* is the lss. Then There is only 1 unique solution, x^*, s.t $Ax^* = b$. $\nexists$ another $\tilde{x}$ s.t $A\tilde{x} = b$ Bc. since There is a pivot in every Row of A, There is 1 Solution x^*.

(f) If $b \in$ Col(A), then the least square solution x^* to system $Ax = b$ satisfies $Ax^* = b$.

lss x^* is projection of b onto col(A)

$\|b - A\vec{x}^*\| \leq \|b - A\vec{x}\|$

in This case $0 \leq 0$ which is TRue

(g) If $AA^T = A^T A$ for a square matrix, then A must be orthogonal.

false consider $A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$

$A^T A = A A^T$

$\begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix} = \begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix} \Rightarrow$ Not orthogonal

Since $\left\{ \begin{bmatrix} 8 \\ 8 \end{bmatrix}, \begin{bmatrix} 8 \\ 8 \end{bmatrix} \right\}$

Not ON set.

(h) Let $A \in M_3(\mathbb{R})$ that represents an orthogonal projection with respect to standard basis in $\mathbb{R}^3$. There exists an orthogonal matrix $Q \in M_3(\mathbb{R})$ sucht that $Q^T AQ$ is diagonal.

$\mathcal{E} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

$A = \sum_{i=1}^{3} \varepsilon_i \otimes \varepsilon_i = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} (1\,0\,0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} (0\,1\,0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} (0\,0\,1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

consider $Q = \begin{pmatrix} 1/3 & 2/3 & -2/3 \\ -2/3 & 2/3 & 1/3 \\ 2/3 & 1/3 & 2/3 \end{pmatrix}$ $Q^T = \begin{pmatrix} 1/3 & -2/3 & 2/3 \\ 2/3 & 2/3 & 1/3 \\ -2/3 & 1/3 & 2/3 \end{pmatrix}$

Then $Q^T A Q = \frac{1}{9}\begin{pmatrix} 1 & -2 & 2 \\ \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \end{pmatrix} \begin{pmatrix} 1 & 2 & -2 \\ \end{pmatrix}$

Then $Q^T A Q = \frac{1}{3}\begin{pmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{1}{3}\begin{pmatrix} -2 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix}$

$$= \frac{1}{9}\begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix} \quad \underline{\text{Diagonal}}$$

<u>TRUE</u>

Consider orthonormal set $\{q_1, \ldots, q_3\}$

whos vectors combined form $Q = \begin{bmatrix} | & | & | \\ q_1 & q_2 & q_3 \\ | & | & | \end{bmatrix}$

if A is such where $\text{col}(A) = \text{span}\{q_1, q_2\}$

$\text{null} A = \text{span}\{q_3\}$.

Then $AQ = \begin{bmatrix} Aq_1 & Aq_2 & Aq_3 \end{bmatrix} = \begin{bmatrix} q_1 & q_2 & 0 \end{bmatrix}$

left multiplyg this by Q^T produces

$$\begin{bmatrix} — q_1 — \\ — q_2 — \\ — q_3 — \end{bmatrix} \begin{bmatrix} | & | & \\ q_1 & q_2 & \vec{0} \\ | & | & \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \underline{\text{Diagonal}}$$

<u>TRUE</u>

Question 7

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7. Consider $A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} = (v_1\,|v_2|\,v_3)$ with v_1, v_2, v_3 are the columns of A.

(a) Use Gram-Schmidt process to construct an orthonormal set $\{q_1, q_2, q_3\}$ such that for $j = 1, 2, 3$,

$$\text{span}\,\{q_1, \dots, q_j\} = \text{span}\,\{v_1, \dots, v_j\}.$$

$$\left\{ \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} -1/2 \\ -1/2 \\ -1/2 \\ -1/2 \end{bmatrix} \right\}$$

(b) Use the answer from (i), find r_{ij}, for $1 \le i \le j \le 3$ such that

$$v_1 = r_{11}q_1, \quad v_2 = r_{12}q_1 + r_{22}q_2, \quad v_3 = r_{13}q_1 + r_{23}q_2 + r_{33}q_3.$$

$$\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = R_{11} \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix} \qquad R_{11} = 2$$

$$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = R_{12} \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix} + R_{22} \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} \qquad \begin{matrix} R_{12} = 1 \\ R_{22} = 1 \end{matrix}$$

$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = R_{13} \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix} + R_{23} \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} + R_{33} \begin{bmatrix} -1/2 \\ -1/2 \\ -1/2 \\ -1/2 \end{bmatrix}$$

$$R_{13} = 0 \qquad R_{23} = 1 \qquad R_{33} = 1$$

(c) Denote $Q = (q_1\,|q_2|\,q_3)$ and $R = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{pmatrix}$. Show that indeed $A = QR$ and

$\mathrm{Col}(A) = \mathrm{Col}(Q)$.

$$A = [a_1\ a_2\ a_3] = [q_1, q_2, q_3][R]$$

$$\begin{pmatrix} 1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & -1/2 \end{pmatrix}\begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}}_{A}$$

$\mathrm{Col}(A) = \mathrm{Col}(Q)$

① $\mathrm{Col}(A) \subseteq \mathrm{Col}(Q)$

every column vector can be written as a l.c of col vectors q_1, q_2, q_3. Conf Part B.

② $\mathrm{Col}(Q) \subseteq \mathrm{Col}(A)$

$$q_1 = \frac{1}{2}a_1$$

$$q_2 = a_2 - \frac{1}{2}a_1$$

$$q_3 = a_3 - \left(a_2 - \frac{1}{2}a_1\right)$$

$\therefore\ \mathrm{Col}(Q) = \mathrm{Col}(A)$

(d) Show that $Q^TQ = I_3$ and $QQ^T = q_1q_1^T + q_2q_2^T + q_3q_3^T$. Therefore QQ^T is the orthogonal projection onto $\mathrm{Col}(Q) = \mathrm{Col}(A)$.

$$\underset{3\times 4}{\begin{pmatrix} 1/2 & -1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \end{pmatrix}}\underset{4\times 3}{\begin{pmatrix} 1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & -1/2 \end{pmatrix}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{pmatrix} 1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 \\ -1/2 & 1/2 & -1/2 \end{pmatrix} \begin{pmatrix} 1/2 & -1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{pmatrix} \begin{pmatrix} 1/2 & -1/2 & 1/2 & -1/2 \end{pmatrix}$$

4×3 $\qquad$ 3×4

$$+ \begin{pmatrix} 1/2 \\ 1/2 \\ -1/2 \\ 1/2 \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 & 1/2 & 1/2 \end{pmatrix}$$

$$+ \begin{pmatrix} 1/2 \\ -1/2 \\ -1/2 \\ -1/2 \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 & -1/2 & -1/2 \end{pmatrix}$$

$$= \begin{pmatrix} 3/4 & 1/4 & 1/4 & -1/4 \\ 1/4 & 3/4 & -1/4 & 1/4 \\ 1/4 & -1/4 & 3/4 & 1/4 \\ -1/4 & 1/4 & 1/4 & 3/4 \end{pmatrix}$$