Coherent functional forecasts of mortality rates and life expectancy

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Joint work with

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Mortality rates

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- Expected that groups will behave similarly.
- We want to forecast **whole curve** $f_{t,j}(x)$ for future years.
- **Coherent** forecasts do not diverge over time.
- Existing functional models do not impose coherence.

Functional time series model

(Hyndman and Ullah, CSDA, 2007)

$$\log[f_{t,j}(x)] = \mu_j(x) + \sum_{k=1}^{K} \beta_{t,j,k}\, \phi_{k,j}(x) + e_{t,j}(x)$$

where $e_{t,j}(x) \stackrel{\text{iid}}{\sim} \mathrm{N}(0, v(x))$.

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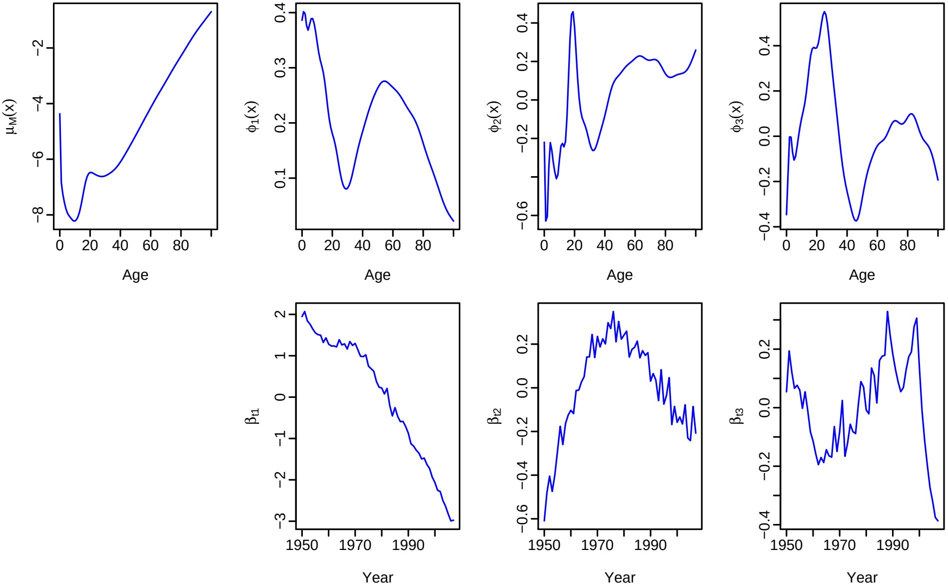
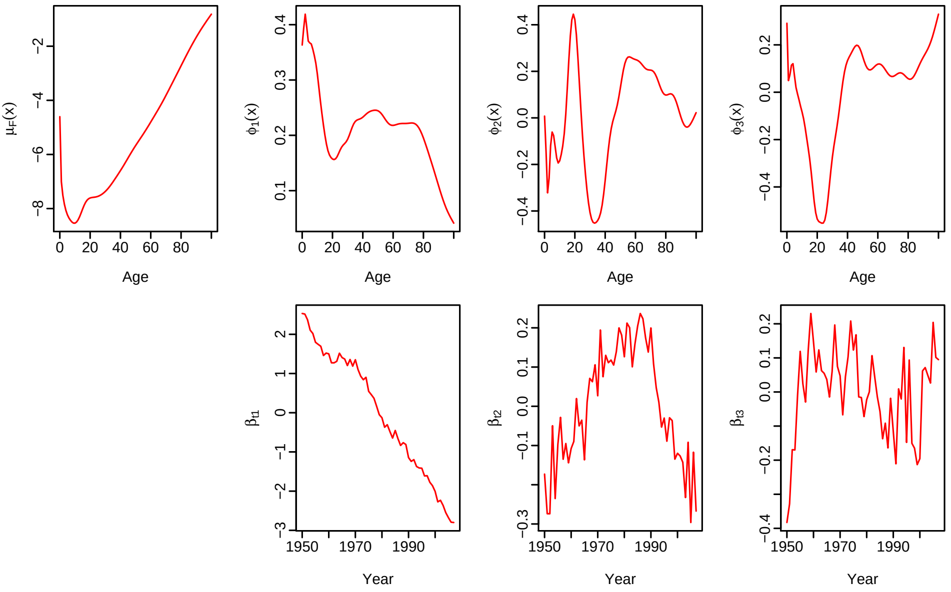
4. Forecast $\beta_{t,j,k}$ using time series models.
5. Put it all together to get forecasts of $f_{t,j}(x)$.

