Bagplots, boxplots and outlier detection for functional data

Han Lin Shang & Rob J Hyndman

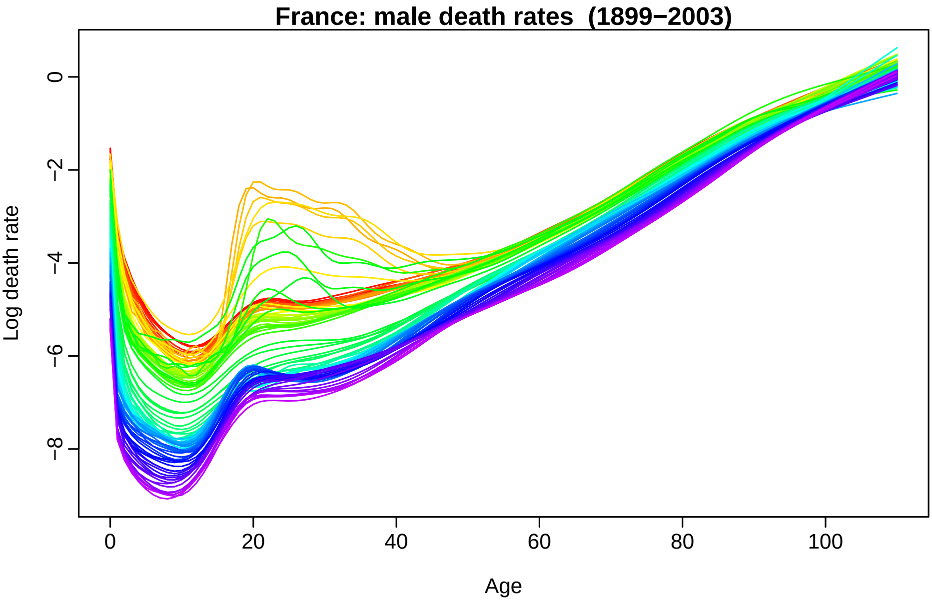
Business & Economic Forecasting Unit

MONASH University

Outline

Outline

French male mortality rates



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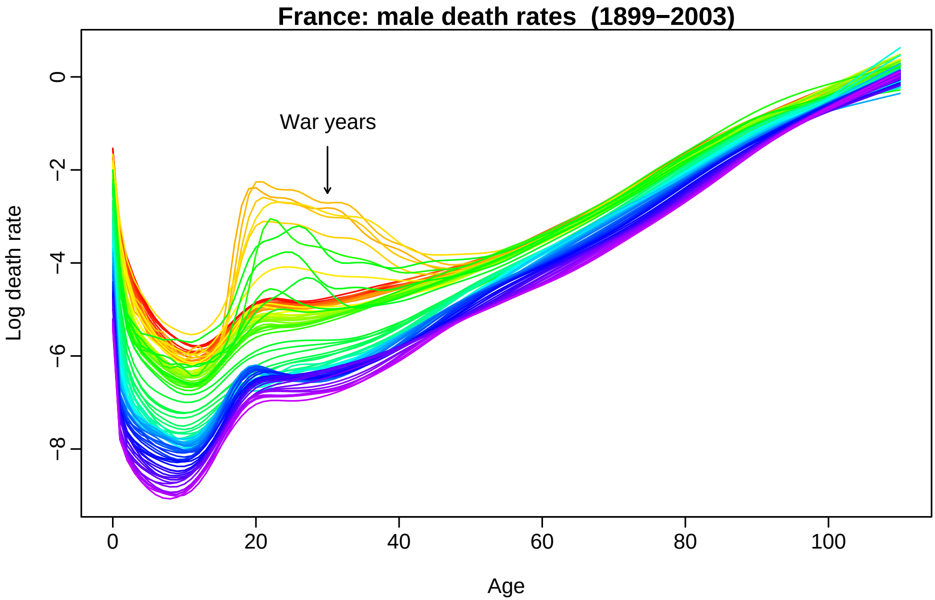

