The covariance matrix $\mathbf{C_U}$ is given by

$$\mathbf{C_U} = \begin{bmatrix} \mathrm{Var}(X) & \mathrm{Cov}(X,Y) \\ \mathrm{Cov}(Y,X) & \mathrm{Var}(Y) \end{bmatrix} = \begin{bmatrix} \frac{73}{960} & -\frac{1}{96} \\ -\frac{1}{96} & \frac{11}{144} \end{bmatrix}.$$

Properties of the Covariance Matrix:

The covariance matrix is the generalization of the variance to random vectors. It is an important matrix and is used extensively. Let's take a moment and discuss its properties. Here, we use concepts from linear algebra such as eigenvalues and positive definiteness. First note that, for any random vector **X**, the covariance matrix $\mathbf{C_X}$ is a symmetric matrix. This is because if $\mathbf{C_X} = [c_{ij}]$, then

$$c_{ij} = \mathrm{Cov}(X_i, X_j) = \mathrm{Cov}(X_j, X_i) = c_{ji}.$$

Thus, the covariance matrix has all the nice properties of symmetric matrices. In particular, $\mathbf{C_X}$ can be diagonalized and all the eigenvalues of $\mathbf{C_X}$ are real. Here, we assume **X** is a real random vector, i.e., the X_i's can only take real values. A special important property of the covariance matrix is that it is positive semi-definite (PSD). Remember from linear algebra that a symmetric matrix **M** is **positive semi-definite (PSD)** if, for all vectors **b**, we have

$$\mathbf{b^T M b} \geq 0.$$

Also, **M** is said to be **positive definite (PD)**, if for all vectors **b**$\neq 0$, we have

$$\mathbf{b^T M b} > 0.$$

By the above definitions, we note that every PD matrix is also PSD, but the converse is not generally true. Here, we show that covariance matrices are always PSD.
Theorem 6.2. Let $\mathbf{X}$ be a random vector with n elements. Then, its covariance matrix $\mathbf{C_X}$ is positive semi-definite(PSD).
Proof. Let **b** be any fixed vector with n elements. Define the random variable Y as

$$Y = \mathbf{b^T (X - EX)}.$$

We have

$$
\begin{aligned}
0 \leq EY^2 & \\
&= E(YY^T) \\
&= \mathbf{b^T} E\big[(\mathbf{X}-\mathbf{EX})(\mathbf{X}-\mathbf{EX})^\mathbf{T}\big]\mathbf{b} \\
&= \mathbf{b^T C_X b}.
\end{aligned}
$$

Note that the eigenvalues of a PSD matrix are always larger than or equal to zero. If all the eigenvalues are strictly larger than zero, then the matrix is positive definite. From linear algebra, we know that a real symmetric matrix is positive definite if and only if all its eigenvalues are positive. Since $\mathbf{C_X}$ is a real symmetric matrix, we can state the following theorem.

Theorem 6.3. Let $\mathbf{X}$ be a random vector with n elements. Then its covariance matrix $\mathbf{C_X}$ is positive definite (PD), if and only if all its eigenvalues are larger than zero. Equivalently, $\mathbf{C_X}$ is positive definite (PD), if and only if $\det(\mathbf{C_X}) > 0$.

Note that the second part of the theorem is implied by the first part. This is because the determinant of a matrix is the product of its eigenvalues, and we already know that all eigenvalues of $\mathbf{C_X}$ are larger than or equal to zero.

Example 6.14

Let X and Y be two independent $Uniform(0,1)$ random variables. Let the random vectors **U** and **V** be defined as

$$
\mathbf{U} = \begin{bmatrix} X \\ X+Y \end{bmatrix}, \quad \mathbf{V} = \begin{bmatrix} X \\ Y \\ X+Y \end{bmatrix}.
$$

Determine whether $\mathbf{C_U}$ and $\mathbf{C_V}$ are positive definite.

Solution

Let us first find $\mathbf{C_U}$. We have

$$
\mathbf{C_U} = \begin{bmatrix} \mathrm{Var}(X) & \mathrm{Cov}(X, X+Y) \\ \mathrm{Cov}(X+Y, X) & \mathrm{Var}(X+Y) \end{bmatrix}.
$$

Since X and Y are independent $Uniform(0,1)$ random variables, we have

$$
\begin{aligned}
\mathrm{Var}(X) = \mathrm{Var}(Y) &= \frac{1}{12}, \\
\mathrm{Cov}(X, X+Y) &= \mathrm{Cov}(X,X) + \mathrm{Cov}(X,Y) \\
&= \frac{1}{12} + 0 = \frac{1}{12}, \\
\mathrm{Var}(X+Y) &= \mathrm{Var}(X) + \mathrm{Var}(Y) = \frac{1}{6}.
\end{aligned}
$$

Thus,

$$\mathbf{C_U} = \begin{bmatrix} \frac{1}{12} & \frac{1}{12} \\ \frac{1}{12} & \frac{1}{6} \end{bmatrix}.$$

So we conclude

$$\begin{aligned} \det(\mathbf{C_U}) &= \frac{1}{12} \cdot \frac{1}{6} - \frac{1}{12} \cdot \frac{1}{12} \\ &= \frac{1}{144} > 0. \end{aligned}$$

Therefore, $\mathbf{C_U}$ is positive definite. For $\mathbf{C_V}$, we have

$$\begin{aligned} \mathbf{C_V} &= \begin{bmatrix} \mathrm{Var}(X) & \mathrm{Cov}(X,Y) & \mathrm{Cov}(X,X+Y) \\ \mathrm{Cov}(Y,X) & \mathrm{Var}(Y) & \mathrm{Cov}(Y,X+Y) \\ \mathrm{Cov}(X+Y,X) & \mathrm{Cov}(X+Y,Y) & \mathrm{Var}(X+Y) \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{12} & 0 & \frac{1}{12} \\ 0 & \frac{1}{12} & \frac{1}{12} \\ \frac{1}{12} & \frac{1}{12} & \frac{1}{6} \end{bmatrix}. \end{aligned}$$

So we conclude

$$\begin{aligned} \det(\mathbf{C_V}) &= \frac{1}{12}\left(\frac{1}{12} \cdot \frac{1}{6} - \frac{1}{12} \cdot \frac{1}{12}\right) - 0 + \frac{1}{12}\left(0 - \frac{1}{12} \cdot \frac{1}{12}\right) \\ &= 0. \end{aligned}$$

Thus, $\mathbf{C_V}$ is not positive definite (we already know that it is positive semi-definite).

Finally, if we have two random vectors, $\mathbf{X}$ and $\mathbf{Y}$, we can define the **cross correlation matrix** of $\mathbf{X}$ and $\mathbf{Y}$ as

$$\mathbf{R_{XY}} = \mathbf{E}[\mathbf{X}\mathbf{Y}^\mathbf{T}].$$

Also, the **cross covariance matrix** of **X** and **Y** is

$$\mathbf{C_{XY}} = \mathbf{E}[(\mathbf{X} - \mathbf{EX})(\mathbf{Y} - \mathbf{EY})^\mathbf{T}].$$
