MATH-1564-K1,K2,K3 Linear Algebra with Abstract Vector Spaces

Homework 11

1. Prove or disprove the following claims.
   (a) The representation of quadratic form $q(\vec{x}) = \vec{x}^T A\vec{x}$, where A is a symmetric matrix, is unique.
   (b) Consider $q(\vec{x}) = \vec{x}^T A\vec{x}$ where $A \in M_2(\mathbb{R})$. $q(\vec{x})$ is not a quadratic form if A is not symmetric.
   (c) If $A \succ 0$ then $A^7 \succ 0$.
   (d) If $A \prec 0$, then $A^4 \prec 0$.
   (e) If $A \succ 0$ and $B \prec 0$ then $A - B \succ 0$.
   (f) Every matrix has a singular value decomposition.
   (g) Similar matrices must have the same singular values.
   (h) If A and B are real symmetric matrices such that $A^3 = B^3$, then A must be equal to B.

2. Orthogonally diagonalize the following matrices

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$$

3. Determine all matrices C such that $C^2 = B$ in problem 2.

4. Let $A \in M_n(\mathbb{R})$ be a symmetric matrix, with eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p$. Show that

$$\max_{\|\vec{x}\| \neq 0} \frac{\vec{x}^T A\vec{x}}{\vec{x}^T \vec{x}} = \lambda_1$$

$$\min_{\|\vec{x}\| \neq 0} \frac{\vec{x}^T A\vec{x}}{\vec{x}^T \vec{x}} = \lambda_p$$

   Comment at what vectors x the max and min values are attained.

5. Consider matrix

$$A = \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}$$

   (a) Determine an SVD of A.
   (b) Write A in the form of $\sum_{i=1}^{r} \sigma_i u_i v_i^T$, a sum of several rank-1 matrices.
   (c) Notice that A is invertible. Determine an SVD of A^{-1}. Do you need to start from scratch?
   (d) Determine an SVD of A^T. Do you need to start from scratch?

6. What can one say about a matrix's eigenvalues and its singular values? Consider the following. Note that here σ_1 is the largest singualr value and σ_n the smallest of matrix A.

(a) Let $A \in M_2(\mathbb{R})$. Let $\boldsymbol{w} \in \mathbb{R}^2$ be a unit vector. Show that

$$\sigma_2 \leq \|A\boldsymbol{w}\| \leq \sigma_1$$

by tracing what happens to $\boldsymbol{w}$ under the three matrices in A's SVD.

(b) Show part (a) algebraically.

(c) Let $A \in M_{m \times n}(\mathbb{R})$. Show that

$$\sigma_n \|v\| \leq \|Av\| \leq \sigma_1 \|v\|$$

for any $v \in \mathbb{R}^n$.

(d) Let λ be a real eigenvalue of matrix $A \in M_n(\mathbb{R})$. Show that

$$\sigma_n \leq |\lambda| \leq \sigma_1$$

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(e) Consider the matrix A in Q5. Determine $\min_{\|x\|=1} \|Ax\|$. Comment at what vectors x the min value is attained.

(f) Consider the matrix A in Q5. Determine $\max_{\|x\|=1} \|Ax\|$. Comment at what vectors x the max value is attained.