Question 1

Tuesday, November 14, 2023 9:55 PM

1. Show that similarity defines an equivalence relation on $M_n(\mathbb{R})$.

Reflexive: $\forall A \in M_n(\mathbb{R})$

if $A = SAS^{-1}$, $S = I_n$, $\Rightarrow \exists I_n$ s.t

$A = I_n A I_n^{-1} \Rightarrow I_n A = A$ $\quad A \sim A$

Symmetric: $\forall A, B \in M_n(\mathbb{R})$

if $A = SBS^{-1}$, $S \in GL(\mathbb{R})$

$S^{-1}A = S^{-1}SBS^{-1} \Rightarrow S^{-1}A = BS^{-1}$

$\Rightarrow S^{-1}AS = BS^{-1} = B = S^{-1}AS$

$A \sim B$

Transitive: $\forall A, B, C \in M_n(\mathbb{R})$

if $A = SBS^{-1}$ and $B = PCP^{-1}$

$\Rightarrow A = S(PCP^{-1})S^{-1} \Rightarrow A = SPCP^{-1}S^{-1}$

let $D = SP$ $\quad D^{-1} = P^{-1}S^{-1} \Rightarrow A = DCD^{-1}$

$n \times n$ $\quad n \times n$

$\underline{A \sim C}$

Question 2

Sunday, November 19, 2023 4:48 PM

2. Let A, B be matrices similar to each other.
   (a) Show that they have the same eigenvalues. Do they have the same eigenvectors?
   (b) Show that they have the same rank.
   (c) Show that they have the same trace.

a) if $A = SBS^{-1}$ for $S \in GL_n(\mathbb{R})$

and if $A\vec{v} = \lambda\vec{v}$

consider $SBS^{-1}\vec{v} = \lambda\vec{v}$

$\Rightarrow BS^{-1}\vec{v} = \lambda S^{-1}\vec{v}$

$\therefore$ λ is an eigenvalue of A for vector $\vec{v}$

and thus λ is also an eigenvalue for B

But w/ dif eigenvector $S^{-1}\vec{v} \neq \vec{v}$ $\forall \vec{v} \in \mathbb{R}^n$

$S \in GL_n(\mathbb{R})$

B) consider for matrix A, there is set

$\lambda_1, \ldots, \lambda_p$, Distinct eigenvalues. Then there are at least

p linearly independent eigenvectors, eigenbasis.

Same for B where $\lambda_1, \ldots, \lambda_p$ are Distinct e-vals

and There are at least p lin ind e-vectors of B.

To form an eigenbasis of B.

Thus # lin. ind e-vects for A $(>p)$ $\Rightarrow$

A - - - - . . for B ($>P$)

$$\Rightarrow \underbrace{\dim\{v_1, v_2, v_3, \dots, v_n\}}_{A \text{ eigenbasis}} = \underbrace{\dim\{x_1, x_2, \dots, x_n\}}_{B \text{ eigenbasis}}$$

c) lemma $\quad tr(AB) = tr(BA)$

if $A = PDP^{-1}$

$tr(A) = tr(PDP^{-1})$, let $DP^{-1} = C$

$\Rightarrow tr(PC) \Rightarrow tr(CP) \Rightarrow tr(DP^{-1}P) \Rightarrow tr(DI_n)$

$\Rightarrow tr(D) = tr(A)$

Question 3

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3. Consider the following matrices.

$$A = \begin{pmatrix} 8 & 3 & -3 \\ -6 & -1 & 3 \\ 12 & 6 & -4 \end{pmatrix} \quad B = \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} \quad C = \begin{pmatrix} 4 & 0 & 3 \\ 0 & 5 & 0 \\ 3 & 0 & -4 \end{pmatrix}$$

$$E = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{pmatrix} \quad F = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \quad G = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

For each of these matrices solve the following:

(a) Find all of the eigenvalues of the matrix, determine the geometric multiplicity and the algebraic multiplicity of each eigenvalue.

