MATH-1564-K1,K2,K3 Linear Algebra with Abstract Vector Spaces

Homework 3

1. Prove the following statements.
   (a) Let $A \in M_{m\times n}(\mathbb{R})$. If an echelon form of A has a row of zeroes then there exists $b \in \mathbb{R}^m$ such that $(A|b)$ has no solution.
   (b) If $m > n$ and $A \in M_{m\times n}(\mathbb{R})$ then there exists $b \in \mathbb{R}^m$ such that $(A|b)$ has no solution.
   (c) If $A \in M_n(\mathbb{R})$ is a square $n \times n$ matrix such that the homogenous system $(A|0)$ has infinity many solutions, then there exists $b \in \mathbb{R}^n$ such that $(A|b)$ has no solution.

2. The following statements are **false**. Prove that they are **false** by providing a counterexample in each case (you may choose the numbers m and n to be whatever is convenient, try to work with small numbers).
   (a) If $A \in M_{m\times n}(\mathbb{R})$ is a matrix such that the homogenous system $(A|0)$ has infinity many solutions, then there exists $b \in \mathbb{R}^n$ such that $(A|b)$ has no solution.
   (b) Let $A, B \in M_{m\times n}(\mathbb{R})$ and $b \in \mathbb{R}^m$. If A and B are row equivalent then $(A|b)$ and $(B|b)$ have the same amount of solutions.
   (c) If $A, B \in M_{m\times n}(\mathbb{R})$ are row equivalent then B can be obtained from A by performing **column** operations (that is, by performing a sequence of operations of the form 'swapping two columns', 'multiplying a column by a scalar different from zero', 'adding to a column another column multiplied by a scalar').
   (d) Let $A \in M_{m\times n}(\mathbb{R})$ and $b \in \mathbb{R}^m$. If $u, v \in \mathrm{Sol}(A|b)$ then $u + v \in \mathrm{Sol}(A|b)$.
   (e) Let $A \in M_{m\times n}(\mathbb{R})$ and $b \in \mathbb{R}^m$. If $u \in \mathrm{Sol}(A|b)$ and $t \in \mathbb{R}$ then $tu \in \mathrm{Sol}(A|b)$.
   (f) If matrix-vector equation $A\vec{x} = \vec{0}$, then either A is a zero matrix or $\vec{x}$ is a zero vector.

3. Decide if the following two matrices are row equivalent: The matrix

$$\begin{pmatrix} 1 & 0 & 2 \\ 3 & -1 & 1 \\ 5 & -1 & 5 \end{pmatrix}$$

   and the matrix

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 10 \\ 2 & 0 & 4 \end{pmatrix}$$

4. In each of the following you are given a set and two operations: A 'sum', acting between two elements in the set, and a 'multiplication by scalar', acting between one element in the set and a scalar from $\mathbb{R}$. In each case determine whether the set with these two operations gives a vector space over $\mathbb{R}$. If it is a vector space then prove this fact. If it is not a vector space then show this by giving a counterexample. **In this question you are allowed to use only the definition of a vector space, not any other claim given in class.**

   (a) The set $P_2(\mathbb{R})$ with the usual operations of summation and multiplication by scalar defined for polynomials.

(b) The set $\mathbb{R}^2$ with the operations

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \oplus \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ 0 \end{pmatrix}$$

and

$$\alpha \odot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ 0 \end{pmatrix}.$$

(c) The set $\mathbb{R}^2$ with the operations

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \oplus \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 - 3 \\ x_2 + y_2 - 2 \end{pmatrix}$$

and

$$\alpha \odot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \alpha x_1 - 3\alpha + 3 \\ \alpha x_2 - 2\alpha + 2 \end{pmatrix}.$$

(d) The set $\{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R} : x_1 > 0, x_2 > 0\}$ with the operations

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \oplus \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 y_1 \\ x_2 y_2 \end{pmatrix}$$

and

$$\alpha \odot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1^\alpha \\ x_2^\alpha \end{pmatrix}.$$

5. Let V be a vector space over $\mathbb{R}$ and let $W \subset V$ and $U \subset V$ be two subspaces of V. The following claims are either true or false. Determine whether they are true or false and prove or disprove using a counterexample accordingly.

   (a) $U \cap W$ is also a subspace of V.

   (b) $U \cup W$ is also a subspace of V.

   (c) We define the following subset of V:

   $$U + W := \{u + w : u \in U, w \in W\}.$$

   In this part of the question the claim is: $U + W$ is a subspace of V.