Homework 1

Problem 1

Starting with the spin 1/2 relation $\langle \mathbf{a}\cdot\mathbf{S}\rangle_{\mathbf{n}} = (\hbar/2)(\mathbf{a}\cdot\mathbf{n})$, where $\mathbf{a}$ is an arbitrary reference vector and $\langle\cdots\rangle_{\mathbf{n}}$ denotes the expectation value evaluated for the ensemble specified by the condition $\mathrm{Prob}_{\mathbf{n}}(S_{\mathbf{n}} = \hbar/2) = 1$ [see, e.g., Eq. (1.18a) and (1.20) in the Lecture Notes],

(a) compute the uncertainty $\Delta_{\mathbf{n}}(\mathbf{a}\cdot\mathbf{S})$.

(b) show that $\mathrm{Prob}_{\mathbf{n}}(S_{\mathbf{n}'} = \hbar/2) = \dfrac{1}{2}(1 + \mathbf{n}\cdot\mathbf{n}')$.

Solution

(a) Let $\mathbf{a} = a\mathbf{n}_a$, where $a = |\mathbf{a}|$ and $\mathbf{n}_a = \mathbf{a}/a$ is a unit dimensionless vector. Although $S_{\mathbf{n}_a} = \mathbf{n}_a\cdot\mathbf{S}$ takes on two possible values, $S_{\mathbf{n}_a} = \pm\hbar/2$, we have

$$(\mathbf{a}\cdot\mathbf{S})^2 = (aS_{\mathbf{n}_a})^2 = (\hbar a/2)^2$$

for both values of $S_{\mathbf{n}_a}$, hence $\big\langle(\mathbf{a}\cdot\mathbf{S})^2\big\rangle_{\mathbf{n}} = (\hbar a/2)^2$ for all $\mathbf{n}$. The uncertainty of $\mathbf{a}\cdot\mathbf{S}$ therefore is

$$\Delta_{\mathbf{n}}(\mathbf{a}\cdot\mathbf{S}) = \sqrt{\langle(\mathbf{a}\cdot\mathbf{S})^2\rangle_{\mathbf{n}} - \langle\mathbf{a}\cdot\mathbf{S}\rangle_{\mathbf{n}}^2} = \frac{\hbar}{2}\sqrt{a^2 - (\mathbf{a}\cdot\mathbf{n})^2} = \frac{\hbar}{2}|\mathbf{a}\times\mathbf{n}|.$$

(b) The probabilities $P_\pm \equiv \mathrm{Prob}_{\mathbf{n}}(S_{\mathbf{n}'} = \pm\hbar/2)$ satisfy the equations

$$P_+ + P_- = 1, \quad \langle S_{\mathbf{n}'}\rangle_{\mathbf{n}} = (\hbar/2)(\mathbf{n}\cdot\mathbf{n}') = (\hbar/2)P_+ + (-\hbar/2)P_- \implies P_+ - P_- = \mathbf{n}\cdot\mathbf{n}'.$$

Solving these equations, we obtain $\mathrm{Prob}_{\mathbf{n}}(S_{\mathbf{n}'} = \hbar/2) = P_+ = \dfrac{1}{2}(1 + \mathbf{n}\cdot\mathbf{n}')$. ■

Problem 2

Is it possible to prepare a pure ensemble of spin 1/2 particles for which

(a) $\langle S_{\mathbf{x}}\rangle = \langle S_{\mathbf{y}}\rangle = \langle S_{\mathbf{z}}\rangle = 0$?

(b) $\langle S_{\mathbf{x}}\rangle + \langle S_{\mathbf{y}}\rangle + \langle S_{\mathbf{z}}\rangle = 0$?

(If your answer is *yes*, provide an example of an ensemble with the claimed property. If your answer is *no*, explain why not.)

Solution

(a) **NO!** Any pure ensemble of spin 1/2 particles is uniquely specified by the unit dimensionless vector (the Bloch vector) $\mathbf{n}$ such that $\mathrm{Prob}\,(S_{\mathbf{n}} = \hbar/2) = 1$. The expectation value of $\mathbf{a}\cdot\mathbf{S}$ for this ensemble is given by $\langle\mathbf{a}\cdot\mathbf{S}\rangle_{\mathbf{n}} = (\hbar/2)(\mathbf{a}\cdot\mathbf{n})$. Replacing $\mathbf{a}$ here with the unit vectors $\mathbf{x}, \mathbf{y}$, and $\mathbf{z}$ forming the Cartesian basis, we obtain the exact "sum rule"

$$\langle S_{\mathbf{x}}\rangle_{\mathbf{n}}^2 + \langle S_{\mathbf{y}}\rangle_{\mathbf{n}}^2 + \langle S_{\mathbf{z}}\rangle_{\mathbf{n}}^2 = (\hbar/2)^2.$$

This rule would be violated if $\langle S_{\mathbf{x}}\rangle_{\mathbf{n}}$, $\langle S_{\mathbf{y}}\rangle_{\mathbf{n}}$, and $\langle S_{\mathbf{z}}\rangle_{\mathbf{n}}$ were allowed to vanish simultaneously.

