Question 1

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1. Prove or disprove the following claims.
   (a) The representation of quadratic form $q(\vec{x}) = \vec{x}^T A\vec{x}$, where A is a symmetric matrix, is unique.

$$x_1^2 + 2x_1x_2 + x_2^2 \qquad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$Q(\vec{x}) = \vec{x}^T A\vec{x}$$
$$Q(\vec{x}) = \vec{x}^T B\vec{x}$$

$$\Rightarrow \vec{x}^T A\vec{x} = \vec{x}^T B\vec{x}$$
$$\Rightarrow \vec{x}^T QDQ^T\vec{x} = \vec{x}^T PHP^T\vec{x}$$
$$\Rightarrow [x]_Q^T D[x]_Q = [x]_P^T H [x]_P$$

D + H Share Same Eigenvalues along main Diagonal $\Rightarrow$ A = B if They are Real + Symmetric, They are one in the other
$\Rightarrow$ Unique $\Rightarrow$ Statement TRUE

   (b) Consider $q(\vec{x}) = \vec{x}^T A\vec{x}$ where $A \in M_2(\mathbb{R})$. $q(\vec{x})$ is not a quadratic form if A is not symmetric.

Counter-example

$$A = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}$$

$$Q(\vec{x}) = \vec{x}^T A\vec{x}$$
$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$
$$= \begin{matrix} x_1 + 3x_2 \\ 0x_1 + 2x_2 \end{matrix} = \underline{x_1^2 + 3x_2x_1 + 2x_2^2}$$

False

(c) If $A \succ 0$ then $A^7 \succ 0$.

- if A pos. def. $\Rightarrow$ A^7 pos. def

TRUE

A is symmetric $\Rightarrow$ $A = QDQ^{-1}$ for Q orthogonal
& D diagonal

$$D = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}$$ where $\lambda_i > 0$ e-vals $\forall i \in \{1, \ldots, n\}$

$A^7 = QD^7Q^{-1} \Rightarrow D = \begin{bmatrix} \lambda_1^7 & & \\ & \ddots & \\ & & \lambda_n^7 \end{bmatrix} \Rightarrow$ all $\lambda_i^7 > 0$

$\Rightarrow$ $A \succ 0$ since still symmetric & all
evals > 0

TRUE

(d) If $A \prec 0$, then $A^4 \prec 0$.

- Negative Definite

$A = QDQ^{-1}$ $\quad D = \begin{bmatrix} \lambda_1 & & 0 \\ 0 & \ddots & \\ & & \lambda_n \end{bmatrix}$ $\quad \lambda_i < 0$ $\forall i$

But any $\alpha \in \mathbb{R}$ s.t. $\alpha < 0$

$\Rightarrow$ $\alpha^4 > 0$

$\Rightarrow$ $\lambda_i^4 > 0$ $\quad \forall i$

$\Rightarrow$ $A^4 = QD^4Q^{-1}$ $\quad D^4 = \begin{bmatrix} \lambda_1^4 & 0 \end{bmatrix}$

$$\begin{bmatrix} 0 & \ddots & \lambda_n^4 \end{bmatrix}$$

Where The eigenvalues are all > 0

$\Rightarrow$ A^4 is P.D. $\Rightarrow$ Statement false

(e) If $A \succ 0$ and $B \prec 0$ then $A - B \succ 0$.

Consider $\underset{(PD)}{A} = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}$ $\underset{(ND)}{B} = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}$

$Q(x) = \vec{x}^T (A-B) \vec{x} \Rightarrow \vec{x}^T A \vec{x} - \vec{x}^T B \vec{x} \Rightarrow$ This can be Said as $Q(\vec{x}) = g(\vec{x}) + h(\vec{x})$

Since $g(\vec{x})$ is > 0 $\forall \vec{x} \neq 0$ and $h(\vec{x})$ is < 0 $\forall \vec{x} \neq 0$ $\Rightarrow$ $Q(x) > 0$ $\forall x \neq 0$ $\Rightarrow$ $A-B$ is PD $\forall \vec{x} \neq 0$. Statement TRUE

(f) Every matrix has a singular value decomposition.

$A \in M_{m \times n}(\mathbb{R})$. $A^T A$ is Symmetric $\Rightarrow$ $A^T A = Q D Q^T$ for Q orth + D Diagonal and $A^T A$ is PSD meaning all eigenvalues are ≥ 0. if $\text{Rank}(A^T A) = r \Rightarrow r = \text{Rank}(A)$ (By Rank-nullity). For each $\lambda_i = \sigma_i^2$, $\exists$ a unit eigenvector for the eigenspace s.t $E_{\sigma_i^2} = \text{Span}\left\{\frac{q}{\|q\|}\right\}$

Thus $\{q_1, \ldots, q_r\}$ col vectors of Q form

$\{q_1, \ldots, q_n\}$, col vectors of Q form

an orthonormal Basis where $\{q_1, \ldots q_r\}$ are

e-vectors of all $\sigma_i \neq 0$ and $q_{r+1}, \ldots, q_n$

are e-vectors of all $\sigma_i = 0$.

Since $A^T A q_i = \sigma_i^2 q_i \Rightarrow$ (q_i e-vector of $A^T A$)

we can consider $AA^T = U\Sigma V^T V \Sigma^T U^T \Rightarrow U(\Sigma \Sigma^T)U^T$

$\Rightarrow$ where for e-vector u_i of AA^T, $AA^T u_i = \sigma_i^2 u_i$

$u_i = \frac{1}{\sigma_i} A q_i \Rightarrow U_R = A Q_R \Sigma_R^{-1}$

$\Rightarrow A Q_R = U_R \Sigma_R$ But for $1, \ldots, R$ where

$q_{i \to R}$ is for $\sigma_i \neq 0$. for $R \to n$, this ON Basis

is extended to include it

s.t. $AQ = U\Sigma \Rightarrow A = U\Sigma Q^{-1} \Rightarrow \boxed{A = U\Sigma V^T}$

SVD

TRUE

(g) Similar matrices must have the same singular values.

false. if $A = P^{-1} B P$ where $A \cong B$, then

A & B share the same eigenvalues But

singular values are derived from AA^T or $A^T A$/

$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix}$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} A \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

But $\sigma_A = 3 \,\&\, 2$

$\sigma_B = 6 \,\&\, 4$

$\sigma_B \neq \sigma_A$ **False**

(h) If A and B are real symmetric matrices such that $A^3 = B^3$, then A must be equal to B.

$A = QDQ^{-1}$ & $B = PXP^{-1}$ & $A^3 = B^3$

$\Rightarrow QD^3Q^{-1} = PX^3P^{-1}$

$A^3 = QD^3Q^T = B^3 = PX^3P^T \Rightarrow$ cube rooting each element in D^3 & X^3 reveals that the eigenvalues are the same $\Rightarrow$ regardless of Q & P, $A = B$ since it is real & symmetric $\Rightarrow$ **Statement TRUE**