Phys 3143 final Exam

Q1

$$\hat{H}(t) = \begin{cases} \omega \hat{S}_x & t < \tilde{\tau} \\ \omega \hat{S}_y & t > \tilde{\tau} \end{cases}$$ * product of 2 Rotations

$$|\psi(0)\rangle = |+z\rangle = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$\hat{S}_z |\psi(0)\rangle = \hbar/2$ $\quad \hat{S}_z \hat{T}(2\tau)|\psi(0)\rangle = -\hbar/2$

$$P(t) = |\langle \psi(0)|\hat{S}_z|\psi(2\tilde{\tau})\rangle|^2$$

@ $t = 2\tau$ $|\psi\rangle$ will be $\hat{T}(2\tau)|\psi(0)\rangle = e^{-it H(t)/\hbar}|\psi(0)\rangle$

$$\Rightarrow |\psi(2\tilde{\tau})\rangle = e^{-i2\tau\omega S_y/\hbar}|\psi(0)\rangle$$

start $\rightarrow$ $|\psi(t)\rangle = \hat{T}(2\tilde{\tau},\tilde{\tau})\hat{T}(\tilde{\tau},0)|\psi(0)\rangle = e^{-i(\tilde{\tau})\omega S_y/\hbar} e^{-i(\tilde{\tau})\omega S_x/\hbar}|\psi(0)\rangle$

$$= P(t) = |\langle +z| \left(1 - \frac{i\tilde{\tau}}{\hbar}\omega S_y - \frac{\tilde{\tau}^2}{2\hbar^2}\omega^2 S_y^2\right)\left(1 - \frac{i\tilde{\tau}}{\hbar}\omega S_x - \frac{\tilde{\tau}^2}{2\hbar^2}\omega^2 S_x^2 + \dots\right)|\psi_0\rangle|^2$$

$$\langle +z|\Big[1 - \frac{i\tilde{\tau}}{\hbar}\omega S_x - \frac{\tilde{\tau}^2}{2\hbar^2}\omega^2 S_x^2 - \frac{i\tilde{\tau}}{\hbar}\omega S_y + \frac{-\tilde{\tau}^2}{\hbar^2}\omega^2 S_y S_x + \frac{i\tilde{\tau}^3}{2\hbar^3}\omega^3 S_y S_x^2 + \frac{-\tilde{\tau}^2}{2\hbar^2}\omega^2 S_y^2 + \frac{i\tilde{\tau}^3}{2\hbar^3}\omega^3 S_y^2 S_x + \frac{\tilde{\tau}^4}{4\hbar^4} S_y^2 S_x^2 + \dots\Big]|\psi_0\rangle$$

$$\Rightarrow \left|1 - \frac{i\tilde{\tau}}{2}\omega - \frac{\tilde{\tau}^2\omega^2}{8} - \frac{i\tilde{\tau}\omega}{2} - \frac{\tilde{\tau}^2}{4}\omega^2 + \frac{i\tilde{\tau}^3\omega^3}{16} - \frac{\tilde{\tau}^2\omega^2}{8} + \frac{i\tilde{\tau}^3\omega^3}{16} + \frac{\tilde{\tau}^4}{64}\omega^4\right|^2$$

$$\Rightarrow 1 + \frac{\tilde{\tau}^2\omega^2}{4} + \frac{\tilde{\tau}^4\omega^4}{64} + \frac{\tilde{\tau}^2\omega^2}{4} + \frac{\tilde{\tau}^4\omega^4}{16} - \frac{\tilde{\tau}^6\omega^6}{16^2} + \frac{\tilde{\tau}^4\omega^4}{64} - \frac{\tilde{\tau}^6\omega^6}{16^2} + \frac{\tilde{\tau}^8\omega^8}{64^2} + \dots$$

But Thats for $+\hbar/2$. Since spin 1/2 has 2 States, Prob will be 1 minus it.

