Question 8

Tuesday, October 24, 2023 9:22 AM

8. Consider the subspace of $\mathbb{R}^4$

$$U = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} : a - b + c - d = 0 \text{ and } a + d = 0 \right\}$$

$-s - b + c - s = 0$

$b = c - 2s$

of $\mathbb{R}^4$.

(a) Find a basis for U.

(b) Find an orthonormal basis $\mathcal{C}$ for U.

(c) Let $x = (1, 2, 3, 4)^T \in \mathbb{R}^4$. Find the orthogonal projection of x onto the space U : $\text{Proj}_U\, x$.

(d) Find an orthonormal basis of $\mathbb{R}^4$ that contains the vectors from $\mathcal{C}$ from (b).

(e) Find the matrix representation of the orthogonal projection of $\mathbb{R}^4$ onto the space U with respect to the basis that you obtained from (d).

(f) Find the matrix representation of the orthogonal projection of $\mathbb{R}^4$ onto the space U with respect to the standard basis of $\mathbb{R}^4$.

(g) Use the answer from (f) to calculate $\text{Proj}_U\, x$

a) $a = -s$
$b = t - 2s$
$c = t$
$d = s$

$$\left\langle \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \\ 0 \\ 1 \end{pmatrix} \right\rangle$$

b) GS Process

$$q_1 = \frac{1}{\|q_1\|} q_1 = \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$$

$$q_2 = \frac{v_2 - \langle v_2, q_1 \rangle q_1}{\| - - - - - \|}$$

$\langle v_2, q_1 \rangle$

$$(-1,-2,0,1)\begin{pmatrix}0\\1/\sqrt{2}\\1/\sqrt{2}\\0\end{pmatrix} = 0 + \frac{-2}{\sqrt{2}} + 0 + 0$$

$$= -\frac{2}{\sqrt{2}}\begin{pmatrix}0\\1/\sqrt{2}\\1/\sqrt{2}\\0\end{pmatrix}$$

$$= \begin{pmatrix}0\\-2/2\\-2/2\\0\end{pmatrix} = \begin{pmatrix}0\\-1\\-1\\0\end{pmatrix}$$

$$V_2 - \begin{pmatrix}0\\-1\\-1\\0\end{pmatrix} = \begin{bmatrix}-1\\-1\\1\\1\end{bmatrix} = \frac{\begin{pmatrix}-1\\-1\\1\\1\end{pmatrix}}{\|\cdot\cdot\|}$$

$$= \begin{pmatrix}-1/2\\-1/2\\1/2\\1/2\end{pmatrix}$$

c) (c) Let $x = (1,2,3,4)^T \in \mathbb{R}^4$. Find the orthogonal projection of x onto the space U : $\mathrm{Proj}_U\, x$.

$\langle q_1, q_2 \rangle$ ON set Span $\bar{U}$

$$\mathrm{proj}_{\bar{U}}\, x = \sum_{i=1}^{2} \mathrm{proj}_{q_i}\, x$$

$$\frac{\langle x, q_1 \rangle}{\langle q_1, q_1 \rangle} q_1 + \frac{\langle x, q_2 \rangle}{\langle q_2, q_2 \rangle} q_2$$

$$\begin{pmatrix}0\\5/2\\5/2\\0\end{pmatrix} + \begin{pmatrix}-1\\-1\\1\\1\end{pmatrix} = \begin{bmatrix}-1\\3/2\\7/2\\\end{bmatrix}$$

$$\begin{bmatrix} 1 \end{bmatrix}$$

(d) Find an orthonormal basis of $\mathbb{R}^4$ that contains the vectors from $\mathcal{C}$ from (b).

$$C = \left\langle \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}, \begin{pmatrix} -1/2 \\ -1/2 \\ 1/2 \\ 1/2 \end{pmatrix} \right\rangle$$

let vectors $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ Be an extension

Then $q_3 = \dfrac{v_3 - \langle v_3, q_1 \rangle q_1 - \langle v_3, q_2 \rangle q_2}{\| \quad - - - \quad \|}$

$$v_3 - 0 - \left(-\frac{1}{2}\begin{pmatrix} -1/2 \\ -1/2 \\ 1/2 \\ 1/2 \end{pmatrix}\right) \Rightarrow v_3 + \frac{1}{2}\begin{pmatrix} -1/2 \\ -1/2 \\ 1/2 \\ 1/2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{matrix} -1/4 \\ -1/4 \\ 1/4 \\ 1/4 \end{matrix}$$

