$\rangle\langle x|$

$\int dx\,|x\rangle f(x)\langle x|\,\hat{P}$

$x|\left(\hat{f}(\hat{x})\hat{P}\right)$

$p)\langle p|\int dx\,|x\rangle x\langle x|$

$f(p)\langle p|x\rangle x\langle x|$

$|x\rangle x\langle x|\,\Big]\mathbb{1}$

$|p\rangle\langle p$

$\hbar\left[\frac{d}{dp}x - x\frac{d}{dp}\right]\hat{f}(p)$

HW 8 - QM - Rodra Goel

Q 33 $[\hat{P}, \hat{f}(x)] = -i\hbar f'(x) \qquad \hat{f}(\hat{x}) = \int dx\,|x\rangle f(x)\langle x|$

$[\hat{P}, \hat{f}(\hat{x})] = \mathbb{1}[\hat{P}, \hat{f}(\hat{x})] = \int dx\,|x\rangle\langle x|[\hat{P}, \hat{f}(\hat{x})]$

Since $[\hat{x}, \hat{P}] = i\hbar\mathbb{1}$ + $[\hat{P}, \hat{x}] = -i\hbar\mathbb{1}$

and $\hat{P}$ is derivative in position space

$\Rightarrow \int dx\,|x\rangle\langle x|\left(-i\hbar\frac{d}{dx}\right)\hat{f}(\hat{x}) \Rightarrow \int dx\,|x\rangle\langle x| -i\hbar\,\hat{f}'(\hat{x}) \Rightarrow$

$\mathbb{1}(-i\hbar)\,\hat{f}'(\hat{x}) \Rightarrow -i\hbar\,\hat{f}'(x)$

b) $\cancel{[\hat{x}, \hat{f}(\hat{p})] = i\hbar f'(\hat{p}) \qquad 1 = \int dx\,|x\rangle\langle x|} \qquad \hat{f}(\hat{p}) = \int dp\,|p\rangle\langle p|$

$\int dx\,|x\rangle\langle x|[\hat{x}, \hat{f}(\hat{p})] \Rightarrow \int dx\,|x\rangle\langle x|\left(\hat{x}\hat{f}(\hat{p}) - \hat{f}(\hat{p})\hat{x}\right)$

$\Rightarrow \int dx\,|x\rangle\langle x|\,\hat{x}\int dp\,|p\rangle f(p)\langle p| - \int dx\,|x\rangle\langle x|\int dp\,|p\rangle f(p)\langle p|\,\hat{x}$

$\Rightarrow \int dx\left[|x\rangle\cancel{\langle x|}x\rangle x\langle x|\int dp\,|p\rangle f(p)\langle p| - |x\rangle\langle x|\int dp\,|p\rangle f(p)\langle p|\int dx\,|x\rangle x\langle x|\right]$

$\Rightarrow \int dx\left[\int dp\,|x\rangle x\langle x|p\rangle f(p)\langle p| - \int dp\int dx\,|x\rangle\langle x|p\rangle f(p)\langle p|x\rangle x\langle x|\right]$

$\int dx\int dp\left[|x\rangle x\langle x|p\rangle f(p)\langle p| - \int dx\,|x\rangle\langle x|p\rangle f(p)\langle p|x\rangle x\langle x|\right]$

$i\hbar\frac{\partial}{\partial p}$

$$\hat{P}\int dx\,|x\rangle f(x)\langle x| - \int dx\,|x\rangle f(x)\langle x|\hat{P}$$

$$\int dx\,|x\rangle\langle x|\hat{P}\int dx\,|x\rangle f(x)\langle x| - \int dx\,|x\rangle f(x)\langle x|\hat{P}\int dx\,|x\rangle\langle x|$$

$$\int dx\,|x\rangle - i\hbar\frac{d}{dx}f(x)\langle x|$$

$$\int dx\,|x\rangle\langle x|\left(\hat{P}\hat{f}(\hat{x}) - \hat{f}(\hat{x})\hat{P}\right)$$

$$\int dx\,|x\rangle\langle x|P\int dx\,|x\rangle f(x)\langle x| - \int dx\,|x\rangle\langle x|\int dx\,|x\rangle f(x)\langle x|\hat{P}$$

$$\int dx\int dx\,|x\rangle\langle x|\hat{P}|x\rangle f(x)\langle x| - \int dx\,|x\rangle\langle x|\left(\hat{f}(\hat{x})\hat{P}\right)$$

