MONASH University

MONASH BUSINESS SCHOOL

Forecasting using R

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2.5 Seasonal ARIMA models

Outline

1 Backshift notation reviewed

2 Seasonal ARIMA models

3 ARIMA vs ETS

4 Lab session 12

Backshift notation

A very useful notational device is the backward shift operator, B, which is used as follows:

$$By_t = y_{t-1} \ .$$

In other words, B, operating on y_t, has the effect of **shifting the data back one period**. Two applications of B to y_t **shifts the data back two periods:**

$$B(By_t) = B^2 y_t = y_{t-2} \ .$$

For monthly data, if we wish to shift attention to "the same month last year," then B^{12} is used, and the notation is $B^{12} y_t = y_{t-12}$.

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Backshift notation

- First difference: $1 - B$.
- Double difference: $(1 - B)^2$.
- dth-order difference: $(1 - B)^d y_t$.
- Seasonal difference: $1 - B^m$.
- Seasonal difference followed by a first difference: $(1 - B)(1 - B^m)$.
- Multiply terms together together to see the combined effect:

$$\begin{aligned}(1 - B)(1 - B^m)y_t &= (1 - B - B^m + B^{m+1})y_t \\ &= y_t - y_{t-1} - y_{t-m} + y_{t-m-1}.\end{aligned}$$

Backshift notation for ARIMA

- ARMA model:

$$\begin{aligned} y_t &= c + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} + e_t + \theta_1 e_{t-1} + \cdots + \theta_q e_{t-q} \\ &= c + \phi_1 B y_t + \cdots + \phi_p B^p y_t + e_t + \theta_1 B e_t + \cdots + \theta_q B^q e_t \end{aligned}$$

$$\phi(B) y_t = c + \theta(B) e_t$$

where $\phi(B) = 1 - \phi_1 B - \cdots - \phi_p B^p$

and $\theta(B) = 1 + \theta_1 B + \cdots + \theta_q B^q$.

- ARIMA(1,1,1) model:

$$\underset{\substack{\uparrow \\ \text{AR(1)}}}{(1-\phi_1 B)} \quad \underset{\substack{\uparrow \\ \text{First} \\ \text{difference}}}{(1-B) y_t} = c + \underset{\substack{\uparrow \\ \text{MA(1)}}}{(1+\theta_1 B) e_t}$$

Backshift notation for ARIMA

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$$\begin{aligned} y_t &= c + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} + e_t + \theta_1 e_{t-1} + \cdots + \theta_q e_{t-q} \\ &= c + \phi_1 B y_t + \cdots + \phi_p B^p y_t + e_t + \theta_1 B e_t + \cdots + \theta_q B^q e_t \\ \phi(B) y_t &= c + \theta(B) e_t \\ &\quad \text{where } \phi(B) = 1 - \phi_1 B - \cdots - \phi_p B^p \\ &\quad \text{and } \theta(B) = 1 + \theta_1 B + \cdots + \theta_q B^q. \end{aligned}$$

- ARIMA(1,1,1) model:

$$\underset{\substack{\uparrow \\ \text{AR(1)}}}{(1 - \phi_1 B)} \quad \underset{\substack{\uparrow \\ \text{First} \\ \text{difference}}}{(1 - B)} y_t \quad = \quad c + \underset{\substack{\uparrow \\ \text{MA(1)}}}{(1 + \theta_1 B)} e_t$$

Backshift notation for ARIMA

- ARIMA(p, d, q) model:

$$\underset{\substack{\uparrow\\ \text{AR}(p)}}{(1 - \phi_1 B - \cdots - \phi_p B^p)}\ \underset{\substack{\uparrow\\ d\text{ differences}}}{(1 - B)^d}\, y_t = c + \underset{\substack{\uparrow\\ \text{MA}(q)}}{(1 + \theta_1 B + \cdots + \theta_q B^q)}\, e_t$$

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Seasonal ARIMA models

ARIMA	$\underbrace{(p, d, q)}$	$\underbrace{(P, D, Q)_m}$
	$\uparrow$ Non-seasonal part of the model	$\uparrow$ Seasonal part of of the model

where m = number of observations per year.

