Question 3

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3. Let V be a vector space over $\mathbb{R}$ and $U, W \subseteq V$ be two subspaces of V.
   (a) Prove that there exist a basis B of U and a basis C of W such that $B \cap C$ is a basis for $U \cap W$.

Consider the space $\bar{U} \cap \bar{W}$ where set B and set C where $B \cap C$ is a basis $\{v_1, \dots v_n\}$. Consider vectors $\{b_1, \dots, b_n\}$ and $\{c_1, \dots, c_p\}$ where any element $y \in \bar{U}$ and $z \in \bar{W}$

$$y = \sum_{i=1}^{n} a_i v_i + \sum_{i=1}^{m} g_i b_i \quad \{v_1, \dots, v_n, b_1, \dots, b_n\}$$

$$z = \sum_{i=1}^{n} a_i v_i + \sum_{i=1}^{p} x_i c_i \quad \{v_1, \dots, v_n, c_1, \dots, c_p\}$$

Such sets form a basis since $B \cap C \subset B \wedge C$

All $\{b_1, \dots, b_m\}$ cannot be written as l.c. of $B \cap C$ and likewise for $\{c_1, \dots, c_p\}$

as such since every vector in $\bar{U}$ is a l.c. of $\{v_1, \dots, v_n, b_1, \dots, b_n\}$ and every vector in $\bar{W}$ is a l.c. of $\{v_1, \dots, v_n, c_1, \dots, c_p\}$, There exists B and C where $\text{Span}\{B\} = \bar{U}$ and $\text{Span}\{C\} = \bar{W}$.

(b) Is it true that for every basis B of U and every basis C of W the set $B \cap C$ is a basis for $U \cap W$?

false

Consider $\bar{V}$ a v.s. over $\mathbb{R}^3$

U is a subspace and $\bar{W}$ is a $\subseteq \bar{V}$

where $B = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right\}$ and $C = \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

The vector $\begin{bmatrix} 3 \\ 3 \end{bmatrix} \in \bar{U} \cap \bar{W}$

But $B \cap C = \{\emptyset\}$

(c) Recall the definition of $U + W$. Prove the following dimension formula:

$$\dim(U+W)=\dim(U)+\dim(W)-\dim(U\cap W).$$

$U+W = \{x+y \mid x \in \bar{U}, y \in \bar{W}\}$

considering the set above, any element in $\bar{U}+\bar{W}$ is a l.c. of the basis vectors of $\bar{U}$ and $\bar{W}$.

That is any $\vec{v} \in \bar{U}+\bar{W}$, $\vec{v} = \sum c_i b_i + \sum g_i d_i$ where $\{b_1 \to b_n\}$ and $\{d_1 \to d_n\}$ form a basis with $\bar{U}$ and $\bar{W}$ respectively. as such $\bar{U}+\bar{W}$ has a basis of $\{b_1, d_1, \dots, b_n, d_n\}$. However all basis vectors that exist in both $\bar{U}$ and $\bar{W}$ exist twice in the union of bases.

as such repeated vectors ($\dim(\bar{U}\cap\bar{W})$) can be omitted.

$$\dim(\bar{U}+\bar{W}) = \underset{|\{1,\dots,n\}|}{\dim(\bar{U})} + \underset{|\{1,\dots,p\}|}{\dim(\bar{W})} - \underbrace{\dim(\bar{U}\cap\bar{W})}_{\text{Repeated basis vectors}}$$

(d) Let $U, W \subset \mathbb{R}_4[x]$ be two subspaces which satisfy $\dim(U) = \dim(W) = 3$ prove that $U \cap W \neq \{\mathbf{0}\}$.

considering that $\dim(U+W) = \dim(U) + \dim(\bar{W}) - \dim(\bar{U}\cap\bar{W})$

since $U+W = \{x+y \mid x \in \bar{U}, y \in \bar{W}\}$, then any element in $\bar{U}+\bar{W}$ can be expressed as l.c. of basis of $\bar{U}$ and $\bar{W}$.

That is $\forall z \in \bar{U} + \bar{w}$, $z = \sum' c_i b_i + \sum' g_i d_i$ where $\{b_1 \to b_n\}$ and $\{d_1 \to d_p\}$ are Basis for $\bar{U} + \bar{w}$.

Then at most $\dim(\bar{U} + \bar{w})$ can be 4.

as such $4 \geq 3 + 3 - \dim(\bar{U} \cap \bar{w})$.

Thus $\dim(\bar{U} \cap \bar{w}) \geq 2$. Thus $\bar{U} \cap \bar{w} \neq \{0\}$

(e) Find the dimension of the following space:

$$\text{span}\{\begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ -3 \\ 1 \end{pmatrix}\} \cap \text{span}\{\begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 4 \\ 1 \end{pmatrix}\}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 0 \\ -1 & 2 & -3 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 2 & 1 & 2 & 3 & 0 \\ 1 & 2 & 4 & 1 & 0 \end{array}\right] \xrightarrow{\text{REF}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 5/7 \\ 0 & 0 & 7/3 & 4/3 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

zero row $\rightarrow$

$$= \text{Span}\{\begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ -3 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \\ 3 \end{pmatrix}\}$$

with dim = 4