$$\Rightarrow \int dx\int dx - i\hbar\frac{d}{dx}|x\rangle f(x)\langle x|$$

$$= \int dx\; -i\hbar f'(\hat{x})$$

Q33 b) $[\hat{x}, \hat{f}(\hat{P})] = i\hbar f'(\hat{P})$

$$\hat{x}\hat{f}(\hat{P}) - \hat{f}(\hat{P})\hat{x}$$

$$\int dx\,|x\rangle x\langle x|\int dP\,|P\rangle f(P)\langle P| - \int dP\,|P\rangle f(P)\langle P|\int dx\,|x\rangle x\langle x|$$

$$\int dx\int dP\,|x\rangle x\langle x|P\rangle f(P)\langle P| - \int dP\int dx\,|P\rangle f(P)\langle P|x\rangle x\langle x|$$

$$\mathbb{1}\int dx\int dP\left[|x\rangle x\langle x|P\rangle f(P)\langle P| - |P\rangle f(P)\langle P|x\rangle x\langle x|\right]\mathbb{1}$$

$$\int dx\int dP\int dP\,|P\rangle\cancel{\langle P|x\rangle} x\cancel{\langle x|P\rangle} f(P)\langle P| - \int dP\,|P\rangle f(P)\cancel{\langle P|x\rangle} x\cancel{\langle x|P\rangle}\langle P$$

$$i\hbar\frac{d}{dP}x f(P) \qquad - i\hbar x\frac{d}{dP}f(P) = i\hbar\left[\frac{d}{dP}x - x\frac{d}{dP}\right]f(P)$$

$$= \underline{i\hbar f'(\hat{P})}$$

$$\hat{H} = \frac{p^2}{2m} + \text{Const}$$

$$\frac{d}{dt}\hat{A}_t = \frac{i}{\hbar}[\hat{H}_t, \hat{A}_t]$$

let $\hat{A}_t = x_t$

$$\frac{d}{dt}\hat{x}_t = \frac{i}{\hbar}[\hat{H}_t, \hat{x}_t]$$

But $\hat{H}$ is independent of time

$$\frac{d}{dt}\hat{x}_t = \frac{i}{\hbar}[H, \hat{x}]_t$$

$$\frac{d}{dt}\hat{x}_t = i/\hbar(\hat{H}\hat{x} - \hat{x}H)$$

$$\frac{i}{\hbar}\left(\frac{P^2\hat{x}}{2m} - \frac{x\hat{P}^2}{2m}\right)$$ $\therefore$ Const cancel

But $\hat{x}$ in momentum space is $i\hbar\frac{\partial}{\partial P}$ operator

$$= \frac{-i}{\hbar}[\hat{H}, i\hbar\frac{\partial}{\partial P}] \Rightarrow -i^2[\hat{H}, \frac{\partial}{\partial P}] = \frac{\partial}{\partial P}\frac{P^2}{2m} =$$

$$\frac{d}{dt}x_t = \frac{\hat{P}}{m}$$

Q35 $\langle p^2 \rangle_\psi$ for $\psi(x) \propto e^{-|x/a|}$

a $\psi \propto e^{-|x/a|}$ $\int_{-\infty}^{\infty} dx\, C_n^2 e^{-2|\frac{x}{a}|} = 1$

Normalization Coeff $C_n = \frac{1}{\sqrt{a}}$

$$\psi = \frac{1}{\sqrt{a}} e^{-|x/a|}$$

$$\langle p^2 \rangle_\psi = \hbar^2 \int dx \left| \frac{\partial}{\partial x} \psi(x) \right|^2 \Rightarrow \hbar^2 \int dx \left| -\frac{e^{-|\frac{x}{a}|}}{a\sqrt{a}} \right|^2$$

$$\Rightarrow \frac{\hbar^2}{a^3} \int dx\, e^{-2|\frac{x}{a}|} = \frac{\hbar^2}{a^3} \left(\frac{|a||x|\, e^{-2|\frac{x}{a}|}}{2x} \right)$$

$$\Rightarrow \boxed{\frac{\hbar^2}{a^2(2)} e^{-2|\frac{x}{a}|}}$$

b) $\langle p^2 \rangle$ for $\psi(x) \propto (1-|\frac{x}{a}|)\underline{\theta(1-|\frac{x}{a}|)}$ $\delta(1-|\frac{x}{a}|) = \frac{\partial}{\partial x}\theta(1-|\frac{x}{a}|)$

$$\theta(1-|\tfrac{x}{a}|) = \begin{cases} 1 & 1-|\frac{x}{a}|>0 \\ 0 & 1-|\frac{x}{a}|<0 \end{cases} \Rightarrow \begin{cases} 1 & -a<x<a \\ 0 & \text{otherwise} \end{cases}$$

