Question 3

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3. Let $A \in M_{m\times n}(\mathbb{R})$ and consider the linear map T_A associated with A defined as $T_A : \mathbb{R}^n \to \mathbb{R}^m$ where $T_A(x) = Ax$. Show that

   (a) T_A is injective if there is a pivot in every col of rref(A).

If There is a pivot in every col of RREf(A), Then There are n pivots. $\Rightarrow$ The Col(A) creates a subspace with dimension n. That means any vector x in the Domain $\mathbb{R}^n$, $x \in \mathbb{R}^n$ is sent via T to a unique vector $y \in \mathbb{R}^m$. essentially y is a linear combination of the columns of A, specified by A. Since There is a pivot in every column of A (Ref), Then the nullspace of A, i.e. the $\ker(T_A) = \{\vec{0}\}$. Thus, This implies T_A is injective.

(b) T_A is surjective if there is a pivot in every row of rref(A).

If There is a pivot in every Row of RREf(A), Then There are m pivots. This implies that any augmented matrix representing the equation $A\vec{x} = \vec{b}$ has only one solution.

This implies that for any vector $\vec{b} \in \mathbb{R}^m$, there is at least one $\vec{x}$ that exists in $\mathbb{R}^n$.

as such the image of $T = \{y \mid y = Ax, x \in \mathbb{R}^n\}$

and $\dim(\operatorname{im} T) = m$, # pivots. Thus

T_A is surjective.