$$\text{Prob}_\psi\left(S_z = -\frac{\hbar}{2}\right) = 1 - \left(1 + \frac{\tilde{\tau}^2\omega^2}{2} + \frac{3\tilde{\tau}^4\omega^4}{32} - \frac{\tilde{\tau}^6\omega^6}{256} + \frac{\tilde{\tau}^8\omega^8}{64^2} + \dots\right)$$

Q2 $\psi(x) \propto \begin{cases} \left(\frac{a}{2}\right)^2 - x^2 & -\frac{a}{2} < x < \frac{a}{2} \\ 0 & x > \frac{a}{2} \text{ or } x < -\frac{a}{2} \end{cases}$

$\int dx\, \psi^2(x) = 1$

$\int_{-\frac{a}{2}}^{a/2} dx\, c_1\left(\frac{a^2}{4} - x^2\right)\left(\frac{a^2}{4} - x^2\right) = 1 \Rightarrow c_1^2 \int_{-\frac{a}{2}}^{a/2} dx\left[\frac{a^4}{16} - \frac{a^2x^2}{4} - \frac{a^2x^2}{4} + x^4\right] = 1$

$\Rightarrow c_1^2 \int_{-\frac{a}{2}}^{a/2} dx\left[\frac{a^4}{16} - \frac{a^2}{2}x^2 + x^4\right] \Rightarrow c_1^2\left\{\int_{-\frac{a}{2}}^{a/2} dx\,\frac{a^4}{16} - \int_{-\frac{a}{2}}^{a/2} dx\,\frac{a^2}{2}x^2 + \int_{-\frac{a}{2}}^{a/2} dx\, x^4\right\} = 1$

$\Rightarrow c_1^2\left\{\frac{a^5}{30}\right\} = 1 \Rightarrow c_1 = \sqrt{\frac{30}{a^5}}$

$P = \left(\int_{-a/2}^{a/2} dx\left[\frac{\sqrt{30}}{\sqrt{a^5}}\left(\frac{a^2}{4} - x^2\right)\right]\left[\frac{\sqrt{2}}{\sqrt{a}}\cos\left(\frac{\pi}{a}x\right)\right]\right)^2 = \underline{.99855501}$

Q3 $\hat{H}(t) = H_0 - P_0 \hat{x}\,\delta(t) \qquad H_0 = \frac{\hat{P}^2}{2m} + V(\hat{x})$

$\langle E \rangle_{t>0} = \langle E \rangle_{t=+0} = \underbrace{\langle \Psi(+0)|}\, H_0 \,\underbrace{|\Psi(+0)\rangle}$

$\langle \Psi(-0)| T^{\dagger}(+0,-0)\, H_0\, T(+0,-0)\, |\Psi(-0)\rangle$

$\langle \Psi(-0)| \hat{T}^{\dagger} \frac{\hat{P}^2}{2m} T |\Psi(-0)\rangle \to \langle \Psi(-0)| \frac{1}{2m}(\hat{P} + P_0 \mathbb{1})^2 | + \hat{V}(x) |\Psi(-0)\rangle$

$\Rightarrow \langle \Psi(-0)| \underbrace{\hat{P}^2}_{E_0} + \underbrace{2\hat{P}P_0}_{0} + P_0^2 + \hat{V}(\hat{x}) |\Psi(-0)\rangle \left(\frac{1}{2m}\right)$

$E_0 + 0 + \frac{P_0^2}{2m}$

Since $T^{\dagger} H_0 T = \frac{1}{2m} T^{\dagger} \hat{P}^2 T + V(\hat{x}) = \frac{1}{2m}\left(\underbrace{T^{\dagger}\hat{P}T}\right)^2 + V(x)$

$\frac{1}{2m}\left(\hat{P} + P_0 \mathbb{1}\right)^2 + V(x)$

$= \frac{1}{2m}\left(\hat{P}^2 + 2\hat{P}P_0 + P_0^2\right) + V(x)$

$\therefore \Rightarrow \langle \Psi | T^{\dagger} H_0 T | \Psi \rangle = \boxed{E_0 + \frac{P_0^2}{2m}} = \langle E \rangle_{t>0}$