Question 1

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1. Compute the determinants of the following matrices:

$$\begin{pmatrix} 2 & 6 & 16 \\ -3 & -6 & 18 \\ 5 & 12 & 35 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ -1 & 5 & 2 \\ 3 & 2 & 0 \end{pmatrix} \quad \begin{pmatrix} 4 & 0 & 1 \\ -2 & 2 & -1 \\ 0 & 4 & -3 \end{pmatrix}, \begin{pmatrix} 4 & -4 & 2 & 1 \\ 1 & 2 & 0 & 3 \\ 2 & 0 & 3 & 4 \\ 0 & -1 & 2 & 1 \end{pmatrix}$$

$$\begin{matrix} 2 & 6 & 16 & 2 & 6 \\ -3 & -6 & 18 & -3 & -6 \\ 5 & 12 & 35 & 5 & 12 \end{matrix}$$

$$-420 + 540 + -576 + 486 - 432 + 630 = \boxed{222}$$

$$\begin{matrix} 1 & 2 & 3 & 1 & 2 \\ -1 & 5 & 2 & -1 & 5 \\ 3 & 2 & 0 & 3 & 2 \end{matrix}$$

$$0 + 12 - 6 - 45 - 4 - 0 = \boxed{-43}$$

$$\begin{matrix} 4 & 0 & 1 & 4 & 0 \\ -2 & 2 & -1 & -2 & 2 \\ 0 & 4 & -3 & 0 & 4 \end{matrix}$$

$$-24 + 0 + -8 - 0 + 16 - 0 = \boxed{-16}$$

$$\begin{pmatrix} 4 & -4 & 2 & 1 \\ 1 & 2 & 0 & 3 \\ 2 & 0 & 3 & 4 \\ 0 & -1 & 2 & 1 \end{pmatrix} \to \begin{matrix} 1 & 2 & 0 & 3 \\ 4 & -4 & 2 & 1 \\ 2 & 0 & 3 & 4 \\ 0 & -1 & 2 & 1 \end{matrix} \quad (-1)$$

$$\to \begin{matrix} 1 & 2 & 0 & 3 \\ 0 & -12 & 2 & -11 \end{matrix} \quad \to \begin{matrix} 1 & 2 & 0 & 3 \\ 0 & -12 & 2 & -11 \end{matrix}$$

$$1 + \frac{11}{12}$$

$$\frac{12}{12} + \frac{11}{12} = 23/12$$

$0 \quad -1 \quad 2 \quad 1 \qquad\qquad 0 \quad 0 \quad 4/6 \quad 23/12$

$$\begin{matrix} 1 & 2 & 0 & 3 \\ 0 & -12 & 2 & -11 \\ 0 & 0 & 7 & 5 \\ 0 & 0 & 22 & 23 \end{matrix} \xrightarrow{(36)} \begin{vmatrix} 1 & 2 & 0 & 3 \\ 0 & -12 & 2 & -11 \\ 0 & 0 & 7 & 5 \\ 0 & 0 & 0 & 51/7 \end{vmatrix}$$

$$22 - \frac{22}{7}(7)$$

$$23 - \frac{22}{7}(5)$$

$$1 \cdot -12 \cdot 7 \cdot \frac{51}{7} = -612\left(-\frac{1}{1}\right)\left(\frac{1}{36}\right)$$

$$\boxed{= 17}$$

Question 2

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2. (a) Let $a, b, c \in \mathbb{R}$. Prove that

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (c-a)(c-b)(b-a)$$

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$ By method of Cofactor expansion using the 1^{st} column

$$= 1\begin{vmatrix} b & b^2 \\ c & c^2 \end{vmatrix} + 1\begin{vmatrix} a & a^2 \\ c & c^2 \end{vmatrix} + 1\begin{vmatrix} a & a^2 \\ b & b^2 \end{vmatrix}$$

$1(bc^2 - b^2c) + 1(ac^2 - a^2c) + 1(ab^2 - a^2b)$

$= bc^2 - c^2b + c^2a - a^2c + ab^2 - a^2b$

$(c-a)(b-c)(b-a) = (c-a)(b^2 - cb - ab + ca)$

$\Rightarrow cb^2 - c^2b \cancel{-abc} + c^2a - ab^2 \cancel{+acb} + a^2b - ca^2$

