3.2.5 Solved Problems:
More about Discrete Random Variables

Problem 1

Let X be a discrete random variable with the following PMF

$$P_X(x) = \begin{cases} 0.3 & \text{for } x = 3 \\ 0.2 & \text{for } x = 5 \\ 0.3 & \text{for } x = 8 \\ 0.2 & \text{for } x = 10 \\ 0 & \text{otherwise} \end{cases}$$

Find and plot the CDF of X.

Solution

The CDF is defined by $F_X(x) = P(X \leq x)$. We have

$$F_X(x) = \begin{cases} 0 & \text{for } x < 3 \\ P_X(3) = 0.3 & \text{for } 3 \leq x < 5 \\ P_X(3) + P_X(5) = 0.5 & \text{for } 5 \leq x < 8 \\ P_X(3) + P_X(5) + P_X(8) = 0.8 & \text{for } 8 \leq x < 10 \\ 1 & \text{for } x \geq 10 \end{cases}$$

Problem 2

Let X be a discrete random variable with the following PMF

$$P_X(k) = \begin{cases} 0.1 & \text{for } k = 0 \\ 0.4 & \text{for } k = 1 \\ 0.3 & \text{for } k = 2 \\ 0.2 & \text{for } k = 3 \\ 0 & \text{otherwise} \end{cases}$$

a. Find EX.
b. Find Var(X).
c. If $Y = (X-2)^2$, find EY.

Solution

a.

$$\begin{aligned} EX &= \textstyle\sum_{x_k \in R_X} x_k P_X(x_k) \\ &= 0(0.1) + 1(0.4) + 2(0.3) + 3(0.2) \\ &= 1.6 \end{aligned}$$

b. We can use Var$(X) = EX^2 - (EX)^2 = EX^2 - (1.6)^2$. Thus we need to find EX^2. Using LOTUS, we have

$$EX^2 = 0^2(0.1) + 1^2(0.4) + 2^2(0.3) + 3^2(0.2) = 3.4$$

Thus, we have

$$\text{Var}(X) = (3.4) - (1.6)^2 = 0.84$$

c. Again, using LOTUS, we have

$$E(X-2)^2 = (0-2)^2(0.1) + (1-2)^2(0.4) + (2-2)^2(0.3) + (3-2)^2(0.2) = 1.$$

Problem 3

Let X be a discrete random variable with PMF

$$P_X(k) = \begin{cases} 0.2 & \text{for } k = 0 \\ 0.2 & \text{for } k = 1 \\ 0.3 & \text{for } k = 2 \\ 0.3 & \text{for } k = 3 \\ 0 & \text{otherwise} \end{cases}$$

Define $Y = X(X-1)(X-2)$. Find the PMF of Y.

Solution

First, note that $R_Y = \{x(x-1)(x-2) | x \in \{0, 1, 2, 3\}\} = \{0, 6\}$. Thus,

$$\begin{aligned} P_Y(0) &= P(Y = 0) = P\big((X = 0) \text{ or } (X = 1) \text{ or } (X = 2)\big) \\ &= P_X(0) + P_X(1) + P_X(2) \\ &= 0.7; \\ P_Y(6) &= P(X = 3) = 0.3 \end{aligned}$$

Thus,

$$P_Y(k) = \begin{cases} 0.7 & \text{for } k = 0 \\ 0.3 & \text{for } k = 6 \\ 0 & \text{otherwise} \end{cases}$$

Problem 4

Let $X \sim Geometric(p)$. Find $E\left[\frac{1}{2^X}\right]$.