Seasonal ARIMA models

E.g., ARIMA$(1, 1, 1)(1, 1, 1)_4$ model (without constant)

$$(1-\phi_1 B)(1-\Phi_1 B^4)(1-B)(1-B^4)y_t \;=\; (1+\theta_1 B)(1+\Theta_1 B^4)e_t.$$

(Non-seasonal AR(1))

(Seasonal AR(1))

(Non-seasonal difference)

(Seasonal difference)

(Non-seasonal MA(1))

(Seasonal MA(1))

Seasonal ARIMA models

E.g., ARIMA$(1,1,1)(1,1,1)_4$ model (without constant)

$$(1-\phi_1 B)(1-\Phi_1 B^4)(1-B)(1-B^4)y_t = (1+\theta_1 B)(1+\Theta_1 B^4)e_t.$$

$\left(\begin{array}{c}\text{Non-seasonal}\\ \text{AR(1)}\end{array}\right)$
$\left(\begin{array}{c}\text{Seasonal}\\ \text{AR(1)}\end{array}\right)$
$\left(\begin{array}{c}\text{Non-seasonal}\\ \text{difference}\end{array}\right)$
$\left(\begin{array}{c}\text{Seasonal}\\ \text{difference}\end{array}\right)$
$\left(\begin{array}{c}\text{Non-seasonal}\\ \text{MA(1)}\end{array}\right)$
$\left(\begin{array}{c}\text{Seasonal}\\ \text{MA(1)}\end{array}\right)$

Seasonal ARIMA models

E.g., ARIMA(1, 1, 1)(1, 1, 1)$_4$ model (without constant)

$$\underbrace{(1-\phi_1 B)}_{\text{Non-seasonal AR(1)}}\underbrace{(1-\Phi_1 B^4)}_{\text{Seasonal AR(1)}}\underbrace{(1-B)}_{\text{Non-seasonal difference}}\underbrace{(1-B^4)}_{\text{Seasonal difference}}y_t = \underbrace{(1+\theta_1 B)}_{\text{Non-seasonal MA(1)}}\underbrace{(1+\Theta_1 B^4)}_{\text{Seasonal MA(1)}}e_t.$$

Seasonal ARIMA models

E.g., ARIMA$(1, 1, 1)(1, 1, 1)_4$ model (without constant)

$$(1-\phi_1 B)(1-\Phi_1 B^4)(1-B)(1-B^4)y_t = (1+\theta_1 B)(1+\Theta_1 B^4)e_t.$$

All the factors can be multiplied out and the general model written as follows:

$$\begin{aligned} y_t &= (1+\phi_1)y_{t-1} - \phi_1 y_{t-2} + (1+\Phi_1)y_{t-4} \\ &\quad - (1+\phi_1+\Phi_1+\phi_1\Phi_1)y_{t-5} + (\phi_1+\phi_1\Phi_1)y_{t-6} \\ &\quad - \Phi_1 y_{t-8} + (\Phi_1+\phi_1\Phi_1)y_{t-9} - \phi_1\Phi_1 y_{t-10} \\ &\quad + e_t + \theta_1 e_{t-1} + \Theta_1 e_{t-4} + \theta_1\Theta_1 e_{t-5}. \end{aligned}$$

Common ARIMA models

In the US Census Bureau uses the following models most often:

ARIMA$(0,1,1)(0,1,1)_m$	with log transformation
ARIMA$(0,1,2)(0,1,1)_m$	with log transformation
ARIMA$(2,1,0)(0,1,1)_m$	with log transformation
ARIMA$(0,2,2)(0,1,1)_m$	with log transformation
ARIMA$(2,1,2)(0,1,1)_m$	with no transformation

Seasonal ARIMA models

The seasonal part of an AR or MA model will be seen in the seasonal lags of the PACF and ACF.

