MONASH University

Rob J Hyndman

Functional time series

with applications in demography

6. Coherent functional forecasting

Outline

The problem

Let $s_{t,j}(x)$ be the smoothed mortality rate for age x in group j in year t.

- Groups may be males and females.
- Groups may be states within a country.
- Expected that groups will behave similarly.
- Coherent forecasts do not diverge over time.
- Existing functional models do not impose coherence.

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Forecasting the coefficients

$$y_{t,x} = s_t(x) + \sigma_t(x)\varepsilon_{t,x},$$
$$s_t(x) = \mu(x) + \sum_{k=1}^{K} \beta_{t,k}\,\phi_k(x) + r_t(x)$$

- We use ARIMA or ETS models for each coefficient $\{\beta_{1,j,k}, \dots, \beta_{n,j,k}\}$.
- The ARIMA models are non-stationary for the first few coefficients ($k = 1, 2$). All ETS models are non-stationary.
- Non-stationary forecasts will diverge. Hence the mortality forecasts are not coherent.

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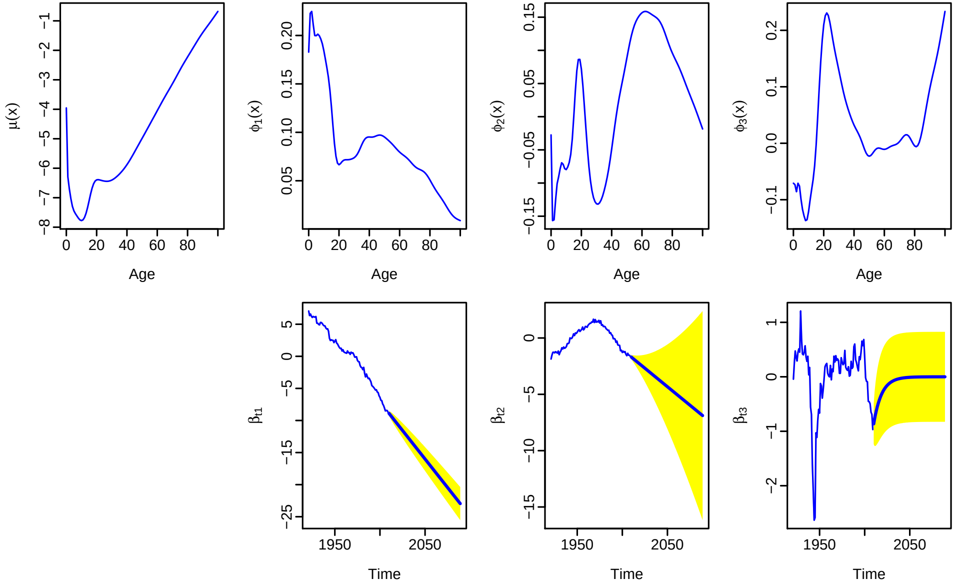
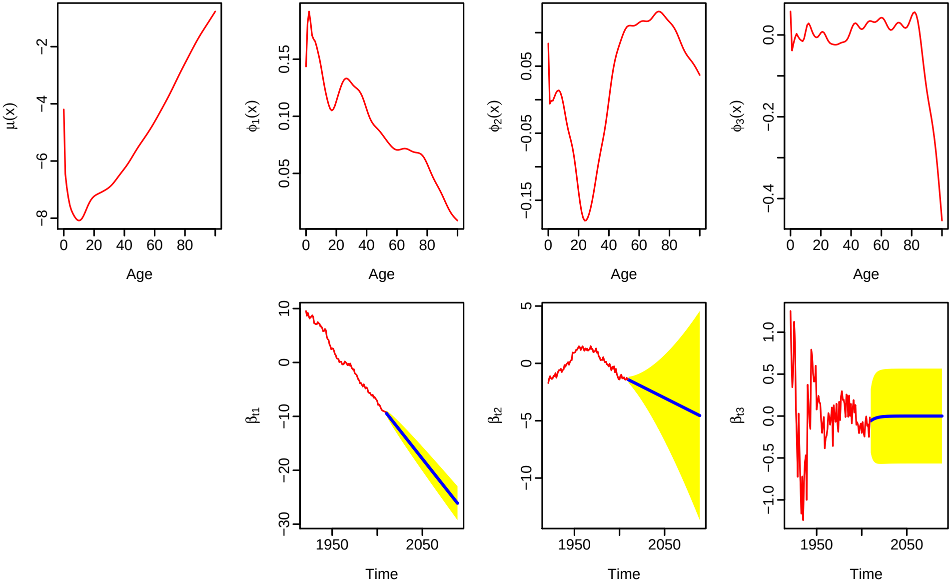
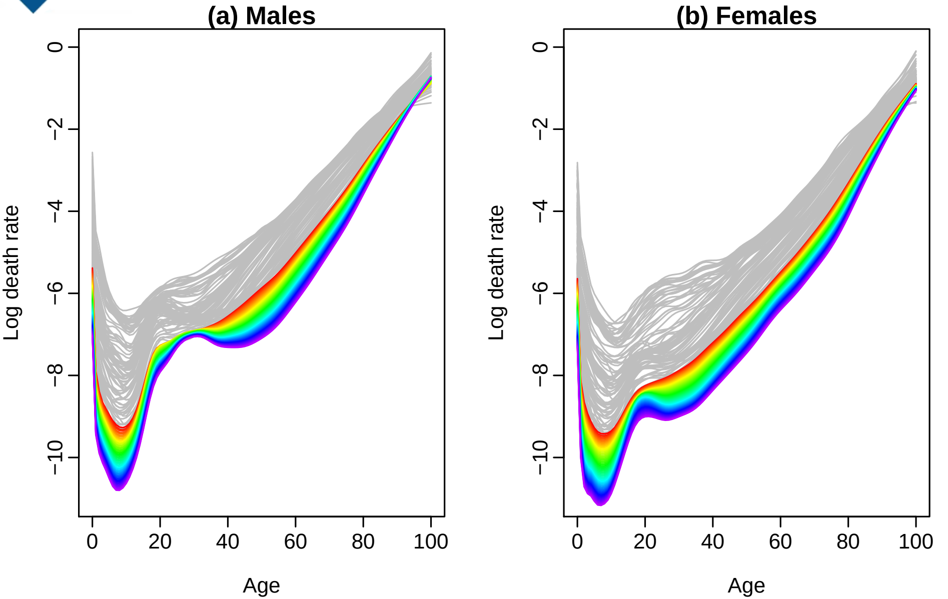
Male fts model



Female fts model



Australian mortality forecasts



Mortality product and ratios

Key idea

Model the geometric mean and the mortality ratio instead of the individual rates for each sex separately.

$$p_t(x) = \sqrt{s_{t,M}(x)s_{t,F}(x)} \quad \text{and} \quad r_t(x) = \sqrt{s_{t,M}(x)\big/s_{t,F}(x)}.$$

- Product and ratio are approximately independent
- Ratio should be stationary (for coherence) but product can be non-stationary.

