Question 1

Monday, October 2, 2023 9:49 AM

1. Answer the following questions.
   (a) Do elementary row operations affect a matrix's column space? Justify your response.
   (b) Do elementary row operations affect a matrix's null space? Justify your response.

a) Yes they do. Consider a matrix with more than 1 row and 1 column. We if the second row is nonzero, then the span of the column space of matrix exists in R^n $n>1$. But ERO can reduce elements in subsequent columns down to 1 and eliminate all other rows to zero making the first row only non-zero. This makes col space span R^1, changing it.

B) No. The matrices Nullspace is the set of vectors where $A\vec{x}=\vec{0}$.
Consider matrix A and its $RREF(A)=B$.
Since A and B are row-equivalent, meaning $\exists$ a set of EROs to turn $B \rightleftarrows A$, then $A\vec{x}=\vec{0}$ also means $B\vec{x}=0$. As such, the EROs maintain the relationship between rows and the nullspace is unchanged.

$Ax=0$ & $Bx=0$

Question 2

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2. Find a basis and then the dimension of the following subspace:

$$\text{span}\{2 + x^2 - 2x^3, 1 - 2x + x^2 - x^3, 5 + 2x + 2x^2 - 5x^3, 3 + 6x - 3x^3\}$$

$$\left[\begin{array}{cccc|c} 2 & 0 & 1 & -2 & 0 \\ 1 & -2 & 1 & -1 & 0 \\ 5 & 2 & 2 & -5 & 0 \\ 3 & 6 & 0 & -3 & 0 \end{array}\right]$$

switch R_2 & R_1

Redo

$$\left[\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 0 \\ 2 & 0 & 1 & -2 & 0 \\ 5 & 2 & 2 & -5 & 0 \\ 3 & 6 & 0 & -3 & 0 \end{array}\right]$$

$$\begin{bmatrix} 3 \\ 6 \\ 0 \\ -3 \end{bmatrix} = 3\begin{bmatrix} 2 \\ 0 \\ 1 \\ -2 \end{bmatrix} - 3\begin{bmatrix} 1 \\ -2 \\ 1 \\ -1 \end{bmatrix}$$

Elim last Row

$$\left[\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 0 \\ 2 & 0 & 1 & -2 & 0 \\ 5 & 2 & 2 & -5 & 0 \end{array}\right]$$

But

$$\begin{bmatrix} 5 \\ 2 \\ 2 \\ -5 \end{bmatrix} = 3\begin{bmatrix} 2 \\ 0 \\ 1 \\ -2 \end{bmatrix} - \begin{bmatrix} 1 \\ -2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 6-1 \\ 0+2 \\ 3-1 \\ -6+1 \end{bmatrix}$$

Thus eliminate last Row

$$\left[\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 0 \\ 2 & 0 & 1 & -2 & 0 \end{array}\right]$$

Thus

$\{1-2x+x^2-x^3,\ 2+x^2-2x^3\}$ forms a Basis and has $\dim = 2$

Question 3

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3. Let V be a vector space over $\mathbb{R}$ and $U, W \subseteq V$ be two subspaces of V.
   (a) Prove that there exist a basis B of U and a basis C of W such that $B \cap C$ is a basis for $U \cap W$.

Consider the space $\bar{U} \cap \bar{W}$ where set B and set C where $B \cap C$ is a basis $\{v_1 \dots v_n\}$. Consider vectors $\{b_1, \dots, b_n\}$ and $\{c_1, \dots, c_p\}$ where any element $y \in \bar{U}$ and $z \in \bar{W}$

$$y = \sum_{i=1}^{n} a_i v_i + \sum_{i=1}^{m} g_i b_i \; \{v_1, \dots, v_n, b_1, \dots, b_n\}$$

$$z = \sum_{i=1}^{n} a_i v_i + \sum_{i=1}^{p} x_i c_i \; \{v_1, \dots, v_n, c_1, \dots, c_p\}$$

Such sets form a basis since $B \cap C \subset B \wedge C$

All $\{b_1, \dots, b_m\}$ cannot be written as l.c. of $B \cap C$ and likewise for $\{c_1, \dots, c_p\}$

as such since every vector in $\bar{U}$ is a l.c. of $\{v_1, \dots, v_n, b_1, \dots, b_n\}$ and every vector in $\bar{W}$ is a l.c. of $\{v_1, \dots, v_n, c_1, \dots, c_p\}$,

There exists B and C where Span$\{B\} = \bar{U}$
and Span$\{C\} = \bar{W}$.

