The Second Definition of the Poisson Process

Let $\lambda > 0$ be fixed. The counting process $\{N(t), t \in [0, \infty)\}$ is called a **Poisson process** with **rate** λ if all the following conditions hold:

1. $N(0) = 0$;
2. $N(t)$ has independent and stationary increments
3. we have

$$\begin{aligned}P(N(\Delta) = 0) &= 1 - \lambda\Delta + o(\Delta),\\ P(N(\Delta) = 1) &= \lambda\Delta + o(\Delta),\\ P(N(\Delta) \geq 2) &= o(\Delta).\end{aligned}$$

We have already shown that any Poisson process satisfies the above definition. To show that the above definition is equivalent to our original definition, we also need to show that any process that satisfies the above definition also satisfies the original definition. A method to show this is outlined in the End of Chapter Problems.

Arrival and Interarrival Times:

Let $N(t)$ be a Poisson process with rate λ. Let X_1 be the time of the first arrival. Then,

$$\begin{aligned}P(X_1 > t) &= P\big(\text{no arrival in } (0, t]\big)\\ &= e^{-\lambda t}.\end{aligned}$$

We conclude that

$$F_{X_1}(t) = \begin{cases} 1 - e^{-\lambda t} & t > 0 \\ & \\ 0 & \text{otherwise} \end{cases}$$

Therefore, $X_1 \sim Exponential(\lambda)$. Let X_2 be the time elapsed between the first and the second arrival (Figure 11.4).

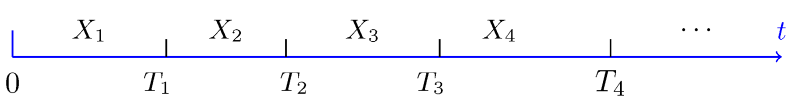

Figure 11.4 - The random variables X_1, X_2, $\cdots$ are called the interarrival times of the counting process $N(t)$.

Let $s > 0$ and $t > 0$. Note that the two intervals $(0, s]$ and $[s, s+t]$ are independent. We can write

$$\begin{aligned} P(X_2 > t | X_1 = s) &= P\big(\text{no arrival in } (s, s+t] | X_1 = s\big) \\ &= P\big(\text{no arrivals in } (s, s+t]\big) \text{ (independent increments)} \\ &= e^{-\lambda t}. \end{aligned}$$

We conclude that $X_2 \sim Exponential(\lambda)$, and that X_1 and X_2 are independent. The random variables X_1, X_2, $\cdots$ are called the **interarrival times** of the counting process $N(t)$. Similarly, we can argue that all X_i's are independent and $X_i \sim Exponential(\lambda)$ for $i = 1, 2, 3, \cdots$.

Interarrival Times for Poisson Processes

If $N(t)$ is a Poisson process with rate λ, then the interarrival times X_1, X_2, $\cdots$ are independent and

$$X_i \sim Exponential(\lambda), \quad \text{for } i = 1, 2, 3, \cdots.$$

Remember that if X is exponential with parameter $\lambda > 0$, then X is a *memoryless* random variable, that is

$$P(X > x + a | X > a) = P(X > x), \quad \text{for } a, x \geq 0.$$

Thinking of the Poisson process, the memoryless property of the interarrival times is consistent with the independent increment property of the Poisson distribution. In some sense, both are implying that the number of arrivals in non-overlapping intervals are independent. To better understand this issue, let's look at an example.

Example 11.2

Let $N(t)$ be a Poisson process with intensity $\lambda = 2$, and let X_1, X_2, $\cdots$ be the corresponding interarrival times.

a. Find the probability that the first arrival occurs after $t = 0.5$, i.e., $P(X_1 > 0.5)$.
b. Given that we have had no arrivals before $t = 1$, find $P(X_1 > 3)$.
c. Given that the third arrival occurred at time $t = 2$, find the probability that the fourth arrival occurs after $t = 4$.

d. I start watching the process at time $t = 10$. Let T be the time of the first arrival that I see. In other words, T is the first arrival after $t = 10$. Find ET and $\mathrm{Var}(T)$.

e. I start watching the process at time $t = 10$. Let T be the time of the first arrival that I see. Find the conditional expectation and the conditional variance of T given that I am informed that the last arrival occurred at time $t = 9$.

