1a) $\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \in \text{Span}\left\{ \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix} \right\}$

$$s\begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix} + t\begin{bmatrix} -1 & 1 \\ 3 & 0 \end{bmatrix} + r\begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

$$\begin{array}{c} \begin{array}{ccc} s & t & r \end{array} \\ \left[\begin{array}{ccc|c} 2 & -1 & -2 & 1 \\ 0 & 1 & 1 & 2 \\ 1 & 3 & 2 & 2 \\ -1 & 0 & -1 & -1 \end{array}\right] \end{array}$$

$R_1 = \frac{1}{2}R_1$ $\quad R_3 = R_3 - \frac{7}{2}R_2$

$R_3 = R_3 + R_1$ $\quad R_4 = R_4 + \frac{1}{2}R_2$

$R_1 = R_1 + \frac{1}{2}R_2$ $\quad R_3 = -2R_3$

$R_1 = R_1 + \frac{1}{2}R_2$ $\quad R_2 = R_2 - R_3$

$R_4 = R_4 + \frac{3}{2}R_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -9 \\ 0 & 0 & 1 & 11 \\ 0 & 0 & 0 & \boxed{17} \end{array}\right]$$

inconsistent Thus Not in Span.

1b) $2 + 3x + 2x^2 - x^3 \in \text{Span}\{1 - x^3, 2 + x + x^2, 3 - x\}$

$c_1(1-x^3) + c_2(2+x+x^2) + c_3(3-x) = 2 + 3x + 2x^2 - x^3$

$\cancel{c_1 - c_1x^3} \cancel{+2c_2} + \cancel{c_2x} \cancel{+c_2x^2} + \cancel{3c_3} \cancel{-c_3x} = 2 + 3x + 2x^2 - x^3$

for x^3: $-c_1 = -1$ for x^2: $c_2 = 2$ for x: $3 = c_2 - c_3$

for const: $2 = c_1 + 2c_2 + 3c_3$

$2 = 1 + 4 - 3 = 2$

$$\boxed{\begin{array}{c} c_1 = 1 \\ c_2 = 2 \\ c_3 = -1 \end{array}}$$

in Span B/c $\exists$ a linear comb

1c) $\text{Span}\left\{\begin{bmatrix}5 & -2\\ -5 & -3\end{bmatrix}, \begin{bmatrix}1 & 1\\ 4 & -1\end{bmatrix}\right\} \subseteq \text{Span}\left\{\begin{bmatrix}2 & 0\\ 1 & 1\end{bmatrix}, \begin{bmatrix}-1 & 1\\ 3 & 0\end{bmatrix}, \begin{bmatrix}-2 & 1\\ 2 & -1\end{bmatrix}\right\}$

$$\left[\begin{array}{ccc|c}2 & -1 & -2 & 5\\ 0 & 1 & 1 & -2\\ 1 & 3 & 2 & -5\\ 1 & 0 & -1 & -3\end{array}\right] \qquad \left[\begin{array}{ccc|c}2 & -1 & -2 & 1\\ 0 & 1 & 1 & 1\\ 1 & 3 & 2 & 4\\ 1 & 0 & -1 & -1\end{array}\right]$$

$R_1 = \frac{1}{2}R_1,\ R_3 = R_3 - R_1,\ R_4 = R_4 - R_1$

$$\left[\begin{array}{ccc|c}1 & -\frac{1}{2} & -1 & 5/2\\ 0 & 1 & 1 & -2\\ 0 & 7/2 & 3 & -15/2\\ 0 & -1/2 & 0 & -11/2\end{array}\right]$$

$R_1 = R_1 + \frac{1}{2}R_2,\ R_3 = R_3 - \frac{7}{2}R_2$

$R_4 = R_4 + \frac{1}{2}R_1$

$$\left[\begin{array}{ccc|c}1 & 0 & 0 & 2\\ 0 & 1 & 0 & -3\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0\end{array}\right]$$ consistent + lin. comb exists

$R_1 = \frac{1}{2}R_1,\ R_3 = R_3 - R_1,\ R_4 = R_4 - R_1$

$R_1 = R_1 + \frac{1}{2}R_2,\ R_3 = R_3 - \frac{7}{2}R_2$

$R_4 = R_4 + \frac{1}{2}R_1$ ...

