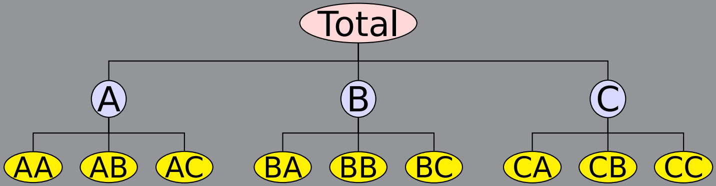
MONASH BUSINESS SCHOOL

Rob J Hyndman

Optimal forecast reconciliation for big time series data



Follow along using R

Requirements

Install the hts package and its dependencies.

Outline

Labour market participation

Australia and New Zealand Standard Classification of Occupations

- 8 major groups
  - 43 sub-major groups
    - 97 minor groups
      - – 359 unit groups
        - * 1023 occupations

Example: statistician

2 Professionals

22 Business, Human Resource and Marketing Professionals

224 Information and Organisation Professionals

2241 Actuaries, Mathematicians and Statisticians

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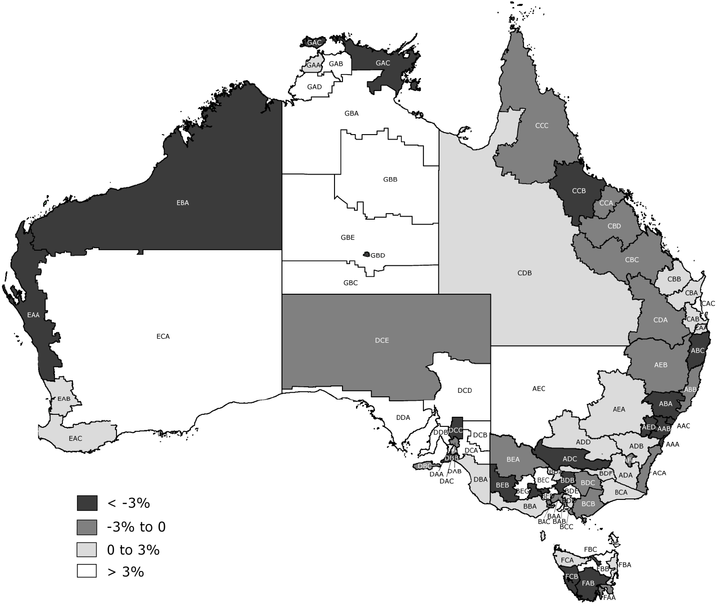
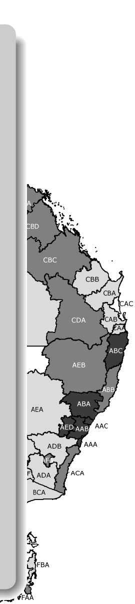
Australian tourism demand



Australian tourism demand

- Quarterly data on visitor night from 1998:Q1 – 2013:Q4
- From: *National Visitor Survey*, based on annual interviews of 120,000 Australians aged 15+, collected by Tourism Research Australia.
- Split by 7 states, 27 zones and 76 regions (a geographical hierarchy)
- Also split by purpose of travel
  - Holiday
  - Visiting friends and relatives (VFR)
  - Business
  - Other
- 304 bottom-level series



Hierarchical time series

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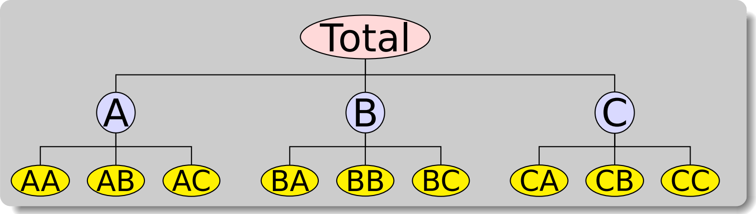

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- Labour turnover by occupation
- Tourism by state and region

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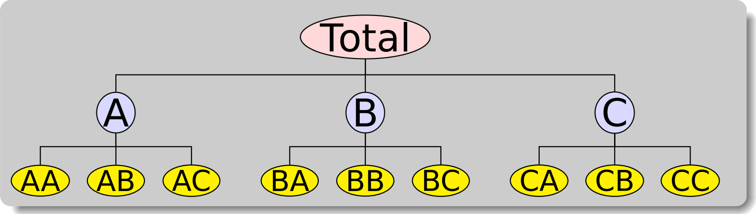

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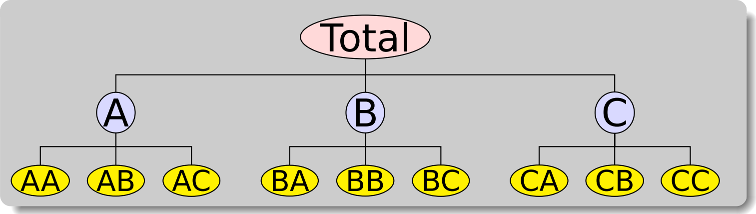

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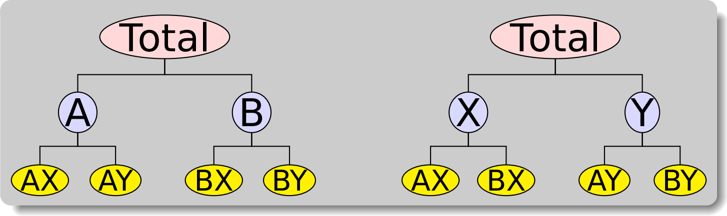

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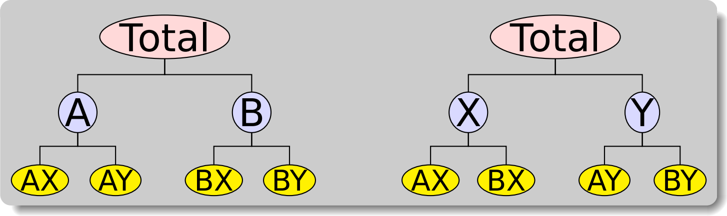

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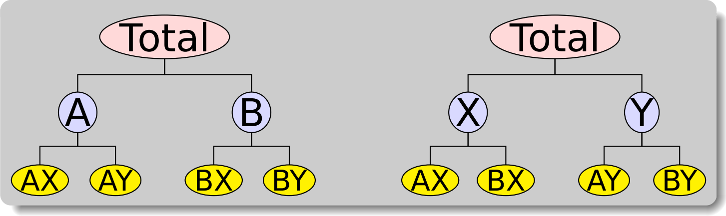

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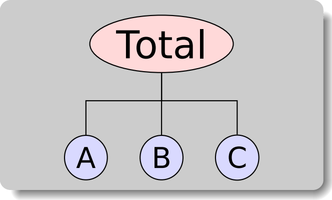

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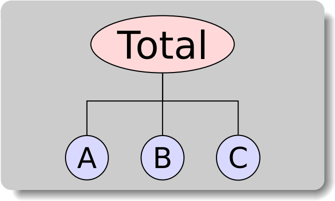

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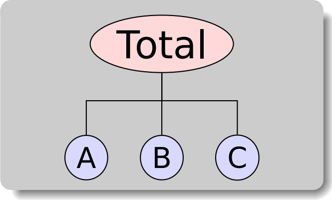

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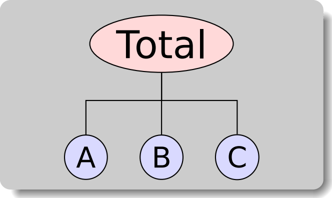

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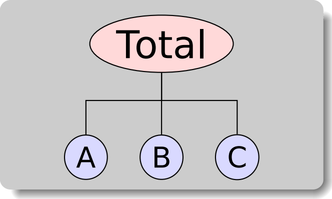

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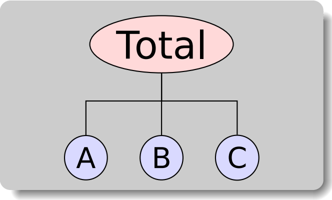

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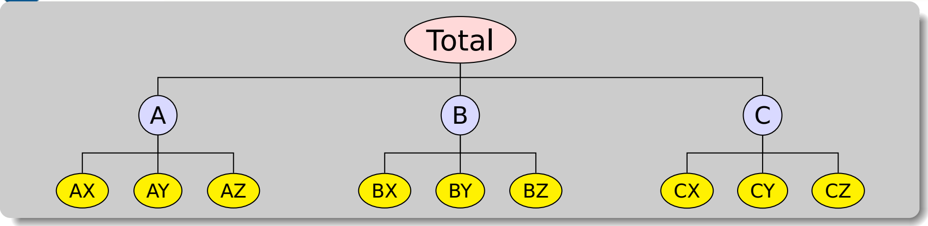

