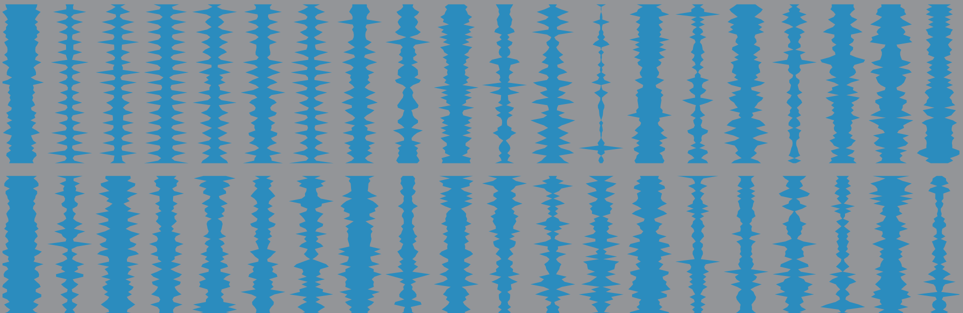
MONASH BUSINESS SCHOOL

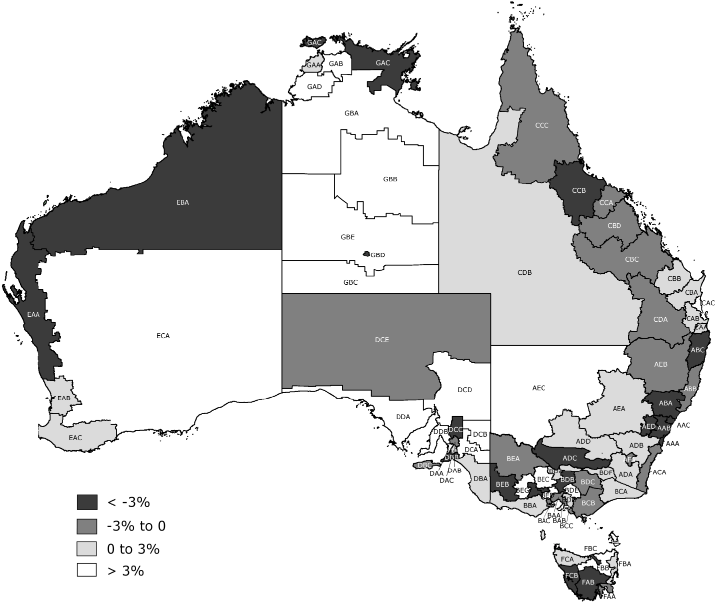
Rob J Hyndman

Visualizing and forecasting big time series data



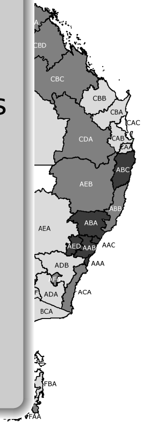
Outline

1. Australian tourism demand



1. Australian tourism demand

- Quarterly data on visitor night from 1998:Q1 – 2013:Q4
- From: *National Visitor Survey*, based on annual interviews of 120,000 Australians aged 15+, collected by Tourism Research Australia.
- Split by 7 states, 27 zones and 76 regions (a geographical hierarchy)
- Also split by purpose of travel
  - Holiday
  - Visiting friends and relatives (VFR)
  - Business
  - Other
- 304 bottom-level series



2. Labour market participation

Australia and New Zealand Standard Classification of Occupations

- 8 major groups
  - 43 sub-major groups
    - 97 minor groups
      - – 359 unit groups
        - * 1023 occupations

Example: statistician

2 Professionals

22 Business, Human Resource and Marketing Professionals

224 Information and Organisation Professionals

2241 Actuaries, Mathematicians and Statisticians

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3. Spectacle sales

- Monthly UK sales data from 2000 – 2014
- Provided by a large spectacle manufacturer
- Split by brand (26), gender (3), price range (6), materials (4), and stores (600)
- About 1 million bottom-level series

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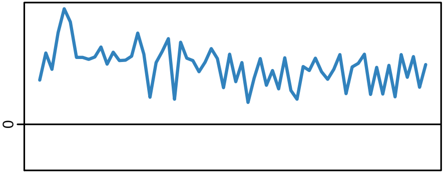
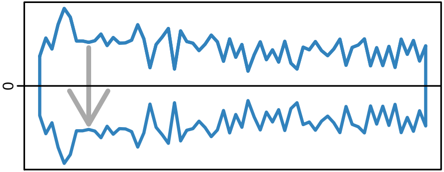
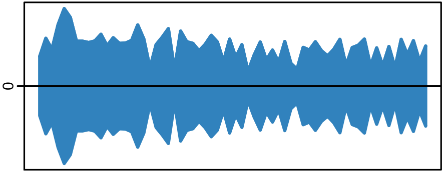
Kite diagrams



Line graph profile

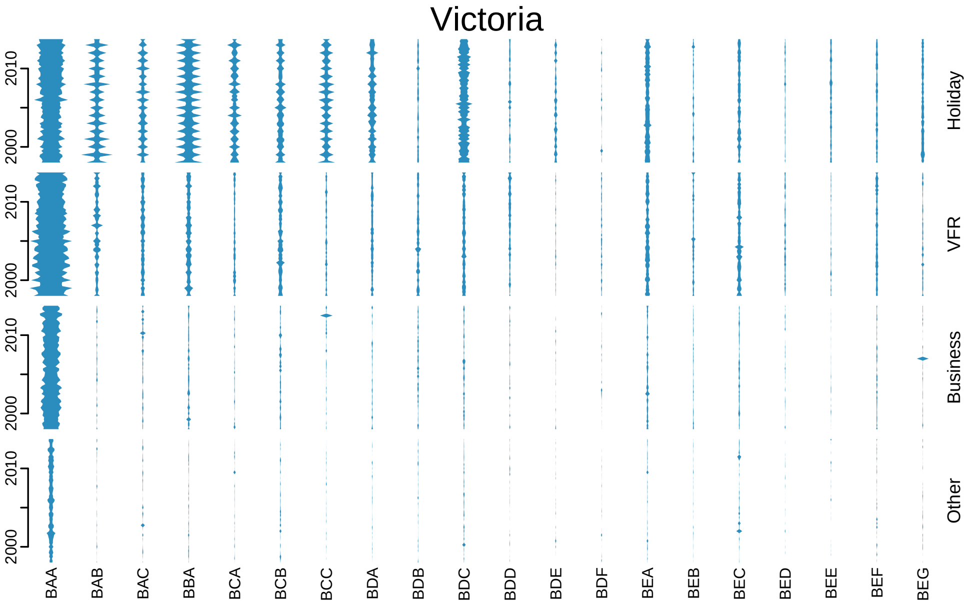


Duplicate & flip around the horizontal axis



Fill the colour

Kite diagrams: Victorian tourism



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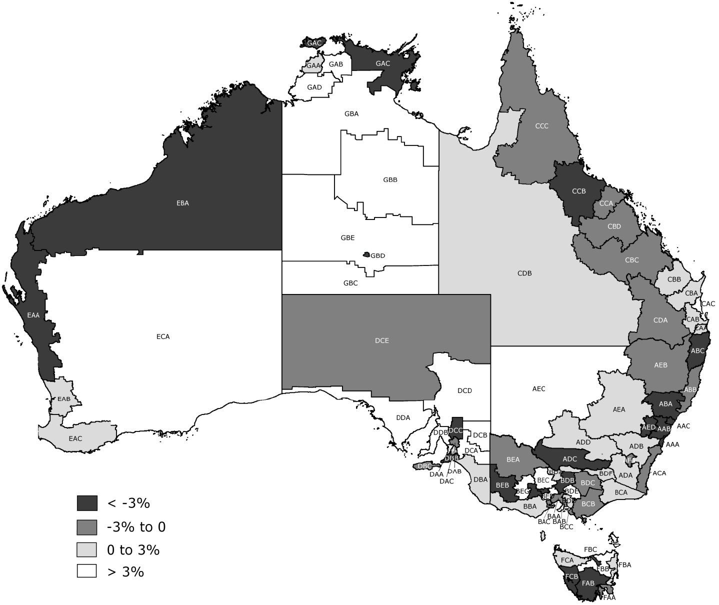

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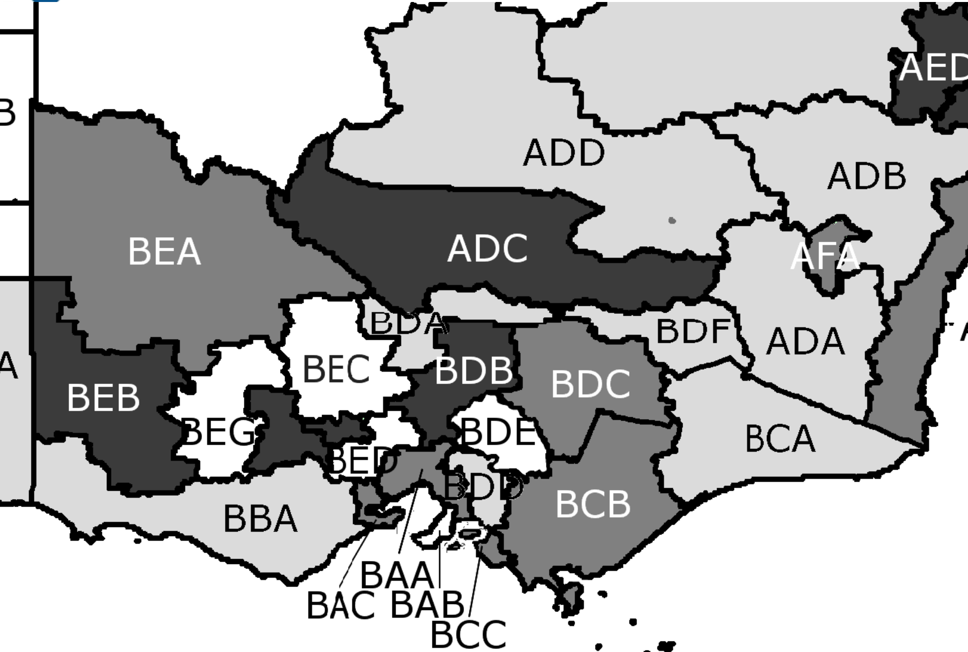