$$\left[\begin{array}{ccc|c}1 & 0 & 0 & 1\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{array}\right]$$ consistent + lin. comb exists

1D) $\text{Span}\left\{\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}1\\1\\0\end{bmatrix}\right\} = \text{span}\left\{\begin{bmatrix}3\\2\\1\end{bmatrix},\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}1\\2\\-1\end{bmatrix}\right\}$

$$\begin{bmatrix}3\\2\\1\end{bmatrix} = 2\begin{bmatrix}1\\0\\1\end{bmatrix} + 1\begin{bmatrix}1\\2\\-1\end{bmatrix}$$

$$= \begin{bmatrix}3\\2\\1\end{bmatrix}$$

as such, $\begin{bmatrix}3\\2\\1\end{bmatrix}$ is Not linearly independent from $\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}1\\2\\-1\end{bmatrix}$ as it can be a lin. com. of ↗.

The span is rewritten as $\text{Span}\left\{\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}1\\2\\-1\end{bmatrix}\right\}$

in order to prove $\text{Span}\left\{\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}1\\1\\0\end{bmatrix}\right\}$ is = ↗

each vector in ↗ can be a lin. com. of

and the other way (subset - equality)

as such the spans are equal

1e) To show 2 vectors span $\mathbb{R}^2$, they must be linearly independent, as such $\begin{bmatrix}1\\-1\end{bmatrix}\begin{bmatrix}-1\\2\end{bmatrix}$ are lin. ind. bc in $A(\vec{x})=\vec{0}$

$\vec{x} = \begin{bmatrix}0\\0\end{bmatrix}$, the trivial solution illustrating their linear independence, thus being a spanning set for $\mathbb{R}^2$

1F) false, because $\begin{bmatrix}1\\-1\end{bmatrix}$ and $\begin{bmatrix}-1\\1\end{bmatrix}$ are linearly dependent as one vector is a scalar multiple of the other. as such a solution to its homogeneous system is nontrivial and it is Not the spanning set of $\mathbb{R}^2$.

19) $R_3[x] = \{a_0 + a_1x + a_2x^2 + a_3x^3 \mid a_0, a_1, a_2, a_3 \in \mathbb{R}\}$

$\{1-x+x^2,\ x-x^2+x^3,\ 1+x^2-x^3,\ x^3\}$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \end{array}\right] \quad \begin{array}{l} R_2 = R_2 + R_1 \\ R_3 = R_3 - R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \end{array}\right] \quad \begin{array}{l} R_3 = R_3 + R_2 \\ R_4 = R_4 - R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \end{array}\right] \quad \begin{array}{ll} R_3 = -R_3 & R_2 = R_2 + R_3 \\ R_1 = R_1 - R_3 & R_4 = R_4 + R_3 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \rightarrow \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus the set $\{\}$ is linearly independent and thus spans $R_3[x]$

2a) $\left\{ \begin{bmatrix} a+b+c \\ a-2b \\ 3a-2c \\ 4c-b \end{bmatrix} \middle| \; a,b,c \in \mathbb{R} \right\}$

$\left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 4 \end{bmatrix} \right\}$ ← Spanning Set

2b) $1\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + 2\begin{bmatrix} a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{pmatrix} 2a \\ 2b \end{pmatrix} \quad \begin{pmatrix} -a \\ -b \end{pmatrix} \quad \forall a,b \in \mathbb{R}$

$a\begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix}, b\begin{pmatrix} 0 & 0 \\ 2 & -1 \end{pmatrix} \Rightarrow \left\{ \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 2 & -1 \end{bmatrix} \right\}$

2c) $\{ p(x) \in \mathbb{R}_3[x] : p'(1) = 0 \}$ Spanning Set ↑

$p'(x) = a_1 + 2a_2x + 3a_3x^2$ $\qquad a_0 = s$

$p'(1) = a_1 + 2a_2 + 3a_3 = 0$ $\qquad a_1 = -2t - 3r$

$a_1 = -2a_2 - 3a_3$ $\qquad a_2 = t$

$a_0 = s \quad a_2 = t \quad a_3 = r$ $\qquad a_3 = r$

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + r\begin{bmatrix} 0 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