Male fts model



Female fts model



Forecasting the coefficients

$$\log[f_{t,j}(x)] = \mu_j(x) + \sum_{k=1}^{K} \beta_{t,j,k}\,\phi_{k,j}(x) + e_{t,j}(x)$$

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- Non-stationary ARIMA forecasts will diverge. Hence the mortality forecasts are not coherent.

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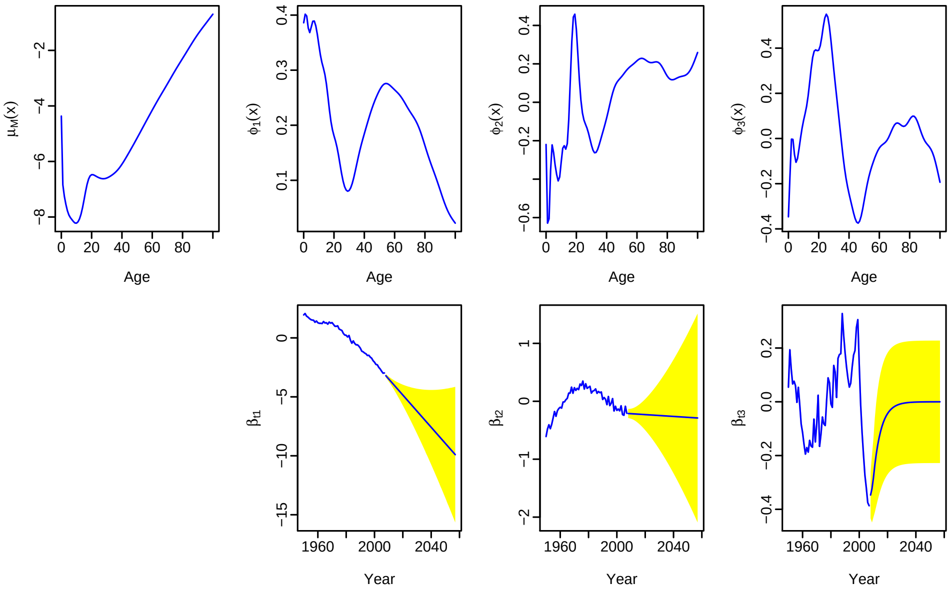