ARIMA(0,0,0)(0,0,1)$_{12}$ will show:

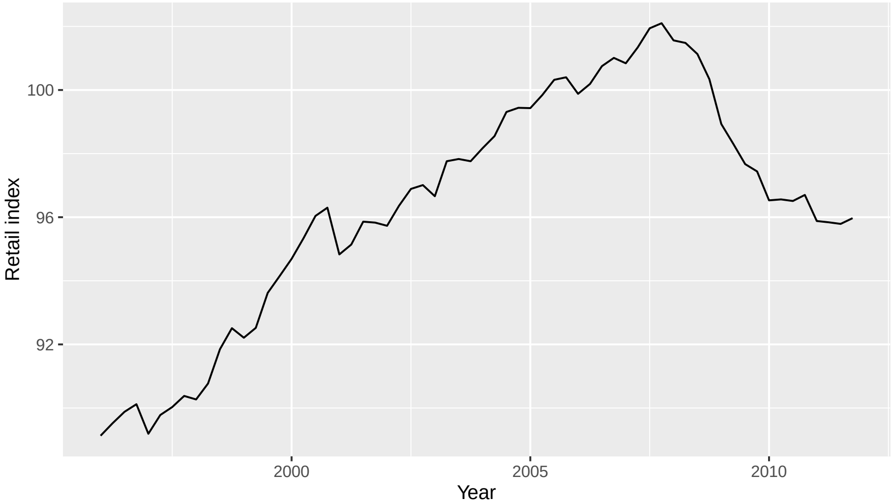
- a spike at lag 12 in the ACF but no other significant spikes.
- The PACF will show exponential decay in the seasonal lags; that is, at lags 12, 24, 36, ...

ARIMA(0,0,0)(1,0,0)$_{12}$ will show:

- exponential decay in the seasonal lags of the ACF
- a single significant spike at lag 12 in the PACF.

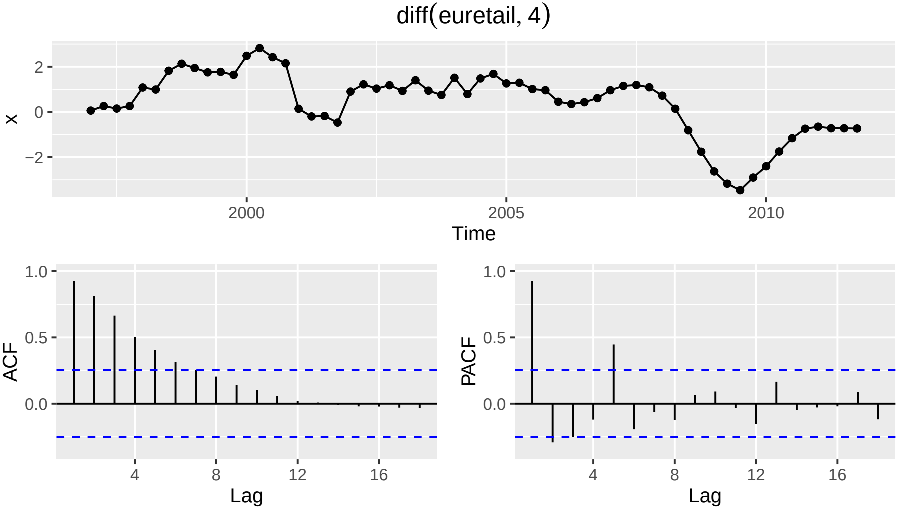
European quarterly retail trade

```r
autoplot(euretail) + xlab("Year") + ylab("Retai
```



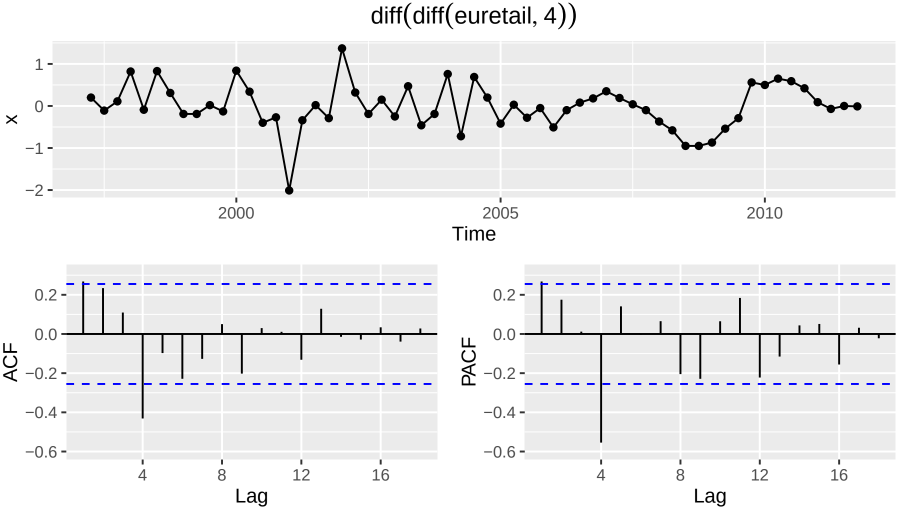
European quarterly retail trade

```
ggtsdisplay(diff(euretail,4))
```