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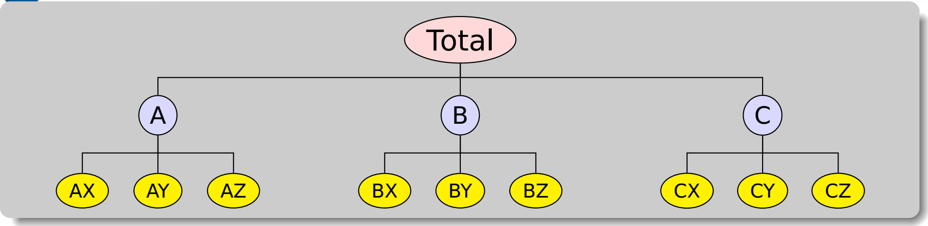

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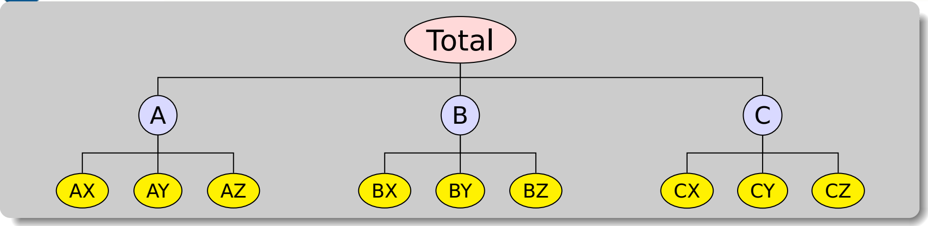

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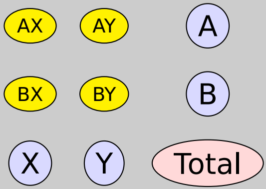

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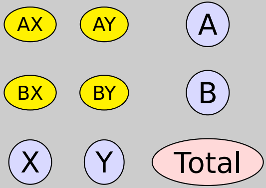

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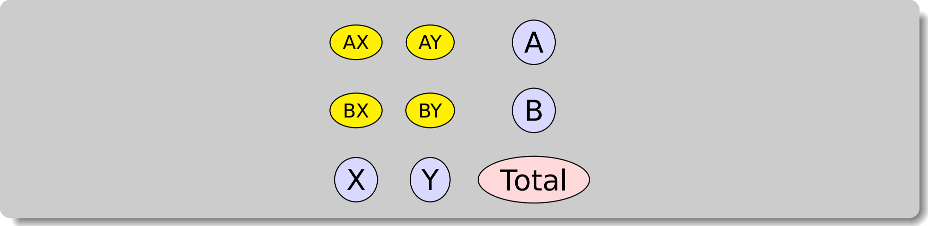

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Hierarchical and grouped time series

Every collection of time series with aggregation constraints can be written as

$$\boldsymbol{y}_t = \boldsymbol{S}\boldsymbol{b}_t$$

where

- $\boldsymbol{y}_t$ is a vector of all series at time t
- $\boldsymbol{b}_t$ is a vector of the most disaggregated series at time t
- $\boldsymbol{S}$ is a "summing matrix" containing the aggregation constraints.

hts package for R

hts: Hierarchical and grouped time series

Methods for analysing and forecasting hierarchical and grouped time series

Version:	4.5
Depends:	forecast ($\geq$ 5.0), SparseM
Imports:	parallel, utils
Published:	2015-06-29
Author:	Rob J Hyndman, Earo Wang and Alan Lee with contributions from Shanika Wickramasuriya
Maintainer:	Rob J Hyndman <Rob.Hyndman at monash.edu>
BugReports:	`https://github.com/robjhyndman/hts/issues`
License:	GPL ($\geq$ 2)

Example using R

```
library(hts)

# bts is a matrix containing the bottom level time series
# nodes describes the hierarchical structure
y <- hts(bts, nodes=list(2, c(3,2)))
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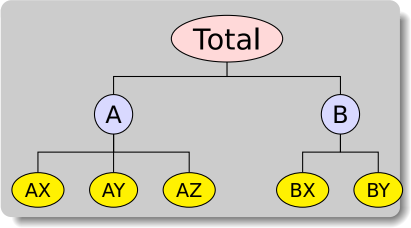

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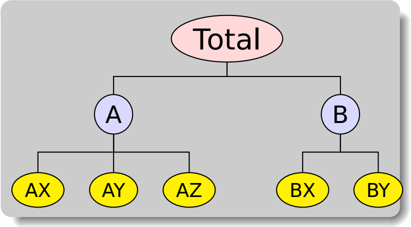


```
summary(y)
smatrix(y)
```

Example using R

Example 1 illustrating the usage of the "groups" argument

```
abc <- ts(5 + matrix(sort(rnorm(1600)), ncol = 16, nrow = 100))

sex <- rep(c("female", "male"), each = 8)
state <- rep(c("NSW","VIC","QLD","SA","WA","NT","ACT","TAS"), 2)

gc <- rbind(sex, state)  # a matrix consists of strings.
gn <- rbind(rep(1:2, each = 8), rep(1:8, 2))
rownames(gc) <- rownames(gn) <- c("Sex", "State")

x <- gts(abc, groups = gc)
y <- gts(abc, groups = gn)

plot(x,level=3)
plot(x,level=2)
plot(x,level=1)
plot(x,level=0)
```

Example using R

Example 2 with two simple hierarchies (geography and product) to show the argument "characters"

```
bnames1 <- c("VICMelbA1", "VICMelbA2", "VICGeelA1", "VICGeelA2",
             "VICMelbB1", "VICMelbB2", "VICGeelB1", "VICGeelB2",
             "NSWSyndA1", "NSWSyndA2", "NSWWollA1", "NSWWollA2",
             "NSWSyndB1", "NSWSyndB2", "NSWWollB1", "NSWWollB2")
bts1 <- matrix(ts(rnorm(160)), ncol = 16)
colnames(bts1) <- bnames1
x1 <- gts(bts1, characters = list(c(3, 4), c(1, 1)),
  gnames = c("State","State.City","Product","Product.Size",
  "State.Product","State.Product.Size","State.City.Product"))

plot(x1, level=1)
plot(x1, level=7)
```

Example using R

Example 3 with a non-hierarchical grouped time series of 3 grouping variables (state, age and sex)

```
bnames2 <- c("VIC1F","VIC1M","VIC2F","VIC2M","VIC3F","VIC3M",
             "NSW1F","NSW1M","NSW2F","NSW2M","NSW3F","NSW3M")
bts2 <- matrix(ts(rnorm(120)), ncol = 12)
colnames(bts2) <- bnames2
x2 <- gts(bts2, characters = c(3, 1, 1),
  gnames=c("State","Age","Sex",
           "State.Age","State.Sex","Age.Sex"))