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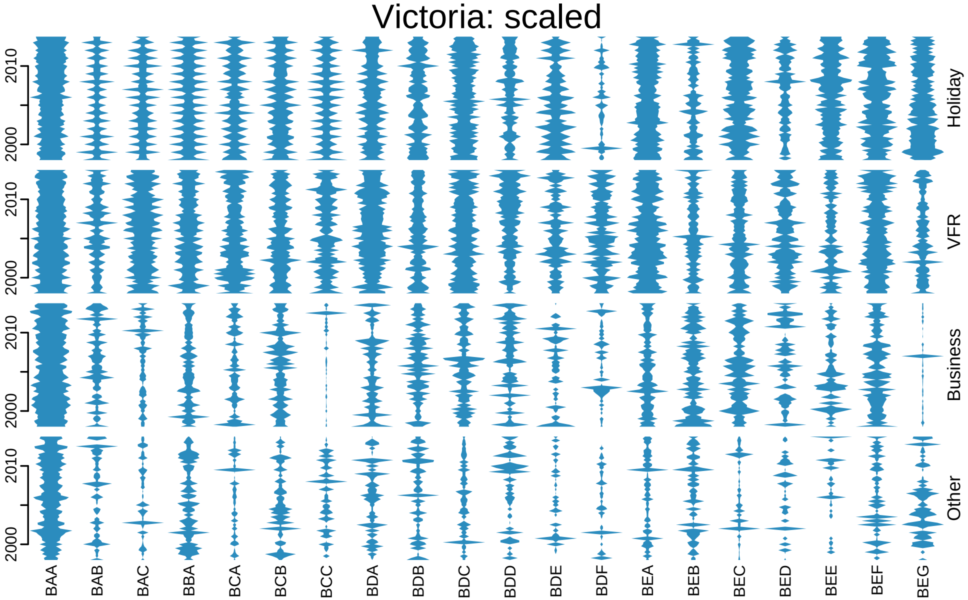
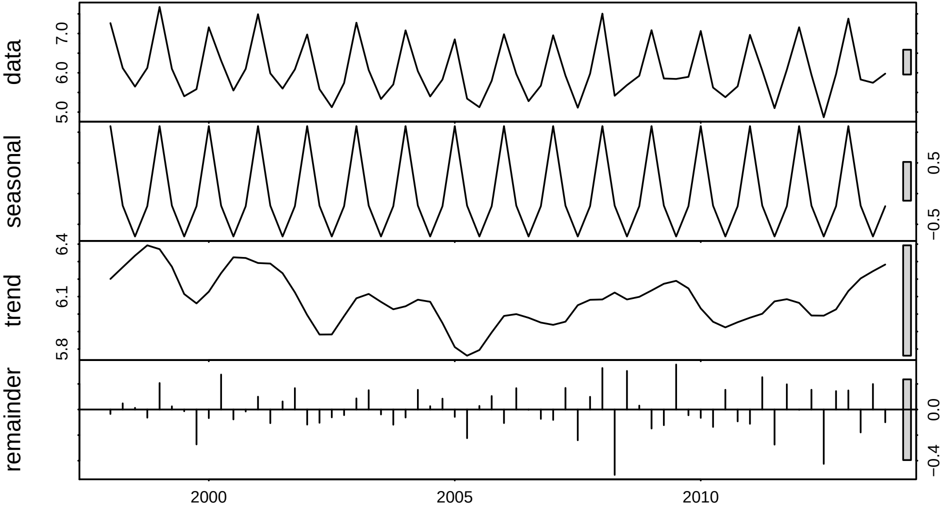


An STL decomposition

STL decomposition of tourism demand for holidays in Peninsula



Seasonal stacked bar chart

- Place positive values above the origin while negative values below the origin
- Map the bar length to the magnitude
- Encode quarters by colours

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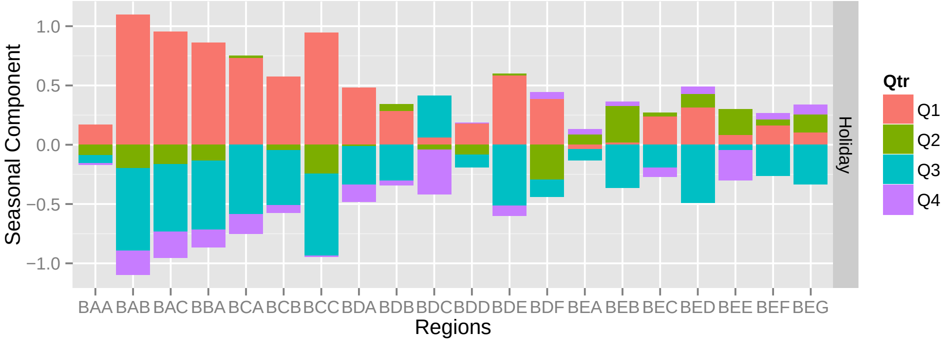
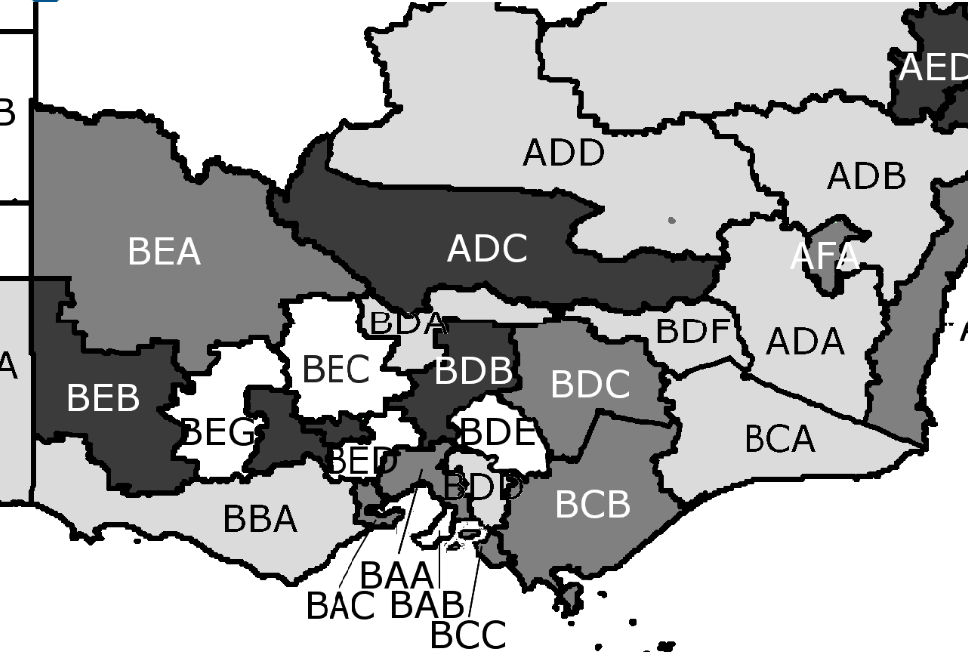


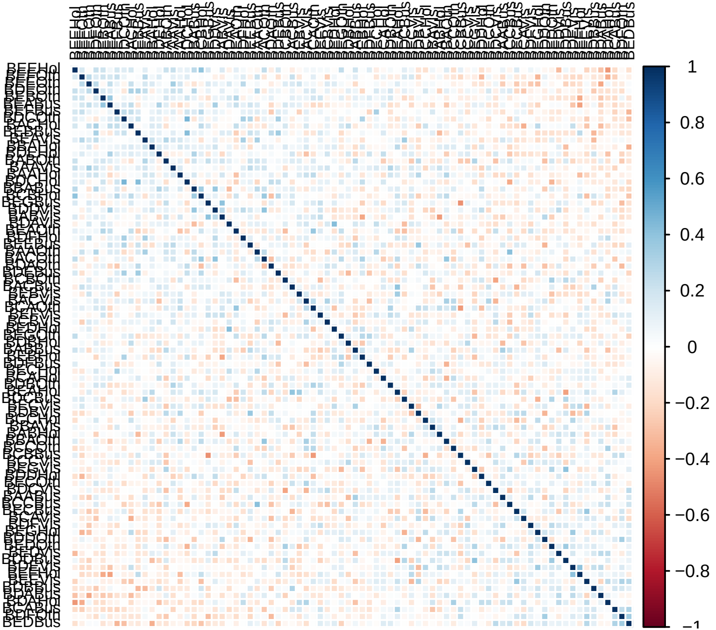
Seasonal stacked bar chart: VIC



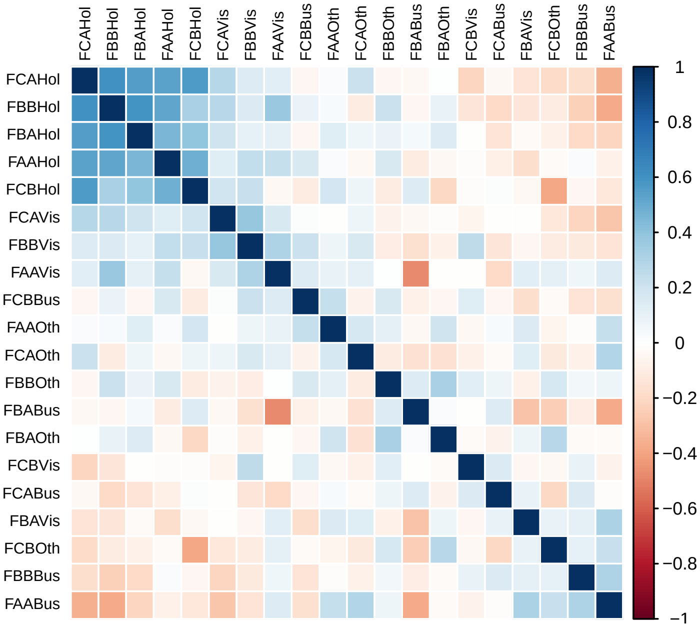
Corrgram of remainder

- Compute the correlations among the remainder components
- Render both the sign and magnitude using a colour mapping of two hues
- Order variables according to the first principal component of the correlations.