European quarterly retail trade

```r
ggtsdisplay(diff(diff(euretail,4)))
```

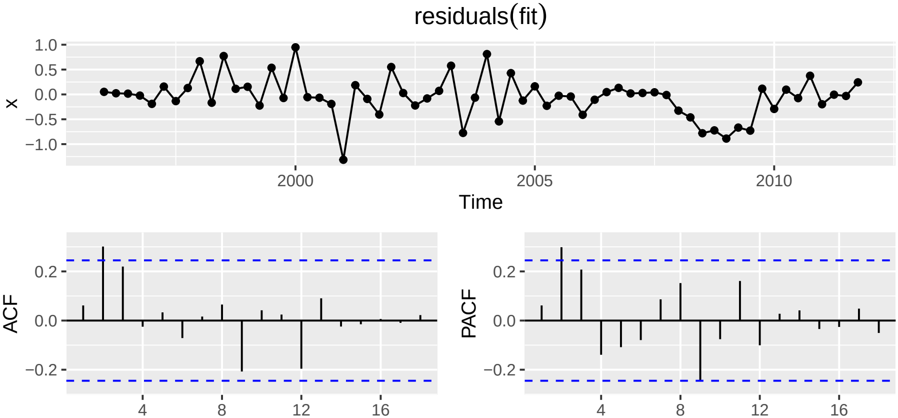


European quarterly retail trade

- $d = 1$ and $D = 1$ seems necessary.
- Significant spike at lag 1 in ACF suggests non-seasonal MA(1) component.
- Significant spike at lag 4 in ACF suggests seasonal MA(1) component.
- Initial candidate model: ARIMA$(0,1,1)(0,1,1)_4$.
- We could also have started with ARIMA$(1,1,0)(1,1,0)_4$.

European quarterly retail trade

```r
fit <- Arima(euretail, order=c(0,1,1),
  seasonal=c(0,1,1))
ggtsdisplay(residuals(fit))
```



European quarterly retail trade

- ACF and PACF of residuals show significant spikes at lag 2, and maybe lag 3.
- AICc of ARIMA(0,1,2)$(0,1,1)_4$ model is 74.36.
- AICc of ARIMA(0,1,3)$(0,1,1)_4$ model is 68.53.

```
fit <- Arima(euretail, order=c(0,1,3),
  seasonal=c(0,1,1))
ggtsdisplay(residuals(fit))
```

