6.2.4 Cauchy-Schwarz Inequality

You might have seen the **Cauchy-Schwarz inequality** in your linear algebra course. The same inequality is valid for random variables. Let us state and prove the Cauchy-Schwarz inequality for random variables.

> Cauchy-Schwarz Inequality
>
> For any two random variables X and Y, we have
>
> $$|EXY| \leq \sqrt{E[X^2]E[Y^2]},$$
>
> where equality holds if and only if $X = \alpha Y$, for some constant $\alpha \in \mathbb{R}$.

You can prove the Cauchy-Schwarz inequality with the same methods that we used to prove $|\rho(X,Y)| \leq 1$ in Section 5.3.1. Here we provide another proof. Define the random variable $W = (X - \alpha Y)^2$. Clearly, W is a nonnegative random variable for any value of $\alpha \in \mathbb{R}$. Thus, we obtain

$$\begin{aligned} 0 \leq EW &= E(X - \alpha Y)^2 \\ &= E[X^2 - 2\alpha XY + \alpha^2 Y^2] \\ &= E[X^2] - 2\alpha E[XY] + \alpha^2 E[Y^2]. \end{aligned}$$

So, if we let $f(\alpha) = E[X^2] - 2\alpha E[XY] + \alpha^2 E[Y^2]$, then we know that $f(\alpha) \geq 0$, for all $\alpha \in \mathbb{R}$. Moreover, if $f(\alpha) = 0$ for some α, then we have $EW = E(X - \alpha Y)^2 = 0$, which essentially means $X = \alpha Y$ with probability one. To prove the Cauchy-Schwarz inequality, choose $\alpha = \frac{EXY}{EY^2}$. We obtain

$$\begin{aligned} 0 &\leq E[X^2] - 2\alpha E[XY] + \alpha^2 E[Y^2] \\ &= E[X^2] - 2\frac{EXY}{EY^2}E[XY] + \frac{(EXY)^2}{(EY^2)^2}E[Y^2] \\ &= E[X^2] - \frac{(E[XY])^2}{EY^2}. \end{aligned}$$

Thus, we conclude

$$(E[XY])^2 \leq E[X^2]E[Y^2],$$

which implies

$$|EXY| \leq \sqrt{E[X^2]E[Y^2]}.$$

Also, if $|EXY| = \sqrt{E[X^2]E[Y^2]}$, we conclude that $f(\frac{EXY}{EY^2}) = 0$, which implies $X = \frac{EXY}{EY^2}Y$ with probability one.

Example 6.23

Using the Cauchy-Schwarz inequality, show that for any two random variables X and Y

$$|\rho(X, Y)| \leq 1.$$

Also, $|\rho(X, Y)| = 1$ if and only if $Y = aX + b$ for some constants $a, b \in \mathbb{R}$.

Solution

Let

$$U = \frac{X - EX}{\sigma_X}, \qquad V = \frac{Y - EY}{\sigma_Y}.$$

Then $EU = EV = 0$, and $Var(U) = Var(V) = 1$. Using the Cauchy-Schwarz inequality for U and V, we obtain

$$|EUV| \leq \sqrt{E[U^2]E[V^2]} = 1.$$

But note that $EUV = \rho(X, Y)$, thus we conclude

$$|\rho(X, Y)| \leq 1,$$

where equality holds if and only if $V = \alpha U$ for some constant $\alpha \in \mathbb{R}$. That is

$$\frac{Y - EY}{\sigma_Y} = \alpha \frac{X - EX}{\sigma_X},$$

which implies

$$Y = \frac{\alpha \sigma_Y}{\sigma_X} X + (EY - \frac{\alpha \sigma_Y}{\sigma_X} EX).$$

In the Solved Problems section, we provide a generalization of the Cauchy-Schwrarz inequality, called *Hölder's inequality*.