MATH1564-K1/K2/K3 – Linear Algebra with Abstract Vector Spaces
Homework 1

1. Let $A = \{1, 2, 3, 4\}$, $B = \{1, 3, 5\}$ and $C = \{2, 4, 5\}$.
   i. Find the following sets:
   $A \cup B$, $A \cap B \cap C$, $(A \cap C) \cup B$.
   ii. Assume that A, B, and C all belong to the universal set $U = \{x \in \mathbb{N} : x \leq 6\}$. Find the following sets:
   $(B \cup C)^c$, $A^c \cap B^c \cap C^c$, U^c.
   iii. Find how many elements are in each one of the following sets (these are called the *cardinalities* of the sets):
   $(B \cap C)^c$, $(B \cup C)^c$, $\{X : X \subseteq B\}$,
   $\{X : X \subseteq A$ and X has at most two elements $\}$.
2. Draw a sketch of the following subsets of $\mathbb{R}^2$:
   i. $\{(x, y) : x \leq 2y + 1\}$.
   ii. $\{(x, y) : x^2 + y^2 \leq 1\}^c$.
   iii. $\{(x, f(x)) : f(x) = e^x\}$.
   iv. $\{t(1, 2) : t \in \mathbb{R}\}$.
   v. $\{(-1, 1) + t(1, 2) : t \in \mathbb{R}\}$.
   vi. $\{s(1, 2) + t(-1, 1) : s, t \in \mathbb{R}\}$.
   vii. $\{s(1, 2) + t(2, 4) : s, t \in \mathbb{R}\}$.
   viii. $\{t(1, 2) : 0 < t\}$.
3. Express the following sets without the use of 'three dots' nor the use of $\cap, \cup$, or complement.
   i. $\{4, 16, 36, 64, 100, ...\}$
   ii. $\{..., \frac{2}{9}, \frac{2}{3}, 2, 6, 18, ...\}$
   iii. $\{3n : n \in \mathbb{Z}\} \cap \{2n + 1 : n \in \mathbb{Z}\}$.
4. In each of the following parts you are given a basic definition regarding operations on sets. Use this definition to solve the questions that follow it.
   i. Given two sets, A and B, the *difference between A and B* is the set
   $$A \setminus B = \{x \in A : x \notin B\}.$$
      a. For the sets A, B and C given in Q1, find the following: $B \setminus C$, $C \setminus B$, $A \setminus (B \cup C)$.
      b. Draw a sketch in $\mathbb{R}^2$ of the set
   $$\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4\} \setminus \{(x, y) \in \mathbb{R}^2 : (x - 2)^2 + y^2 \leq 1\}$$

ii. Given two sets, A and B, the *Cartesian product of A and B* is the set

$$A \times B = \{(x, y) : x \in A \text{ and } y \in B\}.$$

a. For the sets A, B and C given in Q1, find the following: $C \times B$, $B \times C$, $A \times \emptyset$.

b. Draw a sketch in $\mathbb{R}^2$ of the set $[0,1] \times [0,1]$. (Recall that $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$).

c. Draw a sketch in $\mathbb{R}^2$ of the set $([0,4] \times [0,4]) \setminus ([1,2] \times [2,5])$

5. Let A,B and C all be sets with a universal set U. In each of the following parts, a statement is written about these sets (or some of them). We consider such a statement **true** if it is true for every possible sets A,B, C and U. We consider it **false** if there is at least one example of sets A,B, C and U for which the statement does not hold. Determine for each one of the following statements if it is **true or false**. If you claim that it is **false** then provide an example for which the statement fails ("counterexample"). If you claim that the statement is **true** then **prove your claim as best you can**.

   i. $(A \cap B) \cup C = A \cap (B \cup C)$.

   ii. $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$.

   iii. $B = A \cup B$ if and only if $A \subseteq B$.

   iv. $(A \setminus B) \cup (B \setminus A) = A \cup B$.

   v. If $B \subseteq C$ then $(A \times B) \subseteq (A \times C)$.

6. Negate the following statements
   i. For every integer n there exists a rational number a such that $a = n$.
   ii. For every $x > 0, x^2 + y^2 > 0$ for all y.
7. Rewrite each of the following statements using the quan- tifiers "for all" and "there exists" as appropriate.
   i. Not all continuous functions are differentiable.
   ii. There is no largest real number.
   iii. There are infinitely many primes.
8. Let $A = \mathbb{R}^2 \setminus \{(0,0)\}$. For $p, q \in A$, define $p \sim q$ if $p = q$ or the line through the distinct points p and q passes through the origin.
   i. Prove that $\sim$ defines an equivalence relation on A.
   ii. Find the equivalence classes of $\sim$