9A) $\langle\psi| = \langle\psi_1| + \lambda^*\langle\psi_2| \qquad A|\psi\rangle = A|\psi_1\rangle + \lambda A|\psi_2\rangle$

$\langle\psi|A|\psi\rangle = \langle\psi_1|A|\psi_1\rangle + \lambda^*\langle\psi_2|A|\psi_1\rangle + \langle\psi_1|A|\lambda\psi_2\rangle + \lambda^*\langle\psi_2|A|\lambda\psi_2\rangle$

$\langle\psi|B|\psi\rangle = \langle\psi_1|B|\psi_1\rangle + \lambda^*\langle\psi_2|B|\psi_1\rangle + \langle\psi_1|B|\lambda\psi_2\rangle + \lambda^*\langle\psi_2|A|\lambda\psi_2\rangle$

when $\lambda=1$ $\quad \langle\psi_1|A|\psi_1\rangle + \langle\psi_2|A|\psi_1\rangle + \langle\psi_1|A|\psi_2\rangle + \langle\psi_2|A|\psi_2\rangle$

$\therefore \Rightarrow \quad \langle\psi_1|B|\psi_1\rangle + \langle\psi_2|B|\psi_1\rangle + \langle\psi_1|B|\psi_2\rangle + \langle\psi_2|B|\psi_2\rangle$

when $\lambda = i$ $\quad \cancel{\langle\psi_1|A|\psi_1\rangle} + -i\langle\psi_2|A|\psi_1\rangle + i\langle\psi_1|A|\psi_2\rangle - i^2\langle\psi_2|A|\psi_2\rangle$

$\cancel{\langle\psi_1|B|\psi_1\rangle} + -i\langle\psi_2|B|\psi_1\rangle + i\langle\psi_1|B|\psi_2\rangle - i^2\langle\psi_2|B|\psi_2\rangle$

- Divide out i and $\quad \langle\psi_1|A|\psi_2\rangle = \langle\psi_1|B|\psi_2\rangle$

10 $\Rightarrow \langle\psi|A|\psi\rangle = \langle\psi_1|A|\psi_1\rangle + \lambda^*\langle\psi_2|A|\psi_1\rangle + \lambda\langle\psi_1|A|\psi_2\rangle + \lambda^*\lambda\langle\psi_2|A|\psi_2\rangle$

$\langle\psi|A^+|\psi\rangle^*: \qquad |\psi\rangle^* = (|\psi_1\rangle + \lambda|\psi_2\rangle)^* \Rightarrow |\psi_1\rangle + \lambda^*|\psi_2\rangle$

$\langle\psi|^* = \langle\psi_1| + \lambda\langle\psi_2|$

$\langle\psi|A^+|\psi\rangle^* = (\langle\psi_1| + \lambda\langle\psi_2|)(A^+|\psi_1\rangle + \lambda^* A^+|\psi_2\rangle)$

$= \cancel{\langle\psi_1|A|\psi_1\rangle} + \lambda\langle\psi_2|A^+|\psi_1\rangle + \lambda^*\langle\psi_1|A^+|\psi_2\rangle + \cancel{\lambda^*\lambda\langle\psi_2|A^+|\psi_2\rangle}$

$\cancel{\langle\psi_1|A|\psi_1\rangle} + \lambda^*\langle\psi_2|A|\psi_1\rangle + \lambda\langle\psi_1|A|\psi_2\rangle + \cancel{\lambda^*\lambda\langle\psi_2|A|\psi_2\rangle}$