$$1 > |\tfrac{x}{a}| \Rightarrow 1 > \tfrac{x}{a} > -1$$

Normalization Coeff C_n

$$\int \psi^2(x)\,dx \Rightarrow C_n^2 \int_{-\infty}^{\infty} dx\, (1-|\tfrac{x}{a}|)^2 (\theta(1-|\tfrac{x}{a}|))^2$$

$$\Rightarrow \underbrace{\int_{-\infty}^{-a} dx (1-|\tfrac{x}{a}|)^2 \theta^2(1-|\tfrac{x}{a}|)}_{0} + \underbrace{\int_{-a}^{a} dx (1-|\tfrac{x}{a}|)^2 \theta^2(1-|\tfrac{x}{a}|)}_{\Downarrow} + \underbrace{\int_{a}^{\infty} dx (1-|\tfrac{x}{a}|)^2 (\theta(1-|\tfrac{x}{a}|))^2}_{0}$$

$$C_n^2 \left(\tfrac{2}{3}\right) a = 1$$

$$C_n = \sqrt{\tfrac{3}{2}a}$$

$$\langle p^2 \rangle = \hbar^2 \int_{-a}^{a} dx \left(\frac{\partial}{\partial x} \frac{\sqrt{3}\sqrt{a}}{\sqrt{2}} (1-|\tfrac{x}{a}|)(\theta(1-|\tfrac{x}{a}|)) \right)^2$$

$$\frac{d}{dx}\left(1-\left|\frac{x}{a}\right|\right)^2 = \left(-\frac{1}{a}\right)^2 = \frac{1}{a^2}$$

$$\langle p^2\rangle_\psi = \hbar^2\sqrt{\tfrac{3}{2}a}\int_{-a}^{a} dx\left[\frac{d}{dx}\left(1-\left|\frac{x}{a}\right|\right)\right]^2$$ Since $\theta(1-|\frac{x}{a}|)$ is 1 Between $-a$ $+a$

Q36

a)

$$\frac{\hbar^2\sqrt{\frac{3}{2}a}}{a^2}\int_{-a}^{a} dx$$

$$\Rightarrow \frac{\hbar^2}{a^2}\sqrt{\tfrac{3}{2}a}\,(2a) \Rightarrow \boxed{\frac{2\hbar^2}{a}\sqrt{\tfrac{3}{2}a}}$$

b)

$\theta(1-|\frac{x}{a}|)$ is
1 Between $-a + a$

Q36 $\psi_\pm(x) = \langle x | \psi_\pm \rangle = \frac{1}{\sqrt{2}}[\varphi_1(x) \pm i\,\varphi_2(x)]$

a) $|\psi_+\rangle$ orth to $|\psi_-\rangle$

$\langle \psi_- | \psi_+ \rangle = \int dx \langle \psi_- | x \rangle \underbrace{\langle x | \psi_+ \rangle}$

$\int dx \left[\frac{1}{\sqrt{2}}\varphi_1(x) + i\frac{1}{\sqrt{2}}\varphi_2(x)\right]\left[\frac{1}{\sqrt{2}}\varphi_1(x) + i\frac{1}{\sqrt{2}}\varphi_2(x)\right]$

$\int dx \left\{\frac{1}{2}\varphi_1(x)^2 + \frac{i}{2}\varphi_1(x)\varphi_2(x) + \frac{i}{2}\varphi_1(x)\varphi_2(x) + \frac{i^2}{2}\varphi_2(x)^2\right\}$

$\Rightarrow$ take each integral separately

$\frac{1}{2} + i/2\,(0) + i/2\,(0) - \frac{1}{2} = 0$, since $\langle \cdot,\cdot \rangle = 0 \Rightarrow$ orth.

b) $|\psi_+(x)|^2 = P_+(x) = \psi_+^*(x)\,\psi_+(x)$

$|\psi_-(x)|^2 = P_-(x) = \psi_-^*(x)\,\psi_-(x)$

But * denotes conjugates + can be rewritten by chang sign of i

$\therefore\ \psi_+^*(x) = \psi_-(x) \Rightarrow |\psi_+(x)|^2 = \psi_+^*(x)\,\psi_+(x) = \psi_-(x)\,\psi_+(x)$

