Phys 3143 HW2

5a) Derive Triangle inequality $\|\phi + \psi\| \leq \|\phi\| + \|\psi\|$

$\|\phi + \psi\|^2 = \langle \phi + \psi | \phi + \phi \rangle$

$\|\phi + \psi\| = \sqrt{\langle \phi + \psi | \phi + \psi \rangle} \leq \sqrt{\langle \phi | \phi \rangle} + \sqrt{\langle \psi | \psi \rangle}$

consider square of LHS $\|\phi + \psi\|^2 = \langle \phi + \psi | \phi + \psi \rangle$

$\langle \phi + \psi | \phi + \psi \rangle \Rightarrow |\phi\rangle^2 + 2\langle \phi | \psi \rangle + |\psi\rangle^2$

$\langle \phi | \phi \rangle + \langle \psi | \psi \rangle + 2\langle \phi | \psi \rangle^2 \leq$ Square RHS

$(\|\phi\| + \|\psi\|)^2 \Rightarrow$

$\cancel{\langle \phi | \phi \rangle} + \cancel{\langle \psi | \psi \rangle} + 2\langle \phi | \psi \rangle^2 \leq \cancel{\langle \phi | \phi \rangle} + 2\|\phi\| \cdot \|\psi\| + \cancel{\langle \psi | \psi \rangle}$

By Schwartz inequality

$\langle \phi | \psi \rangle^2 \leq \|\phi\| \cdot \|\psi\|$

$\sqrt{\langle \phi | \phi \rangle} \cdot \sqrt{\langle \psi | \psi \rangle} \Rightarrow \langle \phi | \psi \rangle \leq \langle \phi | \phi \rangle \langle \psi | \psi \rangle$

Thus Triangle inequality holds

5b) Triangle inequality is equality when Schwartz inequality yields $0 \leq 0 \Rightarrow 0 = 0$. This occurs when either $|\phi\rangle$ or $|\psi\rangle$ is $|null\rangle$ holding

That $\|null + \psi\| = \|null\| + \|\psi\|$

$\Rightarrow \|\psi\| = \|\psi\|$

5c) $\|\phi - \psi\| \geq (\|\phi\| - \|\psi\|)^2$ Square both sides

$\langle \phi | \phi \rangle + \langle \psi | \psi \rangle - 2\langle \phi | \psi \rangle^2 \geq \langle \phi | \phi \rangle + \langle \psi | \psi \rangle - 2\|\phi\| \cdot \|\psi\|$

By Schwartz inequality

$|\langle \phi | \psi \rangle|^2 \leq \langle \phi | \phi \rangle \langle \psi | \psi \rangle$, multiplying $\# < 0$, flips inequality

$-|\langle \phi | \psi \rangle|^2 \geq -(\langle \phi | \phi \rangle \langle \psi | \psi \rangle)$

$\therefore$ original inequality holds.

6) $|\psi\rangle = C_1|\psi_1\rangle + C_2|\psi_2\rangle + C_3|\psi_3\rangle + \dots + C_N|\psi_N\rangle$

State vector in N Dimensional $\mathcal{H}$

if we factor out a Phase factor $e^{i\theta}$

$|\psi\rangle = e^{i\theta}(C_1'|\psi_1\rangle + \dots + C_N'|\psi_N\rangle)$

$\{C_i\}_{i=N}$ Remains as a set of Complex Numbers where $C_i = a_i + b_i i$

$|\psi\rangle = \psi_n \{\phi_i\}_{i=1}^{N}$

$\psi_n = |\psi_n| e^{i\theta}$ Phase factor

coeff

ψ_n - Prob Amplitude

$\sum_{n=1}^{N} |\psi_n|^2 = 1$ ← Normalized

probability

factored out phase $(N-1)$

$|\psi\rangle = \sum^{N} \underbrace{|\psi_n\rangle e^{i\theta_n}}_{\text{coefficients}} |\phi_n\rangle = e^{i\theta_1}\left(|\psi_1||\phi_1\rangle + \sum_{n=2} \psi_n|\phi_n\rangle\right)$

$|\psi_1| = \sqrt{1 - \underbrace{\sum_{n\neq 1} |\psi_n|^2}_{\text{all other prob}}}$

$\boxed{2(N-1)}$

2N-2 independent Real parameters

$\cos(x) + i\sin(x)$
$\cos(x) - i\sin(x)$

7) $|\psi\rangle = \frac{1}{\sqrt{2}}|+x\rangle + \frac{e^{\frac{2\pi}{3}i}}{\sqrt{2}}|-\vec{x}\rangle$

$|\pm x\rangle = \frac{1}{\sqrt{2}}(|+z\rangle \pm |-z\rangle$

$\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}|+z\rangle + \frac{1}{\sqrt{2}}|-z\rangle\right) + \frac{e^{2\pi i/3}}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}|+z\rangle - \frac{1}{\sqrt{2}}|-z\rangle\right)$