(b) Determine if the matrix is diagonalizable. If it is then find a diagonal matrix D and an invertible matrix P so that the matrix is equal to PDP^{-1}.

a) $\begin{bmatrix} 8 & 3 & -3 \\ -6 & -1 & 3 \\ 12 & 6 & -4 \end{bmatrix}$ $\det\left(\begin{bmatrix} 8-\lambda & 3 & -3 \\ 6 & -1-\lambda & 3 \\ 12 & 6 & -4-\lambda \end{bmatrix}\right) = P_A(\lambda)$

$P_A(\lambda) = -\lambda^3 + 3\lambda^2 - 4$

$(\lambda+1)(\lambda-2)^2 = P_A(\lambda)$

B) yes

$A = \begin{bmatrix} -1 & 1 & 1 \\ 2 & 0 & -1 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 \\ 2 & 0 & -1 \\ 0 & 2 & 2 \end{bmatrix}^{-1}$

for $\lambda = -1$, $AM = 1$, $GM = 1$

$E_{\lambda=-1} = \text{null}\begin{bmatrix} 9 & 3 & -3 \\ -6 & 0 & 3 \\ 12 & 6 & -3 \end{bmatrix} = \text{span}\left\{\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}\right\}$

for $\lambda = 2$, $AM = 2$, $GM = 2$

$E_{\lambda=2} = \text{null}\begin{bmatrix} 6 & 3 & -3 \\ -6 & -3 & 3 \\ 12 & 6 & -6 \end{bmatrix} = \text{span}\left\{\begin{pmatrix} -1/2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 0 \\ 1 \end{pmatrix}\right\}$

$6x_1 = 3x_3 - 3x_2$

$x_1 = \frac{1}{2}x_3 - \frac{1}{2}x_2$

$x_2 = x_2$

$x_3 = x_3$

$\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$ $P_B(\lambda) = -\lambda^3 - 3\lambda^2 + 9\lambda + 27$

$(\lambda+3)^2(\lambda-3)$

for $\lambda=-3$, $Am=2$, $Gm=2$

$$E_{\lambda=-3} = \text{null}\left\{\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}\right\} = \text{span}\left\{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right\}$$

for $\lambda=3$, $Am=1$, $Gm=1$

$$E_{\lambda=3} = \text{null}\begin{bmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{bmatrix} = \text{Span}\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}$$

B) yes

$$B = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}^{-1}$$

$$C = \begin{pmatrix} 4 & 0 & 3 \\ 0 & 5 & 0 \\ 3 & 0 & -4 \end{pmatrix}$$

$$p_C(\lambda) = 5-\lambda\begin{vmatrix} 4-\lambda & 3 \\ 3 & -4-\lambda \end{vmatrix} = (5-\lambda)\left[(4-\lambda)(-4-\lambda)-9\right]$$

$-16+\lambda^2-9$

λ^2-25

$(5-\lambda)(\lambda-5)(\lambda+5)=0$

$(\lambda-5)^2(\lambda+5)$

B) $$C = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -3 \end{bmatrix}\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -5 \end{bmatrix}\begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -3 \end{bmatrix}^{-1}$$

for $\lambda=5$, $Am=2$, $Gm=2$

$$E_{\lambda=5} = \text{null}\left\{\begin{matrix} -1 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & -9 \end{matrix}\right\} = \text{Span}\left\{\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right\}$$

for $\lambda=-5$, $Am=1$, $Gm=1$

$$E_{\lambda=-5} = \text{null}\begin{bmatrix} 9 & 0 & 3 \\ 0 & 10 & 0 \\ 3 & 0 & 1 \end{bmatrix} = \text{Span}\left\{\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}\right\}$$

$$E = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

B) $$E = \begin{bmatrix} 1 & 1/6 & 1 \\ 0 & -1/3 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1/6 & 1 \\ 0 & -1/3 & 1 \\ 0 & 1 & 0 \end{bmatrix}^{-1}$$

