HW2 - Rudra Goel

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i.

$$\begin{cases} 5x + 4y - z = 0 \\ 2x - 4y + z = 1 \\ -7x - 14y + 5z = 10 \end{cases}$$

..

$$\left[\begin{array}{ccc|c} 5 & 4 & -1 & 0 \\ 2 & -4 & 1 & 1 \\ -7 & -14 & 5 & 10 \end{array}\right]$$

$R_1 \to R_1\left(\frac{1}{5}\right)$

$$\left[\begin{array}{ccc|c} 1 & 4/5 & -1/5 & 0 \\ 2 & -4 & 1 & 1 \\ -7 & -14 & 5 & 10 \end{array}\right]$$

$-\frac{8}{5} - \frac{20}{5}$

$\frac{2}{5} + \frac{5}{5}$

$R_2 \leftarrow R_2 + (-2)R_1$

$$\left[\begin{array}{ccc|c} 1 & 4/5 & -1/5 & 0 \\ 0 & -\frac{28}{5} & 7/5 & 1 \\ -7 & -14 & 5 & 10 \end{array}\right]$$

$R_3 \to R_3 + 7R_1$

$$\left[\begin{array}{ccc|c} 1 & 4/5 & -1/5 & 0 \\ 0 & -\frac{28}{5} & 7/5 & 1 \\ 0 & -\frac{42}{5} & \frac{18}{5} & 10 \end{array}\right]$$

$\frac{28}{5} + \frac{-70}{5}$

$-\frac{7}{5} + \frac{25}{5}$

$$R_2 \leftarrow R_2\left(-\frac{5}{28}\right)$$

$$\frac{7}{5} \cdot \frac{-5}{28} = -\frac{7}{28}$$

$$\left[\begin{array}{ccc|c} 1 & 4/5 & -1/5 & 0 \\ 0 & 1 & -7/28 & -5/28 \\ 0 & -42/5 & 18/5 & 10 \end{array}\right]$$

$$R_3 \leftarrow R_3 + \frac{42}{5} R_1$$

$$-\frac{1}{4}\left(\frac{42}{5}\right)$$

$$\frac{18}{5} + \frac{-42}{20}$$

$$\frac{36}{10} + \frac{-21}{10} =$$

$$\frac{15}{10}$$

$$\frac{3}{2}$$

$$\left[\begin{array}{ccc|c} 1 & 4/5 & -1/5 & 0 \\ 0 & 1 & -1/4 & -5/28 \\ 0 & 0 & 3/2 & 17/2 \end{array}\right]$$

$$\frac{42}{5}\left(-\frac{5}{28}\right)$$

$$\frac{-42}{28}$$

$$-\frac{6}{4} = -\frac{3}{2} + \frac{20}{2}$$

$$\frac{17}{2}$$

$$\downarrow \; R_3 \to 2R_3$$

$$\left[\begin{array}{ccc|c} 1 & 4/5 & -1/5 & 0 \\ 0 & 1 & -1/4 & -5/28 \\ 0 & 0 & 3 & 17 \end{array}\right]$$

Pivots

$$z = \boxed{\frac{17}{3}}$$

$$y - \frac{1}{4}\left(\frac{17}{3}\right) = -\frac{5}{28}$$

$$y = -\frac{5}{28} + \frac{17}{12}$$

$$\frac{-15}{84} + \frac{119}{84}$$

$$\frac{104}{84} = \boxed{\frac{26}{21}}$$

$$x + \left(\frac{4}{5}\right)\left(\frac{26}{21}\right) - \left(\frac{1}{5}\right)\left(\frac{17}{3}\right) = 0$$

$$x = \frac{17}{15} - \frac{104}{105}$$

$$\frac{119 - 104}{105} = \frac{15}{105} = \boxed{\frac{1}{7}}$$

One Unique Solution

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1/7 \\ 26/21 \\ 17/3 \end{bmatrix}$$

ii.

$$\begin{cases} 2x + y + 2z = 2 \\ -x + y - z = 2 \\ 3x + 2y + z = 2 \\ 5x + 4y - z = 2 \end{cases}$$

...

