Question 1

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1. Consider bijective map $f : \mathcal{D} \to \mathcal{C}$. In class, we have stated that there exists map $g : \mathcal{C} \to \mathcal{D}$ such that $f \circ g = \mathrm{id}_{\mathcal{D}}$ and $g \circ f = \mathrm{id}_{\mathcal{D}}$. g in this case is called the inverse of f, denoted by f^{-1}. In the context of linear maps, we say linear map $T \in \mathcal{L}(V, W)$ is **invertible** if there exists linear map $S \in \mathcal{L}(W, V)$ such that $T \circ S = \mathrm{id}_W$ and $S \circ T = \mathrm{id}_V$. In this case, we denote S by T^{-1}.

   (a) Let $T \in \mathcal{L}(V, W)$. Show that its inverse T^{-1}, if it exists, is unique.

Consider Set $V = \{v_1, \ldots, v_n\}$

and $W = \{w_1, \ldots, w_n\}$ Basis for $\bar{V}$ & $\bar{W}$ resp.

Then $T \in \mathcal{L}(V, W)$ is such that $T(\vec{V}) = \vec{W}$

That is $T\left(\sum_i^n c_i v_i\right) = \sum_i^n c_i w_i$.

Assume G and h are inverses of T

Such That $G, h \in \mathcal{L}(\bar{W}, \bar{V})$

as such $T \circ G = \mathrm{id}_W$ and $T \circ h = \mathrm{id}_W$.

$G \circ T = \mathrm{Id}_V$ and $h \circ T = \mathrm{Id}_V$

$h = h \circ \mathrm{Id}_W$

$= h \circ (T \circ G)$

$(h \circ T) \circ G$

$\mathrm{Id}_V \circ G$

$= G$ $\Rightarrow$ $\underline{h = G}$ inverse is $\underline{\text{unique}}$

(b) Show that $T \in \mathcal{L}(V, W)$ is invertible iff it is bijective, i.e., an isomorphism.

① if T Bijective $\Rightarrow$ T invertible

Since T is Bijective, $\ker(T) = \{\vec{0}\}$

and Since T Bijective, Then for every $v_i \in \{v_1, \dots, v_n\}$

Basis for $\bar{V}$ $T(v_i) = w_i$, $w_i \in \{w_1, \dots, w_n\}$ Basis for $\bar{W}$.

likewise $\dim(\mathrm{Im}(T)) = \dim(\bar{W})$ and $\exists$ a one-to one map Between $\bar{V}$ and $\bar{W}$. T is linear

$T(\alpha x + \lambda y) = \alpha T(x) + \lambda T(y)$

assume $\exists$ T^{-1} s.t. $T(T^{-1}(x))$ Id_W.

and Say $T(x) = x'$, $T(y) = y'$

Then $T(\alpha T^{-1}(x') + \lambda T^{-1}(y')) = \alpha x' + \lambda y'$

$\Rightarrow T^{-1}$ linear and T is invertible

② if T invertible $\Rightarrow$ T bijective?

Since T invertible, $\exists$ $T^{-1} = S$

Thus $S \circ T = Id_V$ implies injectivity where every element in $\bar{V}$ is mapped to one in $\bar{W}$. and $T \circ S = Id_W$ meaning T surjective since any and all elements in $\bar{W}$ are mapped to some $\bar{V}$. Thus T Bijective.

(c) Give two examples of linear maps that are not invertible for two different reasons.

① $f : \mathbb{R} \to 0$

$x \mapsto 0$

This is surjective
But not injective ∴ Not invertible

② $T : \mathbb{R}^3 \to \mathbb{R}^2$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} x \\ y \end{bmatrix}$ Not invertible