$\Rightarrow cb^2 - c^2b + c^2a - ab^2 + a^2b - ca^2$

$= bc^2 - c^2b + c^2a - a^2c + ab^2 - a^2b$

(b) Find the values of a for which the following set is a basis for $\mathbb{R}^3$:

$$\left\{ \begin{pmatrix} a-1 \\ -3 \\ \ldots \end{pmatrix}, \begin{pmatrix} 3 \\ a+5 \\ \ldots \end{pmatrix}, \begin{pmatrix} -3 \\ -3 \\ \ldots \end{pmatrix} \right\}$$

$$-6 \qquad 6 \qquad a-4$$

$$\begin{array}{ccccc} a-1 & 3 & -3 & a-1 & 3 \\ -3 & a+5 & -3 & -3 & a+5 \\ -6 & 6 & a-4 & -6 & 6 \end{array}$$

$$(a-1)(a+5)(a-4)+54+54-18(a+5)+18(a-1)+9(a-4)$$

$$(a-1)(a^2+a-20)+108\cancel{-18a}-90\cancel{+18a}-18+9a-36$$

$$a^3\cancel{+a^2}-20a\cancel{-a^2}-a+20+108-90-18+9a-36$$

$$a^3-12a-16 \neq 0 \quad \forall a \in \mathbb{R}$$

$$a(a^2-12)-16=0$$

$$a(a^2-12)=16$$

$$a^2=28$$

$$a=-2,4$$

$$\boxed{a \in \mathbb{R},\ a \neq -2,4}$$

(c) Assume that,

$$\begin{vmatrix} a & x & l \\ b & y & m \\ c & z & n \end{vmatrix} = 2$$

Find:

$$\begin{vmatrix} 2a+3x & 2b+3y & 2c+3z \\ l+x & m+y & n+z \\ 7l & 7m & 7n \end{vmatrix}$$

$$\begin{pmatrix} 2a+3x & l+x & 7l \\ 2b+3y & m+y & 7m \\ 2c+3z & n+z & 7n \end{pmatrix}$$

$$2(7)(2) = \boxed{28}$$

Question 3

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3. Let $A, B \in M_n(\mathbb{R})$ and $\lambda \in \mathbb{R}$. Prove or disprove the following claims:

(a) $|A + B| = |A| + |B|$

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ False

$\det(A) = 1$ $\det(B) = +1$

$\det(A+B) = 0$ since $A+B = [0]$, singular

$\boxed{0 \neq 2}$

(b) $|\lambda A| = \lambda |A|$

False

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ Consider $\lambda = 7$

$|\lambda A| = 49$ $\lambda |A| = 7$

$49 \neq 7$

(c) $|\lambda A| = \lambda^n |A|$

TRUE

Consider the expansion of λA via columns

$$\lambda A = \lambda\begin{bmatrix} a_1, \dots, a_n \end{bmatrix} \Rightarrow \begin{bmatrix} \lambda a_1, \dots, \lambda a_n \end{bmatrix}$$

Taking det of ↗ can factor out each Scalar λ from the cols $\Rightarrow \det(\lambda a_1, \dots, \lambda a_n)$

$= \lambda \det(a_1, \dots, \lambda a_n) \Rightarrow (\lambda \cdot \lambda \cdot \dots \cdot \lambda) \det(a_1, \dots, a_n)$

$\Rightarrow \lambda^n(\det(a_1, \dots, a_n)) \Rightarrow \underline{\lambda^n |A|}$

(d) If A is anti-symmetric (that is, $A^T = -A$) and n is odd then A is not invertible.

any anti-symmetric matrix is such That The elements down The main diagonal are zero.

$$\begin{bmatrix} 0 & a_2 & \cdots & a_n \\ a_2 & 0 & & \vdots \\ \vdots & & 0 \; 0 & a_{n-1} \\ a_n & \cdots & a_{n-1} & 0 \end{bmatrix} \qquad A_{ij} = -A_{ji}$$

if you Reduce A To echelon form s.t it is upper triangular, you can Take The total Product $\prod_{i,j}^{n} A_{ij}$ of The main Diagonal.

Note: Since This A has an odd # of Columns,

The $n-1$ column n middle col/Row is

the $\frac{n-1}{2}+1$ column, middle col/row is a row that remains with a zero in its $\frac{n-1}{2}+1$ entry which falls on the main diagonal in echelon form. Thus $\prod_{i,j}^{n}$ will include a 0 in the term where $i,j = \frac{n-1}{2}+1$ making $\det = 0 \Rightarrow A$ is singular.