(b) Is it true that for every basis B of U and every basis C of W the set $B \cap C$ is a basis for $U \cap W$?

false

Consider $\bar{V}$ a v.s. over $\mathbb{R}^3$

U is a subspace and $\bar{W}$ is a $\subseteq \bar{V}$

where $B = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right\}$ and $C = \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

The vector $\begin{bmatrix} 3 \\ 3 \end{bmatrix} \in \bar{U} \cap \bar{W}$

But $B \cap C = \{\emptyset\}$

(c) Recall the definition of $U + W$. Prove the following dimension formula:

$$\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W).$$

$U + W = \{x + y \mid x \in \bar{U}, y \in \bar{W}\}$

Considering the set above, any element in $\bar{U} + \bar{W}$ is a l.c. of the Basis vectors of $\bar{U}$ and $\bar{W}$. That is any $\vec{v} \in \bar{U} + \bar{W}$, $\vec{v} = \sum c_i b_i + \sum g_i d_i$ where $\{b_1 \to b_n\}$ and $\{d_1 \to d_n\}$ form a Basis with $\bar{U}$ and $\bar{W}$ respectively. as such $\bar{U} + \bar{W}$ has a basis of $\{b_1, d_1, \ldots, b_n, d_n\}$. However all Basis vectors that exist in Both $\bar{U}$ and $\bar{W}$ exist twice in the union of Bases.

as such Repeated vectors ($\dim(\bar{U} \cap \bar{W})$) can Be omitted.

$$\dim(\bar{U} + \bar{W}) = \underset{|\{1,\ldots,n\}|}{\dim(\bar{U})} + \underset{|\{1,\ldots,p\}|}{\dim(\bar{W})} - \underbrace{\dim(\bar{U} \cap \bar{W})}_{\text{Repeated Basis vectors}}$$

(d) Let $U, W \subset \mathbb{R}_4[x]$ be two subspaces which satisfy $\dim(U) = \dim(W) = 3$ prove that $U \cap W \neq \{\mathbf{0}\}$.

Considering that $\dim(U + W) = \dim(U) + \dim(\bar{W}) - \dim(\bar{U} \cap \bar{W})$

since $U + W = \{x + y \mid x \in \bar{U}, y \in \bar{W}\}$, Then any element in $\bar{U} + \bar{W}$ can Be expressed as l.c. of Basis of $\bar{U}$ and $\bar{W}$.

That is $\forall\, z \in \bar{U}+\bar{W}$, $z = \sum' c_i b_i + \sum' g_i d_i$ where

$\{b_1 \to b_n\}$ and $\{d_1 \to d_p\}$ are Basis for $\bar{U}+\bar{W}$.

Then at most $\dim(\bar{U}+\bar{W})$ can be 4.

as such $4 \geq 3+3-\dim(\bar{U}\cap\bar{W})$.

Thus $\dim(\bar{U}\cap\bar{W}) \geq 2$. Thus $\bar{U}\cap\bar{W} \neq \{0\}$

(e) Find the dimension of the following space:

$$\text{span}\{\begin{pmatrix}1\\1\\2\\1\end{pmatrix}, \begin{pmatrix}-1\\2\\-3\\1\end{pmatrix}\} \cap \text{span}\{\begin{pmatrix}0\\1\\2\\1\end{pmatrix}, \begin{pmatrix}2\\1\\2\\3\end{pmatrix}, \begin{pmatrix}1\\2\\4\\1\end{pmatrix}\}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 0\\ -1 & 2 & -3 & 1 & 0\\ 0 & 1 & 2 & 1 & 0\\ 2 & 1 & 2 & 3 & 0\\ 1 & 2 & 4 & 1 & 0 \end{array}\right] \xrightarrow{\text{REF}} \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 5/7\\ 0 & 0 & 7/3 & 4/3\\ 0 & 0 & 0 & 2\\ 0 & 0 & 0 & 0 \end{bmatrix}$$

zero row $\rightarrow$

$$= \text{Span}\{\begin{pmatrix}1\\1\\2\\1\end{pmatrix}, \begin{pmatrix}-1\\2\\-3\\1\end{pmatrix}, \begin{pmatrix}0\\1\\2\\1\end{pmatrix}, \begin{pmatrix}2\\1\\2\\3\end{pmatrix}\}$$

with dim = 4

Question 4

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4. Let V, W be vector spaces over $\mathbb{R}$. Consider $(\mathcal{L}(V,W), +, \cdot)$ where $+$ is defined as

$$(f_1 + f_2)(v) = f_1(v) + f_2(v), \forall v \in V$$

for any two linear maps $f_1, f_2 \in \mathcal{L}(V,W)$ and $\cdot$ is defined to be

$$(kf)(v) = k(f(v), \forall v \in V$$

where k is any scalar from $\mathbb{R}$ and f is any linear map in $\mathcal{L}(V,W)$. Show that $(\mathcal{L}(V,W), +, \cdot)$ is a vector space over $\mathbb{R}$.

The vector Space $(\mathcal{L}(\bar{V}, \bar{W}), +, \cdot)$.

① Closed under Addition

let f_1 and $f_2 \in \mathcal{L}(\bar{V}, \bar{W})$, The v.s. Then

$$f(\vec{v}) + f_2(\vec{v}) = (f_1 + f_2)(\vec{v}) \in \mathcal{L}(\bar{V}, \bar{W})$$

$$(f_1 + f_2)(\sum c_i v_i) \Rightarrow f_1(\sum c_i v_i) + f_2(\sum c_i v_i) \in \mathcal{L}(\bar{V}, \bar{W})$$

② Commutative

$f_1 + f_2 \in \mathcal{L}(\bar{V}, \bar{W})$

$$f_1(\vec{v}) + f_2(\vec{v}) = f_2(\vec{v}) + f_1(\vec{v})$$

③ associative

$f_1, f_2, f_3 \in \mathcal{L}(\bar{V}, \bar{W})$

$$f_1 + (f_2 + f_3) = (f_1 + f_2) + f_3$$

④ $\exists\ \vec{0} \in \mathcal{L}(\bar{V}, \bar{W})$. because $\forall f \in \mathcal{L}(\bar{V}, \bar{W})$

consider $f_0 \in \mathcal{L}(\bar{V}, \bar{W})$ where

$$f_0 : \bar{V} \to \bar{W}$$
$$\vec{v} \mapsto 0 \cdot \vec{v} \Rightarrow \vec{0}_W$$

Then for any $f \in \mathcal{L}(\bar{V}, \bar{W})$

$$f(\vec{v}) + f_0(\vec{v}) = f(\vec{v})$$

⑤ additive inverse where

$\forall f \in \mathcal{L}(\bar{V}, \bar{W})$, $\exists\ f_0(\vec{v}) = -f(\vec{v})$

s.t. $f(\vec{v}) + f_0(\vec{v}) = \vec{0}$

⑥ $\forall \alpha \in \mathbb{R}$ and $\forall f \in \mathcal{L}(\bar{V}, \bar{W})$

$\alpha f(\vec{v}) \in \mathcal{L}(\bar{V}, \bar{W})$

⑦ $f_1, f_2 \in \mathcal{L}(\bar{V}, \bar{W})$ and $\forall \alpha \in \mathbb{R}$