$$\left[\begin{array}{ccc|c} 2 & 1 & 2 & 2 \\ -1 & 1 & -1 & 2 \\ 3 & 2 & 1 & 2 \\ 5 & 4 & -1 & 2 \end{array}\right] \xrightarrow{R_1 = \frac{R_1}{2}} \left[\begin{array}{ccc|c} 1 & 1/2 & 1 & 1 \\ -1 & 1 & -1 & 2 \\ 3 & 2 & 1 & 2 \\ 5 & 4 & -1 & 2 \end{array}\right]$$

$R_2 = R_1 + R_2$

$$\left[\begin{array}{ccc|c} 1 & 1/2 & 1 & 1 \\ 0 & 3/2 & 0 & 3 \\ 3 & 2 & 1 & 2 \\ 5 & 4 & -1 & 2 \end{array}\right] \xrightarrow{R_2 = R_2/3} \left[\begin{array}{ccc|c} 1 & 1/2 & 1 & 1 \\ 0 & 1/2 & 0 & 1 \\ 3 & 2 & 1 & 2 \\ 5 & 4 & -1 & 2 \end{array}\right]$$

$R_1 = R_2 + R_1$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1/2 & 0 & 1 \\ 3 & 2 & 1 & 2 \\ 5 & 4 & -1 & 2 \end{array}\right] \quad R_3 = -3R_1 + R_3 \quad \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1/2 & 0 & 1 \\ 0 & 2 & -2 & 2 \\ 5 & 4 & -1 & 2 \end{array}\right]$$

$R_4 - -5R_1 + R_4$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & \frac{1}{2} & 0 & 1 \\ 0 & 2 & -2 & 2 \\ 0 & 4 & -6 & 2 \end{array}\right] \xrightarrow{R_2 = 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 2 & -2 & 2 \\ 0 & 4 & -6 & 8 \end{array}\right]$$

$R_3 = R_3 + -2R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -2 & -2 \\ 0 & 4 & -6 & 2 \end{array}\right] \xrightarrow{R_4 = R_4 - 4R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & -6 & -6 \end{array}\right]$$

$R_3 = -\frac{1}{2}R_3$ & $R_4 = -\frac{1}{6}R_4$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$z = 1$ one
$y = 2$ particular
$x = -1$ solution

iii)
$$\begin{cases} x_1 - x_2 + 2x_3 = 0 \\ 2x_1 - 2x_2 + 4x_3 + x_4 + 2x_5 = 4 \\ 3x_1 + x_2 + 6x_3 + x_5 = -3 \\ x_1 + 2x_3 + 2x_4 + x_5 = 4 \end{cases}$$

$$\left[\begin{array}{ccccc|c} 1 & -1 & 2 & 0 & 0 & 0 \\ 2 & -2 & 4 & 1 & 2 & 4 \\ 3 & 1 & 6 & 0 & 1 & -3 \\ 1 & 0 & 2 & 2 & 1 & 4 \end{array}\right]$$

L $R_4 = R_4 + R_1$

$$\left[\begin{array}{ccccc|c} 1 & -1 & 2 & 0 & 0 & 0 \\ 2 & -2 & 4 & 1 & 2 & 4 \\ 3 & 1 & 6 & 0 & 1 & -3 \\ 0 & 1 & 0 & 2 & 1 & 4 \end{array}\right]$$

$R_2 = R_2 - 2R_1 \wedge R_3 = R_3 + -3R_1$

$$\left[\begin{array}{ccccc|c} 1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 4 \\ 0 & 4 & 0 & 0 & 1 & -3 \\ 0 & 1 & 0 & 2 & 1 & 4 \end{array}\right]$$

Swith R_2 + R_4

$$\left[\begin{array}{ccccc|c} 1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 4 \\ 0 & 4 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 & 2 & 4 \end{array}\right]$$

$R_3 = -4R_2 + R_3$

$$\left[\begin{array}{ccccc|c} 1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 4 \\ 0 & 0 & 0 & -8 & -3 & -19 \\ 0 & 0 & 0 & 1 & 2 & 4 \end{array}\right]$$

Swith R_3 + R_4

$$\left[\begin{array}{ccccc|c} 1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 4 \\ 0 & 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & -8 & -3 & -19 \end{array}\right]$$

$R_4 = R_4 + 8R_3$

$$\left[\begin{array}{ccccc|c} 1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 4 \\ 0 & 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 13 & 13 \end{array}\right]$$

