8.4.4 P-Values

In the above discussions, we only reported an "accept" or a "reject" decision as the conclusion of a hypothesis test. However, we can provide more information using what we call *P-values*. In other words, we could indicate how close the decision was. More specifically, suppose we end up rejecting H_0 at at significance level $\alpha = 0.05$. Then we could ask: "How about if we require significance level $\alpha = 0.01$?" Can we still reject H_0? More specifically, we can ask the following question:

> What is the lowest significance level α that results in rejecting the null hypothesis?

The answer to the above question is called the *P-value*.

> **P-value** is the lowest significance level α that results in rejecting the null hypothesis.

Intuitively, if the P-value is small, it means that the observed data is very unlikely to have occurred under H_0, so we are more confident in rejecting the null hypothesis. How do we find P-values? Let's look at an example.

Example 8.29
You have a coin and you would like to check whether it is fair or biased. More specifically, let θ be the probability of heads, $\theta = P(H)$. Suppose that you need to choose between the following hypotheses:

H_0 (the null hypothesis): The coin is fair, i.e., $\theta = \theta_0 = \frac{1}{2}$.

H_1 (the alternative hypothesis): The coin is not fair, i.e., $\theta > \frac{1}{2}$.

We toss the coin 100 times and observe 60 heads.

1. Can we reject H_0 at significance level $\alpha = 0.05$?
2. Can we reject H_0 at significance level $\alpha = 0.01$?

3. What is the P-value?

Solution

Let X be the random variable showing the number of observed heads. In our experiment, we observed $X = 60$. Since $n = 100$ is relatively large, assuming H_0 is true, the random variable

$$W = \frac{X - n\theta_0}{\sqrt{n\theta_0(1-\theta_0)}} = \frac{X - 50}{5}$$

is (approximately) a standard normal random variable, $N(0,1)$. If H_0 is true, we expect X to be close to 50, while if H_1 is true, we expect X to be larger. Thus, we can suggest the following test: We choose a threshold c. If $W \leq c$, we accept H_0; otherwise, we accept H_1. To calculate $P(\text{type I error})$, we can write

$$\begin{aligned} P(\text{type I error}) &= P(\text{Reject } H_0 \mid H_0) \\ &= P(W > c \mid H_0). \end{aligned}$$

Since $W \sim N(0,1)$ under H_0, we need to choose

$$c = z_\alpha,$$

to ensure significance level α. In this example, we obtain

$$W = \frac{X - 50}{5} = \frac{60 - 50}{5} = 2.$$

1. If we require significance level $\alpha = 0.05$, then

   $$c = z_{0.05} = 1.645$$

   The above value can be obtained in MATLAB using the following command: `norminv(1 − 0.05)`. Since we have $W = 2 > 1.645$, we reject H_0, and accept H_1.
2. If we require significance level $\alpha = 0.01$, then

   $$c = z_{0.01} = 2.33$$

   The above value can be obtained in MATLAB using the following command: `norminv(1 − 0.01)`. Since we have $W = 2 \leq 2.33$, we fail to reject H_0, so we accept H_0.
3. P-value is the lowest significance level α that results in rejecting H_0. Here, since $W = 2$, we will reject H_0 if and only if $c < 2$. Note that $z_\alpha = c$, thus

   $$\alpha = 1 - \Phi(c).$$

If $c = 2$, we obtain

$$\alpha = 1 - \Phi(2) = 0.023$$

Therefore, we reject H_0 for $\alpha > 0.023$. Thus, the P-value is equal to 0.023.

The above example suggests the following way to compute P-values:

Computing P-Values

Consider a hypothesis test for choosing between H_0 and H_1. Let W be the test statistic, and w_1 be the observed value of W.

1. Assume H_0 is true.
2. The P-value is $P(\text{type I error})$ when the test threshold c is chosen to be $c = w_1$.

To see how we can use the above method, again consider Example 8.29. Here,

$$W = \frac{X - 50}{5},$$

which is approximately $N(0, 1)$ under H_0. The observed value of W is

$$w_1 = \frac{60 - 50}{5} = 2.$$

Thus,

$$\begin{aligned} P - \text{value} &= P(\text{type I error when } c = 2) \\ &= P(W > 2) \\ &= 1 - \Phi(2) = 0.023 \end{aligned}$$