MONASH University

MONASH BUSINESS SCHOOL

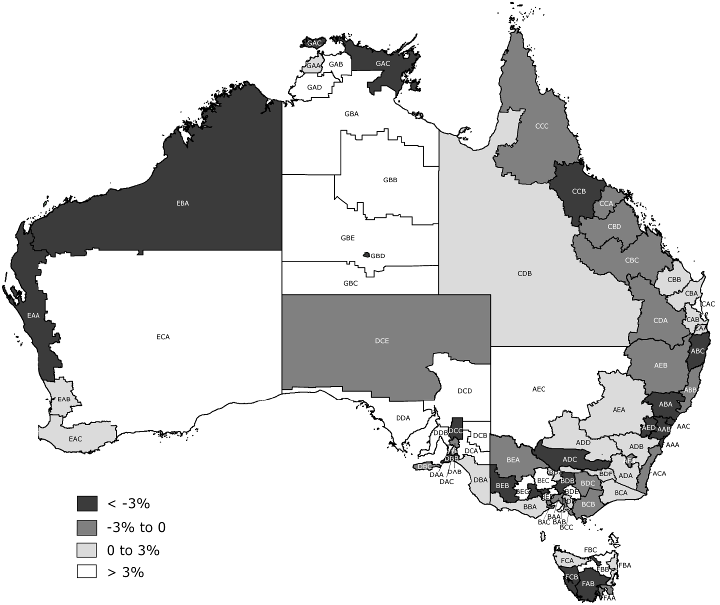
Forecasting: principles and practice

Rob J Hyndman

3.3 Hierarchical forecasting

Outline

Australian tourism demand



Australian tourism demand

- Quarterly data on visitor night from 1998:Q1 – 2013:Q4
- From: *National Visitor Survey*, based on annual interviews of 120,000 Australians aged 15+, collected by Tourism Research Australia.
- Split by 7 states, 27 zones and 76 regions (a geographical hierarchy)
- Also split by purpose of travel
  - Holiday
  - Visiting friends and relatives (VFR)
  - Business
  - Other
- 304 bottom-level series

> 3%

Spectacle sales

- Monthly UK sales data from 2000 – 2014
- Provided by a large spectacle manufacturer
- Split by brand (26), gender (3), price range (6), materials (4), and stores (600)
- About 1 million bottom-level series

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Hierarchical time series

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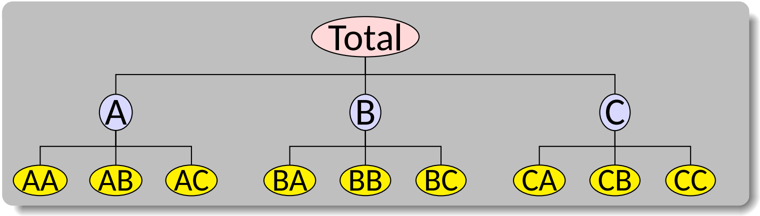

Examples

- Tourism by state and region

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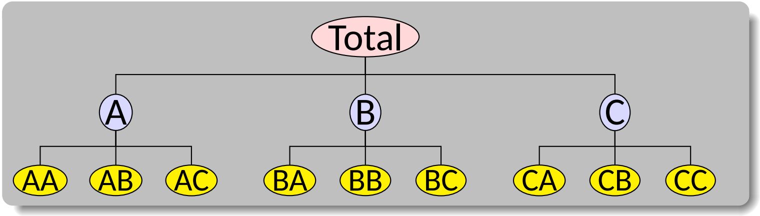

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Grouped time series

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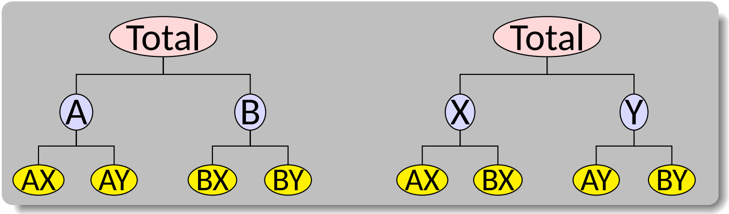

Examples

- Labour turnover by occupation and state
- Spectacle sales by brand, gender, stores, etc.

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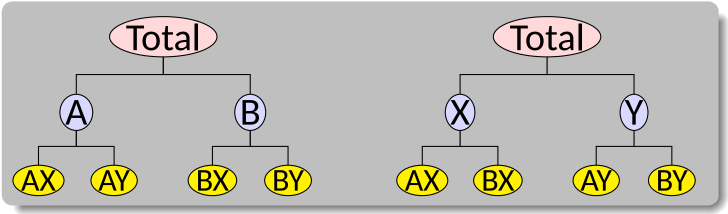

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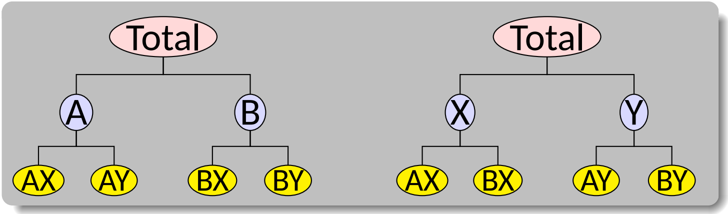

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The problem

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1. Forecast all series at all levels of aggregation using an automatic forecasting algorithm (e.g., `ets`, `auto.arima`, `...`)
2. Reconcile the resulting forecasts so they add up correctly using least squares optimization (i.e., find closest reconciled forecasts to the original forecasts).
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hts package for R

hts: Hierarchical and Grouped Time Series

Methods for analysing and forecasting hierarchical and grouped time series

Version:	5.0
Depends:	R ($\geq$ 3.0.2), forecast ($\geq$ 5.0), SparseM, Matrix, matrixcalc
Imports:	parallel, utils, methods, graphics, grDevices, stats
LinkingTo:	Rcpp ($\geq$ 0.11.0), RcppEigen
Suggests:	testthat
Published:	2016-04-06
Author:	Rob J Hyndman, Earo Wang, Alan Lee, Shanika Wickramasuriya
Maintainer:	Rob J Hyndman <Rob.Hyndman at monash.edu>
BugReports:	`https://github.com/robjhyndman/hts/issues`
License:	GPL ($\geq$ 2)

Example using R

```
library(hts)

# bts is a matrix containing the bottom level time series
# nodes describes the hierarchical structure
y <- hts(bts, nodes=list(2, c(3,2)))
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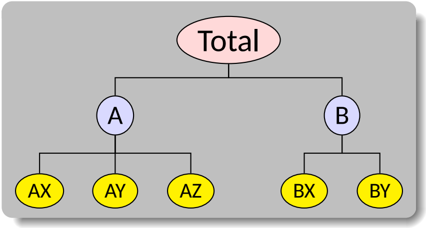
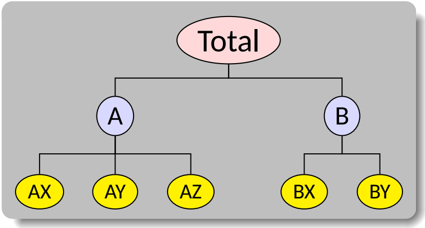


Example using R

```
library(hts)

# bts is a matrix containing the bottom level time series
# nodes describes the hierarchical structure
y <- hts(bts, nodes=list(2, c(3,2)))

# Forecast 10-step-ahead using WLS combination method
# ETS used for each series by default
fc <- forecast(y, h=10)
```



gts function

Usage

```
gts(y, characters)
```

Arguments

`y`	Multivariate time series containing the bottom level series
`characters`	Vector of integers, or list of vectors, showing how column names indicate group structure.

Example

```
bnames <-
  c("VIC1F","VIC1M","VIC2F","VIC2M","VIC3F","VIC3M",
    "NSW1F","NSW1M","NSW2F","NSW2M","NSW3F","NSW3M")
bts <- matrix(ts(rnorm(120)), ncol = 12)
colnames(bts) <- bnames
x <- gts(bts, characters = c(3, 1, 1))
```

gts function

Example 2

```
bnames <-
 c("VICMelbAA","VICMelbAB",
   "VICGeelAA","VICGeelAB",
   "VICMelbBA","VICMelbBB",
   "VICGeelBA","VICGeelBB",
   "NSWSyndAA","NSWSyndAB",
   "NSWWollAA","NSWWollAB",
   "NSWSyndBA","NSWSyndBB",
   "NSWWollBA","NSWWollBB")
bts <- matrix(ts(rnorm(160)), ncol = 16)
colnames(bts) <- bnames
x <- gts(bts, characters = list(c(3, 4), c(1, 1)))
```

forecast.gts function

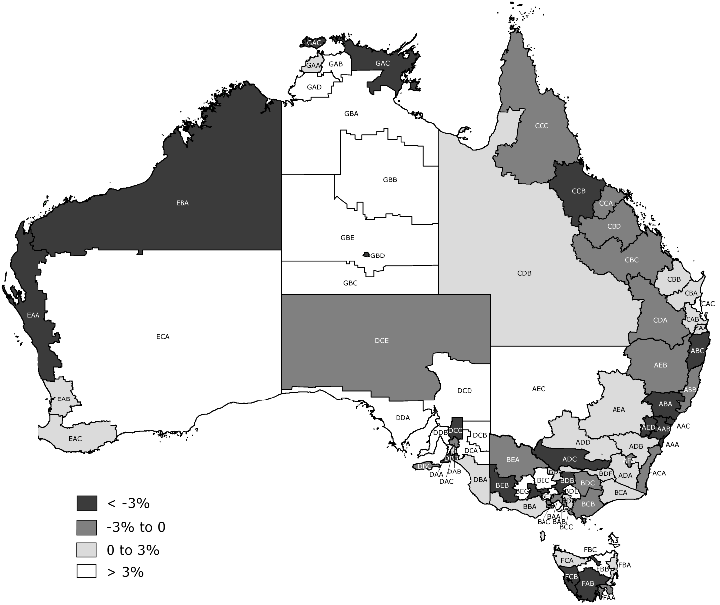
Usage

