Test 2

Problem t3

Observable A is represented by the operator

$$\hat{A} = |\alpha\rangle\langle\alpha| - i\sqrt{2}\,|\alpha\rangle\langle\beta| + i\sqrt{2}\,|\beta\rangle\langle\alpha|,$$

where vectors $|\alpha\rangle$ and $|\beta\rangle$ form an orthonormal basis for a two-dimensional Hilbert space.

(a) What are the possible outcomes of measurement of A?

(b) Find the expectation value of A in the state $|\psi\rangle \propto |\alpha\rangle + |\beta\rangle$, $\langle\psi|\psi\rangle = 1$.

(c) For the state $|\psi\rangle$ of part (b), compute the probabilities of each of the possible outcomes found in part (a).

Solution

(a) Possible measurement outcomes are eigenvalues $a_{1,2}$ of $\hat{A}$. We have

$$\left.\begin{aligned} a_1 + a_2 &= \operatorname{tr}\hat{A} = \langle\alpha|\alpha\rangle = 1 \\ a_1^2 + a_2^2 &= \operatorname{tr}\hat{A}^2 = \operatorname{tr}\big\{3|\alpha\rangle\langle\alpha| + 2|\beta\rangle\langle\beta|\big\} = 5 \end{aligned}\right\} \implies a_1 = -1,\ a_2 = 2.$$

(b) The expectation value of A in the state $|\psi\rangle = \dfrac{1}{\sqrt{2}}\big(|\alpha\rangle + |\beta\rangle\big)$ (notice the normalization coefficient!) is

$$\langle A\rangle_\psi = \langle\psi|\hat{A}|\psi\rangle = \frac{1}{2}\Big\{\langle\alpha|\hat{A}|\alpha\rangle + \langle\alpha|\hat{A}|\beta\rangle + \langle\beta|\hat{A}|\alpha\rangle + \langle\beta|\hat{A}|\beta\rangle\Big\} = \frac{1}{2}.$$

(c) The probabilities of the two outcomes $p_{1,2} = \mathrm{Prob}_\psi(A = a_{1,2})$ satisfy the equations

$$p_1 + p_2 = 1, \quad a_1 p_1 + a_2 p_2 = \langle A\rangle_\psi.$$

Substituting here $a_{1,2}$ and $\langle A\rangle_\psi$ found in parts (a) and (b), we get

$$\left.\begin{aligned} p_1 + p_2 &= 1 \\ -p_1 + 2p_2 &= 1/2 \end{aligned}\right\} \implies p_1 = p_2 = \frac{1}{2}.$$

Problem t4

Spin 2 particles are in the state $|\psi\rangle = \dfrac{1}{\sqrt{2}}\Big\{|+1\rangle - i\,|-1\rangle\Big\}$, where $|\pm1\rangle$ are eigenvectors of $\hat{S}_{\mathbf{z}}$ with eigenvalues $\pm\hbar$.

Evaluate the uncertainty of $S_{\mathbf{n}} = \mathbf{n}\cdot\mathbf{S}$ in this state (here $\mathbf{n}$ is a unit dimensionless vector).
For which $\mathbf{n}$ the uncertainty attains the smallest possible value? What is this value?
Feel free to use the relations

$$\langle S_{\mathbf{n}}\rangle_{\pm1} = \langle\pm1|\hat{S}_{\mathbf{n}}|\pm1\rangle = \pm\hbar n_{\mathbf{z}}, \quad \langle S_{\mathbf{n}}^2\rangle_{\pm1} = \langle\pm1|\hat{S}_{\mathbf{n}}^2|\pm1\rangle = (\hbar^2/2)\big(5 - 3n_{\mathbf{z}}^2\big)$$

that can be obtained by setting $j = 2$ and $m = \pm1$ in the expressions derived in Problem 26.

Solution

Since, obviously, $\langle+1|\hat{S}_{\mathbf{n}}|-1\rangle = \langle-1|\hat{S}_{\mathbf{n}}|+1\rangle = 0$, the expectation value of $S_{\mathbf{n}}$ vanishes:

$$\langle S_{\mathbf{n}}\rangle_\psi = \frac{1}{2}\big[\langle S_{\mathbf{n}}\rangle_{+1} + \langle S_{\mathbf{n}}\rangle_{-1}\big] = 0.$$

On the contrary, the expectation value of $S_{\mathbf{n}}^2$ is finite for all $\mathbf{n}$. Indeed, substitution of

$$\begin{aligned} &\langle S_{\mathbf{n}}^2\rangle_{\pm1} = (\hbar^2/2)\big(5 - 3n_{\mathbf{z}}^2\big) = (\hbar^2/2)\big[2 + 3\big(n_{\mathbf{x}}^2 + n_{\mathbf{y}}^2\big)\big], \\ &\langle+1|\hat{S}_{\mathbf{n}}^2|-1\rangle = \frac{1}{4}n_-^2\langle+1|\hat{S}_+^2|-1\rangle = \frac{1}{4}n_-^2\langle+1|\hat{S}_+|0\rangle\langle0|\hat{S}_+|-1\rangle = \frac{1}{4}n_-^2\big(\hbar\sqrt{2\cdot3}\big)^2 = \frac{3}{2}\,\hbar^2 n_-^2 \end{aligned}$$

into

$$\langle S_{\mathbf{n}}^2\rangle_\psi = \frac{1}{2}\Big[\langle S_{\mathbf{n}}^2\rangle_{+1} + \langle S_{\mathbf{n}}^2\rangle_{-1} + i\langle -1|\hat{S}_{\mathbf{n}}^2|+1\rangle - i\langle +1|\hat{S}_{\mathbf{n}}^2|-1\rangle\Big] = \frac{1}{2}\Big[\langle S_{\mathbf{n}}^2\rangle_{+1} + \langle S_{\mathbf{n}}^2\rangle_{-1} + 2\,\mathrm{Im}\,\langle +1|\hat{S}_{\mathbf{n}}^2|-1\rangle\Big]$$

yields

$$\langle S_{\mathbf{n}}^2\rangle_\psi = \frac{\hbar^2}{2}\Big\{2 + 3\big(n_{\mathbf{x}}^2 + n_{\mathbf{y}}^2\big) - 6n_{\mathbf{x}}n_{\mathbf{y}}\Big\} = \hbar^2\big[1 + (3/2)\,(n_{\mathbf{x}} - n_{\mathbf{y}})^2\big],$$

so that $\langle \mathbf{S}^2\rangle_\psi = \langle S_{\mathbf{x}}^2\rangle_\psi + \langle S_{\mathbf{y}}^2\rangle_\psi + \langle S_{\mathbf{z}}^2\rangle_\psi = 6\hbar^2$, as it should be for spin 2. The uncertainty of $S_{\mathbf{n}}$ is given by

$$\Delta S_{\mathbf{n}} = \sqrt{\langle S_{\mathbf{n}}^2\rangle_\psi - \langle S_{\mathbf{n}}\rangle_\psi^2} = \hbar\sqrt{1 + (3/2)\,(n_{\mathbf{x}} - n_{\mathbf{y}})^2}\,.$$

The uncertainty never vanishes and reaches its minimum $\min\big\{\Delta S_{\mathbf{n}}\big\} = \hbar$ when $\mathbf{n}$ lies in the plane $n_{\mathbf{x}} = n_{\mathbf{y}}$.