MATH-1564-K1,K2,K3 Linear Algebra with Abstract Vector Spaces

Homework 5

1. Answer the following questions.
   (a) Do elementary row operations affect a matrix's column space? Justify your response.
   (b) Do elementary row operations affect a matrix's null space? Justify your response.

2. Find a basis and then the dimension of the following subspace:

$$\text{span}\{2 + x^2 - 2x^3, 1 - 2x + x^2 - x^3, 5 + 2x + 2x^2 - 5x^3, 3 + 6x - 3x^3\}$$

3. Let V be a vector space over $\mathbb{R}$ and $U, W \subseteq V$ be two subspaces of V.
   (a) Prove that there exist a basis B of U and a basis C of W such that $B \cap C$ is a basis for $U \cap W$.
   (b) Is it true that for every basis B of U and every basis C of W the set $B \cap C$ is a basis for $U \cap W$?
   (c) Recall the definition of $U + W$. Prove the following dimension formula:

$$\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W).$$

   (d) Let $U, W \subset \mathbb{R}_4[x]$ be two subspaces which satisfy $\dim(U) = \dim(W) = 3$ prove that $U \cap W \neq \{\mathbf{0}\}$.
   (e) Find the dimension of the following space:

$$\text{span}\{\begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ -3 \\ 1 \end{pmatrix}\} \cap \text{span}\{\begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 4 \\ 1 \end{pmatrix}\}$$

4. Let V, W be vector spaces over $\mathbb{R}$. Consider $(\mathcal{L}(V, W), +, \cdot)$ where $+$ is defined as

$$(f_1 + f_2)(v) = f_1(v) + f_2(v), \forall v \in V$$

   for any two linear maps $f_1, f_2 \in \mathcal{L}(V, W)$ and $\cdot$ is defined to be

$$(kf)(v) = k(f(v), \forall v \in V$$

   where k is any scalar from $\mathbb{R}$ and f is any linear map in $\mathcal{L}(V, W)$. Show that $(\mathcal{L}(V, W), +, \cdot)$ is a vector space over $\mathbb{R}$.

5. Let $T \in \mathcal{L}(V, W)$. Show that
   (a) $\ker T$ is a subspace of V.
   (b) $\text{Im} T$ is a subspace of W.
   (c) If $T \in \mathcal{L}(V, V)$, is it possible that $\ker T \cap \text{Im} T \neq \{\mathbf{0}\}$? Explain your answer.

6. In each of the following you are given two vector spaces and a function between them. Determine whether the function is a linear transformation or not. Prove your claim.

(a)
$$T : \mathbb{R}^3 \to M_2(\mathbb{R})$$
given by,
$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y & y-2z \\ 3x+z & 0 \end{pmatrix}$$

(b)
$$T : \mathbb{R}_2[x] \to \mathbb{R}^3$$
given by,
$$Tp = \begin{pmatrix} p(2) \\ p'(2) \\ p''(2) \end{pmatrix}$$

(c)
$$T : M_2(\mathbb{R}) \to M_2(\mathbb{R})$$
given by,
$$TA = A^2$$

(d) Fix $B \in M_3(\mathbb{R})$ and consider the function:
$$T : M_3(\mathbb{R}) \to M_3(\mathbb{R})$$
given by,
$$TA = AB$$

(e)
$$T : M_2(\mathbb{R}) \to M_2(\mathbb{R})$$
given by,
$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a-c+1 & 2a+3b+2 \\ d-b-8 & 2a \end{pmatrix}$$

7. In each of the following you are given a linear transformation (you don't need to prove that it is a linear transformation). Follow the following directions for each such transformation:

- Find a basis for the kernel and the image of this transformation.
- Find the dimension of the kernel and the image of this transformation. (Remark: This question will continue in the next HW, you may want to keep a copy of your solution to this part of the question).
- Determine whether the transformation is surjective. Explain your answer.
- Determine whether the transformation is injective. Explain your answer.

(a) For
$$A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 1 & 2 \\ 1 & 7 & -6 \\ 0 & 5 & -5 \end{pmatrix}$$
consider the linear map
$$T_A : \mathbb{R}^3 \to \mathbb{R}^4$$
where the notation T_A was defined in class.

(b)

$$S : M_2(\mathbb{R}) \to \mathbb{R}^2$$

given by

$$SA = A \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

(c)

$$L : \mathbb{R}_3[x] \to \mathbb{R}^2$$

given by

$$Lp = \begin{pmatrix} p(2) - p(1) \\ p'(0) \end{pmatrix}$$

(d)

$$\Phi : \mathbb{R}^3 \mapsto \mathbb{R}_3[x]$$

given by

$$\Phi \begin{pmatrix} a \\ b \\ c \end{pmatrix} = (a + b) + (a - 2b + c)x + (b - 3c)x^2 + (a + b + c)x^3$$