$P_E(\lambda) = (\lambda-2)(\lambda-1)(\lambda+1)$

for $\lambda=2$ $Am=1$, $Gm=1$

for $\lambda=1$ $Am=1$ $Gm=1$

for $\lambda = -1$ $Am = 1$, $Gm = 1$

$$F = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

B) Not Diagonizable

$$P_F(\lambda) = (\lambda - 2)^3 (\lambda - 3)$$

for $\lambda = 2$, $Am = 3$, $Gm = 1$

$$E_{\lambda=2} = \text{null}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

for $\lambda = 3$ $Am = 1$, $Gm = 1$

$$E_{\lambda=3} = \text{null}\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \text{Span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$G = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

B) $$G = \begin{bmatrix} 0 & 0 & -i & i \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{bmatrix} \begin{bmatrix} 0 & 0 & -i & i \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}^{-1}$$

$$P_G(\lambda) = (\lambda - 2)(\lambda - 3)(\lambda^2 + 1)$$

for $\lambda = 2$, $Am = 1$, $Gm = 1$

$$\text{null}\begin{bmatrix} -2 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \text{Span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

for $\lambda = 3$, $Am = 1$, $Gm = 1$

$\text{null}[-3 \quad 1 \quad 0 \quad 0]$

$$\text{Nul}\begin{bmatrix} -1 & -3 & 0 & 0 \\ 0 & 0 & -1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \text{Span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

for $\lambda = i$, $Am = 1$, $Gm = 1$

for $\lambda = -i$, $Am = 1$, $Gm = 1$

Question 4

Sunday, November 19, 2023 6:50 PM

4. Consider the matrix E from Q1

$$E = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

(a) Find the eigenvalues of E^2. Is E^2 diagonalizable?

$$E = \underbrace{\begin{bmatrix} 1 & 1/6 & 1 \\ 0 & -1/3 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & 1/6 & 1 \\ 0 & -1/3 & 1 \\ 0 & 1 & 0 \end{bmatrix}^{-1}}_{P^{-1}}$$

$$E^2 = (PDP^{-1})(PDP^{-1}) \Rightarrow PDP^{-1}PDP^{-1} = PD^2P^{-1}$$

$$D^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Diagonizable ✓, $\boxed{\text{e-vals: } 1, 4}$

(b) Find the eigenvalues of E^{10}. Is E^{10} diagonalizable?

yes E^{10} Diagonizable

e-vals = $1, 2^{10} \rightarrow 1, 1024$

(c) Find the eigenvalues of $E^3 - 5E^2 + 2E + 3I$. Is $E^3 - 5E^2 + 2E + 3I$ diagonalizable?

$$\begin{bmatrix} 1 & & \\ & -1 & \\ & & 8 \end{bmatrix} - \begin{bmatrix} 5 & & \\ & 5 & \\ & & 20 \end{bmatrix} + \begin{bmatrix} 2 & & \\ & -2 & \\ & & 4 \end{bmatrix} + \begin{bmatrix} 3 & & \\ & 3 & \\ & & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{bmatrix} \rightarrow \begin{array}{l} PD^3P^{-1} - 5PD^2P^{-1} + 2PDP^{-1} + 3I_n \\ P\underbrace{(D^3 - 5D^2 + 2D + 3I_n)}P^{-1} \end{array}$$

Matrix e-vals: 1, -5 $\Rightarrow$ Diagonizable ✓

(d) Is E invertible? If so, find the eigenvalues of E^{-1}. Is E^{-1} diagonalizable?

$E = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{pmatrix}$ $\det E = -2 \neq 0 \therefore$ E is invertible

If $E\vec{v} = \lambda \vec{v}$

$[E = PDP^{-1}]^{-1} \Rightarrow E^{-1} = P^{-1^{-1}} D^{-1} P^{-1} \Rightarrow \underline{E^{-1} = P D^{-1} P^{-1}}$

Diagonizable

$D^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$ e-vals : $1, -1, 1/2$

$E = \begin{bmatrix} 1 & 1/6 & 1 \\ 0 & -1/3 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1/6 & 1 \\ 0 & -1/3 & 1 \\ 0 & 1 & 0 \end{bmatrix}^{-1}$

$P \qquad D \qquad P^{-1}$

(e) Compute E^5.