$R_4 = \frac{1}{13} R_4$

free variable

$$\left[\begin{array}{ccccc|c} 1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 4 \\ 0 & 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

pivots

To get to RREF

$R_3 = R_3 - 2R_4$

$$\left[\begin{array}{ccccc|c} 1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 4 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

$R_2 = R_2 - R_4 \quad \wedge \quad R_2 = R_2 - 2R_3$

$$\left[\begin{array}{ccccc|c} 1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

$R_1 = R_1 + R_2$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

let $x_3 = s$

$x_1 = -2s - 1$

$x_2 = -1$

$x_3 = s$

$x_4 = 2$

$x_5 = 1$

where $s \in \mathbb{R}$

$$x_1 + 2x_3 = -1 \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = s\begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

Infinite Solutions

$$\begin{cases} x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 = 5 \\ x_1 + 4x_2 + 6x_3 + 8x_4 + 10x_5 = 10 \\ x_1 - x_2 + x_3 - x_4 + x_5 = 0 \\ -x_1 + 4x_2 + x_3 + 6x_4 + 3x_5 = 5 \end{cases}$$

$$\left[\begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & 5 \\ 1 & 4 & 6 & 8 & 10 & 10 \\ 1 & -1 & 1 & -1 & 1 & 0 \\ -1 & 4 & 1 & 6 & 3 & 5 \end{array}\right]$$

$R_2 = R_2 - 2R_1$

$$\left[\begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & 5 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 1 & 0 \\ -1 & 4 & 1 & 6 & 3 & 5 \end{array}\right]$$

Switch R_1 & R_2 and $R_1 = -R_1$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & 5 & 5 \\ 1 & -1 & 1 & -1 & 1 & 0 \\ -1 & 4 & 1 & 6 & 3 & 5 \end{array}\right]$$

$R_2 = R_2 - R_1$

$R_3 = R_3 - R_1$

$R_4 = R_4 + R_1$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right.$$

$R_2 = R_2 + 2R_2$

$$\left[\begin{array}{ccccc|c} 0 & 2 & 3 & 4 & 5 & 5 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & 4 & 1 & 6 & 3 & 5 \end{array}\right]$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 2 & 7 & 5 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & 4 & 1 & 6 & 3 & 5 \end{array}\right] \quad R_4 = R_4 + 4R_3$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 2 & 7 & 5 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 5 & 2 & 7 & 5 \end{array}\right] \quad R_4 = R_4 - R_2$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 2 & 7 & 5 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \text{Switch } R_2 \,\&\, R_3$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 5 & 2 & 7 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} R_3 = \frac{1}{5}R_3 \\ \wedge \\ R_2 = -R_2 \end{array}$$

Infinite solutions

$$\begin{array}{cccccc} & & & s & t & \\ \end{array}$$
$$\left[\begin{array}{ccccc|c} \boxed{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \boxed{1} & -1 & 1 & -1 & 0 \\ 0 & 0 & \boxed{1} & 2/5 & 7/5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

free variables

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ -7/5 \\ -2/5 \\ 1 \end{bmatrix} + t\begin{bmatrix} 0 \\ -2/5 \\ -7/5 \\ 0 \end{bmatrix}$$

a.

$$\begin{cases} x + y + az = -1 \\ -x + (a-1)y + (2-a)z = a+1 \\ 6x + (5a+6)y + (7a+7)z = a^2 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & a & -1 \\ -1 & (a-1) & (2-a) & a+1 \\ 6 & (5a+6) & (7a+7) & a^2 \end{array}\right]$$

$R_2 = R_2 + R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & a & -1 \\ 0 & a & 2 & a \\ 6 & 5a+6 & 7a+7 & a^2 \end{array}\right]$$

$R_3 = R_3 - 6R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & a & -1 \\ 0 & a & 2 & a \\ 0 & 5a & a+7 & a^2+6 \end{array}\right]$$

$R_3 = R_3 - 5R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & a & -1 \\ 0 & a & 2 & a \\ 0 & 0 & a-3 & a^2-5a+6 \end{array}\right]$$

for when $a=3$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & -1 \\ 0 & 3 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

z becomes a free variable & ∞ sols

for when $a=0$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -3 & 6 \end{array}\right] \quad R_3 = 2R_3 \quad \left[\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -6 & 12 \end{array}\right]$$

$R_3 = R_3 + 3R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 12 \end{array}\right] \rightarrow$$