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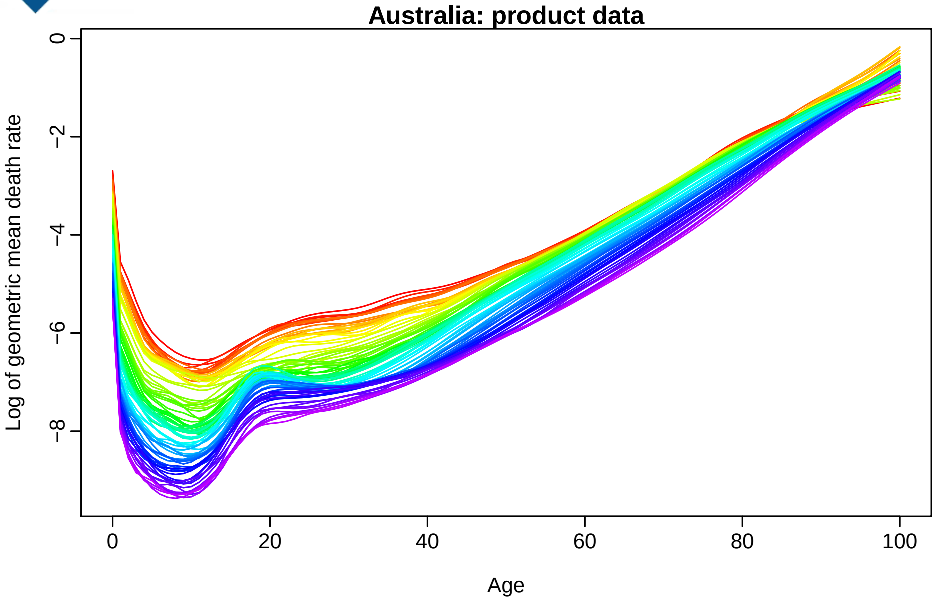
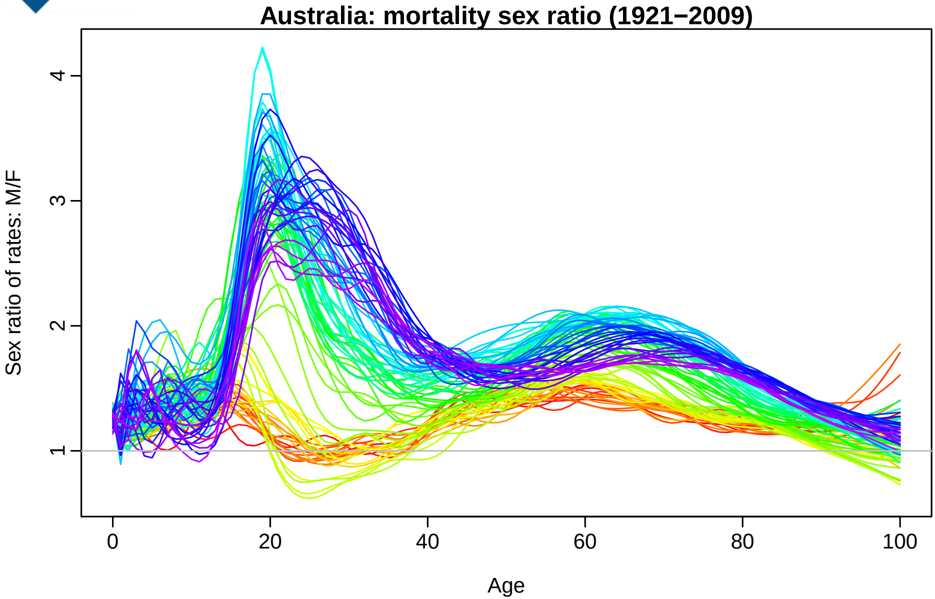
Product data



Ratio data


Australia: mortality sex ratio (1921−2009)
Sex ratio of rates: M/F
Age
0
20
40
60
80
100
1
2
3
4

Model product and ratios

$$p_t(x) = \sqrt{s_{t,M}(x)s_{t,F}(x)} \quad \text{and} \quad r_t(x) = \sqrt{s_{t,M}(x)\big/s_{t,F}(x)}.$$

$$\log[p_t(x)] = \mu_p(x) + \sum_{k=1}^{K} \beta_{t,k}\phi_k(x) + e_t(x)$$

$$\log[r_t(x)] = \mu_r(x) + \sum_{\ell=1}^{L} \gamma_{t,\ell}\psi_\ell(x) + w_t(x).$$

- $\{\gamma_{t,\ell}\}$ restricted to be stationary processes: either ARMA(p,q) or ARFIMA(p,d,q).
- No restrictions for $\beta_{t,1},\dots,\beta_{t,K}$.
- **Forecasts:** $s_{n+h|n,M}(x) = p_{n+h|n}(x)r_{n+h|n}(x)$

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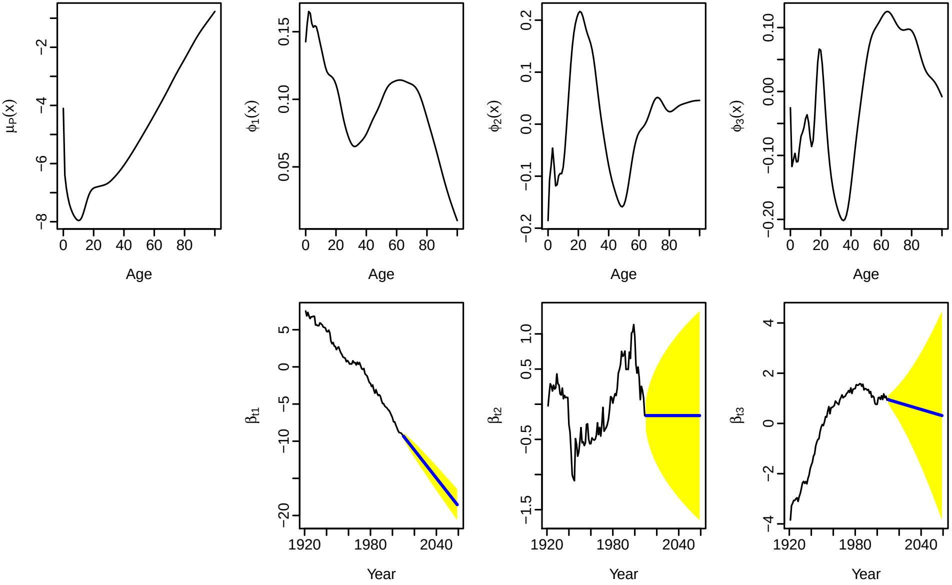
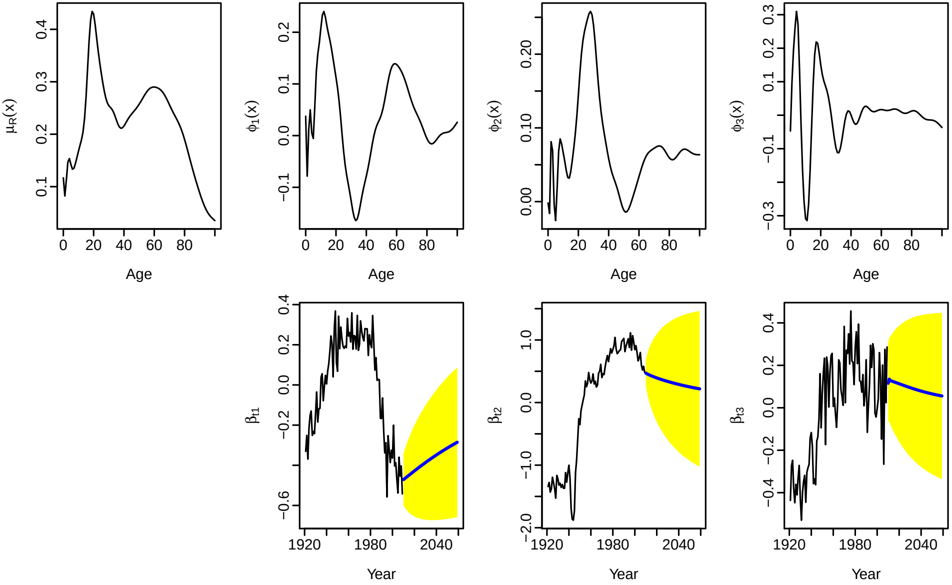
Product model