$\{ 1, -2x + x^2, -3x + x^3 \}$ Spanning set

2D) $\{ P(x) \in \mathbb{R}_n[x] \mid P(1) = P(-1) \}$

all even-degree polynomials

~~$\{ P(x) \in \mathbb{R}_n[x] : n = 2k, k \in \mathbb{Z} \}$ spanning~~ set

for $\mathbb{R}_3[x] = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

for $P_3(1) = a_0 + a_1 + a_2 + a_3$

for $P_3(-1) = a_0 - a_1 + a_2 - a_3$ if $a_1 + a_3 = 0$, Then $P_3(1) = P_3(-1)$

$x^3 - x$ $a_1 = -a_3$ $a_3 = a_3$

for $P_5(x)$: $a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5$ $\begin{bmatrix} a_1 \\ a_3 \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$P_5(1) = a_0 + a_1 + a_2 + a_3 + a_4 + a_5 \Rightarrow a_0 + a_2 x + a_4 x^2$

$P_5(-1) = a_0 - a_1 + a_2 - a_3 + a_4 - a_5$

if $a_1 + a_3 + a_5 = 0$ $a_1 = -a_3 - a_5$

Then $P_5(1) = P_5(-1)$ $a_3 = a_3$

$a_5 = a_5$

Spanning set

$\{ \{ x^n \mid n = 2k, n \in \mathbb{Z}, n \geq 0 \},$

$\{ x^{2k+1} - x \mid k \in \mathbb{Z}, k \geq 1 \} \}$

$\begin{bmatrix} a_1 \\ a_3 \\ a_5 \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$x.$

3a) $\left\{ \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix} \right\}$

$$\left[\begin{array}{ccc|c} 2 & -1 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 3 & 2 & 0 \\ -1 & 0 & -1 & 0 \end{array}\right]$$

lin. ind if $A\vec{x} = \vec{0}$ only when $\vec{x}_1 = \ldots = \vec{x}_n = \vec{0}$

$$\left[\begin{array}{ccc|c} 1 & -1/2 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 7/2 & 3 & 0 \\ 0 & -1/2 & -2 & 0 \end{array}\right]$$

$R_1 = \frac{1}{2}R_1$, $R_3 = R_3 - R_1$, $R_4 = R_4 + R_1$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1/2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1/2 & 0 \\ 0 & 0 & -3/2 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

only solution is $\vec{x} = \vec{0}$

thus lin. ind.

3b) $$\left[\begin{array}{ccc|c} 3 & 1 & 1 & 0 \\ 2 & 0 & 2 & 0 \\ 1 & 1 & -1 & 0 \end{array}\right]$$

Switch R_3 + R_1

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 2 & 0 & 2 & 0 \\ 3 & 1 & 1 & 0 \end{array}\right]$$

$R_2 = R_2 - 2R_1$

$R_3 = R_3 - 3R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -2 & 4 & 0 \\ 0 & -2 & 4 & 0 \end{array}\right]$$

$R_2 = -\frac{1}{2}R_2$

$R_3 = R_3 + R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

free variable thuse $\exists$ Non-trivial solution

Thus set is lin. dep.

3c) $\{1-x^3,\ 2+x+x^2,\ 3-x,\ 1+x+x^2+x^3\}$

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 & 0 \end{array}\right] \quad R_4 = R_4 + R_1$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{array}\right] \quad \begin{array}{l} R_1 = R_1 - 2R_2 \\ R_3 = R_3 - R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 5 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{array}\right] \quad \begin{array}{l} R_3 = -R_3 \\ R_2 = R_3 + R_2 \\ R_1 = R_1 - 5R_3 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{array}\right] \quad \begin{array}{l} R_4 = \frac{1}{2}R_4 \\ R_2 = R_2 - R_4 \\ R_1 = R_1 - R_4 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Only trivial solution exists thus set is lin. ind. since $A\vec{x}=\vec{0}$ where $\vec{x}_1 = \dots = \vec{x}_n = \vec{0}$

5D) $C_1 f(x) + C_2 g(x) + C_3 (h(x)) = \vec{0}$

$C_1 \sin^2 x + C_2 \cos^2 x + C_3 = 0$

if C_1 and $C_2 = 1$ then

$1(\sin^2 x + \cos^2 x) + C_3 = 0$

$1 + C_3 = 0 \therefore C_3 = -1$

where $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ thus $A\vec{x} = \vec{0}$ has a Nontrivial solution + is lin. dep.