```
forecast(object, h,
  method = c("comb", "bu", "mo","tdgsa", "tdgsf", "tdfp"),
  weights = c("wls", "ols", "mint", "nseries"),
  fmethod = c("ets", "arima", "rw"),
  algorithms = c("lu", "cg", "chol", "recursive", "slm"),
  covariance = c("shr", "sam"),
  positive = FALSE,
  parallel = FALSE, num.cores = 2, ...)
```

Arguments

`object`	Hierarchical time series object of class `gts`.
`h`	Forecast horizon
`method`	Method for distributing forecasts within the hierarchy.
`weights`	Weights used for "optimal combination" method. When weights = "sd", it takes account of the standard deviation of forecasts.
`fmethod`	Forecasting method to use
`algorithm`	Method for solving regression equations
`positive`	If TRUE, forecasts are forced to be strictly positive
`parallel`	If TRUE, allow parallel processing
`num.cores`	If parallel = TRUE, specify how many cores are going to be used

Example: Australian tourism



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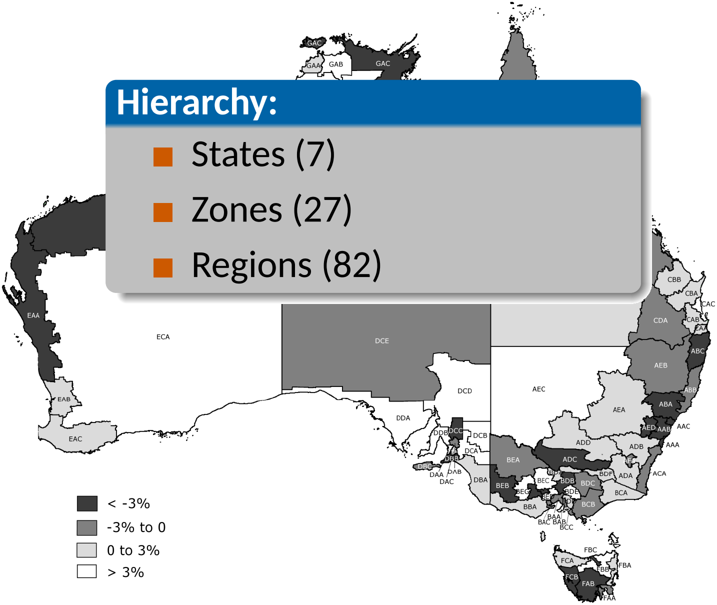

Hierarchy:

- States (7)
- Zones (27)
- Regions (82)

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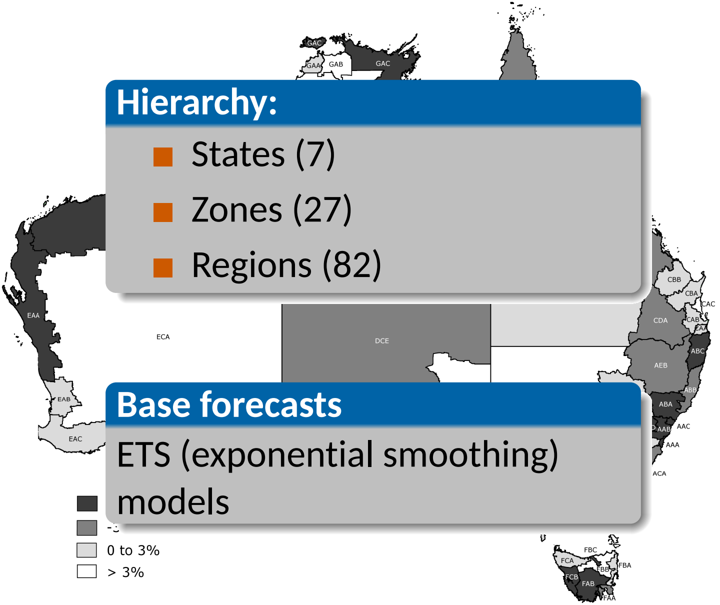

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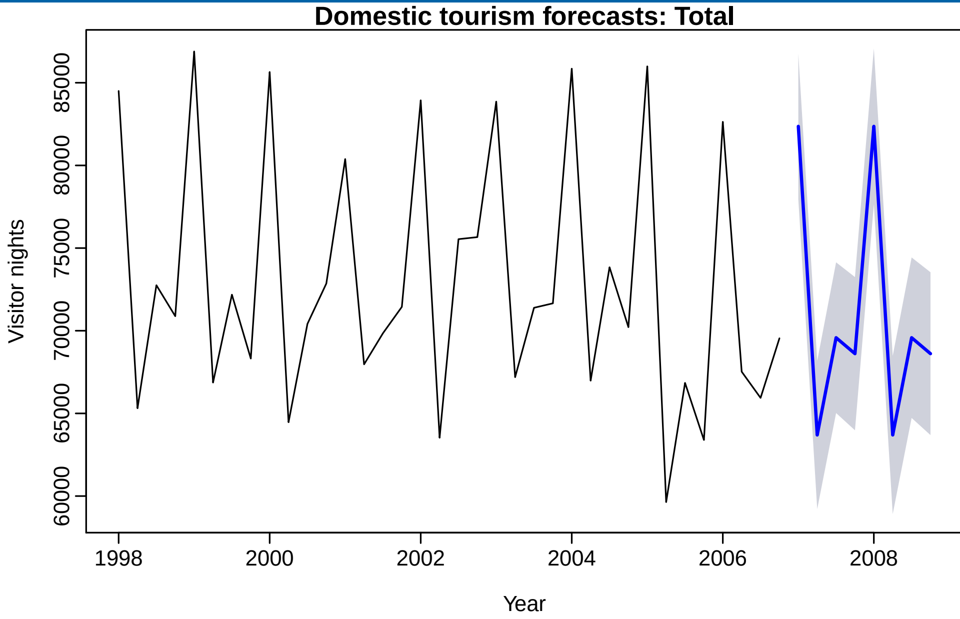
Base forecasts

ETS (exponential smoothing) models

Base forecasts



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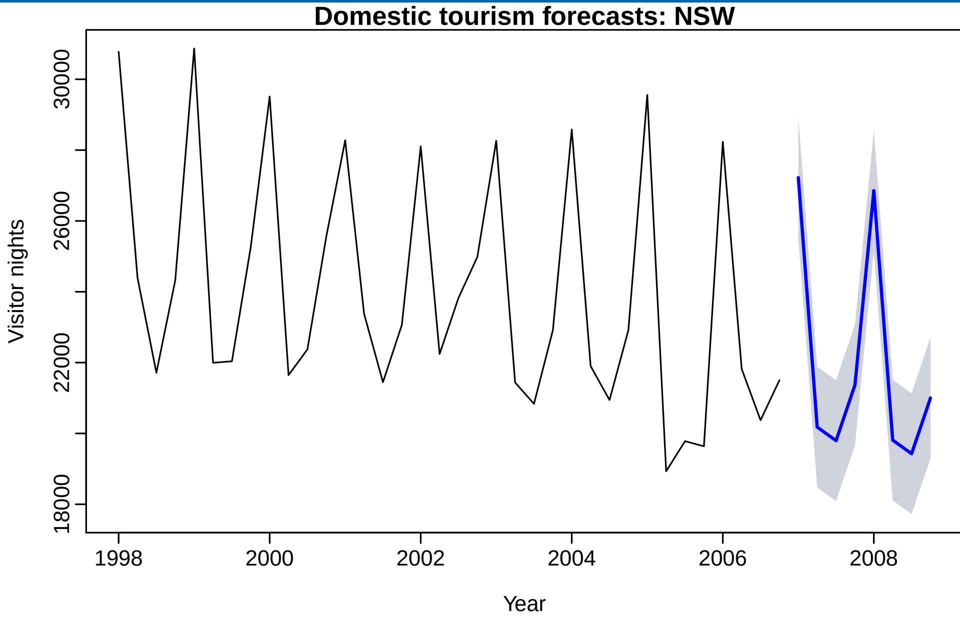

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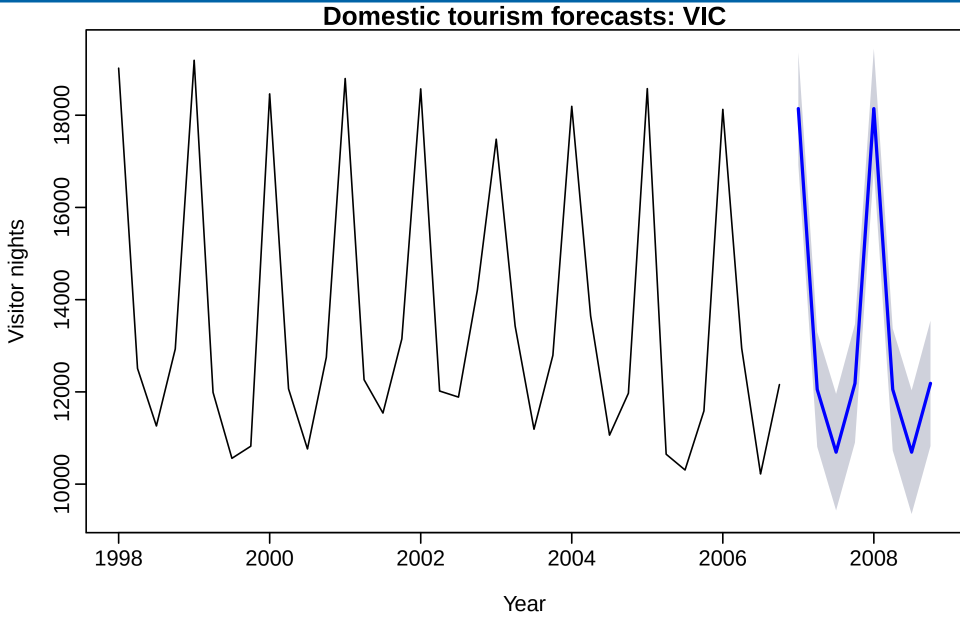

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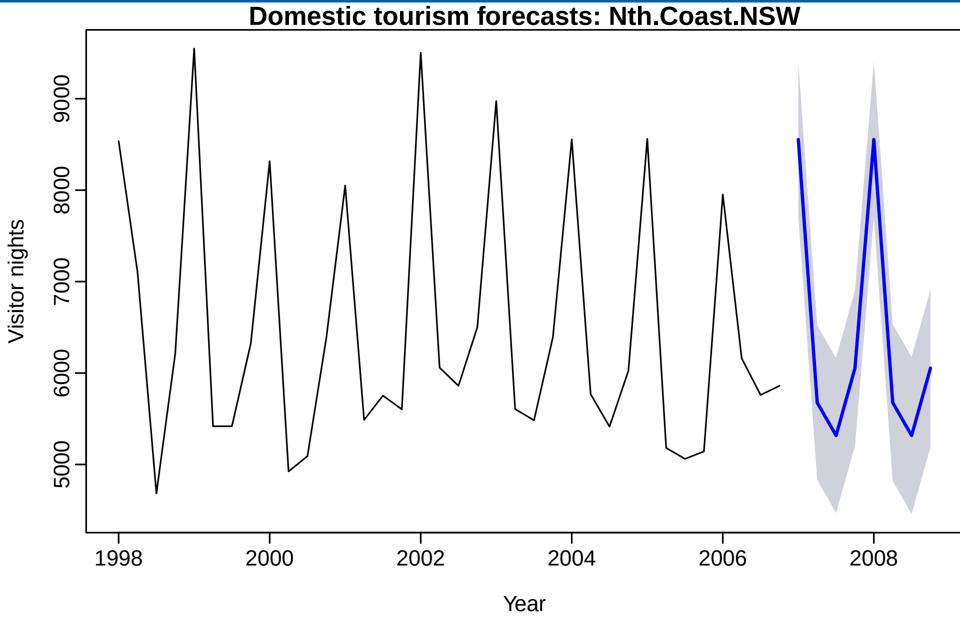

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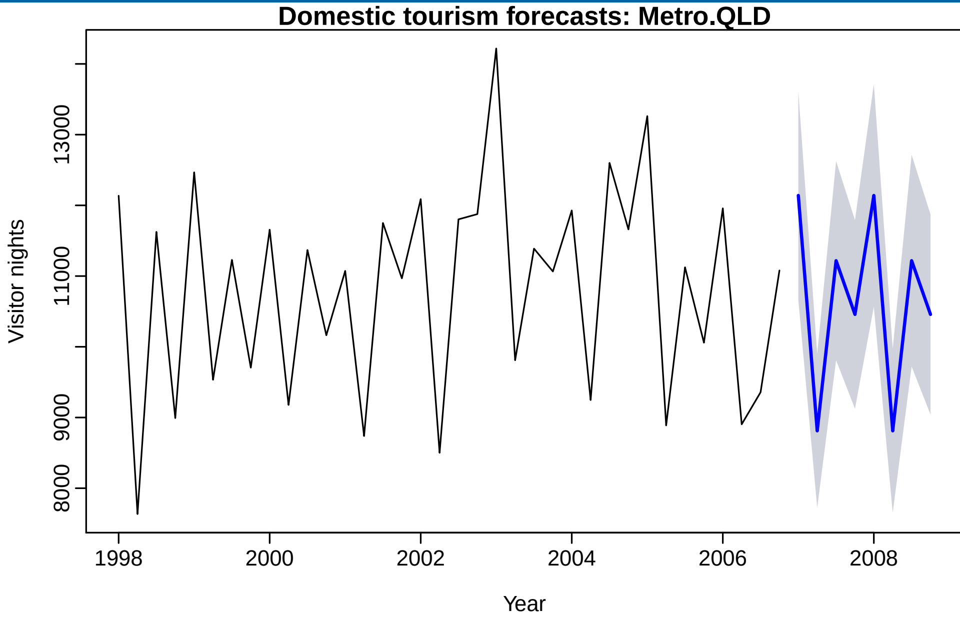

