Question 2

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2. Consider $C[-\pi, \pi]$, vector space of continuous functions defined on interval $[-\pi, \pi]$. Define inner product

$$< f, g >= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x)dx$$

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(a) Show that the set

$$F_n = \{\frac{1}{\sqrt{2}}, \sin(x), \cos(x), \sin(2x), \cos(2x), \ldots, \sin(nx), \cos(nx)\}$$

where $n \in \mathbb{N}$ is an orthonormal set.

F_n is an orthonormal set

① $\langle q_i, q_j \rangle = 0$ for $i \neq j$

$$\langle \frac{1}{\sqrt{2}}, \sin(x) \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{\sqrt{2}}(\sin(x))dx = \frac{1}{\pi\sqrt{2}}\int_{-\pi}^{\pi} \sin(x)dx = \frac{1}{\pi\sqrt{2}} \cos(x)\Big]_{-\pi}^{\pi} = 0$$

② $\langle q_i, q_j \rangle = 1$ for $i = j$

$$\langle \sin(x), \sin(x) \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin^2 x \, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1-\cos(2x)}{2} dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} 1 - \cos(2x) \, dx \Rightarrow \frac{1}{2\pi}\left(\int_{-\pi}^{\pi} 1 - \int_{-\pi}^{\pi} \cos(2x) dx\right)$$

$$\frac{1}{2\pi}\left(2\pi - \frac{1}{2}\sin(2x)\Big]_{-\pi}^{\pi}\right) = \frac{1}{2\pi}\left(2\pi - \frac{1}{2}(\sin(2\pi) - \sin(-2\pi))\right)$$

$$\Rightarrow \frac{1}{2\pi}\left(2\pi - \frac{1}{2}(0 - 0)\right) = \frac{1}{2\pi}(2\pi) = \boxed{1}$$

(b) Determine the orthogonal projection of function $f(x) = x$ onto the space spanned by F_n. This is usually called the n-th order Fourier approximation of function $f(x)$. If we represent this projection as

$$a_0 \frac{1}{\sqrt{2}} + b_1 \sin(x) + c_1 \cos(x) +, \ldots, + b_n \sin(nx) + c_n \cos(nx)$$

then $a_0, b_1, c_1, \ldots, b_n, c_n$ are called the Fourier coefficients of function $f(x)$.

The Projection of $f(x)=x$ onto F_n is

$$\text{proj}_{F_n} f(x) = \sum_{i=1}^{n} \text{proj}_{q_i} f(x) \Rightarrow \sum_{i=1}^{n} \langle f(x), q_i \rangle q_i$$

for $q_i = \frac{1}{\sqrt{2}}$, $\langle x, \frac{1}{\sqrt{2}} \rangle \frac{1}{\sqrt{2}} = \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\pi}\right)\left(\frac{1}{\sqrt{2}}\right) \int_{-\pi}^{\pi} x \, dx = \frac{1}{2\pi} \left(\frac{1}{2}x^2\right)\Big]_{-\pi}^{\pi}$

$$\Rightarrow \frac{1}{2\pi}\left(\frac{\pi^2}{2} - \frac{\pi^2}{2}\right) = 0$$

for $i=2$ $\langle x, \sin(x) \rangle \sin x = \frac{\sin(x)}{\pi} \int_{-\pi}^{\pi} x \sin(x) \, dx = \frac{2\pi \sin(x)}{\pi}$

$$= \frac{2}{1} \sin(x)$$

for $i=3$ $\langle x, \cos(x) \rangle \cos(x) = \frac{\cos(x)}{\pi} \int_{-\pi}^{\pi} x \cos(x) \, dx = \boxed{0}$

for $i=4$ $\langle x, \sin(2x) \rangle \sin 2x = \frac{\sin(2x)}{\pi} \int_{-\pi}^{\pi} x \sin(2x) \, dx = -\frac{2}{2} \sin(2x)$

for $i=5$ $\langle x, \cos(2x) \rangle \cos(2x) = 0$

for $i=6$ $\langle x, \sin(3x) \rangle \sin(3x) = \frac{2}{3} \sin(3x)$

for $i=8$ $\langle x, \sin(4x) \rangle \sin(4x) = -\frac{2}{4} \sin(4x)$

for $i=10$ $\langle x, \sin(5x) \rangle \sin(5x) = \frac{2}{5} \sin(5x)$

$$\sum_{i=1}^{n} \frac{2\sin(ix)}{c_i}$$

where $c \in \{1, -2, 3, -4, \ldots, n\}$