plot(x2, level=3)
```

Outline

Forecasting notation

Let $\hat{\boldsymbol{y}}_n(h)$ be vector of initial h-step forecasts, made at time n, stacked in same order as $\boldsymbol{y}_t$.
(They may not add up.)

Reconciled forecasts must be of the form:

$$\tilde{\boldsymbol{y}}_n(h) = \boldsymbol{S}\boldsymbol{P}\hat{\boldsymbol{y}}_n(h)$$

for some matrix $\boldsymbol{P}$.

- $\boldsymbol{P}$ extracts and combines base forecasts $\hat{\boldsymbol{y}}_n(h)$ to get bottom-level forecasts.

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General properties: bias

$$\tilde{\boldsymbol{y}}_n(h) = \boldsymbol{S}\boldsymbol{P}\hat{\boldsymbol{y}}_n(h)$$

Assume: base forecasts $\hat{\boldsymbol{y}}_n(h)$ are unbiased:

$$\mathrm{E}[\hat{\boldsymbol{y}}_n(h) \mid \boldsymbol{y}_1, \dots, \boldsymbol{y}_n] = \mathrm{E}[\boldsymbol{y}_{n+h} \mid \boldsymbol{y}_1, \dots, \boldsymbol{y}_n]$$

- Let $\hat{\boldsymbol{b}}_n(h)$ be bottom level base forecasts with $\boldsymbol{\beta}_n(h) = \mathrm{E}[\hat{\boldsymbol{b}}_n(h) \mid \boldsymbol{y}_1, \dots, \boldsymbol{y}_n]$.
- Then $\mathrm{E}[\hat{\boldsymbol{y}}_n(h)] = \boldsymbol{S}\boldsymbol{\beta}_n(h)$.
- We want the revised forecasts to be unbiased: $\mathrm{E}[\tilde{\boldsymbol{y}}_n(h)] = \boldsymbol{SPS}\boldsymbol{\beta}_n(h) = \boldsymbol{S}\boldsymbol{\beta}_n(h)$.

Revised forecasts are unbiased iff $\boldsymbol{SPS} = \boldsymbol{S}$.

General properties: bias

$$\tilde{\boldsymbol{y}}_n(h) = \boldsymbol{S}\boldsymbol{P}\hat{\boldsymbol{y}}_n(h)$$

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General properties: variance

$$\tilde{\boldsymbol{y}}_n(h) = \boldsymbol{S}\boldsymbol{P}\hat{\boldsymbol{y}}_n(h)$$

Let error variance of h-step base forecasts $\hat{\boldsymbol{y}}_n(h)$ be

$$\boldsymbol{W}_h = \text{Var}[\boldsymbol{y}_{n+h} - \hat{\boldsymbol{y}}_n(h) \mid \boldsymbol{y}_1, \dots, \boldsymbol{y}_n]$$

Then the error variance of the corresponding revised forecasts is

$$\text{Var}[\boldsymbol{y}_{n+h} - \tilde{\boldsymbol{y}}_n(h) \mid \boldsymbol{y}_1, \dots, \boldsymbol{y}_n] = \boldsymbol{S}\boldsymbol{P}\boldsymbol{W}_h\boldsymbol{P}'\boldsymbol{S}'$$

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BLUF via trace minimization

Theorem

For any $\boldsymbol{P}$ satisfying $\boldsymbol{SPS} = \boldsymbol{S}$, then

$$\min_{\boldsymbol{P}} = \text{trace}\left[\boldsymbol{SPW}_h\boldsymbol{P}'\boldsymbol{S}'\right]$$

has solution $\boldsymbol{P} = (\boldsymbol{S}'\boldsymbol{W}_h^{\dagger}\boldsymbol{S})^{-1}\boldsymbol{S}'\boldsymbol{W}_h^{\dagger}$.

- $\boldsymbol{W}_h^{\dagger}$ is generalized inverse of $\boldsymbol{W}_h$.
- **Problem:** $\boldsymbol{W}_h$ hard to estimate, especially for $h > 1$.

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Optimal combination forecasts

$$\tilde{\boldsymbol{y}}_n(h) = \boldsymbol{S}(\boldsymbol{S}'\boldsymbol{W}_h^{\dagger}\boldsymbol{S})^{-1}\boldsymbol{S}'\boldsymbol{W}_h^{\dagger}\hat{\boldsymbol{y}}_n(h)$$

Revised forecasts

Base forecasts

Solution 1: OLS

- Assume forecast errors have the same aggregation constraints as the data.
- Then $\boldsymbol{W}_h = \boldsymbol{S}\Omega_h\boldsymbol{S}'$ where Ω_h is covariance matrix of bottom level errors.
- If Moore-Penrose generalized inverse used, then $(\boldsymbol{S}'\boldsymbol{W}_h^{\dagger}\boldsymbol{S})^{-1}\boldsymbol{S}'\boldsymbol{W}_h^{\dagger} = (\boldsymbol{S}'\boldsymbol{S})^{-1}\boldsymbol{S}'$.

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Revised forecasts

Base forecasts

Solution 2: WLS

- Suppose we approximate $\boldsymbol{W}_1$ by its diagonal and assume that $\boldsymbol{W}_h \propto \boldsymbol{W}_1$.
- Easy to estimate, and places weight where we have best forecasts.
- Empirically, WLS gives better forecasts than OLS.
- Still need to estimate covariance matrix to produce prediction intervals.

Optimal combination forecasts

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Revised forecasts

Base forecasts

Solution 3: GLS

- Estimate $\boldsymbol{W}_1$ using shrinkage to the diagonal and assume that $\boldsymbol{W}_h \propto \boldsymbol{W}_1$.
- Allows for covariances.
- Empirically, GLS gives better forecasts than WLS or OLS.
- Difficult to compute for large numbers of time series.

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Example using R

```
library(hts)

# bts is a matrix containing the bottom level time series
# nodes describes the hierarchical structure
y <- hts(bts, nodes=list(2, c(3,2)))

# Forecast 10-step-ahead using WLS combination method
# ETS used for each series by default
fc <- forecast(y, h=10)

# Select your own methods
ally <- aggts(y)
allf <- matrix(, nrow=10, ncol=ncol(ally))
for(i in 1:ncol(ally))
  allf[,i] <- mymethod(ally[,i], h=10)
allf <- ts(allf, start=2004)
# Reconcile forecasts so they add up
fc2 <- combinef(allf, nodes=y$nodes)
```

forecast.gts function

Usage