Corrgram of remainder: VIC



Corrgram of remainder: TAS



Feature analysis

- Summarize each time series with a feature vector:
  - strength of trend
  - lumpiness (variance of annual variances of remainder)
  - strength of seasonality
  - size of seasonal peak
  - size of seasonal trough
  - ACF1
  - linearity of trend
  - curvature of trend
  - spectral entropy
- Do PCA on feature matrix

Feature analysis

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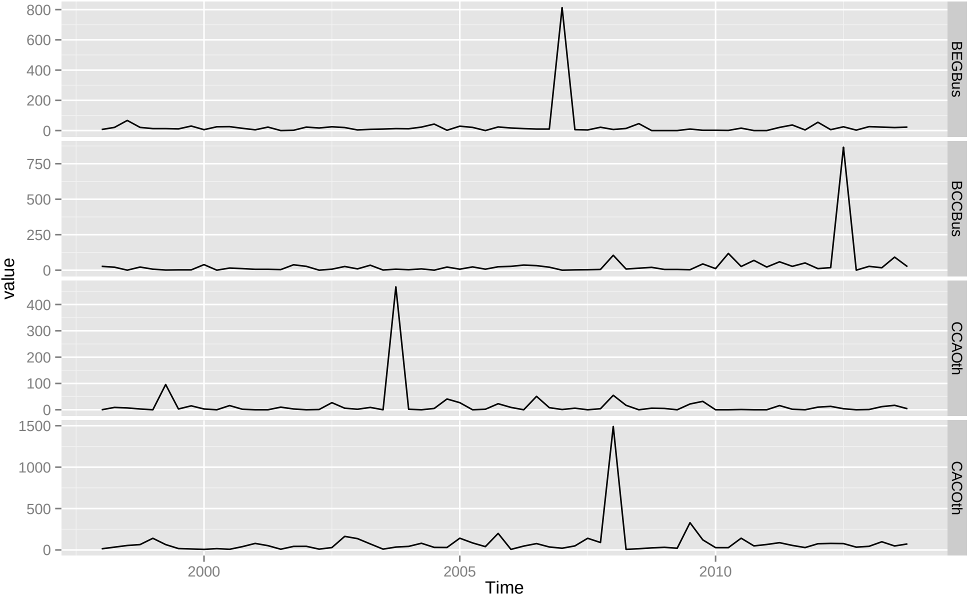

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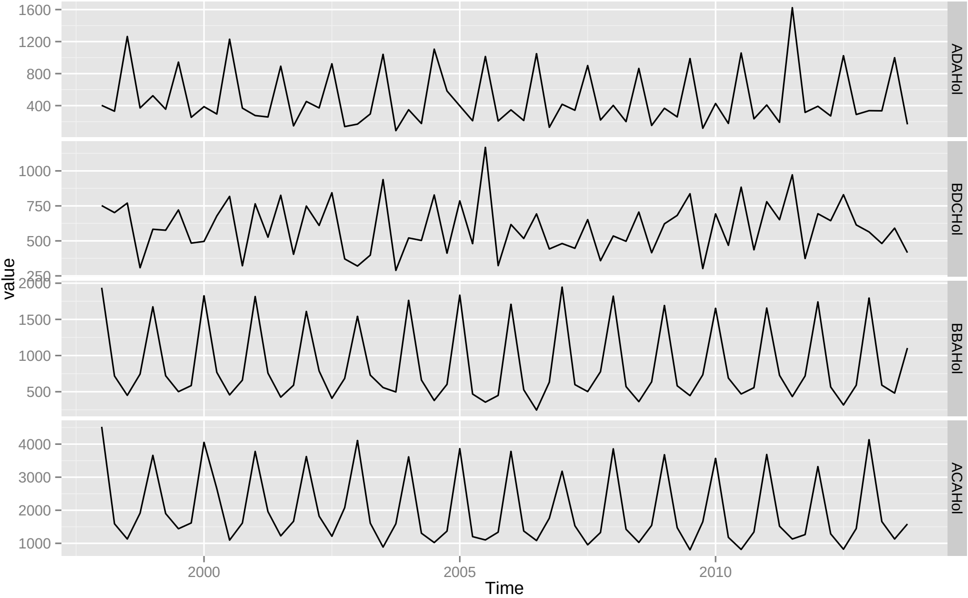

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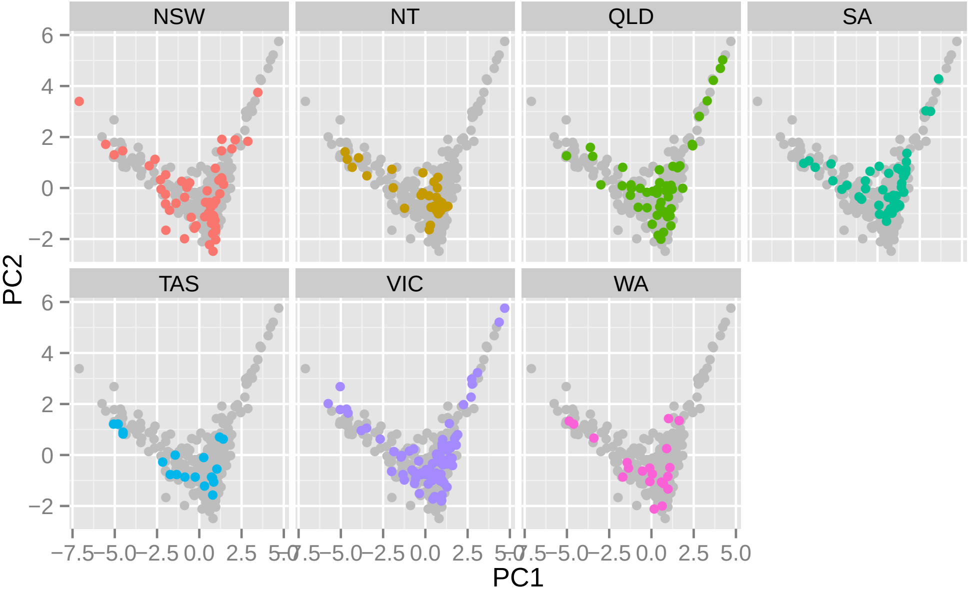

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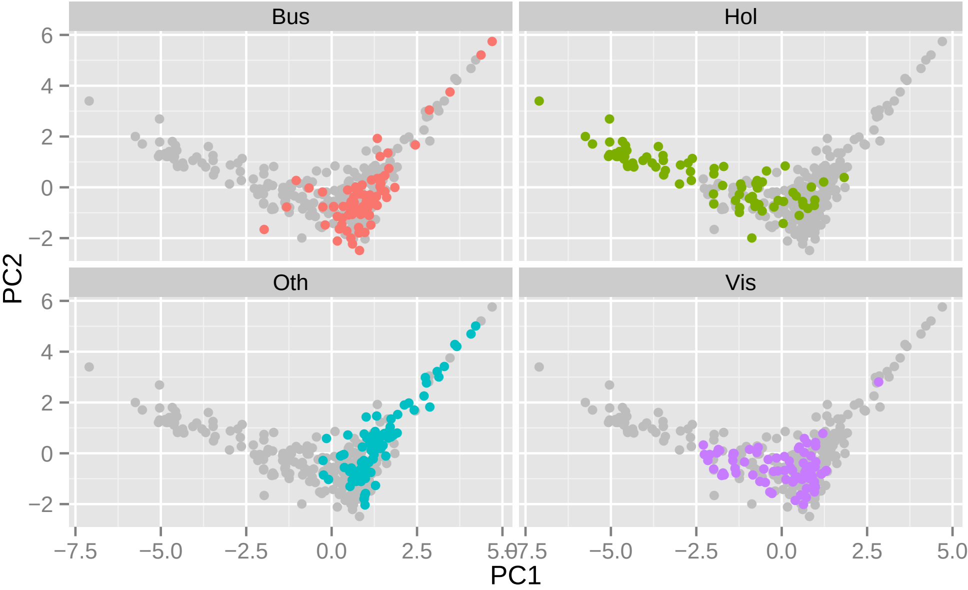
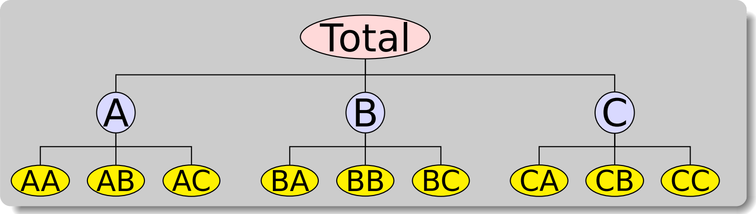


Outline

Hierarchical time series

A **hierarchical time series** is a collection of several time series that are linked together in a hierarchical structure.