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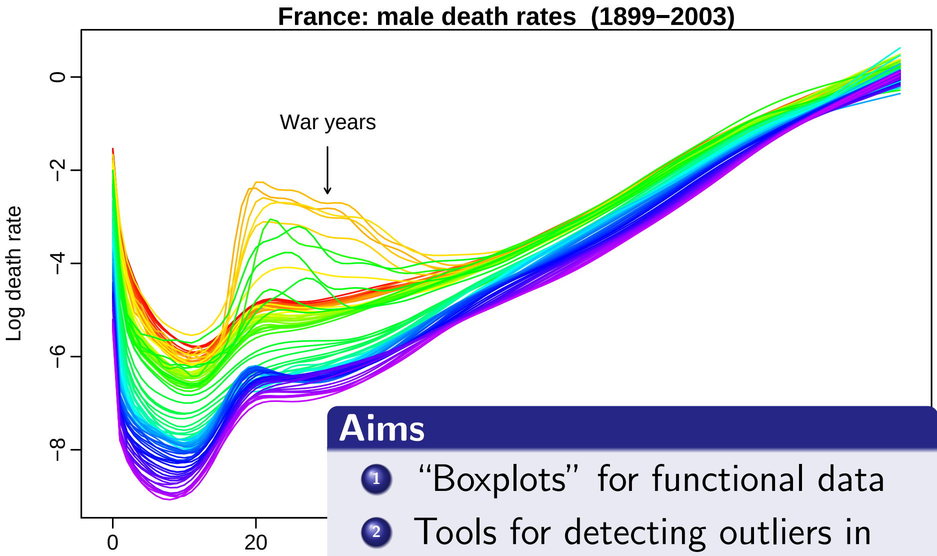

Aims

1. "Boxplots" for functional data
2. Tools for detecting outliers in functional data

Robust principal components

Let $\{y_i(x)\}$, $i = 1, \ldots, n$, be a set of curves.

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➥ Outliers show up in bivariate score space.

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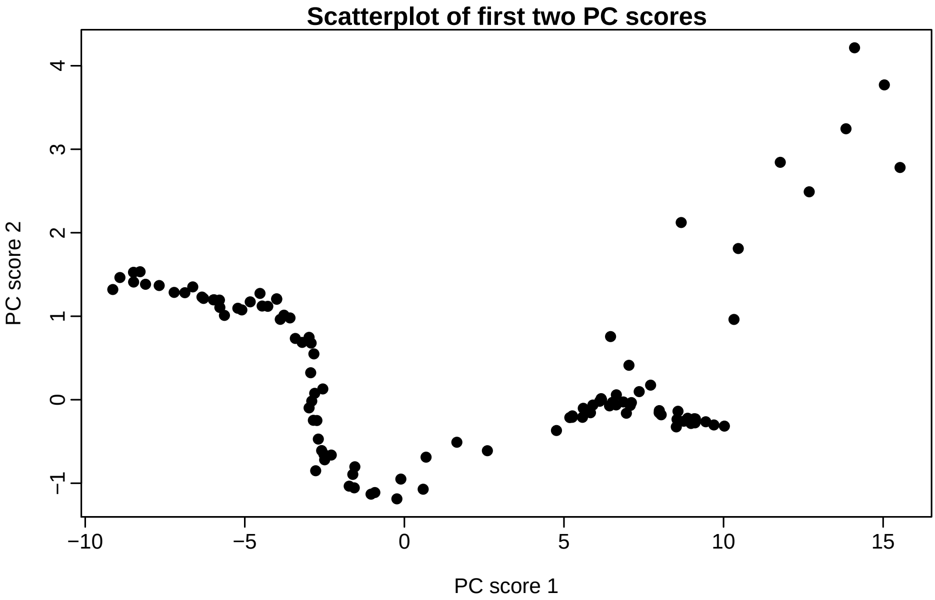