European quarterly retail trade

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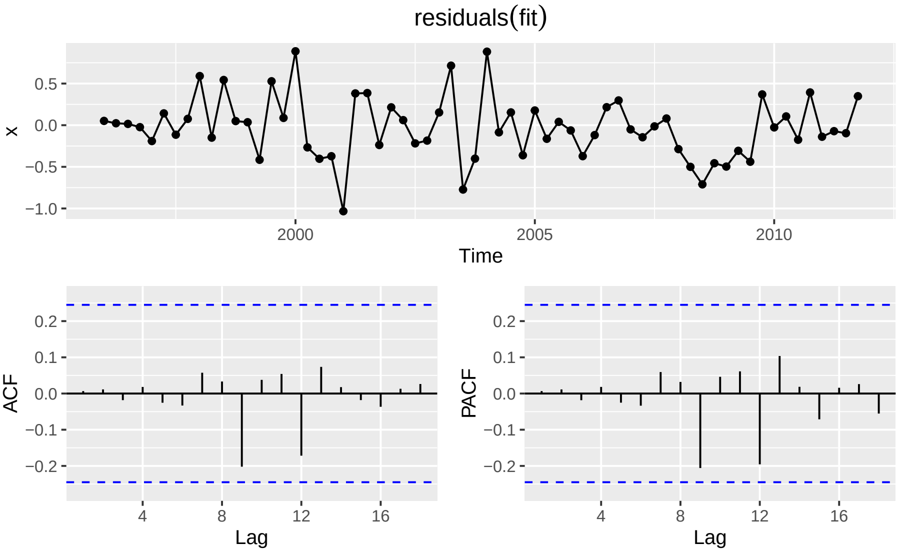
fit <- Arima(euretail, order=c(0,1,3),
  seasonal=c(0,1,1))
ggtsdisplay(residuals(fit))
```

European quarterly retail trade



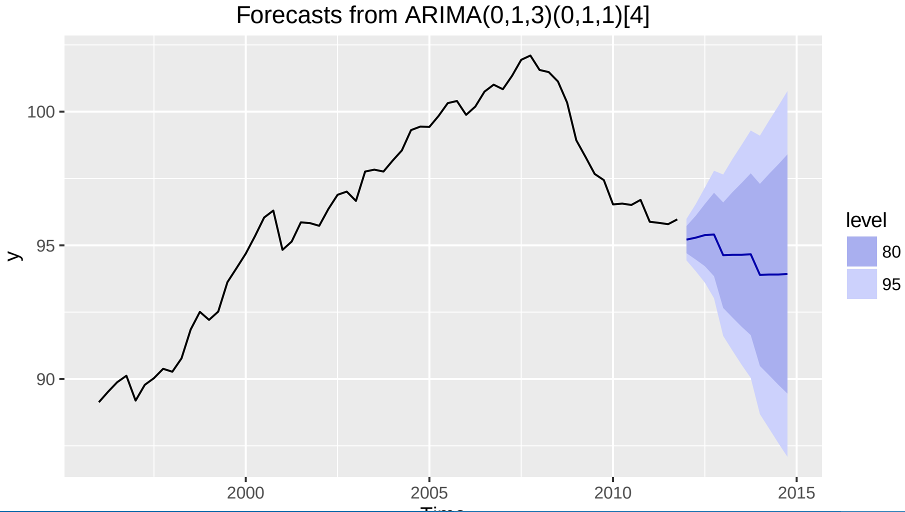
European quarterly retail trade

```r
res <- residuals(fit)
Box.test(res, lag=16, fitdf=4, type="Ljung")
```

```
##
##  Box-Ljung test
##
## data:  res
## X-squared = 7.0105, df = 12, p-value = 0.8569
```

European quarterly retail trade