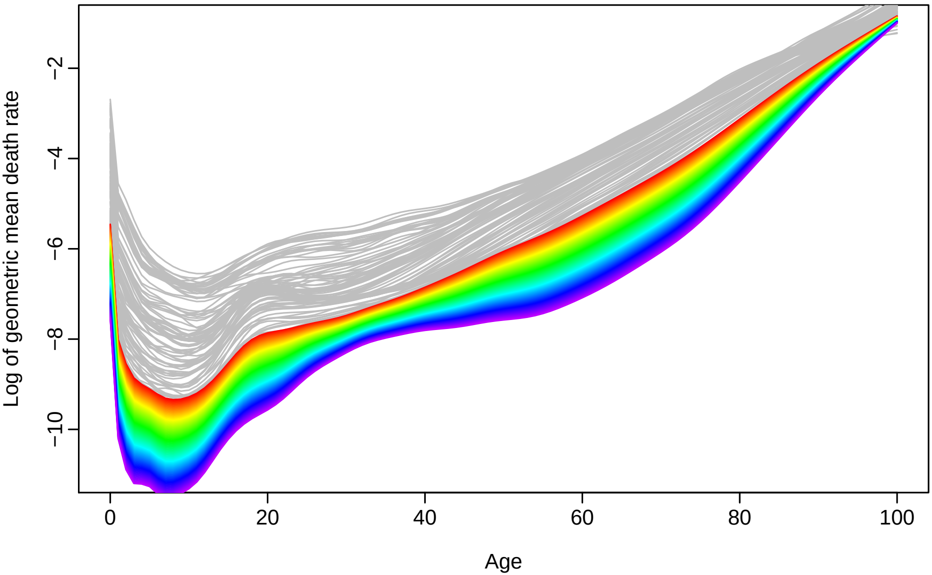


Ratio model



$\mu_R(x)$
0.1
0.2
0.3
0.4
0
20
40
60
80
Age
$\phi_1(x)$
−0.1
0.0
0.1
0.2
Age
$\phi_2(x)$
0.00
0.10
0.20
Age
$\phi_3(x)$
−0.3
−0.1
0.1
0.2
0.3
Age
β_{t1}
−0.6
−0.2
0.0
0.2
0.4
1920
1980
2040
Year
β_{t2}
−2.0
−1.0
0.0
1.0
Year
β_{t3}
−0.4
−0.2
0.0
0.2
0.4
Year

Product forecasts



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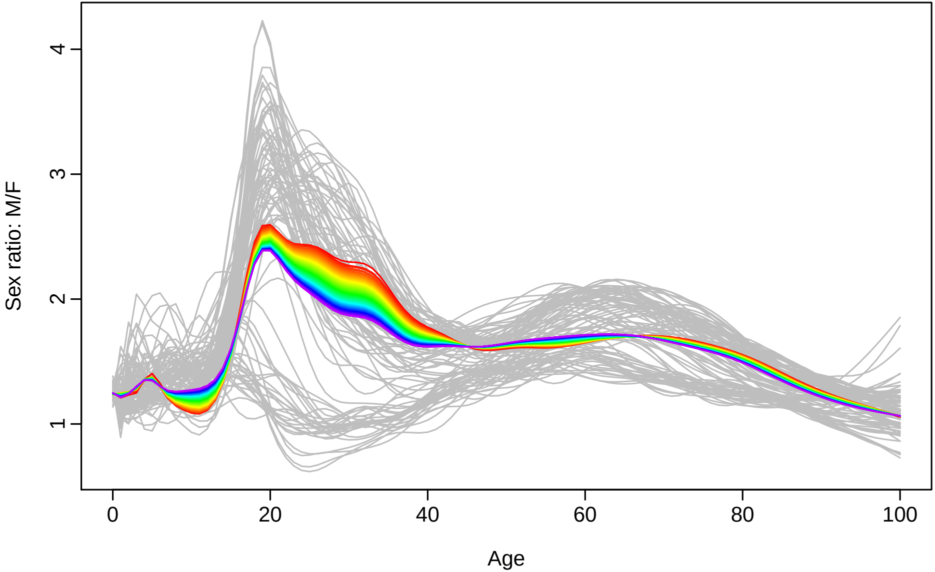
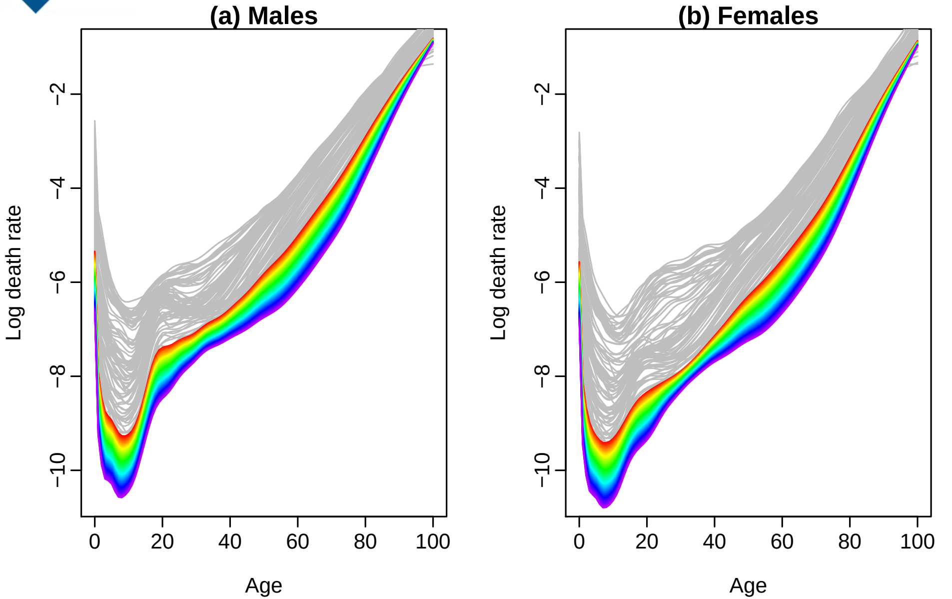