Solution

The PMF of X is given by

$$P_X(k) = \begin{cases} pq^{k-1} & \text{for } k = 1, 2, 3, \ldots \\ 0 & \text{otherwise} \end{cases}$$

where $q = 1 - p$. Thus,

$$\begin{aligned} E\left[\frac{1}{2^X}\right] &= \sum_{k=1}^{\infty} \frac{1}{2^k} P_X(k) \\ &= \sum_{k=1}^{\infty} \frac{1}{2^k} q^{k-1} p \\ &= \frac{p}{2} \sum_{k=1}^{\infty} \left(\frac{q}{2}\right)^{k-1} \\ &= \frac{p}{2} \frac{1}{1 - \frac{q}{2}} \\ &= \frac{p}{1+p}. \end{aligned}$$

Problem 5

If $X \sim Hypergeometric(b, r, k)$, find EX.

Solution

The PMF of X is given by

$$P_X(x) = \begin{cases} \frac{\binom{b}{x}\binom{r}{k-x}}{\binom{b+r}{k}} & \text{for } x \in R_X \\ 0 & \text{otherwise} \end{cases}$$

where $R_X = \{\max(0, k-r), \max(0, k-r)+1, \max(0, k-r)+2, \ldots, \min(k, b)\}$. Finding EX directly seems to be very complicated. So let's try to see if we can find an easier way to find EX. In particular, a powerful tool that we have is linearity of expectation. Can we write X as the sum of simpler random variables X_i? To do so, let's remember the random experiment behind the hypergeometric distribution. You have a bag that contains b blue marbles and r red marbles. You choose $k \leq b+r$ marbles at random (without replacement) and let X be the number of blue marbles in your sample. In particular, let's define the indicator random variables X_i as follows:

$$X_i = \begin{cases} 1 & \text{if the } i\text{th chosen marble is blue} \\ 0 & \text{otherwise} \end{cases}$$

Then, we can write

$$X = X_1 + X_2 + \cdots + X_k.$$

Thus,

$$EX = EX_1 + EX_2 + \cdots + EX_k.$$

To find $P(X_i = 1)$, we note that for any particular X_i all marbles are equally likely to be chosen. This is because of symmetry: no marble is more likely to be chosen than the i th marble as any other marbles. Therefore,

$$P(X_i = 1) = \frac{b}{b+r} \text{ for all } i \in \{1, 2, \cdots, k\}.$$

We conclude

$$\begin{aligned} EX_i &= 0 \cdot p(X_i = 0) + 1 \cdot P(X_i = 1) \\ &= \tfrac{b}{b+r}. \end{aligned}$$

Thus, we have

$$EX = \frac{kb}{b+r}.$$

Problem 6

In Example 3.14 we showed that if $X \sim Binomial(n, p)$, then $EX = np$. We found this by writing X as the sum of n $Bernoulli(p)$ random variables. Now, find EX directly using $EX = \sum_{x_k \in R_X} x_k P_X(x_k)$. *Hint:* Use $k\binom{n}{k} = n\binom{n-1}{k-1}$.

Solution

First note that we can prove $k\binom{n}{k} = n\binom{n-1}{k-1}$ by the following combinatorial interpretation: Suppose that from a group of n students we would like to choose a committee of k students, one of whom is chosen to be the committee chair. We can do this

1. by choosing k people first (in $\binom{n}{k}$ ways), and then choosing one of them to be the chair (k ways), or
2. by choosing the chair first (n possibilities and then choosing $k-1$ students from the remaining $n-1$ students (in $\binom{n-1}{k-1}$ ways)).

Thus, we conclude

$$k\binom{n}{k} = n\binom{n-1}{k-1}.$$

Now, let's find EX for $X \sim Binomial(n,p)$.

$$\begin{aligned} EX &= \textstyle\sum_{k=0}^{n} k\binom{n}{k}p^k q^{n-k} \\ &= \textstyle\sum_{k=1}^{n} k\binom{n}{k}p^k q^{n-k} \\ &= \textstyle\sum_{k=1}^{n} n\binom{n-1}{k-1}p^k q^{n-k} \\ &= np\textstyle\sum_{k=1}^{n} \binom{n-1}{k-1}p^{k-1} q^{n-k} \\ &= np\textstyle\sum_{l=0}^{n-1} \binom{n-1}{l}p^{l} q^{(n-1)-l} \\ &= np. \end{aligned}$$

Note that the last line is true because the $\sum_{l=0}^{n-1} \binom{n-1}{l}p^{l} q^{(n-1)-l}$ is equal to $\sum_{l=0}^{n-1} P_Y(l)$ for a random variable Y that has $Binomial(n-1,p)$ distribution, hence it is equal to 1.

