Homework 2

Problem 5

(a) Derive the triangle inequality $\|\varphi+\psi\| \le \|\varphi\| + \|\psi\|$. (Here $\|\Psi\| = \sqrt{\langle\Psi|\Psi\rangle}$ is the norm of vector $|\Psi\rangle$.)
Suggestion: write $\|\varphi+\psi\|^2 = \langle\varphi+\psi|\varphi+\psi\rangle$ and use the relation $\mathrm{Re}\,\langle\varphi|\psi\rangle \le |\langle\varphi|\psi\rangle|$ and the Schwartz inequality.

(b) When the triangle inequality becomes an equality?

(c) Show that $\|\varphi-\psi\| \ge \big|\|\varphi\| - \|\psi\|\big|$.

Solution

(a) We have

$$\|\varphi+\psi\|^2 = \langle\varphi+\psi|\varphi+\psi\rangle = \langle\varphi|\varphi\rangle + \langle\psi|\psi\rangle + \langle\varphi|\psi\rangle + \langle\psi|\varphi\rangle = \|\varphi\|^2 + \|\psi\|^2 + 2\,\mathrm{Re}\,\langle\varphi|\psi\rangle.$$

The last term here satisfies

$$\mathrm{Re}\,\langle\varphi|\psi\rangle \le |\langle\varphi|\psi\rangle| \le \sqrt{\langle\varphi|\varphi\rangle\langle\psi|\psi\rangle} = \|\varphi\|\,\|\psi\|,$$

where we used the Schwartz inequality. Therefore,

$$\|\varphi+\psi\|^2 \le \|\varphi\|^2 + \|\psi\|^2 + 2\,\|\varphi\|\,\|\psi\| = \big(\|\varphi\| + \|\psi\|\big)^2.$$

Taking a square root, we end up with the triangle inequality $\|\varphi+\psi\| \le \|\varphi\| + \|\psi\|$. ■

(b) It is obvious from the derivation in part (a) that $\|\varphi+\psi\| = \|\varphi\| + \|\psi\|$ if and only if

$$\mathrm{Re}\,\langle\varphi|\psi\rangle = |\langle\varphi|\psi\rangle| = \|\varphi\|\,\|\psi\|,$$

which happens when $|\varphi\rangle = c|\psi\rangle$ with a ***real and non-negative*** coefficient c.

(c) Proceeding as in part (a), we write

$$\|\varphi-\psi\|^2 = \langle\varphi-\psi|\varphi-\psi\rangle = \|\varphi\|^2 + \|\psi\|^2 - 2\,\mathrm{Re}\,\langle\varphi|\psi\rangle \ge \|\varphi\|^2 + \|\psi\|^2 - 2\,\|\varphi\|\,\|\psi\| = \big(\|\varphi\| - \|\psi\|\big)^2,$$

which gives

$$\|\varphi-\psi\| = \sqrt{\|\varphi-\psi\|^2} \ \ge \sqrt{\big(\|\varphi\| - \|\psi\|\big)^2} = \big|\|\varphi\| - \|\psi\|\big|. \quad ■$$

Problem 6

What is the number of independent *real* parameters $N(\mathcal{N})$ needed to specify (up to a phase factor) a state vector in $\mathcal{N}$-dimensional Hilbert space?

Solution

Let $\{|\phi_n\rangle\}$ be an arbitrary orthonormal basis. Any state vector $|\psi\rangle$ can be expanded in this basis as $|\psi\rangle = \sum_n \psi_n|\phi_n\rangle$. Substituting here $\psi_n = |\psi_n|\,e^{i\theta_n}$ and taking into account that $\|\psi\| = \sum_n |\psi_n|^2 = 1$, we obtain

$$|\psi\rangle = \sum_{n=1}^{n=\mathcal{N}} |\psi_n|\,e^{i\theta_n}|\phi_n\rangle = e^{i\theta_1}\left\{\left(1 - \sum_{n=2}^{n=\mathcal{N}} |\psi_n|^2\right)^{1/2} |\phi_1\rangle + \sum_{n=2}^{n=\mathcal{N}} |\psi_n|\,e^{i(\theta_n-\theta_1)}|\phi_n\rangle\right\}.$$

If we ignore the overall phase factor $e^{i\theta_1}$ here, the remaining parameters are $\mathcal{N}-1$ numbers $|\psi_2|, |\psi_3|, \ldots, |\psi_{\mathcal{N}}|$ and $\mathcal{N}-1$ relative phases $\theta_2-\theta_1, \theta_3-\theta_1, \ldots, \theta_{\mathcal{N}}-\theta_1$. That is, we need

$$N(\mathcal{N}) = 2(\mathcal{N}-1)$$

real parameters altogether. For example, for $\mathcal{N}=2$ (as it is the case for spin 1/2), we need only 2 parameters, but for $\mathcal{N}=3$ (e.g., spin 1) the corresponding figure is $N(3)=4$.

Problem 7

For any pure state ψ of spin 1/2 there exists a unique Bloch vector $\mathbf{n}$ such that $\mathrm{Prob}_\psi(S_\mathbf{n} = \hbar/2) = 1$. Find the angles θ and ϕ specifying the Bloch vector in the spherical coordinates for the state represented by the state vector

$$|\psi\rangle = \frac{1}{\sqrt{2}}|+\mathbf{x}\rangle + \frac{e^{2\pi i/3}}{\sqrt{2}}|-\mathbf{x}\rangle.$$

Suggestion: write $|\psi\rangle$ in $|\pm\mathbf{z}\rangle$ basis and compare with $|\psi\rangle = \cos(\theta/2)|+\mathbf{z}\rangle + e^{i\phi}\sin(\theta/2)|-\mathbf{z}\rangle$ (up to a phase factor).

