Homework 2

Problem 5

(a) Derive the triangle inequality $\|\varphi+\psi\| \le \|\varphi\| + \|\psi\|$. (Here $\|\Psi\| = \sqrt{\langle\Psi|\Psi\rangle}$ is the norm of vector $|\Psi\rangle$.)

Suggestion: write $\|\varphi+\psi\|^2 = \langle\varphi+\psi|\varphi+\psi\rangle$ and use the relation $\mathrm{Re}\,\langle\varphi|\psi\rangle \le |\langle\varphi|\psi\rangle|$ and the Schwartz inequality.

(b) When the triangle inequality becomes an equality?

(c) Show that $\|\varphi-\psi\| \ge \big|\|\varphi\| - \|\psi\|\big|$.

Problem 6

What is the number of independent *real* parameters $N(\mathcal{N})$ needed to specify (up to a phase factor) a state vector in $\mathcal{N}$-dimensional Hilbert space?

Problem 7

For any pure state ψ of spin 1/2 there exists a unique Bloch vector $\mathbf{n}$ such that $\mathrm{Prob}_\psi(S_\mathbf{n} = \hbar/2) = 1$. Find the angles θ and ϕ specifying the Bloch vector in the spherical coordinates for the state represented by the state vector

$$|\psi\rangle = \frac{1}{\sqrt{2}}|+\mathbf{x}\rangle + \frac{e^{2\pi i/3}}{\sqrt{2}}|-\mathbf{x}\rangle.$$

Suggestion: write $|\psi\rangle$ in $|\pm\mathbf{z}\rangle$ basis and compare with $|\psi\rangle = \cos(\theta/2)|+\mathbf{z}\rangle + e^{i\phi}\sin(\theta/2)|-\mathbf{z}\rangle$ (up to a phase factor).

Problem 8

Verify that the inner product of two spin 1/2 state vectors

$$|\mathbf{n}_1\rangle = \cos(\theta_1/2)|+\mathbf{z}\rangle + e^{i\phi_1}\sin(\theta_1/2)|-\mathbf{z}\rangle, \qquad |\mathbf{n}_2\rangle = \cos(\theta_2/2)|+\mathbf{z}\rangle + e^{i\phi_2}\sin(\theta_2/2)|-\mathbf{z}\rangle$$

with arbitrary angles $\theta_{1,2}$ and $\phi_{1,2}$ satisfies

$$\big|\langle\mathbf{n}_1|\mathbf{n}_2\rangle\big|^2 = \frac{1}{2}\big(1 + \mathbf{n}_1\cdot\mathbf{n}_2\big).$$

For reference: Cartesian components of a unit vector $\mathbf{n}$ specified by angles θ and ϕ is spherical polar coordinates read

$$n_\mathbf{x} = \sin\theta\cos\phi, \quad n_\mathbf{y} = \sin\theta\sin\phi, \quad n_\mathbf{z} = \cos\theta.$$

You will also need the trigonometric identities

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta, \quad 1+\cos(2\alpha) = 2\cos^2\alpha, \quad 1-\cos(2\alpha) = 2\sin^2\alpha, \quad \sin(2\alpha) = 2\sin\alpha\cos\alpha.$$