```r
autoplot(forecast(fit, h=12))
```



European quarterly retail trade

```r
auto.arima(euretail)
```

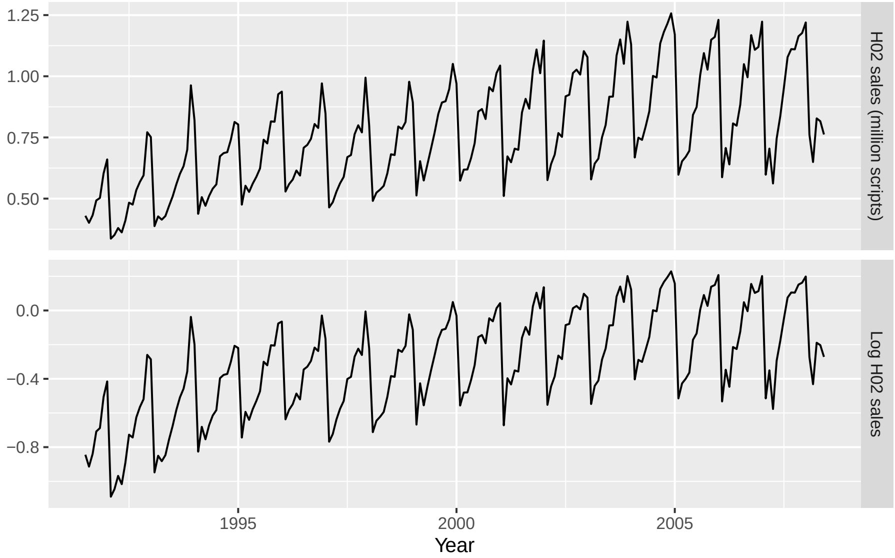
```
## Series: euretail
## ARIMA(1,1,2)(0,1,1)[4]
##
## Coefficients:
##          ar1      ma1     ma2     sma1
##       0.7345  -0.4655  0.2162  -0.8413
## s.e.  0.2239   0.1995  0.2096   0.1869
##
## sigma^2 estimated as 0.1592:  log likelihood=-29.69
## AIC=69.37   AICc=70.51   BIC=79.76
```

European quarterly retail trade

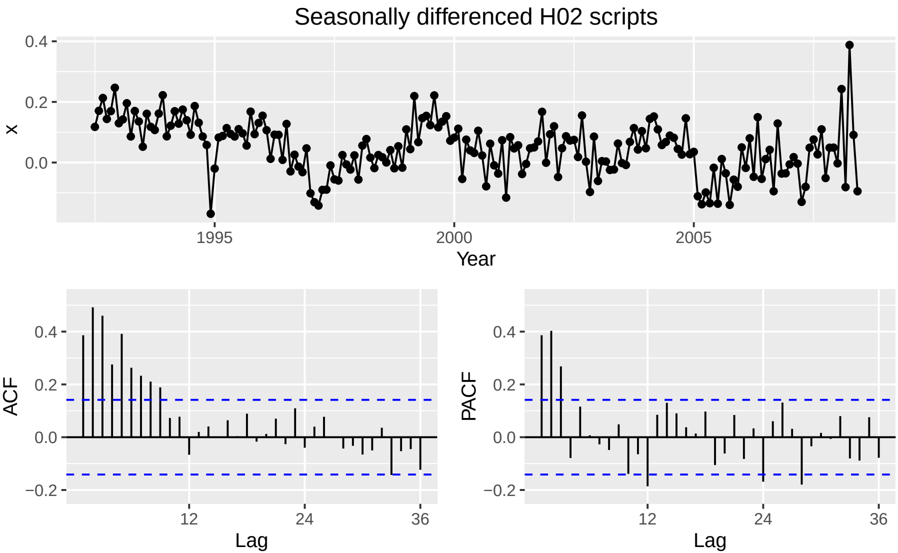
```r
auto.arima(euretail, stepwise=FALSE, approximation=FALSE)
```

```
## Series: euretail
## ARIMA(0,1,3)(0,1,1)[4]
##
## Coefficients:
##          ma1     ma2     ma3     sma1
##       0.2625  0.3697  0.4194  -0.6615
## s.e.  0.1239  0.1260  0.1296   0.1555
##
## sigma^2 estimated as 0.1564:  log likelihood=-28.7
## AIC=67.4   AICc=68.53   BIC=77.78
```

Cortecosteroid drug sales



Cortecosteroid drug sales



Cortecosteroid drug sales

- Choose $D = 1$ and $d = 0$.
- Spikes in PACF at lags 12 and 24 suggest seasonal AR(2) term.
- Spikes in PACF suggests possible non-seasonal AR(3) term.
- Initial candidate model: ARIMA$(3,0,0)(2,1,0)_{12}$.

Cortecosteroid drug sales

Model	AICc
ARIMA$(3,0,0)(2,1,0)_{12}$	-475.12
ARIMA$(3,0,1)(2,1,0)_{12}$	-476.31
ARIMA$(3,0,2)(2,1,0)_{12}$	-474.88
ARIMA$(3,0,1)(1,1,0)_{12}$	-463.40
ARIMA$(3,0,1)(0,1,1)_{12}$	-483.67
ARIMA$(3,0,1)(0,1,2)_{12}$	-485.48
ARIMA$(3,0,1)(1,1,1)_{12}$	-484.25

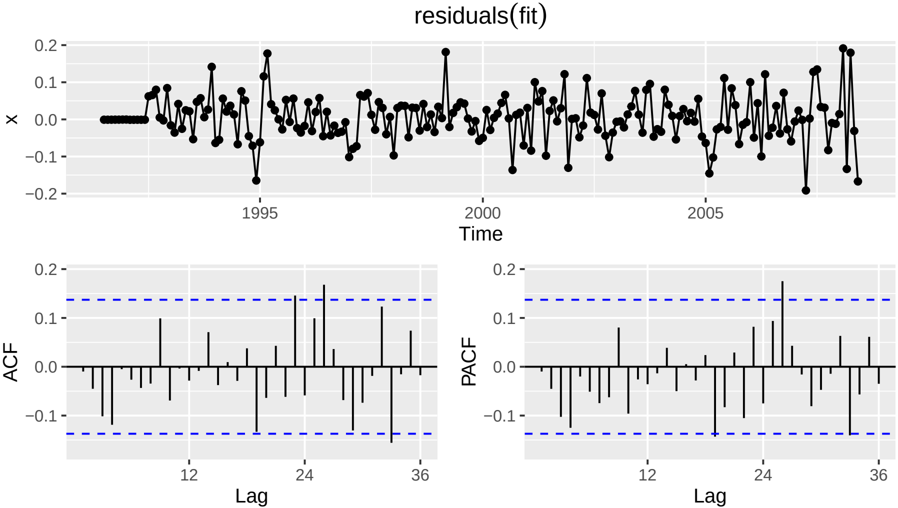
Cortecosteroid drug sales