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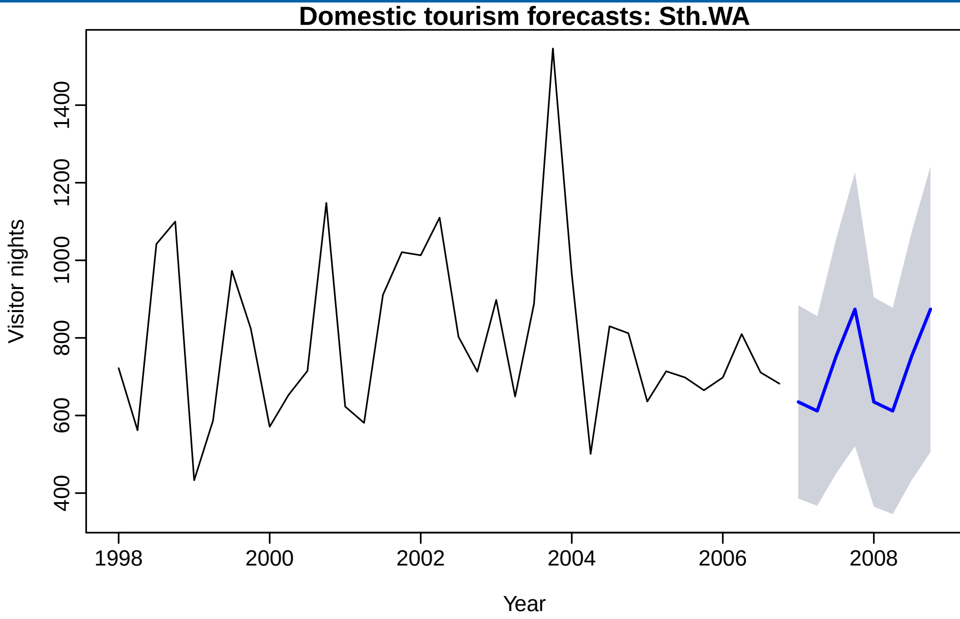

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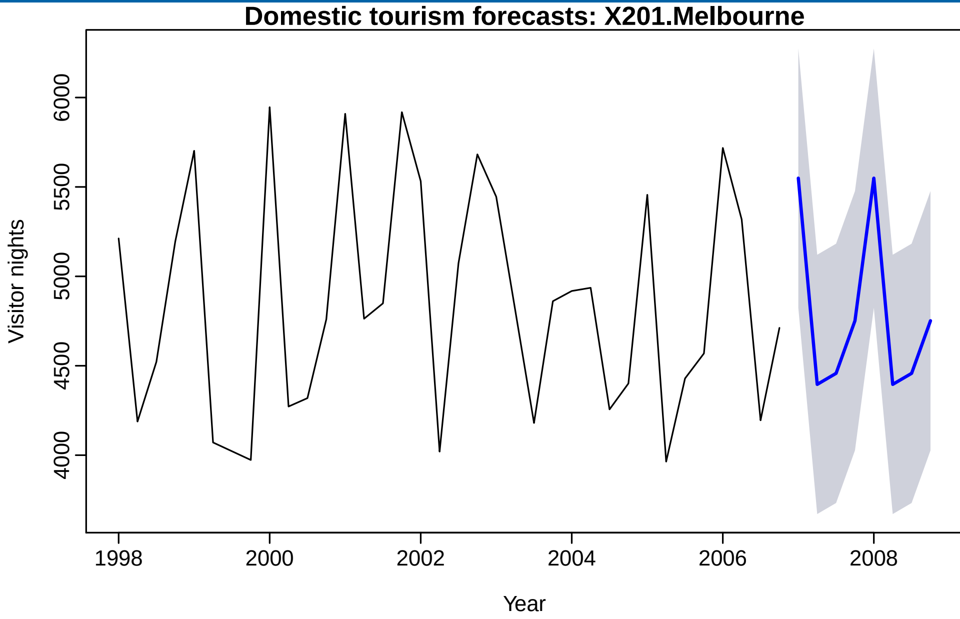

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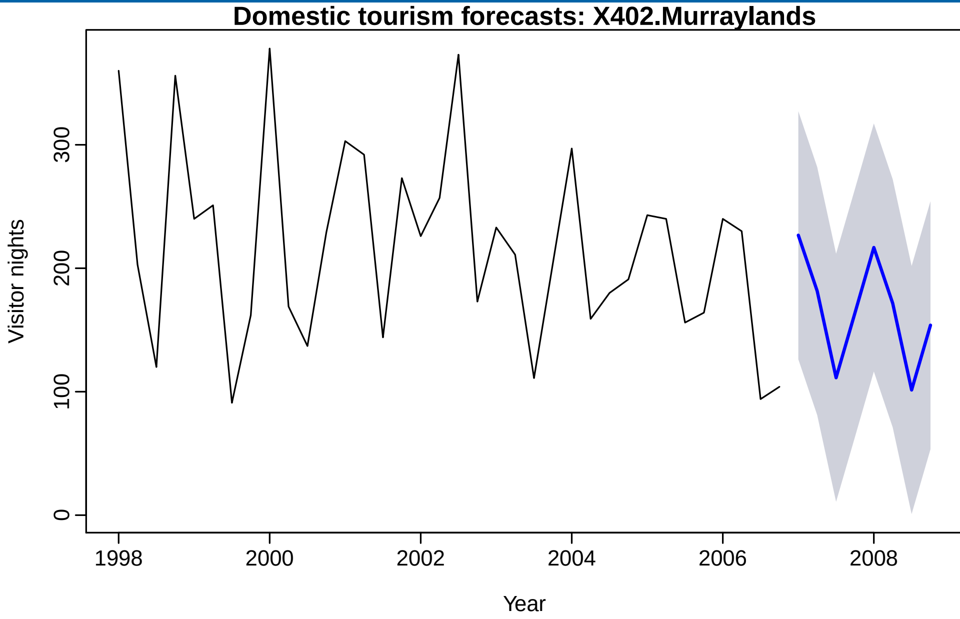

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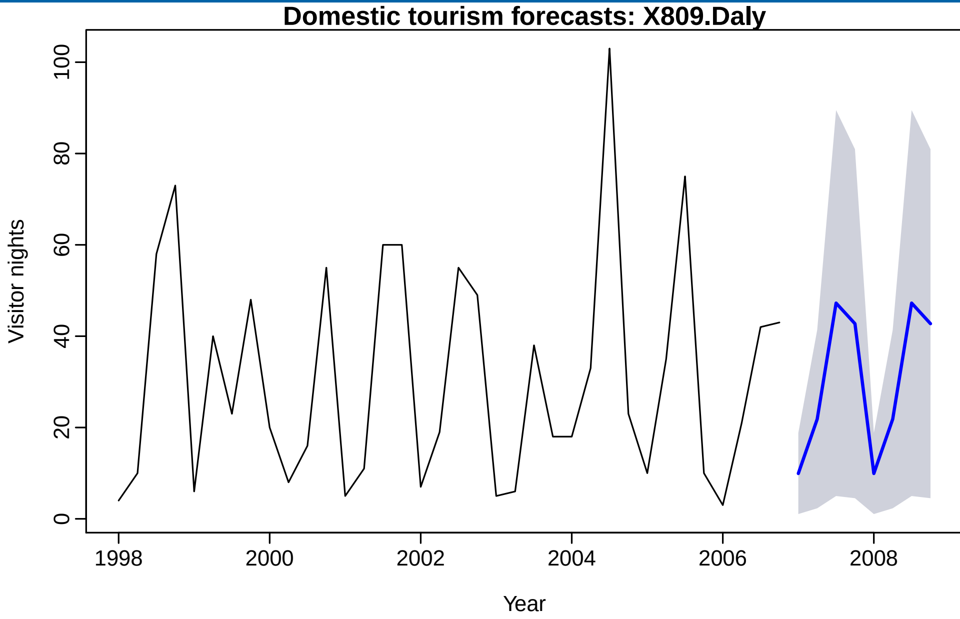

Forecast evaluation

Training sets **Test sets $h = 1$**

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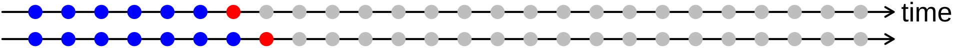

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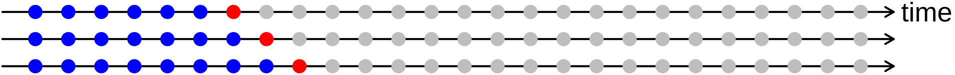

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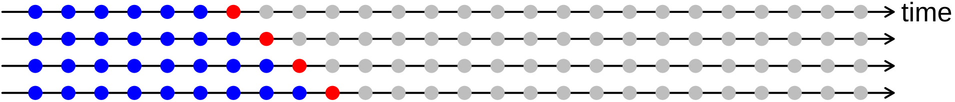

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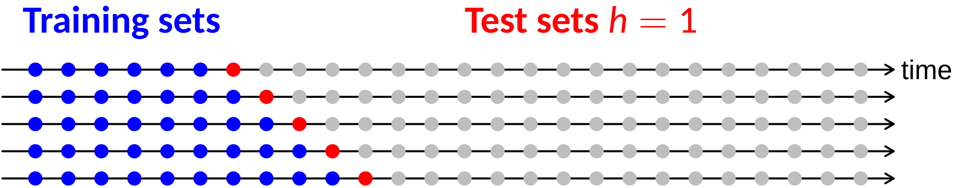

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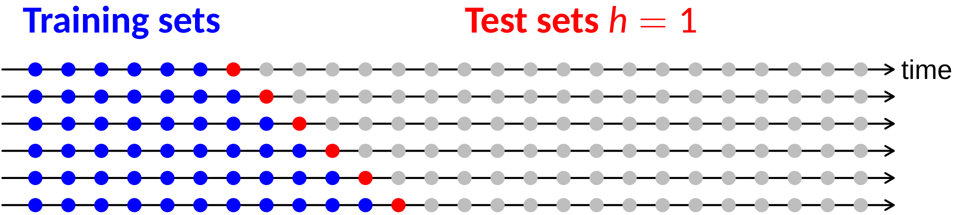

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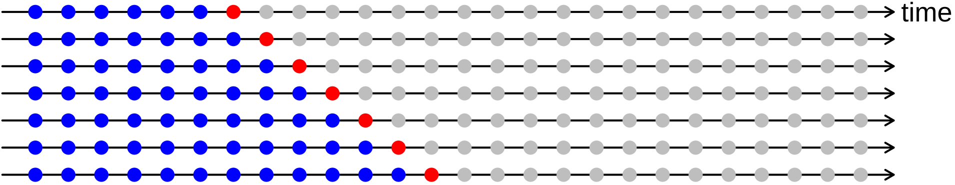

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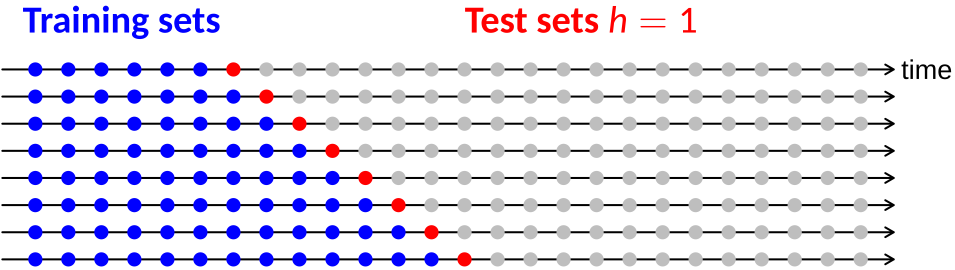

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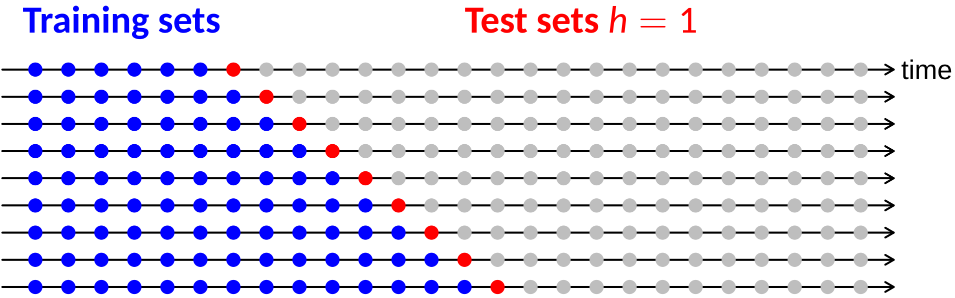

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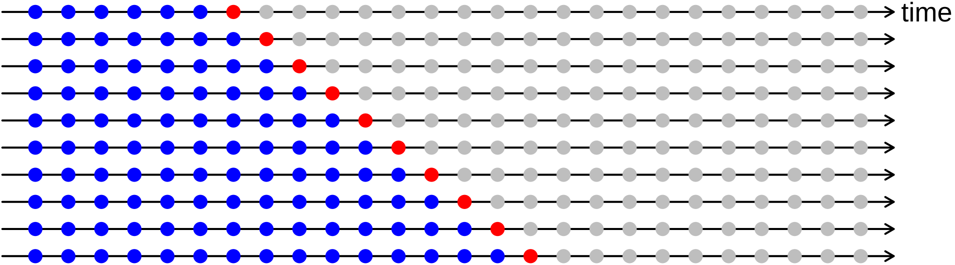

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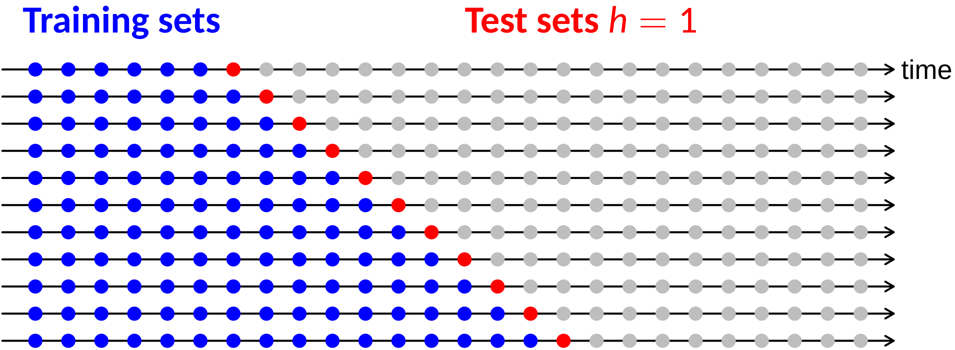

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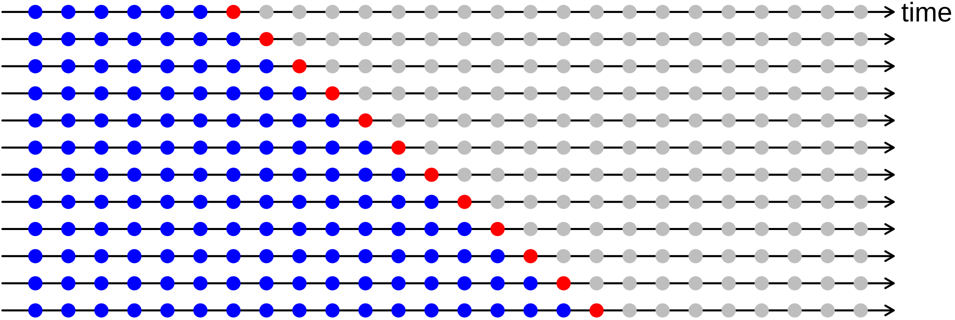

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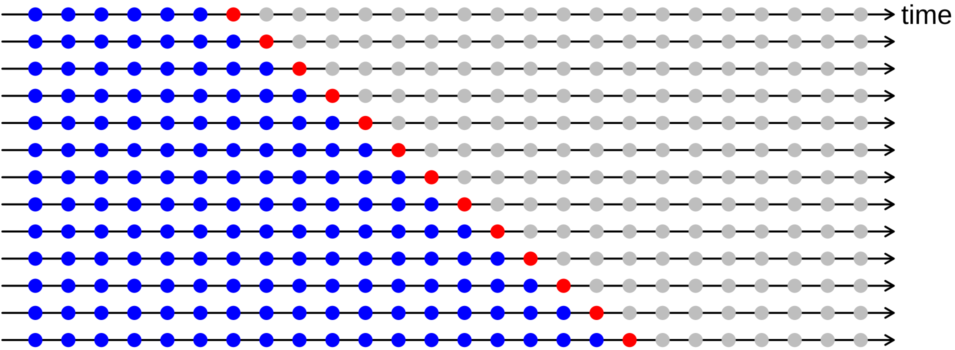

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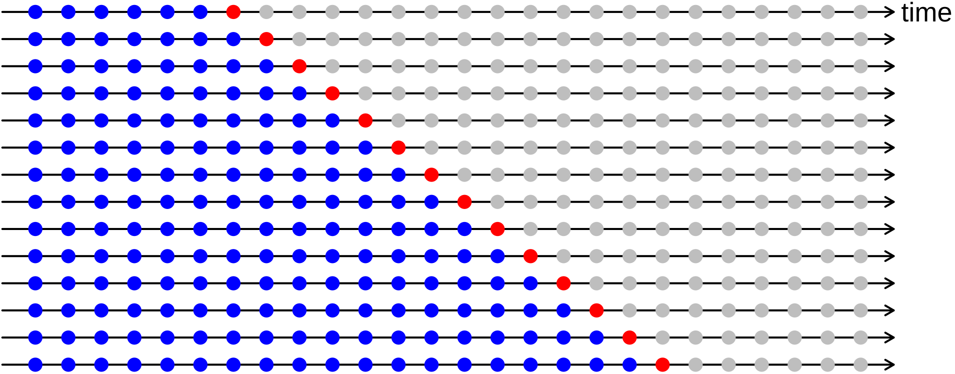

