Homework 5

Problem 19

Matrix elements of spin 1/2 operator $\hat{S}_{\mathbf{n}} = \mathbf{n} \cdot \hat{\mathbf{S}}$ in $|\pm\mathbf{x}\rangle$ basis form 2×2 matrix

$$\hat{S}_{\mathbf{n}} = \begin{pmatrix} \langle +\mathbf{x}|\hat{S}_{\mathbf{n}}|+\mathbf{x}\rangle & \langle +\mathbf{x}|\hat{S}_{\mathbf{n}}|-\mathbf{x}\rangle \\ \langle -\mathbf{x}|\hat{S}_{\mathbf{n}}|+\mathbf{x}\rangle & \langle -\mathbf{x}|\hat{S}_{\mathbf{n}}|-\mathbf{x}\rangle \end{pmatrix}.$$

Find this matrix and verify that it yields the correct commutator $[\hat{S}_{\mathbf{y}}, \hat{S}_{\mathbf{z}}] = i\hbar\hat{S}_{\mathbf{x}}$.

Problem 20

Because spin 1/2 operator $\hat{S}_{\mathbf{n}}$ has only two eigenvalues, any function of this operator can be written as the first-degree polynomial $\hat{f}(\hat{S}_{\mathbf{n}}) = c_0\hat{\mathbb{1}} + c_1\hat{S}_{\mathbf{n}}$. Find this polynomial for $\hat{f}(\hat{S}_{\mathbf{n}}) = e^{-i\theta\hat{S}_{\mathbf{n}}/\hbar}$, where θ is a real dimensionless number.

Problem 21

Show that for spin 1/2 operators $(\mathbf{a}\cdot\hat{\mathbf{S}})(\mathbf{b}\cdot\hat{\mathbf{S}}) = \alpha\hat{\mathbb{1}} + \boldsymbol{\beta}\cdot\hat{\mathbf{S}}$, where α and $\boldsymbol{\beta}$ are, respectively, a scalar and a vector that you need to find.

Problem 22

(a) Evaluate $\Delta S_{\mathbf{n}_1}$, $\Delta S_{\mathbf{n}_2}$, and $\big\langle[\hat{S}_{\mathbf{n}_1}, \hat{S}_{\mathbf{n}_2}]\big\rangle \equiv \langle\mathbf{n}|[\hat{S}_{\mathbf{n}_1}, \hat{S}_{\mathbf{n}_2}]|\mathbf{n}\rangle$ for spin 1/2 in the state $|\mathbf{n}\rangle$.
Feel free to use the relations

$$\langle\mathbf{a}\cdot\mathbf{S}\rangle_{\mathbf{n}} = \langle\mathbf{n}|\mathbf{a}\cdot\hat{\mathbf{S}}|\mathbf{n}\rangle = \frac{\hbar}{2}(\mathbf{a}\cdot\mathbf{n}), \quad \Delta_{\mathbf{n}}(\mathbf{a}\cdot\mathbf{S}) = \frac{\hbar}{2}|\mathbf{a}\times\mathbf{n}|, \quad \big[\mathbf{a}\cdot\hat{\mathbf{S}}, \mathbf{b}\cdot\hat{\mathbf{S}}\big] = i\hbar(\mathbf{a}\times\mathbf{b})\cdot\hat{\mathbf{S}}.$$

(b) Verify that the expressions obtained in part (a) obey the uncertainty relation

$$\Delta S_{\mathbf{n}_1}\Delta S_{\mathbf{n}_2} \geq \frac{1}{2}\Big|\big\langle[\hat{S}_{\mathbf{n}_1}, \hat{S}_{\mathbf{n}_2}]\big\rangle\Big|.$$

Suggestion: As you will discover, part (b) reduces to demonstrating a certain relation between unit dimensionless vectors $\mathbf{n}$, $\mathbf{n}_1$, and $\mathbf{n}_2$. This can be done, for example, by switching to the spherical polar coordinates with $\mathbf{z} = \mathbf{n}$.