MATH-1564-K1,K2,K3 Linear Algebra with Abstract Vector Spaces

Homework 7

1. Consider the following linear transformations $T, S : \mathbb{R}_3[x] \to \mathbb{R}_3[x]$ given by

$$Tp(x) = p'(x) \quad \text{and} \quad Sp(x) = p(x+1)$$

and consider the following bases of $\mathbb{R}^3[x]$:

$$\mathcal{E} = \left\{1, x, x^2, x^3\right\}$$

and

$$\mathcal{B} = \left\{1, 1+x, (1+x)^2, (1+x)^3\right\}$$

   (a) Find $[T]_{\mathcal{E}\to\mathcal{B}}, [T]_{\mathcal{B}\to\mathcal{B}}, [T]_{\mathcal{B}\to\mathcal{E}}, [T]_{\mathcal{E}\to\mathcal{E}}$.
   (b) Find $[S]_{\mathcal{E}\to\mathcal{E}}, [S]_{\mathcal{B}\to\mathcal{B}}$.
   (c) Find $[T \circ S]_{\mathcal{E}\to\mathcal{E}}$.
   (d) $[T \circ S]_{\mathcal{B}\to\mathcal{B}}$.
   (e) Use $[T]_{\mathcal{E}\to\mathcal{E}}$ to find a basis for the kernel and image of T.

2. Consider the following ordered bases of $\mathbb{R}^3$:

$$\mathcal{B} =< \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} >$$
$$\mathcal{C} =< \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} >$$
$$\mathcal{E} =< \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} >$$

   Find the following matrices of transition from basis to basis:

$$[id]_{\mathcal{B}\to\mathcal{E}}, [id]_{\mathcal{C}\to\mathcal{E}}, [id]_{\mathcal{E}\to\mathcal{B}}, [id]_{\mathcal{E}\to\mathcal{C}}, [id]_{\mathcal{B}\to\mathcal{C}}, [id]_{\mathcal{C}\to\mathcal{E}}.$$

3. Consider the transformations S and T and the bases B and E from Q1. Find the following matrices of transition from basis to basis:

$$[id]_{\mathcal{B}\to\mathcal{E}}, [id]_{\mathcal{E}\to\mathcal{B}}$$

   Check that the formula of transition from basis to basis holds in the following cases:

$$[T]_{\mathcal{B}} = [id]_{\mathcal{E}\to\mathcal{B}}[T]_{\mathcal{E}}[id]_{\mathcal{B}\to\mathcal{E}}$$
$$[S]_{\mathcal{E}} = [id]_{\mathcal{B}\to\mathcal{E}}[S]_{\mathcal{B}}[id]_{\mathcal{E}\to\mathcal{B}}$$

4. Consider curve defined by $49x^2 - 30\sqrt{3}xy + 19y^2 = 64$ on $\mathbb{R}^2$.

(a) Show that with respect to basis

$$\mathcal{B} =< \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}, \begin{pmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix} >$$

the curve is an ellipse.

(b) Show that with respect to basis

$$\mathcal{C} =< \begin{pmatrix} 2 \\ 2\sqrt{3} \end{pmatrix}, \begin{pmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix} >$$

the curve is the unit circle.

5. True or false. Remember to justify your answer.
   (a) There exists a non-zero upper-triangular matrix $A \in M_2(\mathbb{R})$ such that A^2 is the zero matrix.
   (b) Let $A \in M_n(\mathbb{R})$. If $AB = BA$ for every $B \in M_n(\mathbb{R})$ then $A = \lambda I_n$ for some $\lambda \in \mathbb{R}$.
   (c) Let $A, B \in M_n(\mathbb{R})$. Then AB is invertible if and only if both A and B are invertible.
   (d) Let $A \in M_n(\mathbb{R})$. A is NOT invertible if and only if there exists $B \in M_n(\mathbb{R})$ such that $AB = 0$.
   (e) Let $A, B \in M_n(\mathbb{R})$. If both A and B are invertible then $AB = BA$.
   (f) Let $A \in M_n(\mathbb{R})$. If A is invertible then $A + I$ is also invertible.
   (g) If $A^2 - I$ is invertible then $A - I$ is invertible.

6. In class we mentioned that $< A, B >_1 = tr(AB^T)$ defines an inner product on on $M_{m\times n}(\mathbb{R})$ and in studio covered that $< A, B >_2 = tr(A^T B)$ is an inner product. Is there a typo in terms of where the transpose operation is on?

7. Let $M \in M_n(\mathbb{R})$. Characterize M such that $< \cdot, \cdot >_M : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by $< x, y >_M = (Mx)^T (My)$ is an inner product. Justify your answer.

8. Consider the subspace of $\mathbb{R}^4$

$$U = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} : a - b + c - d = 0 \text{ and } a + d = 0 \right\}$$

of $\mathbb{R}^4$.

   (a) Find a basis for U.
   (b) Find an orthonormal basis $\mathcal{C}$ for U.
   (c) Let $x = (1, 2, 3, 4)^T \in \mathbb{R}^4$. Find the orthogonal projection of x onto the space U : $\mathrm{Proj}_U\, x$.
   (d) Find an orthonormal basis of $\mathbb{R}^4$ that contains the vectors from $\mathcal{C}$ from (b).
   (e) Find the matrix representation of the orthogonal projection of $\mathbb{R}^4$ onto the space U with respect to the basis that you obtained from (d).
   (f) Find the matrix representation of the orthogonal projection of $\mathbb{R}^4$ onto the space U with respect to the standard basis of $\mathbb{R}^4$.
   (g) Use the answer from (f) to calculate $\mathrm{Proj}_U\, x$