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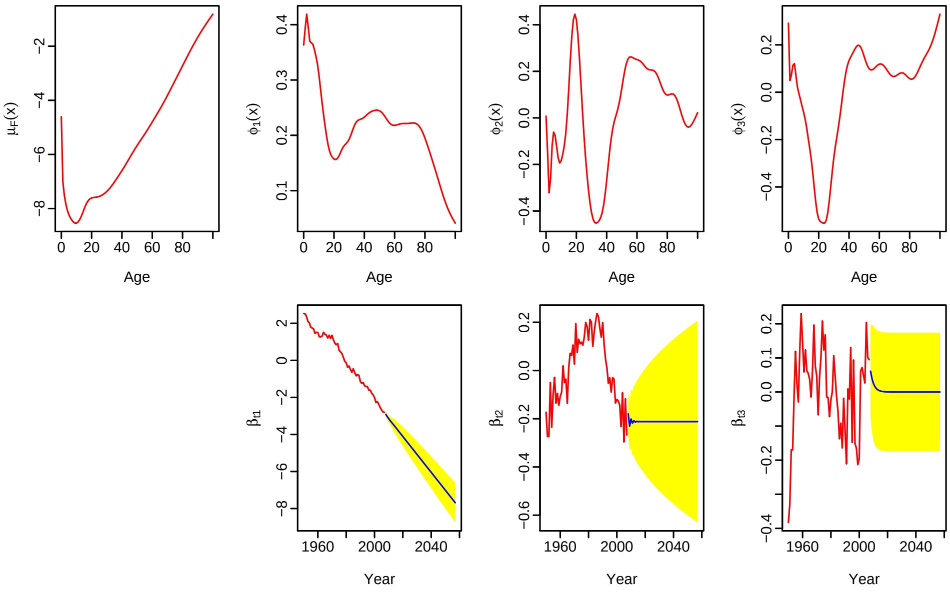
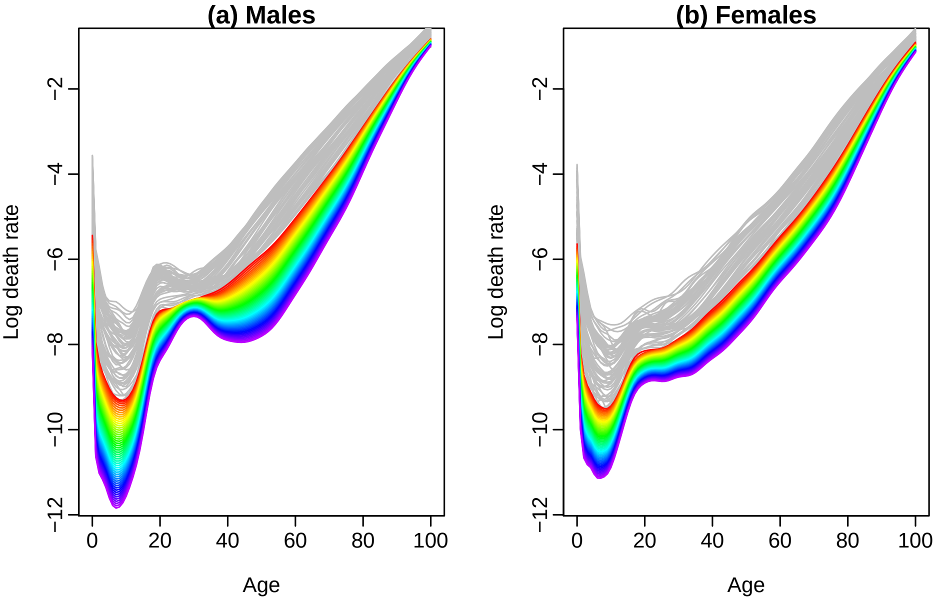


Australian mortality forecasts



Mortality product and ratios

Key idea

Model the geometric mean and the mortality ratio instead of the individual rates for each sex separately.

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$$p_t(x) = \sqrt{f_{t,M}(x) f_{t,F}(x)} \quad \text{and} \quad r_t(x) = \sqrt{f_{t,M}(x) \big/ f_{t,F}(x)}.$$

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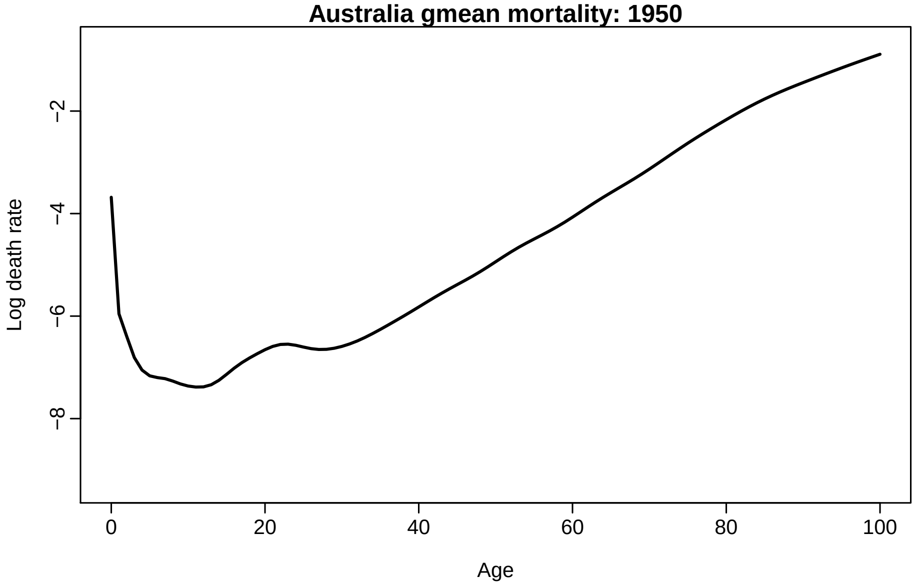
- Product and ratio are approximately independent
- Ratio should be stationary (for coherence) but product can be non-stationary.