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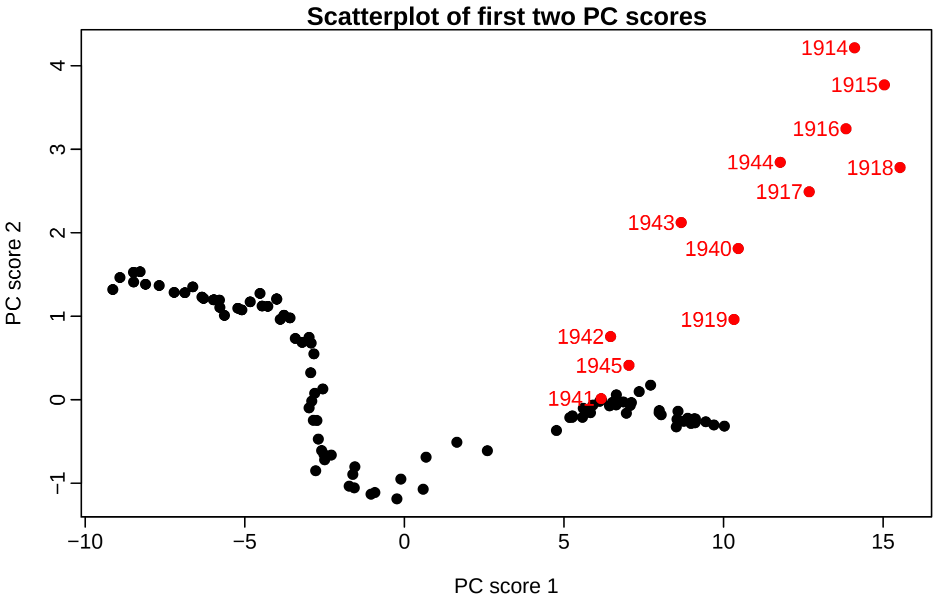
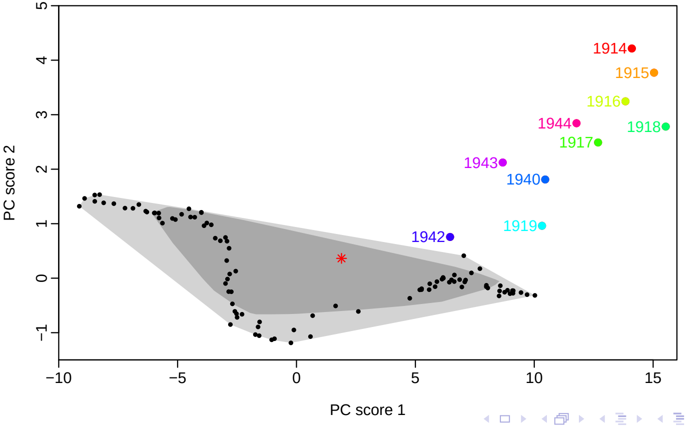


Outline

Functional bagplot

- Bivariate bagplot due to Rousseeuw et al. (1999).
- Rank points by halfspace location depth.
- Display median, 50% convex hull and outer convex hull (with 99% coverage if bivariate normal).



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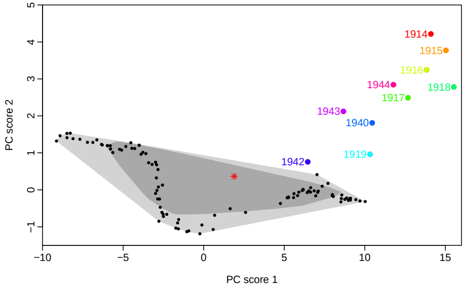
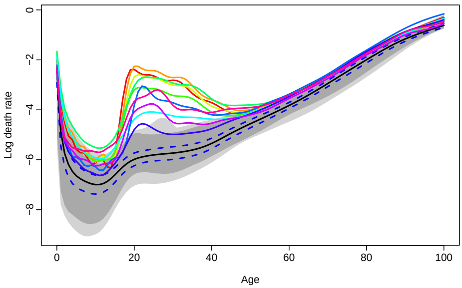
- Rank points by halfspace location depth.
- Display median, 50% convex hull and outer convex hull (with 99% coverage if bivariate normal).
- Boundaries contain all curves inside bags.
- 95% CI for median curve also shown.





