HW 6 - Phys 3143

23 a $\hat{A}, \hat{P}$ operators, $\hat{C}$ Scalar

$[A*B, C] = A*BC - CA*B$

add in identity $A*CB = AC*B$

$A*BC - CA*B = A*CB + AC*B$

$A*BC - A*CB \quad -CA*B + AC*B \Rightarrow AC*B - CA*B$

$\Rightarrow A*[B,C] + [A,C]*B$

b. $[A\cdot B, a\cdot \mathcal{L}] \Rightarrow A\cdot[B, a\cdot \mathcal{L}] + [A, a\cdot \mathcal{L}]\cdot B$

$A\cdot(i\hbar(a\times B)) + (i\hbar(a\times A))\cdot B \Rightarrow i\hbar(A\cdot(a\times B) + (a\times A)\cdot B) = 0$

Since $\alpha\cdot(B\times\gamma) = (\alpha\times B)\cdot\gamma$

$\Rightarrow \alpha(B\times\gamma) - (\alpha\times B)\cdot\gamma = 0$

$\Rightarrow i\hbar(0) = 0 \quad \therefore$ commutes

24 a $[A\times B, a\cdot\mathcal{L}] = A\times[B, a\cdot\mathcal{L}] + [A, a\cdot\mathcal{L}]\times B \Rightarrow$

$A\times(i\hbar(a\times B)) + (i\hbar(a\times A))\times B \Rightarrow i\hbar(A\times(a\times B) + (a\times A)\times B)$

Consider identities: $A\times(a\times B) + a\times(B\times A) + B\times(A\times a) = 0$

$\Rightarrow a\times(B\times A) = -A\times(a\times B) - B\times(A\times a) \Rightarrow a\times(A\times B) = A\times(a\times B) + B\times(A\times a)$

$\therefore i\hbar(A\times(a\times B) + (-(-((a\times A)\times B)))) \Rightarrow \boxed{i\hbar(a\times(A\times B))}$

b) $[\mathcal{L}^2, A] = \alpha A + \beta A\times\mathcal{L}$

$\Rightarrow [\mathcal{L}*\mathcal{L}, A] = \mathcal{L}[\mathcal{L}, A] + [\mathcal{L}, A]\mathcal{L}$

$\mathcal{L}(i\hbar(a\times A)) + (i\hbar(a\times A))\mathcal{L} \Rightarrow i\hbar(\mathcal{L}\times(a\times A) + (a\times A)\times\mathcal{L})$

$\Rightarrow i\hbar(a\times(\mathcal{L}\times A))$ Note $\mathcal{L}\times A = 2i\hbar\hat{A} - A\times\mathcal{L}$

$i\hbar(a\times(2i\hbar\hat{A} - A\times\mathcal{L}))$

$$\boxed{\begin{aligned}\alpha &= -2\hbar^2\\ \beta &= -i\hbar\end{aligned}}$$

$S_y = \frac{1}{2i}(S_+ - S_-)$ $\qquad$ $S_+|{+z}\rangle = |{+z}\rangle$ $\qquad$ $S_+ = S_x + iS_y$

$S_-|{+z}\rangle = |{-z}\rangle$ $\qquad$ $S_- = S_x - iS_y$

$\frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix} + \frac{i\hbar}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix}$

$\frac{\hbar}{2}\left[\begin{pmatrix}0&1\\1&0\end{pmatrix} + \begin{pmatrix}0&1\\-1&0\end{pmatrix}\right]$

$\frac{\hbar}{2}\begin{pmatrix}1&1\\-1&1\end{pmatrix} = S_+$

$\frac{\hbar}{2}\begin{pmatrix}1&-1\\1&1\end{pmatrix} = S_-$

$\frac{\hbar}{2}\left(\begin{pmatrix}1&1\\-1&1\end{pmatrix} - \begin{pmatrix}1&-1\\1&1\end{pmatrix}\right)$

$\frac{\hbar}{2}\begin{pmatrix}0&2\\-2&0\end{pmatrix}$

$\frac{\hbar}{2i}\begin{pmatrix}0&1\\-1&0\end{pmatrix}$

25 $R(\phi z)\,R(\theta y)\,|{\pm z}\rangle$

for any $R(\alpha n) = e^{-i\alpha S_n/\hbar} = \cos\frac{\alpha}{2}\mathbb{1} - i\sin\frac{\alpha}{2}\left(\frac{2}{\hbar}S_n\right)$

$R(\theta y) = e^{-i\theta S_y/\hbar} = \cos\frac{\theta}{2}\mathbb{1} - i\sin\frac{\theta}{2}\left(\frac{2}{\hbar}S_y\right)$

$R(\phi z) = e^{-i\phi S_z/\hbar} = \cos\frac{\phi}{2}\mathbb{1} - i\sin\frac{\phi}{2}\left(\frac{2}{\hbar}S_z\right)$.