(e) If A is anti-symmetric (that is, $A^T = -A$) and n is even then A is not invertible.

consider $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $n=2$

$\det(A) = 1 \neq 0 \Rightarrow A$ is invertible

False

(f) If $AB = 0$ then $|A^2| + |B^2| = 0$.

if $AB = [0] \Rightarrow (AB)_{ij} = \sum_{k}^{n} A_{ik} B_{kj} = 0$
$\forall i,j \in \{1, \dots, n\}$

$(A^2)_{ij} = \sum_{k}^{n} A_{ik} A_{kj}$ $(B^2)_{ij} = \sum_{k}^{n} B_{ik} B_{kj}$

$A = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \end{bmatrix}$

$$\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \det(A^2) = 1 \qquad AB = 0$$

$$\det(B^2) = 0 \qquad 0 \neq 1 \Rightarrow \text{false}$$

(g) If $|A + B| = |A|$ then B is the zero matrix.

$$A = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A+B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \det(A+B) = 0 \qquad \text{False}$$

$$\det(A) = 0$$

Question 4

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4. (a) Compute the determinant of the following $n \times n$ matrix:

$$\begin{pmatrix} 4 & 1 & 1 & \dots & 1 \\ 1 & 4 & 1 & \dots & 1 \\ 1 & 1 & 4 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 4 \end{pmatrix}$$

for $n = 2$ $\Delta = 15$

$n = 3$ $\Delta = 54$

$n = 4$ $\Delta = 189$

$n = 5$ $\Delta = 648$

$4, \underset{n=2}{15/4}, \underset{n=3}{18/5}, \underset{n=4}{21/6}, \underset{n=5}{24/7}$

$$4 \cdot \left(\prod_{p=2}^{n} \frac{3(p+3)}{p+2} \right)$$

(b) For the matrix,

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

Prove that $|A| = 1 + (-1)^{(n+1)}$. (Note the 1 on the left lowest corner)

Prove that $|A| = 1 + (-1)^{n+1}$. (Note the 1 on the left lowest corner).

for This matrix to be reduced to Echelon form, we first subtract the 1^{st} Row from the last. This turns the A_{n2} entry to -1. Then Add the Second Row to the last Row to turn $A_{n2} = 0$. But this Results in $A_{n3} = 1$. Perform this process iteratively until the A_{nn-1} Term is Non-zero. Note, if n is odd, this term will be -1 since you subtracted the 1^{st} Row and every other subsequent Rows from the last to turn 1 to zero. likewise if n is even.

Performing a Subtraction or addition to the n^{th} Row when $A_{nn-1} = \pm 1$ will Result in $A_{nn} = 0, 2$. At this point since A is in echelon form, the total product of the main diagonals is determinant A_{nn} making $|A| = 0$ or 2. 0 if n is even and 2 if n is odd $\Rightarrow |A| = 1 + (-1)^{n+1}$

Question 5

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5. For the following exercises $i = \sqrt{-1}$. And the field we are working with is $\mathbb{C}$.

(a) Determine all complex numbers z such that $z^2 = i$.

$$(a+bi)(a+bi) = a^2 + abi + abi + b^2 i^2$$

$$= a^2 + 2abi - b^2 = i$$

$$a^2 - b^2 = i(1-2ab)$$

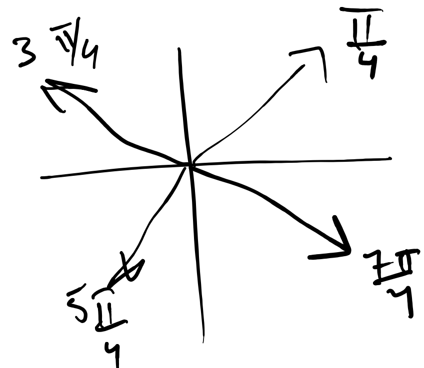
let $b = 1$

$$a^2 - 1 = i(1-2a)$$

$$a^2 = i - 2ai + 1$$

$$a^2 + 2ai = 1 + i$$

$$a(a+2i) = 1+i$$



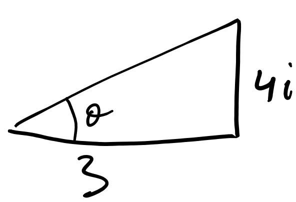
$$z = \cos\theta + i\sin\theta$$

$$\cos^2\theta + 2i\cos\theta\sin\theta - \sin^2\theta$$

$$\boxed{\begin{aligned} &\theta = \frac{\pi}{4} + \pi k, k \in \mathbb{Z} \\ &z = \cos\theta + i\sin\theta \end{aligned}}$$

(b) Write $3 + 4i$ in its polar form.

