Question 6

Monday, December 4, 2023 11:10 AM

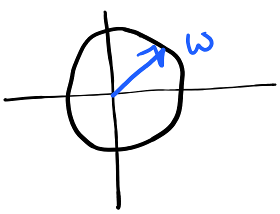
6. What can one say about a matrix's eigenvalues and its singular values? Consider the following. Note that here σ_1 is the largest singualr value and σ_n the smallest of matrix A.

   (a) Let $A \in M_2(\mathbb{R})$. Let $\boldsymbol{w} \in \mathbb{R}^2$ be a unit vector. Show that

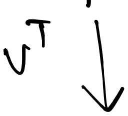
$$\sigma_2 \leq \|A\boldsymbol{w}\| \leq \sigma_1$$

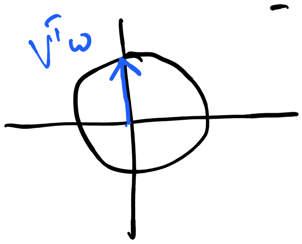
   by tracing what happens to $\boldsymbol{w}$ under the three matrices in A's SVD.

- if we say $A = U\Sigma V^T$ w is transformed By A as such

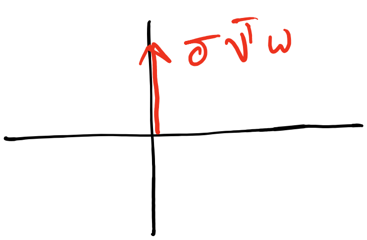


for some $\vec{w}$ of unit length, it exists on the unit circle.

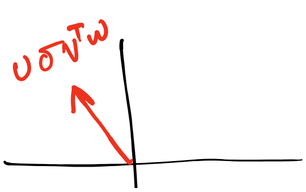




- Under V^T, w gets transformed into a different Basis



Applying Σ to $V^T x$ scales it By singular values



- Applying U to $\Sigma V^T x$ Rotates it again to the Left-singular

Vectors of A

The Norm of This new Vector Can Be analyzed as follows:

- each component is stretched By σ_i

So since $\vec{w}$ is of length 1, the stretching & contraction of $\vec{w}$ is Bounded By σ_i for Both the largest & Smallest singular values.

(b) Show part (a) algebraically.

$\|Aw\| \Rightarrow \|U\Sigma V^T w\|$

$\|V^T w\| = \sqrt{\langle V^T w, V^T w\rangle}$, in $\mathbb{R}^n$ $\langle \cdot,\cdot \rangle$ is $x^T x$

$\Rightarrow \sqrt{(V^T w)^T (V^T w)} \Rightarrow \sqrt{w^T V V^T w} = \sqrt{w^T w} \Rightarrow$

$\sqrt{\langle w, w\rangle} \Rightarrow \|w\|$ $\qquad \sigma_1 \left[\begin{matrix} w \\ \\ \end{matrix} \right.$

$\therefore$

$\Rightarrow \|U\Sigma V^T w\| \Rightarrow \|U\Sigma w\| \Rightarrow \|\Sigma w\|$ since U is orthogonal too

$\Rightarrow$ since $\vec{w}$ is of unit length, we essentially scale w's components By σ_1 & σ_2 Respectively. As a Result the maximum length That $\begin{bmatrix} \sigma_1 w_1 \\ \sigma_2 w_2 \end{bmatrix}$ Can Be is σ_1 & likewise for its lower Bound which is

σ_2

(c) Let $A \in M_{m\times n}(\mathbb{R})$. Show that

$$\sigma_n\|v\| \leq \|Av\| \leq \sigma_1\|v\|$$

for any $v \in \mathbb{R}^n$.

for any $Av \Rightarrow \|U\Sigma V^T v\|$. Since V^T is orthogonal, it is norm-preserving indicating $\|V^T v\| = \|v\|$ and scaling $V^T v$ by each singular value Σ would be analogous to scaling the vector itself by Σ. One can consider the bounds of this scaling where it scales maximally by σ_1, the largest singular value and scales minimally by σ_n, the smallest....

$$\Rightarrow \quad \sigma_n\|v\| \leq \|U\Sigma V^T v\| \leq \sigma_1\|v\|$$

$$\Rightarrow \quad \sigma_n\|v\| \leq \|Av\| \leq \sigma_1\|v\|$$

(d) Let λ be a real eigenvalue of matrix $A \in M_n(\mathbb{R})$. Show that

$$\sigma_n \leq |\lambda| \leq \sigma_1$$

.

$Av = \lambda v$ for eigenvector $\vec{v}$ in $\mathbb{R}^n$

Taking norm of both sides $\Rightarrow \|Av\| = |\lambda|\,\|\vec{v}\|$

$\Rightarrow |\lambda| = \dfrac{\|Av\|}{\|v\|}$. By proof 6 c)

$\sigma_n\|v\| \leq \|Av\| \leq \sigma_1\|v\| \Rightarrow \sigma_n \leq \dfrac{\|Av\|}{\|v\|} \leq \sigma_1$

$$\Rightarrow \sigma_n \le |\lambda| \le \sigma_1$$

(e) Consider the matrix A in Q5. Determine $\min_{\|x\|=1} \|Ax\|$. Comment at what vectors x the min value is attained.

$$A = \begin{pmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{5}/5 & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}^T$$

mapping vector $\begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$

will minimize $\|Av\|$ to 1

(f) Consider the matrix A in Q5. Determine $\max_{\|x\|=1} \|Ax\|$. Comment at what vectors x the max value is attained.

$\|Av\|$ maximized to $\sigma_1 = 4$

Since $Av_i = \sigma_i u_i$

$\|Av_i\| = \sigma_i \|u_i\|$ @ $\sigma_i = 4$

$i = 1$

v_1 Thus vector is $\begin{bmatrix} \sqrt{5}/5 \\ 2/\sqrt{5} \end{bmatrix}$