Question 4

Sunday, October 22, 2023 10:51 PM

4. Consider curve defined by $49x^2 - 30\sqrt{3}xy + 19y^2 = 64$ on $\mathbb{R}^2$.

(a) Show that with respect to basis

$$\mathcal{B} = < \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}, \begin{pmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix} >$$

the curve is an ellipse.

$E = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ $\quad x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Consider $[id]_{B \to E} = \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix}$ Change of Basis mtrx

$$\begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1/2\, x + \sqrt{3}/2\, y \\ -\frac{\sqrt{3}}{2}x + \frac{1}{2}y \end{bmatrix} = \begin{bmatrix} U \\ S \end{bmatrix}$$

$U = \frac{1}{2}x + \frac{\sqrt{3}}{2}y$

$x = \left(U - \frac{\sqrt{3}}{2}y\right)(2)$

$x = 2U - \sqrt{3}y$

$x = 2U - \sqrt{3}\left(\frac{1}{2}S + \frac{\sqrt{3}}{2}U\right)$

$x = 2U - \frac{\sqrt{3}}{2}S - \frac{3}{2}U$

$\boxed{x = \frac{1}{2}U - \frac{\sqrt{3}}{2}S}$

$S = -\frac{\sqrt{3}}{2}x + \frac{1}{2}y$

$S = \frac{-\sqrt{3}}{2}\left(2U - \sqrt{3}y\right) + \frac{1}{2}y$

$S = -\sqrt{3}U + \frac{3}{2}y + \frac{1}{2}y$

$S = -\sqrt{3}U + \cancel{\frac{4}{2}}y$ $2y$

$\boxed{y = \frac{1}{2}S + \frac{\sqrt{3}}{2}U}$

$$49\left(\tfrac{1}{2}u-\tfrac{\sqrt{3}}{2}s\right)\left(\tfrac{1}{2}u-\tfrac{\sqrt{3}}{2}s\right)-30\sqrt{3}\left(\tfrac{1}{2}u-\tfrac{\sqrt{3}}{2}s\right)\left(\tfrac{\sqrt{3}}{2}u+\tfrac{1}{2}s\right)+19\left(\tfrac{\sqrt{3}}{2}u+\tfrac{1}{2}s\right)\left(\tfrac{\sqrt{3}}{2}u+\tfrac{1}{2}s\right)=64$$

$$49\left(\tfrac{1}{4}u^2-\tfrac{\sqrt{3}}{2}su+\tfrac{3}{4}s^2\right)-30\sqrt{3}\left(\tfrac{\sqrt{3}}{4}u^2-\tfrac{1}{2}su-\tfrac{\sqrt{3}}{4}s^2\right)+19\left(\tfrac{3}{4}u^2+\tfrac{\sqrt{3}}{2}su+\tfrac{1}{4}s^2\right)=64$$

$\underline{u^2}$

$$\frac{49}{4}u^2-\frac{90}{4}u^2+\frac{57}{4}u^2=4u^2$$

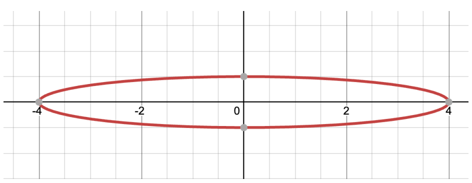
$\underline{su}$

$$-\frac{49\sqrt{3}}{2}su+\frac{30\sqrt{3}}{2}su+\frac{19\sqrt{3}}{2}su=0$$

$\underline{s^2}$

$$\frac{147}{4}s+\frac{90}{4}s^2+\frac{19}{4}s^2=256s^2$$

$$4u^2+256s^2=64$$



ellipse

(b) Show that with respect to basis

$$\mathcal{C} =< \begin{pmatrix} 2 \\ 2\sqrt{3} \end{pmatrix}, \begin{pmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix} >$$

the curve is the unit circle.

Consider $\begin{bmatrix} x \\ y \end{bmatrix} = \alpha \begin{bmatrix} 2 \\ 2\sqrt{3} \end{bmatrix} + \beta \begin{bmatrix} -\sqrt{3}/2 \\ 1/2 \end{bmatrix}$

$$\begin{cases} x = 2\alpha - \frac{\sqrt{3}}{2}\beta \\ y = 2\sqrt{3}\alpha + \frac{1}{2}\beta \end{cases}$$

$$49\left(2\alpha-\tfrac{\sqrt{3}}{2}\beta\right)^2-30\sqrt{3}\left(2\alpha-\tfrac{\sqrt{3}}{2}\beta\right)\left(2\sqrt{3}\alpha+\tfrac{1}{2}\beta\right)+19\left(2\sqrt{3}\alpha+\tfrac{1}{2}\beta\right)^2=64$$

when simplifying, we get

$$64\alpha^2 + 64\beta^2 = 64$$

$$\Rightarrow \alpha^2 + \beta^2 = 1 \quad \pi \text{ unit } \underline{\text{circle}}$$