Mortality rates



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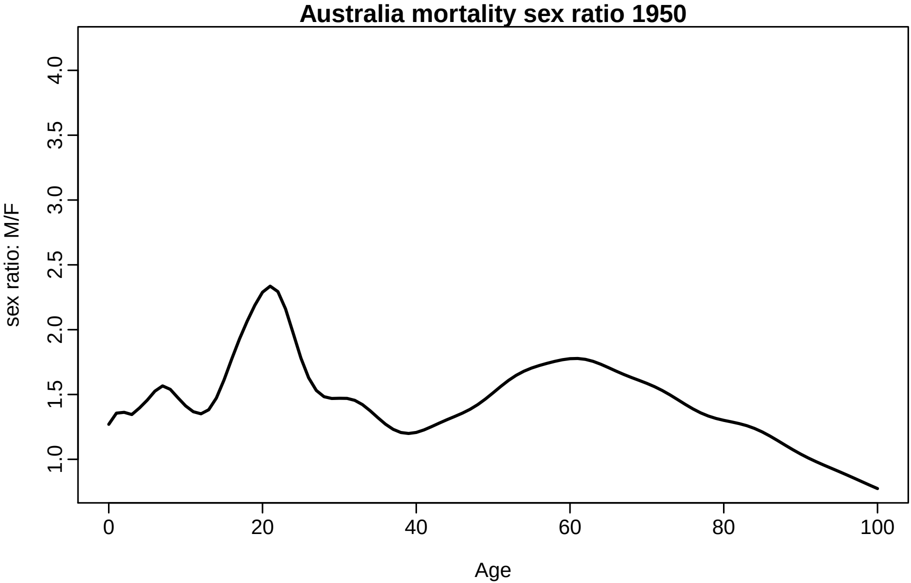