```
forecast(object, h,
  method = c("comb", "bu", "mo", "tdgsf", "tdgsa", "tdfp"),
  fmethod = c("ets", "rw", "arima"),
  weights = c("sd", "none", "nseries"),
  positive = FALSE,
  parallel = FALSE, num.cores = 2, ...)
```

Arguments

`object`	Hierarchical time series object of class `gts`.
`h`	Forecast horizon
`method`	Method for distributing forecasts within the hierarchy.
`fmethod`	Forecasting method to use
`positive`	If TRUE, forecasts are forced to be strictly positive
`weights`	Weights used for "optimal combination" method. When weights = "sd", it takes account of the standard deviation of forecasts.
`parallel`	If TRUE, allow parallel processing
`num.cores`	If parallel = TRUE, specify how many cores are going to be used

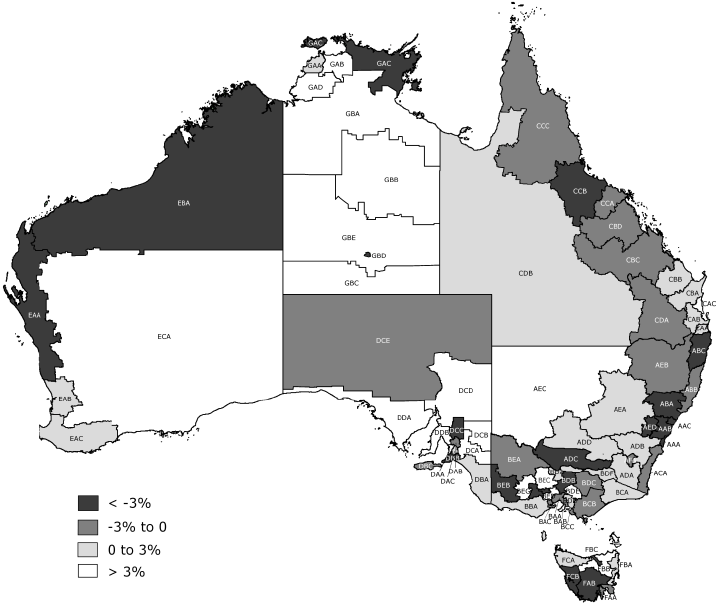
Exercise

Use the `infantgts` data to:

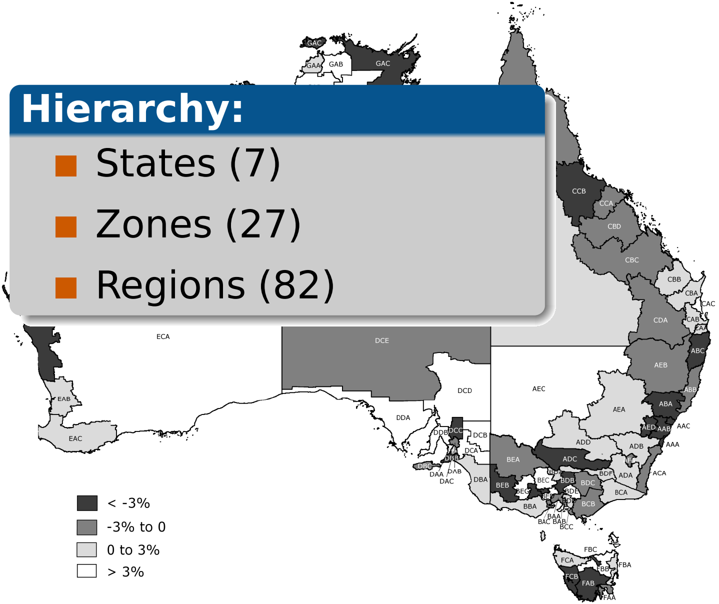
1. Plot various levels of aggregation.
2. Forecast the series using `auto.arima` with the default WLS reconciliation method.
3. Plot the reconciled forecasts at various levels of aggregation.

Outline

Australian tourism



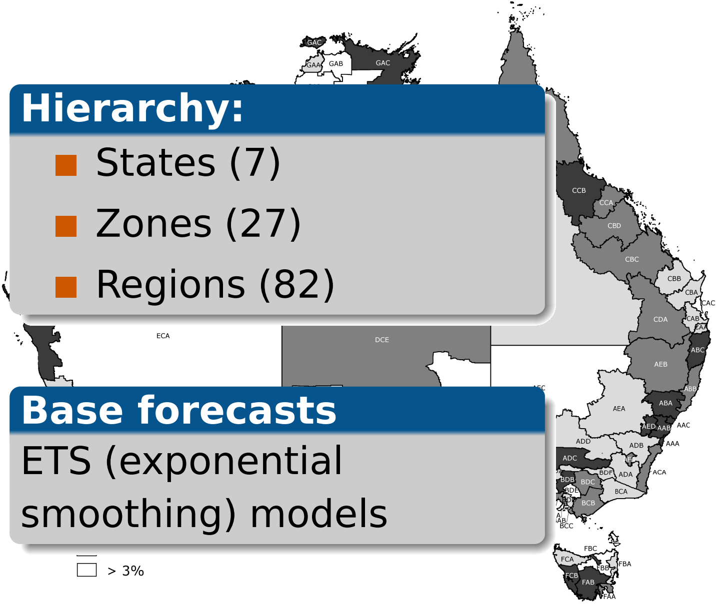
Australian tourism



Hierarchy:

- States (7)
- Zones (27)
- Regions (82)

Australian tourism



Hierarchy:

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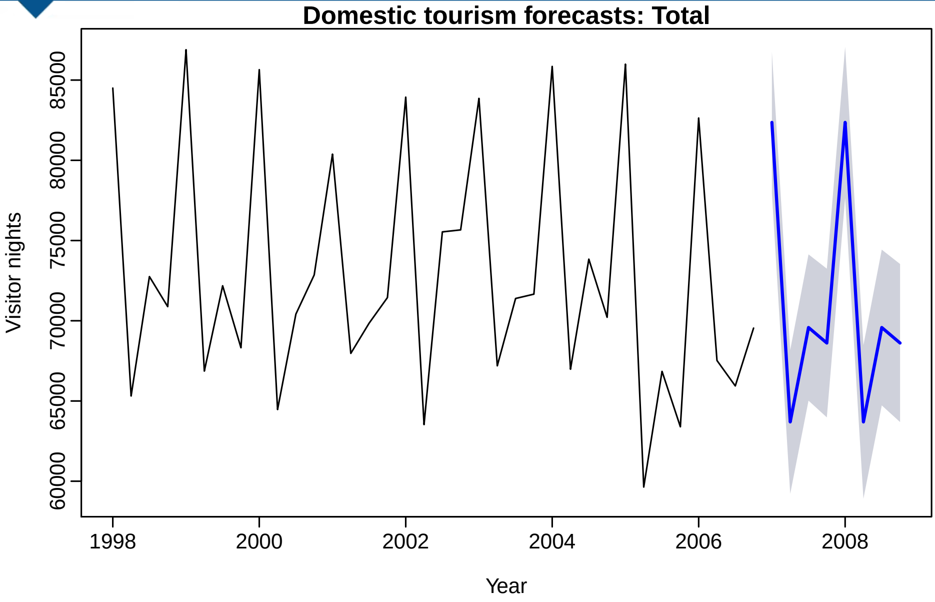
Base forecasts

ETS (exponential smoothing) models

Base forecasts



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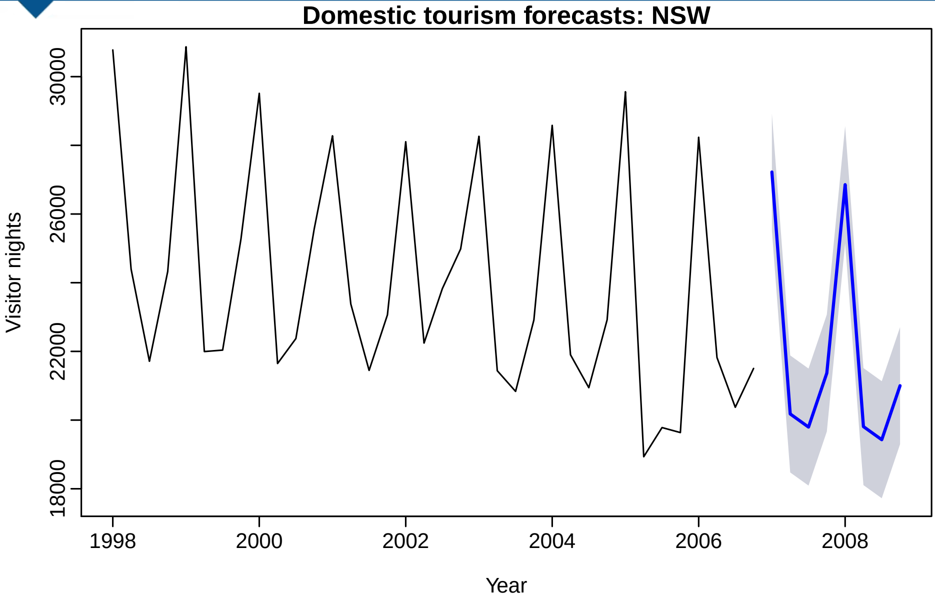

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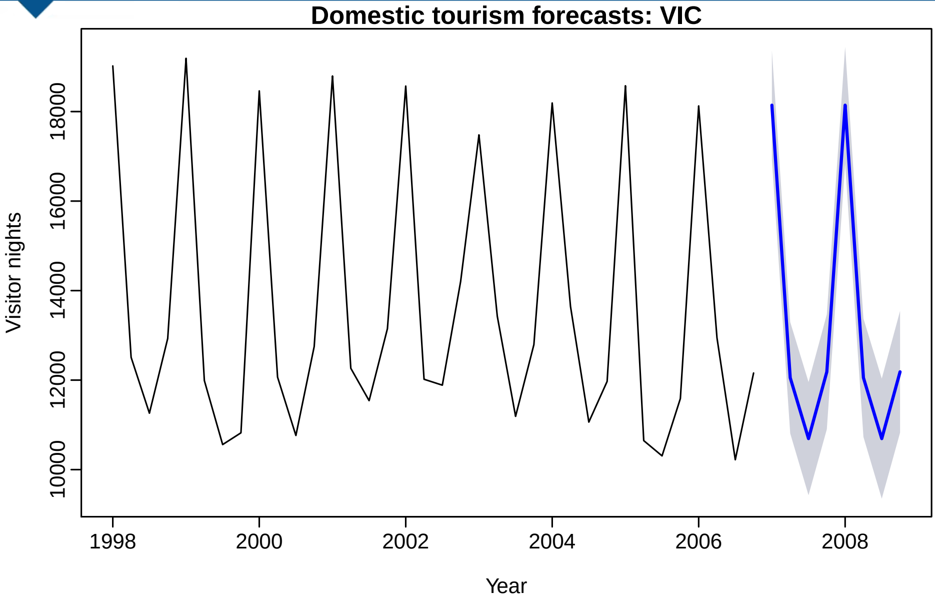