$(f_1 + f_2)\alpha = \alpha f_1 + \alpha f_2$

⑧ $\alpha, \beta \in \mathbb{R}$ and $f \in \mathcal{L}(\bar{V}, \bar{W})$

$(\alpha + \beta) f = \alpha f + \beta f$

⑨ $\alpha, \beta \in \mathbb{R}$ and $f \in \mathcal{L}(\bar{V}, \bar{W})$

$(\alpha \beta) \cdot f = \alpha(\beta f)$

⑩ $\exists$ The multiplicative identity: 1

s.t. $\forall f \in \mathcal{L}(\bar{V}, \bar{W})$

$$1 \cdot f(\vec{v}) = f(\vec{v})$$

Question 5

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5. Let $T \in \mathcal{L}(V, W)$. Show that

(a) $\ker T$ is a subspace of V.

The kernel of T is a subset of $\vec{V}$

such that for $x \in \ker T$ $T(x) = \vec{0}_W$

The kerT forms a subspace of $\vec{V}$ because

① it is non empty since it contains the zero vector ($\vec{0}_V$) since T must map $\vec{0}_V \mapsto \vec{0}_W$

thus $\vec{0}_V \in \ker T$

② it is closed under addition. Since T is a linear map.

$x, y \in \ker T$: $\underbrace{T(x) + T(y)}_{T(x+y)} = \vec{0}_W + \vec{0}_W = \underline{\vec{0}_W}$

③ it is CUSM.

$\forall \alpha \in \mathbb{R}$ and $\forall x \in \ker T$

$T(\alpha x) \Rightarrow \alpha T(x) = \alpha \cdot (\vec{0}_W) = \vec{0}_W$

(b) ImT is a subspace of W.

$\text{Im}T = \{ y \mid y = T(\vec{v}),\ \vec{v} \in \bar{V} \}$

Subspace Because

① Not empty because ImT contains the $\vec{0}_W$, as a preimage

② CUA Since

any $a, B \in \text{Im}T$

$\underbrace{a + B}_{\in \text{IMT}} = T(\vec{v}_1) + T(\vec{v}_2)$ CUA

③ CUSM

$\forall \alpha \in \mathbb{R}$ and $y \in \text{IMT}$

$\alpha y = \alpha T(\vec{v}) = \underbrace{T(\alpha \vec{v})}_{\in \text{ of IMT}}$

(c) If $T \in \mathcal{L}(V, V)$, is it possible that $\ker T \cap \text{Im}T \neq \{\mathbf{0}\}$? Explain your answer.

Suppose $T: \mathbb{R}^2 \to \mathbb{R}^2$

$T(x) = Ax$ $\quad x \mapsto Ax$ $\quad$ where $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Then $\ker T = \left\{\begin{pmatrix}-1\\1\end{pmatrix}\right\}$

and image of $T = \{y \mid y = T(x) = Ax, x \in \mathbb{R}^2\}$

$$\left[\begin{array}{cc|c} 1 & 1 & -1 \\ -1 & -1 & 1 \end{array}\right] \quad R_2 = R_2 + R_1 \quad \left[\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right]$$

$\begin{pmatrix}-1\\1\end{pmatrix}$ exists in the image of T since

$\exists\, x$ in $\mathbb{R}^2$ where $\begin{pmatrix}-1\\1\end{pmatrix} = A\vec{x}$.

as such $\ker T \cap \operatorname{im}(T) = \left\{\begin{pmatrix}-1\\1\end{pmatrix}\right\} \neq \{\vec{0}\}$

it is possible

Question 6

Monday, October 2, 2023 10:00 PM

6. In each of the following you are given two vector spaces and a function between them. Determine whether the function is a linear transformation or not. Prove your claim.

