6.1.4 Characteristic Functions

There are random variables for which the moment generating function does not exist on any real interval with positive length. For example, consider the random variable X that has a *Cauchy* distribution

$$f_X(x) = \frac{\frac{1}{\pi}}{1+x^2}, \quad \text{for all } x \in \mathbb{R}.$$

You can show that for any nonzero real number s

$$M_X(s) = \int_{-\infty}^{\infty} e^{sx} \frac{\frac{1}{\pi}}{1+x^2} dx = \infty.$$

Therefore, the moment generating function does not exist for this random variable on any real interval with positive length. If a random variable does not have a well-defined MGF, we can use the characteristic function defined as

$$\phi_X(\omega) = E[e^{j\omega X}],$$

where $j = \sqrt{-1}$ and ω is a real number. It is worth noting that $e^{j\omega X}$ is a complex-valued random variable. We have not discussed complex-valued random variables. Nevertheless, you can imagine that a complex random variable can be written as $X = Y + jZ$, where Y and Z are ordinary real-valued random variables. Thus, working with a complex random variable is like working with two real-valued random variables. The advantage of the characteristic function is that it is defined for all real-valued random variables. Specifically, if X is a real-valued random variable, we can write

$$|e^{j\omega X}| = 1.$$

Therefore, we conclude

$$\begin{aligned} |\phi_X(\omega)| &= |E[e^{j\omega X}]| \\ &\leq E[|e^{j\omega X}|] \\ &\leq 1. \end{aligned}$$

The characteristic function has similar properties to the MGF. For example, if X and Y are independent

$$
\begin{aligned}
\phi_{X+Y}(\omega) &= E[e^{j\omega(X+Y)}] \\
&= E[e^{j\omega X}e^{j\omega Y}] \\
&= E[e^{j\omega X}]E[e^{j\omega Y}] \quad \text{(since } X \text{ and } Y \text{ are independent)} \\
&= \phi_X(\omega)\phi_Y(\omega).
\end{aligned}
$$

More generally, if X_1, X_2, ..., X_n are n independent random variables, then

$$\phi_{X_1+X_2+\cdots+X_n}(\omega) = \phi_{X_1}(\omega)\phi_{X_2}(\omega)\cdots\phi_{X_n}(\omega).$$

Example 6.10

If $X \sim Exponential(\lambda)$, show that

$$\phi_X(\omega) = \frac{\lambda}{\lambda - j\omega}.$$

Solution

Recall that the PDF of X is

$$f_X(x) = \lambda e^{-\lambda x}u(x),$$

where $u(x)$ is the unit step function. We conclude

$$
\begin{aligned}
\phi_X(\omega) &= E[e^{j\omega X}] \\
&= \int_0^\infty \lambda e^{-\lambda x}e^{j\omega x}dx \\
&= \left[\frac{\lambda}{j\omega - \lambda}e^{(j\omega-\lambda)x}\right]_0^\infty \\
&= \frac{\lambda}{\lambda - j\omega}.
\end{aligned}
$$

Note that since $\lambda > 0$, the value of $e^{(j\omega-\lambda)x}$, when evaluated at $x = +\infty$, is zero.