Examples

- Net labour turnover
- Tourism by state and region

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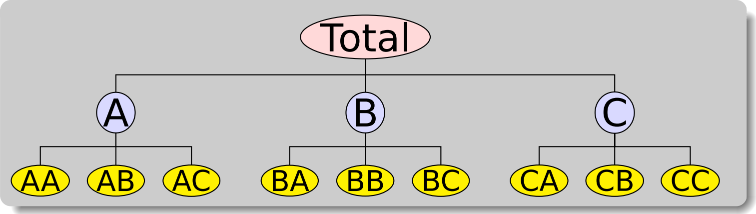

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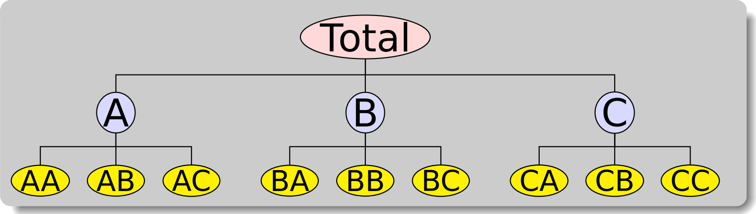

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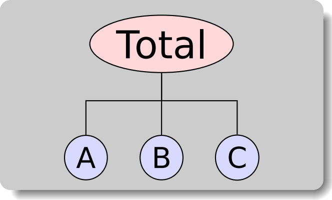

Y_t : observed aggregate of all series at time t.

$Y_{X,t}$: observation on series X at time t.

$\boldsymbol{B}_t$: vector of all series at bottom level in time t.

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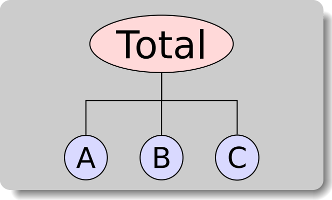

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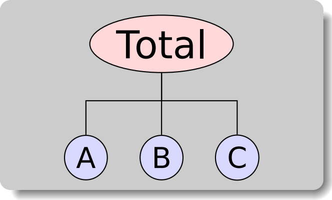

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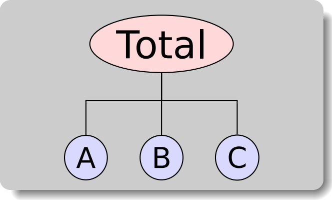

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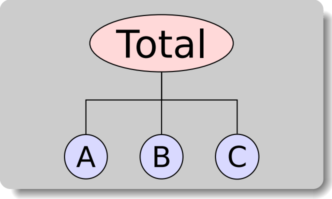

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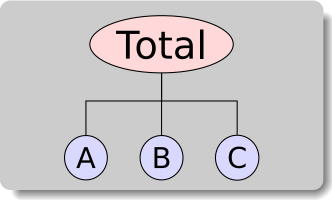

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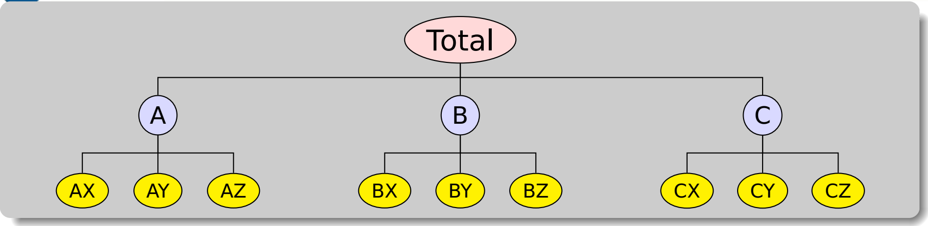

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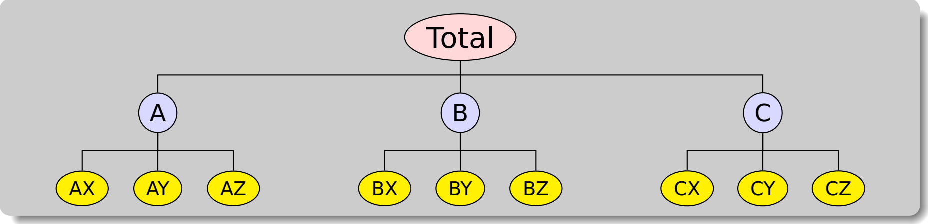

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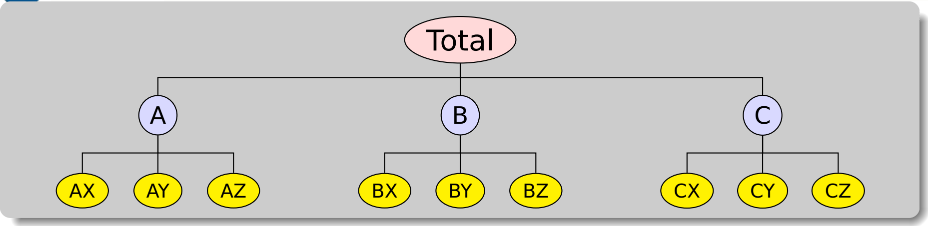

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$$\boldsymbol{y}_t = \boldsymbol{S}\boldsymbol{B}_t$$

Forecasting notation

Let $\hat{\boldsymbol{y}}_n(h)$ be vector of initial h-step forecasts, made at time n, stacked in same order as $\boldsymbol{y}_t$.

(They may not add up.)

Reconciled forecasts are of the form:

$$\tilde{\boldsymbol{y}}_n(h) = \boldsymbol{S}\boldsymbol{P}\hat{\boldsymbol{y}}_n(h)$$

for some matrix $\boldsymbol{P}$.

- $\boldsymbol{P}$ extracts and combines base forecasts $\hat{\boldsymbol{y}}_n(h)$ to get bottom-level forecasts.

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General properties: bias

$$\tilde{\boldsymbol{y}}_n(h) = \boldsymbol{S}\boldsymbol{P}\hat{\boldsymbol{y}}_n(h)$$

Assume: base forecasts $\hat{\boldsymbol{y}}_n(h)$ are unbiased:

$$\mathrm{E}[\hat{\boldsymbol{y}}_n(h)|\boldsymbol{y}_1,\dots,\boldsymbol{y}_n] = \mathrm{E}[\boldsymbol{y}_{n+h}|\boldsymbol{y}_1,\dots,\boldsymbol{y}_n]$$

- Let $\hat{\boldsymbol{B}}_n(h)$ be bottom level base forecasts with $\beta_n(h) = \mathrm{E}[\hat{\boldsymbol{B}}_n(h)|\boldsymbol{y}_1,\dots,\boldsymbol{y}_n]$.
- Then $\mathrm{E}[\hat{\boldsymbol{y}}_n(h)] = \boldsymbol{S}\beta_n(h)$.
- We want the revised forecasts to be unbiased: $\mathrm{E}[\tilde{\boldsymbol{y}}_n(h)] = \boldsymbol{S}\boldsymbol{P}\boldsymbol{S}\beta_n(h) = \boldsymbol{S}\beta_n(h)$.

Revised forecasts are unbiased iff $\boldsymbol{S}\boldsymbol{P}\boldsymbol{S} = \boldsymbol{S}$.

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Assume: base forecasts $\hat{\boldsymbol{y}}_n(h)$ are unbiased:

$$\mathrm{E}[\hat{\boldsymbol{y}}_n(h)|\boldsymbol{y}_1,\dots,\boldsymbol{y}_n] = \mathrm{E}[\boldsymbol{y}_{n+h}|\boldsymbol{y}_1,\dots,\boldsymbol{y}_n]$$

- Let $\hat{\boldsymbol{B}}_n(h)$ be bottom level base forecasts with $\beta_n(h) = \mathrm{E}[\hat{\boldsymbol{B}}_n(h)|\boldsymbol{y}_1,\dots,\boldsymbol{y}_n]$.
- Then $\mathrm{E}[\hat{\boldsymbol{y}}_n(h)] = \boldsymbol{S}\beta_n(h)$.
- We want the revised forecasts to be unbiased: $\mathrm{E}[\tilde{\boldsymbol{y}}_n(h)] = \boldsymbol{S}\boldsymbol{P}\boldsymbol{S}\beta_n(h) = \boldsymbol{S}\beta_n(h)$.

Revised forecasts are unbiased iff $\boldsymbol{S}\boldsymbol{P}\boldsymbol{S} = \boldsymbol{S}$.

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$$\Sigma_h = \text{Var}[\hat{\boldsymbol{y}}_n(h)|\boldsymbol{y}_1, \dots, \boldsymbol{y}_n]$$

Then the variance of the revised forecasts is given by

$$\text{Var}[\tilde{\boldsymbol{y}}_n(h)|\boldsymbol{y}_1, \dots, \boldsymbol{y}_n] = \boldsymbol{S}\boldsymbol{P}\Sigma_h\boldsymbol{P}'\boldsymbol{S}'.$$

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BLUF via trace minimization

Theorem

For any $\boldsymbol{P}$ satisfying $\boldsymbol{SPS} = \boldsymbol{S}$, then
$$\min_{\boldsymbol{P}} = \text{trace}[\boldsymbol{SP}\Sigma_h\boldsymbol{P}'\boldsymbol{S}']$$
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- $\Sigma_h^{\dagger}$ is generalized inverse of Σ_h.

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Revised forecasts | Base forecasts

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Base forecasts

- Equivalent to GLS estimate of regression $\hat{\boldsymbol{y}}_n(h) = \boldsymbol{S}\beta_n(h) + \varepsilon_h$ where $\varepsilon \sim N(0, \Sigma_h)$.

