Homework 8

Problem 33

(a) Derive the operator identity

$$[\hat{p}, \hat{f}(\hat{x})] = -i\hbar \hat{f}'(\hat{x}),$$

where $\hat{f}(\hat{x})$ and $\hat{f}'(\hat{x})$ are operator-valued functions of $\hat{x}$ defined by their spectral decompositions

$$\hat{f}(\hat{x}) = \int dx\, |x\rangle f(x) \langle x|, \quad \hat{f}'(\hat{x}) = \int dx\, |x\rangle f'(x) \langle x|,$$

where $f'(x) = df(x)/dx$. *Hint*: see Eqs. (5.48) in the Lecture Notes for a similar derivation.

(b) Show that

$$[\hat{x}, \hat{f}(\hat{p})] = i\hbar \hat{f}'(\hat{p}),$$

where $\hat{f}(\hat{p})$ and $\hat{f}'(\hat{p})$ are functions of $\hat{p}$ defined similarly to $\hat{f}(\hat{x})$ and $\hat{f}'(\hat{x})$ of part (a).

Problem 34

Use the relations established in Problem 36 to show that

$$\hat{H} = \frac{\hat{p}^2}{2m} + \text{const}$$

is the only translationally invariant (i.e., commuting with $\hat{p}$) Hamiltonian yielding the Heisenberg equation of motion (see, e.g., Eqs. (4.14) in the Lecture Notes) for the position operator

$$\frac{d}{dt}\hat{x}_t = \frac{\hat{p}_t}{m}.$$

Problem 35

Evaluate $\langle p^2 \rangle_\psi$ for

(a) $\psi(x) \propto e^{-|x/a|}$, **(b)** $\psi(x) \propto \big(1 - |x/a|\big)\theta\big(1 - |x/a|\big)$.

(Don't forget the normalization coefficients.)

Problem 36

States $\psi_\pm$ correspond to the position-space wave functions

$$\psi_\pm(x) = \langle x|\psi_\pm\rangle = \frac{1}{\sqrt{2}}\big[\varphi_1(x) \pm i\varphi_2(x)\big],$$

where $\varphi_1(x)$ and $\varphi_2(x)$ are real functions that satisfy

$$\int_{-\infty}^{\infty} dx\, \varphi_1^2(x) = \int_{-\infty}^{\infty} dx\, \varphi_2^2(x) = 1, \quad \int_{-\infty}^{\infty} dx\, \varphi_1(x)\varphi_2(x) = 0, \quad \varphi_1(x) = \varphi_1(-x), \quad \varphi_2(x) = \varphi_2(-x).$$

(a) Show that state vectors $|\psi_+\rangle$ and $|\psi_-\rangle$ are orthogonal.

(b) Show that states $\psi_\pm$ are characterized by the same probability densities in the position and momentum spaces,

$$|\psi_+(x)|^2 = |\psi_-(x)|^2, \quad |\psi_+(p)|^2 = |\psi_-(p)|^2.$$

(This example shows that the position- and the momentum-space probability densities do not specify the state.)

Problem 37

(a) $|\psi\rangle$ is an eigenvector of Hermitian operator $\hat{H}$. Show that $\langle\psi|[\hat{A}, \hat{H}]|\psi\rangle = 0$ for all linear operators $\hat{A}$.

(b) A one-dimensional motion of a particle of mass m in the potential $V(x)$ is described by the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \hat{V}(\hat{x}), \quad \hat{V}(\hat{x}) = \int dx\, |x\rangle V(x) \langle x|.$$

By substituting $\hat{A} = \hat{x}\hat{p}$ into the identity proved in part (a), show that

$$\langle\psi|\hat{p}^2/m|\psi\rangle = \alpha \langle\psi|\hat{x}\hat{V}'(\hat{x})|\psi\rangle,$$

where $|\psi\rangle$ is a bound-state eigenvector of $\hat{H}$, $\hat{V}'(\hat{x}) = \int dx\, |x\rangle V'(x) \langle x|$ with $V'(x) = dV(x)/dx$, and α is a numerical coefficient that you need to find. You may find useful the relations derived in Problem 33.

(c) Using the relation established in part (b), evaluate the momentum uncertainty Δp for bound states in the logarithmic potential well $V(x) = \varepsilon \ln|x/a|$.