(a)

$$T : \mathbb{R}^3 \to M_2(\mathbb{R})$$

given by,

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y & y-2z \\ 3x+z & 0 \end{pmatrix}$$

① closed under addition

$$T\left(\begin{matrix} x_1 \\ y_1 \\ z_1 \end{matrix} + \begin{matrix} x_2 \\ y_2 \\ z_2 \end{matrix}\right) = \begin{pmatrix} (x_1+x_2)+(y_1+y_2) & (y_1+y_2)-2(z_1+z_2) \\ 3(x_1+x_2)+(z_1+z_2) & 0 \end{pmatrix}$$

$$T\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + T\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1+y_1 & y_1-2z_1 \\ 3x_1+z_1 & 0 \end{pmatrix} + \begin{pmatrix} x_2+y_2 & y_2-2z_2 \\ 3x_2+z_2 & 0 \end{pmatrix}$$

✓

② Closed under Scalar multiplication

$$T\left(\alpha\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} \alpha(x+y) & \alpha(y-2z) \\ \alpha(3x+z) & 0 \end{pmatrix}$$

$$\parallel$$

$$\alpha T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \alpha\begin{pmatrix} x+y & y-2z \\ 3x+z & 0 \end{pmatrix}$$

✓

(b)

$T : \mathbb{R}_{\ldots}[x] \to \mathbb{R}^3$

$T : \mathbb{R}_2[x] \to \mathbb{R}^3$

given by,

$$Tp = \begin{pmatrix} p(2) \\ p'(2) \\ p''(2) \end{pmatrix}$$

(c)

let $p(x)$ and $q(x) \in \mathbb{R}_2[x]$

$p(x) = a_0 + a_1 x + a_2 x^2$ $\quad$ $q(x) = b_0 + b_1 x + b_2 x^2$

$p'(x) = a_1 + 2a_2 x$ $\quad$ $q'(x) = b_1 + 2b_2 x$

$p''(x) = 2a_2$ $\quad$ $q''(x) = 2b_2$ $\quad$ CVA

① $T(p(x) + q(x)) = \begin{pmatrix} a_0 + 2a_1 + 4a_2 + b_0 + 2b_1 + 4b_2 \\ a_1 + 4a_2 + b_1 + 4b_2 \\ 2a_2 + 2b_2 \end{pmatrix}$ ✓

$T(p(x)) + T(q(x)) = \begin{pmatrix} a_0 + 2a_1 + 4a_2 \\ a_1 + 4a_2 \\ 2a_2 \end{pmatrix} + \begin{pmatrix} b_0 + 2b_1 + 4b_2 \\ b_1 + 4b_2 \\ 2b_2 \end{pmatrix}$

② $\forall \alpha \in \mathbb{R}$ ✓ $\quad$ CUSM

$T(\alpha p(x)) = \begin{pmatrix} \alpha(a_0 + 2a_1 + 4a_2) \\ \alpha(4a_2 + a_1) \\ \alpha(2a_2) \end{pmatrix}$

$\alpha T(p(x)) = \alpha \begin{pmatrix} a_0 + 2a_1 + 4a_2 \\ a_1 + 4a_2 \\ 2a_2 \end{pmatrix}$

(c)

$$T : M_2(\mathbb{R}) \to M_2(\mathbb{R})$$

given by,

$$TA = A^2$$

let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$

$$T(A) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} (a^2 + ac) & (b^2 + bd) \\ (c^2 + ac) & (d^2 + db) \end{bmatrix}$$

$$T(A+B) = \begin{bmatrix} (a+e) & (b+f) \\ (c+g) & (d+h) \end{bmatrix}\begin{bmatrix} (a+e) & (b+f) \\ (c+g) & (d+h) \end{bmatrix} =$$

$$\begin{bmatrix} (a^2 + 2ae + e^2 + ac + ce + ga + eg) & (\dots) \\ (\dots) & (\dots) \end{bmatrix} \neq$$

$T(A) + T(B)$ Not a lin. map

$TA = A$

(d) Fix $B \in M_3(\mathbb{R})$ and consider the function:

$$T : M_3(\mathbb{R}) \to M_3(\mathbb{R})$$

given by,

$$TA = AB$$

(e)