No sols since it is inconsistent

i) $a=0$

ii) $a \neq 0, 3$

iii) $a=3$

$$\begin{cases} (a+1)x + ay - az = 2 + a \\ (a+1)x + (a+2)y - (a+2)z = a+4 \\ (a+1)x + ay + (a^2-6)z = a^2 - 2a + 4 \\ (2a+2)x + 2ay + (a^2-a-6)z = a^2 - a + 6 \end{cases}$$

$$\left[\begin{array}{ccc|c} (a+1) & a & -a & (2+a) \\ (a+1) & (a+2) & -(a+2) & (a+4) \\ (a+1) & a & a^2-6 & a^2-2a+4 \\ 2a+2 & 2a & a^2-a-6 & a^2-a+6 \end{array}\right]$$

$-a - 2 + a$

$a + a - 2 - a$

$R_2 = R_2 - R_1$ and $R_3 = R_3 - R_1$ and $R_4 = R_4 - 2R_1$

$$\left[\begin{array}{ccc|c} a+1 & a & -a & 2+a \\ 0 & 2 & -2 & 2 \\ 0 & 0 & \overset{(a+3)(a-2)}{a^2+a-6} & \overset{(a-1)(a-2)}{a^2-3a+2} \\ 0 & 0 & \overset{(a+3)(a-2)}{a^2+a-6} & \underset{(a-2)(a-1)}{a^2-3a+2} \end{array}\right]$$

$-5 \quad -4$

for $a = -3$

$$\left[\begin{array}{ccc|c} -2 & -3 & 3 & -1 \\ 0 & 2 & -2 & 2 \\ 0 & 0 & 0 & 20 \\ 0 & 0 & 0 & 20 \end{array}\right]$$

inconsistent & NO Sols

0

for $a = 2$

$$\left[\begin{array}{ccc|c} \underline{3} & 2 & -2 & 4 \\ 0 & \underline{2} & -2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

pivots < # vars

free variable

$\therefore$ ∞ sols

for $a = -1$

for $a = -1$

$$\left[\begin{array}{ccc|c} 0 & -1 & 1 & 1 \\ 0 & 1 & -1 & 3 \\ 0 & -1 & -5 & 7 \\ 0 & -2 & -4 & 8 \end{array}\right]$$

free variable

$\emptyset$ sols

i) $a = -3$

ii) $a \neq -3, 2, -1$

iii) $a = 2, -1$

3. Let $A \in M_{m\times n}(\mathbb{R})$ and denote

$$L(A) := \{b \in \mathbb{R}^m : (A|b) \text{ has at least one solution }\}.$$

Prove

i. If $b \in L(A)$ and $d \in L(A)$ then $b + d \in L(A)$.

Because The linear System b is an element of $\mathcal{L}(A)$, it lives in $\mathbb{R}^m$, and likewise for The linear system d.

Since The Sum of two vectors does Not change the size or Their dimension, $b+d$ is still part of R^m thus making it an element of $\mathcal{L}(A)$.

let $S_b = \{x \in \mathbb{R}^n \mid Ax = b\}$ and

$$S_d = \{x \in \mathbb{R}^n \mid Ax = d\}$$

$x_1 \in S_b$ and $x_2 \in S_d$. Since x_1 and x_2 are Solutions $Ax_1 + Ax_2 = b+d$

By factoring out A, $A(x_1 + x_2) = b+d$

and since $x_1 + x_2$ is a Solution to $(A|b+d)$

$b+d$ exists in $\mathcal{L}(A)$

ii. If $b \in L(A)$ and $t \in \mathbb{R}$ then $t \cdot b \in L(A)$

if b is a linear system in $\mathcal{L}(A)$, then it has m dimensions.

The Scalar multiple of $t \cdot b$ is also an element of $\mathcal{L}(A)$ since the dimensions of b are not changed and remain as a dimension of m.

let $S_b = \{ x \in R^n \mid Ax = b \}$ and $x_1 \in S_b$

$$t \cdot A x_1 = t \cdot b$$

meaning $A(t \cdot x_1) = t \cdot b$

$\therefore$ $t \cdot b$ has one solution $(t \cdot x_1)$ and thus $t \cdot b \in \mathcal{L}(A)$

4. In each of the following parts appears a description of a matrix $A \in M_{m \times n}$. Determine whether or not such a matrix exists (where m and n can be any numbers you want; try to work with small numbers). If you claim that such a matrix exists then provide a concrete example. If you claim that such a matrix does not exist then explain carefully why this is true.