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Revised forecasts

Base forecasts

Solution 1: OLS

- Assume $\varepsilon_h \approx \boldsymbol{S}\varepsilon_{B,h}$ where $\varepsilon_{B,h}$ is the forecast error at bottom level.
- Then $\Sigma_h \approx \boldsymbol{S}\Omega_h\boldsymbol{S}'$ where $\Omega_h = \text{Var}(\varepsilon_{B,h})$.
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Revised forecasts

Base forecasts

Solution 2: WLS

- Suppose we approximate Σ_1 by its diagonal.
- Easy to estimate, and places weight where we have best forecasts.
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Challenges

$$\tilde{\boldsymbol{y}}_n(h) = \boldsymbol{S}(\boldsymbol{S}'\Lambda\boldsymbol{S})^{-1}\boldsymbol{S}'\Lambda\hat{\boldsymbol{y}}_n(h)$$

- Computational difficulties in big hierarchies due to size of the $\boldsymbol{S}$ matrix and singular behavior of $(\boldsymbol{S}'\Lambda\boldsymbol{S})$.
- Loss of information in ignoring covariance matrix in computing point forecasts.
- Still need to estimate covariance matrix to produce prediction intervals.

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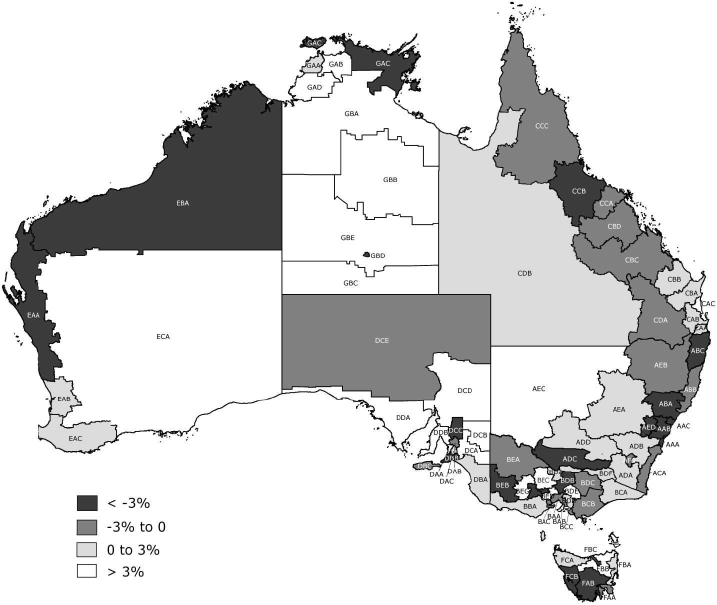
Outline

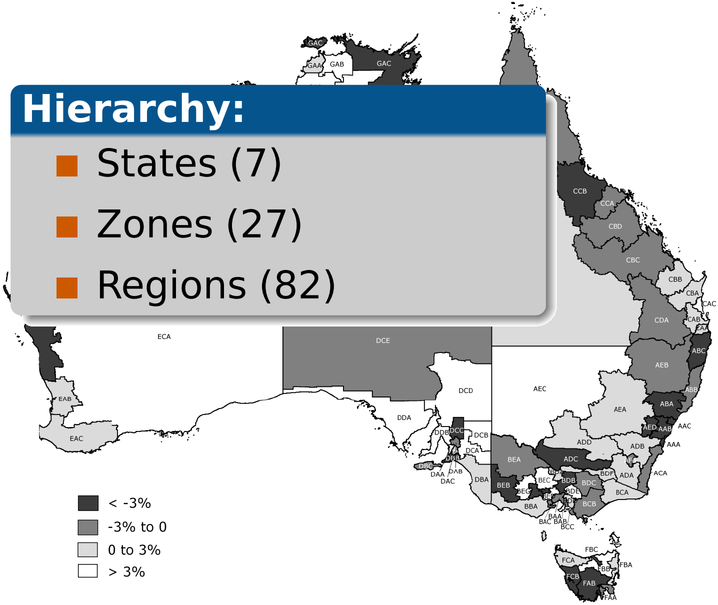
Australian tourism



Australian tourism

Hierarchy:

- States (7)
- Zones (27)
- Regions (82)



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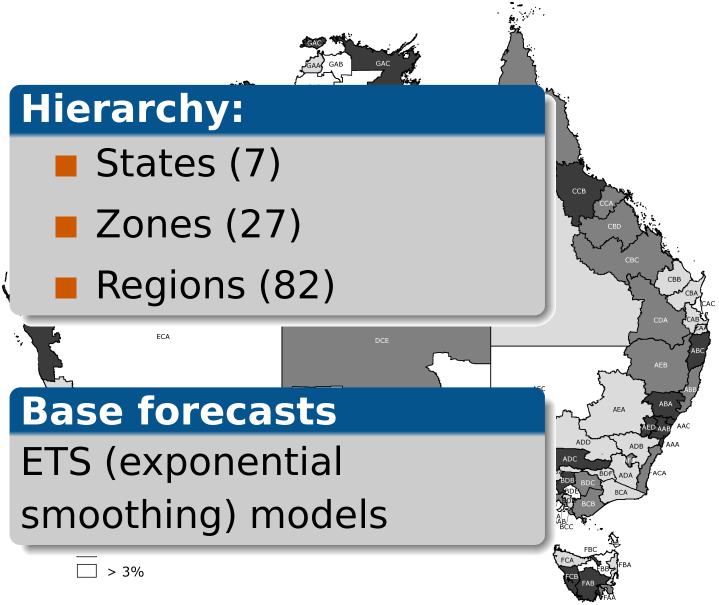

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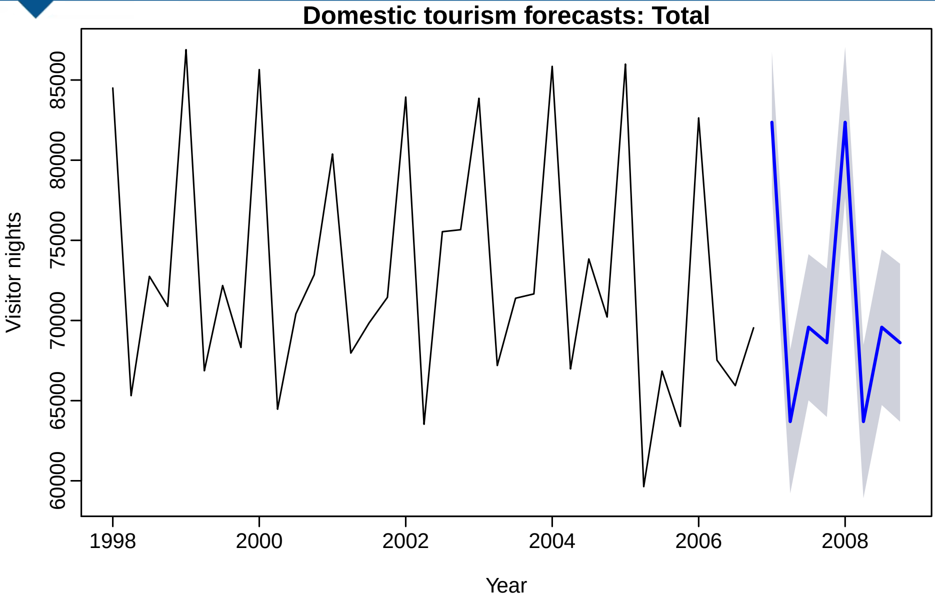
Base forecasts

ETS (exponential smoothing) models

Base forecasts



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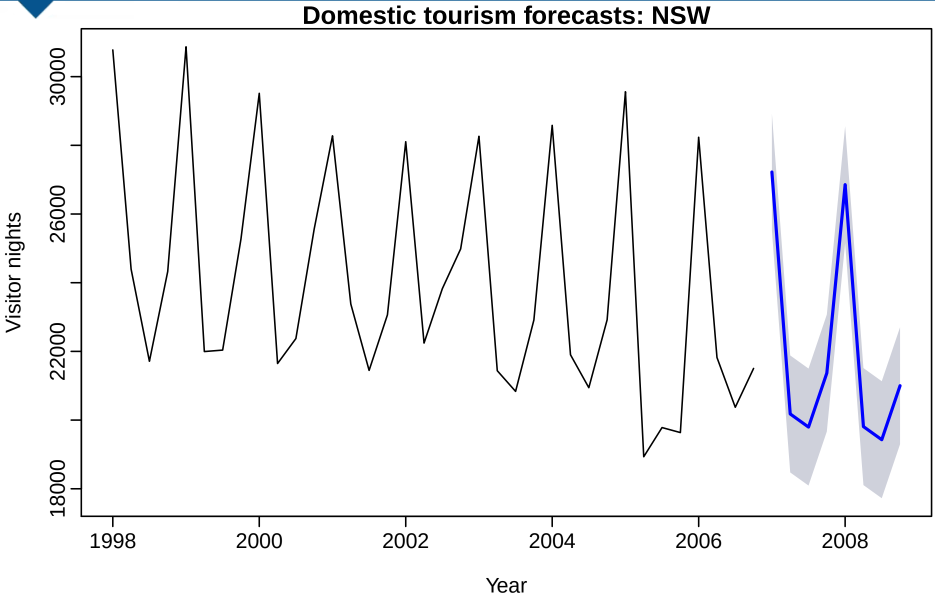