$\lambda^*\langle\psi_2|A|\psi_1\rangle + \lambda\langle\psi_1 A|\psi_2\rangle = \lambda\langle\psi_2|A^+|\psi_1\rangle + \lambda^*\langle\psi_1|A^+|\psi_2\rangle$

for $\lambda=1$; They equal eachother

for $\lambda = i$ $\quad -i\langle\psi_2|A|\psi_1\rangle + i\langle\psi_1|A|\psi_2\rangle - i\langle\psi_2|A^+|\psi_1\rangle - i\langle\psi_1|A^+|\psi_2\rangle$

multiply

Thus equal

9B) Since in 9a we proved that $\langle \psi_1 | B | \psi_2 \rangle = \langle \psi_1 | A | \psi_2 \rangle$ for all $|\psi\rangle$ written as a l.c. of basis $|\psi\rangle = |\psi_1\rangle + \lambda |\psi_2\rangle$ for $\lambda = 1 + i$ (cmplx conj).

as such for all $|\psi\rangle$ $\langle \psi | A | \psi \rangle = \langle \psi | B | \psi \rangle$ meaning $\hat{A}$ and $\hat{B}$ act on $|\psi\rangle$ the same such that the inner product of $\langle \psi |$ and $A|\psi\rangle / B|\psi\rangle$ is the same as the other.

Thus, if A acts on $|\psi\rangle$ the same as B acts on $|\psi\rangle$ $\forall |\psi\rangle$, then $A|\psi\rangle = B|\psi\rangle$ and thus $\hat{A} = \hat{B}$

11A $\frac{\partial}{\partial\lambda}[e^{\lambda A} B e^{-\lambda A}] = (e^{\lambda A})(B(-A)e^{-\lambda A}) + (Be^{-\lambda A})(Ae^{\lambda A})$

$e^{\lambda A}[A,B]e^{-\lambda A} = e^{\lambda A}(AB-BA)e^{-\lambda A}$

$e^{\lambda A}(B(-A)e^{-\lambda A} + A(Be^{-\lambda A}))$

$\Rightarrow e^{\lambda A}(-BA+AB)e^{-\lambda A} \Rightarrow e^{\lambda A}(AB-BA)e^{-\lambda A} = e^{\lambda A}[A,B]e^{-\lambda A}$

11B $\hat{f}(\lambda) = e^{\lambda A} B e^{-\lambda A}$

The Taylor series for $\hat{f}(\lambda) = \sum \frac{1}{n!} \hat{f}^n(\lambda)\lambda^n$

$f(\lambda) = \frac{1}{1}(e^{\lambda A}Be^{-\lambda A}) + 1(e^{\lambda A}Be^{-\lambda A})'\lambda + \frac{1}{2}(e^{\lambda A}Be^{-\lambda A})''\lambda^2 + \ldots$

$f(0) = 1(e^{0A}Be^{0A}) + 1(e^{0A}Be^{0A})' + \frac{1}{2}(e^{0}Be^{0})''\lambda^2 + \ldots$

$f(1) = B + (\underbrace{\cancel{e^{\lambda A}}(AB-BA)\cancel{e^{-\lambda A}}}_{=1}) + \frac{1}{2}(e^{\lambda A}(AB-BA)e^{-\lambda A})''$

$\frac{\partial}{\partial\lambda}[e^{\lambda A}(AB-BA)e^{-\lambda A}] = (e^{\lambda A})((AB-BA)(-A)e^{-\lambda A}) + Ae^{\lambda A}(AB-BA)e^{-\lambda A}$

$e^{\lambda A}((AB-BA)(-A)e^{-\lambda A} + A(AB-BA)e^{-\lambda A})$

$e^{\lambda A}\underbrace{((AB-BA)(-A) + A(AB-BA))}_{[A,[A,B]]}e^{-\lambda A}$

Sub Back in Taylor expansion $\longrightarrow$ $[A,[A,B]]$

$f(1) = B + [A,B] + \frac{1}{2}[A,[A,B]] + \frac{1}{6}\underbrace{(e^{\lambda A}Be^{-\lambda A})'''}_{[A,A,[A,B]]}\underbrace{\lambda^3}_{\to 1}$

and thus

$e^{A}Be^{-A} = B + [A,B] + \frac{1}{2}[A,[A,B]] + \frac{1}{6}[A,A,[A,B]] + \ldots$