   a. For **every** $b \in \mathbb{R}^n$ there exist infinitely many solutions to the linear system $(A|b)$. $b \in R^n$

a) Yes, such a matrix can exist where n > m, and there are less pivots than there are variables to the system where for every possible value of b, there are infinite solutions

free variable

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ and $\forall b \in R^3$, there are infinite solutions

b) let $b_1 \in R^3$ $\wedge$ $b_2 \in R^3$

$$\left[\begin{array}{cccc|c} 1 & 3 & 2 & 5 & 0 \\ 0 & 0 & 4 & 10 & 0 \end{array}\right] \quad \left[\begin{array}{cccc|c} 1 & 3 & 2 & 5 & 0 \\ 0 & 0 & 4 & 10 & 0 \end{array}\right]$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 & | & 7 \end{bmatrix}$$

Infinite sols

Inconsistence & No Solution

matrix Can exist

c) c. There exists $b_1 \in \mathbb{R}^n$ such that the system $(A|b_1)$ has exactly one solution, there exists $b_2 \in \mathbb{R}^n$ such that the system $(A|b_2)$ has an infinite amount of solutions and there exists $b_3 \in \mathbb{R}^n$ such that the system $(A|b_3)$ has no solution.

Such a matrix cannot exist. By definition, for a linear system to have one particular solution, the number of pivots within the augmented matrix must equal the number of unknown variables (n)

$$\begin{array}{cccc} & & & b_1 \\ \left[\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right| & & & \left.\begin{matrix} \# \\ \# \\ \# \end{matrix}\right] \end{array}$$

However, for the same augmented matrix, except with the coefficient matrix b2, to have infinite solutions, the number of pivots would have to change such that there are less pivots than the number of unknown variables, which is impossible without changing matrix A (either by adding more variables or by adding more equations, which in itself changes the linear system entirely.

d) d. The homogeneous system $(A|0)$ has exactly one solution and the echelon form of A has a row of zeroes.

"Trivial Solution"

One Solution

$$\left[\begin{array}{ccc|c} 3 & 2 & 3 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

for the linear System

$$\begin{cases} 3x + 2y + 3z = 0 \\ 4y + z = 0 \end{cases}$$

Such a matrix

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$t = 0$

$6x + 4y + 6z = 0$

exists

e) e. The echelon form of the matrix has a row of zeroes and for every $b \in \mathbb{R}^n$ there exists exactly one solution to the linear system $(A|b)$.

Such a matrix cannot exist because since there is a row of zeros, there must be an associated 0 as the last element in the constant matrix b for there to exist a particular solution. If for every possible value of the constant matrix b, there can exist some matrix where the final row of zeros on the LHS of the augmented matrix is associated with a nonzero value in the RHS of the matrix making the system inconsistent and thus having no solutions

f) f. The 3-tuple $(1, 2, 3)$ is a solution to the homogeneous system $(A|0)$ and the echelon form of A has at least three rows which are not zero rows.

Such a matrix cannot exist because for the 3rd row, there is only one non zero element multiplied by one of the parts of the solution (1, 2, or 3) and there is no non-zero coefficient when multiplied by 1, 2, or 3 to produce a product of 0 to satisfy the homogenous system of equations.

G) g. The 3-tuple $(1, 2, 3)$ is a solution to the homogeneous system $(A|0)$ and the echelon form of A has at least two rows which are not zero rows.

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{3} & 0 \end{array}\right]$$

Such a matrix can exist

H) h. The set of solutions of the homogeneous system $(A|0)$ is exactly the set $\{(s + t, s, t) : s, t \in \mathbb{R}\}$.

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 2 & -2 & -2 & 0 \\ 1 & -1 & -1 & 0 \end{array}\right]$$

Exists

$$(s+t) - s - t = 0$$

$$2s+2t - 2s - 2t = 0$$

$$s+t - s - t = 0$$

i. The set of solutions of the homogeneous system $(A|0)$ is exactly the set $\{(s+t, s-t) : s, t \in \mathbb{R}\}$.

Such a matrix does exist

$$\begin{bmatrix} x \\ y \end{bmatrix} = s\begin{pmatrix} 1 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

if you parameterize both x & y

the solution set becomes

$\{R^2\}$