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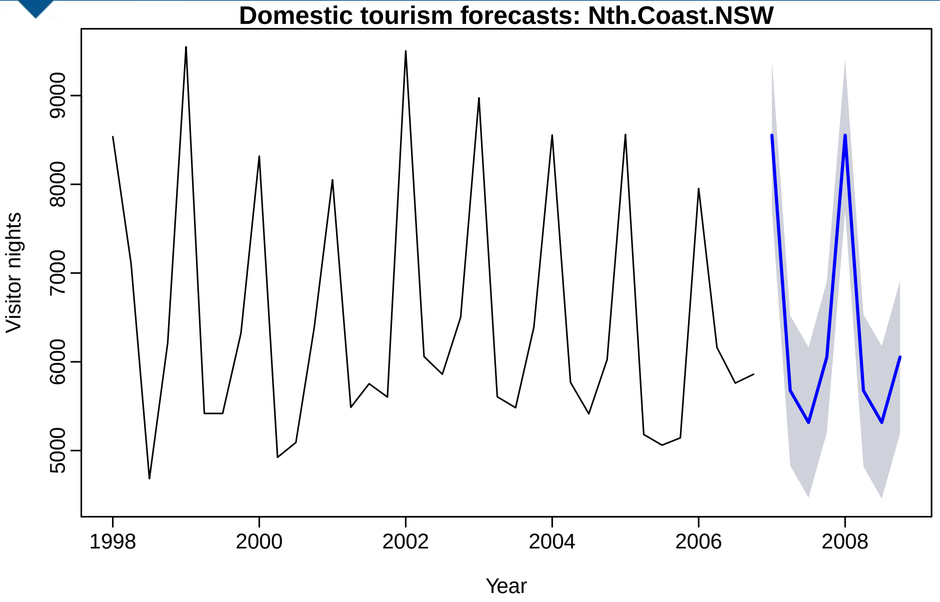

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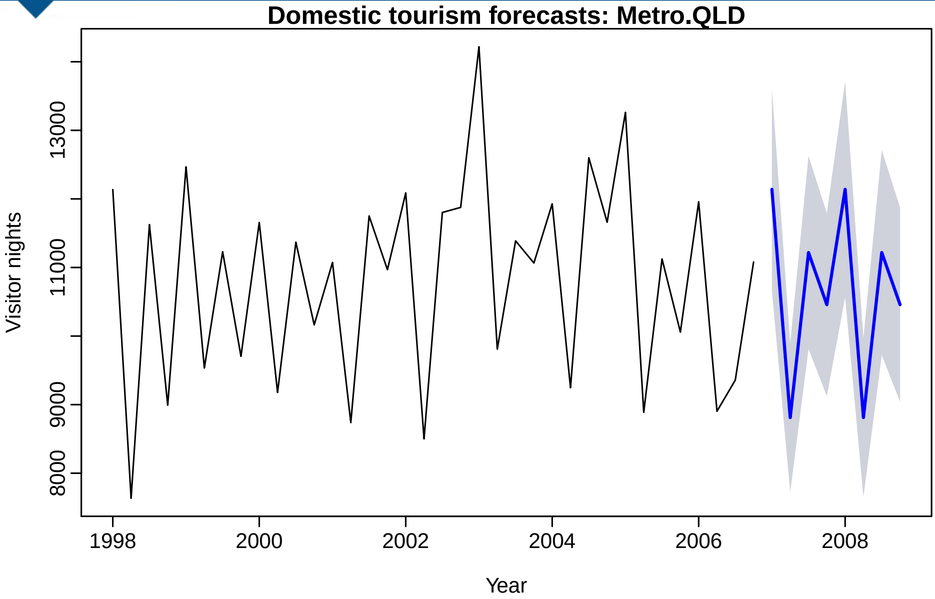

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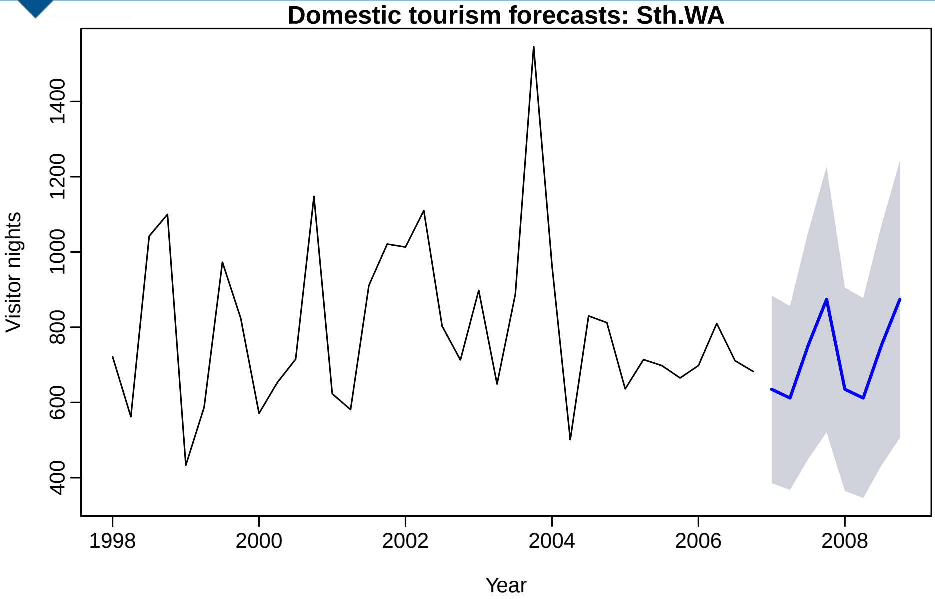

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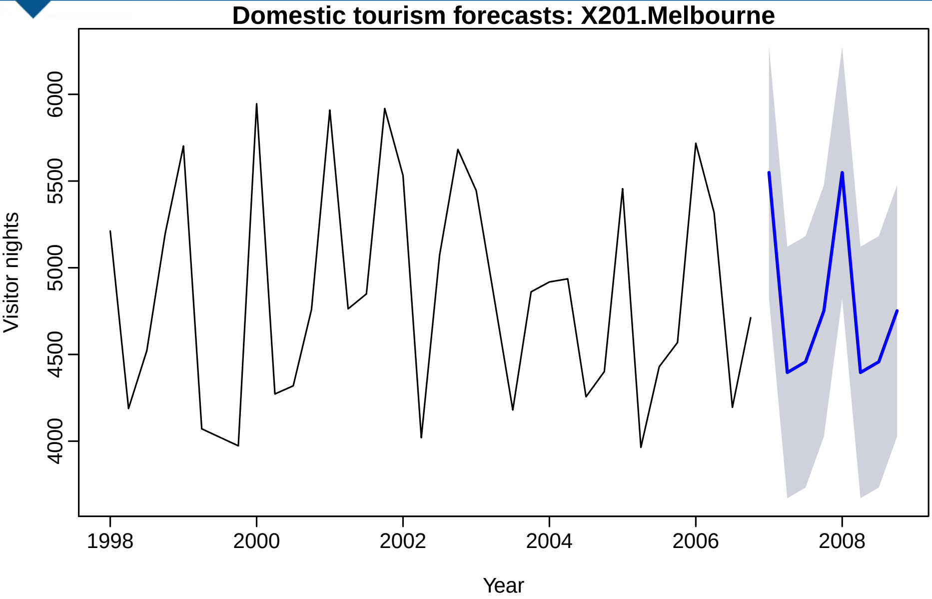

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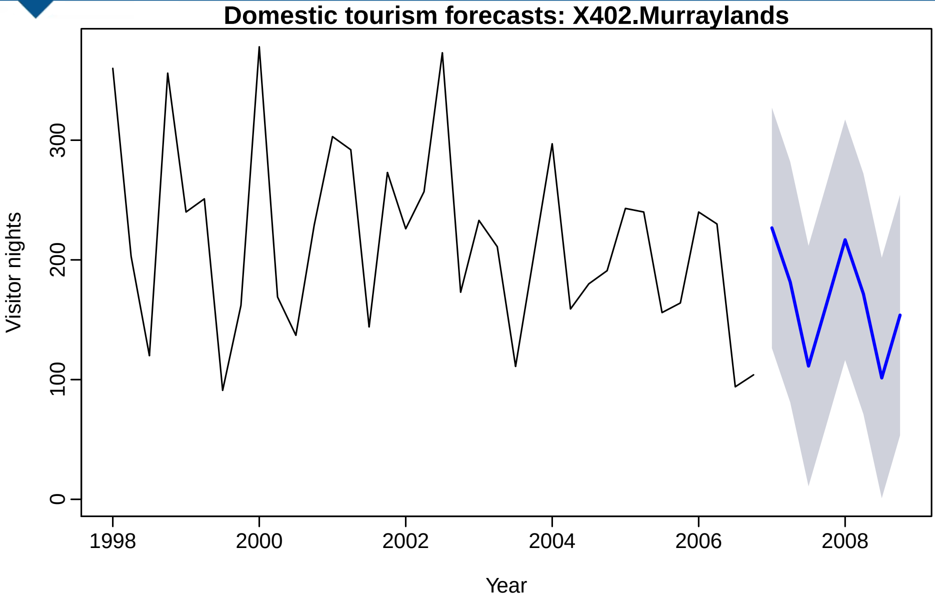

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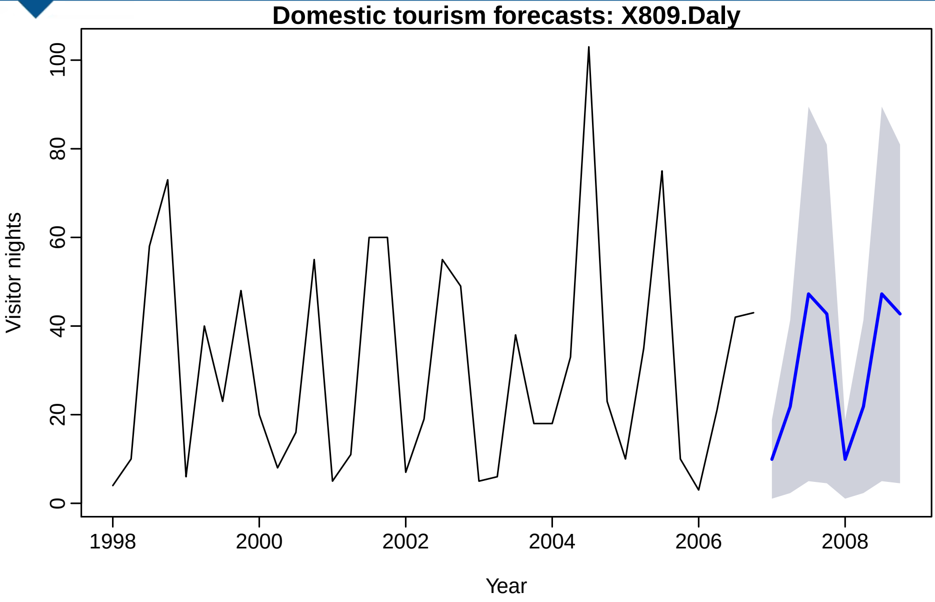
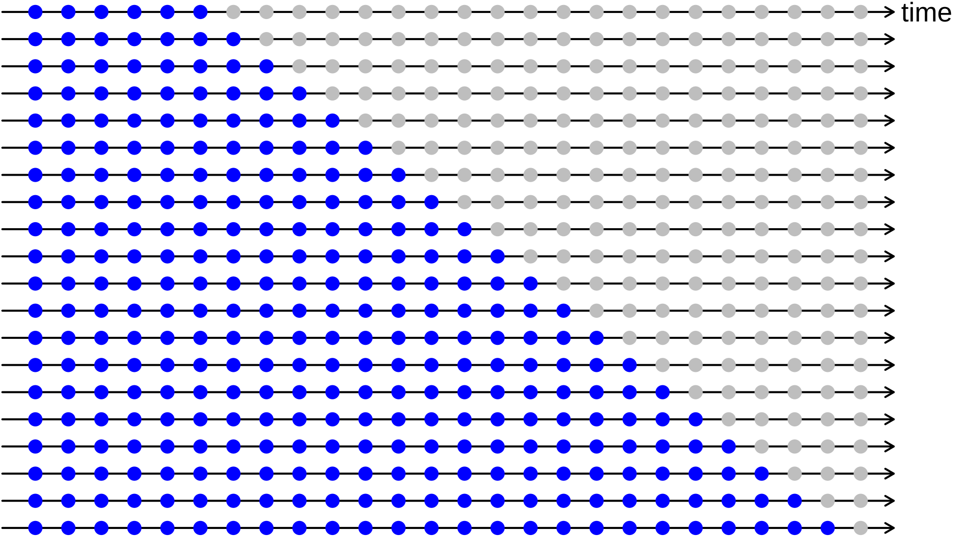


Forecast evaluation

Training sets



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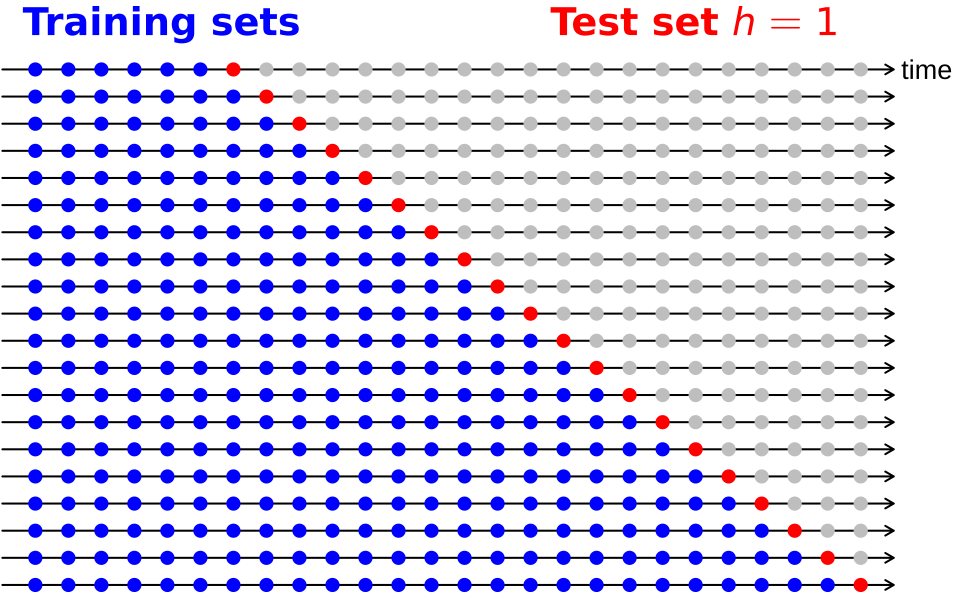
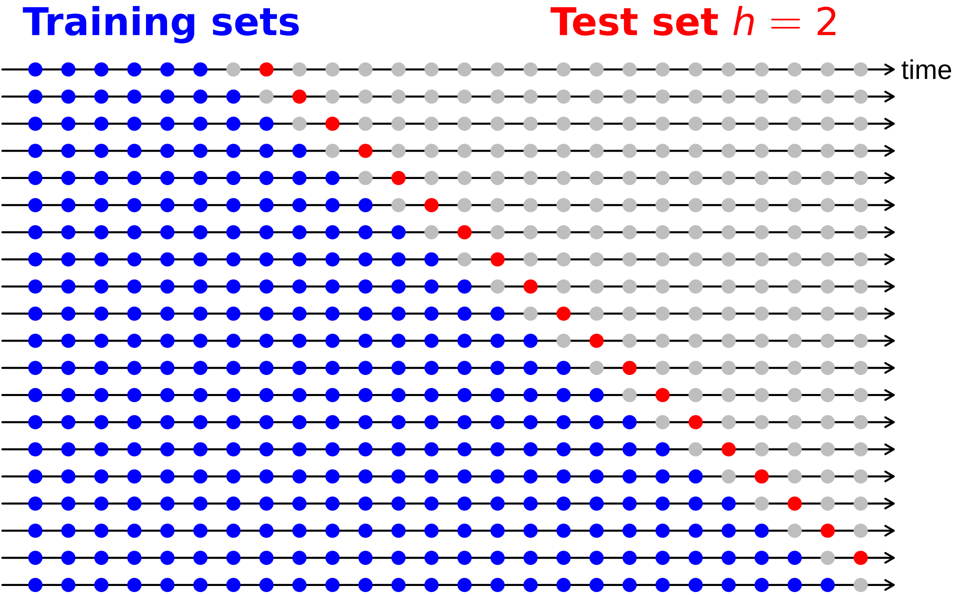


Forecast evaluation



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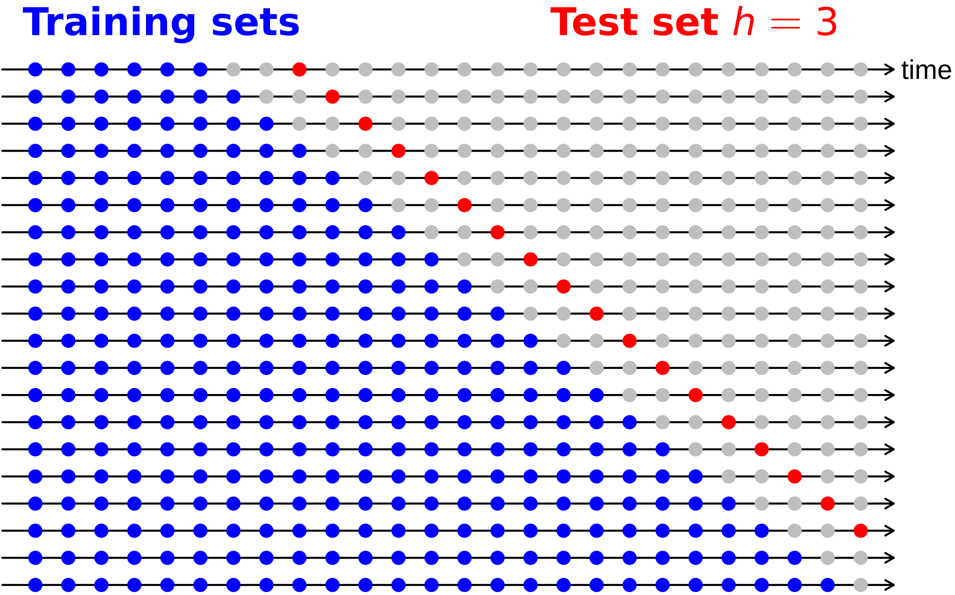

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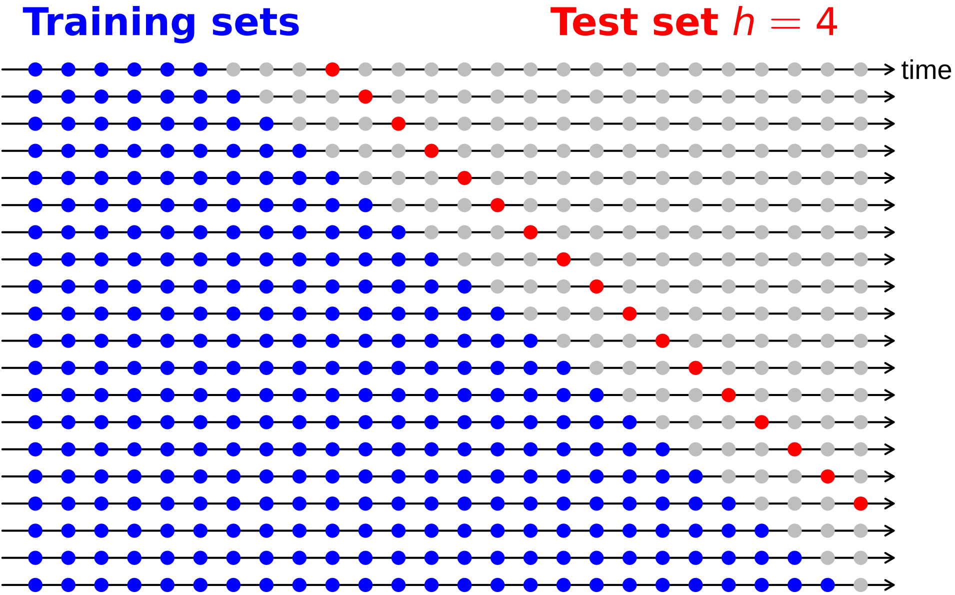

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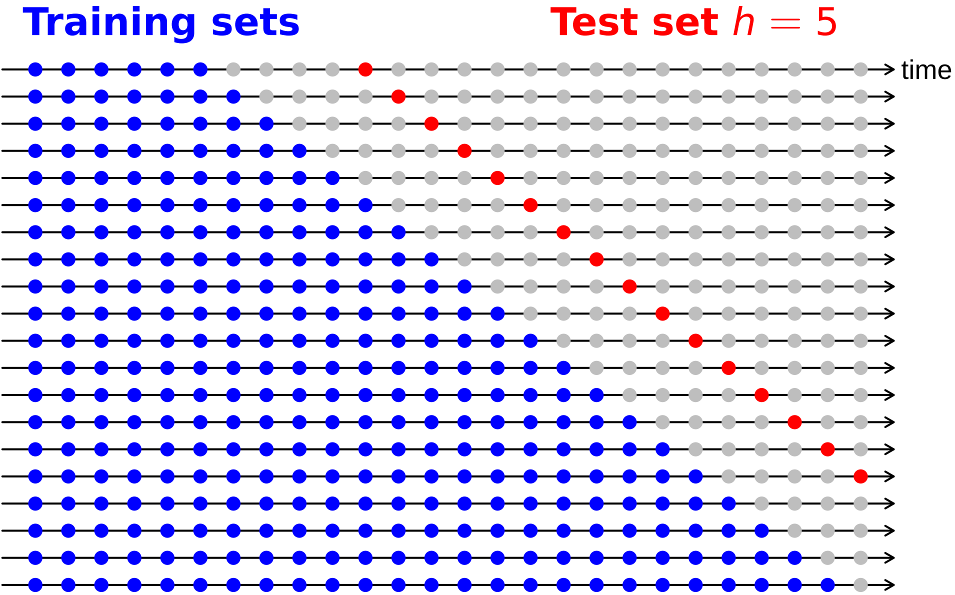

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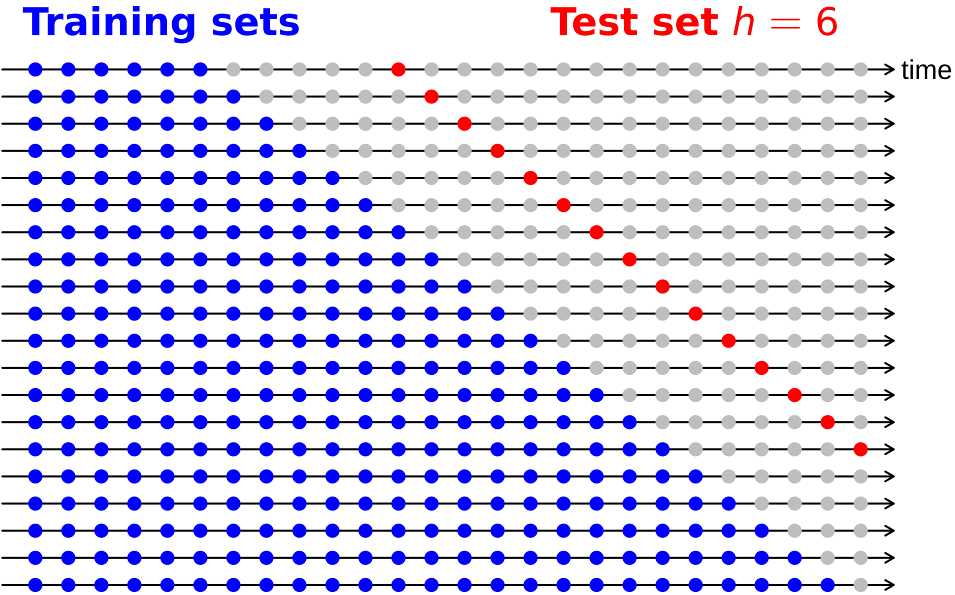
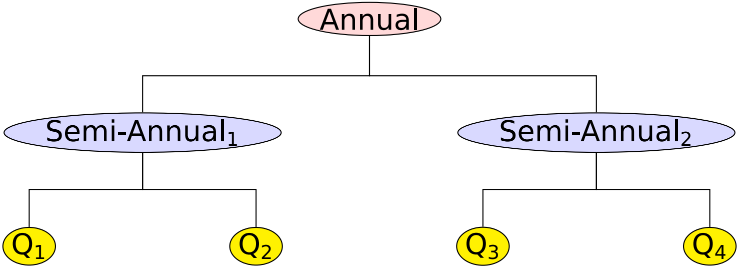


Hierarchy: states, zones, regions

			Forecast horizon				
RMSE	$h = 1$	$h = 2$	$h = 3$	$h = 4$	$h = 5$	$h = 6$	**Ave**
			Australia				
Base	1762.04	1770.29	1766.02	1818.82	1705.35	1721.17	**1757.28**
Bottom	1736.92	1742.69	1722.79	1752.74	1666.73	1687.43	**1718.22**
OLS	1747.60	1757.68	1751.77	1800.67	1686.00	1706.45	**1741.69**
WLS	1705.21	1715.87	1703.75	1729.56	1627.79	1661.24	**1690.57**