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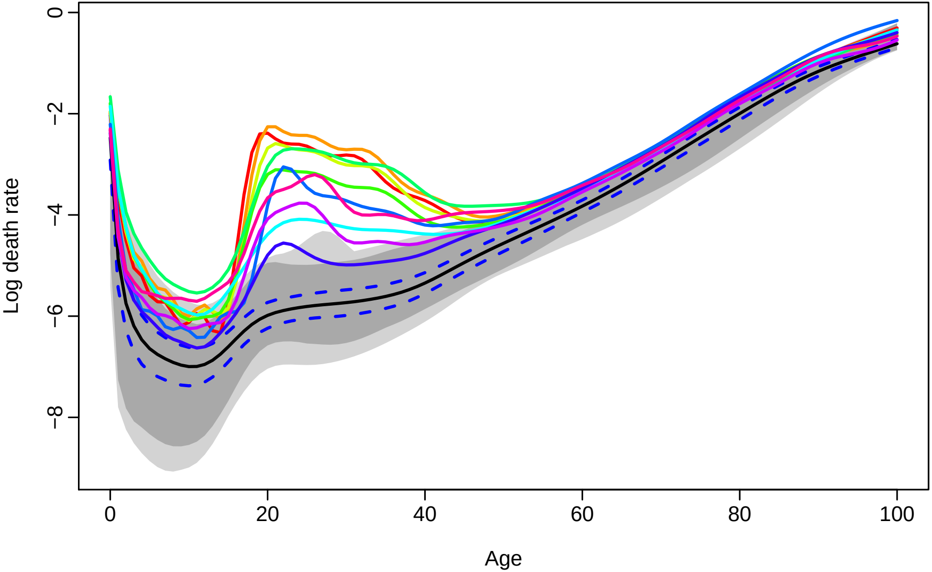
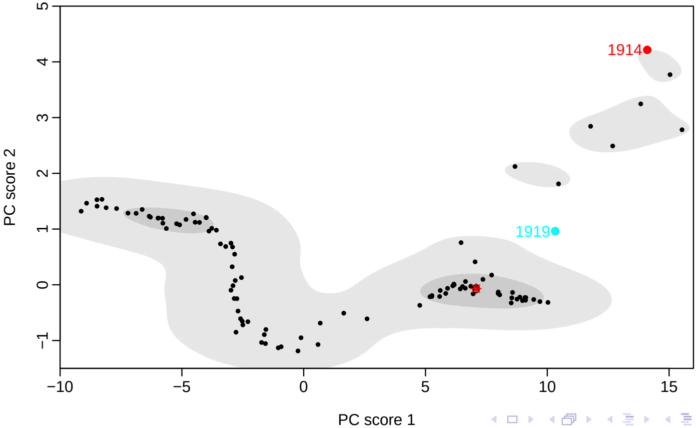


Functional HDR boxplot

- Bivariate HDR boxplot due to Hyndman (1996).
- Rank points by value of kernel density estimate.
- Display mode, 50% and (usually) 99% highest density regions (HDRs) and mode.



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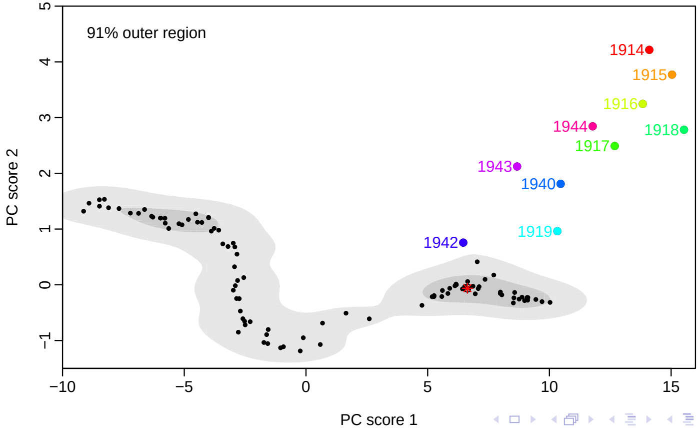

