Question 7

Monday, October 2, 2023 10:00 PM

7. In each of the following you are given a linear transformation (you don't need to prove that it is a linear transformation). Follow the following directions for each such transformation:

- Find a basis for the kernel and the image of this transformation.
- Find the dimension of the kernel and the image of this transformation. (Remark: This question will continue in the next HW, you may want to keep a copy of your solution to this part of the question).
- Determine whether the transformation is surjective. Explain your answer.
- Determine whether the transformation is injective. Explain your answer.

(a) For

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 1 & 2 \\ 1 & 7 & -6 \\ 0 & 5 & -5 \end{pmatrix}$$

consider the linear map

$$T_A : \mathbb{R}^3 \to \mathbb{R}^4$$

where the notation T_A was defined in class.

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 3 & 1 & 2 & 0 \\ 1 & 7 & -6 & 0 \\ 0 & 5 & -5 & 0 \end{array}\right] \text{ in RREF is}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \ker(T_A) = \left\{ \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \right\}$$

Dim = 1

$$x_1 + 0x_2 + x_3 = 0$$
$$0x_1 + x_2 - x_3 = 0$$

$$T_A(x) = \vec{x}A$$

?3?

$\text{im } T_A = \{ y \mid y = \vec{x}A,\ x \in \mathbb{R} \}$

or any l.c. of columns A

$\text{im } T_a = \left\{ \begin{pmatrix} 1 \\ 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 7 \\ 5 \end{pmatrix} \right\}$ Dim = 2

Not injective since $\ker T_A \neq \vec{0}$

Not surjective since $\text{span}(\text{im } T_A) \neq \mathbb{R}^4$

(b)

$$S : M_2(\mathbb{R}) \to \mathbb{R}^2$$

given by

$$SA = A \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

$\left[\begin{array}{cc|c} 3 & -2 & 0 \\ 3 & -2 & 0 \end{array}\right] \to$ $3x_1 - 2x_2 = 0$

$\ker S = \left\{ \begin{pmatrix} 1/3 & -1/2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1/3 & -1/2 \end{pmatrix} \right\}$

$\text{Dim}(\ker(S)) = 2$

$\text{im } S = \{ y \mid y = A \begin{pmatrix} 3 \\ -2 \end{pmatrix},\ A \in M_2(\mathbb{R}) \}$

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} a_1 \cdot 3 + a_2 \cdot 2 \\ \hline a_3 \cdot 3 + a_4 \cdot -2 \end{pmatrix}$$

$\text{im}(S) = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ dim = 2

Not injective since $\ker(S) \neq \{\vec{0}\}$

is surjective since $\mathrm{Span}(\mathrm{Im}(S)) = \mathbb{R}^2$

(c)

$$L : \mathbb{R}_3[x] \to \mathbb{R}^2$$

given by

$$Lp = \begin{pmatrix} p(2) - p(1) \\ p'(0) \end{pmatrix}$$

$P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ $\quad$ $p'(x) = a_1 + 2a_2 x + 3a_3 x^2$

$$L(P(x)) = \begin{pmatrix} a_0 + 2a_1 + 4a_2 + 8a_3 - (a_0 + a_1 + a_2 + a_3) \\ a_1 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 + 3a_2 + 7a_3 \\ a_1 \end{pmatrix}$$

$$\left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 7 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 7 & 0 \end{array}\right]$$

$\{1, x^3 - \frac{7}{3}x^2\}$

$\ker(S)$ $\quad$ $a_1 = 0$ $\quad$ $a_0 = s$

$a_2 = -\frac{7}{3}a_3 \to a_2 = -\frac{7}{3}t$

$a_3 = t$

$$S = \left\{ s\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, t\begin{pmatrix} 0 \\ 0 \\ -7/3 \\ 1 \end{pmatrix} \right\}$$

Dim = 2, Not injective

$\ker(S) \neq \{\vec{0}\}$

$\operatorname{im}(S) = \left\{ y \mid y = \begin{pmatrix} p(2) - p(1) \\ p'(0) \end{pmatrix}, \ p(x) \in \mathbb{R}_3[x] \right\}$

$y = \begin{pmatrix} a_1 + 3a_2 + 7a_3 \\ a_1 \end{pmatrix}$

$y_1 = a_1 + 3a_2 + 7a_3$

$y_1 = 3 + 3a_2 + 7a_3$

$\operatorname{im}(S) = \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$ dim = 2

Surjective since $\operatorname{span}(\operatorname{im}(S)) = \mathbb{R}^2$

(d)

$$\Phi : \mathbb{R}^3 \mapsto \mathbb{R}_3[x] \longrightarrow$$

given by

$$\Phi \begin{pmatrix} a \\ b \\ c \end{pmatrix} = (a+b) + (a-2b+c)x + (b-3c)x^2 + (a+b+c)x^3$$

$a + b = 0$

$a - 2b + c = 0$

$b - 3c = 0$

$a + b + c = 0$

$\rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right]$

Ref $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\ker(\Phi) = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

dim = 0

injective since $\ker(\phi)$ is Trivial

$$im(\phi) = \left\{ y \mid y = \dots, \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 \right\}$$

cnst $a+b$

x $a-2b+c$

x^2 $b-3c$

x^3 $a+b+c$

$$y = a+b+ax-2bx+cx+bx^2-3cx^2$$
$$+ax^3+bx^3+cx^3$$

$$a(1+x+x^3)+$$
$$b(1-2x+x^2+x^3)+$$
$$c(x-3x^2+x^3)$$

$$im(\phi) = \{ 1+x+x^3, 1-2x+x^2+x^3, x-3x^2+x^3 \}$$

Dim $=3$

Not surjective Bc $\text{Span}(im(\phi)) \neq \mathbb{R}_3[x]$