Question 6

Monday, October 23, 2023 7:09 PM

6. In class we mentioned that $< A, B >_1 = tr(AB^T)$ defines an inner product on on $M_{m\times n}(\mathbb{R})$ and in studio covered that $< A, B >_2 = tr(A^TB)$ is an inner product. Is there a typo in terms of where the transpose operation is on?

$$\langle A,B\rangle_1 : M_{m\times n}(\mathbb{R}) \to \mathbb{R}$$

$$A,B \longmapsto tr(AB^T)$$

① $\langle x,y\rangle = \langle y,x\rangle \ \forall x,y \in V$

② $\langle x, ky\rangle = k\langle x,y\rangle \ \forall x,y \in V \ \forall k \in \mathbb{R}$

③ $\langle x, y+z\rangle = \langle x,y\rangle + \langle x,z\rangle \ \forall x,y,z \in V$

④ $\langle x,x\rangle \geq 0$ and $\langle x,x\rangle = 0$ iff $x=0$

$$(AB)_{m,n} = \sum_{j=1}^{n} A_{mj} B_{jn}$$

$$\underset{A}{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}} \underset{B}{\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

$$(AB^T)_{i,j} = \sum_{k=1}^{n} A_{ik} B_{jk}$$

$$tr(\underset{m\times n}{A}\,\underset{n\times m}{B^T}) = \sum_{p=1}^{m}\sum_{k=1}^{n} A_{pk} B_{pk}$$

$$tr(\underset{n\times m}{A^T}\,\underset{m\times n}{B}) = \sum_{k=1}^{n}\sum_{p=1}^{m} A_{kp} B_{kp}$$

Consider the definitions of $tr(AB^T)$ and $tr(A^TB)$

The ips $\langle A,B\rangle_1 : M_{m\times n}(\mathbb{R}) \to \mathbb{R}$

$$A, B \longmapsto tr(AB^T)$$

is an inner product space since

① $\langle A, B \rangle = tr(AB^T) = \sum_{p=1}^{m} \sum_{k=1}^{n} A_{pk} B_{pk}$

$BA^T \quad = \sum_{p=1}^{m} \sum_{k=1}^{n} B_{pk} A_{pk}$

$\Rightarrow tr(BA^T) = \langle B, A \rangle$

$\forall A, B \in M_{n \times n}(\mathbb{R})$

② $\langle A, \alpha B \rangle = tr(A(\alpha B)^T)$

$= \alpha \, tr(AB^T)$

$= \alpha \langle A, B \rangle \quad \forall \alpha \in \mathbb{R}$

③ $\langle A, B+C \rangle = tr(A(B+C)^T)$

$\Rightarrow \sum_{p=1}^{m} \sum_{k=1}^{n} A_{pk} (B_{pk} + C_{pk})$

$\Rightarrow \sum_{p=1}^{m} \sum_{k=1}^{n} A_{pk} B_{pk} + A_{pk} C_{pk}$

$\Rightarrow \sum_{p=1}^{m} \sum_{k=1}^{n} A_{pk} B_{pk} + \sum_{p=1}^{m} \sum_{k=1}^{n} A_{pk} C_{pk}$

$\Rightarrow tr(AB^T) + tr(AC^T)$

$\Rightarrow \langle A, B \rangle + \langle A, C \rangle$ ✓

④ $\langle A, A\rangle > 0 \wedge \langle A, A\rangle = 0$ iff $A = 0$

$\langle A, A\rangle = tr(AA^T) = \sum_{p=1}^{m}\sum_{k=1}^{n} A_{pk} A_{pk} = 0$ when $A @ pk = 0$

and $p \in \{1, \ldots, m\}, k \in \{1, \ldots, n\}$

every element in A is zero ✓

Similarly $\langle A, B\rangle_2 : M_{m \times n}(\mathbb{R}) \longrightarrow \mathbb{R}$

$A, B \longmapsto tr(A^T B)$

is an inner product space as it satisfies

① Symmetry ② Homogeneity ③ Distributivity ④ Pos. Def.

$\therefore$ NOT A TYPO