Numerical methods in Biomedical Engineering

Tutorial V: Jacobi method to solve system of linear equations

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1 Introduction

Jacobi method is an iterative algorithm to solve a system of linear equations. This algorithm is particularly useful for a diagonally dominant system of linear equations. The algorithm starts with a guess for each unknown and the iteration is done till the subsequent guesses are within the tolerance limit.

Consider a system of linear equations with coefficient matrix A, the right hand side vector $\mathbf{b}$.

$$A\mathbf{x} = \mathbf{b} \tag{1}$$

where,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \tag{2}$$

The coefficient matrix A can be decomposed into a diagonal matrix, D, and a non-diagonal matrix,

R. Therefore,

$$A = D + R \tag{3}$$

where,

$$D = \begin{bmatrix} a_{11} & 0 & 0 & \dots & 0 \\ 0 & a_{22} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{nn} \end{bmatrix}, \quad and \quad R = \begin{bmatrix} 0 & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & 0 & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & 0 \end{bmatrix} \tag{4}$$

Then,

$$(D + R)\mathbf{x} = \mathbf{b} \tag{5}$$

The solution of the above equation can be iterated using,

$$\mathbf{x}^{(k+1)} = D^{-1}(\mathbf{b} - R\mathbf{x}^{(k)}) \tag{6}$$

The element based formula is,

$$\mathbf{x}_i^{(k+1)} = \frac{1}{a_{ii}}(\mathbf{b}_i - \sum_{i \neq j} a_{ij}\mathbf{x}_j^{(k)}), \quad i = 1, 2, \dots, n. \tag{7}$$

2 Pseudo-code

Input: Coefficient matrix A, RHS matrix $\mathbf{b}$, and initital guess matrix $\mathbf{x}^0$.

Output: Solution if the convergence criterion is met.

```
iteration number, $k = 0$
while not convergent do
    for $i = 1 : n$
    $sum = 0$
        for $j = 1 : n$
            if $i \neq j$
                $sum = sum + a_{ij}x_j^{(k)}$
            end
        end
        $x_i^{(k+)} = \frac{1}{a_{ii}}(b_i - sum)$
        end
    $k = k + 1$
end
```