6.2.2 Markov and Chebyshev Inequalities

Let X be any positive continuous random variable, we can write

$$\begin{aligned} EX &= \int_{-\infty}^{\infty} x f_X(x)dx \\ &= \int_{0}^{\infty} x f_X(x)dx \\ &\geq \int_{a}^{\infty} x f_X(x)dx \\ &\geq \int_{a}^{\infty} a f_X(x)dx \\ &= a \int_{a}^{\infty} f_X(x)dx \\ &= aP(X \geq a). \end{aligned}$$

Thus, we conclude

$$P(X \geq a) \leq \frac{EX}{a}, \qquad \text{for any } a > 0.$$

We can prove the above inequality for discrete or mixed random variables similarly (using the generalized PDF), so we have the following result, called **Markov's inequality**.

Markov's Inequality

If X is any nonnegative random variable, then

$$P(X \geq a) \leq \frac{EX}{a},$$

Example 6.19

Prove the union bound using Markov's inequality.

Solution

Similar to the discussion in the previous section, let $A_1, A_2, \ldots, A_n$ be any events and X be the number events A_i that occur. We saw that

$$EX = P(A_1) + P(A_2) + \ldots + P(A_n) = \sum_{i=1}^{n} P(A_i).$$

Since X is a nonnegative random variable, we can apply Markov's inequality. Choosing $a = 1$, we have

$$P(X \geq 1) \leq EX = \sum_{i=1}^{n} P(A_i).$$

But note that $P(X \geq 1) = P\left(\bigcup_{i=1}^{n} A_i\right)$.

Example 6.20
Let $X \sim Binomial(n, p)$. Using Markov's inequality, find an upper bound on $P(X \geq \alpha n)$, where $p < \alpha < 1$. Evaluate the bound for $p = \frac{1}{2}$ and $\alpha = \frac{3}{4}$.

Solution

Note that X is a nonnegative random variable and $EX = np$. Applying Markov's inequality, we obtain

$$P(X \geq \alpha n) \leq \frac{EX}{\alpha n} = \frac{pn}{\alpha n} = \frac{p}{\alpha}.$$

For $p = \frac{1}{2}$ and $\alpha = \frac{3}{4}$, we obtain

$$P(X \geq \frac{3n}{4}) \leq \frac{2}{3}.$$

Chebyshev's Inequality:

Let X be any random variable. If you define $Y = (X - EX)^2$, then Y is a nonnegative random variable, so we can apply Markov's inequality to Y. In particular, for any positive real number b, we have

$$P(Y \geq b^2) \leq \frac{EY}{b^2}.$$

But note that

$$
\begin{aligned}
&EY = E(X-EX)^2 = Var(X),\\
&P(Y \geq b^2) = P\big((X-EX)^2 \geq b^2\big) = P\big(|X-EX| \geq b\big).
\end{aligned}
$$

Thus, we conclude that

$$P\big(|X-EX| \geq b\big) \leq \frac{Var(X)}{b^2}.$$

This is **Chebyshev's inequality**.

Chebyshev's Inequality

If X is any random variable, then for any $b>0$ we have

$$P\big(|X-EX| \geq b\big) \leq \frac{Var(X)}{b^2}.$$

Chebyshev's inequality states that the difference between X and EX is somehow limited by $Var(X)$. This is intuitively expected as variance shows on average how far we are from the mean.

Example 6.21

Let $X \sim Binomial(n,p)$. Using Chebyshev's inequality, find an upper bound on $P(X \geq \alpha n)$, where $p<\alpha<1$. Evaluate the bound for $p=\frac{1}{2}$ and $\alpha=\frac{3}{4}$.

Solution

One way to obtain a bound is to write

$$
\begin{aligned}
P(X \geq \alpha n) \quad &= P(X-np \geq \alpha n - np)\\
&\leq P\big(|X-np| \geq n\alpha - np\big)\\
&\leq \frac{Var(X)}{(n\alpha-np)^2}\\
&= \frac{p(1-p)}{n(\alpha-p)^2}.
\end{aligned}
$$

For $p=\frac{1}{2}$ and $\alpha=\frac{3}{4}$, we obtain

$$P(X \geq \frac{3n}{4}) \leq \frac{4}{n}.$$