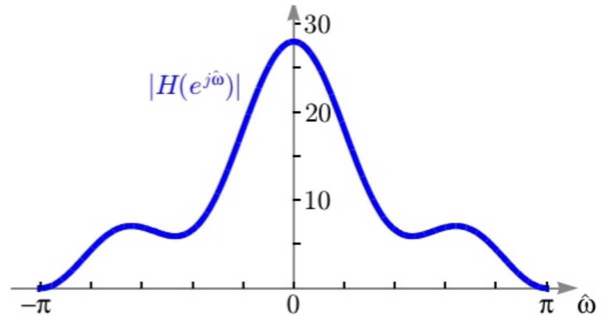
PROBLEM 8.1.* Consider an LTI system whose impulse response is $h[n]$ and whose frequency response is:

$$H(e^{j\hat{\omega}}) = e^{-3j\hat{\omega}}\big(10 + 10\cos(\hat{\omega}) + 4\cos(2\hat{\omega}) + 4\cos(3\hat{\omega})\big),$$

with magnitude response $|H(e^{j\hat{\omega}})|$ shown here:



(a) Compare the sum of the impulse response coefficients $\sum_{k=-\infty}^{\infty} h[k]$ to the frequency response evaluated at zero frequency $H(e^{j0})$.

(b) Find the system output $y[n]$ when the system input is:

$$x[n] = 0.25 + 2\cos(\tfrac{\pi}{3}n) + 2\cos(\tfrac{\pi}{2}n) + 4\cos(\tfrac{2\pi}{3}n).$$

Find α and β so that the difference equation relating the input $x[n]$ to the output $y[n]$ can be written as:

$$y[n] = \alpha x[n] + \alpha x[n-1] + \beta x[n-2] + (\alpha\beta)x[n-3] + \beta x[n-4] + \alpha x[n-5] + \alpha x[n-6].$$

a) $h[n]$ can be derived from expanding $H(e^{j\omega})$ and finding the coefficients of each exponential

$$e^{-3j\omega}\left(10 + 5e^{j\omega} + 5e^{-j\omega} + 2e^{j2\omega} + 2e^{-j2\omega} + 2e^{j3\omega} + 2e^{-j3\omega}\right)$$

$$10\cancel{e^{-j3\omega}} + \cancel{5e^{-j2\omega}} + \cancel{5e^{-j4\omega}} + \cancel{2e^{-j\omega}} + \cancel{2e^{-j5\omega}} + \cancel{2e^{0}} + \cancel{2e^{-j6\omega}}$$

$h[n] = \{\underline{2}, 2, 5, 10, 5, 2, 2\} \Rightarrow \sum h[n] = 28$ Same

$$H(e^{j\omega})\big|_{\hat{\omega}=0} = e^{3j0}\big(10 + 10\cos(0) + 4\cos(2(0)) + 4\cos(3(0))\big) = 28$$

b) if $x[n]$ is of $A\cos(\hat{\omega}n + \phi) \Rightarrow$

$y[n] = |H(e^{j\omega})| \cdot A\cos(\hat{\omega}n + \phi + \angle H(e^{j\omega}))$

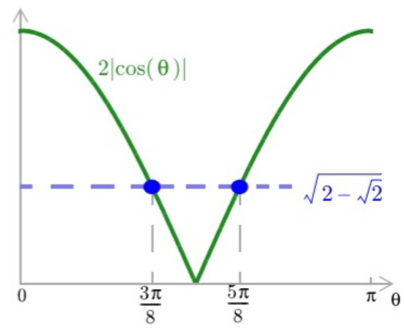
$\hat{\omega}$	$\lvert H(e^{j\omega})\rvert$	$\angle H(e^{j\omega})$
0	28	0
$\pi/3$	9	$-\pi$
$\pi/2$	6	$-3\pi/2$
$2\pi/3$	7	-2π

$y[n] = 7 - 18\cos(\tfrac{\pi}{3}n) + 12\cos(\tfrac{\pi}{2}n - \tfrac{3\pi}{2}) + 28\cos(\tfrac{2\pi}{3}n)$

Solved in Part (a) $\boxed{\alpha = 2 \quad \beta = 5}$

PROBLEM 8.2.* Consider an FIR filter defined by the following difference equation:

$$y[n] = x[n] + \sqrt{2-\sqrt{2}}\,x[n-1] + x[n-2].$$



Hint: The accompanying plot of $2|\cos(\theta)|$ versus θ shows how the constant $\sqrt{2-\sqrt{2}} \approx 0.765$ is related to $3\pi/8$ and $5\pi/8$:

(a) Find the filter output $y[n]$ when the filter input is the sinusoid $x[n] = \dfrac{1}{\sqrt{2-\sqrt{2}}}\sin(\pi n/2)$.