Coherent forecasts



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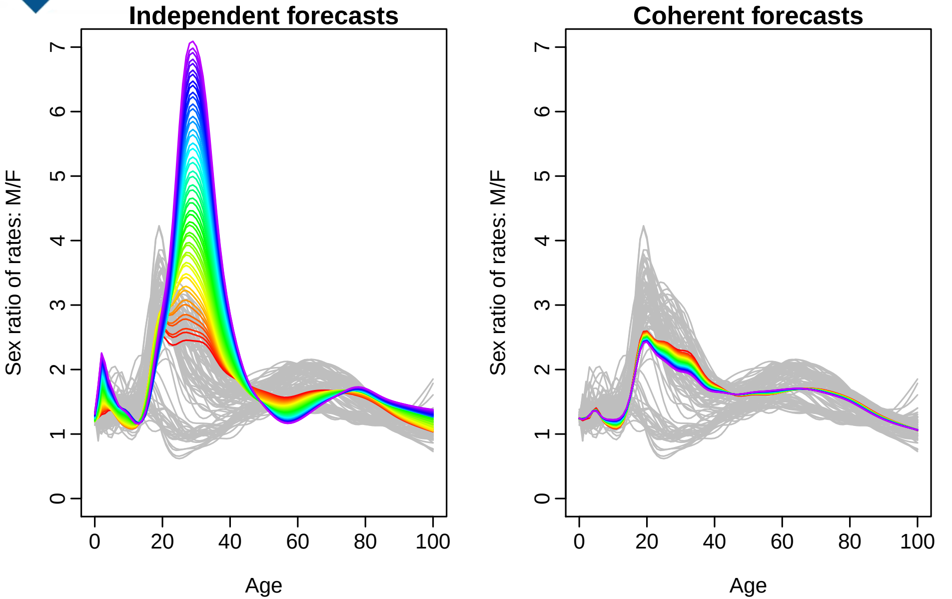
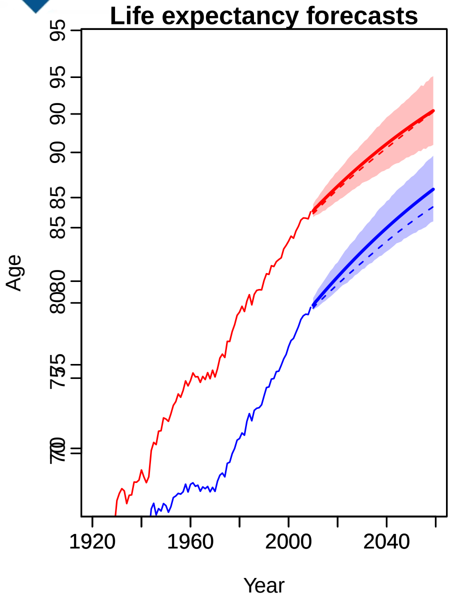
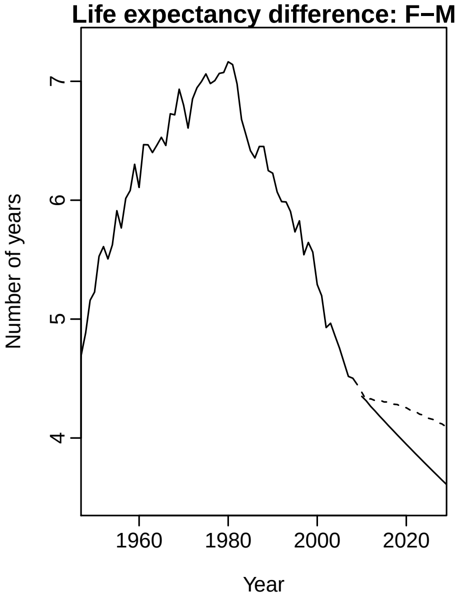


Life expectancy forecasts





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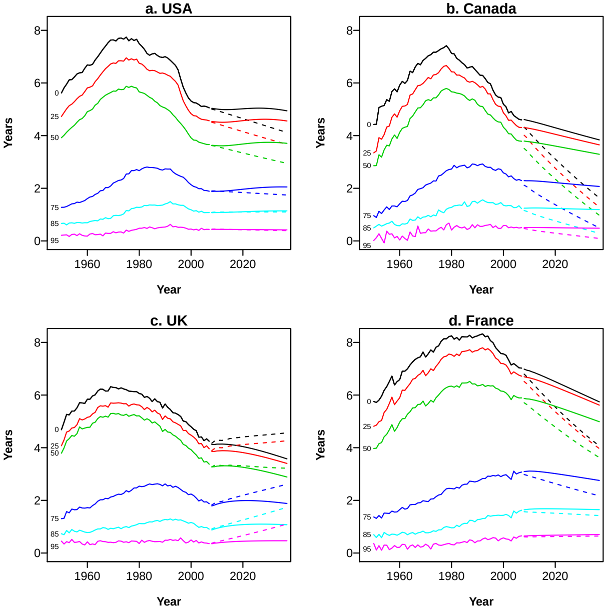

Li-Lee method

Li & Lee (*Demography*, 2005) method is a special case of our approach.

$$s_{t,j}(x) = \mu_j(x) + \beta_t\phi(x) + \gamma_{t,j}\psi_j(x) + e_{t,j}(x)$$

where f is *unsmoothed* log mortality rate, β_t is a random walk with drift and $\gamma_{t,j}$ is AR(1) process.

- No smoothing.
- Only one basis function for each part,
- Random walk with drift very limiting.
- AR(1) very limiting.
- The $\gamma_{t,j}$ coefficients will be highly correlated with each other, and so independent models are not appropriate.

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ARFIMA model

Non-seasonal ARIMA model

$$\phi_p(B)(1-B)^d y_t = c + \theta_q(B)\varepsilon_t$$

p, d and q are integer.

- An ARFIMA model is identical except that $d > -\frac{1}{2}$ can be non-integer.
- $(1-B)^d$ is given by the binomial expansion ($|d| < \frac{1}{2}$)

$$(1-B)^d = 1 + \sum_{j=1}^{\infty} \frac{\pi_j}{j!} B^j \qquad \pi_j = \prod_{k=1}^{j} (k-1-d)$$

$-\frac{1}{2} < d < 0$	y_t is stationary with "anti-persistence"
$0 < d < \frac{1}{2}$	y_t is stationary with "long memory"
$d \geq \frac{1}{2}$	y_t is non-stationary

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ARFIMA model identification

Long-memory ARFIMA model

$$\phi_p(B)(1-B)^d y_t = c + \theta_q(B)\varepsilon_t$$

p and q are integer, $0 < d < \frac{1}{2}$.

- d can be estimated by MLE if p and q are known (Haslett & Raftery, 1989).
- So we set $p = 2$ and $q = 0$, and estimate d constrained to $(0, 0.5)$.
- Then fix d, and select p and q using `auto.arima()`.
- Automated in `arfima()`.

