Question 6

Sunday, November 5, 2023 6:23 PM

6. True or False.

(a) If A, B are symmetric matrices, then so are their product AB.

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 7 \\ 5 & 8 \end{bmatrix}$$ false

(b) If A admits a QR factorization, i.e., $A = QR$, then $R = Q^T A$.

$A = QR \longrightarrow$ left multiply Q^T in

$Q^T A = Q^T Q R$ Note, Q has ON. columns

That span col(A). Thus $Q^T Q = I_n$

$Q^T A = R$ TRUE

(c) If $A \in M_{m \times n}(\mathbb{R})$, then $\text{rank}(A) = \text{rank}(A^T A)$.

By Rank Nullity $\text{Rank}(A) + \dim(\text{null}\,A) = n$

$\text{Rank}(A^T A) + \dim(\text{null}\,A^T A) = n \rightarrow$ since $A^T A$ is $n \times n$

consider $Ax = 0$. left multiply A^T

$= A^T A x = A^T 0 = 0$

$\Rightarrow x \in \text{null}(A^T A)$ and null(A)

Thus $\dim(\text{null}\,A) = \dim[\text{null}(A^T A)]$ $\therefore$ $\text{Rank}\,A = \text{Rank}\,A^T A$

(d) Least square solution x^* to system $Ax = b$ is chosen so that Ax^* is as close as possible to b.

TRue.

$\|b - Ax^*\| \leq \|b - \vec{v}\| \quad \forall \vec{v} \in col(A)$

(e) If the cols of A are linearly independent, then the least square solution to system $Ax = b$ is unique.

TRue, if A is linearly independent and x^* is the lss. Then There is only 1 unique solution, x^*, s.t $Ax^* = b$. $\nexists$ another $\tilde{x}$ s.t $A\tilde{x} = b$ Bc. since There is a pivot in every Row of A, There is 1 Solution x^*.

(f) If $b \in$ Col(A), then the least square solution x^* to system $Ax = b$ satisfies $Ax^* = b$.

lss x^* is Projection of b onto col(A)

$\|b - A\vec{x}^*\| \leq \|b - Ax\|$

in This case $0 \leq 0$ which is TRue

(g) If $AA^T = A^TA$ for a square matrix, then A must be orthogonal.

false consider $A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$

$A^T A = AA^T$

$\begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix} = \begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix} \Rightarrow$ Not orthogonal

Since $\left\{ \begin{bmatrix} 8 \\ 8 \end{bmatrix}, \begin{bmatrix} 8 \\ 8 \end{bmatrix} \right\}$

Not ON set.

(h) Let $A \in M_3(\mathbb{R})$ that represents an orthogonal projection with respect to standard basis in $\mathbb{R}^3$. There exists an orthogonal matrix $Q \in M_3(\mathbb{R})$ sucht that $Q^T AQ$ is diagonal.

$$\mathcal{E} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$A = \sum_{i=1}^{3} \varepsilon_i \otimes \varepsilon_i = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} (1\ 0\ 0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} (0\ 1\ 0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} (0\ 0\ 1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

consider $Q = \begin{pmatrix} 1/3 & 2/3 & -2/3 \\ -2/3 & 2/3 & 1/3 \\ 2/3 & 1/3 & 2/3 \end{pmatrix}$ $Q^T = \begin{pmatrix} 1/3 & -2/3 & 2/3 \\ 2/3 & 2/3 & 1/3 \\ -2/3 & 1/3 & 2/3 \end{pmatrix}$

Then $Q^T A Q = \frac{1}{?}\begin{pmatrix} 1 & -2 & 2 \\ \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & -2 \\ \end{pmatrix}$

Then $Q^TAQ = \frac{1}{3}\begin{pmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{1}{3}\begin{pmatrix} -2 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix}$

$= \frac{1}{9}\begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix}$ Diagonal

TRUE

Consider orthonormal set $\{q_1, \dots, q_3\}$

whos vectors combined form $Q = \begin{bmatrix} | & | & | \\ q_1 & q_2 & q_3 \\ | & | & | \end{bmatrix}$

if A is such where $\text{col}(A) = \text{span}\{q_1, q_2\}$

$\text{null}A = \text{span}\{q_3\}$.

Then $AQ = \begin{bmatrix} Aq_1 & Aq_2 & Aq_3 \end{bmatrix} = \begin{bmatrix} q_1 & q_2 & 0 \end{bmatrix}$

left multiplying this by Q^T produces

$$\begin{bmatrix} — q_1 — \\ — q_2 — \\ — q_3 — \end{bmatrix} \begin{bmatrix} | & | & \\ q_1 & q_2 & \vec{0} \\ | & | & \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ Diagonal

TRue