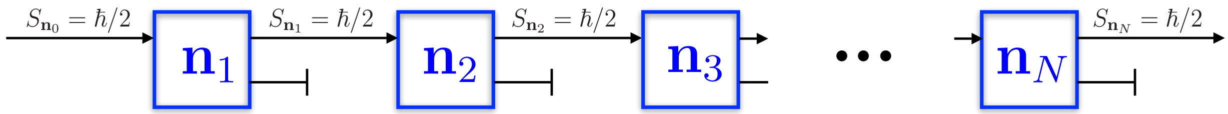
(b) **YES!** Indeed,

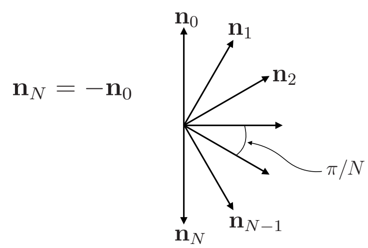
$$\langle S_{\mathbf{x}}\rangle_{\mathbf{n}} + \langle S_{\mathbf{y}}\rangle_{\mathbf{n}} + \langle S_{\mathbf{z}}\rangle_{\mathbf{n}} = \big\langle(\mathbf{x}+\mathbf{y}+\mathbf{z})\cdot\mathbf{S}\big\rangle_{\mathbf{n}} = (\hbar/2)(\mathbf{x}+\mathbf{y}+\mathbf{z})\cdot\mathbf{n},$$

vanishes for any pure ensemble specified by the Bloch vector $\mathbf{n}$ perpendicular to $\mathbf{x}+\mathbf{y}+\mathbf{z}$, such as $\mathbf{n} = \mathbf{x}-\mathbf{y}$.

Problem 3

A filtered beam of spin 1/2 particles with $S_{\mathbf{n}_0} = \hbar/2$ is sent through N consecutive Stern-Gerlach filters oriented in the directions of the unit vectors $\mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_N = -\mathbf{n}_0$ such that the angle between $\mathbf{n}_i$ and $\mathbf{n}_{i+1}$ is π/N.





Let $\mathcal{P}(N)$ be the probability that particles in the initial beam successfully pass all N filters.

(a) Evaluate $\mathcal{P}(N)$ for $N=2$ and $N=3$.

(b) Find $\mathcal{P}(\infty) = \lim_{N\to\infty} \mathcal{P}(N)$ and evaluate the leading term in the expansion of $\mathcal{P}(\infty) - \mathcal{P}(N)$ in $1/N \ll 1$.
Suggestion: expand $\ln \mathcal{P}(N)$ to the lowest non-vanishing order in $1/N$.

Solution

The probability P_i that that particles entering ith filter will pass is given by

$$P_i = \mathrm{Prob}_{\mathbf{n}_{i-1}}(S_{\mathbf{n}_i} = \hbar/2) = \frac{1}{2}\big(1 + \mathbf{n}_{i-1}\cdot\mathbf{n}_i\big) = \frac{1}{2}\big[1 + \cos(\pi/N)\big] = \big[\cos(\pi/2N)\big]^2$$

for all i. The probability $\mathcal{P}(N)$ we are interested in is the product of all p_i, i.e.,

$$\mathcal{P}(N) = \prod_{i=1}^{i=N} p_i = \big[\cos(\pi/2N)\big]^{2N}.$$

(a) Using $\cos(\pi/4) = 1/\sqrt{2}$ and $\cos(\pi/6) = \sqrt{3}/2$, we obtain

$$\mathcal{P}(2) = \big(1/\sqrt{2}\big)^4 = \frac{1}{4}, \qquad \mathcal{P}(3) = \big(\sqrt{3}/2\big)^6 = \frac{27}{64}.$$

Notice that $\mathcal{P}(N)$ increases with N.

