k	f_0	ϕ
1	0	$\pi/2$
2	98	$2\pi/3$
3	140	π
4	168	$4\pi/3$

PROBLEM 4.1.* Let $x(t) = 8\sum_{k=1}^{4}\cos\left(2\pi(7k^3 - 70k^2 + 259k - 196)t + k\pi/3\right)$.

(a) Sketch the two-sided spectrum for $x(t)$, carefully labeling the frequency (in Hz) and complex coefficient (in polar form) of each line.

(b) Find the smallest positive real T for which the following equation is true for all time t:

$$x(t) = x(t + T).$$

(c) The signal $x(t)$ is periodic. Find its fundamental period T_0.

(d) The signal $x(t)$ is periodic. Find its fundamental frequency f_0, in Hz.

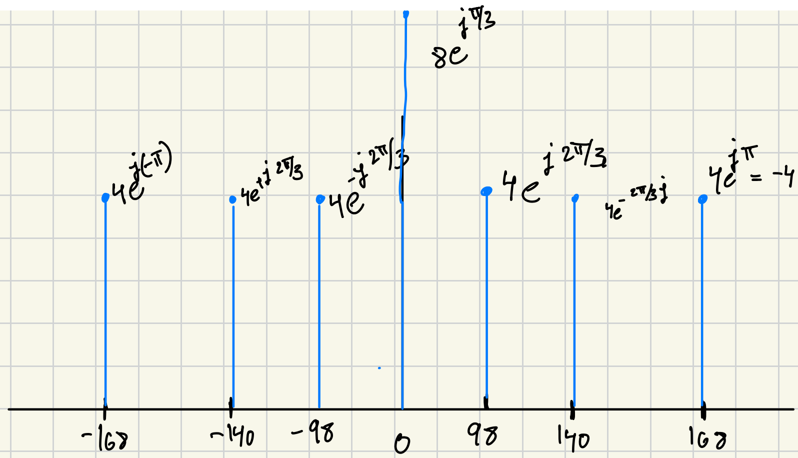
(e) Which harmonics are present in $x(t)$? (The k-th harmonic is *present* when its spectrum has a line at kf_0).

(f) The signal can be written as the Fourier series $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t}$.

Specify all of the nonzero FS coefficients $\{a_k\}$ in a table in the shown format, with the first column specifying the integer k, and the second column specifying the corresponding FS coefficient a_k. Only nonzero coefficients should appear in the table.

k	a_k
⋮	⋮

a)



b) $2\pi(7k^3 - 70k^2 + 259k - 196)T = 2\pi \quad \forall k \in \{1, 2, 3, 4\}$

$\Rightarrow T(7k^3 - 70k^2 + 259k - 196) = 1$

$k=1$: $0T = 0$

$k=2$: $98T = \alpha$ St These vars are integers

$k=3$: $140T = \beta$

$k=4$: $168T = \lambda$ $\Rightarrow f_0 = \frac{1}{T} = GCD(98, 140, 168)$

$\Rightarrow \boxed{T = 1/14\,s}$

c) $GCD(98, 140, 168) \Rightarrow 14 = f_0$

$T_0 = 1/f_0 \Rightarrow \boxed{1/14\,s}$

D) $f_0 = 1/T_0 = \boxed{14\,Hz}$

E) The 0^{th}, 7^{th}, 10^{th}, and 12^{th} Harmonics and -7^{th}, -10^{th}, -12^{th} harmonics are presed

f)

k	a_k
-12	$4e^{j(-\pi)}$
-10	$4e^{j(2\pi/3)}$
-7	$4e^{j(-2\pi/3)}$
0	$8e^{j(\pi/3)}$
7	$4e^{j(2\pi/3)}$
10	$4e^{j(-2\pi/3)}$
12	$4e^{j(\pi)}$

PROBLEM 4.2.* Consider the signal $s(t) = \sqrt{\pi e} + \sqrt{\pi}\cos(2.5\sqrt{\pi^3 e}\,t) + \sqrt{e^3}\cos(3\sqrt{\pi^3 e}\,t)$.

(a) Is it periodic? (YES or NO.)

(b) If NO, explain why not.
If YES, specify its fundamental frequency f_0 (in Hz).