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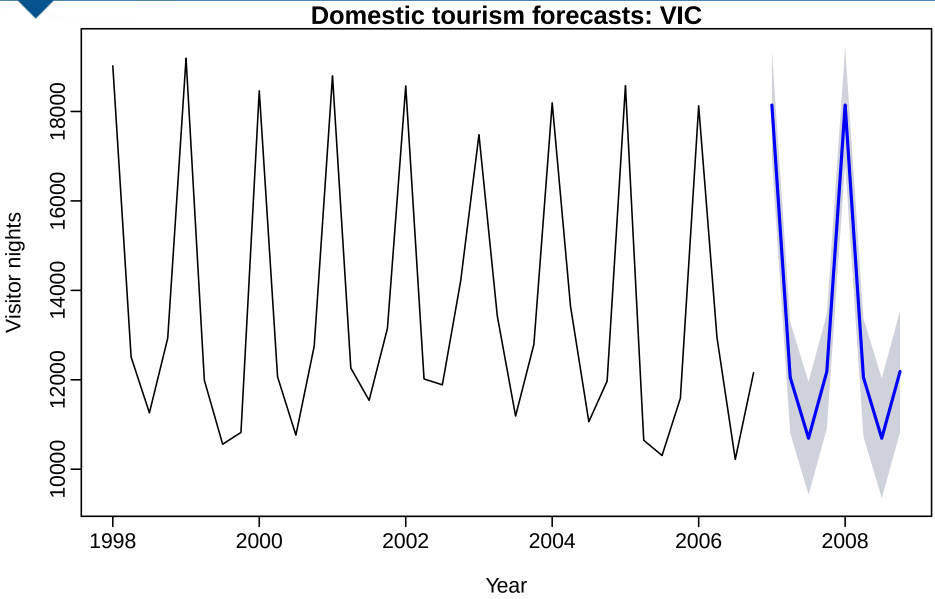

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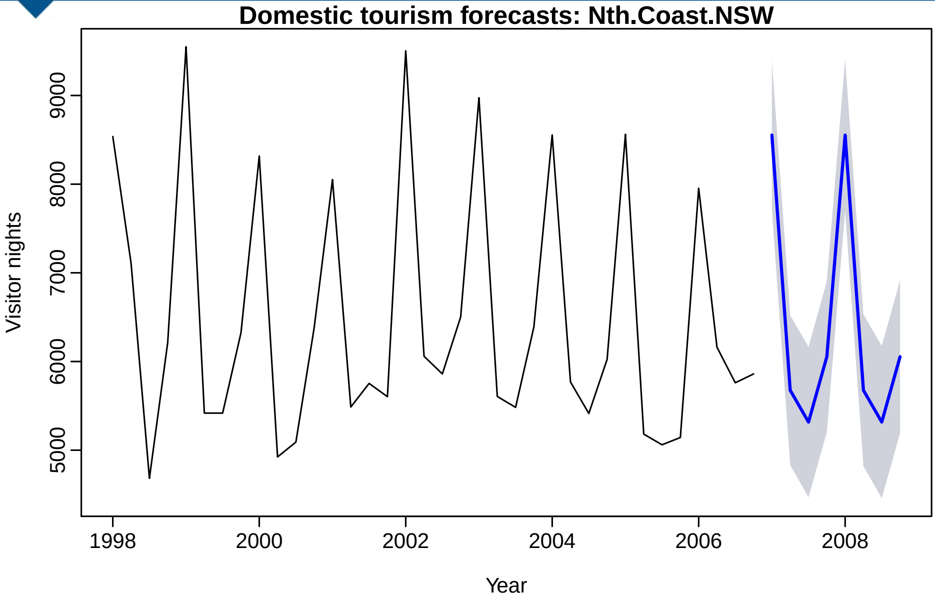

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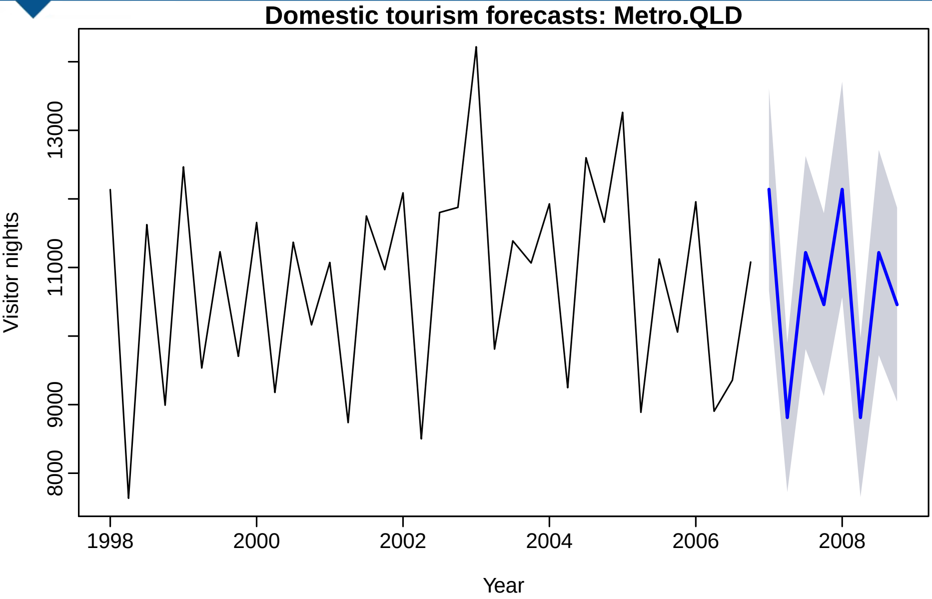

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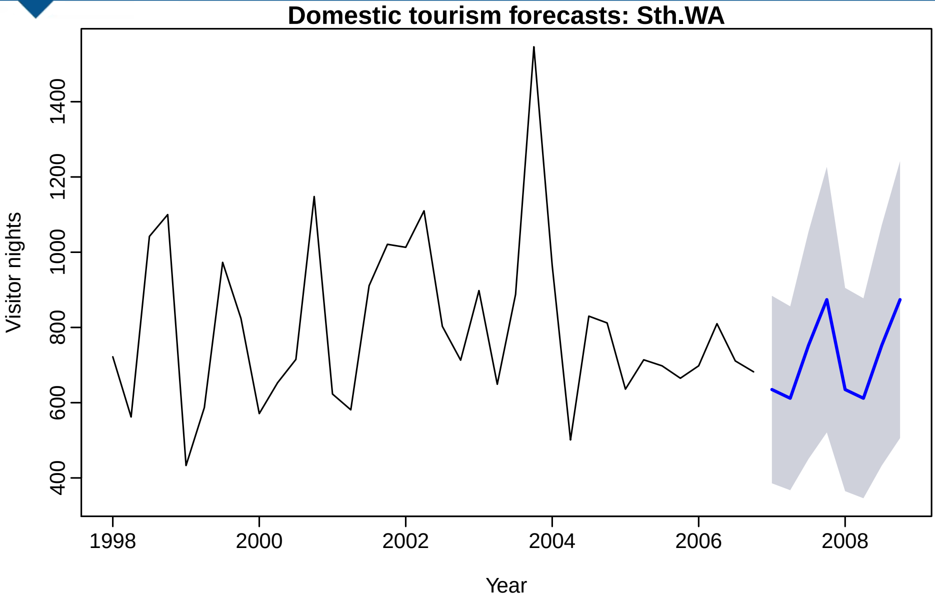

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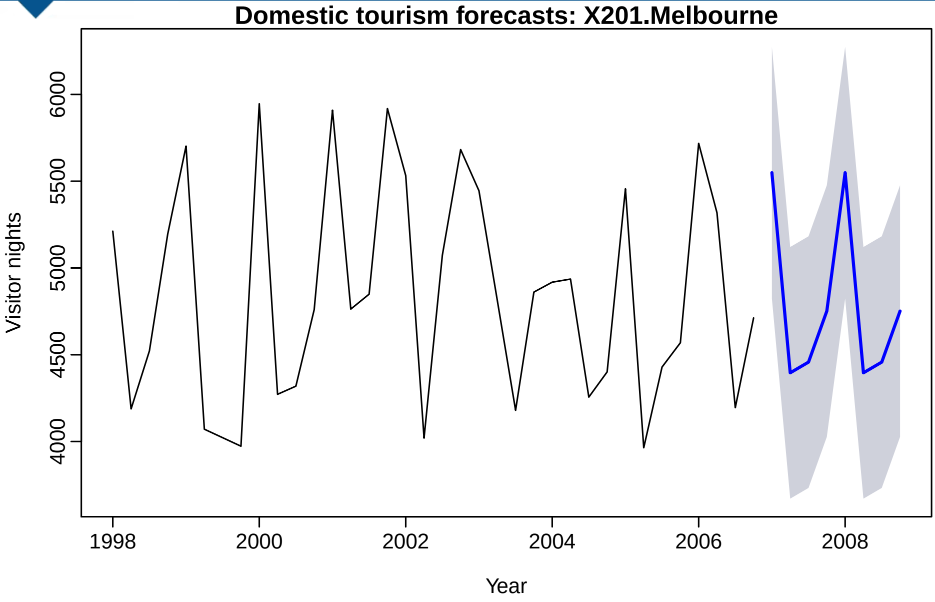