$E^5 = PD^5P^{-1} \Rightarrow \begin{bmatrix} 1 & 1/6 & 1 \\ 0 & -1/3 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 32 \end{bmatrix} \begin{bmatrix} 1 & 1/6 & 1 \\ 0 & -1/3 & 1 \\ 0 & 1 & 0 \end{bmatrix}^{-1}$

$= \begin{bmatrix} 1 & 31 & 10 \\ 0 & 32 & 11 \\ 0 & 0 & -1 \end{bmatrix}$

Question 5

Sunday, November 19, 2023 8:42 PM

5. Each of the following you are given a linear map. Determine whether it is diagonalizable.

(a) $T : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ given by

$$TA = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} A$$

$$P_T(\lambda) = (1-\lambda)(4-\lambda) - 4 \Rightarrow 4 - 5\lambda + \lambda^2 - 4 \Rightarrow \lambda^2 - 5\lambda \; ; \lambda = 0 \;\&\; 5$$

for $\lambda = 0$, $Am = 1$, $Gm = 1$

$$E_{\lambda=0} = \text{null}\begin{Bmatrix} 1 & 2 \\ 2 & 4 \end{Bmatrix} = \text{span}\left\{\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right\}$$

for $\lambda = 5$, $Am = 1$, $Gm = 1$

$$E_{\lambda=5} = \text{null}\begin{Bmatrix} -4 & 2 \\ 2 & -1 \end{Bmatrix} = \text{span}\left\{\begin{pmatrix} 1 \\ -2 \end{pmatrix}\right\}$$

$\therefore \; \forall \lambda_i \quad Am_{\lambda_i} = Gm_{\lambda_i} \Rightarrow$ Diagonizible

(b) $T : \mathbb{R}_2[x] \to \mathbb{R}_2[x]$ given by $Tp(x) = x(p(x+1) - p(x))$

$\mathcal{B} = \langle 1, x, x^2 \rangle$

$p(x) = 1 \longrightarrow x(p(x+1) - p(x)) \Rightarrow x(0) = 0$

$p(x) = x \longrightarrow x(x+1-x) = x$

$p(x) = x^2 \longrightarrow x((x+1)^2 - x^2) = 2x^2 + x$

$x^2 + 2x + 1 - x^2$

$$[T]_B = \begin{bmatrix} [Tb_1]_B & [Tb_2]_B & [Tb_3]_B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$P_T(\lambda) = -\lambda \begin{vmatrix} 1-\lambda & 1 \\ \end{vmatrix} \; -\lambda[(1-\lambda)(2-\lambda) - 0] \Rightarrow (-\lambda)(1-\lambda)(2-\lambda) = P_T(\lambda)$

$$P_B(\lambda) = \begin{vmatrix} \ldots \\ 0 & 2-\lambda \end{vmatrix} = -\lambda\left[(1-\lambda)(2-\lambda) - 0\right] \Rightarrow (\lambda)(1-\lambda)(2-\lambda) = P_B(\lambda)$$

$\lambda = 0, 1, 2$

for $\lambda = 0$ $\quad E_{\lambda=0} = \text{null}\left\{\begin{matrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{matrix}\right\} = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}$ $\quad AM = GM = 1$

for $\lambda = 1$ $\quad E_{\lambda=1} = \text{null}\left\{\begin{matrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{matrix}\right\} = \text{span}\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right\}$ $\quad AM = GM = 1$

for $\lambda = 2$ $\quad E_{\lambda=2} = \text{null}\left\{\begin{matrix} -2 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{matrix}\right\} = \text{span}\left\{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\}$ $\quad AM = GM = 1$