Solution

a. Since $X_1 \sim Exponential(2)$, we can write

$$\begin{aligned} P(X_1 > 0.5) &= e^{-(2\times 0.5)} \\ &\approx 0.37 \end{aligned}$$

Another way to solve this is to note that

$$P(X_1 > 0.5) = P(\text{no arrivals in } (0, 0.5]) = e^{-(2\times 0.5)} \approx 0.37$$

b. We can write

$$\begin{aligned} P(X_1 > 3|X_1 > 1) &= P(X_1 > 2) \text{ (memoryless property)} \\ &= e^{-2\times 2} \\ &\approx 0.0183 \end{aligned}$$

Another way to solve this is to note that the number of arrivals in $(1, 3]$ is independent of the arrivals before $t = 1$. Thus,

$$\begin{aligned} P(X_1 > 3|X_1 > 1) &= P\big(\text{no arrivals in } (1, 3] \mid \text{no arrivals in } (0, 1]\big) \\ &= P\big(\text{no arrivals in } (1, 3]\big) \text{ (independent increments)} \\ &= e^{-2\times 2} \\ &\approx 0.0183 \end{aligned}$$

c. The time between the third and the fourth arrival is $X_4 \sim Exponential(2)$. Thus, the desired conditional probability is equal to

$$\begin{aligned} P(X_4 > 2|X_1 + X_2 + X_3 = 2) &= P(X_4 > 2) \text{ (independence of the } X_i\text{'s)} \\ &= e^{-2\times 2} \\ &\approx 0.0183 \end{aligned}$$

d. When I start watching the process at time $t = 10$, I will see a Poisson process. Thus, the time of the first arrival from $t = 10$ is $Exponential(2)$. In other words, we can write

$$T = 10 + X,$$

where $X \sim Exponential(2)$. Thus,

$$\begin{aligned} ET &= 10 + EX \\ &= 10 + \frac{1}{2} = \frac{21}{2}, \end{aligned}$$

$$\begin{aligned} \text{Var}(T) &= \text{Var}(X) \\ &= \frac{1}{4}. \end{aligned}$$

e. Arrivals before $t = 10$ are independent of arrivals after $t = 10$. Thus, knowing that the last arrival occurred at time $t = 9$ does not impact the distribution of the first arrival after $t = 10$. Thus, if A is the event that the last arrival occurred at $t = 9$, we can write

$$\begin{aligned} E[T|A] &= E[T] \\ &= \frac{21}{2}, \end{aligned}$$

$$\begin{aligned} \text{Var}(T|A) &= \text{Var}(T) \\ &= \frac{1}{4}. \end{aligned}$$

Now that we know the distribution of the interarrival times, we can find the distribution of arrival times

$$\begin{aligned} T_1 &= X_1, \\ T_2 &= X_1 + X_2, \\ T_3 &= X_1 + X_2 + X_3, \\ &\vdots \end{aligned}$$

More specifically, T_n is the sum of n independent $Exponential(\lambda)$ random variables. In previous chapters we have seen that if $T_n = X_1 + X_2 + \cdots + X_n$, where the X_i's are independent $Exponential(\lambda)$ random variables, then $T_n \sim Gamma(n, \lambda)$. This has been shown using MGFs. Note that here $n \in \mathbb{N}$. The $Gamma(n, \lambda)$ is also called **Erlang** distribution, i.e, we can write

$$T_n \sim Erlang(n, \lambda) = Gamma(n, \lambda), \quad \text{for } n = 1, 2, 3, \cdots.$$

The PDF of T_n, for $n = 1, 2, 3, \cdots$, is given by

$$f_{T_n}(t) = \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!}, \quad \text{for } t > 0.$$

Remember that if $X \sim Exponential(\lambda)$, then

$$E[X] = \frac{1}{\lambda},$$
$$\mathrm{Var}(X) = \frac{1}{\lambda^2}.$$

Since $T_n = X_1 + X_2 + \cdots + X_n$, we conclude that

$$E[T_n] = nEX_1 = \frac{n}{\lambda},$$
$$\mathrm{Var}(T_n) = n\mathrm{Var}(X_n) = \frac{n}{\lambda^2}.$$

Note that the arrival times are not independent. In particular, we must have $T_1 \leq T_2 \leq T_3 \leq \cdots$.

Arrival Times for Poisson Processes

If $N(t)$ is a Poisson process with rate λ, then the arrival times T_1, T_2, $\cdots$ have $Gamma(n, \lambda)$ distribution. In particular, for $n = 1, 2, 3, \cdots$, we have

$$E[T_n] = \frac{n}{\lambda}, \text{ and } \mathrm{Var}(T_n) = \frac{n}{\lambda^2}.$$

The above discussion suggests a way to simulate (generate) a Poisson process with rate λ. We first generate i.i.d. random variables X_1, X_2, X_3, $\cdots$, where $X_i \sim Exponential(\lambda)$. Then the arrival times are given by

$$\begin{aligned} T_1 &= X_1, \\ T_2 &= X_1 + X_2, \\ T_3 &= X_1 + X_2 + X_3, \\ &\vdots \end{aligned}$$