$\cos(\frac{\pi}{2}n - \pi/2)$

(b) This FIR filter is called a "nulling" filter because it nulls a sinusoidal input for a particular sinusoid frequency. Which frequency does this filter null? In other words, for what positive value of $\hat{\omega}_0$ (in the range 0 to π) will an input of the form $x[n] = A\cos(\hat{\omega}_0 n + \varphi)$ result in the all-zero output, $y[n] = 0$ for all n?

1-2-1 filter $\Rightarrow H(e^{j\omega}) = e^{-j\omega}(b_1 + 2\cos(\omega))$

a) Given $\hat{\omega}$ of $x[n]$ is $\pi/2 \Rightarrow |H(e^{j\omega})| = \sqrt{2-\sqrt{2}} + 2\cancel{\cos(\pi/2)}^{\,0}$

$= .765$

$\angle H(e^{j\omega}) = -\pi/2$

$\Rightarrow y[n] = \dfrac{\sqrt{2-\sqrt{2}}}{\sqrt{2-\sqrt{2}}}\cos(\frac{\pi}{2}n - \pi) \Rightarrow y[n] = \cos(\frac{\pi}{2}n - \pi)$

$\Rightarrow \boxed{-\cos(\frac{\pi}{2}n)}$

b) $b_1 + 2\cos(\omega) = 0$

$\Rightarrow \sqrt{2-\sqrt{2}} = -2\cos(\hat{\omega})$

$\omega = \cos^{-1}\left(\dfrac{-\sqrt{2-\sqrt{2}}}{2}\right) \Rightarrow \boxed{.625\pi}$

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PROBLEM 8.3.* Consider an LTI system whose frequency response is $H(e^{j\hat{\omega}}) = e^{-2j\hat{\omega}}(2 - 2\cos(2\hat{\omega}))$.

$2 - 2\cos(\pi)$

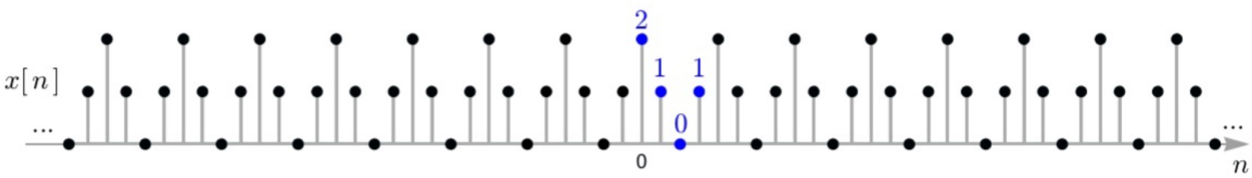
(a) Is this system an FIR filter?
If *YES*, find the difference equation that defines the FIR filter.
If *NO*, explain why not.

(b) Find the *dc gain* of the system.

(c) Find the output $y[n]$ in response to the constant input sequence $x[n] = 16$, for all n.

(d) Find the output in response to the "plus-minus" input sequence $x[n] = \cos(\pi n) = (-1)^n$.

(e) Find the output $y[n]$ in response to the "periodic extension" of [2 1 0 1], *i.e.*, an input $x[n]$ that is periodic with period 4 and satisfies $[x[0], \ldots x[3]] = [2, 1, 0, 1]$, as illustrated below:



(f) Specify a sinusoidal input $x[n]$ in *standard form*[1] that would result in the following output:

$$y[n] = 12\cos(\tfrac{\pi}{3}n).$$

(g) Specify numeric values for the constants a and b so that a system input of the form:

$$x[n] = a\delta[n] + b\delta[n-2] + a\delta[n-4] + \frac{a}{b} + a\cos(\pi n + b)$$

will result in the following system output:

$$y[n] = -\delta[n] + 2\delta[n-4] - \delta[n-9].$$

a) $H(e^{j\omega}) = e^{-j2\omega}(2 - e^{j2\omega} - e^{-j2\omega}) = 2e^{-j2\omega} - e^{0} - e^{-j4\omega}$