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$T(A) = \begin{bmatrix} ab_{11} + ab_{21} + ab_{31} & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{bmatrix} \Rightarrow \begin{bmatrix} a(b_{11} + b_{21} + b_{31}) & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{bmatrix}$$

$$T(A + Q) = \begin{bmatrix} (a_{11} + q_{11})(b_{11} + b_{21} + b_{31}) & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{bmatrix}$$

$$T(A) + T(Q) = \begin{bmatrix} a(b_{11} + b_{21} + b_{31}) & \dots & \dots \\ & \dots & \dots \end{bmatrix} + \begin{bmatrix} q_{11}(b_{11} + b_{21} + b_{31}) & \dots & \dots \\ \dots & \dots & \dots \end{bmatrix}$$

$$\begin{bmatrix} \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix} \begin{bmatrix} \cdots & \cdots & \cdots \end{bmatrix}$$

CUA ✓ $\Rightarrow \begin{bmatrix} (a+q)(b_{11}+b_{21}+b_{31}) & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}$

② CUSM ✓

$$T(\alpha A) = \alpha, \begin{bmatrix} a(b_{11}+b_{21}+b_{31}) & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix} = \begin{bmatrix} \alpha a(b_{11}+b_{21}+b_{31}) & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}$$

$$\alpha T(A) = \alpha, \begin{bmatrix} a(b_{11}+b_{21}+b_{31}) & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}$$ ✓

(e)

$$T : M_2(\mathbb{R}) \to M_2(\mathbb{R})$$

given by,

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a-c+1 & 2a+3b+2 \\ d-b-8 & 2a \end{pmatrix}$$

① Not CUA

let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$

$$T(A+B) = \begin{bmatrix} (a+e)-(c+g)+1 & 2(a+e)+3(b+f)+2 \\ (d+h)-(b+f)-8 & 2(a+e) \end{bmatrix}$$

$$T(A)+T(B) = \begin{bmatrix} a-c+1 & 2a+3b+2 \\ d-b-8 & 2a \end{bmatrix} + \begin{bmatrix} e-g+1 & 2e+3f+2 \\ h-f-8 & 2e \end{bmatrix}$$

$$\neq \begin{bmatrix} a+e-c-g+2 & \cdots \\ \cdots & \cdots \end{bmatrix}$$

Question 7

Monday, October 2, 2023 10:00 PM

7. In each of the following you are given a linear transformation (you don't need to prove that it is a linear transformation). Follow the following directions for each such transformation:

- Find a basis for the kernel and the image of this transformation.
- Find the dimension of the kernel and the image of this transformation. (Remark: This question will continue in the next HW, you may want to keep a copy of your solution to this part of the question).
- Determine whether the transformation is surjective. Explain your answer.
- Determine whether the transformation is injective. Explain your answer.

(a) For

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 1 & 2 \\ 1 & 7 & -6 \\ 0 & 5 & -5 \end{pmatrix}$$

consider the linear map

$$T_A : \mathbb{R}^3 \to \mathbb{R}^4$$

where the notation T_A was defined in class.

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 3 & 1 & 2 & 0 \\ 1 & 7 & -6 & 0 \\ 0 & 5 & -5 & 0 \end{array}\right] \text{ in RREF is}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \ker(T_A) = \left\{ \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \right\}$$

Dim = 1

$x_1 + 0x_2 + x_3 = 0$

$0x_1 + x_2 - x_3 = 0$

$T_A(x) = \vec{x}A$

3?

$\text{im}\, T_A = \{ y \mid y = \vec{x}A,\ x \in \mathbb{R} \}$

or any l.c. of columns A

$\text{im}\, T_a = \left\{ \begin{pmatrix} 1 \\ 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 7 \\ 5 \end{pmatrix} \right\}$ Dim = 2

Not injective since $\ker T_A \neq \vec{0}$

Not surjective since $\text{span}(\text{im}\, T_A) \neq \mathbb{R}^4$

(b)

$$S : M_2(\mathbb{R}) \to \mathbb{R}^2$$

given by

$$SA = A \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

$\left[\begin{array}{cc|c} 3 & -2 & 0 \\ 3 & -2 & 0 \end{array}\right] \to$ $3x_1 - 2x_2 = 0$

