Question 2

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2. Let A, B be matrices similar to each other.
   (a) Show that they have the same eigenvalues. Do they have the same eigenvectors?
   (b) Show that they have the same rank.
   (c) Show that they have the same trace.

a) if $A = SBS^{-1}$ for $S \in GL_n(\mathbb{R})$

and if $A\vec{v} = \lambda\vec{v}$

consider $SBS^{-1}\vec{v} = \lambda\vec{v}$

$\Rightarrow BS^{-1}\vec{v} = \lambda S^{-1}\vec{v}$

$\therefore$ λ is an eigenvalue of A for vector $\vec{v}$

and thus λ is also an eigenvalue for B

But w/ dif eigenvector $S^{-1}\vec{v} \neq \vec{v}$ $\forall \vec{v} \in \mathbb{R}^n$

$S \in GL_n(\mathbb{R})$

B) consider for matrix A, there is set

$\lambda_1, \dots, \lambda_p$, distinct eigenvalues. Then there are at least

p linearly independent eigenvectors, eigenbasis.

Same for B where $\lambda_1, \dots, \lambda_p$ are distinct e-vals

and there are at least p lin ind e-vectors of B.

To form an eigenbasis of B.

Thus # lin. ind e-vects for A $(>p) \Rightarrow$

A - - - - - for B (> P)

$$\Rightarrow \underbrace{\dim\{v_1, v_2, v_3, \dots, v_n\}}_{A \text{ eigenbasis}} = \underbrace{\dim\{x_1, x_2, \dots, x_n\}}_{B \text{ eigenbasis}}$$

c) lemma $tr(AB) = tr(BA)$

if $A = PDP^{-1}$

$tr(A) = tr(PDP^{-1})$, let $DP^{-1} = C$

$\Rightarrow tr(PC) \Rightarrow tr(CP) \Rightarrow tr(DP^{-1}P) \Rightarrow tr(DI_n)$

$\Rightarrow tr(D) = tr(A)$