Theorem 11.1

Let $Y_n \sim Binomial\big(n, p = p(n)\big)$. Let $\mu > 0$ be a fixed real number, and $\lim_{n\to\infty} np = \mu$. Then, the PMF of Y_n converges to a $Poisson(\mu)$ PMF, as $n \to \infty$. That is, for any $k \in \{0, 1, 2, \ldots\}$, we have

$$\lim_{n\to\infty} P_{Y_n}(k) = \frac{e^{-\mu}\mu^k}{k!}.$$

Poisson Process as the Limit of a Bernoulli Process:

Suppose that we would like to model the arrival of events that happen completely at random at a rate λ per unit time. Here is one way to do this. At time $t = 0$, we have no arrivals yet, so $N(0) = 0$. We now divide the half-line $[0, \infty)$ to tiny subintervals of length δ as shown in Figure 11.2.

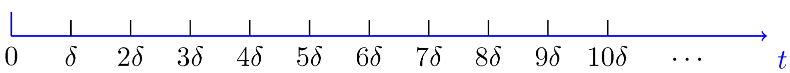

Figure 11.2 - Dividing the half-line $[0, \infty)$ to tiny subintervals of length δ.

Each subinterval corresponds to a time slot of length δ. Thus, the intervals are $(0, \delta]$, $(\delta, 2\delta]$, $(2\delta, 3\delta]$, $\cdots$. More generally, the kth interval is $\big((k-1)\delta, k\delta\big]$. We assume that in each time slot, we toss a coin for which $P(H) = p = \lambda\delta$. If the coin lands heads up, we say that we have an arrival in that subinterval. Otherwise, we say that we have no arrival in that interval. Figure 11.3 shows this process. Here, we have an arrival at time $t = k\delta$, if the kth coin flip results in a heads.

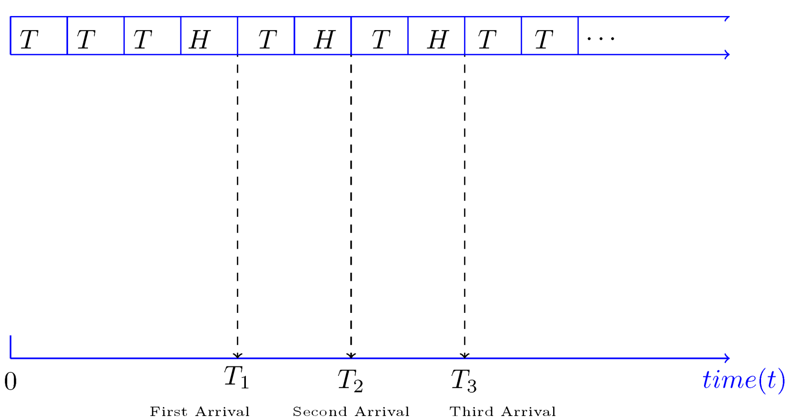

Figure 11.3 - Poisson process as a limit of a Bernoulli process.

Now, let $N(t)$ be defined as the number of arrivals (number of heads) from time 0 to time t. There are $n \approx \frac{t}{\delta}$ time slots in the interval $(0, t]$. Thus, $N(t)$ is the number of