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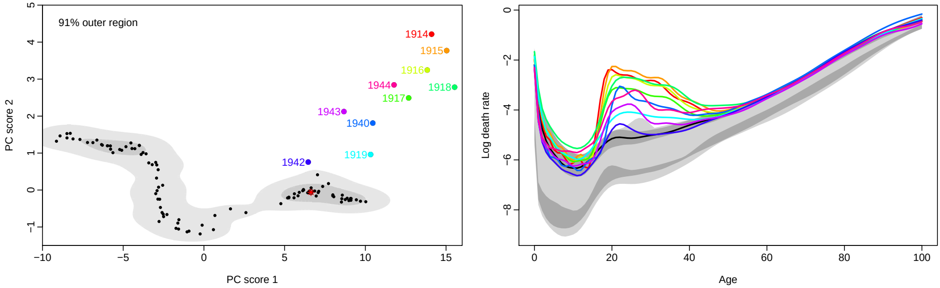

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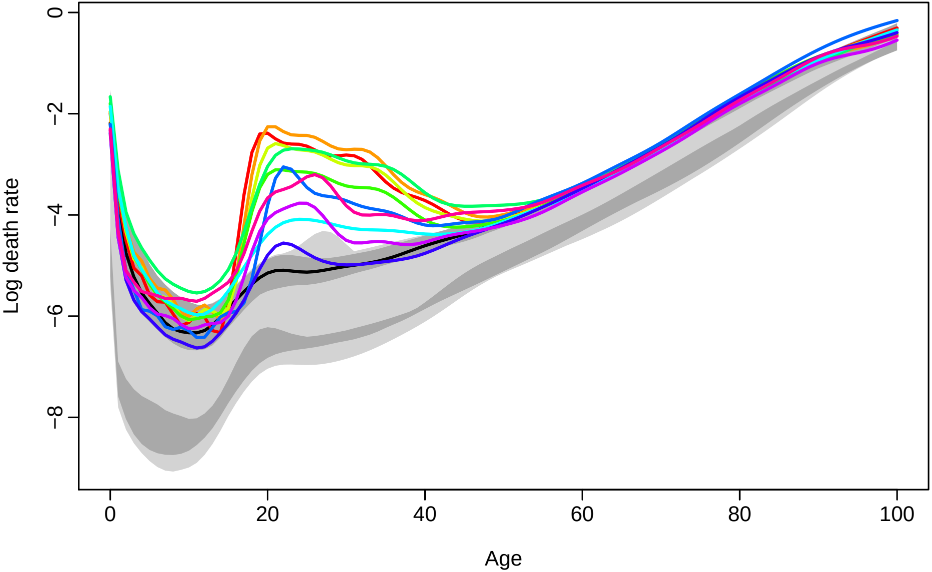

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Outlier detection: existing methods

Likelihood ratio method

- Febrero et al. (2007) find curve that maximizes LRT statistic.
- If LRT $> C$, then curve is considered outlier.
- C is computed via smoothed bootstrap.
- Process continues until no more outliers.

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Disadvantages

- Computationally intensive.
- Ignores shape outliers.
- If trimmed mean is used and there is no outlier, C will be downward biased.

Outlier detection: existing methods

Integrated squared error method

- Hyndman & Ullah (2007) proposed the use of

$$v_i = \int_x \left[\hat{y}_i(x) - \mu(x) - \sum_{k=1}^{K} z_{i,k}\phi_k(x)\right]^2 dx$$

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Disadvantages

- Depends on K and λ.
- If K large, outliers modelled by higher components.

Outlier detection: comparison

French male mortality data set

Based on historical information, the outliers are expected to be 1914–1919 & 1940–1945.