Problem 7

Let X be a discrete random variable with $R_X \subset \{0,1,2,\ldots\}$. Prove

$$EX = \sum_{k=0}^{\infty} P(X > k).$$

Solution

Note that

$$P(X > 0) = P_X(1) + P_X(2) + P_X(3) + P_X(4) + \cdots,$$

$$\begin{aligned} P(X>1) &= P_X(2)+P_X(3)+P_X(4)+\cdots, \\ P(X>2) &= P_X(3)+P_X(4)+P_X(5)+\cdots. \end{aligned}$$

Thus

$$\begin{aligned} \sum_{k=0}^{\infty} P(X>k) &= P(X>0)+P(X>1)+P(X>2)+\ldots \\ &= P_X(1)+2P_X(2)+3P_X(3)+4P_X(4)+\ldots \\ &= EX. \end{aligned}$$

Problem 8

If $X \sim Poisson(\lambda)$, find Var(X).

Solution

We already know $EX=\lambda$, thus Var$(X)=EX^2-\lambda^2$. You can find EX^2 directly using LOTUS; however, it is a little easier to find $E[X(X-1)]$ first. In particular, using LOTUS we have

$$\begin{aligned} E[X(X-1)] &= \sum_{k=0}^{\infty} k(k-1)P_X(k) \\ &= \sum_{k=0}^{\infty} k(k-1)e^{-\lambda}\frac{\lambda^k}{k!} \\ &= e^{-\lambda}\sum_{k=2}^{\infty}\frac{\lambda^k}{(k-2)!} \\ &= e^{-\lambda}\lambda^2\sum_{k=2}^{\infty}\frac{\lambda^{k-2}}{(k-2)!} \\ &= e^{-\lambda}\lambda^2 e^{\lambda} \\ &= \lambda^2. \end{aligned}$$

So, we have $\lambda^2=E[X(X-1)]=EX^2-EX=EX^2-\lambda$. Thus, $EX^2=\lambda^2+\lambda$ and we conclude

$$\begin{aligned} \mathrm{Var}(X) &= EX^2-(EX)^2 \\ &= \lambda^2+\lambda-\lambda^2 \\ &= \lambda. \end{aligned}$$

Problem 9

Let X and Y be two independent random variables. Suppose that we know Var $(2X - Y) = 6$ and Var$(X + 2Y) = 9$. Find Var(X) and Var(Y).

Solution

Let's first make sure we understand what Var$(2X - Y)$ and Var$(X + 2Y)$ mean. They are Var(Z) and Var(W), where the random variables Z and W are defined as $Z = 2X - Y$ and $W = X + 2Y$. Since X and Y are independent random variables, then $2X$ and $-Y$ are independent random variables. Also, X and $2Y$ are independent random variables. Thus, by using Equation 3.7, we can write

$$\mathrm{Var}(2X - Y) = \mathrm{Var}(2X) + \mathrm{Var}(\text{-Y}) = 4\mathrm{Var}(X) + \mathrm{Var}(\mathrm{Y}) = 6,$$

$$\mathrm{Var}(X + 2Y) = \mathrm{Var}(X) + \mathrm{Var}(2\mathrm{Y}) = \mathrm{Var}(X) + 4\mathrm{Var}(\mathrm{Y}) = 9.$$

By solving for Var(X) and Var(Y), we obtain Var$(X) = 1$ and Var$(Y) = 2$.