✓ **Diagonizable**

(c) Let V be a vector space and $B = (v_1, v_2, v_3)$ a basis for V. Here we consider the linear transformation $T : V \to V$ which satisfies $Tv_1 = 5v_1$, $Tv_2 = v_2 + 2v_3$ and $Tv_3 = 2v_2 + v_3$.

$$[T]_B = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

$$P_{T_B}(\lambda) = (5-\lambda)\left[(1-\lambda)(1-\lambda) - 4\right] \Rightarrow (5-\lambda)(1 - 2\lambda + \lambda^2 - 4) \Rightarrow$$
$$(5-\lambda)(\lambda-3)(\lambda+1) = P(\lambda)$$

for $\lambda = 5$

$E_5 = \text{null}\left\{\begin{matrix} 0 & 0 & 0 \\ 0 & -4 & 2 \\ 0 & 2 & -4 \end{matrix}\right\} = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}$ $\quad AM = GM = 1$

for $\lambda = 3$

$E_3 = \text{null}\left\{\begin{matrix} 2 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 2 & -2 \end{matrix}\right\} = \text{span}\left\{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\}$ $\quad AM = GM = 1$

for $\lambda = -1$

$E_{-1} = \text{null}\left\{\begin{matrix} 6 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{matrix}\right\} = \text{span}\left\{\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}\right\}$ $\quad AM = GM = 1$

Diagonizable since $\forall \lambda_i$ $AM_i = GM_i$ ✓

Question 6

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6. Let V be a vector space of dimension 5. Does there exist a linear map $T : V \to V$ such that $dim Im T = 3$ and:

(a) T has 5 distinct eigenvalues?

No. if T were to have 5 Distinct eigenvalues Then $\exists$ a matrix D, Diagonal and P inv. s.t $T = PDP^{-1}$. if There are 5 e-vals Then D has a pivot in every column meaning It has Rank = to 5. since $T \sim D$, This contradicts $\dim Im T = 3$ BC $\dim Im T = 5$, since T has Rank equal to D which is 5.

(b) T has 4 distinct eigenvalues?

Possible. By Rank-nullity, $\dim(null(T)) = 2 \Rightarrow \lambda = 0 \Rightarrow E_{\lambda=0}$ has dim 2. additionally $\lambda_2, \lambda_3, \lambda_4$ are Non zero + Distinct + have E-vectors v_2, v_3, v_4 which are linearly independent. These are $\notin$ Null T $\Rightarrow$ are $\in$ Im T $\Rightarrow \dim(Im T) = 3$ ✓

(c) T has 4 distinct eigenvalues and T is not diagonalizable?

False. Not possible. If T is Not Digonaizable $\Rightarrow$ $\sum GM(\lambda_i) \neq n \Longrightarrow \sum_{i=1}^{4} GM(\lambda_i) < n$. But By proof GB, Since Nullity of T = 2 and $\lambda_2, \lambda_3, \lambda_4$ make $\{v_2, v_3, v_4\}$ lin ind $\Rightarrow \sum \dim M(\lambda_i) = 2 + 1 + 1 + 1 = 5$

$= n$, a contradiction to the fact that T is not diagonizable.

Question 7

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7. Prove or disprove the following claims.

(a) If $A \in M_3(\mathbb{R})$ has rows equal to v $2v$ $3v$ for some $v \in \mathbb{R}^3$ and A has a nonzero eigenvalue then A is diagonalizable.

$$\begin{array}{ccccc} a-\lambda & b & c & a-\lambda & b \\ 2a & 2b-\lambda & 2c & 2a & 2b-\lambda \\ 3a & 3b & 3c-\lambda & 3a & 3b \end{array}$$

TRUE

$(a-\lambda)(2b-\lambda)(3c-\lambda) + \cancel{6abc} + \cancel{6abc} - \cancel{6abc} + \cancel{3ac\lambda} - \cancel{6abc} + \cancel{6bc\lambda} - 6abc + 2ab\lambda$