$R(\phi z)R(\theta y)|{+z}\rangle = e^{-i\phi S_z/\hbar}\,e^{-i\theta S_y/\hbar}\,|{+z}\rangle$

$S_x = S_+ - iS_y$

$S_x = S_- + iS_y$ $\qquad$ $S_+ - iS_y = S_- + iS_y$

$S_+ - S_- = 2iS_y$

$\cancel{R(\phi z)R(\theta y)|{-z}\rangle = e^{+i\phi S_z/\hbar}\,e^{+i\theta S_y/\hbar}|{-z}\rangle}$

$\cancel{\left(\cos\frac{\phi}{2}\mathbb{1} - i\sin\frac{\phi}{2}\left(\frac{2}{\hbar}S_z\right)\right)\left(\cos\frac{\theta}{2}\mathbb{1} - i\sin\frac{\theta}{2}\left(\frac{2}{\hbar}S_y\right)\right)}$

$\cancel{\left(\cos\frac{\phi}{2}\cos\frac{\theta}{2} - i\sin\frac{\phi}{2}\cos\frac{\theta}{2}\left(\frac{2}{\hbar}\right)S_z - i\sin\frac{\theta}{2}\cos\frac{\phi}{2}\left(\frac{2}{\hbar}\right)S_y + i^2\sin\frac{\phi}{2}\sin\frac{\theta}{2}\left(\frac{4}{\hbar^2}\right)\overset{-S_x}{S_zS_y}\right)|{+z}\rangle}$

$\cancel{\cos\frac{\phi}{2}\cos\frac{\theta}{2} - i\left(\sin\frac{\phi}{2}\cos\frac{\theta}{2}\left(\frac{2}{\hbar}\right)S_z + \sin\frac{\theta}{2}\cos\frac{\phi}{2}\left(\frac{2}{\hbar}\right)S_y + \sin\frac{\phi}{2}\sin\frac{\theta}{2}\left(\frac{4}{\hbar^2}\right)S_x\right)|{+z}\rangle}$

$\cancel{\left(\cos\frac{\phi}{2}\cos\frac{\theta}{2} - i\left(\sin\frac{\phi}{2}\sin\frac{\theta}{2}\left(\frac{2}{\hbar}\right)\right)\right)|{+z}\rangle \therefore = R(\phi z)R(\theta y)|{+z}\rangle}$

$\cancel{R(\phi z)R(\theta y)|{-z}\rangle = \left(\cos\frac{\phi}{2}\cos\frac{\theta}{2} + i\left(\sin\frac{\phi}{2}\sin\frac{\theta}{2}\left(\frac{2}{\hbar}\right)\right)\right)|{-z}\rangle}$

$\left(\cos\frac{\theta}{2}\mathbb{1} - i\sin\frac{\theta}{2}\left(\frac{2}{\hbar}\right)S_y\right)|{\pm z}\rangle = |{\pm n}\rangle$

$\cos\frac{\theta}{2}|{+z}\rangle - i\sin\frac{\theta}{2}\left(\frac{2}{\hbar}\right)\left(\frac{1}{2i}\right)(S_+ - S_-)|{+z}\rangle$ $\cancel{\cos\frac{\theta}{2}|{+z}\rangle - i\sin\frac{\theta}{2}\left(\frac{2}{\hbar}\right)\left(\frac{1}{2i}\right)(S_+ - S_-)|{+z}\rangle}$

$\cos\frac{\theta}{2}\begin{bmatrix}1\\0\end{bmatrix} - i\sin\frac{\theta}{2}\left(\frac{2}{\hbar}\right)\left(\frac{\hbar}{2}\right)\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} \Rightarrow \cos\frac{\theta}{2}\begin{bmatrix}1\\0\end{bmatrix} + \sin\frac{\theta}{2}\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$

$= R(\theta y)|{+z}\rangle = \cos\frac{\theta}{2}|{+z}\rangle + \sin\frac{\theta}{2}|{-z}\rangle \rightarrow$ introduce $R(\phi z)$