```r
(fit <- Arima(h02, order=c(3,0,1), seasonal=c(0,1,2),
  lambda=0))
```

```
## Series: h02
## ARIMA(3,0,1)(0,1,2)[12]
## Box Cox transformation: lambda= 0
##
## Coefficients:
##           ar1     ar2     ar3     ma1     sma1     sma2
##       -0.1603  0.5481  0.5678  0.3827  -0.5222  -0.1768
## s.e.   0.1636  0.0878  0.0942  0.1895   0.0861   0.0872
##
## sigma^2 estimated as 0.004278:  log likelihood=250.04
## AIC=-486.08   AICc=-485.48   BIC=-463.28
```

Cortecosteroid drug sales

```r
ggtsdisplay(residuals(fit))
```



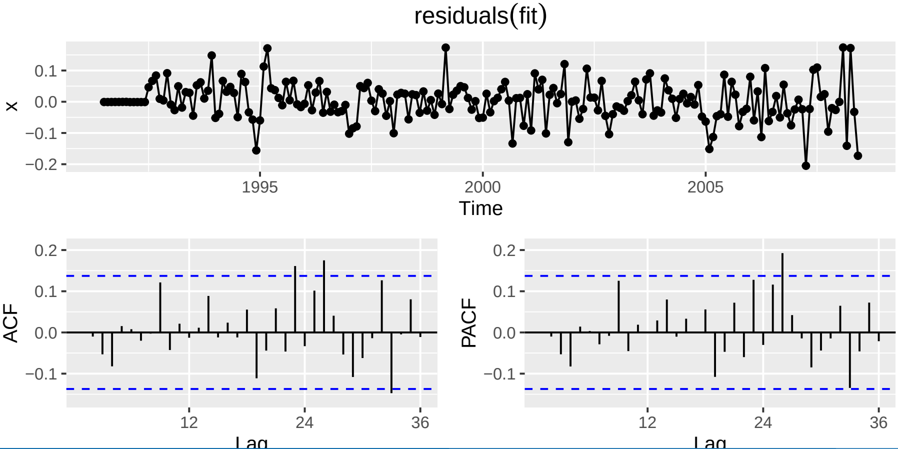
Cortecosteroid drug sales

```
Box.test(residuals(fit), lag=36, fitdf=6,
  type="Ljung")
```

```
##
##  Box-Ljung test
##
## data:  residuals(fit)
## X-squared = 50.712, df = 30, p-value = 0.01045
```

Cortecosteroid drug sales

```r
fit <- auto.arima(h02, lambda=0, d=0, D=1, max.order=9,
  stepwise=FALSE, approximation=FALSE)
ggtsdisplay(residuals(fit))
```



Cortecosteroid drug sales

