

MATH-1564-K1,K2,K3 –Linear Algebra with Abstract Vector Spaces
Homework 2

1. Solve the following linear systems by presenting them in a matrix form and bringing the matrix to echelon form. If the system has an infinite amount of solutions then present a parametrization of these solutions. In any case specify how many solutions the system has.

i.

$$\begin{cases} 5x + 4y - z = 0 \\ 2x - 4y + z = 1 \\ -7x - 14y + 5z = 10 \end{cases}$$

ii.

$$\begin{cases} 2x + y + 2z = 2 \\ -x + y - z = 2 \\ 3x + 2y + z = 2 \\ 5x + 4y - z = 2 \end{cases}$$

iii.

$$\begin{cases} x_1 - x_2 + 2x_3 = 0 \\ 2x_1 - 2x_2 + 4x_3 + x_4 + 2x_5 = 4 \\ 3x_1 + x_2 + 6x_3 + x_5 = -3 \\ x_1 + 2x_3 + 2x_4 + x_5 = 4 \end{cases}$$

iv.

$$\begin{cases} x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 = 5 \\ x_1 + 4x_2 + 6x_3 + 8x_4 + 10x_5 = 10 \\ x_1 - x_2 + x_3 - x_4 + x_5 = 0 \\ -x_1 + 4x_2 + x_3 + 6x_4 + 3x_5 = 5 \end{cases}$$

2. In each of the following, find for which values of a does the linear system have: (i) No solution, (ii) Exactly one solution, (iii) Infinitely many solutions.

a.

$$\begin{cases} x + y + az = -1 \\ -x + (a - 1)y + (2 - a)z = a + 1 \\ 6x + (5a + 6)y + (7a + 7)z = a^2 \end{cases}$$

b

$$\begin{cases} (a + 1)x + ay - az = 2 + a \\ (a + 1)x + (a + 2)y - (a + 2)z = a + 4 \\ (a + 1)x + ay + (a^2 - 6)z = a^2 - 2a + 4 \\ (2a + 2)x + 2ay + (a^2 - a - 6)z = a^2 - a + 6 \end{cases}$$

3. Let $A \in M_{m \times n}(\mathbb{R})$ and denote

$$L(A) := \{b \in \mathbb{R}^m : (A|b) \text{ has at least one solution} \}.$$

Prove

- i. If $b \in L(A)$ and $d \in L(A)$ then $b + d \in L(A)$.
- ii. If $b \in L(A)$ and $t \in \mathbb{R}$ then $t \cdot b \in L(A)$

4. In each of the following parts appears a description of a matrix $A \in M_{m \times n}$. Determine whether or not such a matrix exists (where m and n can be any numbers you want; try to work with small numbers). If you claim that such a matrix exists then provide a concrete example. If you claim that such a matrix does not exist then explain carefully why this is true.
- For **every** $b \in \mathbb{R}^n$ there exist infinitely many solutions to the linear system $(A|b)$.
 - There exists $b_1 \in \mathbb{R}^n$ such that the system $(A|b_1)$ has infinitely many solutions and there exists $b_2 \in \mathbb{R}^n$ such that the system $(A|b_2)$ has no solution.
 - There exists $b_1 \in \mathbb{R}^n$ such that the system $(A|b_1)$ has exactly one solution, there exists $b_2 \in \mathbb{R}^n$ such that the system $(A|b_2)$ has an infinite amount of solutions and there exists $b_3 \in \mathbb{R}^n$ such that the system $(A|b_3)$ has no solution.
 - The homogeneous system $(A|0)$ has exactly one solution and the echelon form of A has a row of zeroes.
 - The echelon form of the matrix has a row of zeroes and for every $b \in \mathbb{R}^n$ there exists exactly one solution to the linear system $(A|b)$.
 - The 3-tuple $(1, 2, 3)$ is a solution to the homogeneous system $(A|0)$ and the echelon form of A has at least three rows which are not zero rows.
 - The 3-tuple $(1, 2, 3)$ is a solution to the homogeneous system $(A|0)$ and the echelon form of A has at least two rows which are not zero rows.
 - The set of solutions of the homogeneous system $(A|0)$ is exactly the set $\{(s + t, s, t) : s, t \in \mathbb{R}\}$.
 - The set of solutions of the homogeneous system $(A|0)$ is exactly the set $\{(s + t, s - t) : s, t \in \mathbb{R}\}$.