9.1.2 Maximum A Posteriori (MAP) Estimation

The posterior distribution, $f_{X|Y}(x|y)$ (or $P_{X|Y}(x|y)$), contains all the knowledge about the unknown quantity X. Therefore, we can use the posterior distribution to find point or interval estimates of X. One way to obtain a point estimate is to choose the value of x that maximizes the posterior PDF (or PMF). This is called the ***maximum a posteriori (MAP) estimation***.

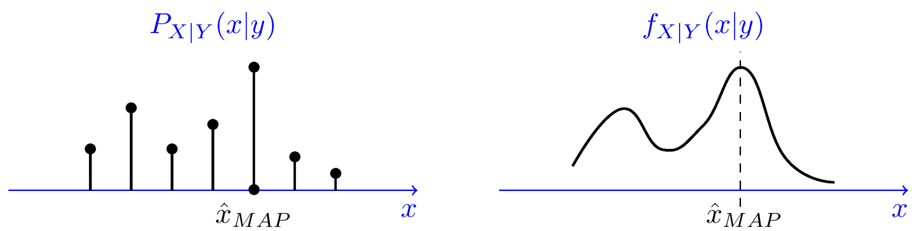

Figure 9.3 - The maximum a posteriori (MAP) estimate of X given $Y = y$ is the value of x that maximizes the posterior PDF or PMF. The MAP estimate of X is usually shown by $\hat{x}_{MAP}$.

<u>Maximum A Posteriori (MAP) Estimation</u>

The MAP estimate of the random variable X, given that we have observed $Y = y$, is given by the value of x that maximizes

$$f_{X|Y}(x|y) \text{ if } X \text{ is a continuous random variable,}$$
$$P_{X|Y}(x|y) \text{ if } X \text{ is a discrete random variable.}$$

The MAP estimate is shown by $\hat{x}_{MAP}$.

To find the MAP estimate, we need to find the value of x that maximizes

$$f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x) f_X(x)}{f_Y(y)}.$$

Note that $f_Y(y)$ does not depend on the value of x. Therefore, we can equivalently find the value of x that maximizes

$$f_{Y|X}(y|x)f_X(x).$$

This can simplify finding the MAP estimate significantly, because finding $f_Y(y)$ might be complicated. More specifically, finding $f_Y(y)$ usually is done using the law of total probability, which involves integration or summation, such as the one in Example 9.3.

> To find the MAP estimate of X given that we have observed $Y = y$, we find the value of x that maximizes
>
> $$f_{Y|X}(y|x)f_X(x).$$
>
> If either X or Y is discrete, we replace its PDF in the above expression by the corresponding PMF.

Example 9.4
Let X be a continuous random variable with the following PDF:

$$f_X(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq 1 \\ \\ 0 & \text{otherwise} \end{cases}$$

Also, suppose that

$$Y \mid X = x \quad \sim \quad Geometric(x).$$

Find the MAP estimate of X given $Y = 3$.

Solution

We know that $Y \mid X = x \quad \sim \quad Geometric(x)$, so

$$P_{Y|X}(y|x) = x(1-x)^{y-1}, \quad \text{for } y = 1, 2, \cdots.$$

Therefore,

$$P_{Y|X}(3|x) = x(1-x)^2.$$

We need to find the value of $x \in [0, 1]$ that maximizes

$$\begin{aligned} P_{Y|X}(y|x) f_X(x) &= x(1-x)^2 \cdot 2x \\ &= 2x^2(1-x)^2. \end{aligned}$$

We can find the maximizing value by differentiation. We obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x^2(1-x)^2\right] = 2x(1-x)^2 - 2(1-x)x^2 = 0.$$

Solving for x (and checking for maximization criteria), we obtain the MAP estimate as

$$\hat{x}_{MAP} = \frac{1}{2}.$$