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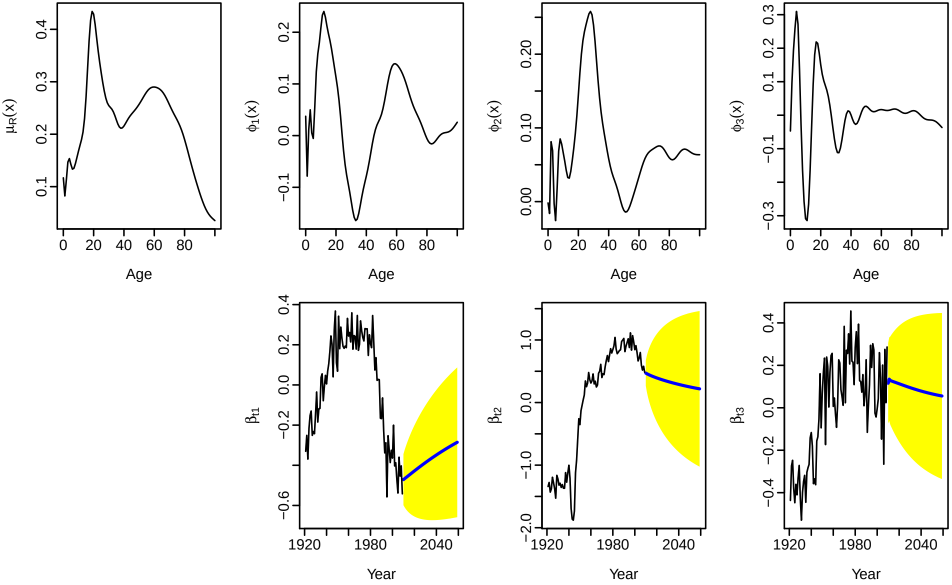

Outline

Life expectancy (recap)

$m(x)$ = mortality rate at age x.

Life expectancy at birth

$$e_0 = \int_0^\infty \exp\left[\int_0^x m(u)du\right] dx$$

- Approximated using life table methods.
- Iterate for $x = 0, 1, \dots,$, starting with $\ell_0 = 1$:

Variations for $x = 0$ and upper age group.

$q_x = m_x/(1 + 0.5m_x)$	Prob of death at age x
$d_x = \ell_x q_x$	Propn deaths at age x
$\ell_{x+1} = \ell_x - d_x$	Propn survive to age x
$L_x = \ell_x - 0.5d_x$	Propn survive to age $x + 0.5$

Approximate life expectancy at birth

$$e_0 = \sum_{x=0}^{\infty} L_x$$

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Approximate remaining life expectancy at age u

$$e_u = \sum_{x=u}^{\infty} L_x$$

Life expectancy (recap)

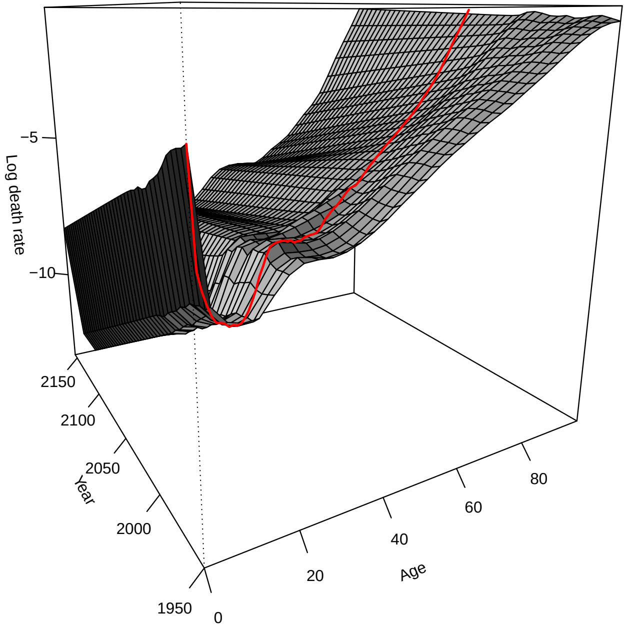
$$\log[m_t(x_i)] = s_t(x_i) + \sigma_t(x_i)\varepsilon_{t,i},$$

$$s_t(x) = \mu(x) + \sum_{k=1}^{K} \beta_{t,k}\,\phi_k(x) + r_t(x)$$

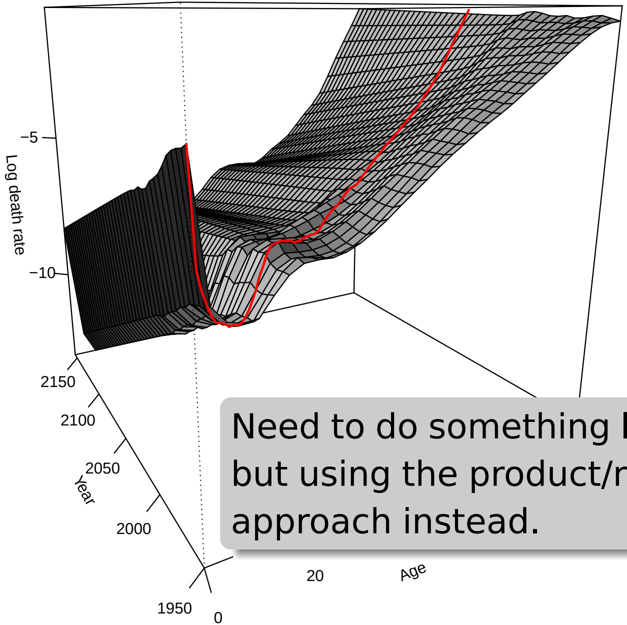
Non-coherent cohort life expectancy

- Computed from $m_{s+x}(x)$ for a given s.
- Combine observed $m_{s+x}(x)$ where $s+x \leq T$ with forecast $m_{s+x}(x)$ for $s+x > T$.
- Compute $e^*_{0,s}$.
- Prediction intervals by simulation
  - $r_t(x)$ resampled
  - $\varepsilon_{t,i} \sim \mathrm{N}(0,1)$
  - $\beta_{t,k}$ simulated from ARIMA model

Cohort life expectancy



Cohort life expectancy



Need to do something like this but using the product/ratio approach instead.

Simulate future mortality rates

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- $\{\gamma_{t,\ell}\}$ and $\{\beta_{t,k}\}$ simulated.
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- Generate many future sample paths for $s_{t,M}(x)$ and $s_{t,F}(x)$ to estimate uncertainty in e_0.