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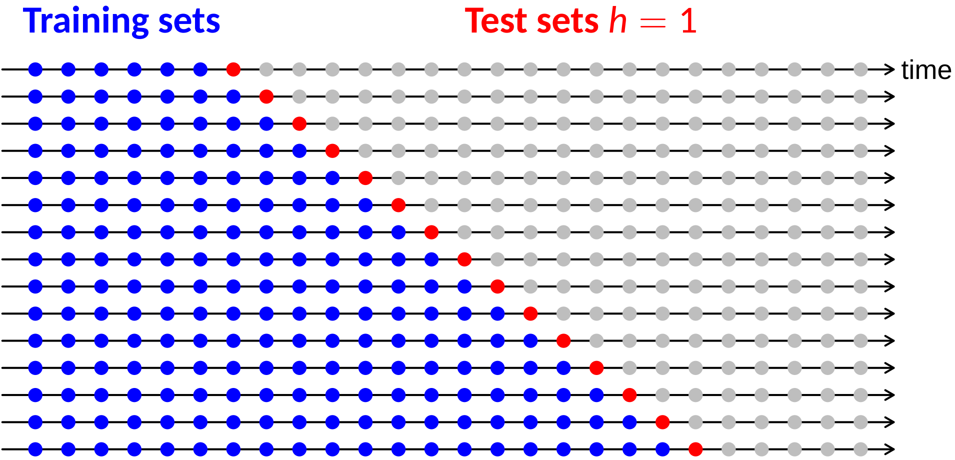

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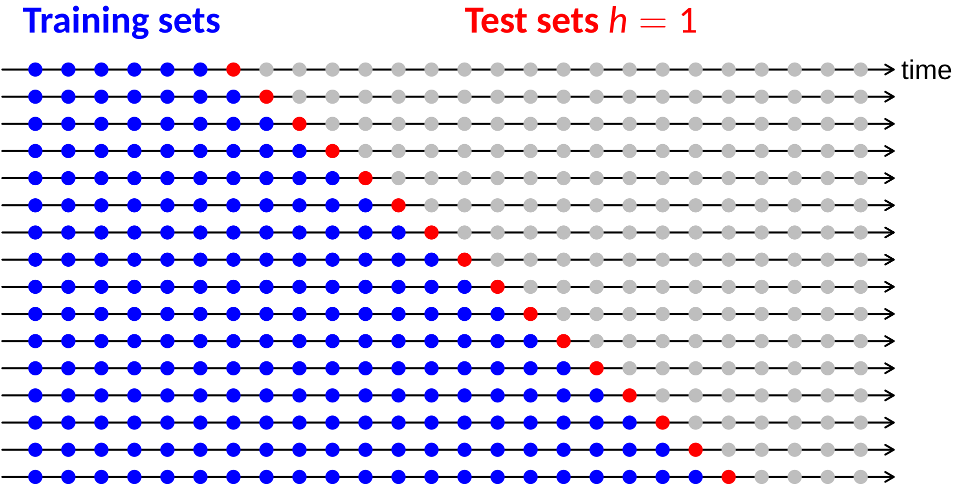

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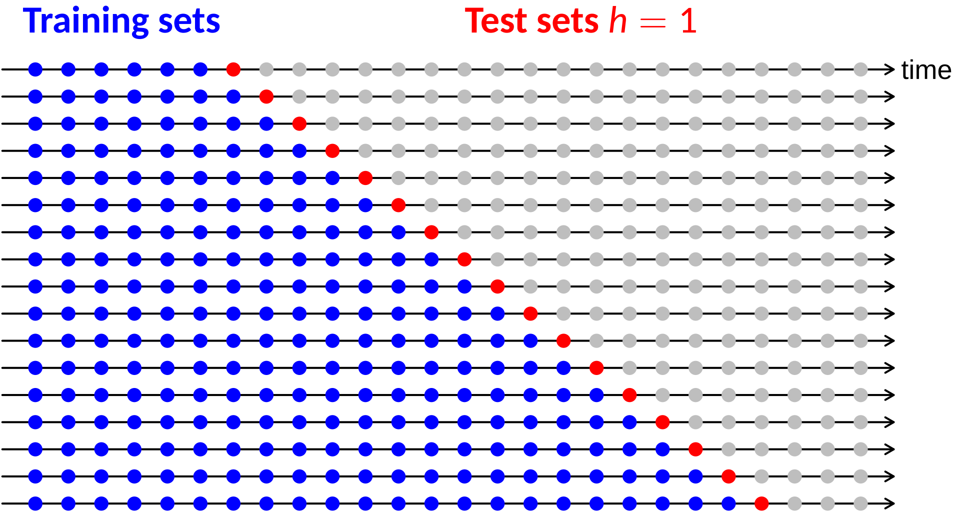

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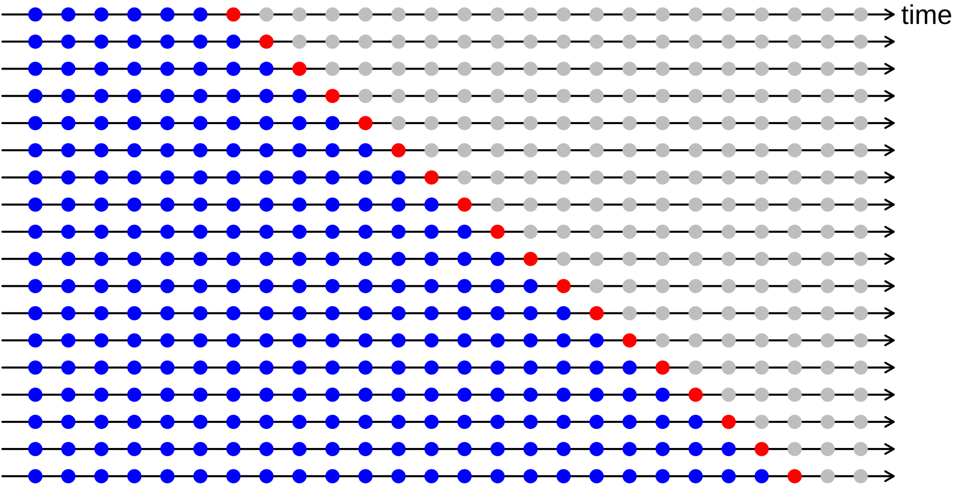

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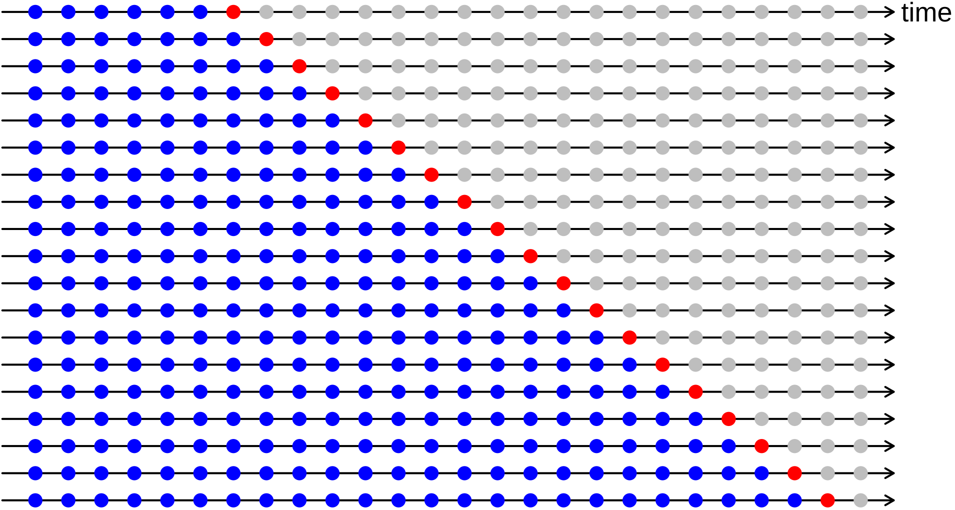

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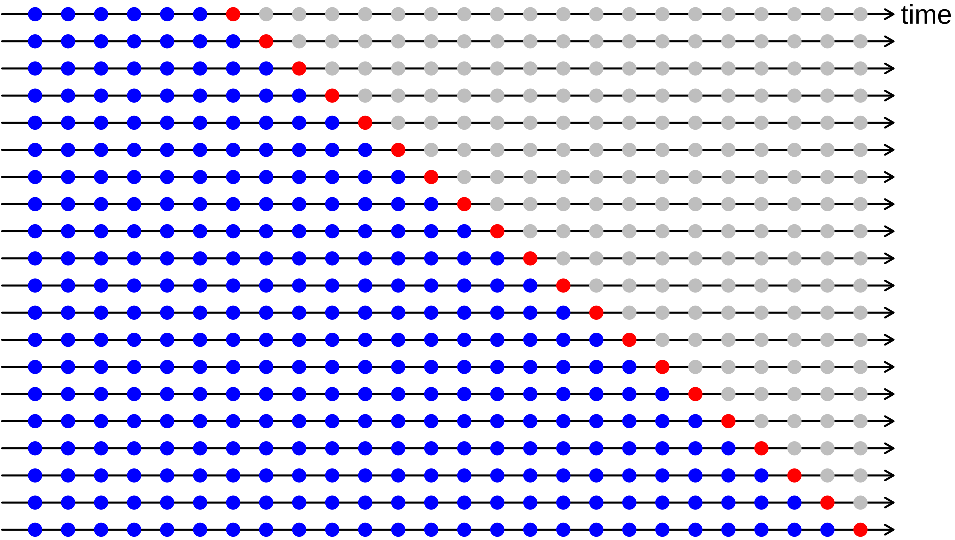
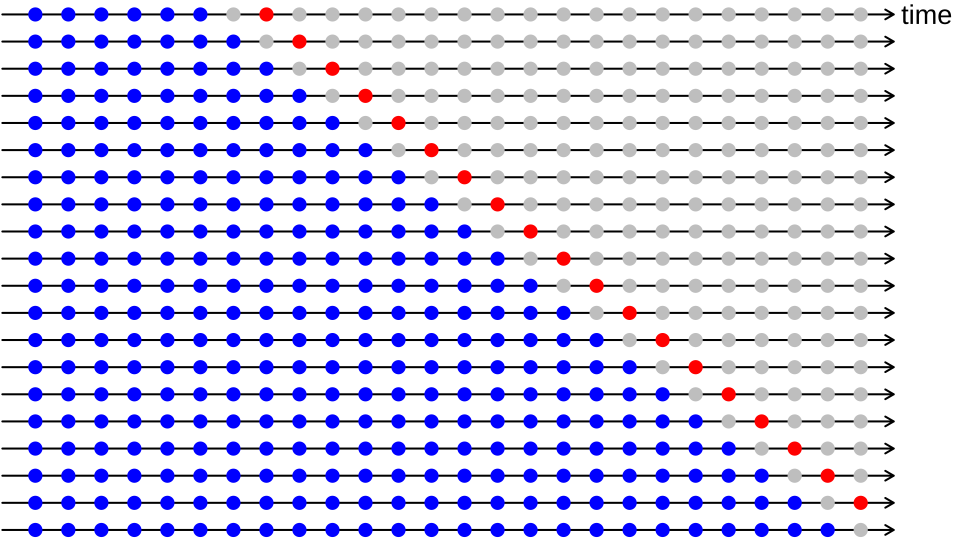


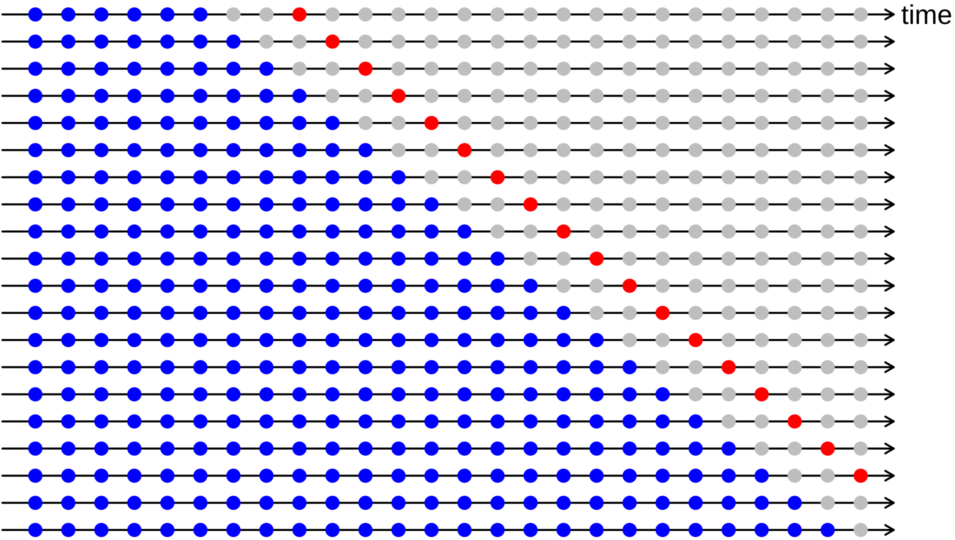
Forecast evaluation

Training sets **Test sets $h = 2$**



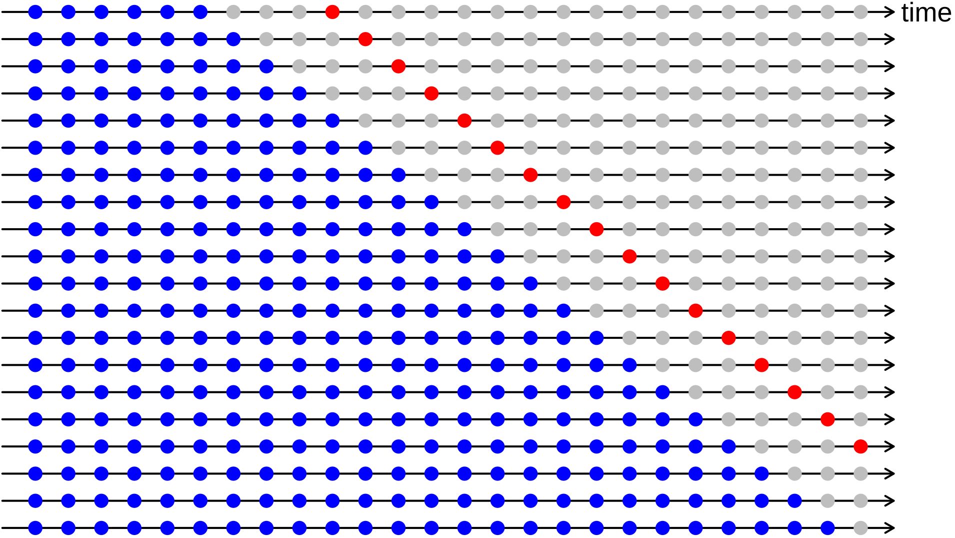
Forecast evaluation

Training sets **Test sets $h = 3$**

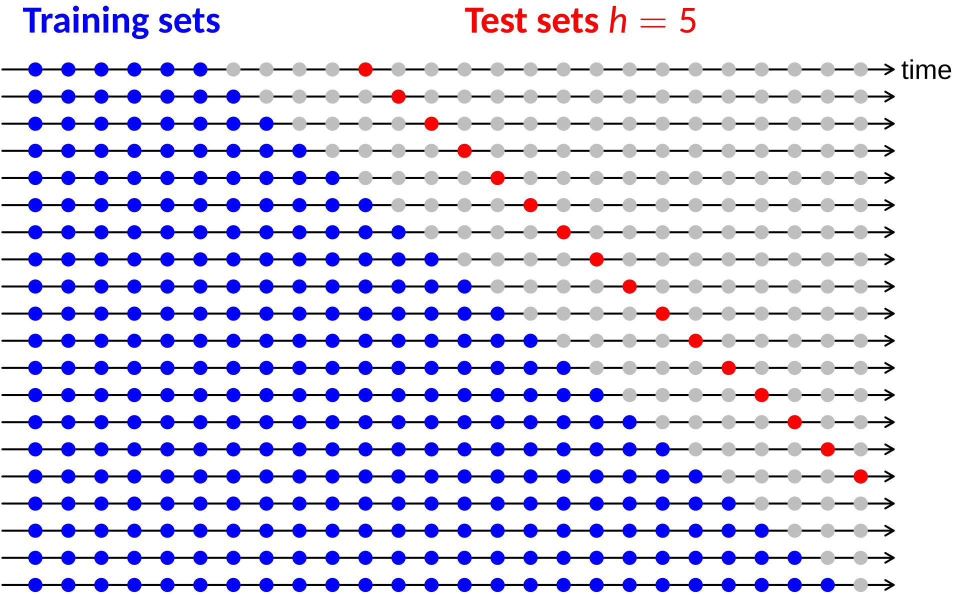


Forecast evaluation

Training sets **Test sets $h = 4$**

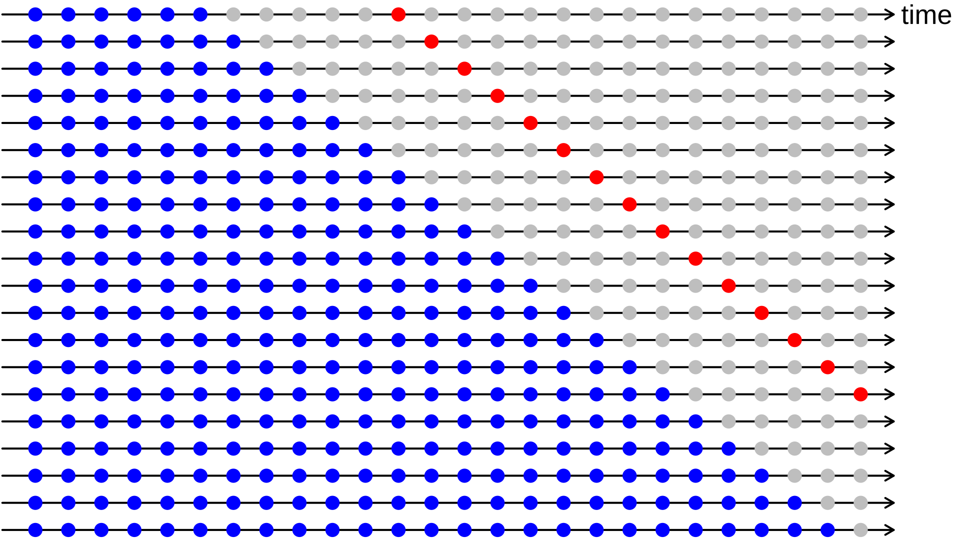


Forecast evaluation



Forecast evaluation

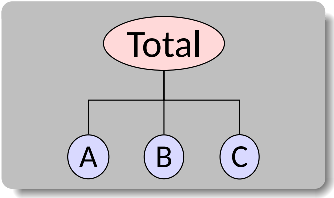
Training sets **Test sets $h = 6$**



Hierarchy: states, zones, regions

RMSE	$h = 1$	$h = 2$	$h = 3$	$h = 4$	$h = 5$	$h = 6$	**Ave**
				Forecast horizon			
				Australia			
Base	1762.04	1770.29	1766.02	1818.82	1705.35	1721.17	**1757.28**
Bottom	1736.92	1742.69	1722.79	1752.74	1666.73	1687.43	**1718.22**
OLS	1747.60	1757.68	1751.77	1800.67	1686.00	1706.45	**1741.69**
WLS	1705.21	1715.87	1703.75	1729.56	1627.79	1661.24	**1690.57**
GLS	1704.64	1715.60	1705.31	1729.04	1626.36	1661.64	**1690.43**
				States			
Base	399.77	404.16	401.92	407.26	395.38	401.17	**401.61**