4a) $\bar{V}$ s.t. $\{w_1, w_2, w_3\}$ are lin. ind. vectors in $\bar{V}$ is $\{w_1 + w_2 + w_3, w_2 + w_3, w_3\}$

for This set to be linearly independent, only The trivial solution can exist for its homogeneous sys

$C_1(w_1 + w_2 + w_3) + C_2(w_2 + w_3) + C_3 w_3 = \vec{0}$

$C_1 w_1 + \cancel{C_1 w_2} + C_1 w_3 + \cancel{C_2 w_2} + C_2 w_3 + C_3 w_3 = \vec{0}$

$C_1 w_1 + w_2(C_1 + C_2) + w_3(C_1 + C_2 + C_3) = 0$

$C_1 = -C_2$ $\quad C_1 = -C_2 - C_3$

if $C_1 \neq 0$, Then C_2 + C_3 are $\neq 0$ which produce a non zero sol, thus only $\vec{0}$ is solution + vectors in set are linearly independent.

4B) $\{\underbrace{w_1 + 2w_2 + w_3}_{\text{vector 1}}, \underbrace{w_2 + w_3}_{\text{vector 2}}, \underbrace{w_1 + w_2}_{\text{vector}_3}\}$

$$\begin{array}{c} w_1 \\ w_2 \\ w_3 \end{array}\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 \\ -1/2 & 1 & 0 & 0 \end{array}\right] \quad \begin{array}{l} R_2 = R_2 - 2R_1 \\ R_3 = R_3 - R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \quad \begin{array}{l} R_3 = R_3 - R_2 \\ -1 - -1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

free variable thus non-trivial solution exists + vectors in $\{\ldots\}$ are linearly dependent.

5. $\bar{V}$ $S \subset \bar{V}$ + $T \subset \bar{V}$

a) if $S \subset T$ + S is lin. ind., Then T is lin. ind.

let Set S Be $\{v_1, v_2\}$ where $v_1 + v_2$ are lin. ind. let T Be $\{V_1, V_2, 2V_1 + V_2\}$, thus S is a subset of T. T is Not lin. ind Because $2V_1 + V_2$ Can Be rewritten as a linear combination of V_1 and V_2, The other elements in T.

b) $S \subset T$, if T is lin. ind, then S is lin. ind.

Consider The set T containing vectors $\{V_1, V_2, \ldots, V_n\}$ Then if S is a subset of T, it can contain at most $\{V_1, V_2, \ldots, V_{n-1}\}$. Since T is linearly independent $c_1V_1 + c_2V_2 + \ldots + c_nV_n = \vec{0}$ where $c_1 = c_2 = \ldots = c_n = 0$, The Trivial solution only exists.

Now consider set S, and its linear combinations $d_1V_1 + d_2V_2 + \ldots + d_{n-1}V_{n-1} = \vec{0}$. Because The set T is linearly independent, its homogeneous solution for all vectors including $V_1 \rightarrow V_n$ is $\vec{0}$ for all $c_1 \rightarrow c_n$ including $c_1 \rightarrow c_{n-1}$. As such $d_1 \rightarrow d_{n-1}$ must also Be only zero to satisfy Their homogeneous system Thus proving set S is linearly independent.

5c) if S + T are lin. ind., Then $S \cap T$ is lin ind. -> TRUE

To prove this, I will Disprove its Negation and prove By way of Contradiction

There are three possible cases for $S \cap T$

$S \cap T$ is $\emptyset$, $S \cap T$ is lin. ind., $S \cap T$ is lin. dep.

- $S \cap T$ can Be true.

Consider set $S: \{v_1, v_2\}$ lin. ind. and $T \{v_3, v_4\}$ lin. ind. Then $S \cap T = \{\} \Rightarrow \emptyset$.

- Otherwise if $S \cap T$ is Non empty, Then it must Be linearly independent. Suppose By way of Contradiction, $S \cap T$ is Not empty and linearly dependent.