Simulate future mortality rates

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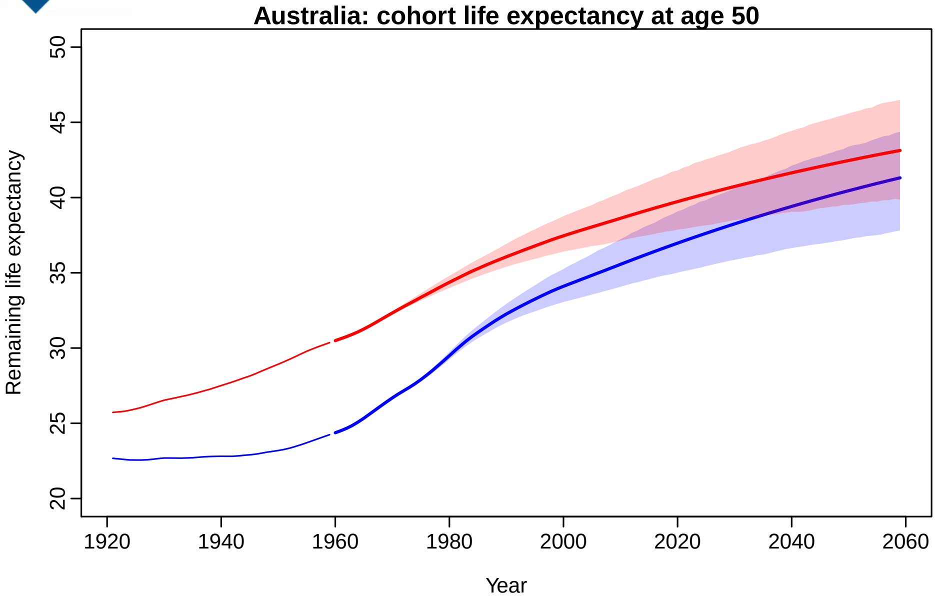
Cohort life expectancy

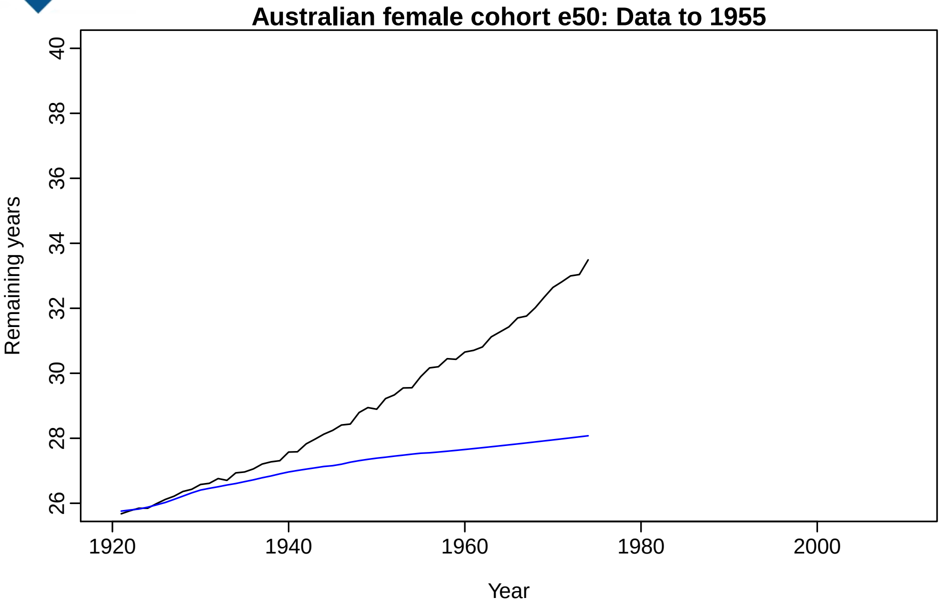


Complete code

```
library(demography)

# Read data
aus <- hmd.mx("AUS","username","password","Australia")

# Smooth data
aus.sm <- smooth.demogdata(aus)

#Fit model
aus.pr <- coherentfdm(aus.sm)

# Forecast
aus.pr.fc <- forecast(aus.pr, h=100)

# Compute life expectancies
e50.m.aus.fc <- flife.expectancy(aus.pr.fc, series="male",
  age=50, PI=TRUE, nsim=1000, type="cohort")
e50.f.aus.fc <- flife.expectancy(aus.pr.fc, series="female",
  age=50, PI=TRUE, nsim=1000, type="cohort")
```

Forecast accuracy evaluation

Forecast accuracy evaluation



Forecast accuracy evaluation

- Compute age 50 remaining cohort life expectancy with a rolling forecast origin beginning in 1921.
- Compare against actual cohort life expectancy where available.
- Compute 80% prediction interval actual coverage.

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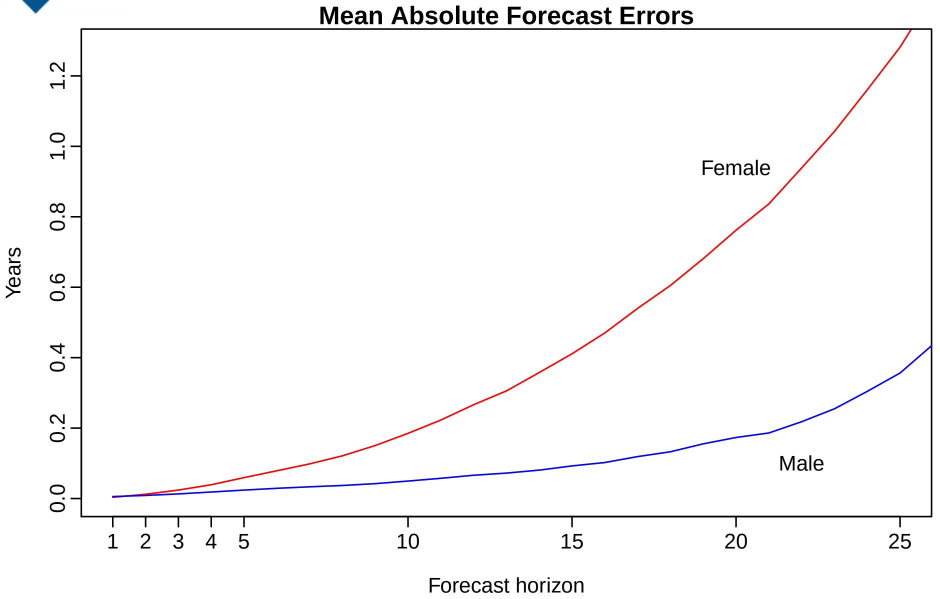

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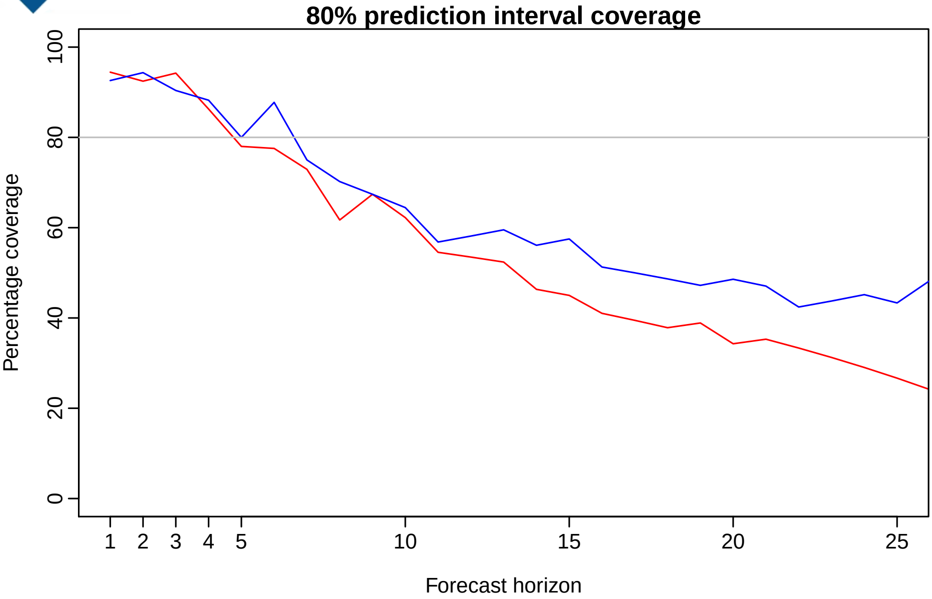