Bottom	404.29	406.95	404.96	409.02	399.80	401.55	**404.43**
OLS	404.47	407.62	405.43	413.79	401.10	404.90	**406.22**
WLS	398.84	402.12	400.71	405.03	394.76	398.23	**399.95**
GLS	398.84	402.16	400.86	405.03	394.59	398.22	**399.95**
				Regions			
Base	93.15	93.38	93.45	93.79	93.50	93.56	**93.47**
Bottom	93.15	93.38	93.45	93.79	93.50	93.56	**93.47**
OLS	93.28	93.53	93.64	94.17	93.78	93.88	**93.71**
WLS	93.02	93.32	93.38	93.72	93.39	93.53	**93.39**
GLS	92.98	93.27	93.34	93.66	93.34	93.46	**93.34**

Hierarchical time series

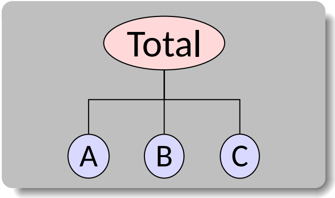


y_t : observed aggregate of all series at time t.

$y_{X,t}$: observation on series X at time t.

$\boldsymbol{b}_t$: vector of all series at bottom level in time t.

Hierarchical time series



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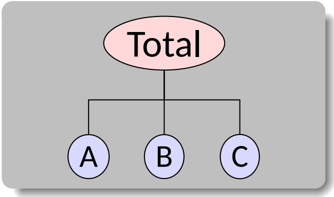

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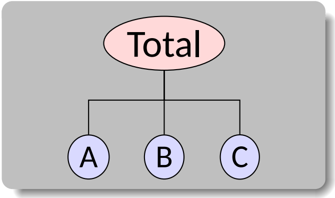

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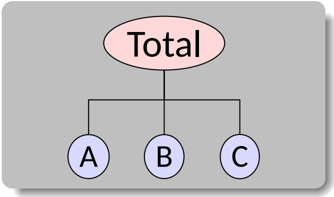

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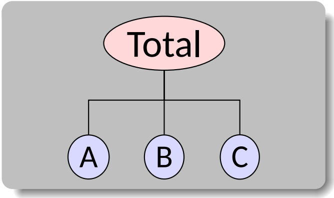

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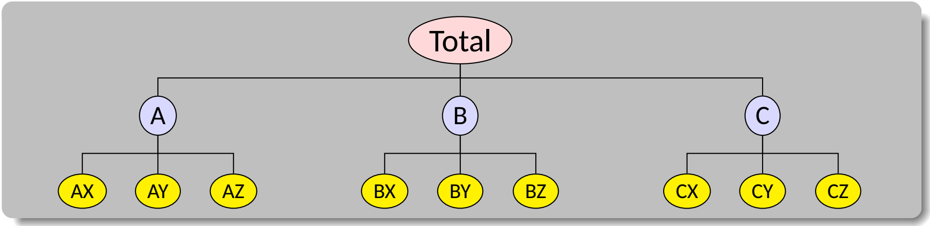

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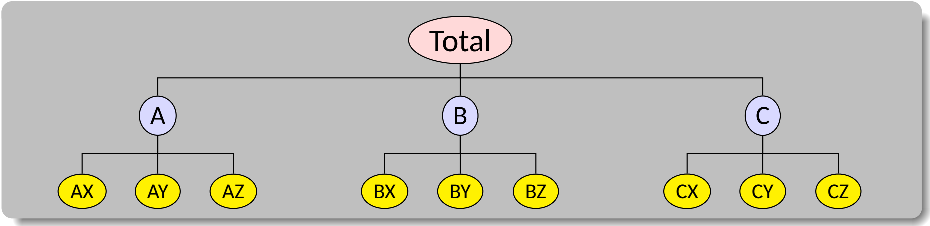

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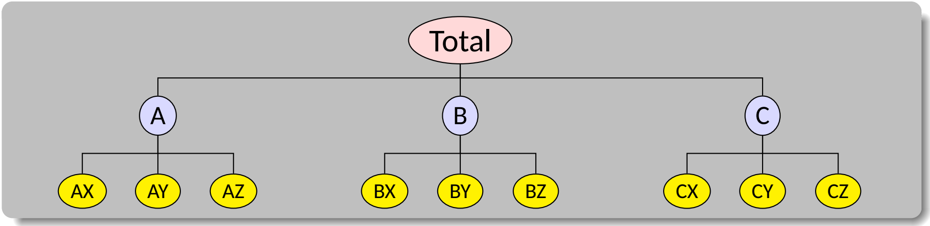

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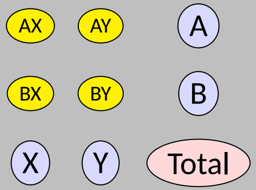
Grouped data



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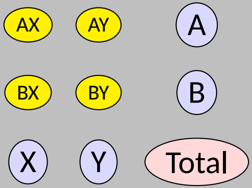

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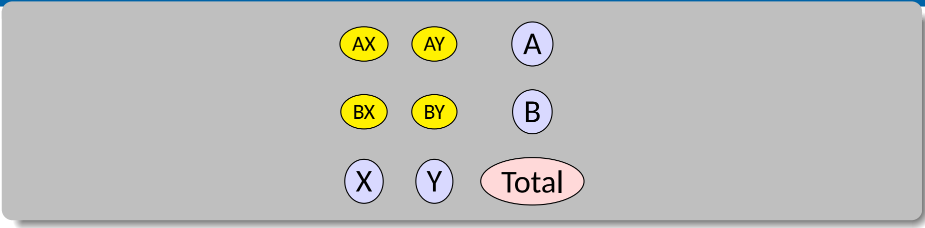

