Final Exam

Problem f1

Spin 1/2 is subject to the time-dependent Hamiltonian

$$\hat{H}(t) = \begin{cases} \omega \hat{S}_{\mathbf{x}}, & t < \tau, \\ \omega \hat{S}_{\mathbf{y}}, & t > \tau, \end{cases}$$

where $\tau > 0$ and ω is independent of time. A non-destructive measurement of $S_{\mathbf{z}}$ at $t = 0$ found $S_{\mathbf{z}} = \hbar/2$. What is the probability that another measurement of $S_{\mathbf{z}}$, at $t = 2\tau$, will yield $S_{\mathbf{z}} = -\hbar/2$?

Problem f2

A quantum particle placed in a hard-wall box (i.e., a rectangular potential well of infinite depth) of width a is in the state described by the wave function

$$\psi(x) \propto \begin{cases} (a/2)^2 - x^2, & |x| < a/2, \\ 0, & |x| > a/2. \end{cases}$$

Find the probability $\mathcal{P}$ that a measurement of energy will find the lowest possible value. (As you will discover, $\mathcal{P}$ is very close to 1.)

For reference: the ground state wave function for a particle in a hard-wall box reads

$$\psi_0(x)\Big|_{|x|<a/2} = (2/a)^{1/2} \cos(kx), \quad k = \pi/a.$$

Problem f3

The Hamiltonian of a kicked particle reads

$$\hat{H}(t) = \hat{H}_0 - p_0 \hat{x}\, \delta(t), \quad \hat{H}_0 = \frac{\hat{p}^2}{2m} + V(\hat{x}).$$

Given that at $t < 0$ (i.e., before the kick) the particle was in a bound state of $H\big|_{t<0} = \hat{H}_0$ with eigenenergy E_0, find the expectation value of energy at $t > 0$.

For reference: the evolution operator for $\hat{H}(t) = \hat{H}_0 + \hat{V}\delta(t/t_0)$ is given by $\hat{T}(+0, -0) = e^{-it_0\hat{V}/\hbar}$ (see Problem 32).

Problem f∗ (bonus)

A linear operator $\hat{\Sigma}_\lambda$ that depends on a positive real parameter λ is defined by its action on the position-space wave functions $\psi(x) = \langle x|\psi\rangle$:

$$\langle x|\hat{\Sigma}_\lambda|\psi\rangle = \sqrt{\lambda}\, \psi(\lambda x).$$

(a) Show that $\hat{\Sigma}_\lambda^\dagger = \hat{\Sigma}_{f(\lambda)}$, where $f(\lambda)$ is a function of λ that you need to find.

(b) Show that $\hat{\Sigma}_\lambda$ is unitary.

(c) Determine the effect of $\hat{\Sigma}_\lambda$ on the momentum-space wave functions. That is, relate $\langle p|\hat{\Sigma}_\lambda|\psi\rangle$ to $\psi(p) = \langle p|\psi\rangle$.