$3 + 4i$



$5\cos(53.13^\circ) + 5i\sin(53.13^\circ)$ $\quad \cos\theta = \frac{-}{5}$

$= 53.13^\circ$

$= 5e^{i\theta}, \quad \theta = 53.13^\circ$

(c) Determine call $z \in \mathbb{C}$ such that $z^4 = 1$.

$(a+bi)^4 = a^4 + 4a^3bi + 6a^2b^2i^2 + 4ab^3i^3 + b^4i^4$

$= a^4 + 4a^3bi - 6a^2b^2 - 4ab^3i + b^4 = 1$

$r^4(\cos\theta + i\sin\theta)^4 = 1$

$R^4(e^{i\theta})^4 = 1 \qquad r^4e^{4i\theta} = 1$

$re^{i\theta} = \sqrt[4]{1}$

$\therefore \left[r(\cos\theta + i\sin\theta)\right]^4 = 1$

$r = 1$ for $\theta = \frac{\pi}{2}k, \ \forall k \in \mathbb{Z}$

$z = \cos\theta + i\sin\theta, \quad \theta = \frac{\pi}{2}k \ \forall k \in \mathbb{Z} \Rightarrow z = \{1, -1, i, -i\}$

(d) If $z \in \mathbb{C}$ is non-zero, what is z^{-1} in polar form? Compare $z, \bar{z}, z^{-1}$ in a sketch.

$z = r(\cos\theta + i\sin\theta) \Rightarrow re^{i\theta}$

$(re^{i\theta})^{-1} = r^{-1}e^{-i\theta} \Rightarrow \frac{1}{r}(\cos\theta - i\sin\theta)$

i

Question 6

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6. Let $f(x)$ be a polynomial with real coefficients. Show that if $z \in \mathbb{C}$ is a root of $f(x)$ then so is its complex conjugate $\bar{z}$.

lemma: $\overline{z^n} = (\bar{z})^n$

Pf

if $z = re^{i\theta}$, complex #

Then $z^n = r^n e^{i\theta n}$

$\overline{z^n} = r^n e^{-i\theta n}$

But $(\bar{z})^n = (re^{-i\theta})^n \Rightarrow r^n e^{-i\theta n}$

$f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_k x^k$

if z is a Solution Then $f(z) = a_0 + a_1 z + \dots + a_k z^k = 0$

consider the complex conjugate of Both sides

$\bar{0} = \overline{a_0 + a_1 z + \dots + a_k z^k} \Rightarrow 0 = a_0 + a_1 \bar{z} + \dots + a_k \overline{z^k}$

But By lemma #1, $\overline{z^k} = \bar{z}^k$

$\Rightarrow 0 = a_0 + a_1 \bar{z} + a_2 \bar{z}^2 + \dots + a_k \bar{z}^k$ $\therefore$ Thus $\bar{z}$ is also a complex conjugate.

Question 7

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7. Let $A \in M_n(\mathbb{R})$ that is invertible with eigen-pair (λ, v).

(a) Is v an eigenvector of A^5? What's its corresponding eigenvalue? Generalize.

$A\vec{v} = \lambda\vec{v}$

$A^2 v \Rightarrow \quad AAv = A\lambda v = \lambda Av = \lambda(\lambda v) = \lambda^2\vec{v}$

Do this iteratively on A^n for $n=5$

The corresponding eigenvalue for A^5 is λ^5

Generally $A^n\vec{v} = \lambda^n\vec{v}$

(b) Is v an eigenvector of A^{-1}? What's its corresponding eigenvalue?

$A^{-1}A\vec{v} = A^{-1}\lambda\vec{v}$

$I_n\vec{v} = \lambda A^{-1}v$

$A^{-1}v = \frac{1}{\lambda}\vec{v}I_n \Rightarrow A^{-1}\vec{v} = \frac{1}{\lambda}\vec{v} \quad \therefore$ yes

$\frac{1}{\lambda}$ is eigenvalue

(c) Is v an eigenvector of $A^2 + 3A + 6I_n$? What's its corresponding eigenvalue?

$(A^2 + 3A + 6I_n)\vec{v}$

$I_n v = \lambda v$
$\lambda = 1$

$A^2\vec{v} + 3A\vec{v} + 6I_n v$

$\lambda^2 v + 3\lambda v + 6$

eigenvalue is

$(\lambda^2 + 3\lambda + 6)$