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Hierarchical and grouped time series

Every collection of time series with aggregation constraints can be written as

$$\boldsymbol{y}_t = \boldsymbol{S}\boldsymbol{b}_t$$

where

- $\boldsymbol{y}_t$ is a vector of all series at time t
- $\boldsymbol{b}_t$ is a vector of the most disaggregated series at time t
- $\boldsymbol{S}$ is a "summing matrix" containing the aggregation constraints.

Forecasting notation

Let $\hat{\boldsymbol{y}}_n(h)$ be vector of initial h-step forecasts, made at time n, stacked in same order as $\boldsymbol{y}_t$.
(In general, they will not "add up".)

Reconciled forecasts must be of the form:

$$\tilde{\boldsymbol{y}}_n(h) = \boldsymbol{S}\boldsymbol{P}\hat{\boldsymbol{y}}_n(h)$$

for some matrix $\boldsymbol{P}$.

- $\boldsymbol{P}$ extracts and combines base forecasts $\hat{\boldsymbol{y}}_n(h)$ to get bottom-level forecasts.

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Optimal combination forecasts

Main result

The best (minimum sum of variances) unbiased forecasts are obtained when $\boldsymbol{P} = (\boldsymbol{S}'\Sigma_h^{-1}\boldsymbol{S})^{-1}\boldsymbol{S}'\Sigma_h^{-1}$, where Σ_h is the h-step base forecast error covariance matrix.

$$\tilde{\boldsymbol{y}}_n(h) = \boldsymbol{S}(\boldsymbol{S}'\Sigma_h^{-1}\boldsymbol{S})^{-1}\boldsymbol{S}'\Sigma_h^{-1}\hat{\boldsymbol{y}}_n(h)$$

Reconciled forecasts — Base forecasts

Problem: Σ_h hard to estimate, especially for $h > 1$.

Solutions:

- Ignore Σ_h (OLS)
- Assume Σ_h diagonal (WLS) [Default in `hts`]
- Try to estimate Σ_h (GLS)

