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Please read all instructions carefully before beginning.

• Total there are 5 problems on pages (excluding this one).

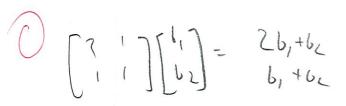
- You have 75 minutes to complete this exam.
- No CALCULATORS.
- There are no aids of any kind (notes, text, computers, phones, etc.) allowed.
- you can assume that \mathbb{R}^n , $\mathcal{P}_n = \{$ polynomials of degree less or equal to $n\}$, and $M_{m \times n}$ with the usual addition and scalar multiplication on the respected spaces are vector spaces.
- All vector spaces are real and finite dimensional.
- Write your answers in the box when provided.
- Show all your work.
- Good luck!

- 1. (4+4pts) Short Answer questions. Show your work to get full credits.
 - (a) Let $\mathcal{B}=\{b_1,b_2\}$ and $\mathcal{D}=\{d_1,d_2\}$ be bases for a vector space $\mathcal{V}.$ Suppose

$$\begin{aligned} d_1 &= 2b_1 + b_2 \\ d_2 &= b_1 + b_2 \end{aligned}$$

Find the change of coordinates matrix $[id]_{\mathcal{B} \to \mathcal{D}}$





(b) Let u and v be vectors in a real inner product space V. Show that if $||u-v||^2 = ||u||^2 + ||v||^2$, then $u \perp v$.

2. (3+3+3+3 pts)

Circle the correct answer for the following. You do not need to give a reasoning. No partial credits. For the **following questions**, let A be an $n \times n$ matrix; B and D be $m \times n$ matrices.

(a) $\det(-A) = -\det(A)$.

der (LA) = L'der(A)

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(b) Let B be an $m \times n$ matrix. Then the least solution of Bx = b is the vector $\hat{x} = (B^T B)^{-1} B^T b$.



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QQTz ortho. proj.



1) o x 1

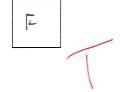
(c) If the columns of A are orthonormal in \mathbb{R}^n , then $\det(A) = 1$.







(d) If a square matrix has orthonormal columns then it aslo has orthonormal rows.



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3. (15 pts)

Let \mathcal{P}_2 be all real polynomials of degree less or equal to 2 with an ordered basis $\mathcal{B}=$ $\{1, x, x^2\}$. Define $T: \mathcal{P}_2 \to \mathcal{P}_2$ to be

$$T(p) = p(x-1)$$

(For example, $T(3+x+x^2) = 3 + (x-1) + (x-1)^2 = 3 - x + x^2$.)

- (i) Find $[T]_{\mathcal{B} \to \mathcal{B}}$ and $([T]_{\mathcal{B} \to \mathcal{B}})^{-1}$ (ii) Find $[T^{-1}]_{\mathcal{B} \to \mathcal{B}}$
- (iii) Use (ii) to find $T^{-1}(x^2 + 2x)$. (Your answer should be a polynomial for (iii)).

(i)
$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

[7] p-78 = [[7(6,)] p | [7(6)] p ... [[6]]

T(x) = X-1

 $[T]_{0-70} = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

$$T(x^2) = (x-1)^2 = x^2 - 2x + 1$$

 $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$$T^{-1}$$
: $T(p)=p(x+1)$

$$= 7 \left[T^{-1} \right]_{0.70} = \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right]$$

$$T(1) = 1$$
 $T(y) = X + 1$

T(x) = x + 1 $T(x^2) = (x+1)^2 = x^2 + 2x + 1$

4. (10 pts)

Let
$$A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$
 and $b = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$. Denote the range of A by \mathcal{R}_A .

- (i) Find an orthonormal basis of \mathcal{R}_A . (Note that the third column of A is the sum of the first two columns of A.)
- (ii) Find $proj_{\mathcal{R}_A}b$.

$$= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} \sqrt{3} \\ \sqrt{3} \\ \sqrt{3} \end{pmatrix} = 0$$

$$= \begin{pmatrix} 1 \\ \sqrt{3} \\$$

$$W_3 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} - \frac{1}{3} \begin{pmatrix} 3 \\ 1 \end{pmatrix} - \frac{1}{3$$

$$\begin{pmatrix} 2 & - & 1 \\ 0 & - & 1 \end{pmatrix}$$

$$\frac{3}{\sqrt{2}}q_1 + \sqrt{3}$$

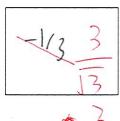
$$= \left(\frac{3}{2}\right) \left(\frac{1}{2}\right)^{\frac{1}{3}}$$

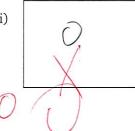
$$= \begin{pmatrix} \frac{3}{2} \\ \frac{3}{2} \\ \frac{3}{2} \\ \frac{1}{3} \\$$

5. (15 pts)

Let
$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$
, $v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, and $L = span\{v_1, v_2\}$.

- (i) Find a unit vector $w \in \mathbb{R}^3$ such that $w \perp L$.
- (ii) Find the volume of the parallelepiped formed by the vectors v_1, v_2, w .
- (iii) Find the area of the triangle formed by the points (0,0,0), (1,1,0), (1,0,1) in \mathbb{R}^3 .





$$(w_1)$$
 (v_2) (v_3) (v_4) (v_5) $(v_5$

$$\left(\begin{array}{c} \omega_{1} \\ \omega_{2} \\ \end{array}\right), \left(\begin{array}{c} 0 \\ \end{array}\right) = 0$$

$$\left(\begin{array}{c} \omega_{1} \\ \end{array}\right), \left(\begin{array}{c} 1 \\ \end{array}\right) = 0$$

$$w_3 = w_2$$

$$w_3 = -\omega_1$$

$$w_3 = -\omega_1$$

$$\frac{1}{2} \int_{0}^{1} \frac{1}{3} \int_$$

$$1(-1/3) - 1(2/52)$$
 $-1/3 - 2/53 = -1/3$