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- $\{\gamma_{t,\ell}\}$ restricted to be stationary processes: either ARMA(p,q) or ARFIMA(p,d,q).
- No restrictions for $\beta_{t,1},\dots,\beta_{t,K}$.
- **Forecasts:** $f_{n+h|n,M}(x) = p_{n+h|n}(x) r_{n+h|n}(x)$
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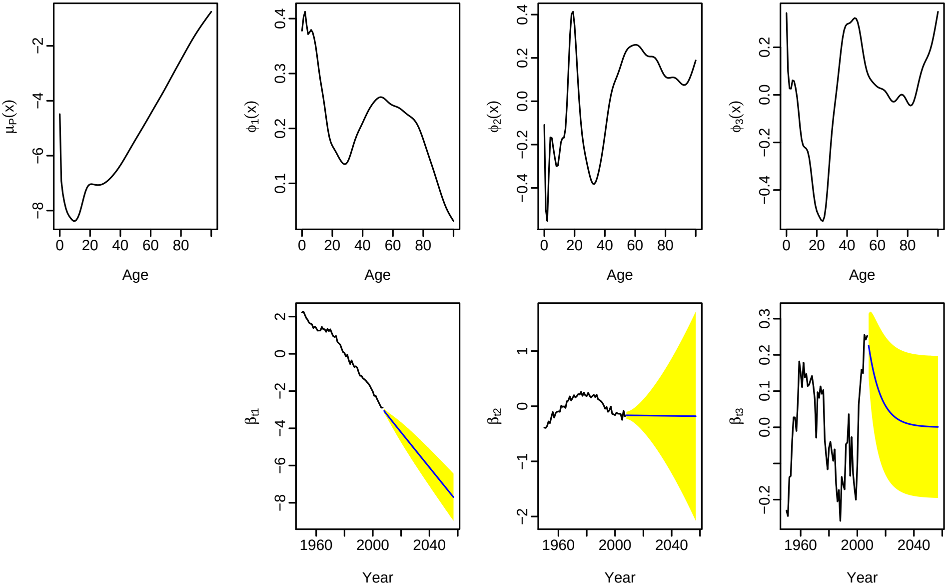
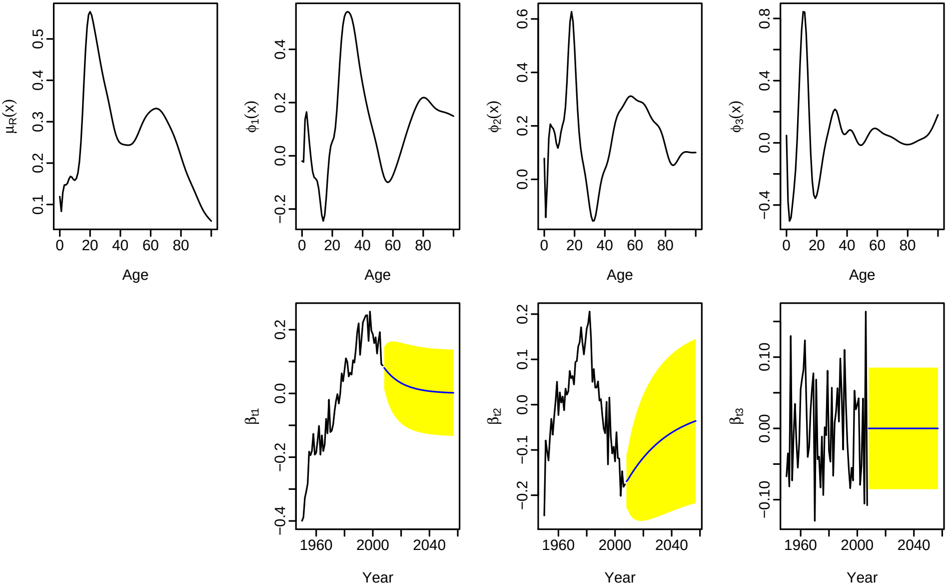
Product model

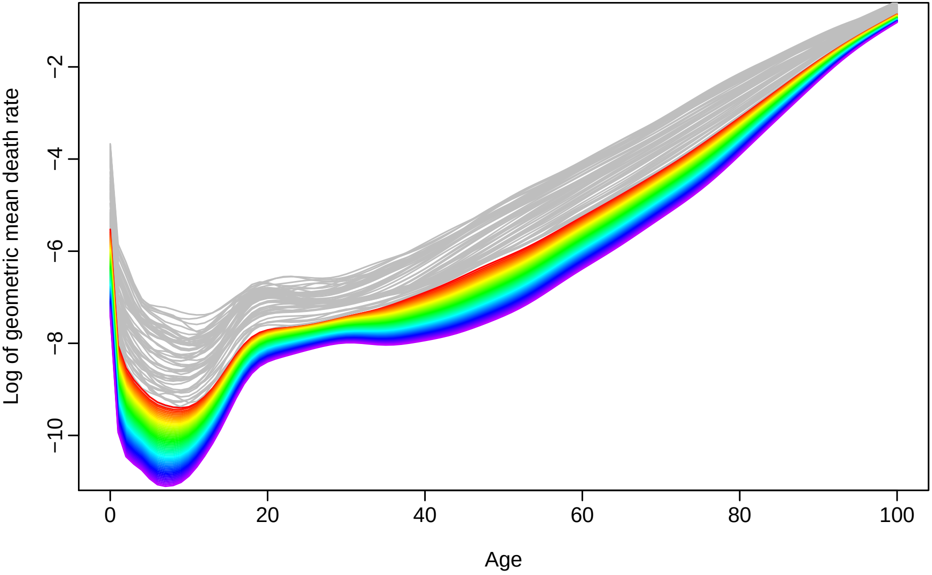


Ratio model



$\mu_R(x)$
Age
$\phi_1(x)$
Age
$\phi_2(x)$
Age
$\phi_3(x)$
Age
β_{t1}
Year
β_{t2}
Year
β_{t3}
Year