```r
Box.test(residuals(fit), lag=36, fitdf=8,
  type="Ljung")
```

```
##
##  Box-Ljung test
##
## data:  residuals(fit)
## X-squared = 44.766, df = 28, p-value = 0.02329
```

Cortecosteroid drug sales

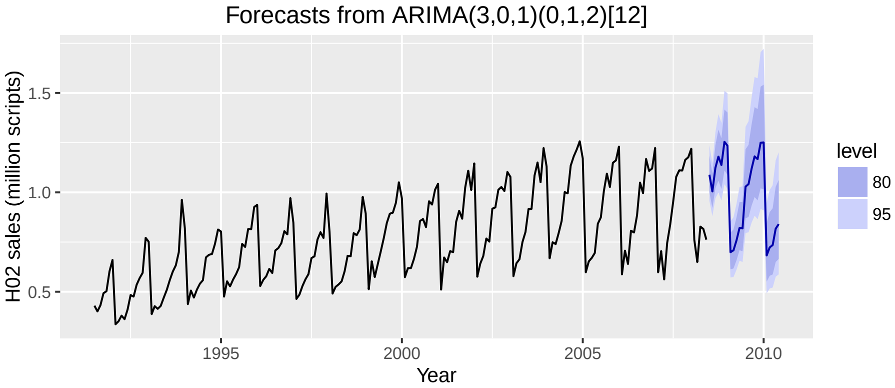
Model	RMSE
ARIMA(3,0,0)(2,1,0)[12]	0.0661
ARIMA(3,0,1)(2,1,0)[12]	0.0646
ARIMA(3,0,2)(2,1,0)[12]	0.0645
ARIMA(3,0,1)(1,1,0)[12]	0.0679
ARIMA(3,0,1)(0,1,1)[12]	0.0644
ARIMA(3,0,1)(0,1,2)[12]	0.0622
ARIMA(3,0,1)(1,1,1)[12]	0.0630
ARIMA(4,0,3)(0,1,1)[12]	0.0648
ARIMA(3,0,3)(0,1,1)[12]	0.0639
ARIMA(4,0,2)(0,1,1)[12]	0.0648
ARIMA(3,0,2)(0,1,1)[12]	0.0644
ARIMA(2,1,3)(0,1,1)[12]	0.0634
ARIMA(2,1,4)(0,1,1)[12]	0.0632
ARIMA(2,1,5)(0,1,1)[12]	0.0640

Cortecosteroid drug sales

- Models with lowest AICc values tend to give slightly better results than the other models.
- AICc comparisons must have the same orders of differencing. But RMSE test set comparisons can involve any models.
- No model passes all the residual tests.
- Use the best model available, even if it does not pass all tests.
- In this case, the ARIMA$(3,0,1)(0,1,2)_{12}$ has the lowest RMSE value and the best AICc value for models with fewer than 6 parameters.

Cortecosteroid drug sales

```r
fit <- Arima(h02, order=c(3,0,1), seasonal=c(0,1,2),
  lambda=0)
autoplot(forecast(fit)) +
  ylab("H02 sales (million scripts)") + xlab("Year")
```



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ARIMA vs ETS

- Myth that ARIMA models are more general than exponential smoothing.
- Linear exponential smoothing models all special cases of ARIMA models.
- Non-linear exponential smoothing models have no equivalent ARIMA counterparts.
- Many ARIMA models have no exponential smoothing counterparts.
- ETS models all non-stationary. Models with seasonality or non-damped trend (or both) have two unit roots; all other models have one unit root.

Equivalences

ETS model	ARIMA model	Parameters
ETS(A,N,N)	ARIMA(0,1,1)	$\theta_1 = \alpha - 1$
ETS(A,A,N)	ARIMA(0,2,2)	$\theta_1 = \alpha + \beta - 2$
		$\theta_2 = 1 - \alpha$
ETS(A,A,N)	ARIMA(1,1,2)	$\phi_1 = \phi$
		$\theta_1 = \alpha + \phi\beta - 1 - \phi$
		$\theta_2 = (1 - \alpha)\phi$
ETS(A,N,A)	ARIMA$(0,0,m)(0,1,0)_m$	
ETS(A,A,A)	ARIMA$(0,1,m+1)(0,1,0)_m$	
ETS(A,A,A)	ARIMA$(1,0,m+1)(0,1,0)_m$	

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