Question 6

Monday, October 2, 2023 10:00 PM

6. In each of the following you are given two vector spaces and a function between them. Determine whether the function is a linear transformation or not. Prove your claim.

(a)

$$T : \mathbb{R}^3 \to M_2(\mathbb{R})$$

given by,

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y & y-2z \\ 3x+z & 0 \end{pmatrix}$$

① closed under addition

$$T\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 \quad z_2 \end{pmatrix} = \begin{pmatrix} (x_1+x_2)+(y_1+y_2) & (y_1+y_2)-2(z_1+z_2) \\ 3(x_1+x_2)+(z_1+z_2) & 0 \end{pmatrix}$$

$$T\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + T\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1+y_1 & y_1-2z_1 \\ 3x_1+z_1 & 0 \end{pmatrix} + \begin{pmatrix} x_2+y_2 & y_2-2z_2 \\ 3x_2+z_2 & 0 \end{pmatrix}$$

✓

② closed under scalar multiplication

$$T\left(\alpha\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} \alpha(x+y) & \alpha(y-2z) \\ \alpha(3x+z) & 0 \end{pmatrix}$$

$$\alpha T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \alpha\begin{pmatrix} x+y & y-2z \\ 3x+z & 0 \end{pmatrix}$$

|| ✓

(b)

$T : \mathbb{R}[x] \to \mathbb{R}^3$

$$T : \mathbb{R}_2[x] \to \mathbb{R}^3$$

given by,

$$Tp = \begin{pmatrix} p(2) \\ p'(2) \\ p''(2) \end{pmatrix}$$

(c)

let $p(x)$ and $q(x) \in \mathbb{R}_2[x]$

$p(x) = a_0 + a_1 x + a_2 x^2$ $\quad$ $q(x) = b_0 + b_1 x + b_2 x^2$

$p'(x) = a_1 + 2a_2 x$ $\quad$ $q'(x) = b_1 + 2b_2 x$

$p''(x) = 2a_2$ $\quad$ $q''(x) = 2b_2$

CVA

① $T(p(x) + q(x)) = \begin{pmatrix} a_0 + 2a_1 + 4a_2 + b_0 + 2b_1 + 4b_2 \\ a_1 + 4a_2 + b_1 + 4b_2 \\ 2a_2 + 2b_2 \end{pmatrix}$ ✓

$T(p(x)) + T(q(x)) = \begin{pmatrix} a_0 + 2a_1 + 4a_2 \\ a_1 + 4a_2 \\ 2a_2 \end{pmatrix} + \begin{pmatrix} b_0 + 2b_1 + 4b_2 \\ b_1 + 4b_2 \\ 2b_2 \end{pmatrix}$

② $\forall \alpha \in \mathbb{R}$ ✓ $\quad$ CVSM

$T(\alpha p(x)) = \begin{pmatrix} \alpha(a_0 + 2a_1 + 4a_2) \\ \alpha(4a_2 + a_1) \\ \alpha(2a_2) \end{pmatrix}$

$\alpha T(p(x)) = \alpha \begin{pmatrix} a_0 + 2a_1 + 4a_2 \\ a_1 + 4a_2 \\ 2a_2 \end{pmatrix}$

(c)

$$T : M_2(\mathbb{R}) \to M_2(\mathbb{R})$$

given by,

$$TA = A^2$$

let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$

$$T(A) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} (a^2 + ac) & (b^2 + bd) \\ (c^2 + ac) & (d^2 + db) \end{bmatrix}$$

$$T(A+B) = \begin{bmatrix} (a+e) & (b+f) \\ (c+g) & (d+h) \end{bmatrix}\begin{bmatrix} (a+e) & (b+f) \\ (c+g) & (d+h) \end{bmatrix} =$$

$$\begin{bmatrix} (a^2 + 2ae + e^2 + ac + ce + ga + eg) & (\dots) \\ (\dots) & (\dots) \end{bmatrix} \neq$$

$T(A) + T(B)$ Not a lin. map

$TA = A$

(d) Fix $B \in M_3(\mathbb{R})$ and consider the function:

$$T : M_3(\mathbb{R}) \to M_3(\mathbb{R})$$

given by,

$$TA = AB$$

(e)

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$T(A) = \begin{bmatrix} ab_{11} + ab_{21} + ab_{31} & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{bmatrix} \Rightarrow \begin{bmatrix} a(b_{11} + b_{21} + b_{31}) & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{bmatrix}$$

$$T(A+Q) = \begin{bmatrix} (a_{11} + q_{11})(b_{11} + b_{21} + b_{31}) & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{bmatrix}$$

$$T(A) + T(Q) = \begin{bmatrix} a(b_{11} + b_{21} + b_{31}) & \dots & \dots \\ \dots & \dots & \dots \end{bmatrix} + \begin{bmatrix} q_{11}(b_{11} + b_{21} + b_{31}) & \dots & \dots \\ \dots & \dots & \dots \end{bmatrix}$$

$$\begin{bmatrix} \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix} \begin{bmatrix} \cdots & \cdots & \cdots \end{bmatrix}$$

CUA ✓ $\Rightarrow \begin{bmatrix} (a+q)(b_{11}+b_{21}+b_{31}) & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & & \end{bmatrix}$

② CUSM ✓

$$T(\alpha A) = \alpha, \begin{bmatrix} a(b_{11}+b_{21}+b_{31}) & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix} = \begin{bmatrix} \alpha a(b_{11}+b_{21}+b_{31}) & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}$$

$$\alpha T(A) = \alpha, \begin{bmatrix} a(b_{11}+b_{21}+b_{31}) & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}$$ ✓

(e)

$$T : M_2(\mathbb{R}) \to M_2(\mathbb{R})$$

given by,

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a-c+1 & 2a+3b+2 \\ d-b-8 & 2a \end{pmatrix}$$

① Not CUA

let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$

$$T(A+B) = \begin{bmatrix} (a+e)-(c+g)+1 & 2(a+e)+3(b+f)+2 \\ (d+h)-(b+f)-8 & 2(a+e) \end{bmatrix}$$

$$T(A)+T(B) = \begin{bmatrix} a-c+1 & 2a+3b+2 \\ d-b-8 & 2a \end{bmatrix} + \begin{bmatrix} e-g+1 & 2e+3f+2 \\ h-f-8 & 2e \end{bmatrix}$$

$$\neq \begin{bmatrix} a+e-c-g+2 & \cdots \\ \cdots & \cdots \end{bmatrix}$$