$\ker S = \left\{ \begin{pmatrix} 1/3 & -1/2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1/3 & -1/2 \end{pmatrix} \right\}$

$\text{Dim}(\ker(S)) = 2$

$\text{im}\, S = \{ y \mid y = A \begin{pmatrix} 3 \\ -2 \end{pmatrix},\ A \in M_2(\mathbb{R}) \}$

$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} a_1 3 + a_2 \cdot 2 \\ \hline a_3 3 + a_4 \cdot -2 \end{pmatrix}$

$\text{im}(S) = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ dim = 2

Not injective since $\ker(S) \neq \{\vec{0}\}$

is surjective since $\operatorname{Span}(\operatorname{Im}(S)) = \mathbb{R}^2$

(c)

$$L : \mathbb{R}_3[x] \to \mathbb{R}^2$$

given by

$$Lp = \begin{pmatrix} p(2) - p(1) \\ p'(0) \end{pmatrix}$$

$P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ $\qquad p'(x) = a_1 + 2a_2 x + 3a_3 x^2$

$$L(P(x)) = \begin{pmatrix} a_0 + 2a_1 + 4a_2 + 8a_3 - (a_0 + a_1 + a_2 + a_3) \\ a_1 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 + 3a_2 + 7a_3 \\ a_1 \end{pmatrix}$$

$$\left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 7 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 7 & 0 \end{array}\right] \qquad \{1, x^3 - \tfrac{7}{3}x^2\}$$

$\ker(S)$ $\qquad a_1 = 0 \qquad a_0 = s$

$a_2 = -\frac{7}{3} a_3 \to a_2 = -\frac{7}{3} t$

$a_3 = t$

$$S = \left\{ s\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, t\begin{pmatrix} 0 \\ 0 \\ -7/3 \\ 1 \end{pmatrix} \right\}$$

Dim = 2, Not injective

$\ker(S) \neq \{\vec{0}\}$

$$\text{im}(S) = \left\{ y \mid y = \begin{pmatrix} p(2) - p(1) \\ p'(0) \end{pmatrix}, \; p(x) \in \mathbb{R}_3[x] \right\}$$

$$y = \begin{pmatrix} a_1 + 3a_2 + 7a_3 \\ a_1 \end{pmatrix}$$

$y_1 = a_1 + 3a_2 + 7a_3$

$y_1 = 3 + \underbrace{3a_2 + 7a_3}$

$$\text{im}(S) = \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\} \quad \dim = 2$$

Surjective since $\text{span}(\text{im}(S)) = \mathbb{R}^2$

(d)

$$\Phi : \mathbb{R}^3 \mapsto \mathbb{R}_3[x] \longrightarrow$$

given by

$$\Phi \begin{pmatrix} a \\ b \\ c \end{pmatrix} = (a+b) + (a-2b+c)x + (b-3c)x^2 + (a+b+c)x^3$$

$$\left.\begin{array}{l} a+b=0 \\ a-2b+c=0 \\ b-3c=0 \\ a+b+c=0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right]$$

Ref $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\ker(\Phi) = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

$\dim = 0$

injective since $\ker(\phi)$ is Trivial

$$im(\phi) = \left\{ y \mid y = \dots, \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 \right\}$$

cnst $a+b$

x $a-2b+c$

x^2 $b-3c$

x^3 $a+b+c$

$$y = a+b+ax-2bx+cx+bx^2-3cx^2$$
$$+ax^3+bx^3+cx^3$$

$a(1+x+x^3)+$

$b(1-2x+x^2+x^3)+$

$c(x-3x^2+x^3)$

$$im(\phi) = \{1+x+x^3, 1-2x+x^2+x^3, x-3x^2+x^3\}$$

Dim $=3$

Not surjective Bc $Span(im(\phi)) \neq \mathbb{R}_3[x]$