$\frac{3\sqrt{\pi^3 e}}{2\pi} = f$

$\frac{2.5\sqrt{\pi^3 e}}{2\pi}$ is f

a) yes, it is periodic

B) The fundamental frequency f_0 of this signal will be the GCD $\left(\frac{2.5\sqrt{\pi^3 e},\ 3\sqrt{\pi^3 e}}{2\pi}\right)$

$\Rightarrow \Rightarrow \frac{.5\sqrt{\pi^3 e}}{2\pi} \Rightarrow \boxed{\frac{\sqrt{\pi^3 e}}{4\pi}\ \text{Hz}}$

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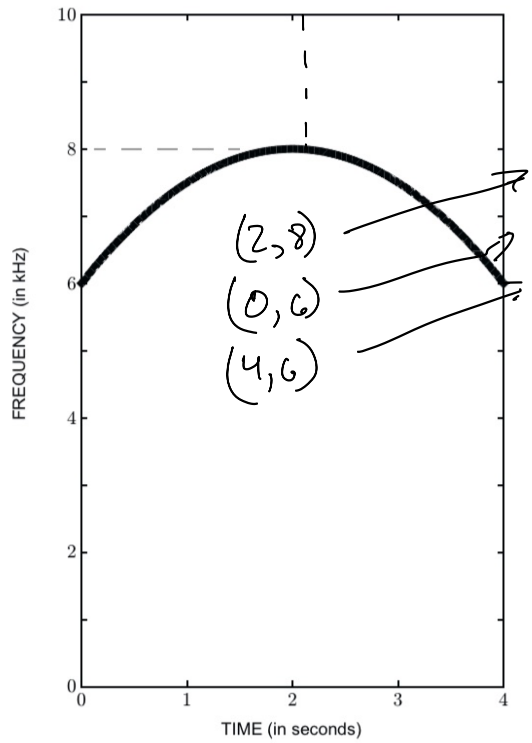
PROBLEM 4.3.* Find positive numerical values for the constants A, B, and C so that the spectrogram of

$$x(t) = \frac{AB}{C}\sin\left(A(t-2) + Bt^2 - Ct^3 - A^2BC + 0.3\pi\right)$$

for time in the range $0 < t < 4$ looks like this:



$$f_i(t) = \frac{\partial}{\partial t}\varphi(t)\frac{1}{2\pi}$$

$$= (A + 2Bt - 3Ct^2)\frac{1}{2\pi}$$

$$f_i(0) = 6 = \frac{A}{2\pi} + 0 + 0$$

$$\Rightarrow A = 12\,000\,\pi$$

$$f_i(2) = 8000 = (12000\pi + 4B - 12C)\frac{1}{2\pi}$$

$$16\,000\pi - 12000\pi = 4B - 12C$$

$$4000\pi = 4B - 12C \to 8000\pi - 4000\pi$$

$$-4000\pi / -12$$

$$f_i(4) = \cancel{(6000)\,2\pi} = \cancel{12000\pi} + 8B - 48C$$

$$0 = 8B - 48C$$

$$B = 2000\pi$$

$$C = \frac{1000\pi}{3}$$

PROBLEM 4.4.* Shown below is the spectrum of a signal $s(t)$:

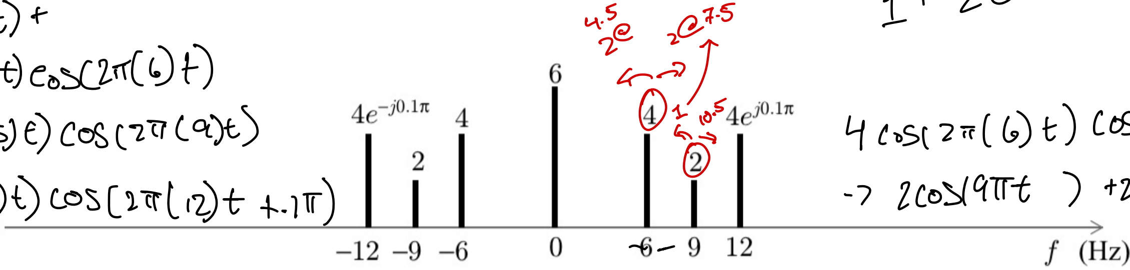
$6\cos(2\pi(1.5)t) +$

$8\cos(2\pi(1.5)t)\cos(2\pi(6)t)$

$4\cos(2\pi(1.5)t)\cos(2\pi(9)t)$

$8\cos(2\pi(1.5)t)\cos(2\pi(12)t + .1\pi)$

$1 + 2e^{j(.1\pi)}$



$4\cos(2\pi(6)t)\cos(2\pi(1.5)t)$

$\to 2\cos(9\pi t) + 2\cos(15\pi t)$

$s(t) = 6 + 8\cos(2\pi(6)t) + 4\cos(2\pi(9)t) + 8\cos(2\pi(12)t + .1\pi)$

(a) Is it periodic? If NO, explain why not. If YES, specify its fundamental frequency f_0 (in Hz).

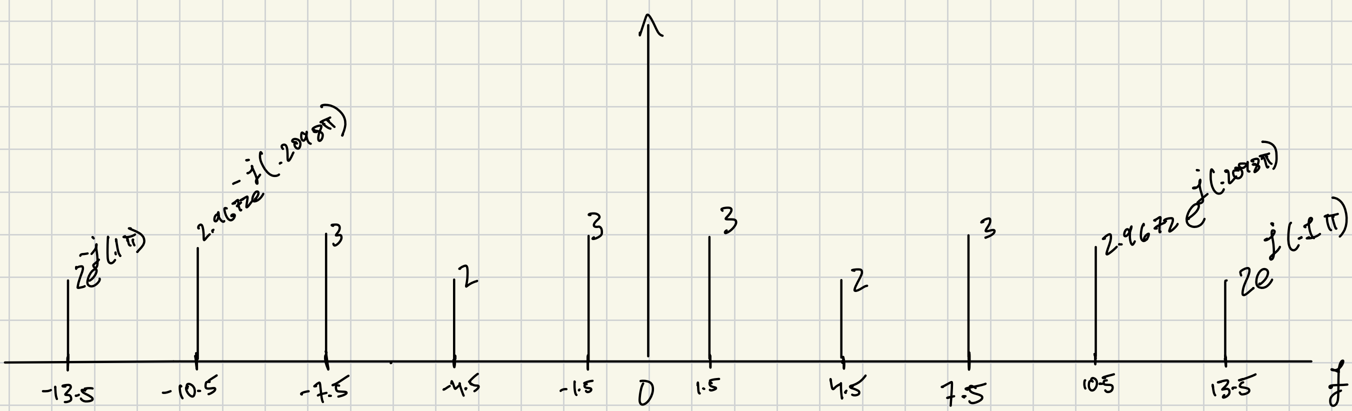
(b) Sketch the spectrum for $x(t) = s(t)\cos(3\pi t)$, carefully labeling the frequency and coefficient of each line.

(c) Sketch the spectrum for $y(t) = s(t)\cos(60\pi(t - 0.01))$, carefully labeling the frequency and coefficient of each line.

(d) Does there exist a frequency f_1 for which $s(t)\cos(2\pi f_1 t)$ is *not* periodic? If YES, specify such an f_1 (in Hz). If NO, explain.

a) yes, all frequencies present are integer multiples of a fundamental frequency $f_0 = 3\,Hz$