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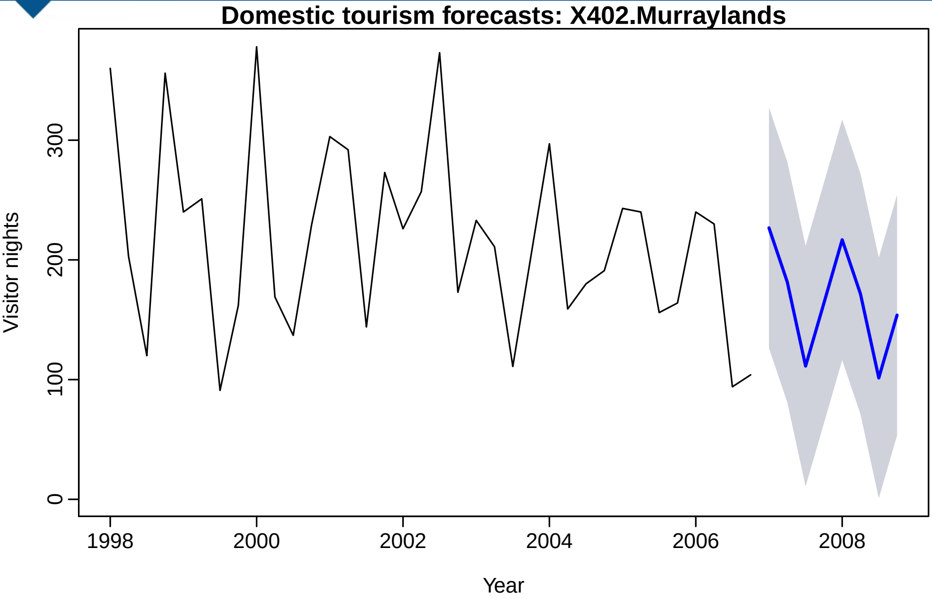

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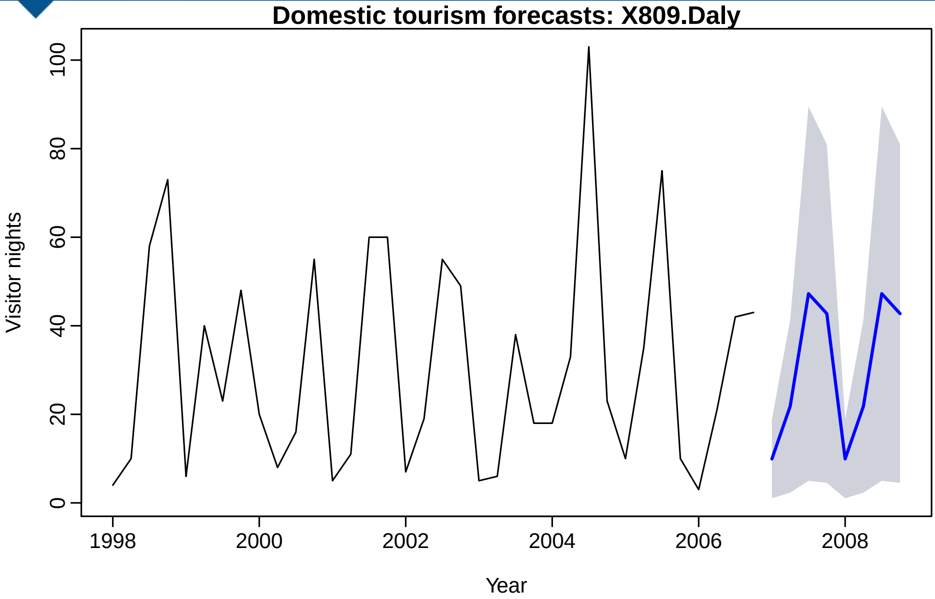
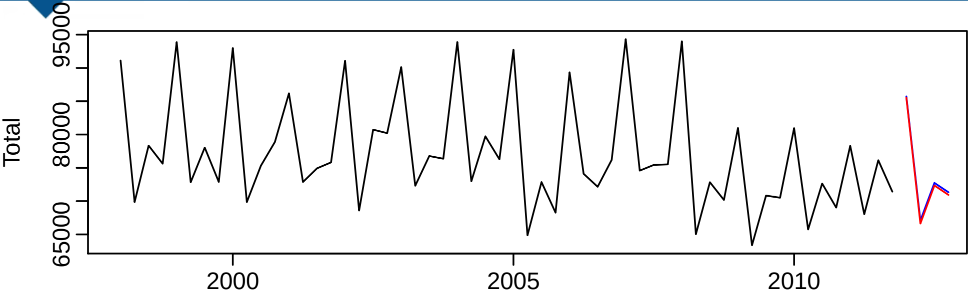


Reconciled forecasts



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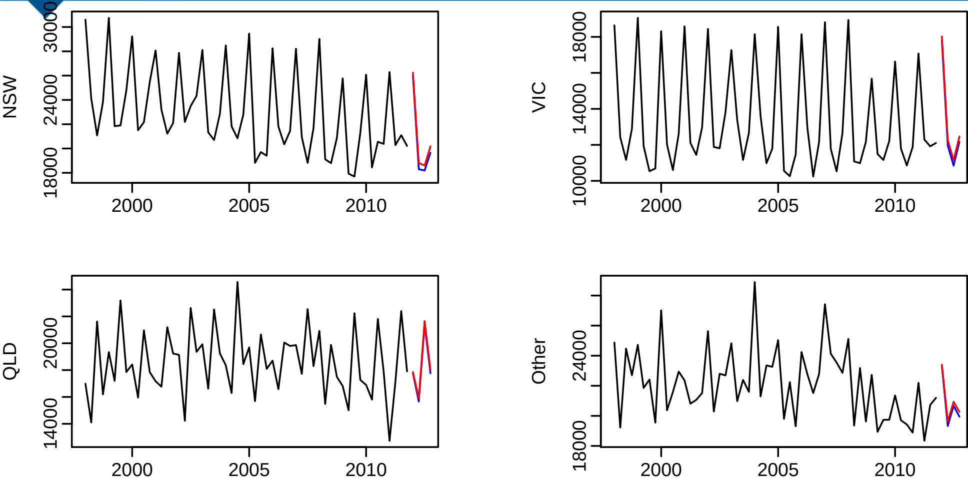

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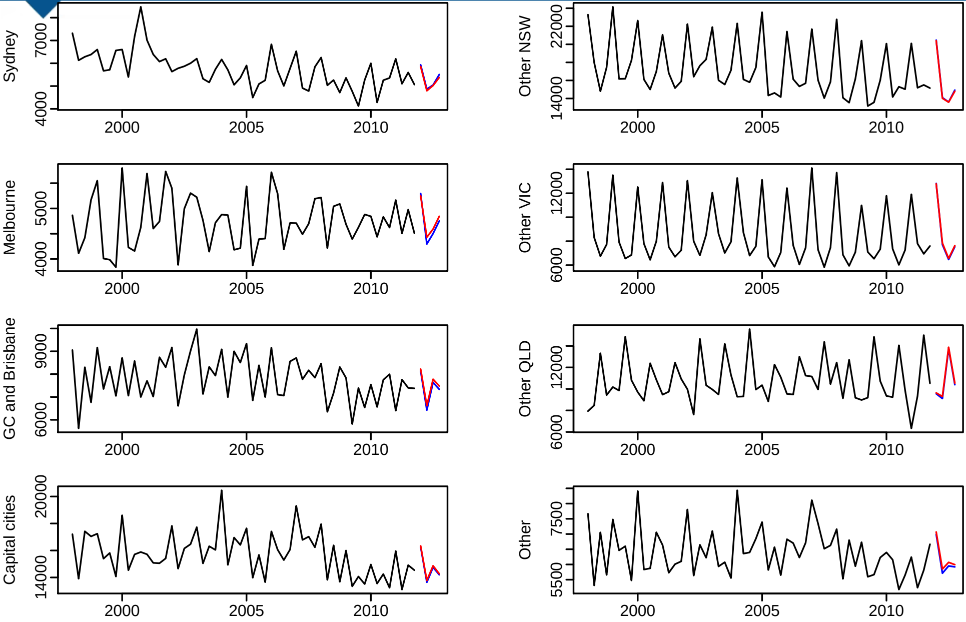

Forecast evaluation

- Select models using all observations;
- Re-estimate models using first 12 observations and generate 1- to 8-step-ahead forecasts;
- Increase sample size one observation at a time, re-estimate models, generate forecasts until the end of the sample;
- In total 24 1-step-ahead, 23 2-steps-ahead, up to 17 8-steps-ahead for forecast evaluation.

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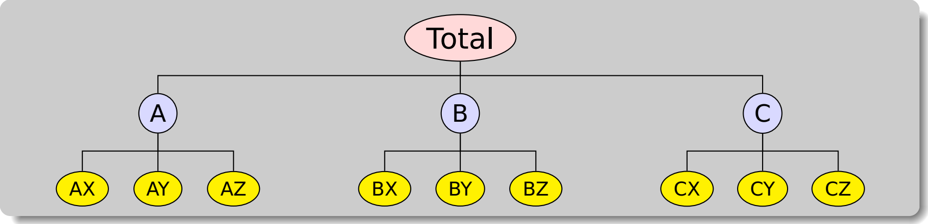
- In total 24 1-step-ahead, 23 2-steps-ahead, up to 17 8-steps-ahead for forecast evaluation.