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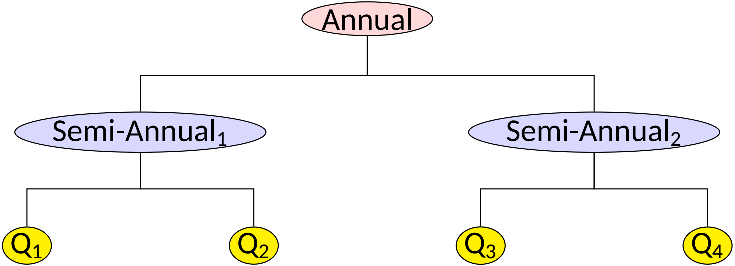
Outline

Lab Session 15

Outline

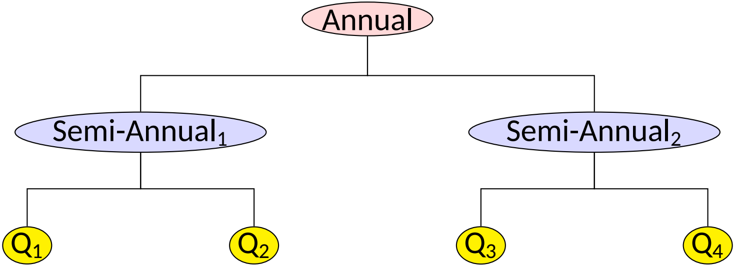
Temporal hierarchies



Basic idea:

- Forecast series at each available frequency.
- Optimally reconcile forecasts within the same year.

Temporal hierarchies



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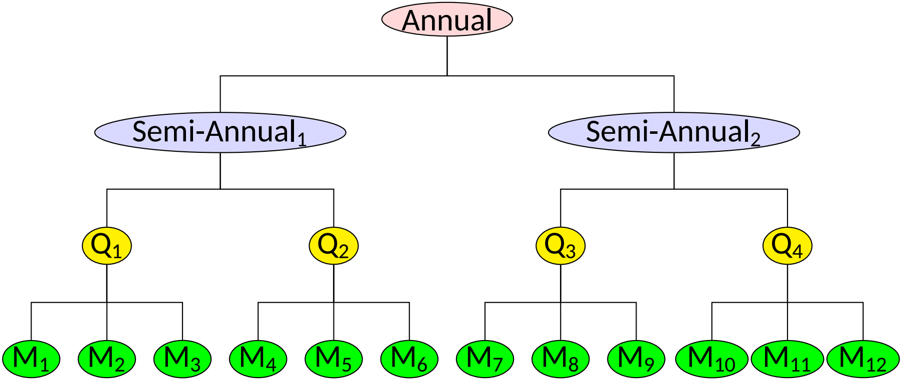
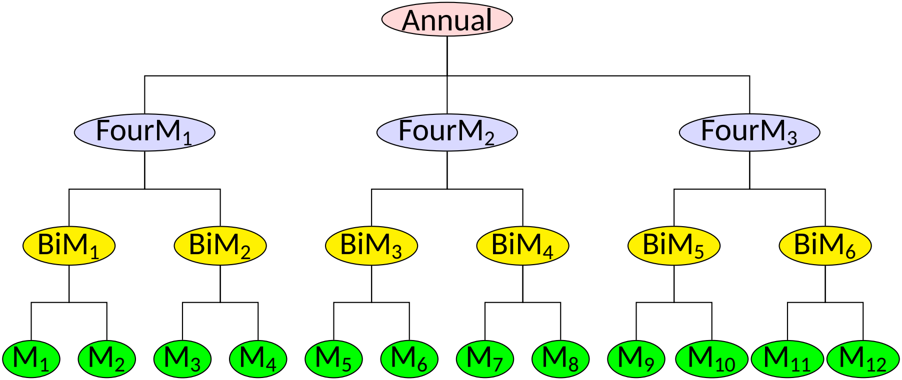


- $k = 2, 4, 12$ nodes
- $k = 3, 6, 12$ nodes
- Why not $k = 2, 3, 4, 6, 12$ nodes?

Monthly series



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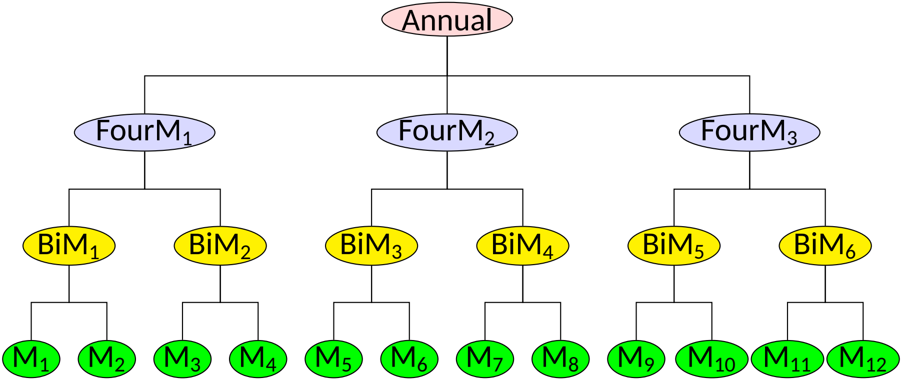

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Monthly data

$$\underbrace{\begin{pmatrix} A \\ SemiA_1 \\ SemiA_2 \\ FourM_1 \\ FourM_2 \\ FourM_3 \\ Q_1 \\ \vdots \\ Q_4 \\ BiM_1 \\ \vdots \\ BiM_6 \\ M_1 \\ \vdots \\ M_{12} \end{pmatrix}}_{(28\times 1)} = \underbrace{\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & & \vdots & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & & \vdots & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ & & & & & & \boldsymbol{I}_{12} & & & & & \end{pmatrix}}_{\boldsymbol{S}} \underbrace{\begin{pmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \\ M_5 \\ M_6 \\ M_7 \\ M_8 \\ M_9 \\ M_{10} \\ M_{11} \\ M_{12} \end{pmatrix}}_{\boldsymbol{B}_t}$$

In general

For a time series $y_1, \dots, y_T$, observed at frequency m, we generate aggregate series

$$y_j^{[k]} = \sum_{t=1+(j-1)k}^{jk} y_t, \qquad \text{for } j = 1, \dots, \lfloor T/k \rfloor$$

- $k \in F(m) = \{\text{factors of } m\}$.
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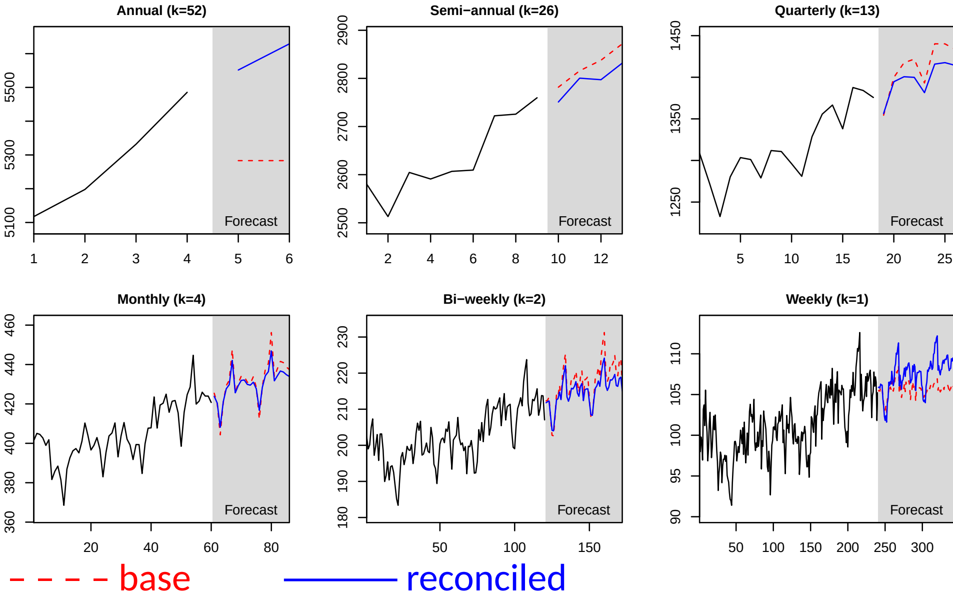