$\Rightarrow h[n] = \{-\underline{1}, 0, 2, 0, -1\}$

$\boxed{\text{Yes}}$ $\underline{y[n] = -x[n] + 2x[n-2] - x[n-4]}$

b) DC Gain is $\sum_{k}^{N} h[k] \Rightarrow 0$

c) $\forall n \in \mathbb{Z}$, $x[n] = 16 \Rightarrow y[n] = 0$ Since @ $\omega = 0$ for x, $f[n] = 0$

D) if $x[n] = (-1)^n$, $y[n] = 0$ Since for $n = 0, 2, 4$, $x[n]$ is even and 1 and thus summing over that interval results in $y[n] = 0$

E) $x[n] = 1 + \cos(\frac{\pi}{2}n)$ Given $\hat{\omega} = \pi/2 \Rightarrow |H(e^{j\omega})| = 4$ \& $\angle H(e^{j\omega}) = -\pi$

$\Rightarrow x[n] = 4\cos(\frac{\pi}{2}n - \pi) = \boxed{-4\cos(\frac{\pi}{2}n)}$

f) $y = 12\cos(\frac{\pi}{3}n)$ $\qquad \hat{\omega} = \frac{\pi}{3}$

$$2 - 2\cos(2\omega) = \pm 12$$

$$2 - 2\cos(\tfrac{2\pi}{3}) = 2 - 2(-\tfrac{1}{2}) = 2 + 1 = 3 = |H(e^{j\omega})|$$

$$\angle H(e^{j\omega}) = -2\omega = -\frac{2\pi}{3}$$

$$\boxed{x[n] = 4\cos(\tfrac{\pi}{3}n + \tfrac{2\pi}{3})}$$

G) $x[n] = \alpha\,\delta[n] + \beta\,\delta[n-2] + \alpha\,\delta[n-4] + \frac{\alpha}{\beta} + \alpha\cos(\pi n + \beta)$

$x[n]$ has frequency $\omega = \pi$ and 0 (0 for a/b and π for $\cos(\dots)$ terms)

$$H(e^{j0}) = 0 \quad \& \quad H(e^{j\pi}) = 0$$

$\Rightarrow$ $x[n]$ is nulled @ $\omega = \pi$ and $0 \Rightarrow x[n] = a\,\delta[n] + \beta\,\delta[n-2]$ + to $\delta[n-4]$

$y[0] = -1 = -a \Rightarrow \boxed{a = 1}$

$y[4] = 2 = -1 + 2b - 1$

$2 = -2 + 2B \Rightarrow \boxed{B = 2}$

x \ $h[n]$	-1	0	2	0	-1
a	$-a$	0	$2a$	0	$-a$
0	0	0	0	0	0
B	$-B$	0	$2b$	0	$-B$
0	0	0	0	0	0
a	$-a$	0	$2a$	0	$-a$

Made with Goodnotes

PROBLEM 8.4.* Consider the following FIR filter:

$$y[n] = x[n] + Ax[n-1] + Bx[n-2] + Ax[n-3] + x[n-4].$$

Find numeric values for the unspecified constants A and B so that the output in response to:

$$x[n] = 2.5\cos(0.25\pi n + 0.25\pi) + 5\cos(0.5\pi n + 0.5\pi)$$

is $\quad y[n] = 0$, for all n. $\qquad \omega = \frac{\pi}{4}$ and $\frac{\pi}{2}$

$$h[n] = \{1, A, B, A, 1\}$$

$$H(e^{j\omega}) = 1 + Ae^{-j\omega} + Be^{-j2\omega} + Ae^{-j3\omega} + e^{-j4\omega}$$

$$e^{-j2\omega}\left(e^{j2\omega} + Ae^{j\omega} + B + Ae^{-j\omega} + e^{-j2\omega}\right) \Rightarrow e^{-j2\omega}\left(B + 2A\cos(\omega) + 2\cos(2\omega)\right)$$

$$|H(e^{j\omega})| = B + 2A\cos(\omega) + 2\cos(2\omega)$$

$$0 = B + 2A\cos\left(\tfrac{\pi}{4}\right) + 2\cancel{\cos\left(\tfrac{\pi}{2}\right)}\to 0 \quad \text{for} \quad \hat{\omega} = \pi/4$$

$$0 = B + 2A\cancel{\cos\left(\tfrac{\pi}{2}\right)}\to 0 + 2\cos(\pi) \quad \text{for} \quad \hat{\omega} = \pi/2$$

$$\boxed{B = 2} \qquad 2 + 2A\left(\tfrac{\sqrt{2}}{2}\right) = 0 \Rightarrow A\sqrt{2} = -2 \Rightarrow \boxed{A = -\frac{2}{\sqrt{2}}}$$