Product forecasts



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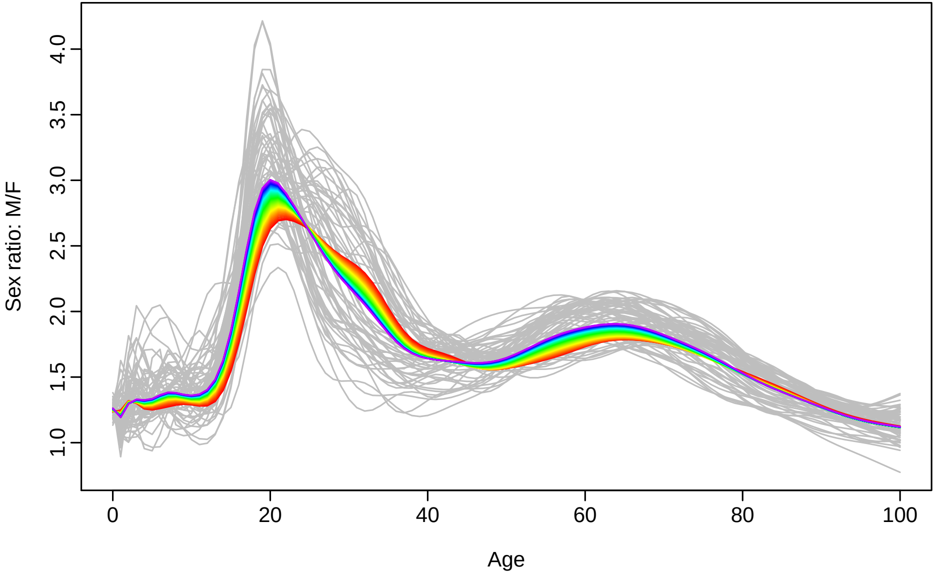
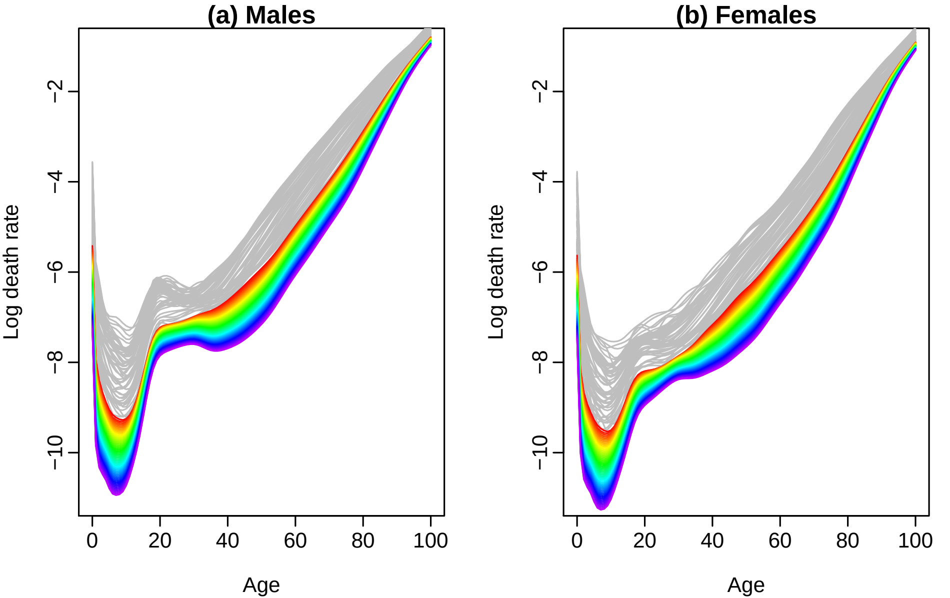