(b) Taking into account that $\cos\alpha = 1 - \alpha^2/2 + \ldots$ and $\ln(1+\beta) = \beta + \ldots$ for small α and β, we obtain

$$\ln \mathcal{P}(N) = 2N \ln\big[\cos(\pi/2N)\big] = 2N \ln\left(1 - \frac{\pi^2}{8N^2} + \ldots\right) = -\frac{\pi^2}{4N} + \ldots,$$

where the ellipsis $(\cdots)$ stands for the higher-order in $1/N \ll 1$ contributions. This gives

$$\mathcal{P}(N \gg 1) = e^{\ln P_N} = 1 - \frac{\pi^2}{4N} + \ldots \quad \Longrightarrow \quad \mathcal{P}(\infty) = 1, \quad \mathcal{P}(\infty) - \mathcal{P}(N) = \frac{\pi^2}{4N} + \ldots.$$

This result shows that multiple consecutive measurements can *nudge* the state vector into its orthogonal. This observation is the essence of the so-called ***quantum Zeno effect***.

Problem 4

Measurements on spin 1/2 particles in a pure quantum state specified by the Bloch vector $\mathbf{n}$ have found

$$\langle S_{\mathbf{n}_1}\rangle_{\mathbf{n}} = \langle S_{\mathbf{n}_2}\rangle_{\mathbf{n}} = \frac{\hbar\alpha}{2}\,, \quad |\alpha| \le 1,$$

where the unit vectors $\mathbf{n}_{1,2}$ satisfy $\mathbf{n}_1\cdot\mathbf{n}_2 = \alpha^2$. Express the Bloch vector $\mathbf{n}$ via $\mathbf{n}_1$, $\mathbf{n}_2$, and α. Make sure that the expression you found has correct $\alpha \to 0$ and $\alpha \to 1$ limits.
Suggestion: write $\mathbf{n}$ as a linear combination of the three mutually orthogonal vectors $\mathbf{n}_1+\mathbf{n}_2$, $\mathbf{n}_1-\mathbf{n}_2$, and $\mathbf{n}_1\times\mathbf{n}_2$.

Solution

Since $\langle S_{\mathbf{n}_{1,2}}\rangle_{\mathbf{n}} = (\hbar/2)\,(\mathbf{n}_{1,2}\cdot\mathbf{n})$, the measurement results translate to the relations

$$\mathbf{n}_1\cdot\mathbf{n} = \mathbf{n}_2\cdot\mathbf{n} = \alpha.$$

This gives $(\mathbf{n}_1-\mathbf{n}_2)\cdot\mathbf{n} = 0$, hence the Bloch vector $\mathbf{n}$ is a linear combination of vectors $\mathbf{n}_1+\mathbf{n}_2$ and $\mathbf{n}_1\times\mathbf{n}_2$,

$$\mathbf{n} = A(\mathbf{n}_1+\mathbf{n}_2) + B(\mathbf{n}_1\times\mathbf{n}_2).$$

Multiplying both sides of this equation by either $\mathbf{n}_1$ or $\mathbf{n}_2$ and using $\mathbf{n}_{1,2}\cdot\mathbf{n} = \alpha$ and $\mathbf{n}_1\cdot\mathbf{n}_2 = \alpha^2$, we obtain

$$\mathbf{n}_1\cdot\mathbf{n} = A(1+\mathbf{n}_1\cdot\mathbf{n}_2) \quad\Longrightarrow\quad \alpha = A(1+\alpha^2) \quad\Longrightarrow\quad A = \frac{\alpha}{1+\alpha^2}\,.$$

The remaining coefficient B can now be found from the normalization condition $|\mathbf{n}| = 1$. We have

$$1 = |\mathbf{n}|^2 = A^2|\mathbf{n}_1+\mathbf{n}_2|^2 + B^2|\mathbf{n}_1\times\mathbf{n}_2|^2 = 2A^2(1+\mathbf{n}_1\cdot\mathbf{n}_2) + B^2\left[1-(\mathbf{n}_1\cdot\mathbf{n}_2)^2\right] = \frac{2\alpha^2}{1+\alpha^2} + B^2(1-\alpha^4)\,,$$

so that

$$B^2 = \frac{1}{1-\alpha^4}\left[1-\frac{2\alpha^2}{1+\alpha^2}\right] = \frac{1}{(1+\alpha^2)^2} \quad\Longrightarrow\quad B = \pm\frac{1}{1+\alpha^2}\,.$$

Thus, the Bloch vector takes on two possible values,

$$\mathbf{n} = \frac{1}{1+\alpha^2}\left[\alpha(\mathbf{n}_1+\mathbf{n}_2) \pm (\mathbf{n}_1\times\mathbf{n}_2)\right].$$

In the limit $\alpha \to 0$ (i.e., for $\mathbf{n}_1\perp\mathbf{n}_2$ and $\mathbf{n}_{1,2}\perp\mathbf{n}$) this expression simplifies to $\mathbf{n} = \pm\,\mathbf{n}_1\times\mathbf{n}_2$, whereas in the limit $\alpha \to 1$ it reduces to $\mathbf{n} = \mathbf{n}_1 = \mathbf{n}_2$. Both limits are obviously correct.