PROBLEM 8.5.* Consider a generalization of the FIR nulling filter from Prob. 8.2, defined by the difference equation:

$$y[n] = x[n] + b_1 x[n-1] + x[n-2].$$

In Prob. 8.2 the coefficient b_1 was $\sqrt{2-\sqrt{2}}$, in this problem it could be anything.

(a) Write an equation for the filter's frequency response in the form:

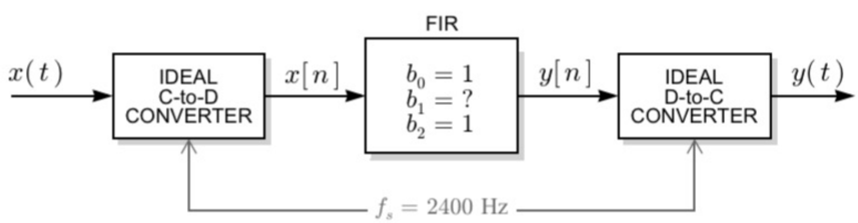
$$H(e^{j\hat{\omega}}) = e^{-Cj\hat{\omega}}\big(B + A\cos(\hat{\omega})\big)$$

for some constants A, B, and C that may depend on the unspecified coefficient b_1.
(*Hint:* factor out $e^{-j\hat{\omega}}$ and then apply Euler's relation to what's left.)

(b) In order for this filter to null a sinusoid with frequency $\hat{\omega}_0$, what must the coefficient b_1 be? Express b_1 as a function of the digital frequency $\hat{\omega}_0$ to be nulled.

When combined with ideal C-D and D-C converters, this filter can also be used to null a *continuous-time* sinusoidal input.

Consider the cascade below, where a continuous-time input is sampled at $f_s = 2400$ Hz, filtered in discrete time, and then converted back to continuous time (also with $f_s = 2400$ Hz):



(c) If $y(t) = 0$ for all t when $x(t) = 0.16\cos(1600\pi t + 0.16\pi)$, what is b_1?

(d) If $y(t) = 0$ for all t when $x(t) = 0.52\cos(5200\pi t + 0.52\pi)$, what is b_1?

a) $h[n] = \{1, b_1, 1\} \Rightarrow H(e^{j\omega}) = 1 + b_1 e^{-j\omega} + e^{-j2\omega}$

$H(e^{j\omega}) = e^{-j\omega}(e^{+j\omega} + b_1 + e^{-j\omega}) \Rightarrow \boxed{e^{-j\omega}(b_1 + 2\cos(\omega))}$

$\boxed{\begin{matrix} C = 1 \\ B = b_1 \\ A = 2 \end{matrix}}$

B) $\boxed{b_1 = -2\cos(\omega_0)}$

c) $x(t) = .16\cos(2\pi(800)t + .16\pi)$

$x[n] = .16\cos\left(2\pi\left(\frac{800}{2400}\right)n + .16\pi\right)$

$x[n] = .16\cos\left(\frac{2\pi}{3}n + .16\pi\right)$ $\quad \hat{\omega} = \frac{2\pi}{3} \Rightarrow$ null $\frac{2\pi}{3}$

$b_1 = -2\cos\left(\frac{2\pi}{3}\right) \Rightarrow -2\left(-\frac{1}{2}\right) = \boxed{1}$

D) $x(t) = .52\cos(2\pi(2600)t + .52\pi) \Rightarrow x[n] = .52\cos\left(2\pi\left(\frac{2600}{2400}\right)n + .52\pi\right)$

$.52\cos\left(2\pi\left(\frac{13}{12}\right)n + .52\pi\right) \Rightarrow \omega = \frac{26}{12}\pi - 2\pi \Rightarrow \frac{26}{12}\pi - \frac{24}{12}\pi = \frac{2}{12}\pi = \pi/6$

$.52\cos(\pi/6\, n + .52\pi) \Rightarrow \omega$ to null is $\pi/6$

$b_1 = -2\cos(\pi/6) = \frac{\sqrt{3}}{2}(-2) = \boxed{-\sqrt{3}}$