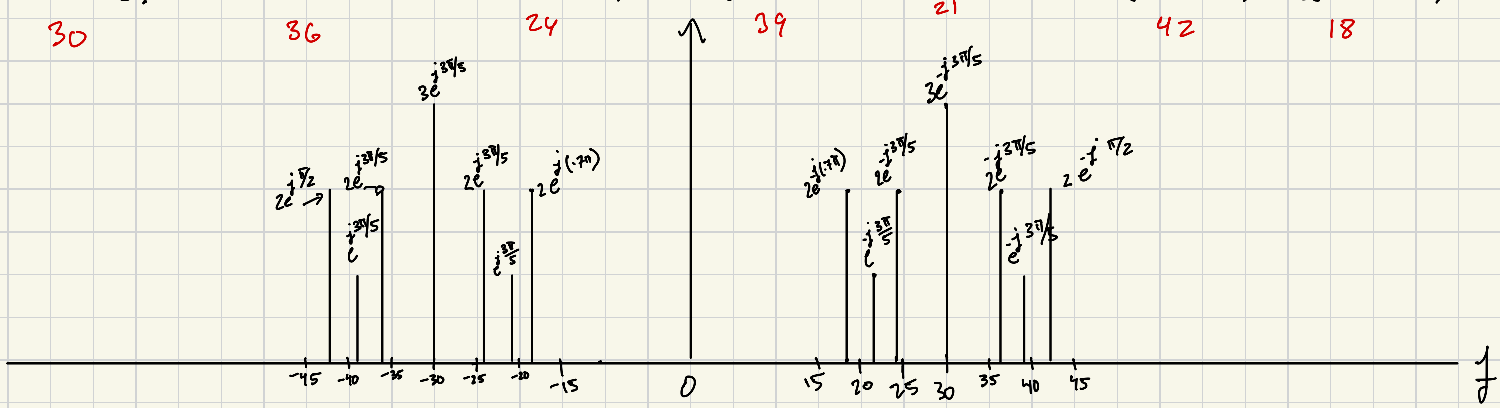
b) $x(t) = \cos(3\pi t)\, s(t) \Rightarrow \cos(2\pi(1.5)t)\, s(t)$



c) $x(t) = \cos(2\pi(30)(t - .01))\, s(t) \Rightarrow \cos(60\pi t - 3\pi/5)\, s(t)$ $f \in \{30, 36, 24, 39, 42, 84, 36\}$

$6\cos(60\pi t - \frac{3\pi}{5}) + 4\cos(72\pi t - 3\pi/5) + 4\cos(48\pi t - 3\pi/5) + 2\cos(78\pi t - 3\pi/5) + 2\cos(42\pi t - 3\pi/5) + 4\cos(84\pi t - .5\pi) + 4\cos(36\pi t - .7\pi)$

30 36 24 39 21 42 18



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D) for the Signal $x(t) = \cos(2\pi f_1 t)\, S(t)$ to break its Periodicity, the Resulting signal $x(t)$, when decomposed as a sum of sinusoids, will not have a Fundamental frequency. in other words, the GCD of all frequencies in $x(t)$ must not exist. Therefore, for the frequencies in $S(t)$ ($f_s \in \{0, 6, 9, 12\}$) the expression $(f_s - f_1)$ or $(f_s + f_1)$* must Result in an irrational number.

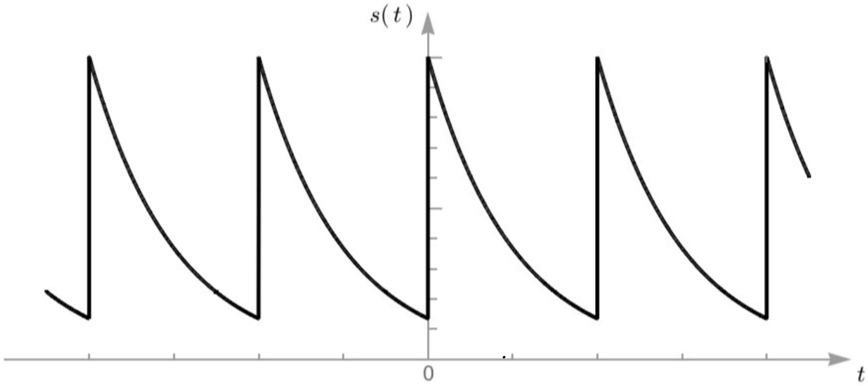
* By identity $\cos\beta \cos\alpha = \left(\cos(\beta-\alpha) + \cos(\beta+\alpha)\right)\frac{1}{2}$

Since the frequencies in $S(t)$ are Rational, having an α that is irrational would make the decomposed frequencies some Rational # $\pm$ an irrational. This would Result in an irrational # $\Rightarrow$ There does exist a f_1 to make $S(t)\cos(2\pi f_1 t)$ Non Periodic.

PROBLEM 4.5.* Consider the periodic signal $s(t) = \sum_{k=-\infty}^{\infty} \frac{e^{jk\pi t}}{1+jk\pi}$ shown below

$1 + jk\pi \Rightarrow \sqrt{1+k^2\pi^2}\, e^{j\tan^{-1}(k\pi)}$

(neither the time-axis scale nor the vertical scale is specified):



(a) Find the fundamental period T_0 (in seconds) and the fundamental frequency f_0 (in Hz).

