1a. The uncertainty $\Delta_n(a \cdot S) = \sqrt{Var_n(a \cdot S)}$

The variance of the ensemble

$$Var_n(\vec{a} \cdot \vec{S}) = \langle ((a \cdot S) - \langle a \cdot S \rangle)^2 \rangle$$

$$= \langle (a \cdot S)^2 \rangle - \langle (a \cdot S) \rangle^2$$

$$= \frac{\hbar^2}{4} a^2 - \left(\frac{\hbar}{2}(a \cdot n)\right)^2$$

$$\frac{\hbar^2}{4} a^2 - \frac{\hbar^2}{4}(\vec{a} \cdot \vec{n})^2$$

$$\Delta_n(\vec{a} \cdot \vec{S}) = \hbar/2 \sqrt{|a|^2 - (\vec{a} \cdot \vec{n})^2}$$

1b. Because we start off with an ensemble n where $Prob_n(S_n = \hbar/2) = 1$,

The spin vector S_n is grounded in the Bloch Vector n, thus $\langle S \rangle_n = (\hbar/2)\, n$

Since S_n = Both $+/- \hbar/2$, the vector's expected value $\langle S_{n'} \rangle_{n'} - \langle S_{n'} \rangle_{n'} = \hbar/2\,(n \cdot n')$ for Both ensembles n and n',

The $Prob_n(S_n = \hbar/2) = Prob_{n'}(S_{n'} = \hbar/2)$

$= \frac{1}{2}(1 + n \cdot n')$ Since expected values are related to probability as a weighted average

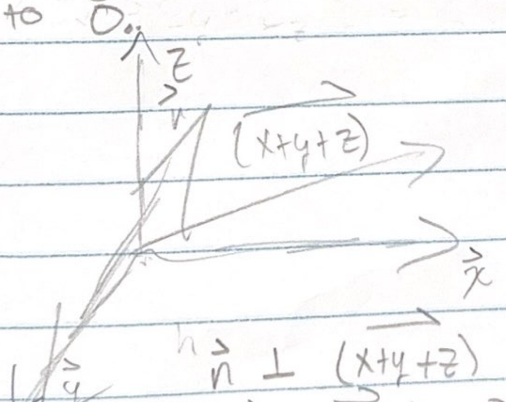
2a. This is Not possible because in the Cartesian basis $\{x, y, z\}$, at least one cardinal direction is bound to contain the expected value of the spin vector s.t. $S = \hbar/2$ in a pure ensemble w/ 1/2 spin particles. Additionally, since space is isotropic, one value of S_x or S_y or S_z cannot be zero for a pure 1/2 spin ensemble.

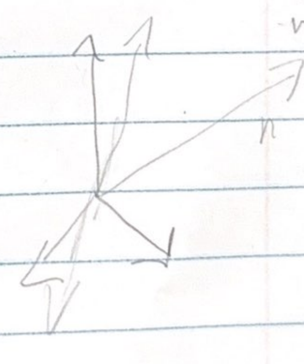
$$\langle S_x\rangle^2 + \langle S_y\rangle^2 + \langle S_z\rangle^2 = \left(\frac{\hbar}{2}\right)^2$$

$$(\hbar/2)^2(n_x^2 + n_y^2 + n_z^2) = 1$$

2B. This is possible to prepare a pure ensemble where $\sum S_{x,y,z}$ are zero. In such an ensemble, $\langle S_x + S_y + S_z\rangle_n = \hbar\langle(\vec{x}+\vec{y}+\vec{z})\cdot S\rangle_n$ implies/equals $= \hbar/2\,(x+y+z)\cdot\vec{n}$ however if the bloch vector $\vec{n}$ is $\perp$ (orthogonal) to $(\vec{x}+\vec{y}+\vec{z})$ or vector $\vec{a}$, then it will evaluate to $\vec{0}$.

$$\rangle_n = \underbrace{\hbar/2\,(x+y+z)\cdot\vec{n}}_{n \perp \text{ to } x+y+z}$$