Coherent forecasts



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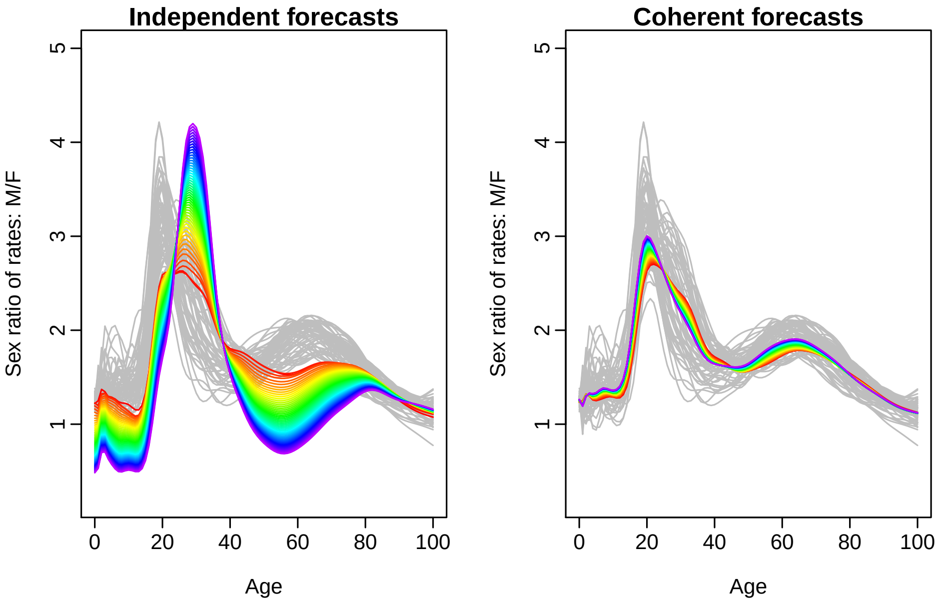
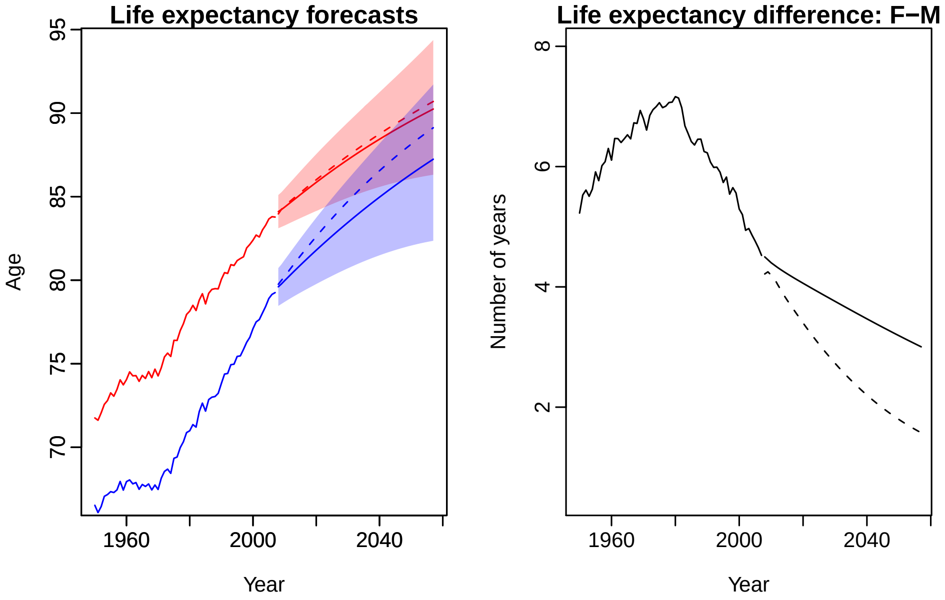


Life expectancy forecasts



Coherent forecasts for J groups

$$p_t(x) = [f_{t,1}(x) f_{t,2}(x) \cdots f_{t,J}(x)]^{1/J}$$

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Li-Lee method

Li & Lee (*Demography*, 2005) method is a special case of our approach.

$$f_{t,j}(x) = \mu_j(x) + \beta_t \phi(x) + \gamma_{t,j} \psi_j(x) + e_{t,j}(x)$$

where f is *unsmoothed* log mortality rate, β_t is a random walk with drift and $\gamma_{t,j}$ is AR(1) process.

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- The $\gamma_{t,j}$ coefficients will be highly correlated with each other, and so independent models are not appropriate.

Some conclusions

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- Easy to compute prediction intervals.