MATH-1564-K1,K2,K3 Linear Algebra with Abstract Vector Spaces

Homework 6

1. Consider bijective map $f : \mathcal{D} \to \mathcal{C}$. In class, we have stated that there exists map $g : \mathcal{C} \to \mathcal{D}$ such that $f \circ g = \mathrm{id}_{\mathcal{D}}$ and $g \circ f = \mathrm{id}_{\mathcal{D}}$. g in this case is called the inverse of f, denoted by f^{-1}. In the context of linear maps, we say linear map $T \in \mathcal{L}(V, W)$ is **invertible** if there exists linear map $S \in \mathcal{L}(W, V)$ such that $T \circ S = \mathrm{id}_W$ and $S \circ T = \mathrm{id}_V$. In this case, we denote S by T^{-1}.
   (a) Let $T \in \mathcal{L}(V, W)$. Show that its inverse T^{-1}, if it exists, is unique.
   (b) Show that $T \in \mathcal{L}(V, W)$ is invertible iff it is bijective, i.e., an isomorphism.
   (c) Give two examples of linear maps that are not invertible for two different reasons.

2. In class, we have shown that if two finite-dimensional vector spaces are of the same dimension, then they are isomorphic. Show the converse is also true.

3. Let $A \in M_{m\times n}(\mathbb{R})$ and consider the linear map T_A associated with A defined as $T_A : \mathbb{R}^n \to \mathbb{R}^m$ where $T_A(x) = Ax$. Show that
   (a) T_A is injective if there is a pivot in every col of rref(A).
   (b) T_A is surjective if there is a pivot in every row of rref(A).

4. Consider the following matrices:

$$A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & 1 \end{pmatrix}, \quad F = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 2 \\ 3 & 1 \\ 2 & -1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 & 3 \\ 1 & 1 & -2 \\ 2 & -1 & 1 \end{pmatrix}, \quad E = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

   Compute the following expressions if defined: AB, BA, D^2, B^2, DC, CB, BC, FE, EF, CE, EC.

5. Here are some facts about matrices that will come in handy in the future.
   (a) A square matrix $A \in M_n(\mathbb{R})$ is called **diagonal** if all of its entries that are not on the main diagonal are equal zero, that is, A is diagonal if $(A)_{ij} = 0$ for all $i \neq j$. Here is an example of a diagonal matrix:
$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$
   Prove that if $A, B \in M_n(\mathbb{R})$ are both diagonal then both $A + B$ and AB are diagonal as well.
   (b) For a square matrix $A \in M_n(\mathbb{R})$ the **trace** of A, denoted $tr(A)$, is the sum of all of its entries on the main diagonal, that is $tr(A) = \sum_{i=1}^{n} (A)_{ii}$. Here is an example of a trace computation:
$$tr \begin{pmatrix} 3 & 5 & 7 \\ 0 & 2 & 4 \\ 8 & 1 & -1 \end{pmatrix} = 3 + 2 + (-1) = 4.$$

i. For $A, B \in M_n(\mathbb{R})$ prove that $tr(AB) = tr(BA)$.

ii. Show that $tr(\cdot) : M_n(\mathbb{R}) \to \mathbb{R}$ is a linear map.

(c) For a square matrix $A \in M_n(\mathbb{R})$ the **transposed** of A, denoted A^T, is the matrix obtained by turning each row of A into a column by order, that is $(A^T)_{i,j} = (A)_{j,i}$. Here is an example of a transposed computation:

$$\begin{pmatrix} 3 & 5 & 7 \\ 0 & 2 & 4 \\ 8 & 1 & -1 \end{pmatrix}^T = \begin{pmatrix} 3 & 0 & 8 \\ 5 & 2 & 1 \\ 7 & 4 & -1 \end{pmatrix}.$$

For $A, B \in M_n(\mathbb{R})$ prove that $(AB)^T = B^T A^T$.

(d) Let $A \in M_{m \times n}(\mathbb{R})$, $B \in M_{n \times k}(\mathbb{R})$, and $C \in M_{k \times \ell}(\mathbb{R})$. Prove that

$$A(BC) = (AB)C.$$

6. Consider each of the following there is a claim, which might be **true or false**. If the claim is true then prove it, and if it is false then provide a counterexample. (For counter examples you may choose any n you wish, but if you want to prove a claim then you should prove it for all possible n's).

   (a) If $A \in M_n(\mathbb{R})$ satisfies $A^2 = 0$ then $A = 0$. (Here 0 is the zero matrix).

   (b) If $A, B \in M_n(\mathbb{R})$ are such that $AB = BA$ then $AB^2 = B^2 A$.

   (c) Let $A, B, C \in M_n(\mathbb{R})$. If $AB = CB$ then $A = C$.

   (d) Let $A \in M_n(\mathbb{R})$, then $(A + I)^2 = A^2 + 2A + I$.

   (e) Let $A, B \in M_n(\mathbb{R})$, then $(A + B)^2 = A^2 + 2AB + B^2$.

7. The following claims are either **true or false**. Determine which case is it for each claim and prove your answer.

   (a) For any two matrices $A \in M_{m \times n}(\mathbb{R})$ and $B \in M_{n \times k}(\mathbb{R})$ we have $\text{rank}(AB) = \text{rank}(A) \cdot \text{rank}(B)$.

   (b) If A is a square matrix then its column space is equal to its null space.

   (c) If $A \in M_{m \times n}(\mathbb{R})$ is such that the vectors in its rows are linearly independent then the vectors in its columns are also linearly independent.

   (d) If $A \in M_n(\mathbb{R})$ is such that the vectors in its rows are linearly independent then the vectors in its columns are also linearly independent

8. For any two matrices $A \in M_{m \times n}(\mathbb{R})$ and $B \in M_{n \times k}(\mathbb{R})$ which satisfy $AB = 0$ prove that $\text{rank}(B) + \text{rank}(A) \leq n$.

9. The following claims are either **true or false**. Determine which case is it for each claim and prove your answer.

   (a) For any two $m \times n$ matrices A and B we have $\text{rank}(A + B) = \text{rank}(A) + \text{rank}(B)$.

   (b) For any two $m \times n$ matrices A and B we have $\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B)$.