$$\Rightarrow \begin{bmatrix} 3/4 \\ -1/4 \\ 1/4 \\ 1/4 \end{bmatrix} \frac{1}{\sqrt{12/16}} \Rightarrow \frac{2}{\sqrt{3}} \begin{bmatrix} 3/4 \\ -1/4 \\ 1/4 \\ 1/4 \end{bmatrix} \Rightarrow \begin{bmatrix} 3/2\sqrt{3} \\ -1/2\sqrt{3} \\ 1/2\sqrt{3} \\ 1/2\sqrt{3} \end{bmatrix} \quad \frac{3}{2\sqrt{3}} \Rightarrow \frac{3\sqrt{3}}{6} \quad \boxed{\frac{\sqrt{3}}{2}}$$

$$\Rightarrow \begin{bmatrix} \sqrt{3}/2 \\ -\sqrt{3}/6 \\ \sqrt{3}/6 \\ \sqrt{3}/6 \end{bmatrix}$$

$$q_4 = \frac{v_4 - \langle v_4, q_1 \rangle q_1 - \langle v_4, q_2 \rangle q_2 - \langle v_4, q_3 \rangle q_3}{\| \quad - - - \quad \|}$$

$$\frac{v_4 - 0 - \left(\frac{1}{2}\begin{pmatrix} -1/2 \\ -1/2 \\ 1/2 \\ 1/2 \end{pmatrix}\right) - \left(\frac{\sqrt{3}}{6}\begin{pmatrix} \sqrt{3}/2 \\ -\sqrt{3}/6 \\ \sqrt{3}/6 \\ \sqrt{3}/6 \end{pmatrix}\right)}{\| \quad - - - - - - - \quad \|}$$

$1 \quad -2 \quad -3 = -4$

$1 \quad -5 \quad = -4$

$\| \quad \begin{pmatrix} -1/4 \end{pmatrix} \quad \begin{pmatrix} 1/4 \end{pmatrix}$

$$v_4 - \begin{pmatrix} -3/12 \\ 3/12 \\ 3/12 \end{pmatrix} - \begin{pmatrix} -4/12 \\ 4/12 \\ 4/12 \end{pmatrix}$$

$$\overline{\| - - - \|}$$

$$\Rightarrow \begin{pmatrix} 0 \\ 1/3 \\ -1/3 \\ -1/3 \end{pmatrix} \frac{3}{\sqrt{3}} = \begin{bmatrix} 0 \\ 3/3\sqrt{3} \\ -3/3\sqrt{3} \\ -3/3\sqrt{3} \end{bmatrix}$$

(e) Find the matrix representation of the orthogonal projection of $\mathbb{R}^4$ onto the space U with respect to the basis that you obtained from (d).

$$C = \left\langle \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}, \begin{pmatrix} -1/2 \\ -1/2 \\ 1/2 \\ 1/2 \end{pmatrix}, \begin{bmatrix} \sqrt{3}/2 \\ -\sqrt{3}/6 \\ \sqrt{3}/6 \\ \sqrt{3}/6 \end{bmatrix}, \begin{bmatrix} 0 \\ 3/3\sqrt{3} \\ -3/3\sqrt{3} \\ -3/3\sqrt{3} \end{bmatrix} \right\rangle$$

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 3 & 7 \\ 0 & 0 & 2 & 8 \end{bmatrix}$$

(f) Find the matrix representation of the orthogonal projection of $\mathbb{R}^4$ onto the space U with respect to the standard basis of $\mathbb{R}^4$.

$$\left[\mathrm{Proj}_U(\cdot)\right]_{E\to E} = \ldots \qquad C = \left\langle \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}, \begin{pmatrix} -1/2 \\ 1/2 \\ 1/2 \end{pmatrix} \right\rangle$$

$$\begin{pmatrix} 0 \\ \sqrt{2}/2 \\ \sqrt{2}/2 \\ 0 \end{pmatrix}\left(0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right) + \begin{pmatrix} -1/2 \\ -1/2 \\ 1/2 \\ 1/2 \end{pmatrix}\left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

$$= \begin{pmatrix} 1/4 & 1/4 & -1/4 & -1/4 \\ 1/4 & 3/4 & 1/4 & -1/4 \\ -1/4 & 1/4 & 3/4 & 1/4 \\ -1/4 & -1/4 & 1/4 & 1/4 \end{pmatrix}$$

(g) Use the answer from (f) to calculate $\mathrm{Proj}_U\, x$

$$\begin{pmatrix} 1/4 & 1/4 & -1/4 & -1/4 \\ 1/4 & 3/4 & 1/4 & -1/4 \\ -1/4 & 1/4 & 3/4 & 1/4 \\ -1/4 & -1/4 & 1/4 & 1/4 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \begin{bmatrix} -1 \\ 3/2 \\ 7/2 \\ 1 \end{bmatrix}$$