Outline

Coherent forecasts for J groups

$$p_t(x) = [s_{t,1}(x)s_{t,2}(x)\cdots s_{t,J}(x)]^{1/J}$$

and $$r_{t,j}(x) = s_{t,j}(x)/p_t(x),$$

$$\log[p_t(x)] = \mu_p(x) + \sum_{k=1}^{K}\beta_{t,k}\phi_k(x) + e_t(x)$$

$$\log[r_{t,j}(x)] = \mu_{r,j}(x) + \sum_{l=1}^{L}\gamma_{t,l,j}\psi_{l,j}(x) + w_{t,j}(x).$$

- $p_t(x)$ and all $r_{t,j}(x)$ are approximately independent.

Coherent forecasts for J groups

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$$\log[p_t(x)] = \mu_p(x) + \sum_{k=1}^{K} \beta_{t,k} \phi_k(x) + e_t(x)$$

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- $p_t(x)$ and all $r_{t,j}(x)$ are approximately independent.
- Ratios satisfy constraint $r_{t,1}(x) r_{t,2}(x) \cdots r_{t,J}(x) = 1$.
- $\log[s_{t,j}(x)] = \log[p_t(x) r_{t,j}(x)]$

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Coherent forecasts for J groups

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\begin{aligned}
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\end{aligned}
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- $\mu_j(x) = \mu_p(x) + \mu_{r,j}(x)$ is group mean
- $z_{t,j}(x) = e_t(x) + w_{t,j}(x)$ is error term.
- $\{\gamma_{t,\ell}\}$ restricted to be stationary processes: either ARMA(p,q) or ARFIMA(p,d,q).
- No restrictions for $\beta_{t,1},\dots,\beta_{t,K}$.

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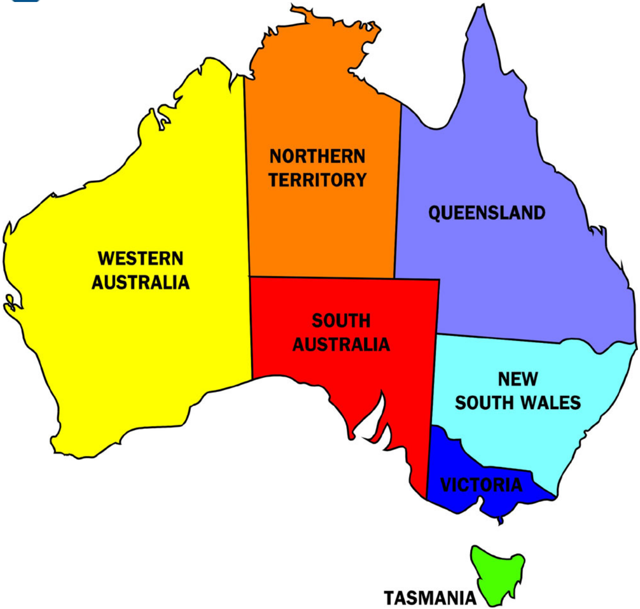
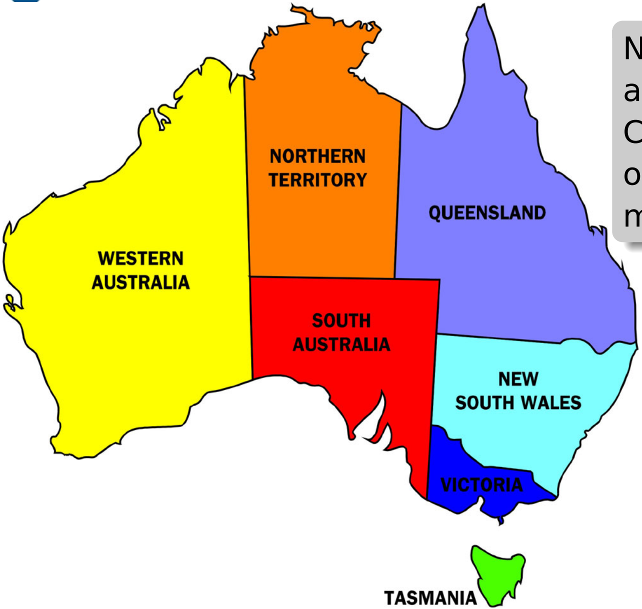
Outline

Forecasting state mortality

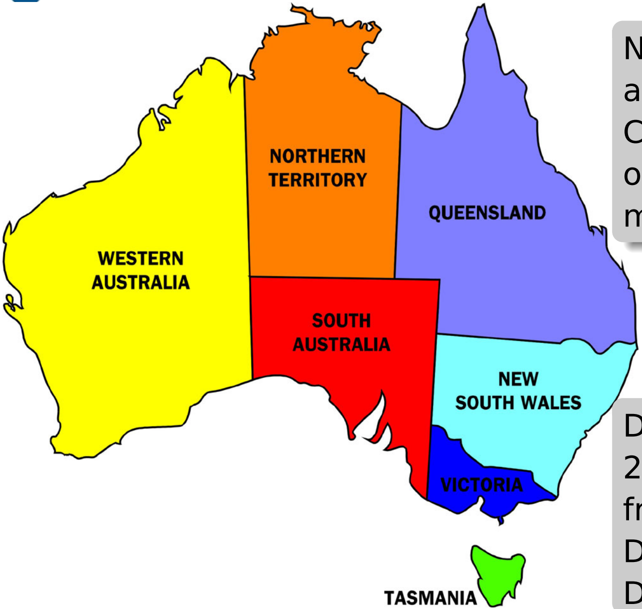


Forecasting state mortality



Northern Territory and Australian Capital Territory omitted due to missing data.