(b) Evaluate the integral:

$$\frac{1}{T_0}\int_0^{T_0} s(t)\,dt.$$

a) $e^{jk\pi t} \Rightarrow e^{j2\pi k(\frac{1}{2})t} \Rightarrow f_0 = \frac{1}{2}$

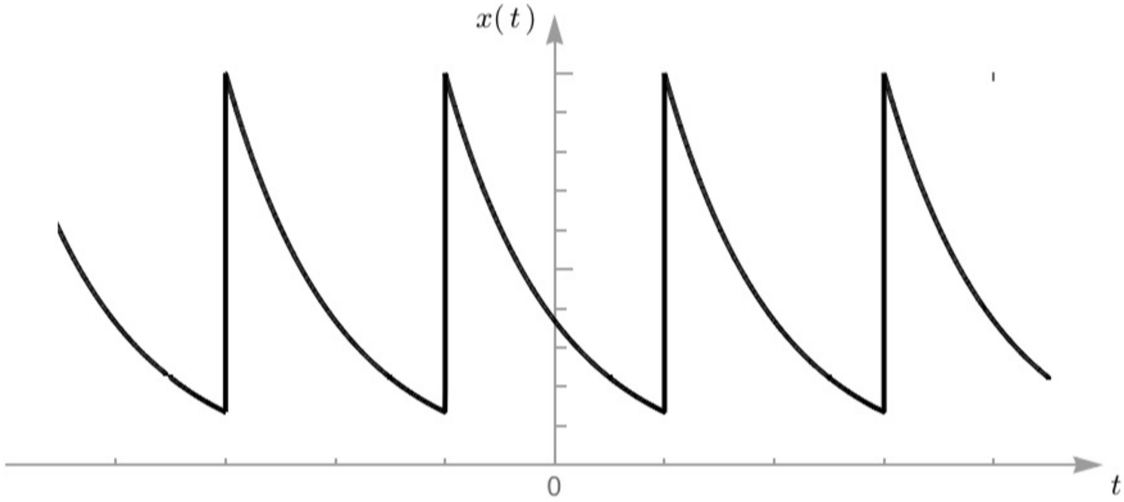
$\frac{1}{1+jk\pi} = a_k \Rightarrow s(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k(\frac{1}{2})t}$

$T_0 = 2$ seconds , $f_0 = \frac{1}{2}$ Hz

B) $a_k = \frac{1}{T_0}\int_0^{T_0} s(t) e^{-j2\pi f_0 k t} dt \Rightarrow$ when $k=0 \Rightarrow \frac{1}{T_0}\int_0^{T_0} s(t) \cancel{e^{-j(2\pi/T_0)0t}}^{\to 1} dt \Rightarrow \underbrace{\frac{1}{T_0}\int_0^{T_0} s(t)dt}_{= a_0}$

$k=0,\ s(t) = \frac{e^{j\pi(0)t}}{1+j(0)\pi} \Rightarrow 1 \Rightarrow \frac{1}{T_0}\int_0^{T_0} 1\,dt \Rightarrow \frac{1}{T_0}[T_0 - 0] \Rightarrow \frac{2}{2} = \boxed{1}$

(c) Shown below is a shifted version $x(t) = s(t - \frac{T_0}{2})$:



Find an expression for the k-th coefficient a_k in its Fourier series $x(t) = \sum_{k=-\infty}^{\infty} a_k \underline{e^{j2\pi k f_0 t}}$.
Simplify as much as possible.

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-j2\pi f_0 t k}\, dt \qquad x(t) = \sum_{k=-\infty}^{\infty} \frac{e^{j\pi k(\underline{t-1})}}{1 + kj\pi} \Rightarrow \sum_{-\infty}^{\infty} \underbrace{\frac{e^{-jk\pi}}{1+kj\pi}}_{a_k} \underbrace{e^{jk\pi t}}_{\downarrow\ e^{jk\pi(2)(\frac{1}{2})t}} = x(t)$$

$\frac{1}{2} \uparrow f_0$

$$\boxed{a_k \Rightarrow \left(\frac{1}{1+kj\pi}\right)\left(e^{-jk\pi\left(\frac{T_0}{2}\right)}\right)}$$