Homework 4

Problem 14

$\hat{A} = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}$ is a linear operator in a two-dimensional Hilbert space.

(a) Verify that $\hat{A} \neq \hat{A}^\dagger$ and $[\hat{A}, \hat{A}^\dagger] \neq \hat{0}$, i.e., that $\hat{A}$ is neither Hermitian nor unitary.

(b) Solve the eigenvalue problem $\hat{A}|\phi\rangle = a|\phi\rangle$, $|\phi\rangle \neq |\text{null}\rangle$. Do vectors $|\phi\rangle$ satisfying these equations span the space?

Problem 15

$\hat{A}$ is a linear operator (not necessarily Hermitian) acting in a two-dimensional Hilbert space.
Express $\det \hat{A}$ via $\operatorname{tr} \hat{A}$ and $\operatorname{tr} \hat{A}^2$.

Problem 16

(a) Observable A corresponds to the Hermitian operator

$$\hat{A} = \begin{pmatrix} 1 & e^{-i\pi/3} \\ e^{i\pi/3} & -1 \end{pmatrix}.$$

Find all possible outcomes of measurement of A.

(b) Find the state vector $|\psi\rangle$ representing the state for which a measurement of A is certain to yield the largest of the possible outcomes found in part (a).

(c) Observable B corresponds to the operator

$$\hat{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Evaluate the expectation value of B in the state found in part (b).

Problem 17

(a) $\hat{A}$ is Hermitian operator on a two-dimensional Hilbert space.
Express eigenvalues of $\hat{A}$ via $\operatorname{tr} \hat{A}$ and $\operatorname{tr} \hat{A}^2$.

(b) $|\varphi\rangle$ and $|\psi\rangle$ are normalized vectors with an inner product $\langle\varphi|\psi\rangle = \alpha$, where α is a complex number.
Using the relation derived in part (a) and the identity $\operatorname{tr}\big(|\Phi\rangle\langle\Psi|\big) = \langle\Psi|\Phi\rangle$ [see Problem 13(c)], find eigenvalues of the operator

$$\hat{A} = |\varphi\rangle\langle\psi| + |\psi\rangle\langle\varphi|.$$

(c) Verify that the eigenvalues found in part (b) are real numbers, as they should be.

Problem 18

In matrix notations, spin 1/2 operator $\hat{S}_{\mathbf{n}} = \mathbf{n} \cdot \hat{\mathbf{S}}$ in $|\pm\mathbf{z}\rangle$ basis is given by

$$\hat{S}_{\mathbf{n}} = \begin{pmatrix} \langle +\mathbf{z}|\hat{S}_{\mathbf{n}}|+\mathbf{z}\rangle & \langle +\mathbf{z}|\hat{S}_{\mathbf{n}}|-\mathbf{z}\rangle \\ \langle -\mathbf{z}|\hat{S}_{\mathbf{n}}|+\mathbf{z}\rangle & \langle -\mathbf{z}|\hat{S}_{\mathbf{n}}|-\mathbf{z}\rangle \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} n_{\mathbf{z}} & n_- \\ n_+ & -n_{\mathbf{z}} \end{pmatrix}, \quad n_\pm = n_{\mathbf{x}} \pm i n_{\mathbf{y}}.$$

Solve the eigenvalue problem for $\hat{S}_{\mathbf{n}}$ and verify that the eigenvectors you found coincide (up to phase factors) with

$$|\mathbf{n}\rangle = \cos(\theta/2)|+\mathbf{z}\rangle + e^{i\phi}\sin(\theta/2)|-\mathbf{z}\rangle, \quad |-\mathbf{n}\rangle = \sin(\theta/2)|+\mathbf{z}\rangle - e^{i\phi}\cos(\theta/2)|-\mathbf{z}\rangle.$$