All of the above results can be proven directly from the definition of covariance. For example, if X and Y are independent, then as we have seen before $E[XY] = EXEY$, so

$$\mathrm{Cov}(X, Y) = E[XY] - EXEY = 0.$$

Note that the converse is not necessarily true. That is, if $\mathrm{Cov}(X, Y) = 0$, X and Y may or may not be independent. Let us prove Item 6 in Lemma 5.3, $\mathrm{Cov}(X + Y, Z) = \mathrm{Cov}(X, Z) + \mathrm{Cov}(Y, Z)$. We have

$$\begin{aligned}
\mathrm{Cov}(X + Y, Z) &= E[(X + Y)Z] - E(X + Y)EZ \\
&= E[XZ + YZ] - (EX + EY)EZ \\
&= EXZ - EXEZ + EYZ - EYEZ \\
&= \mathrm{Cov}(X, Z) + \mathrm{Cov}(Y, Z).
\end{aligned}$$

You can prove the rest of the items in Lemma 5.3 similarly.

Example 5.33

Let X and Y be two independent $N(0, 1)$ random variables and

$$\begin{aligned}
Z &= 1 + X + XY^2, \\
W &= 1 + X.
\end{aligned}$$

Find Cov(Z, W).

Solution

$$\begin{aligned}
\mathrm{Cov}(Z, W) &= \mathrm{Cov}(1 + X + XY^2, 1 + X) \\
&= \mathrm{Cov}(X + XY^2, X) && \text{(by part 5 of Lemma 5.3)} \\
&= \mathrm{Cov}(X, X) + \mathrm{Cov}(XY^2, X) && \text{(by part 6 of Lemma 5.3)} \\
&= \mathrm{Var}(X) + E[X^2Y^2] - E[XY^2]EX && \text{(by part 1 of Lemma 5.3 \& definition of Cov)} \\
&= 1 + E[X^2]E[Y^2] - E[X]^2E[Y^2] && \text{(since } X \text{ and } Y \text{ are independent)} \\
&= 1 + 1 - 0 = 2.
\end{aligned}$$

Variance of a sum:

One of the applications of covariance is finding the variance of a sum of several random variables. In particular, if $Z = X + Y$, then