$$\begin{aligned}\mathrm{Var}(X|Y=1) &= E[X^2|Y=1]-(E[X|Y=1])^2\\ &= \frac{21}{50}-\left(\frac{7}{12}\right)^2\\ &= \frac{287}{3600}.\end{aligned}$$

Independent Random Variables:

When two jointly continuous random variables are independent, we must have

$$f_{X|Y}(x|y) = f_X(x).$$

That is, knowing the value of Y does not change the PDF of X. Since $f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)}$, we conclude that for two independent continuous random variables we must have

$$f_{XY}(x,y) = f_X(x)f_Y(y).$$

Two continuous random variables X and Y are independent if

$$f_{XY}(x,y) = f_X(x)f_Y(y), \qquad \text{for all } x,y.$$

Equivalently, X and Y are independent if

$$F_{XY}(x,y) = F_X(x)F_Y(y), \qquad \text{for all } x,y.$$

If X and Y are independent, we have

$$\begin{aligned}&E[XY] = EXEY,\\ &E[g(X)h(Y)] = E[g(X)]E[h(Y)].\end{aligned}$$

Suppose that we are given the joint PDF $f_{XY}(x,y)$ of two random variables X and Y. If we can write

$$f_{XY}(x,y) = f_1(x)f_2(y),$$

then X and Y are independent.

Example 5.23

Determine whether X and Y are independent:

a. $f_{XY}(x,y) = \begin{cases} 2e^{-x-2y} & x,y>0 \\ \\ 0 & \text{otherwise} \end{cases}$

b. $f_{XY}(x,y) = \begin{cases} 8xy & 0<x<y<1 \\ \\ 0 & \text{otherwise} \end{cases}$

Solution

a. We can write

$$f_{XY}(x,y) = \left[e^{-x}u(x)\right]\left[2e^{-2y}u(y)\right],$$

where $u(x)$ is the unit step function:

$$u(x) = \begin{cases} 1 & x \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

Thus, we conclude that X and Y are independent.

b. For this case, it does not seem that we can write $f_{XY}(x,y)$ as a product of some $f_1(x)$ and $f_2(y)$. Note that the given region $0<x<y<1$ enforces that $x<y$. That is, we always have $X<Y$. Thus, we conclude that X and Y are not independent. To show this, we can obtain the marginal PDFs of X and Y and show that $f_{XY}(x,y) \neq f_X(x)f_Y(y), \text{ for some } x,y$. We have, for $0 \leq x \leq 1$,

$$\begin{aligned} f_X(x) &= \int_x^1 8xydy \\ &= 4x(1-x^2). \end{aligned}$$

Thus,

$$f_X(x) = \begin{cases} 4x(1-x^2) & 0<x<1 \\ \\ 0 & \text{otherwise} \end{cases}$$

Similarly, we obtain

$$f_Y(y) = \begin{cases} 4y^3 & 0<y<1 \\ \\ 0 & \text{otherwise} \end{cases}$$

As we see, $f_{XY}(x,y) \neq f_X(x)f_Y(y)$, thus X and Y are NOT independent.

Example 5.24
Consider the unit disc

$$D = \{(x,y) | x^2 + y^2 \leq 1\}.$$

Suppose that we choose a point (X,Y) uniformly at random in D. That is, the joint PDF of X and Y is given by

$$f_{XY}(x,y) = \begin{cases} c & (x,y) \in D \\ \\ 0 & \text{otherwise} \end{cases}$$

a. Find the constant c.
b. Find the marginal PDFs $f_X(x)$ and $f_Y(y)$.
c. Find the conditional PDF of X given $Y = y$, where $-1 \leq y \leq 1$.
d. Are X and Y independent?

Solution

a. We have

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x,y) dx dy \\ &= \iint\limits_D c \;\; dx dy \\ &= c(\text{area of } D) \\ &= c(\pi). \end{aligned}$$

Thus, $c = \frac{1}{\pi}$.

b. For $-1 \leq x \leq 1$, we have

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{XY}(x,y) dy \\ &= \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi} \;\; dy \\ &= \frac{2}{\pi}\sqrt{1-x^2}. \end{aligned}$$