$R(\phi z)R(\theta y)|{+z}\rangle = \left[\cos\frac{\phi}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix} - \underbrace{i\sin\frac{\phi}{2}\left(\frac{2}{\hbar}\right)\left(\frac{\hbar}{2}\right)\begin{pmatrix}1&0\\0&-1\end{pmatrix}}_{-i\sin\frac{\phi}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}S_z}\right]\left[\cos\frac{\theta}{2}|{+z}\rangle + \sin\frac{\theta}{2}|{-z}\rangle\right]$

$\Rightarrow \cos\frac{\phi}{2}\cos\frac{\theta}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} - i\sin\frac{\phi}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\cos\frac{\theta}{2}\begin{pmatrix}1\\0\end{pmatrix} + \cos\frac{\phi}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}\sin\frac{\theta}{2}\begin{pmatrix}0\\1\end{pmatrix} - i\sin\frac{\phi}{2}\sin\frac{\theta}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}$

$\Rightarrow \cos\frac{\phi}{2}\cos\frac{\theta}{2}|{+z}\rangle - i\sin\frac{\phi}{2}\cos\frac{\theta}{2}|{+z}\rangle + \cos\frac{\phi}{2}\sin\frac{\theta}{2}|{-z}\rangle - i\sin\frac{\phi}{2}\sin\frac{\theta}{2}\begin{pmatrix}0\\-1\end{pmatrix}]$ Pull out negative

$\left(\cos\frac{\phi}{2}\cos\frac{\theta}{2} - i\sin\frac{\phi}{2}\cos\frac{\theta}{2}\right)|{+z}\rangle + \left(\cos\frac{\phi}{2}\sin\frac{\theta}{2} + i\sin\frac{\phi}{2}\sin\frac{\theta}{2}\right)|{-z}\rangle$

$\left(\cos\frac{\theta}{2}\right)\underbrace{\left(\cos\frac{\phi}{2} - i\sin\frac{\phi}{2}\right)}_{e^{-i\phi}}|{+z}\rangle + \left(\sin\frac{\theta}{2}\right)\underbrace{\left(\cos\frac{\phi}{2} + i\sin\frac{\phi}{2}\right)}_{e^{i\phi}}|{-z}\rangle = |n\rangle$

$\cos\frac{\theta}{2}|{+z}\rangle + \sin\frac{\theta}{2}e^{i\phi}|{-z}\rangle$ PRve

The same follows for $|{-z}\rangle \therefore |{-n}\rangle$

$\Rightarrow \sin\frac{\theta}{2}|{+z}\rangle - e^{i\phi}\cos\frac{\theta}{2}|{-z}\rangle = |{-n}\rangle$

27a write projectors of $P_{m,n} = |m;n\rangle\langle m;n|$

$$\begin{bmatrix} P_{-1}+P_0+P_1 = \mathbb{1} \\ \hbar\,(P_1-P_{-1}) = S_n \\ \hbar^2\,(P_1+P_{-1}) = S_n^2 \end{bmatrix} \Rightarrow \begin{array}{cccc} P_{-1} & P_0 & P_1 & \vec{b} \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & \mathbb{1} \\ -\hbar & 0 & \hbar & S_n \\ \hbar^2 & 0 & \hbar^2 & S_n^2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & \mathbb{1} \\ 0 & \hbar & 2\hbar & S_n+\hbar\mathbb{1} \\ 0 & -\hbar^2 & 0 & S_n^2-\hbar^2\mathbb{1} \end{array}\right] \Rightarrow \cancel{\left[\begin{array}{ccc|c} 1 & 1 & 1 & \mathbb{1} \\ 0 & -\hbar^2 & 0 & S_n^2-\hbar^2\mathbb{1} \\ 0 & \hbar & 2\hbar & S_n+\hbar \end{array}\right]}$$

$$-\hbar^2 P_0 = \frac{S_n^2 - \hbar^2\mathbb{1}}{-\hbar^2} \qquad \boxed{P_0 = -\frac{1}{\hbar^2}S_n^2 + \mathbb{1}}$$

$$\hbar P_0 + 2\hbar P_1 = S_n + \hbar\mathbb{1}$$

$$P_1 = \frac{S_n + \hbar\mathbb{1} - \hbar P_0}{2\hbar} \Rightarrow \boxed{\frac{1}{2\hbar}S_n + \frac{1}{2}\mathbb{1} - \frac{1}{2}\left(-\frac{1}{\hbar^2}S_n^2 + \mathbb{1}\right)}$$