$|\psi_-(x)|^2 = \psi_-^*(x)\,\psi_-(x) = \psi_+(x)\,\psi_-(x)$

$\therefore$ equal + same principle for vectors written in momentum - space p

$\langle (P/P\omega)^2 \rangle_n$

$\frac{1}{2} E_n$

$\Rightarrow \Delta x \Delta P = \frac{E_n}{\omega} = \frac{\frac{\varepsilon \pm \omega}{2}}{\omega}$

$(\hat{x})|\psi\rangle = \hat{0}$

Q 37 $|\psi\rangle$ is e-vect of $\hat{H}$

a) $\langle\psi|[\hat{A}, \hat{H}]|\psi\rangle = 0$

$\langle\psi|\hat{A}\hat{H}|\psi\rangle - \langle\psi|\hat{H}\hat{A}|\psi\rangle$

$\Rightarrow \lambda\langle\psi|A|\psi\rangle - \lambda^*\langle\psi|A|\psi\rangle = \lambda\langle\psi|A|\psi\rangle - \lambda\langle\psi|A|\psi\rangle$

$= 0$ since hermitian eigenvalues are Real $\therefore \lambda = \lambda^*$

b) $\hat{H} = \hat{p}^2/2m + V(x)$ $\qquad \hat{V}(x) = \int dx\, |x\rangle V(x) \langle x|$

$\langle\psi|[\hat{x}\hat{p}, \hat{H}]|\psi\rangle \Rightarrow \langle\psi|\hat{x}\hat{p}\hat{H}|\psi\rangle - \langle\psi|\hat{H}\hat{x}\hat{p}|\psi\rangle$

$\langle\psi|\hat{x}\hat{p}\hat{H}|\psi\rangle = \langle\psi|\hat{H}\hat{x}\hat{p}|\psi\rangle$

$\langle\psi|\hat{x}\hat{p}\frac{p^2}{2m}|\psi\rangle + \langle\psi|\hat{x}\hat{p}\hat{V}(\hat{x})|\psi\rangle = \langle\psi|\frac{p^2}{2m}\hat{x}\hat{p}|\psi\rangle + \langle\psi|\hat{V}(\hat{x})\hat{x}\hat{p}|\psi\rangle$

$-i\hbar\frac{\partial}{\partial x}\langle\psi|x\frac{p^2}{2m}|\psi\rangle - i\hbar\frac{\partial}{\partial x}\langle\psi|\hat{x}\hat{V}(\hat{x})|\psi\rangle = i\hbar\frac{\partial}{\partial p}\langle\psi|\frac{p^3}{2m}|\psi\rangle - i\hbar\frac{\partial}{\partial p}\langle\psi|\hat{V}(x)\hat{P}|\psi\rangle$

$-i\hbar\langle\psi|p^2/2m|\psi\rangle - i\hbar\langle\psi|\hat{x}\hat{V}'(\hat{x})|\psi\rangle - i\hbar\langle\psi|\hat{V}(x)|\psi\rangle = i\hbar\langle\psi|\frac{3\hat{p}^2}{2m}|\psi\rangle + \ldots$

$-4i\hbar\langle\psi|\hat{p}^2/2m|\psi\rangle - i\hbar\langle\psi|\hat{x}V'(\hat{x})|\psi\rangle - \cancel{i\hbar\langle\psi|\hat{V}(x)|\psi\rangle} = \cancel{-i\hbar\langle\psi|\hat{V}(x)|\psi\rangle}$

$\langle\psi|\hat{p}^2/2m|\psi\rangle = -\frac{1}{4}\langle\psi|\hat{x}\hat{V}'(x)|\psi\rangle \Rightarrow \alpha = -1/4$

c) $\Delta P = \sqrt{\langle P^2\rangle_\psi}$, $\quad V(x) = \varepsilon \ln|\frac{x}{a}| \quad \frac{dV}{dx} = \varepsilon\left(\frac{1}{x}\right)$

$\frac{1}{2m}\langle\psi|\hat{p}^2|\psi\rangle = -\frac{1}{4}\langle\psi|\hat{x}\,\varepsilon\left(\frac{1}{x}\right)|\psi\rangle$

$\langle p^2\rangle_\psi = -\frac{m}{2}\varepsilon\langle\psi|\hat{x}\int dx\,|x\rangle\frac{1}{x}\langle x|\psi\rangle$

$= -\frac{m}{2}\varepsilon\langle\psi|\int dx\,|x\rangle\frac{x}{x}\langle x|\psi\rangle$

$-\frac{m}{2}\varepsilon\langle\psi|\psi\rangle \Rightarrow -\frac{m}{2}\varepsilon$

$\underline{\Delta P = \sqrt{(-m/2)\,\varepsilon}}$