Question 6

Sunday, October 15, 2023 5:53 PM

6. Consider each of the following there is a claim, which might be **true or false**. If the claim is true then prove it, and if it is false then provide a counterexample. (For counter examples you may choose any n you wish, but if you want to prove a claim then you should prove it for all possible n's).
   (a) If $A \in M_n(\mathbb{R})$ satisfies $A^2 = 0$ then $A = 0$. (Here 0 is the zero matrix).
   (b) If $A, B \in M_n(\mathbb{R})$ are such that $AB = BA$ then $AB^2 = B^2A$.
   (c) Let $A, B, C \in M_n(\mathbb{R})$. If $AB = CB$ then $A = C$.
   (d) Let $A \in M_n(\mathbb{R})$, then $(A+I)^2 = A^2 + 2A + I$.
   (e) Let $A, B \in M_n(\mathbb{R})$, then $(A+B)^2 = A^2 + 2AB + B^2$.

a) if $A \in M_n(\mathbb{R})$, then

$$A^2 = \sum_{k=1}^{n} A_{ik} A_{kj} \quad \text{for arbitrary } i, j \leq n$$

if $A^2 = \sum_{k=1}^{n} A_{ik} A_{kj} = 0$, then $A = [0]$ false

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 + bc & ab + bd \\ ca + dc & cb + d^2 \end{bmatrix}$$

$a^2 + bc = 0$
$ab + bd = 0$
$ca + dc = 0$
$cb + d^2 = 0$

$a = 1$
$1 + bc = 0$
$b + bd = 0$
$c + dc = 0$
$cb + d^2 = 0$

$bc = -1$
$b(d+1) = 0$
$c(d+1) = 0$
$cb + d^2 = 0$

$d = -1$

$bc = -1$
$cb + 1 = 0$

$cb = -1$

$c = 1$
$b = -1$

$\begin{bmatrix} 1 & -1 \end{bmatrix}$ False

$\begin{bmatrix} 1 & -1 \end{bmatrix}$ False

(b) If $A, B \in M_n(\mathbb{R})$ are such that $AB = BA$ then $AB^2 = B^2A$.

b) If $AB = BA$, Then $AB^2 = B^2A$

<u>Proof)</u> $AB^2 \Rightarrow ABB = BBA$

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj} = \sum_{k=1}^{n} B_{ik}A_{kj} = (BA)_{ij}$$

let $C = AB$. Then C also $= BA$

$$ABB = BBA \Rightarrow CB = BC \Rightarrow (CB)_{ij} = (BC)_{ij}$$

$$\Rightarrow \sum_{k=1}^{n} C_{ik}B_{kj} = \sum_{k=1}^{n} B_{ik}C_{kj}$$

But Since $C = AB, BA$

$$\Rightarrow \sum_{k=1}^{n}\left(\sum_{k=1}^{n} A_{ik}B_{kj}\right)B_{kj} = \sum_{k=1}^{n} B_{ik}\left(\sum_{k=1}^{n} A_{ik}B_{kj}\right)$$

Since $AB = BA$ @ ij, Then $B_{kj} = B_{ik}$

$$\Rightarrow \sum_{k=1}^{n}\left(\sum_{k=1}^{n} A_{ik}B_{kj}\right)B_{kj} = \sum_{k=1}^{n}\left(\sum_{k=1}^{n} A_{ik}B_{kj}\right)B_{kj}$$

$$AB^2 = B^2A$$

(c) Let $A, B, C \in M_n(\mathbb{R})$. If $AB = CB$ then $A = C$.

false   if B is the zero matrix, $AB = 0$
$CB = 0$
$\forall A, B \in M_n(\mathbb{R})$

ex)

$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $CB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $A \neq C$

(d) Let $A \in M_n(\mathbb{R})$, then $(A + I)^2 = A^2 + 2A + I$.

Consider the expansion of $(A + I_n)^2$

$(A + I_n)(A + I_n) \Rightarrow A(A + I_n) + I_n(A + I_n)$

$\Rightarrow A^2 + A I_n + I_n A + (I_n)^2$

$A^2 + 2 A I_n + I_n$

$\Rightarrow A^2 + 2A + I_n$ ✓ claim is True

(e) Let $A, B \in M_n(\mathbb{R})$, then $(A + B)^2 = A^2 + 2AB + B^2$.

like above, lets expand $(A+B)^2 \Rightarrow (A+B)(A+B)$

$A(A+B) + B(A+B) \Rightarrow \underbrace{A^2 + AB + BA + B^2}$

Note: $AB \neq BA$ thus $\Downarrow \neq$ $A^2 + 2AB + B^2$ ✓

counterexample

$$\overset{A}{\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}} \overset{B}{\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}} \quad (A+B)^2 = \begin{bmatrix} 3 & 3 \\ 5 & 5 \end{bmatrix}^2 = \begin{bmatrix} 24 & 24 \\ 40 & 40 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 16 & 21 \\ 28 & 37 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = 2 \begin{bmatrix} 5 & 0 \\ 9 & 0 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 18 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

$$\begin{matrix} 1 & 0 & 10 \\ 1 & 0 & 10 \end{matrix} \quad RHS = \begin{bmatrix} 27 & 21 \\ 47 & 37 \end{bmatrix} \neq \begin{bmatrix} 24 & 24 \\ 40 & 40 \end{bmatrix}$$