Homework 3

Problem 9

(a) Prove that if $\langle\psi|\hat{A}|\psi\rangle = \langle\psi|\hat{B}|\psi\rangle$ for all $|\psi\rangle$ then $\langle\psi_1|\hat{A}|\psi_2\rangle = \langle\psi_1|\hat{B}|\psi_2\rangle$ for all $|\psi_1\rangle$ and $|\psi_2\rangle$.

Suggestion. Substitute $|\psi\rangle = |\psi_1\rangle + \lambda|\psi_2\rangle$ into $\langle\psi|\hat{A}|\psi\rangle = \langle\psi|\hat{B}|\psi\rangle$, then set $\lambda = 1$ and $\lambda = i$.

(b) By definition, $\hat{A} = \hat{B}$ if and only if $\hat{A}|\psi\rangle = \hat{B}|\psi\rangle$ for all $|\psi\rangle$.

Use the result of part (a) to show that $\hat{A} = \hat{B}$ if and only if $\langle\psi|\hat{A}|\psi\rangle = \langle\psi|\hat{B}|\psi\rangle$ for all $|\psi\rangle$.

Problem 10

Show that if $\langle\psi|\hat{A}|\psi\rangle = \langle\psi|\hat{A}^\dagger|\psi\rangle^*$ for all $|\psi\rangle$ then $\langle\psi_1|\hat{A}|\psi_2\rangle = \langle\psi_2|\hat{A}^\dagger|\psi_1\rangle^*$ for all $|\psi_1\rangle$ and $|\psi_2\rangle$.

Suggestion. Substitute $|\psi\rangle = |\psi_1\rangle + \lambda|\psi_2\rangle$ into $\langle\psi|\hat{A}|\psi\rangle = \langle\psi|\hat{A}^\dagger|\psi\rangle^*$ then set $\lambda = 1$ and $\lambda = i$ [cf. Problem 9(a)].

Problem 11

(a) Show that $\dfrac{d}{d\lambda}\left[e^{\lambda\hat{A}}\hat{B}e^{-\lambda\hat{A}}\right] = e^{\lambda\hat{A}}[\hat{A},\hat{B}]e^{-\lambda\hat{A}}$ for all $\hat{A}$ and $\hat{B}$.

(b) Derive the expansion $e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + [\hat{A},\hat{B}] + \dfrac{1}{2!}[\hat{A},[\hat{A},\hat{B}]] + \dfrac{1}{3!}\Big[\hat{A},[\hat{A},[\hat{A},\hat{B}]]\Big] + \ldots$

Suggestion: expand $\hat{F}(\lambda) = e^{\lambda\hat{A}}\hat{B}e^{-\lambda\hat{A}}$ in Taylor series about $\lambda = 0$, then set $\lambda = 1$.

Problem 12

Operators $\hat{A}$ and $\hat{B}$ commute with their commutator: $\big[\hat{A},[\hat{A},\hat{B}]\big] = \big[\hat{B},[\hat{A},\hat{B}]\big] = \hat{\mathbb{0}}$.

(a) Show that $\big[\hat{A},e^{\lambda\hat{B}}\big] = \lambda e^{\lambda\hat{B}}[\hat{A},\hat{B}]$. *Suggestion*: use the expansion derived in Problem 11(b).

(b) Show that $\hat{F}(\lambda) = e^{\lambda\hat{A}}e^{\lambda\hat{B}}e^{-\lambda(\hat{A}+\hat{B})}$ obeys the differential equation $\dfrac{d}{d\lambda}\hat{F}(\lambda) = \lambda[\hat{A},\hat{B}]\hat{F}(\lambda)$.

(c) By solving the equation derived in part (b), obtain the Baker-Hausdorff formula $e^{\hat{A}}e^{\hat{B}} = e^{\hat{A}+\hat{B}}e^{\frac{1}{2}[\hat{A},\hat{B}]}$.

Problem 13

By definition, *trace* of a linear operator acting in a finite-dimensional Hilbert space is the sum of its diagonal matrix elements: $\operatorname{tr}\hat{A} = \sum_n\langle\phi_n|\hat{A}|\phi_n\rangle$, where $\{|\phi_n\rangle\}$ is an orthonormal basis. (In an infinitely-dimensional space the sum in $\operatorname{tr}\hat{A}$ turns to an infinite series, which does not have to converge, let alone converge absolutely; equating two such divergent series would be meaningless.)

(a) Show that $\operatorname{tr}\hat{A}$ is independent of the choice of an orthonormal basis.

That is, show that $\sum_n\langle\phi_n|\hat{A}|\phi_n\rangle = \sum_n\langle\varphi_n|\hat{A}|\varphi_n\rangle$, where $\{|\phi_n\rangle\}$ and $\{|\varphi_n\rangle\}$ are two orthonormal basis sets.

(b) Show that $\operatorname{tr}(\hat{A}\hat{B}) = \operatorname{tr}(\hat{B}\hat{A})$.

(c) Find $\operatorname{tr}\hat{A}$ for $\hat{A} = |\varphi\rangle\langle\psi|$, where $|\varphi\rangle$ and $|\psi\rangle$ are arbitrary vectors.