$\Rightarrow \frac{1}{2}(\underbrace{1 + e^{2\pi i/3}}_{})|+z\rangle + \frac{1}{2}(1 - e^{i2\pi/3})|-z\rangle$

$\frac{1}{2}e^{i\pi/3}(e^{i\pi/3} + e^{-i\pi/3}) = 2e^{i\pi/3}\cos(\pi/3)$

$e^{i\pi/3}\{\cos\pi/3|+z\rangle + \underbrace{\tfrac{1}{2}e^{-i\pi/3}(1 - e^{2\pi i/3})}_{}|-z\rangle\}$

$\frac{1}{2}(e^{-i\pi/3} - e^{i\pi/3}) = -i\sin\frac{\pi}{3} = \theta$

orthonormal identity $\phi = e^{+i3\pi/2}\sin\pi/3$

$|\psi\rangle = e^{i\pi/3}\{\cos\pi/3|+z\rangle + e^{+i3\pi/2}\sin\pi/3|-z\rangle\}$

$\boxed{\theta = \frac{2\pi}{3} \quad \phi = 3\pi/2}$

θ

$\frac{\pi}{6}$ 2 $\frac{\pi}{12}$

$\frac{2\pi/3}{2}$

$\frac{\pi}{3}$

$\frac{2\pi}{3} = \frac{\theta}{2}$

$\frac{\theta}{2}$ $\frac{x}{2} = \frac{\pi}{3}$

$x = \frac{2\pi}{3}$

$\frac{2\pi/3}{2} = \frac{\pi}{3}$

8) $\langle n_1|n_2\rangle = \langle \cos\frac{\theta_1}{2}|z\rangle + e^{i\phi_1}\sin\frac{\theta_1}{2}|{-z}\rangle|\cos\frac{\theta_2}{2}|z\rangle + e^{i\phi_2}\sin\frac{\theta_2}{2}|z\rangle$

$\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} + e^{i(\phi_2-\phi_1)}\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2} = \langle n_1|n_2\rangle$

$|\langle n_1|n_2\rangle|^2 = \left(\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} + e^{i(\phi_2-\phi_1)}\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\right)(\ldots)^*$ ← cmplx conj.

APPLY euler's formula

$\left|\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} + (\cos(\phi_2-\phi_1) + i\sin(\phi_2-\phi_1))\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\right|^2$

Distribute

$\left|\underbrace{\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} + \cos(\phi_2-\phi_1)\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}}_{a} + \underbrace{i\sin(\phi_2-\phi_1)\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}}_{bi}\right|^2 \quad |a+bi|^2 = a^2+b^2$

$\left(\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} + \cos(\phi_2-\phi_1)\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\right)^2 + \left[\sin(\phi_2-\phi_1)\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\right]^2$

$\cos^2\frac{\theta_1}{2}\cos^2\frac{\theta_2}{2} + 2\cos(\phi_2-\phi_1)\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} + \cos^2(\phi_2-\phi_1)\sin^2\frac{\theta_1}{2}\sin^2\frac{\theta_2}{2} +$

factor out → $\left[\sin^2(\phi_2-\phi_1)\sin^2\frac{\theta_1}{2}\sin^2\frac{\theta_2}{2}\right]$

$\cos^2\frac{\theta_1}{2}\cos^2\frac{\theta_2}{2} + \sin^2\frac{\theta_1}{2}\sin^2\frac{\theta_2}{2} + 2\cos(\phi_2-\phi_1)\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}$

$\frac{1}{4}(1+\cos\theta_1)(1+\cos\theta_2) + \frac{1}{4}(1-\cos\theta_1)(1-\cos\theta_2) + \frac{1}{2}\cos(\phi_2-\phi_1)\sin\theta_1\sin\theta_2$

$\frac{1}{2}(1 + \sin\theta_1\sin\theta_2\cos(\phi_2-\phi_1) + \cos\theta_1\cos\theta_2) = |\langle n_2|n_1\rangle|^2$

for dot product $\frac{1}{2}(1 + n_1\cdot n_2)$

$\sin\theta_1\sin\theta_2\cos\phi_1\cos\phi_2 + \sin\theta_1\sin\theta_2\sin\phi_1\sin\phi_2 + \cos\theta_1\cos\theta_2$

$\sin\theta_1\sin\theta_2(\underbrace{\cos\phi_1\cos\phi_2 + \sin\phi_1\sin\phi_2}) + \cos\theta_1\cos\theta_2$

$(1 + \sin\theta_1\sin\theta_2\cos(\phi_2-\phi_1) + \cos\theta_1\cos\theta_2)\frac{1}{2} = \frac{1}{2}(1 + n_1\cdot n_2)$

Thus $|\langle n_2|n_1\rangle|^2 = \frac{1}{2}(1 + n_1\cdot n_2)$