Question 4

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4. Let V, W be vector spaces over $\mathbb{R}$. Consider $(\mathcal{L}(V,W), +, \cdot)$ where $+$ is defined as

$$(f_1 + f_2)(v) = f_1(v) + f_2(v), \forall v \in V$$

for any two linear maps $f_1, f_2 \in \mathcal{L}(V,W)$ and $\cdot$ is defined to be

$$(kf)(v) = k(f(v), \forall v \in V$$

where k is any scalar from $\mathbb{R}$ and f is any linear map in $\mathcal{L}(V,W)$. Show that $(\mathcal{L}(V,W), +, \cdot)$ is a vector space over $\mathbb{R}$.

The vector Space $(\mathcal{L}(\bar{V},\bar{W}), +, \cdot)$.

① Closed under Addition

let f_1 and $f_2 \in \mathcal{L}(\bar{V},\bar{W})$, The v.s. Then

$$f(\vec{v}) + f_2(\vec{v}) = (f_1 + f_2)(\vec{v}) \in \mathcal{L}(\bar{V},\bar{W})$$

$$(f_1 + f_2)\left(\sum c_i v_i\right) \Rightarrow f_1\left(\sum c_i v_i\right) + f_2\left(\sum c_i v_i\right) \in \mathcal{L}(\bar{V},\bar{W})$$

② Commutative

$f_1 + f_2 \in \mathcal{L}(\bar{V},\bar{W})$

$f_1(\vec{v}) + f_2(\vec{v}) = f_2(\vec{v}) + f_1(\vec{v})$

③ associative

$f_1, f_2, f_3 \in \mathcal{L}(\bar{V},\bar{W})$

$f_1 + (f_2 + f_3) = (f_1 + f_2) + f_3$

④ $\exists\ \vec{0} \in \mathcal{L}(\bar{V},\bar{W})$. because $\forall f \in \mathcal{L}(\bar{V},\bar{W})$

consider $f_0 \in \mathcal{L}(\bar{V}, \bar{\omega})$ where

$$f_0 : \bar{V} \to \bar{\omega}$$
$$\vec{v} \mapsto 0 \cdot \vec{v} \Rightarrow \vec{0}_\omega$$

Then for any $f \in \mathcal{L}(\bar{V}, \bar{\omega})$

$$f(\vec{v}) + f_0(\vec{v}) = f(\vec{v})$$

⑤ additive inverse where

$\forall f \in \mathcal{L}(\bar{V}, \bar{\omega})$, $\exists\ f_0(\vec{v}) = -f(\vec{v})$

s.t. $f(\vec{v}) + f_0(\vec{v}) = \vec{0}$

⑥ $\forall \alpha \in \mathbb{R}$ and $\forall$ $f \in \mathcal{L}(\bar{V}, \bar{\omega})$

$\alpha f(\vec{v}) \in \mathcal{L}(\bar{V}, \bar{\omega})$

⑦ $f_1, f_2 \in \mathcal{L}(\bar{V}, \bar{\omega})$ and $\forall \alpha \in \mathbb{R}$

$(f_1 + f_2)\alpha = \alpha f_1 + \alpha f_2$

⑧ $\alpha, \beta \in \mathbb{R}$ and $f \in \mathcal{L}(\bar{V}, \bar{\omega})$

$(\alpha + \beta) f = \alpha f + \beta f$

⑨ $\alpha, \beta \in \mathbb{R}$ and $f \in \mathcal{L}(\bar{V}, \bar{\omega})$

$(\alpha \beta) \cdot f = \alpha (\beta f)$

⑩ $\exists$ The multiplicative identity: 1

s.t. $\forall f \in \mathcal{L}(\bar{V}, \bar{\omega})$

$$1 \cdot f(\vec{v}) = f(\vec{v})$$