Question 5

Monday, October 23, 2023 6:22 PM

5. True or false. Remember to justify your answer.
   (a) There exists a non-zero upper-triangular matrix $A \in M_2(\mathbb{R})$ such that A^2 is the zero matrix.
   (b) Let $A \in M_n(\mathbb{R})$. If $AB = BA$ for every $B \in M_n(\mathbb{R})$ then $A = \lambda I_n$ for some $\lambda \in \mathbb{R}$.
   (c) Let $A, B \in M_n(\mathbb{R})$. Then AB is invertible if and only if both A and B are invertible.
   (d) Let $A \in M_n(\mathbb{R})$. A is NOT invertible if and only if there exists $B \in M_n(\mathbb{R})$ such that $AB = 0$.
   (e) Let $A, B \in M_n(\mathbb{R})$. If both A and B are invertible then $AB = BA$.
   (f) Let $A \in M_n(\mathbb{R})$. If A is invertible then $A + I$ is also invertible.
   (g) If $A^2 - I$ is invertible then $A - I$ is invertible.

a) $\begin{bmatrix} x & y \\ 0 & z \end{bmatrix}\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} = \begin{bmatrix} x^2 & xy+yz \\ 0 & z^2 \end{bmatrix}$

$x^2 = 0$ $\quad x = 0$

$xy + yz = 0$ $\quad z = 0 \quad y \in \mathbb{R}$ ✓

$z^2 = 0$

$\begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$ ✓

TRUE

(b) Let $A \in M_n(\mathbb{R})$. If $AB = BA$ for every $B \in M_n(\mathbb{R})$ then $A = \lambda I_n$ for some $\lambda \in \mathbb{R}$.

let $B = E_{i,j}$ The matrix that is 1 @ i,j and zero everywhere else.

Then $AE_{ij} = \begin{pmatrix} | & & | & & | \\ 0 & \cdots & b_{\cdot,i} & \cdots & 0 \\ | & & | & & | \end{pmatrix}$

$\begin{array}{cc} & 1,2 \\ \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} & \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \end{array} = \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix}$

$E_{ij}A = \begin{pmatrix} — & 0 & — \\ — & b_{j,\cdot} & — \\ — & 0 & — \end{pmatrix}$

if $AE_{ij} = E_{ij}A$, then $a_{,i} = a_{j,}$

$\therefore$ True for Diagonal matrix

$E_{i,i}A = AE_{ii}$ $\quad \therefore = \lambda I_n$

(c) Let $A, B \in M_n(\mathbb{R})$. Then AB is invertible if and only if both A and B are invertible.

$AB(AB)^{-1} = I_n$ $\quad (AB)^{-1}AB$ $\quad \Longrightarrow$

$A\underbrace{BB^{-1}}_{I_n}A^{-1} = I_n$ $\quad B^{-1}\underbrace{A^{-1}A}_{I_n}B = I_n$

$I_n \Rightarrow \underbrace{AI_n}A^{-1}$ $\quad B^{-1}B = in$

$\Rightarrow AA^{-1} = I_n$ $\therefore$ True

if A and B inv $\Longleftarrow$

Then AB is invertible

$AA^{-1} = I_n$

$I_n \Rightarrow \underbrace{AB}\underbrace{B^{-1}A^{-1}} = I_n$

$(AB)(AB)^{-1} \to AB$ is invertible

(d) Let $A \in M_n(\mathbb{R})$. A is NOT invertible if and only if there exists $B \in M_n(\mathbb{R})$ such that $AB = 0$.

False

Consider $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $B = [0]$ zero matrix

$AB = 0$ and A is invertible.

(e) Let $A, B \in M_n(\mathbb{R})$. If both A and B are invertible then $AB = BA$.

false $\quad A = \begin{bmatrix} 2 & 7 \\ 2 & 8 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$

inv $\quad$ inv

$$AB = \begin{bmatrix} -3 & 23 \\ -4 & 26 \end{bmatrix} \neq \begin{bmatrix} 6 & 27 \\ 4 & 17 \end{bmatrix} = BA$$

(f) Let $A \in M_n(\mathbb{R})$. If A is invertible then $A + I$ is also invertible.

let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $A + I_n = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

which is NOT invertible - false

(g) If $A^2 - I$ is invertible then $A - I$ is invertible.

$(A - I)(A + I)$

$A^2 - IA + IA - I^2$

$= A^2 - I$

$$[(A - I)(A + I)][(A - I)(A + I)]^{-1} = I_n$$

$$(A - I)\underbrace{(A + I)(A + I)^{-1}}_{I_n}(A - I)^{-1}$$

$(A - I) I_n (A - I)^{-1} \quad I_n$

$(A-I)\, I_n (A-I)^{-1} = I_n$

$(A-I)(A-I)^{-1} = I_n$

Thus $A-I$ invertible if A^2-I inv