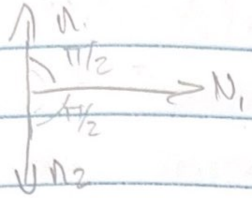




$$\vec{n} \perp \overrightarrow{(x+y+z)}$$

$$\vec{n}\cdot(x+y+z) = 0$$

3a for $P(N)$ where $N=2$



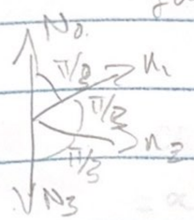
$$P(N) = \left(\cos\frac{\pi}{2N}\right)^{2N}$$

$$P(2) = \frac{1}{2}\left(1+\cos\frac{\pi}{2}\right)$$

$$= \cos^2 \pi/2N \Rightarrow \cos^2\frac{\pi}{4}$$

$$= \boxed{1/4}$$

for $N=3$



$$P(3) = \left(\cos\pi/6\right)^{3(2)} = \left(\cos\pi/6\right)^6$$

$$\boxed{\approx .421875}$$

3b) for $N=\infty$

$$P(\infty) = \left(\cos\pi/\infty\right)^{\infty} \Rightarrow \lim_{N\to\infty}\left(\cos\frac{\pi}{2N}\right)^{2N}$$

$$\ln(P(\infty)) = \lim_{N\to\infty}\left(\ln\left(\cos^{2N}\left(\frac{\pi}{2N}\right)\right)\right)$$

$$e^{\ln(P\infty)} = \lim_{N\to\infty}\left(e^{\ln(\cos^{2N}(\pi/2N))}\right)$$

$$P(\infty) = \lim_{N\to\infty}\cos^{2N}(\pi/2N)$$

$$\lim_{N\to\infty}\cos(\pi/2n) = 1$$

$$\lim_{N\to\infty}(1)^{2N} = \boxed{1}$$

$$\boxed{P(\infty) = 1}$$

4) $\vec{n}_1 \cdot n = \alpha$ $\quad \vec{n}_2 \cdot \vec{n} = \alpha \quad n_1 \cdot n_2 = \alpha^2$

$\vec{n} \cdot (\vec{n}_1 - \vec{n}_2) = 0$

Bloch vector $\vec{n}$ can be rewritten as a linear combination of n_1 and n_2

$$\vec{n} = A(n_1 + n_2) + B(n_1 - n_2) + C(n_1 \times n_2)$$

for coefficient A:

$\vec{n}_1 \cdot \vec{n} = \alpha = \vec{n}_2 \cdot \vec{n} \implies = A$

$A(\underbrace{n_1 \cdot n_1}_{1} + \underbrace{n_1 \cdot n_2}_{\alpha^2}) = A(1 + \alpha^2)$

But $A(n_1 \cdot n_1 + n_1 \cdot n_2) = \alpha$

$A(1+\alpha^2) = \alpha$

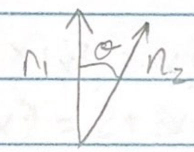
$A = \frac{\alpha}{1+\alpha^2}$

for coeff B.

since $\vec{n} \cdot (n_1 - n_2) = 0$

$\vec{n} \cdot (0) = 0$

$B = 0$



$n_1 \cdot n_2 = \alpha^2$

$\|n_1\|\|n_2\|\cos\theta$

$n_1 \times n_2 = \|n_1\|\|n_2\|\sin\theta$

$\alpha^2 = \|n_1\|\|n_2\|\cos\theta$

$\frac{\alpha^2}{\cos\theta} = \|n_1\|\|n_2\|$

$n_1 \times n_2 = \alpha^2 \frac{\sin\theta}{\cos\theta}$

$\alpha^2 \tan\theta$

for coefficient C:

$C(\vec{n}_1 \times \vec{n}_2) = 0$

Since $n_1 \cdot n_2 > 0$

Cross/orthogonal must be zero

$C = 0$

$$\vec{n} = \frac{\alpha}{1+\alpha^2}(\vec{n}_1 + \vec{n}_2) + \vec{0}(\vec{n}_1 - \vec{n}_2) + \vec{0}(\vec{n}_1 \times \vec{n}_2)$$