Solution

Using $\langle +\mathbf{z}|\pm\mathbf{x}\rangle = 1/\sqrt{2}$ and $\langle -\mathbf{z}|\pm\mathbf{x}\rangle = \pm 1/\sqrt{2}$, we find

$$\langle \pm\mathbf{z}|\psi\rangle = \frac{1}{2}\big(1 \pm e^{2\pi i/3}\big) = e^{i\pi/3} \times \begin{cases} \cos(\pi/3), \\ e^{3\pi i/2}\sin(\pi/3). \end{cases}$$

Substitution into $|\psi\rangle = |+\mathbf{z}\rangle\langle +\mathbf{z}|\psi\rangle + |-\mathbf{z}\rangle\langle -\mathbf{z}|\psi\rangle$ yields the state vector in $|\pm\mathbf{z}\rangle$ basis,

$$|\psi\rangle = e^{i\pi/3}\Big[\cos(\pi/3)|+\mathbf{z}\rangle + e^{3\pi i/2}\sin(\pi/3)|-\mathbf{z}\rangle\Big].$$

Comparison with the "canonical" form quoted above shows that the two angles specifying the Bloch vector are

$$\theta = 2\pi/3, \quad \phi = 3\pi/2.$$

Problem 8

Verify that the inner product of two spin 1/2 state vectors

$$|\mathbf{n}_1\rangle = \cos(\theta_1/2)|+\mathbf{z}\rangle + e^{i\phi_1}\sin(\theta_1/2)|-\mathbf{z}\rangle, \qquad |\mathbf{n}_2\rangle = \cos(\theta_2/2)|+\mathbf{z}\rangle + e^{i\phi_2}\sin(\theta_2/2)|-\mathbf{z}\rangle$$

with arbitrary angles $\theta_{1,2}$ and $\phi_{1,2}$ satisfies

$$\big|\langle \mathbf{n}_1|\mathbf{n}_2\rangle\big|^2 = \frac{1}{2}\big(1 + \mathbf{n}_1\cdot\mathbf{n}_2\big).$$

For reference: Cartesian components of a unit vector $\mathbf{n}$ specified by angles θ and ϕ is spherical polar coordinates read

$$n_\mathbf{x} = \sin\theta\cos\phi, \quad n_\mathbf{y} = \sin\theta\sin\phi, \quad n_\mathbf{z} = \cos\theta.$$

You will also need the trigonometric identities

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta, \quad 1+\cos(2\alpha) = 2\cos^2\alpha, \quad 1-\cos(2\alpha) = 2\sin^2\alpha, \quad \sin(2\alpha) = 2\sin\alpha\cos\alpha.$$

Solution

The inner product of state vectors $|\mathbf{n}_1\rangle$ and $|\mathbf{n}_2\rangle$ is

$$\langle \mathbf{n}_1|\mathbf{n}_2\rangle = \cos(\theta_1/2)\cos(\theta_2/2) + \sin(\theta_1/2)\sin(\theta_2/2)\,e^{-i\phi}, \quad \phi = \phi_1 - \phi_2.$$

This gives

$$\begin{aligned}
\big|\langle \mathbf{n}_1|\mathbf{n}_2\rangle\big|^2 &= \langle \mathbf{n}_1|\mathbf{n}_2\rangle\langle \mathbf{n}_1|\mathbf{n}_2\rangle^* \\
&= \big[\cos(\theta_1/2)\cos(\theta_2/2) + \sin(\theta_1/2)\sin(\theta_2/2)\,e^{-i\phi}\big]\big[\cos(\theta_1/2)\cos(\theta_2/2) + \sin(\theta_1/2)\sin(\theta_2/2)\,e^{+i\phi}\big] \\
&= \cos^2(\theta_1/2)\cos^2(\theta_2/2) + \sin^2(\theta_1/2)\sin^2(\theta_2/2) + 2\sin(\theta_1/2)\cos(\theta_1/2)\sin(\theta_2/2)\cos(\theta_2/2)\cos\phi \\
&= \frac{1}{4}(1+\cos\theta_1)(1+\cos\theta_2) + \frac{1}{4}(1-\cos\theta_1)(1-\cos\theta_2) + \frac{1}{2}\sin\theta_1\sin\theta_2\cos\phi \\
&= \frac{1}{2}\big(1 + \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos\phi\big).
\end{aligned}$$

Comparing this expression with

$$
\begin{aligned}
\mathbf{n}_1 \cdot \mathbf{n}_2 &= n_{1\mathbf{x}} n_{2\mathbf{x}} + n_{1\mathbf{y}} n_{2\mathbf{y}} + n_{1\mathbf{z}} n_{2\mathbf{z}} \\
&= \sin\theta_1 \sin\theta_2 \big(\cos\phi_1 \cos\phi_2 + \sin\phi_1 \sin\phi_2\big) + \cos\theta_1 \cos\theta_2 \\
&= \sin\theta_1 \sin\theta_2 \cos\phi + \cos\theta_1 \cos\theta_2 ,
\end{aligned}
$$

we arrive at

$$
\big|\langle \mathbf{n}_1 | \mathbf{n}_2 \rangle\big|^2 = \frac{1}{2}\big(1 + \mathbf{n}_1 \cdot \mathbf{n}_2\big). \quad \blacksquare
$$