Question 6

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6. Let V be a vector space of dimension 5. Does there exist a linear map $T : V \to V$ such that $dim Im T = 3$ and:

(a) T has 5 distinct eigenvalues?

No. if T were to have 5 Distinct eigenvalues Then $\exists$ a matrix D, Diagonal and P inv. s.t $T = PDP^{-1}$. if there are 5 e-vals, Then D has a pivot in every column meaning It has Rank = to 5. since $T \sim D$, This contradicts $dim Im T = 3$ BC $Dim Im T = 5$, since T has Rank equal to D which is 5.

(b) T has 4 distinct eigenvalues?

Possible. By Rank-nullity, $dim(null(T)) = 2 \Rightarrow \lambda = 0 \Rightarrow E_{\lambda=0}$ has dim 2. additionally $\lambda_2, \lambda_3, \lambda_4$ are Non zero + Distinct + have E-vectors v_2, v_3, v_4 which are linearly independent. These are $\notin$ null T $\Rightarrow$ are $\in$ Im T $\Rightarrow dim(Im T) = 3$ ✓

(c) T has 4 distinct eigenvalues and T is not diagonalizable?

False. Not possible. If T is Not Digonalizable $\Rightarrow$ $\sum GM(\lambda_i) \neq n \Longrightarrow \sum_{i=1}^{4} GM(\lambda_i) < n$. But By proof 6B, Since nullity of $T = 2$ and $\lambda_2, \lambda_3, \lambda_4$ make $\{v_1, .. v_2, .. v_4\}$ lin ind $\Rightarrow \sum dim M(\lambda_i) = 2 + 1 + 1 + 1 = 5$

$= n$, a contradiction to the fact that T is not diagonizable.