Homework 6

Problem 23

(a) Operators $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$ are vectors, operator $\hat{C}$ is a scalar. Show that

$$[\hat{\mathbf{A}} * \hat{\mathbf{B}}, \hat{C}] = \hat{\mathbf{A}} * [\hat{\mathbf{B}}, \hat{C}] + [\hat{\mathbf{A}}, \hat{C}] * \hat{\mathbf{B}},$$

where the symbol $*$ stands for either the dot product $\bullet$ or the cross product $\times$, so that $\hat{\mathbf{A}} * \hat{C}\hat{\mathbf{B}} = \hat{\mathbf{A}}\hat{C} * \hat{\mathbf{B}}$.

(b) Using the formula derived in part (a), vector identity $\boldsymbol{\alpha} \cdot (\boldsymbol{\beta} \times \boldsymbol{\gamma}) = (\boldsymbol{\alpha} \times \boldsymbol{\beta}) \cdot \boldsymbol{\gamma}$ and the operator relation $[\hat{\mathbf{A}}, \mathbf{a} \cdot \hat{\mathbf{J}}] = i\hbar(\mathbf{a} \times \hat{\mathbf{A}})$, show that the dot product of two vector operators $\hat{\mathbf{A}} \cdot \hat{\mathbf{B}}$ commutes with the angular momentum operator $\hat{\mathbf{J}}$.

Problem 24

(a) Use the formula derived in Problem 23(a), the Jacobi identity $\boldsymbol{\alpha} \times (\boldsymbol{\beta} \times \boldsymbol{\gamma}) + \boldsymbol{\beta} \times (\boldsymbol{\gamma} \times \boldsymbol{\alpha}) + \boldsymbol{\gamma} \times (\boldsymbol{\alpha} \times \boldsymbol{\beta}) = \mathbf{0}$, and the relation $[\hat{\mathbf{A}}, \mathbf{a} \cdot \hat{\mathbf{J}}] = i\hbar(\mathbf{a} \times \hat{\mathbf{A}})$ to show that

$$[\hat{\mathbf{A}} \times \hat{\mathbf{B}}, \mathbf{a} \cdot \hat{\mathbf{J}}] = i\hbar\, \mathbf{a} \times (\hat{\mathbf{A}} \times \hat{\mathbf{B}}).$$

(b) Using the formula derived in Problem 23(a) and the relations $[\hat{\mathbf{J}}, \mathbf{a} \cdot \hat{\mathbf{A}}] = i\hbar(\mathbf{a} \times \hat{\mathbf{A}})$ and $\hat{\mathbf{A}} \times \hat{\mathbf{J}} + \hat{\mathbf{J}} \times \hat{\mathbf{A}} = 2i\hbar\hat{\mathbf{A}}$, show that

$$[\hat{\mathbf{J}}^2, \hat{\mathbf{A}}] = \alpha\hat{\mathbf{A}} + \beta\hat{\mathbf{A}} \times \hat{\mathbf{J}}$$

with the coefficients α and β that you need to find.

Problem 25

A unit vector $\mathbf{n}$ specified in the spherical polar coordinates by the angles θ and ϕ can be obtained by first rotating $\mathbf{z}$ by θ about $\mathbf{y}$, and then rotating the resulting vector by ϕ about $\mathbf{z}$. Verify that the rotated spin 1/2 state vectors

$$\hat{R}(\phi\mathbf{z})\hat{R}(\theta\mathbf{y})|\pm\mathbf{z}\rangle$$

coincide (up to phase factors) with the eigenvectors of $\hat{S}_{\mathbf{n}}$

$$|+\mathbf{n}\rangle = \cos(\theta/2)|+\mathbf{z}\rangle + e^{i\phi}\sin(\theta/2)|-\mathbf{z}\rangle, \quad |-\mathbf{n}\rangle = \sin(\theta/2)|+\mathbf{z}\rangle - e^{i\phi}\cos(\theta/2)|-\mathbf{z}\rangle.$$

Suggestion: write the rotation operators as first-degree polynomials of spin 1/2 operators [see Problem 20].

Problem 26

Using the relations

$$\hat{\mathbf{J}}^2 = \frac{1}{2}(\hat{J}_+\hat{J}_- + \hat{J}_-\hat{J}_+) + \hat{J}_{\mathbf{z}}^2, \quad \hat{J}_{\mathbf{n}} = \mathbf{n} \cdot \hat{\mathbf{J}} = \frac{1}{2}(n_+\hat{J}_- + n_-\hat{J}_+) + n_{\mathbf{z}}\hat{J}_{\mathbf{z}},$$

$$\hat{\mathbf{J}}^2|j,m;\mathbf{z}\rangle = \hbar^2 j(j+1)|j,m;\mathbf{z}\rangle, \quad \hat{J}_{\mathbf{z}}|j,m;\mathbf{z}\rangle = \hbar m|j,m;\mathbf{z}\rangle, \quad \hat{J}_\pm|j,m;\mathbf{z}\rangle = \hbar\sqrt{j(j+1) - m(m\pm 1)}\,|j,m\pm 1;\mathbf{z}\rangle,$$

compute the expectation values $\langle J_{\mathbf{n}}\rangle_{m,\mathbf{z}} \equiv \langle j,m;\mathbf{z}|\hat{J}_{\mathbf{n}}|j,m;\mathbf{z}\rangle$ and $\langle J_{\mathbf{n}}^2\rangle_{m,\mathbf{z}} \equiv \langle j,m;\mathbf{z}|\hat{J}_{\mathbf{n}}^2|j,m;\mathbf{z}\rangle$.

Problem 27

Vectors $|m;\mathbf{n}\rangle$ with $m = 0, \pm 1$ are simultaneous eigenvectors of spin 1 operators $\hat{\mathbf{S}}^2$ and $\hat{S}_{\mathbf{n}}$:

$$\hat{\mathbf{S}}^2|m;\mathbf{n}\rangle = 2\hbar^2|m;\mathbf{n}\rangle, \quad \hat{S}_{\mathbf{n}}|m;\mathbf{n}\rangle = \hbar m|m;\mathbf{n}\rangle, \quad \langle m;\mathbf{n}|m';\mathbf{n}\rangle = \delta_{m,m'}.$$

Write (a) the projectors $\hat{\mathcal{P}}_{m,\mathbf{n}} = |m;\mathbf{n}\rangle\langle m;\mathbf{n}|$ and (b) the rotation operator $\hat{R}(\theta\mathbf{n}) = e^{-i\theta\hat{S}_{\mathbf{n}}/\hbar}$ as second-degree polynomials of $\hat{S}_{\mathbf{n}}$.