$\cancel{6abc} - \cancel{3ac\lambda} - \cancel{6bc\lambda} + 2b\lambda^2 + a\lambda^2 + 3c\lambda^2 - \lambda^3$

$\lambda^2(2b + a + 3c - \lambda)$ $\lambda = 0$ and $2b + a + 3c \neq 0$

for $\lambda = 0$, $\dim(E_0) = 2$ $GM = AM = 2$

For $\lambda = a + 2b + 3c \neq 0$ $\dim E = 1$ $AM = GM = 1$

$\forall \lambda_i$ $GM_{\lambda_i} = AM_{\lambda_i} \Rightarrow$ Diagonizable

(b) If $A \in M_4(\mathbb{R})$ has characteristic polynomial $q_A(x) = x^2(x+5)(x+6)$ and

$$\begin{pmatrix} 0 \\ -1 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 7 \\ 5 \\ 3 \\ 4 \end{pmatrix} \in null(A)$$

then A is diagonalizable.

TRUE

Consider each E-Val of A Based on $q_A(\lambda)$

for $\lambda = 0$, The span of E_0 = nullity A; $\dim(E_0) = 2 = AM$

Additionally eigenvalues -5 and -6 each have eigenvectors with eigenspaces of span 1.

$\Rightarrow \forall \lambda_i$ The $GM(\lambda_i) = AM(\lambda_i) \Rightarrow$ A Digonizable

(c) Let $A \in M_n(\mathbb{R})$. If 0 is an eigenvalue of A then its geometric multiplicity is equal to $n - rankA$.

By Rank - nullity

$\dim\{Rank(A)\} + \dim\{Null(A)\} = n \quad \therefore \dim\{null(A)\} = n - \dim(Rank(A))$

and Geometric multiplicity for $\lambda = 0$ is

The nullity of matrix since for any

maindiagonal elements $A - \lambda I_n \Rightarrow A - \vec{0} \Rightarrow A$

$GM(\lambda = 0) = \dim(null(A))$

TRUE

(d) There exists $A \in M_5(\mathbb{R})$ which is diagonalizable and satisfies $rankA = 1$ and $trA = 0$.

if $Rank(A) = 1 < 5 \Rightarrow \det(A) = 0 \Rightarrow 0$ is eigenvector

Note By Rank-nullity $\dim(null\,A) = 4 \Rightarrow \dim\{E_{\lambda=0}\} = 4$

if $\exists$ Diagonal D of $A = PDP^{-1}$ for $P \in GL_5(\mathbb{R})$

Then $tr(A) = tr(D) = 0$

But if D is Digonal of A of The

form $A = PDP^{-1} \Rightarrow$ D has The eigenvalues of

A down its main digonal. Since Rank $A = 1$, $\exists$ 1 Nonzero

eigenvalue as The 5^{th} element on main diagonal in D.

This contradicts $tr(D) = 0$ since all 4 other e-vals $= 0$

and $\lambda_5 \neq 0$. $\Rightarrow$ Statement false.

(e) If $A \in M_n(\mathbb{R})$ is diagonalizable and 2 is the only eigenvalue of A then $A = 2I$.

if $\lambda = 2$ for $A \in M_n(\mathbb{R}) \Rightarrow$ The AM of $\lambda = 2$ is going to be n since if Diagonizable $\Rightarrow \sum AM = n \Rightarrow AM(2) = n$. But Diagonizable also implies $AM = GM$ $\forall \lambda_i \Rightarrow GM$ of $\lambda = 2$ is also n.

$\Rightarrow \dim\{null\{A - \lambda I_n\}\} = n \Rightarrow$ every column is a free variable $\Rightarrow A - \lambda I_n = [0] \Rightarrow \underline{A = 2I_n}$

Statement TRUE

(f) If $A, B \in M_n(\mathbb{R})$ have the same eigenvalues and A is diagonalizable then so is B.

False

consider $A =$ zero matrix

A has 0 as its eigenvalues

Consider also $B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

This matrix only has eigenvalues of 0

B is Not Digonizable.