7.2.2 Sequence of Random Variables

Here, we would like to discuss what we precisely mean by a sequence of random variables. Remember that, in any probability model, we have a sample space S and a probability measure P. For simplicity, suppose that our sample space consists of a finite number of elements, i.e.,

$$S = \{s_1, s_2, \cdots, s_k\}.$$

Then, a random variable X is a mapping that assigns a real number to any of the possible outcomes s_i, $i = 1, 2, \cdots, k$. Thus, we may write

$$X(s_i) = x_i, \qquad \text{for } i = 1, 2, \cdots, k.$$

When we have a sequence of random variables X_1, X_2, X_3, $\cdots$, it is also useful to remember that we have an underlying sample space S. In particular, each X_n is a function from S to real numbers. Thus, we may write

$$X_n(s_i) = x_{ni}, \qquad \text{for } i = 1, 2, \cdots, k.$$

In sum, a sequence of random variables is in fact a sequence of functions $X_n : S \to \mathbb{R}$.

Example 7.3

Consider the following random experiment: A fair coin is tossed once. Here, the sample space has only two elements $S = \{H, T\}$. We define a sequence of random variables X_1, X_2, X_3, $\cdots$ on this sample space as follows:

$$X_n(s) = \begin{cases} \frac{1}{n+1} & \text{if } s = H \\ \\ 1 & \text{if } s = T \end{cases}$$

a. Are the X_i's independent?
b. Find the PMF and CDF of X_n, $F_{X_n}(x)$ for $n = 1, 2, 3, \cdots$.
c. As n goes to infinity, what does $F_{X_n}(x)$ look like?

Solution

a. The X_i's are not independent because their values are determined by the same coin toss. In particular, to show that X_1 and X_2 are not independent, we can write

$$\begin{aligned} P(X_1 = 1, X_2 = 1) &= P(T) \\ &= \frac{1}{2}, \end{aligned}$$

which is different from

$$\begin{aligned} P(X_1 = 1) \cdot P(X_2 = 1) &= P(T) \cdot P(T) \\ &= \frac{1}{4}. \end{aligned}$$

b. Each X_i can take only two possible values that are equally likely. Thus, the PMF of X_n is given by

$$P_{X_n}(x) = P(X_n = x) = \begin{cases} \frac{1}{2} & \text{if } x = \frac{1}{n+1} \\ \\ \frac{1}{2} & \text{if } x = 1 \end{cases}$$

From this we can obtain the CDF of X_n

$$F_{X_n}(x) = P(X_n \leq x) = \begin{cases} 1 & \text{if } x \geq 1 \\ \\ \frac{1}{2} & \text{if } \frac{1}{n+1} \leq x < 1 \\ \\ 0 & \text{if } x < \frac{1}{n+1} \end{cases}$$

c. Figure 7.3 shows the CDF of X_n for different values of n. We see in the figure that the CDF of X_n approaches the CDF of a $Bernoulli\left(\frac{1}{2}\right)$ random variable as $n \to \infty$. As we will discuss in the next sections, this means that the sequence X_1, X_2, X_3, $\cdots$ converges *in distribution* to a $Bernoulli\left(\frac{1}{2}\right)$ random variable as $n \to \infty$.

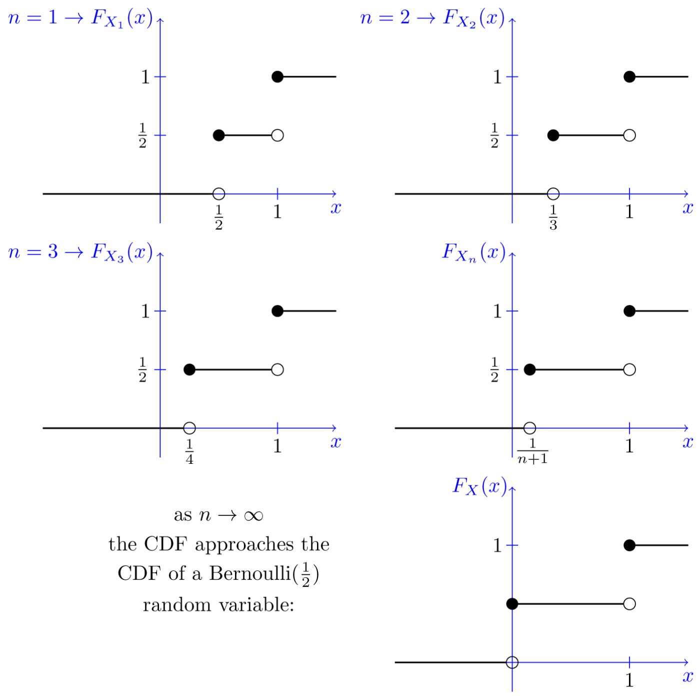

Fig.7.3 - CDFs of X_n for Example 7.12

The previous example was defined on a very simple sample space $S = \{H, T\}$. Let us look at an example that is defined on a more interesting sample space.

Example 7.4

Consider the following random experiment: A fair coin is tossed repeatedly forever. Here, the sample space S consists of all possible sequences of heads and tails. We define the sequence of random variables $X_1, X_2, X_3, \cdots$ as follows:

$$X_n = \begin{cases} 0 & \text{if the } n\text{th coin toss results in a heads} \\ \\ 1 & \text{if the } n\text{th coin toss results in a tails} \end{cases}$$

In this example, the X_i's are independent because each X_i is a result of a different coin toss. In fact, the X_i's are i.i.d. $Bernoulli\left(\frac{1}{2}\right)$ random variables. Thus, when we would like to refer to such a sequence, we usually say, "Let $X_1, X_2, X_3, \cdots$ be a sequence of i.i.d. $Bernoulli\left(\frac{1}{2}\right)$ random variables." We usually do not state the sample space because it is implied that the sample space S consists of all possible sequences of heads and tails.