Method	Outliers detected
Likelihood ratio	—
Integrated squared error	1914–1918, 1940, 1943–1944
Bagplot	1914–1919, 1940, 1942–1944
91% HDR boxplot	1914–1919, 1940, 1942–1944

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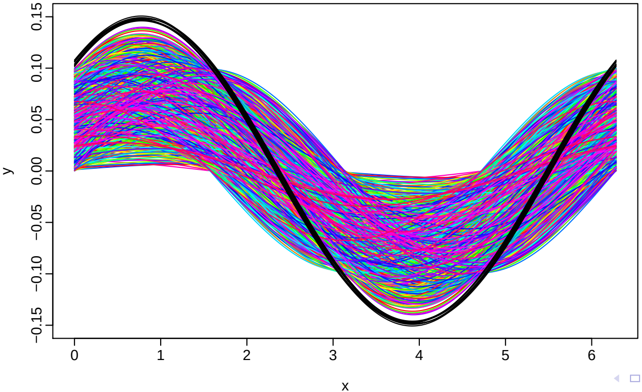
Method	Sensitivity	Specificity	Time (s)
Likelihood ratio	0%	100%	18.8
Integrated squared error	50%	94%	3.4
Bagplot	83%	98%	0.6
91% HDR boxplot	83%	98%	0.3

Outlier detection: comparison

Simulation

- $y_i(x) = a_i \sin(x) + b_i \cos(x), \qquad 0 < x < 2\pi$
- $a_i, b_i \sim \mathrm{Unif}(0, 0.1)$ with probability 99%
- $a_i, b_i \sim \mathrm{Unif}(0.1, 0.108)$ with probability 1%



Outliers shown in black

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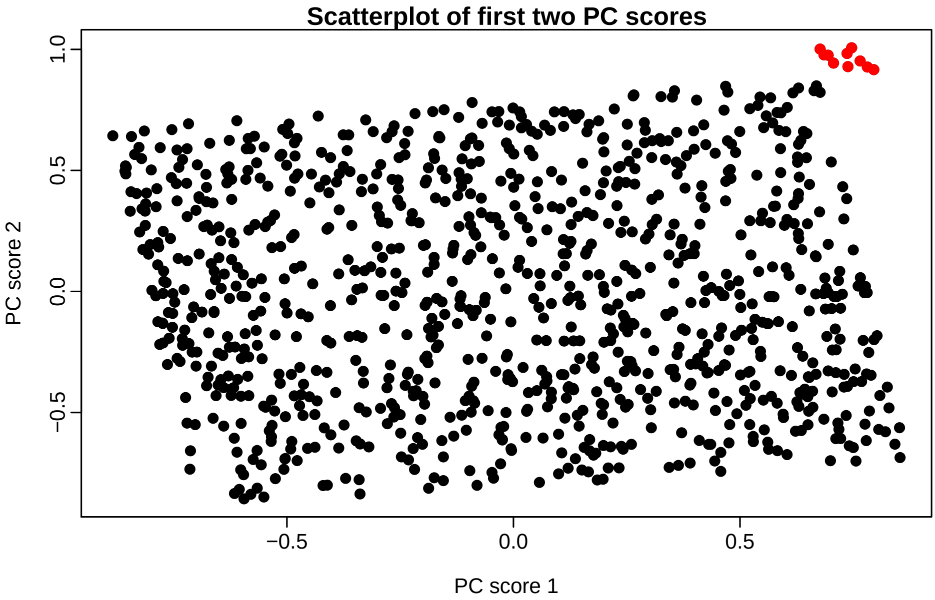

Outlier detection: comparison

Simulation

Method	Outliers detected
Likelihood ratio	—
Integrated squared error	—
Bagplot	—
99% HDR boxplot	All

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Simulation

Method	Outliers detected
Likelihood ratio	—
Integrated squared error	—
Bagplot	—
99% HDR boxplot	All

Method	Sensitivity	Specificity	Time (s)
Likelihood ratio	0%	100%	28.5
Integrated squared error	0%	100%	18.8
Bagplot	0%	100%	7.3
99% HDR boxplot	100%	100%	6.9

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➥ Paper and R code: **www.robhyndman.info**

➥ Comments to: **Han.Shang@buseco.monash.edu**