GLS	1704.64	1715.60	1705.31	1729.04	1626.36	1661.64	**1690.43**
			States				
Base	399.77	404.16	401.92	407.26	395.38	401.17	**401.61**
Bottom	404.29	406.95	404.96	409.02	399.80	401.55	**404.43**
OLS	404.47	407.62	405.43	413.79	401.10	404.90	**406.22**
WLS	398.84	402.12	400.71	405.03	394.76	398.23	**399.95**
GLS	398.84	402.16	400.86	405.03	394.59	398.22	**399.95**
			Regions				
Base	93.15	93.38	93.45	93.79	93.50	93.56	**93.47**
Bottom	93.15	93.38	93.45	93.79	93.50	93.56	**93.47**
OLS	93.28	93.53	93.64	94.17	93.78	93.88	**93.71**
WLS	93.02	93.32	93.38	93.72	93.39	93.53	**93.39**
GLS	92.98	93.27	93.34	93.66	93.34	93.46	**93.34**

Outline

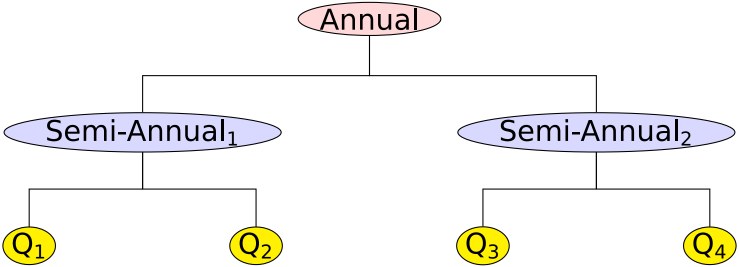
Temporal hierarchies



Basic idea:

- ➥ Forecast series at each available frequency.
- ➥ Optimally combine forecasts within the same year.

Temporal hierarchies



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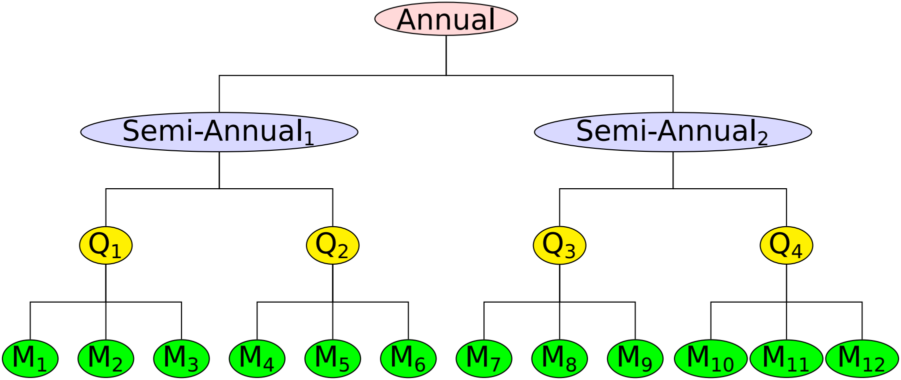
Monthly series



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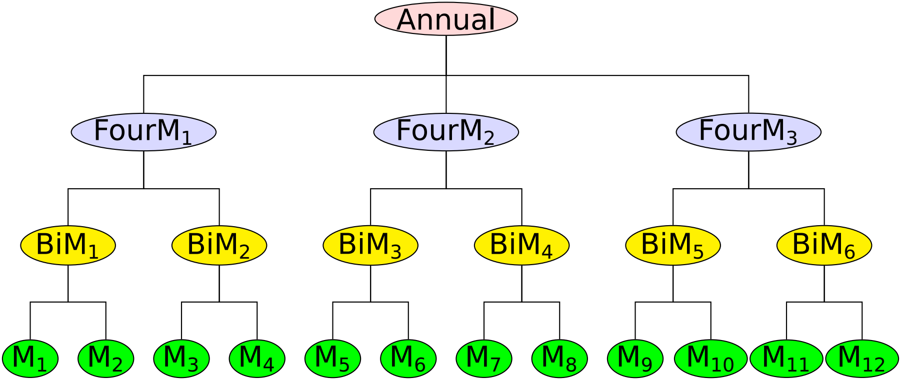
- $k = 3, 6, 12$ nodes
- Why not $k = 2, 3, 4, 6, 12$ nodes?

Monthly series



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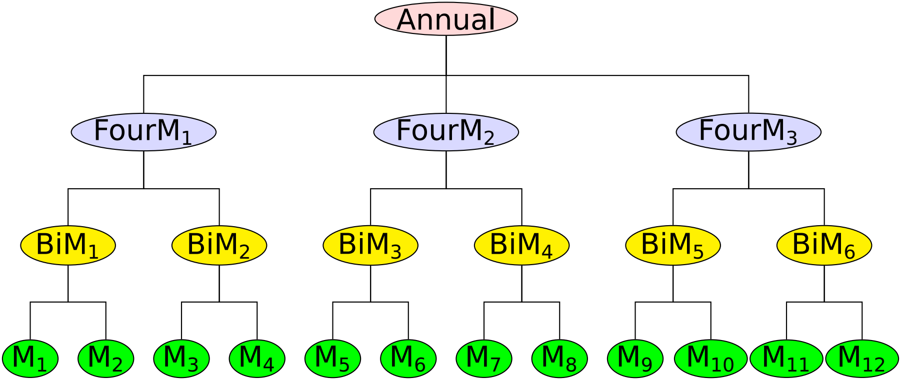

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Monthly data

$$\underbrace{\begin{pmatrix} A \\ SemiA_1 \\ SemiA_2 \\ FourM_1 \\ FourM_2 \\ FourM_3 \\ Q_1 \\ \vdots \\ Q_4 \\ BiM_1 \\ \vdots \\ BiM_6 \\ M_1 \\ \vdots \\ M_{12} \end{pmatrix}}_{(28\times 1)} = \underbrace{\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & & \vdots & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & & \vdots & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ & & & & & & & & & & & \\ & & & & & & \boldsymbol{I}_{12} & & & & & \\ & & & & & & & & & & & \end{pmatrix}}_{\boldsymbol{S}} \underbrace{\begin{pmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \\ M_5 \\ M_6 \\ M_7 \\ M_8 \\ M_9 \\ M_{10} \\ M_{11} \\ M_{12} \end{pmatrix}}_{\boldsymbol{B}_t}$$

In general

For a time series $y_1, \dots, y_T$, observed at frequency m, we generate aggregate series

$$y_j^{[k]} = \sum_{t=1+(j-1)k}^{jk} y_t, \qquad \text{for } j = 1, \dots, \lfloor T/k \rfloor$$

- $k \in F(m) = \{\text{factors of } m\}$.
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WLS weights

Hierarchy variance scaling Λ_H: diagonal.

Series variance scaling Λ_V: elements equal within aggregation level.

Structural scaling $\Lambda_S = \text{diag}(\boldsymbol{S1})$: elements equal to # nodes at each level.

- Depends only on seasonal period m.
- Independent of data and model.
- Allows forecasts where no errors available.

Quarterly example

$$\Lambda_H = \text{diag}\big(\hat\sigma_A^2,\ \hat\sigma_{S_1}^2,\ \hat\sigma_{S_2}^2,\ \hat\sigma_{Q_1}^2,\ \hat\sigma_{Q_2}^2,\ \hat\sigma_{Q_3}^2,\ \hat\sigma_{Q_4}^2\big)$$

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Experimental setup:

- M3 forecasting competition (Makridakis and Hibon, 2000, *IJF*). In total 3003 series.
- 1,428 monthly series with a test sample of 12 observations each.
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Results: Monthly

MAE percent difference relative to base

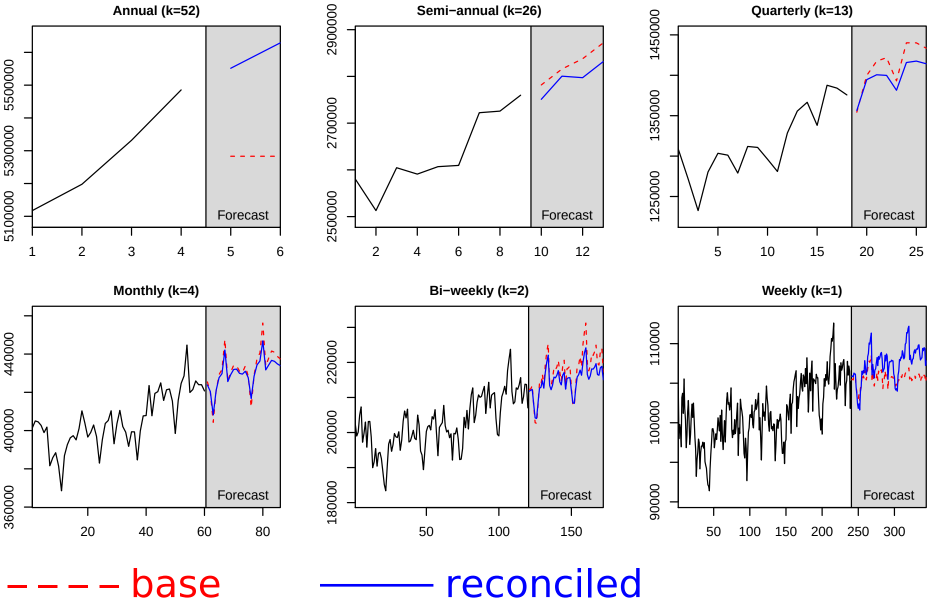
	max h	BU	WLS_H	WLS_V	WLS_S
Annual	1	-19.6	-22.0	-22.0	-25.1