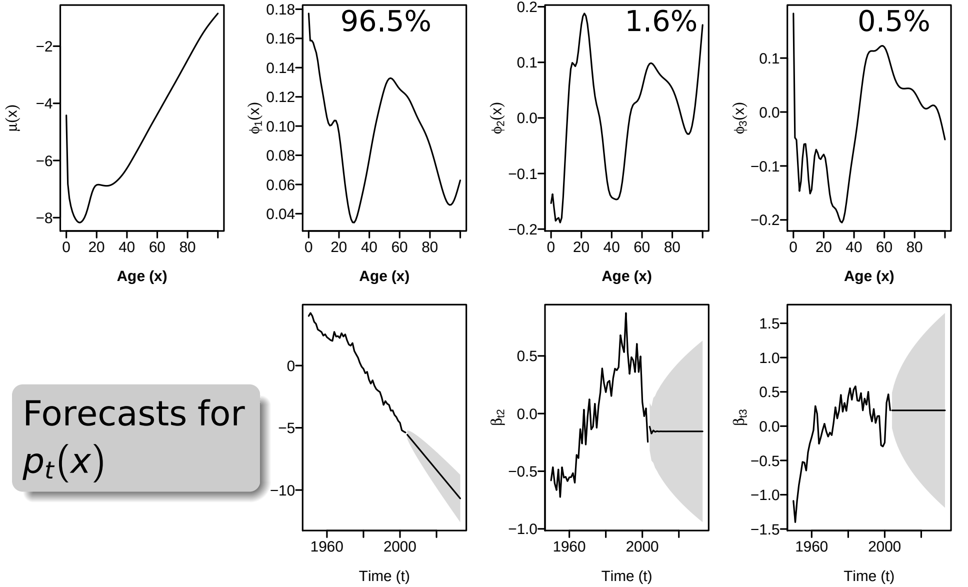
Forecasting state mortality



Northern Territory and Australian Capital Territory omitted due to missing data.

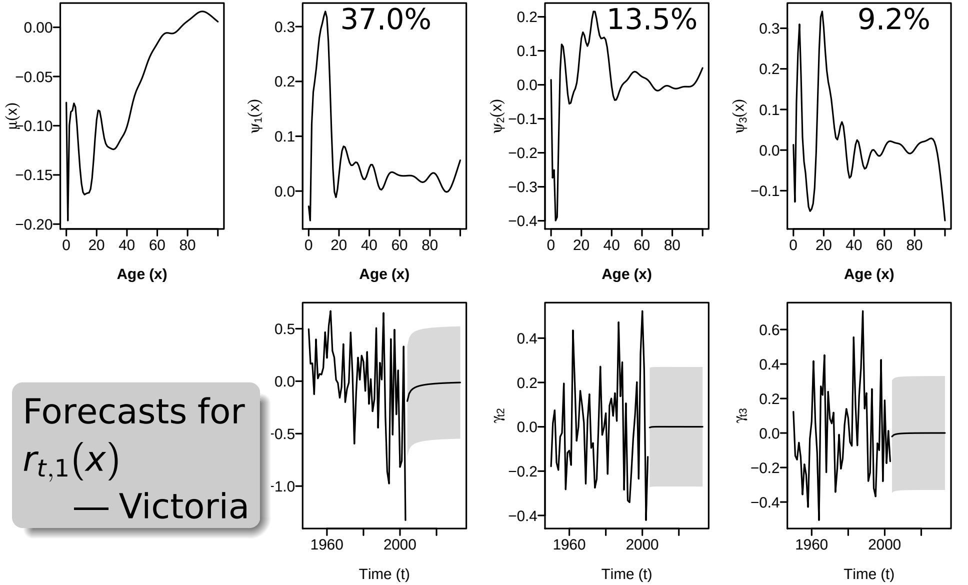
Data for 1950 to 2003 obtained from Australian Demographic Data Bank.

Forecasting state mortality



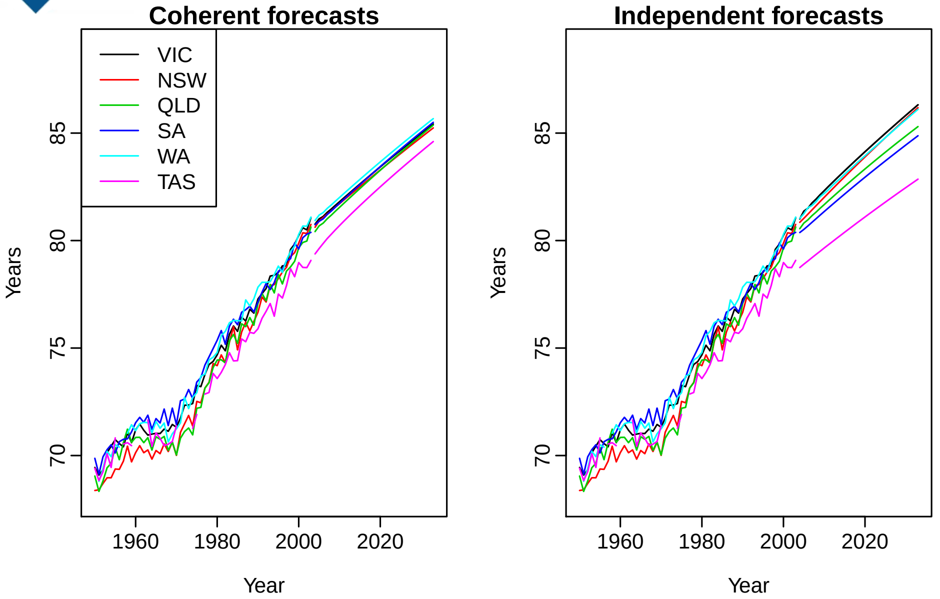
Forecasts for $p_t(x)$

Forecasting state mortality

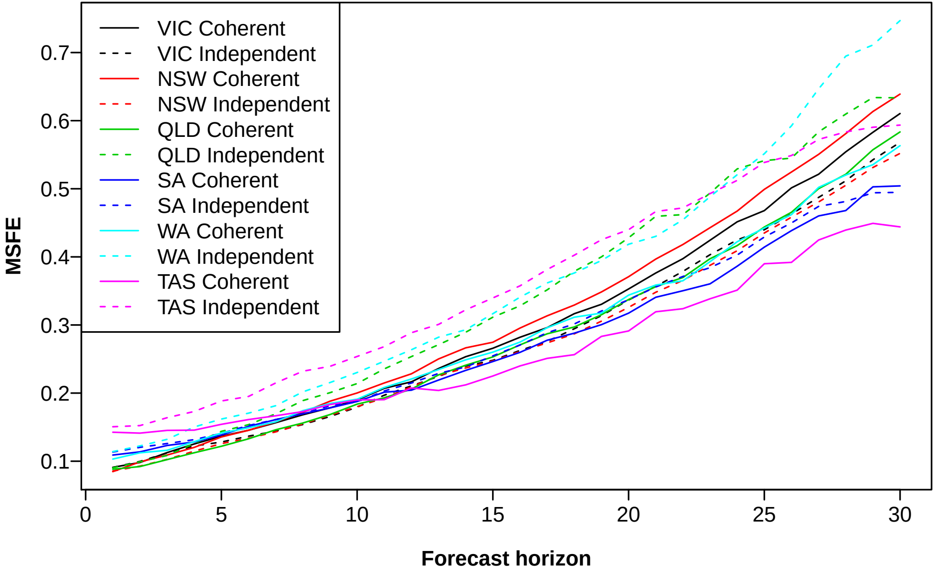


Forecasts for $r_{t,1}(x)$ — Victoria

Forecasting state mortality

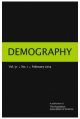


Forecasting state mortality



Outline

Selected references



Hyndman, Booth, Yasmeen (2013). “Coherent mortality forecasting: the product-ratio method with functional time series models”. *Demography* **50**(1), 261–283.

Hyndman (2014). *demography: Forecasting mortality, fertility, migration and population data*. `cran.r-project.org/package=demography`