Hierarchy: states, zones, regions

MAPE	$h = 1$	$h = 2$	$h = 4$	$h = 6$	$h = 8$	Average
Top Level: Australia						
Bottom-up	3.79	3.58	4.01	4.55	4.24	4.06
OLS	3.83	3.66	**3.88**	**4.19**	4.25	**3.94**
WLS	**3.68**	**3.56**	3.97	4.57	4.25	4.04
Level: States						
Bottom-up	10.70	10.52	10.85	11.46	11.27	11.03
OLS	11.07	10.58	11.13	11.62	12.21	11.35
WLS	**10.44**	**10.17**	**10.47**	**10.97**	**10.98**	**10.67**
Level: Zones						
Bottom-up	14.99	14.97	14.98	15.69	15.65	15.32
OLS	15.16	15.06	15.27	15.74	16.15	15.48
WLS	**14.63**	**14.62**	**14.68**	**15.17**	**15.25**	**14.94**
Bottom Level: Regions						
Bottom-up	33.12	32.54	32.26	33.74	33.96	33.18
OLS	35.89	33.86	34.26	36.06	37.49	35.43
WLS	**31.68**	**31.22**	**31.08**	**32.41**	**32.77**	**31.89**

Outline

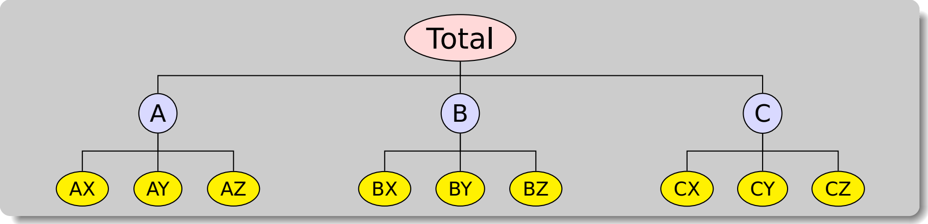
Fast computation: hierarchical data



$$\boldsymbol{y}_t = \begin{pmatrix} Y_t \\ Y_{A,t} \\ Y_{B,t} \\ Y_{C,t} \\ Y_{AX,t} \\ Y_{AY,t} \\ Y_{AZ,t} \\ Y_{BX,t} \\ Y_{BY,t} \\ Y_{BZ,t} \\ Y_{CX,t} \\ Y_{CY,t} \\ Y_{CZ,t} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}}_{\boldsymbol{S}} \underbrace{\begin{pmatrix} Y_{AX,t} \\ Y_{AY,t} \\ Y_{AZ,t} \\ Y_{BX,t} \\ Y_{BY,t} \\ Y_{BZ,t} \\ Y_{CX,t} \\ Y_{CY,t} \\ Y_{CZ,t} \end{pmatrix}}_{\boldsymbol{B}_t}$$

$$\boldsymbol{y}_t = \boldsymbol{S}\boldsymbol{B}_t$$

Fast computation: hierarchical data



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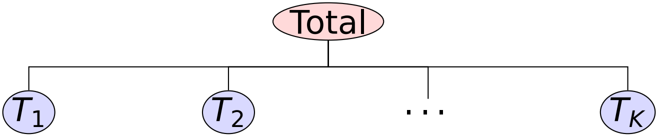
$$\boldsymbol{y}_t = \boldsymbol{S}\boldsymbol{B}_t$$

Fast computation: hierarchies

Think of the hierarchy as a tree of trees:



Then the summing matrix contains k smaller summing matrices:

$$\boldsymbol{S} = \begin{bmatrix} \mathbf{1}'_{n_1} & \mathbf{1}'_{n_2} & \cdots & \mathbf{1}'_{n_K} \\ \boldsymbol{S}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \boldsymbol{S}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \boldsymbol{S}_K \end{bmatrix}$$

where $\mathbf{1}_n$ is an n-vector of ones and tree T_i has n_i terminal nodes.

Fast computation: hierarchies

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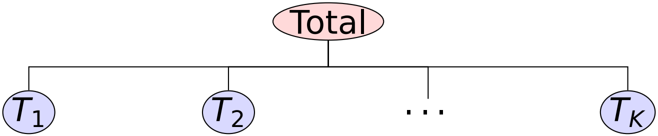

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Fast computation: hierarchies

$$\boldsymbol{S}'\Lambda\boldsymbol{S} = \begin{bmatrix} \boldsymbol{S}_1'\Lambda_1\boldsymbol{S}_1 & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{S}_2'\Lambda_2\boldsymbol{S}_2 & \cdots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{S}_K'\Lambda_K\boldsymbol{S}_K \end{bmatrix} + \lambda_0\boldsymbol{J}_n$$

- λ_0 is the top left element of Λ;
- Λ_k is a block of Λ, corresponding to tree T_k;
- $\boldsymbol{J}_n$ is a matrix of ones;
- $n = \sum_k n_k$.

Now apply the Sherman-Morrison formula . . .

Fast computation: hierarchies

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Fast computation: hierarchies

$$(\boldsymbol{S}'\Lambda\boldsymbol{S})^{-1} = \begin{bmatrix} (\boldsymbol{S}_1'\Lambda_1\boldsymbol{S}_1)^{-1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & (\boldsymbol{S}_2'\Lambda_2\boldsymbol{S}_2)^{-1} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & (\boldsymbol{S}_K'\Lambda_K\boldsymbol{S}_K)^{-1} \end{bmatrix} - c\boldsymbol{S}_0$$

- $\boldsymbol{S}_0$ can be partitioned into K^2 blocks, with the (k,ℓ) block (of dimension $n_k \times n_\ell$) being

$$(\boldsymbol{S}_k'\Lambda_k\boldsymbol{S}_k)^{-1}\boldsymbol{J}_{n_k,n_\ell}(\boldsymbol{S}_\ell'\Lambda_\ell\boldsymbol{S}_\ell)^{-1}$$

- $\boldsymbol{J}_{n_k,n_\ell}$ is a $n_k \times n_\ell$ matrix of ones.
- $c^{-1} = \lambda_0^{-1} + \sum_k \mathbf{1}_{n_k}'(\boldsymbol{S}_k'\Lambda_k\boldsymbol{S}_k)^{-1}\mathbf{1}_{n_k}$.
- Each $\boldsymbol{S}_k'\Lambda_k\boldsymbol{S}_k$ can be inverted similarly.
- $\boldsymbol{S}'\Lambda\boldsymbol{y}$ can also be computed recursively.

Fast computation: hierarchies

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- $\boldsymbol{S}'\Lambda\boldsymbol{y}$ can also be computed recursively.

The recursive calculations can be done in such a way that we never store any of the large matrices involved.

Fast computation

A similar algorithm has been developed for grouped time series with two groups.

When the time series are not strictly hierarchical and have more than two grouping variables:

- Use sparse matrix storage and arithmetic.
- Use iterative approximation for inverting large sparse matrices.

Paige & Saunders (1982)
ACM Trans. Math. Software

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Outline

hts package for R

hts: Hierarchical and grouped time series

Methods for analysing and forecasting hierarchical and grouped time series

Version:	4.5
Depends:	forecast ($\geq$ 5.0), SparseM
Imports:	parallel, utils
Published:	2014-12-09
Author:	Rob J Hyndman, Earo Wang and Alan Lee
Maintainer:	Rob J Hyndman $<$Rob.Hyndman at monash.edu$>$
BugReports:	`https://github.com/robjhyndman/hts/issues`
License:	GPL ($\geq$ 2)

Example using R

```
library(hts)

# bts is a matrix containing the bottom level time series
# nodes describes the hierarchical structure
y <- hts(bts, nodes=list(2, c(3,2)))
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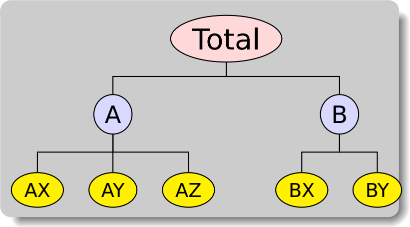

Example using R

```
library(hts)

# bts is a matrix containing the bottom level time series
# nodes describes the hierarchical structure
y <- hts(bts, nodes=list(2, c(3,2)))

# Forecast 10-step-ahead using WLS combination method
# ETS used for each series by default
fc <- forecast(y, h=10)
```

forecast.gts function

Usage

```
forecast(object, h,
  method = c("comb", "bu", "mo", "tdgsf", "tdgsa", "tdfp"),
  fmethod = c("ets", "rw", "arima"),
  weights = c("sd", "none", "nseries"),
  positive = FALSE,
  parallel = FALSE, num.cores = 2, ...)
```

Arguments

`object`	Hierarchical time series object of class `gts`.
`h`	Forecast horizon
`method`	Method for distributing forecasts within the hierarchy.
`fmethod`	Forecasting method to use
`positive`	If TRUE, forecasts are forced to be strictly positive
`weights`	Weights used for "optimal combination" method. When weights = "sd", it takes account of the standard deviation of forecasts.
`parallel`	If TRUE, allow parallel processing
`num.cores`	If parallel = TRUE, specify how many cores are going to be used

Outline

1. Examples of biggish time series
2. Time series visualisation
3. BLUF: Best Linear Unbiased Forecasts
4. Application: Australian tourism
5. Fast computation tricks
6. hts package for R
7. **References**

References

RJ Hyndman, RA Ahmed, G Athanasopoulos, and HL Shang (2011). "Optimal combination forecasts for hierarchical time series". *Computational statistics & data analysis* **55**(9), 2579–2589.

RJ Hyndman, AJ Lee, and E Wang (2014). *Fast computation of reconciled forecasts for hierarchical and grouped time series*. Working paper 17/14. Department of Econometrics & Business Statistics, Monash University

RJ Hyndman, AJ Lee, and E Wang (2014). *hts: Hierarchical and grouped time series*. `cran.r-project.org/package=hts`.

RJ Hyndman and G Athanasopoulos (2014). *Forecasting: principles and practice*. OTexts. `OTexts.org/fpp/`.

References

RJ Hyndman, RA Ahmed, G Athanasopoulos, and HL Shang (2011). "Optimal combination forecasts for hierarchical time series". *Computational statistics & data analysis* **55**(9), 2579–2589.

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RJ Hyndman, AJ Lee, and E Wang (2014). *hts:*

- Papers and R code:
  robjhyndman.com
- Email: **Rob.Hyndman@monash.edu**