$$+\frac{1}{2\hbar^2}S_n^2 - \frac{1}{2}\mathbb{1}$$

$$\boxed{\frac{1}{2\hbar^2}S_n^2 + \frac{1}{2\hbar}S_n}$$

$$\boxed{\frac{1}{2\hbar}\left(\frac{S_n^2}{\hbar} + S_n\right) = P_1}$$

$$\boxed{P_{-1} = \mathbb{1} - \left[\frac{1}{2\hbar}S_n + \frac{1}{2}\mathbb{1} - \frac{1}{2}\left(-\frac{1}{\hbar^2}S_n^2 + \mathbb{1}\right)\right] - \left[-\frac{1}{\hbar^2}S_n^2 + \mathbb{1}\right]}$$

$$\cancel{\mathbb{1}} - \frac{1}{2\hbar}S_n - \cancel{\frac{1}{2}\mathbb{1}} - \frac{1}{2\hbar^2}S_n^2 + \cancel{\frac{1}{2}\mathbb{1}} + \frac{1}{\hbar^2}S_n^2 - \cancel{\mathbb{1}}$$

$$-\frac{1}{2\hbar}S_n - \frac{1}{2\hbar^2}S_n^2 + \frac{1}{\hbar^2}S_n^2 \Rightarrow -\frac{1}{2\hbar}S_n + \frac{2}{2\hbar^2}S_n^2 - \frac{1}{2\hbar^2}S_n^2$$

$$\Rightarrow \boxed{\frac{1}{2\hbar}\left(\frac{S_n^2}{\hbar} - S_n\right) = P_{-1}}$$

27B $R(\theta n) = e^{-i\theta \frac{S_n}{\hbar}} = \cos\frac{\theta}{2}\mathbb{1} - i\sin\left(\frac{\theta}{2}\right)\left(\frac{2}{\hbar}\right)(S_n)$

$R(\theta n)\mathbb{1} = e^{-i\theta S_n/\hbar}(P_{-1}+P_0+P_1) \Rightarrow \left[\cos\frac{\theta}{2}\mathbb{1} - i\sin\frac{\theta}{2}\left(\frac{2}{\hbar}\right)S_n\right]\left(\frac{1}{2\hbar}\left(\frac{S_n^2}{\hbar} - S_n\right) + \mathbb{1} - \frac{S_n^2}{\hbar^2} + \frac{1}{2\hbar}\left(S_n^2/\hbar + S_n\right)\right)$ X

$\cos\frac{\theta}{2}\mathbb{1} - i\sin\frac{\theta}{2}\left(\frac{2}{\hbar}\right)\left(\hbar(P_1 - P_{-1})\right) \Rightarrow \cos\frac{\theta}{2}\mathbb{1} - i\sin\frac{\theta}{2}(2)\left(\frac{1}{2\hbar}\left(\frac{S_n^2}{\hbar} + S_n\right) - \frac{1}{2\hbar}\left(S_n^2/\hbar - S_n\right)\right)$

26 a $\langle J_n \rangle_{M;z} = \langle j, M; z | \hat{J}_n | j, M; z \rangle$

$\hat{J}_n = \frac{1}{2}(n_+ J_- + n_- J_+) + n_z J_z$

$\langle j, M; z | \frac{1}{2}(n_+ J_- + n_- J_+) + n_z J_z | j, M; z \rangle$

$|j, M; z\rangle$ $\cancel{\frac{1}{2}\left(n_+ \hbar\sqrt{j(j+1) - M(M-1)}\, |j, M-1; z\rangle + n_- \hbar\sqrt{j(j+1) - M(M+1)}\, |j, M+1; z\rangle\right) + n_z \hbar M |j, M; z\rangle}$

$$\underbrace{\langle j, M; z | \tfrac{1}{2} n_+ \underbrace{J_- |j, M; z\rangle}_{\hbar\sqrt{j(j+1) - M(M+1)}\,|j, M+1; z\rangle}}_{= 0} + \underbrace{\langle j, M; z | \tfrac{1}{2} n_- \underbrace{J_+ |j, M; z\rangle}_{\leftarrow \text{Same}}}_{= 0} + \underbrace{\langle j, M; z | n_z \underbrace{J_z | j, M; z\rangle}_{\hbar M}}_{\langle j, M; z | n_z \hbar M | j, M; z\rangle}$$

$$= \boxed{\hbar M}$$

b) $\langle J_n^2 \rangle_{M;z} = \langle j, M; z | J_n^2 | j, M; z \rangle$ Since $J^2 |j, M; z\rangle = \hbar^2 j(j+1) |j, M; z\rangle$

Thus bra-ket unit vectors match

$\langle J_n^2 \rangle = \boxed{\hbar^2 j(j+1)}$