Question 5

Sunday, November 12, 2023 5:37 PM

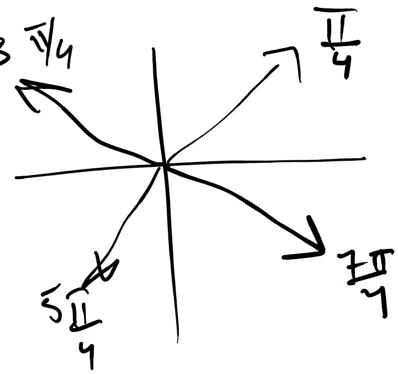
5. For the following exercises $i = \sqrt{-1}$. And the field we are working with is $\mathbb{C}$.

(a) Determine all complex numbers z such that $z^2 = i$.

$$(a+bi)(a+bi) = a^2 + abi + abi + b^2 i^2$$
$$= a^2 + 2abi - b^2 = i$$
$$a^2 - b^2 = i(1-2ab)$$

let $b=1$

$$a^2 - 1 = i(1-2a)$$
$$a^2 = i - 2ai + 1$$
$$a^2 + 2ai = 1 + i$$
$$a(a+2i) = 1+i$$

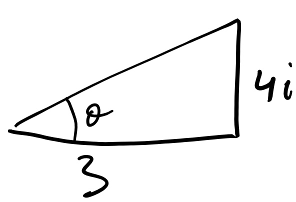


$$z = \cos\theta + i\sin\theta$$
$$\cos^2\theta + 2i\cos\theta\sin\theta - \sin^2\theta$$

$$\boxed{\begin{array}{l}\theta = \frac{\pi}{4} + \pi k, k \in \mathbb{Z} \\ z = \cos\theta + i\sin\theta\end{array}}$$

(b) Write $3 + 4i$ in its polar form.

$3 + 4i$



$5\cos(53.13°) + 5i\sin(53.13°)$ $\quad \cos\theta = \frac{-}{5}$

$= 53.13°$

$= 5e^{i\theta}, \ \theta = 53.13°$

(c) Determine call $z \in \mathbb{C}$ such that $z^4 = 1$.

$(a+bi)^4 = a^4 + 4a^3bi + 6a^2b^2i^2 + 4ab^3i^3 + b^4i^4$

$= a^4 + 4a^3bi - 6a^2b^2 - 4ab^3i + b^4 = 1$

$r^4(\cos\theta + i\sin\theta)^4 = 1$

$R^4(e^{i\theta})^4 = 1 \quad r^4e^{4i\theta} = 1$

$re^{i\theta} = \sqrt[4]{1}$

$\therefore \ [r(\cos\theta + i\sin\theta)]^4 = 1$

$r = 1$ for $\theta = \frac{\pi}{2}k, \ \forall k \in \mathbb{Z}$

$z = \cos\theta + i\sin\theta, \ \theta = \frac{\pi}{2}k \ \forall k \in \mathbb{Z} \Rightarrow z = \{1, -1, i, -i\}$

(d) If $z \in \mathbb{C}$ is non-zero, what is z^{-1} in polar form? Compare $z, \bar{z}, z^{-1}$ in a sketch.

$z = r(\cos\theta + i\sin\theta) \Rightarrow re^{i\theta}$

$(re^{i\theta})^{-1} = r^{-1}e^{-i\theta} \Rightarrow \frac{1}{r}(\cos\theta - i\sin\theta)$