UK Accidents and Emergency Demand



UK Accidents and Emergency Demand

1. Type 1 Departments — Major A&E
2. Type 2 Departments — Single Specialty
3. Type 3 Departments — Other A&E/Minor Injury
4. Total Attendances
5. Type 1 Departments — Major A&E $>$ 4 hrs
6. Type 2 Departments — Single Specialty $>$ 4 hrs
7. Type 3 Departments — Other A&E/Minor Injury $>$ 4 hrs
8. Total Attendances $>$ 4 hrs
9. Emergency Admissions via Type 1 A&E
10. Total Emergency Admissions via A&E
11. Other Emergency Admissions (i.e., not via A&E)
12. Total Emergency Admissions
13. Number of patients spending $>$ 4 hrs from decision to admission

UK Accidents and Emergency Demand

- **Minimum training set**: all data except the last year
- Base forecasts using `auto.arima()`.
- Mean Absolute Scaled Errors for 1, 4 and 13 weeks ahead using a rolling origin.

Aggr. Level	h	Base	Reconciled	Change
Weekly	1	1.6	1.3	−17.2%
Weekly	4	1.9	1.5	−18.6%
Weekly	13	2.3	1.9	−16.2%
Weekly	1–52	2.0	1.9	−5.0%
Annual	1	3.4	1.9	−42.9%

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- **Minimum training set**: all data except the last year
- Base forecasts using `auto.arima()`.
- Mean Absolute Scaled Errors for 1, 4 and 13 weeks ahead using a rolling origin.

Aggr. Level	h	**Base**	**Reconciled**	**Change**
Weekly	1	1.6	1.3	-17.2%
Weekly	4	1.9	1.5	-18.6%
Weekly	13	2.3	1.9	-16.2%
Weekly	1–52	2.0	1.9	-5.0%
Annual	1	3.4	1.9	-42.9%

thief package for R

thief: Temporal HIErarchical Forecasting

Install from CRAN

```
install.packages("thief")
```

Usage

```
library(thief)
thief(y)
```

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Outline

Lab Session 16