# Question 7

Sunday, October 15, 2023 10:09 PM

7. The following claims are either **true or false**. Determine which case is it for each claim and prove your answer.
   (a) For any two matrices $A \in M_{m\times n}(\mathbb{R})$ and $B \in M_{n\times k}(\mathbb{R})$ we have $\text{rank}(AB) = \text{rank}(A) \cdot \text{rank}(B)$.
   (b) If $A$ is a square matrix then its column space is equal to its null space.
   (c) If $A \in M_{m\times n}(\mathbb{R})$ is such that the vectors in its rows are linearly independent then the vectors in its columns are also linearly independent.
   (d) If $A \in M_n(\mathbb{R})$ is such that the vectors in its rows are linearly independent then the vectors in its columns are also linearly independent

a) false.

let $A = \begin{bmatrix} 2 & 3 \\ 7 & 1 \end{bmatrix}$ $\text{RREF}(A) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\text{Rank}(A) = 2$

$B = \begin{bmatrix} 4 & 12 \\ 8 & 1 \end{bmatrix}$ $\text{RREF}(B) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\text{Rank}(B) = 2$

$AB = \begin{bmatrix} 8+24 & 24+3 \\ 28+8 & 84+1 \end{bmatrix} = \begin{bmatrix} 32 & 27 \\ 36 & 85 \end{bmatrix} \overset{\text{RREF}(AB)}{=} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\text{Rank}(AB) = 2$

$\text{Rank}(AB) = 2 \neq 4 = \text{Rank}(A) \cdot \text{Rank}(B)$

b) If $A$ is a square matrix then its column space is equal to its null space.

Consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, Then $\text{Col}(A) = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

But $\text{Nul}(A) = \{\vec{0}\} \neq \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$, Thus false

c)

(c) If $A \in M_{m\times n}(\mathbb{R})$ is such that the vectors in its rows are linearly independent then the vectors in its columns are also linearly independent.

$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$ vectors $\{[1,2,1],[1,2,3]\}$

are linearly independent

But $\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}2\\2\end{bmatrix},\begin{bmatrix}1\\3\end{bmatrix}\right\}$ are LD

false

D)

(d) If $A \in M_n(\mathbb{R})$ is such that the vectors in its rows are linearly independent then the vectors in its columns are also linearly independent

True! If $A$ is such a matrix with lin. ind. Rows

Then the Rank of $A$ $Rank(A) = n$.

$\Rightarrow \dim(Rows(A)) = n$, $n$ linearly independent Rows

Note: $Rank(A)$ implies it is equal to Both the

$\dim(Row(A))$ and $\dim(Col(A))$.

$\Rightarrow n = \dim(col(A))$. Meaning $n$ independent Colums

Thus all colums in $A$ are linearly independent