Question 1

Sunday, October 22, 2023 6:49 PM

1. Consider the following linear transformations $T, S : \mathbb{R}_3[x] \to \mathbb{R}_3[x]$ given by

$$Tp(x) = p'(x) \quad \text{and} \quad Sp(x) = p(x+1)$$

and consider the following bases of $\mathbb{R}^3[x]$:

$$\mathcal{E} = \{1, x, x^2, x^3\}$$

and

$$\mathcal{B} = \{1, 1+x, (1+x)^2, (1+x)^3\}$$

(a) Find $[T]_{\mathcal{E}\to\mathcal{B}}, [T]_{\mathcal{B}\to\mathcal{B}}, [T]_{\mathcal{B}\to\mathcal{E}}, [T]_{\mathcal{E}\to\mathcal{E}}$.

$$[T]_{\mathcal{E}\to\mathcal{B}} = \begin{pmatrix} [T(e_1)]_{\mathcal{B}} & \cdots & [T(e_n)]_{\mathcal{B}} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 & 1 & -2 & 3 \\ 0 & 0 & 2 & -6 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$3x^2 + 6x + 3$

$$[T]_{\mathcal{B}\to\mathcal{B}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$[T]_{\mathcal{B}\to\mathcal{E}} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$[T]_{\mathcal{E}\to\mathcal{E}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(b) Find $[S]$ …

(b) Find $[S]_{\mathcal{E}\to\mathcal{E}}$, $[S]_{\mathcal{B}\to\mathcal{B}}$.

$$[S]_{E\to E} = \left([S(e_1)]_e \ \cdots \ [S(e_n)]_e \right)$$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$[S]_{B\to B} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(c) Find $[T \circ S]_{\mathcal{E}\to\mathcal{E}}$.

$$T\circ S = T(S(p(x)))$$
$$\Rightarrow T(p(x+1))$$
$$\Rightarrow (p(x+1))\frac{d}{dx}$$

$$[T]_{B\to E}\,[S]_{E\to B}$$

$$[T\circ S]_{E\to E} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(d) $[T \circ S]_{\mathcal{B}\to\mathcal{B}}$.

$$T\circ S = T(S(p(x)))$$
$$\Rightarrow T(p(x+1))$$

$$[T]_{E\to B}\,[S]_{B\to E}$$

$\Rightarrow \left(P(x+1)\right)\frac{d}{dx}$

$$[T \circ S]_{B \to B} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(e) Use $[T]_{\mathcal{E}\to\mathcal{E}}$ to find a basis for the kernel and image of L.

$$[T]_{\mathcal{E}\to\mathcal{E}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\to$ Basis

$$\ker(T) = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$\text{image}(T) = \{ y \mid T(p(x)) = p'(x) = y \}$$

$$\text{im}(T) = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 3 \\ 0 \end{pmatrix} \right\}$$