Question 2

Sunday, November 12, 2023 3:47 PM

2. (a) Let $a, b, c \in \mathbb{R}$. Prove that

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (c-a)(c-b)(b-a)$$

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$ By method of cofactor expansion using the 1^{st} column

$$= 1\begin{vmatrix} b & b^2 \\ c & c^2 \end{vmatrix} + 1\begin{vmatrix} a & a^2 \\ c & c^2 \end{vmatrix} + 1\begin{vmatrix} a & a^2 \\ b & b^2 \end{vmatrix}$$

$$1(bc^2 - b^2c) + 1(ac^2 - a^2c) + 1(ab^2 - a^2b)$$

$$= bc^2 - c^2b + c^2a - a^2c + ab^2 - a^2b$$

$$(c-a)(b-c)(b-a) = (c-a)(b^2 - cb - ab + ca)$$

$$\Rightarrow cb^2 - c^2b \cancel{- abc} + c^2a - ab^2 \cancel{+ acb} + a^2b - ca^2$$

$$\Rightarrow cb^2 - c^2b + c^2a - ab^2 + a^2b - ca^2$$

$$= bc^2 - c^2b + c^2a - a^2c + ab^2 - a^2b$$

(b) Find the values of a for which the following set is a basis for $\mathbb{R}^3$:

$$\left\{ \begin{pmatrix} a-1 \\ -3 \\ \end{pmatrix}, \begin{pmatrix} 3 \\ a+5 \\ \end{pmatrix}, \begin{pmatrix} -3 \\ -3 \\ \end{pmatrix} \right\}$$

$$\begin{pmatrix} -6 \end{pmatrix} \quad \begin{pmatrix} 6 \end{pmatrix} \quad \begin{pmatrix} a-4 \end{pmatrix}$$

$$\begin{matrix} a-1 & 3 & -3 & a-1 & 3 \\ -3 & a+5 & -3 & -3 & a+5 \\ -6 & 6 & a-4 & -6 & 6 \end{matrix}$$

$$(a-1)(a+5)(a-4) + 54 + 54 - 18(a+5) + 18(a-1) + 9(a-4)$$

$$(a-1)(a^2+a-20) + 108 \cancel{-18a} - 90 \cancel{+18a} - 18 + 9a - 36$$

$$a^3 \cancel{+a^2} - 20a \cancel{-a^2} - a + 20 + 108 - 90 - 18 + 9a - 36$$

$$a^3 - 12a - 16 \neq 0 \quad \forall a \in \mathbb{R}$$

$$a(a^2-12) - 16 = 0$$

$$a(a^2-12) = 16$$

$$a^2 = 28$$

$$a = -2, 4$$

$$\boxed{a \in \mathbb{R},\ a \neq -2, 4}$$

(c) Assume that,

$$\begin{vmatrix} a & x & l \\ b & y & m \\ c & z & n \end{vmatrix} = 2$$

Find:

$$\begin{vmatrix} 2a+3x & 2b+3y & 2c+3z \\ l+x & m+y & n+z \\ 7l & 7m & 7n \end{vmatrix}$$

$$\begin{pmatrix} 2a+3x & l+x & 7l \\ 2b+3y & m+y & 7m \\ 2c+3z & n+z & 7n \end{pmatrix}$$

$$2(7)(2) = \boxed{28}$$