12a $[A,[A,B]] = A(AB-BA) - (AB-BA)(A) = 0$

$= B(AB-BA) - (AB-BA)B$

Show that $[A, e^{\lambda B}] = \lambda e^{\lambda B}[\hat{A},\hat{B}]$

Since $[A,[A,B]] = \hat{0}$, then Taylor expansion

for $e^{A} B e^{-A} = \hat{B} + [A,B] + 0 + 0 + 0 + \ldots = B + [A,B]$

Thus $[A, e^{\lambda B}] = Ae^{\lambda B} - e^{\lambda B}A$

$\lambda e^{\lambda B}[\hat{A},\hat{B}] = \lambda e^{\lambda B}(AB-BA)$

$e^{\lambda B} A e^{-\lambda B} = \hat{A} + \lambda[A,B] + 0 + \ldots \rightarrow$ Divide out $e^{-\lambda B}$

$e^{\lambda B}A = Ae^{\lambda B} + \lambda e^{\lambda B}[A,B] \rightarrow$ subtract first term

$\underline{e^{\lambda B}A - Ae^{\lambda B}} = \lambda e^{\lambda B}[A,B]$

Rearrange to be $Ae^{\lambda B} - e^{\lambda B}A = [A, e^{\lambda B}]$

$\therefore [A, e^{\lambda B}] = \lambda e^{\lambda B}[A,B]$

~~12b~~ $\hat{F}(\lambda) = e^{\lambda A} e^{\lambda B} e^{-\lambda(A+B)}$

$\cancel{\frac{\partial F}{\partial \lambda} = e^{\lambda A} e^{\lambda B}\left(-(A+B)e^{-\lambda(A+B)}\right) + e^{-\lambda(A+B)}\left(e^{\lambda A} B e^{\lambda B} + A e^{\lambda A} e^{\lambda B}\right)}$

$\cancel{-(A+B)e^{\lambda A}e^{\lambda B}e^{-\lambda(A+B)} + e^{\lambda A}e^{\lambda B}e^{-\lambda(A+B)}(A+B)}$

12B $\hat{F}(\lambda) = e^{\lambda A} e^{\lambda B} e^{-\lambda(A+B)}$

$$\frac{\partial f}{\partial \lambda} = e^{\lambda A} e^{\lambda B} e^{-\lambda(A+B)}\left(-(A+B)\right) + e^{-\lambda(A+B)}\left(e^{\lambda A} B e^{\lambda B} + e^{\lambda B} A e^{\lambda A}\right)$$

$$e^{-\lambda(A+B)} e^{\lambda A} e^{\lambda B}\left(e^{\lambda A} B e^{-\lambda A} + A\right)$$

$$\underbrace{e^{\lambda A} e^{\lambda B} e^{-\lambda(A+B)}}_{f(\lambda)}\left(e^{\lambda A} B e^{-\lambda A} - B + \hat{A} - A\right)$$

$$\left(\underbrace{\cancel{\hat{B}} + \lambda[A,B] \cancel{- B}}_{\lambda[A,B]} + \hat{0}\right)$$

Thus $\underline{\dfrac{\partial f}{\partial \lambda} = \lambda[A,B] f(\lambda)}$

12C By solving equation in 12B, derive BCH formula $e^A e^B = e^{A+B} e^{\frac{1}{2}[A,B]}$

$$\int_0^\lambda \frac{\partial f}{\partial \lambda} = \lambda[A,B] f(\lambda)$$

$$\int f(\lambda)^{-1}\, \partial f = \int \lambda[A,B]\, \partial \lambda$$

$$e^{\ln(f(\lambda))} = [A,B]\int \lambda\, \partial x \Rightarrow [A,B]\left(\tfrac{1}{2}\lambda^2 + C\right) \Rightarrow e^{\lambda^2 \frac{1}{2}[A,B] + C[A,B]}$$