This implies That $|S \cap T| \geq 2$ Because atlest one vector in $S \cap T$ is a lin. comb of the other, a scalar multiple. Call These vectors v_1 and v_2 where $v_1 = c v_2$, $c \in \mathbb{R}$. This also implies That v_1 and v_2 must also exist in Both S and T Based on Definitions of intersection of sets. if v_1 and v_2 exists in S and T, then S and T are also lin. dep, a contradiction.

5d) if $S + T$ are lin ind, then $S \cup T$ is lin ind $\to$ false

let $S: \{v_1, v_2\}$ and $T: \{2v_1 + v_2, v_3\}$ are linearly independent.

$S \cup T: \{v_1, v_2, 2v_1 + v_2, v_3\}$ ϵ linearly dep since $2v_1 + v_2$ is a linear combination of v_1 and v_2

$$\begin{array}{c} v_1 \\ v_2 \\ v_3 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

free variable, thus Non-trivial solution exists, lin. dep.

5e) $w = \text{Span}\{S\}$, $U = \text{Span}\{T\}$

$\bar{\omega} + \bar{U} = \{w + u \mid w \in \bar{\omega}, u \in \bar{U}\} = \text{Span}\{S \cup T\}$

Since $\vec{\omega} \in \bar{\omega}$, $\vec{\omega}$ can be rewritten as a lin. comb. of vectors $\{v_1, \ldots, v_n\}$ in Set S and likewise for $\vec{U} \in \bar{U}$, $\vec{U}$ can be rewritten as a linear combination of vectors $\{x_1, \ldots, x_n\}$ for Set T.

$\vec{\omega} = c_1 v_1 + \ldots + c_n v_n \Rightarrow \vec{w} + \vec{U} = c_1 v_1 + d_1 x_1 + \ldots + c_n v_n + d_n x_n$

$\vec{U} = d_1 x_1 + \ldots + d_n x_n$ $\quad$ where $S \cup T: \{v_1, x_1, \ldots, v_n, x_n\}$

and $\vec{U} + \vec{\omega} \in \text{Span}(S \cup T)$ By This Def.

(6a) let V Be VS. where $\dim \bar{V} = 3$. Then $\exists$ a subspace $\bar{W}$ of $\bar{V}$ and subspace $\bar{U}$ of $\bar{W}$ $(\bar{U} \subset \bar{W} \subset \bar{V})$ s.t. $\dim U = 1$ + $\dim \bar{W} = 2$

if $\bar{V}$ is a vector space with $\dim(V) = 3$, Then a vector V can Be written as $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ where $v \in \bar{V}$.

Consider The subspace $\bar{W}$ with dimension $(\bar{W}) = 2$ Then a vector in $\bar{W}$, $w \in \bar{W}$, can Be expressed as $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ for all $w \in \bar{W}$, Because

$\bar{W}$ is a subspace, it is closed under addition + scalar multiplication as follows.

$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \begin{bmatrix} w_3 \\ w_4 \end{bmatrix} = \begin{bmatrix} w_1 + w_3 \\ w_2 + w_4 \end{bmatrix}$ which is $\dim = 2$

$\alpha \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} \alpha w_1 \\ \alpha w_2 \end{bmatrix}$ $\dim = 2$, where $w_1, w_2, w_3, w_4 \in \bar{V}$

Then consider subspace $\bar{U}$ with $\dim = 1$

$[v_1]$ Since it is a subspace of $\bar{W}$, By transitivity, it can Be expressed as $[v_1]$ and is CUA + CUSM.

6B) $\bar{V}$ is v.s. $\dim(V) = 3$ let $\bar{W}$ Be Non trivial Subspace of $\bar{V}$ and let $\bar{U}$ Be Non-trivial Subspace of $\bar{W}$ $(\bar{U} \subset \bar{W} \subset \bar{V})$. Then $\dim(\bar{U}) = 1$ + $\dim(\bar{W}) = 2$

- in order for a set to Be a subspace of a Parent vector space, a proper subset, it must have $n-1$ basis vectors where n is the number of basis vectors for the Parent vector space.

Proof.

