Homework 7

Problem 28

Matrix elements of spin 1 operator $\hat{S}_{\mathbf{n}} = \mathbf{n}\cdot\hat{\mathbf{S}}$ in $|m;\mathbf{z}\rangle \equiv |j=1,m;\mathbf{z}\rangle$ basis form 3×3 matrix

$$\hat{S}_{\mathbf{n}} = \begin{pmatrix} \langle 1;\mathbf{z}|\hat{S}_{\mathbf{n}}|1;\mathbf{z}\rangle & \langle 1;\mathbf{z}|\hat{S}_{\mathbf{n}}|0;\mathbf{z}\rangle & \langle 1;\mathbf{z}|\hat{S}_{\mathbf{n}}|-1;\mathbf{z}\rangle \\ \langle 0;\mathbf{z}|\hat{S}_{\mathbf{n}}|1;\mathbf{z}\rangle & \langle 0;\mathbf{z}|\hat{S}_{\mathbf{n}}|0;\mathbf{z}\rangle & \langle 0;\mathbf{z}|\hat{S}_{\mathbf{n}}|-1;\mathbf{z}\rangle \\ \langle -1;\mathbf{z}|\hat{S}_{\mathbf{n}}|1;\mathbf{z}\rangle & \langle -1;\mathbf{z}|\hat{S}_{\mathbf{n}}|0;\mathbf{z}\rangle & \langle -1;\mathbf{z}|\hat{S}_{\mathbf{n}}|-1;\mathbf{z}\rangle \end{pmatrix}.$$

Find this matrix and verify that $[\hat{S}_{\mathbf{x}}, \hat{S}_{\mathbf{y}}] = i\hbar\hat{S}_{\mathbf{z}}$.

Problem 29

(a) Unit vectors $\mathbf{n}$ and $\mathbf{n}'$ enter the expectation value $\langle J^k_{\mathbf{n}}\rangle_{m,\mathbf{n}'}$ via their dot product $\mathbf{n}\cdot\mathbf{n}'$ [see, e.g., Sec. 3.2.3 in the Lecture Notes]. Modify the results of Problem 26 to obtain $\langle J_{\mathbf{n}}\rangle_{m,\mathbf{n}'}$ and $\langle J^2_{\mathbf{n}}\rangle_{m,\mathbf{n}'}$ for arbitrary $\mathbf{n}$ and $\mathbf{n}'$.

(b) Verify that for spin 1/2 the relations obtained in part (a) yield $\langle S_{\mathbf{n}}\rangle_{\mathbf{n}'} = (\hbar/2)(\mathbf{n}\cdot\mathbf{n}')$ and $\langle S^2_{\mathbf{n}}\rangle_{\mathbf{n}'} = (\hbar/2)^2$, where $\mathbf{n}'$ is the Bloch vector.

(c) Write down $\langle S_{\mathbf{n}}\rangle_{m,\mathbf{n}'}$ and $\langle S^2_{\mathbf{n}}\rangle_{m,\mathbf{n}'}$ for spin 1.

Problem 30

The Hamiltonian of spin 1 particles placed in a time-independent magnetic field reads

$$H = \omega\,\mathbf{n}\cdot\hat{\mathbf{S}}, \quad \mathbf{n} = \frac{1}{\sqrt{2}}(\mathbf{x}+\mathbf{z}).$$

Given that at $t=0$ the spins are in the state

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\big(|+1\rangle - |-1\rangle\big),$$

where $|\pm 1\rangle$ are eigenvectors of $\hat{S}_{\mathbf{z}}$ with eigenvalues $\pm\hbar$, find the probability $\mathcal{P}(t)$ that a measurement of $S_{\mathbf{z}}$ at arbitrary $t>0$ will find $S_{\mathbf{z}}=0$.

Suggestion: use the relation $\hat{R}(\theta\mathbf{n}) = \hat{\mathbb{1}} - i\sin\theta(\hat{S}_{\mathbf{n}}/\hbar) - (1-\cos\theta)(\hat{S}_{\mathbf{n}}/\hbar)^2$ derived in Problem 28(b).

Problem 31

The Hamiltonian is given by $\hat{H} = \varepsilon\begin{pmatrix} 1 & i\sqrt{3} \\ -i\sqrt{3} & -1\end{pmatrix}$, where ε has units of energy.

(a) Solve the eigenvalue problem for $\hat{H}$ and write the evolution operator $\hat{T}(t) = e^{-it\hat{H}/\hbar}$ in terms of eigenvectors of $\hat{H}$.

(b) Observable A corresponds to the operator $\hat{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}$.

Write the "dressed" operator $\hat{A}_t = \hat{T}^\dagger(t)\hat{A}\hat{T}(t)$ in the basis of eigenvectors of $\hat{H}$ found in part (a).

(c) Evaluate the expectation value $\langle A\rangle_t$ given that $|\psi(0)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$.

Problem 32

A *delta-pulse* is described by the Hamiltonian

$$\hat{H}(t) = \hat{H}_0 + \hat{V}\delta(t/t_0),$$

where the operators $\hat{H}_0$ and $\hat{V}$ are independent of time and the parameter t_0 has units of time.

Find the evolution operator $\hat{T}(+0,-0)$ relating state vectors immediately before/after the pulse.

Suggestion: replace the delta-function $\delta(t/t_0)$ in $\hat{H}(t)$ with

$$f_\tau(t) = \begin{cases} t_0/2\tau, & |t| < \tau, \\ 0, & |t| > \tau \end{cases}$$

[so that $\delta(t/t_0) = \lim_{\tau \to +0} f_\tau(t)$], evaluate $\hat{T}(\tau, -\tau)$, and take the limit $\tau \to +0$.