$$f(\lambda) = e^{\lambda^2 \frac{1}{2}[A,B] + C[A,B]} \Rightarrow e^{\lambda^2 + C[A,B]} e^{\frac{1}{2}[A,B]}$$

$$e^{\lambda A} e^{\lambda B} e^{-\lambda(A+B)} = e^{\lambda^2 + C_1} e^{\frac{1}{2}[A,B]} \Big|_{\lambda=1}, \text{ Divide } e^{-\lambda(A+B)}$$

$$e^A e^B = e^{\frac{1}{2}[A,B]} \underbrace{e^{\lambda^2 + C + \lambda(A+B)}}_{= 1 \in R}\Big|_{\lambda=1}$$

$$e^A e^B = e^{\frac{1}{2}[A,B]} e^{A+B}$$

Thus $\boxed{e^A e^B = e^{A+B} e^{\frac{1}{2}[A,B]}}$

13A $\sum_n \langle \phi_n | A | \phi_n \rangle = \sum_n \langle \psi_n | A | \psi_n \rangle$ where $\{|\phi_n\rangle\}_n$ and $\{|\psi_n\rangle\}$ sets

Consider converting both $|\phi\rangle$ and $|\psi\rangle$ of basis vectors to $|x\rangle$ basis.

$$|\psi\rangle = \mathbb{1}|\psi\rangle = \sum_n |x_n\rangle \underbrace{\langle x_n|\psi\rangle}_{\psi_n} = \sum_n \psi_n |x_n\rangle$$

and

$$|\phi\rangle = \mathbb{1}|\phi\rangle = \sum_n |x_n\rangle \underbrace{\langle x_n|\phi\rangle}_{\phi_n} = \sum_n \phi_n |x_n\rangle$$

and also consider the operator relation $\hat{A} = \mathbb{1}\hat{A}$

$$\hat{A} = \hat{A}\mathbb{1} = \sum_n \underbrace{A|\psi_n\rangle}_{|x_n\rangle}\langle \psi_n| = |x_n\rangle\langle \psi_n|$$

and

$$= \sum_n \underbrace{A|\phi_n\rangle}_{|x_n\rangle}\langle \phi_n| = |x_n\rangle\langle \phi_n|$$

→ inserting it back into def. of $Tr(A)$ for $|\psi\rangle$ and $|\phi\rangle$

$\sum_n \langle \phi_n | x_n \rangle\langle \phi_n | \phi_n \rangle = \sum_n \langle \psi_n | x_n \rangle\langle \psi_n | \psi_n \rangle$ are equal in same basis $|x_n\rangle$ for all matrix elements

13B $Tr(AB) = \sum_n \langle \phi_n | AB | \phi_n \rangle = \sum_{n,k} \langle \phi_n | A | \psi_k \rangle\langle \psi_k | B | \phi_n \rangle$ for orthonormal basis $\{|\psi_k\rangle\}$

$$= \sum_{n,k} \langle \psi_k | B \underbrace{|\phi_n\rangle\langle \phi_n|}_{\text{combine}} A | \psi_k \rangle = \sum_k \langle \psi_k | BA | \psi_k \rangle = Tr(BA)$$

in $\{|\psi_k\rangle\}$

13C Trace (A) where $\hat{A} = |\phi\rangle\langle\psi| \Rightarrow$ Trace$(|\phi\rangle\langle\psi|)$

for $|\phi\rangle$ and $|\psi\rangle$ written in orthonormal basis $\{|\phi_n\rangle\}$

$|\phi\rangle = \phi_n|\phi_n\rangle$ and $|\psi\rangle = \psi_n|\phi_n\rangle \Rightarrow Tr(\langle \phi_n | \phi_n \rangle\langle \psi_n | \phi_n \rangle)$

$= \sum_n \langle \psi_n | \phi_n \rangle\langle \phi_n | \psi_n \rangle$ sum of terms along diagonal of outer product $|\phi\rangle\langle\psi|$