10/2

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Aran, the one major theorem we covered in class is that there is a unique linear map T from V with basis $\{b_1, \ldots, b_n\}$ to W with basis $\{d_1, \ldots, d_n\}$ such that $Tb_i = d_i, \forall i \in \{1, \ldots, n\}$. And this map is bijective. Consequences of this is that two finite-dimensional vector spaces are isormorphic if their dimensions are the same. We also went over the basic terminology of linear maps.

1. Give a counter-example to the statement: There is only one unique isomorphism from vector space V to vector space W if dim V = dim $W < \infty$. Note how this is phrased differently from the theorem above.
2. Consider matrix $A = \begin{pmatrix} 1 & 2 \\ -2 & -4 \\ 3 & 6 \end{pmatrix}$. Determine Col(A), Nul(A). Find bases for these two subspaces and determine their dimensions.
3. Consider $T_A : \mathbb{R}^2 \to \mathbb{R}^3$
   $x \mapsto Ax$
   where A is given in Q1. Determine ImT_a and KerT_a . Find bases for these two subspaces and determine their dimensions. Note ImT_a + KerT_a = 2, which is the dimension of domain of T_a.
4. Give an explicit isomorphism from $M_{3 \times 4}(\mathbb{R})$ to $M_{2 \times 6}(\mathbb{R})$.
5. Let S be a finite set with cardinality $|S| = n$, where $n \in \mathbb{N}$. Consider vector space $\mathbb{R}^S \coloneqq \{f \mid f : S \to \mathbb{R}\}$, set of all functions whose domain is S and codomain is $\mathbb{R}$, with the usual function addition and function-scalar multiplication. (This is a vector space. If you don't believe me, you should verify this is indeed the case.)
   a. Find a basis for $\mathbb{R}^S$.
   b. Show $\mathbb{R}^S \cong \mathbb{R}^n$.
6. Visualize what each transformation does to vectors in $\mathbb{R}^2$. You can draw some vectors and their images under each map. What do its kernel and image look like?
   Consider $T_A : \mathbb{R}^2 \to \mathbb{R}^2$
   $x \mapsto Ax$
   a. $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
   b. $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$
   c. $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
   d. $A = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$
   e. $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
7. Determine true/false.
   Consider $T_A : \mathbb{R}^n \to \mathbb{R}^m$
   $x \mapsto Ax$
   a. If A is 3×2, then T_A cannot be surjective.
   b. If A is 2×3, then T_A cannot be surjective.
8. Let V be a vector space over $\mathbb{R}$ with ordered basis $\mathscr{B} = < b_1, \ldots, b_n >$.
   Consider the map
   $$[\cdot]_{\mathscr{B}} : V \to \mathbb{R}^n$$
   $$v \mapsto [v]_{\mathscr{B}}$$
   i.e., it maps vectors to their coordinates w.r.t. $\mathscr{B}$.
   a. Show this map is a vector space isomorphism. Determine explicitly its inverse map, including its mapping rule.
   b. Show that $w \in V$ is a linear combinaiton of $\{w_1, \ldots, w_m\} \subseteq V$ iff $[w]_{\mathscr{B}}$ is a linear combiniation of $\{[w_1]_{\mathscr{B}} \ldots, [w_m]_{\mathscr{B}}\}$.
   c. Show that $\{w_1, \ldots, w_m\} \subseteq V$ are linear indepent iff $\{[w_1]_{\mathscr{B}} \ldots, [w_m]_{\mathscr{B}}\}$ are linearly independent in $\mathbb{R}^n$.
   d. $\{w_1, \ldots, w_m\}$ spans V iff $\{[w_1]_{\mathscr{B}} \ldots, [w_m]_{\mathscr{B}}\}$ spans $\mathbb{R}^n$

Aran, we have not talked about matrix representation of linear r from first principle.

Aran, we have not talked about matrix can be determined by T directly by plugging in some examples and using definitons.

We have not talked about rank-nullity yet. So this part needs to considering (A|b).

This is a continuation of the last problem from 09/27. You can