Semi-annual	3	0.6	-4.0	-3.6	-5.4
Four-monthly	4	2.0	-2.4	-2.2	-3.0
Quarterly	6	2.4	-1.6	-1.7	-2.8
Bi-monthly	9	0.7	-2.9	-3.3	-4.3
Monthly	18	0.0	-2.2	-3.2	-3.9

Results: Quarterly

MAE percent difference relative to base

	max h	BU	WLS_H	WLS_V	WLS_S
Annual	1	-20.9	-22.7	-22.8	-22.7
Semi-annual	3	-4.5	-6.0	-6.2	-4.8
Quarterly	6	0.0	-0.2	-1.1	-0.3

UK Accidents and Emergency Demand



UK Accidents and Emergency Demand

1. Type 1 Departments — Major A&E
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3. Type 3 Departments — Other A&E/Minor Injury
4. Total Attendances
5. Type 1 Departments — Major A&E $>$ 4 hrs
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8. Total Attendances $>$ 4 hrs
9. Emergency Admissions via Type 1 A&E
10. Total Emergency Admissions via A&E
11. Other Emergency Admissions (i.e., not via A&E)
12. Total Emergency Admissions
13. Number of patients spending $>$ 4 hrs from decision to admission

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- **Minimum training set**: all data except the last year
- Base forecasts using `auto.arima()`.
- Reconciled using WLS_V.
- Mean Absolute Scaled Errors for 1, 4 and 13 weeks ahead using a rolling origin.

Aggr. Level	h	Base	Reconciled	Change
Weekly	1	1.6	1.3	−17.2%
Weekly	4	1.9	1.5	−18.6%
Weekly	13	2.3	1.9	−16.2%
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References

RJ Hyndman, RA Ahmed, G Athanasopoulos, and HL Shang (2011). "Optimal combination forecasts for hierarchical time series". *Computational statistics & data analysis* **55**(9), 2579–2589.

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