BWOC.

assume That $\bar{W} \subseteq \bar{V}$, and $\dim(V) = 3$, then $\dim(\bar{W}) = 3$.

if $\dim(V) = 3$, then $\bar{V}$ Contains 3 basis vectors That are linearly independent. $\{v_1, v_2, v_3\}$. if $\bar{W}$ is a subspace of $\bar{V}$, then it must Share linearly independent Basis vectors, of at least $\{v_1\}$. But since $\dim(\bar{W}) = 3$, it must have Basis vectors $\{v_1, v_2, v_3\}$ These are linearly independent, consider Span of $\bar{W}$ + $\bar{V}$. $\text{Span}(\bar{W}) = \{v_1, v_2, v_3\}$

$\text{Span}(\bar{V}) = \{v_1, v_2, v_3\}$

since each vector in $\{v_1, v_2, v_3\}$ Can Be a lin comb of Basis of $\bar{V}$, $\bar{W} = \bar{V}$, Not a Subspace a contradiction.

as such, the dim of a subspace has to be at most n-1. This is traced for $\bar{w}$ and $\bar{u}$ subspaces of $\bar{v}$ and $\bar{w}$, respectively.

6c) false. in order for a set of vectors to be considered a V.S., they must be linearly independent and their span must equal the V.S.

$$\left.\begin{array}{l}\{v_1, v_2\} \\ \{v_2, v_3\} \\ \{v_1, v_3\}\end{array}\right] \text{lin. ind}$$

What if v_3 is a linear comb. of $v_1 + v_2$?

$$\Big\{\underset{v_1}{\begin{bmatrix}1\\0\end{bmatrix}}, \underset{v_2}{\begin{bmatrix}0\\1\end{bmatrix}}\Big\} \quad \Big\{\underset{v_2}{\begin{bmatrix}0\\1\end{bmatrix}}, \underset{v_3}{\begin{bmatrix}1\\1\end{bmatrix}}\Big\} \quad \Big\{\underset{v_1}{\begin{bmatrix}1\\0\end{bmatrix}}, \underset{v_3}{\begin{bmatrix}1\\1\end{bmatrix}}\Big\}$$

all three cases above are lin. ind.

But $\{v_1, v_2, v_3\}$ is lin dep.

since $v_3 = 1v_1 + 1v_2$

$\Rightarrow \{v_1, v_2, v_3\}$ is NOT basis for $\bar{v}$.

6 i) $v_1 \to v_n \in \bar{V}$, if $\dim(\text{span}\{v_1 \to v_n\}) = n$, then $\{v_1 \to v_n\}$ are lin. ind.

① - if $\dim(\text{span}(v_1 \to v_n))$ is n, this means all the vectors $v_1 \to v_n$ cannot be linear combinations of any the other vectors, otherwise such vectors would be redundant in the span and the dim $< n$.

- This implies that $v_1 \to v_n$ are linearly independent because there are n basis vectors - as implied by $\dim(\ldots) = n$ -. If $v_1 \to v_n$ serve as a basis, then they are linearly independent, through def. of basis

② if $\{v_1 \to v_n\}$ are lin. ind., then $\dim(\text{span}\{v_1 \to v_n\}) = n$.

if $v_1 \to v_n$ are lin ind. it implies that any element $x \in \text{span}\{v_1 \to v_n\}$ needs n many vectors and n many scalars to create a linear combination. it thus implies that you need n basis vectors to create vector $\vec{x}$ in span of $v_1 \to v_n$. as such the dimension of this span is n.

6e) Counter example let $\bar{V}$ be V.S. over $\mathbb{R}^2$

let $V_1 = \text{Span}\left\{\begin{bmatrix}0\\1\end{bmatrix}\right\}$ $V_2 = \text{Span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$

$V_3 = \text{Span}\left\{\begin{bmatrix}\frac{1}{2}\\1\end{bmatrix}\right\}$

$$\text{Span}\left\{\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right\} = \text{Span}\left\{\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1/2\\1\end{pmatrix}\right\}$$

$$V_1 + V_2 = V_1 + V_3$$

$\dim V_2 = 1$

$\dim V_3 = 1$ But $\text{Span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\} \neq \text{Span}\left\{\begin{bmatrix}1/2\\1\end{bmatrix}\right\}$

$$V_2 \neq V_3$$