Question 5

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5. Let $T \in \mathcal{L}(V, W)$. Show that

(a) $\ker T$ is a subspace of V.

The kernel of T is a subset of $\vec{V}$
such that for $x \in \ker T$ $T(x) = \vec{0}_W$
The kerT forms a subspace of $\vec{V}$ because
① it is non empty since it contains the
zero vector ($\vec{0}_V$) since T must map $\vec{0}_V \mapsto \vec{0}_W$
thus $\vec{0}_V \in \ker T$
② it is closed under addition. Since T is
a linear map.

$x, y \in \ker T$: $T(x) + T(y) = \vec{0}_W + \vec{0}_W = \underline{\vec{0}_W}$

$T(x+y)$

③ it is CUSM.
$\forall \alpha \in \mathbb{R}$ and $\forall x \in \ker T$
$T(\alpha x) \Rightarrow \alpha T(x) = \alpha \cdot (\vec{0}_W) = \vec{0}_W$

(b) ImT is a subspace of W.

$\text{ImT} = \{ y \mid y = T(\vec{v}),\ \vec{v} \in \vec{V} \}$

Subspace Because

① Not empty because ImT contains the $\vec{0}_W$, as a preimage

② CUA Since

any $a, B \in \text{ImT}$

$\underbrace{a + B}_{\in \text{IMT}} = T(\vec{v}_1) + T(\vec{v}_2)$ CUA

③ CUSM

$\forall \alpha \in \mathbb{R}$ and $y \in \text{IMT}$

$\alpha y = \alpha T(\vec{v}) = \underbrace{T(\alpha \vec{v})}_{\in \text{ of IMT}}$

(c) If $T \in \mathcal{L}(V, V)$, is it possible that ker$T \cap$ Im$T \neq \{\mathbf{0}\}$? Explain your answer.

Suppose $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$T(x) = Ax$ $\quad x \longmapsto Ax$ $\quad$ where

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

Then $\ker T = \left\{\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\}$

and image of $T = \{ y \mid y = T(x) = Ax, x \in \mathbb{R}^2 \}$

$$\left[\begin{array}{cc|c} 1 & 1 & -1 \\ -1 & -1 & 1 \end{array}\right] \quad R_2 = R_2 + R_1 \quad \left[\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right]$$

$\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ exists in the image of T since

$\exists$ x in $\mathbb{R}^2$ where $\begin{pmatrix} -1 \\ 1 \end{pmatrix} = A\vec{x}$.

as such $\ker T \cap \operatorname{im}(T) = \left\{\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\} \neq \{\vec{0}\}$

it is possible