$$\begin{aligned} P_{X|Y}(i|1) &= \frac{P_{XY}(i,1)}{P_Y(1)} \\ &= \frac{\frac{1}{13}}{\frac{3}{13}} = \frac{1}{3}, \quad \text{for } i = -1, 0, 1. \end{aligned}$$

Thus, we conclude

$$P_{X|Y}(i|1) = \begin{cases} \frac{1}{3} & \text{for } i = -1, 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

By looking at the above conditional PMF, we conclude that, given $Y = 1$, X is uniformly distributed over the set $\{-1, 0, 1\}$.

c. X and Y are **not** independent. We can see this as the conditional PMF of X given $Y = 1$ (calculated above) is not the same as marginal PMF of X, $P_X(x)$.

Conditional Expectation:

Given that we know event A has occurred, we can compute the conditional expectation of a random variable X, $E[X|A]$. Conditional expectation is similar to ordinary expectation. The only difference is that we replace the PMF by the conditional PMF. Specifically, we have

$$E[X|A] = \sum_{x_i \in R_X} x_i P_{X|A}(x_i).$$

Similarly, given that we have observed the value of random variable Y, we can compute the conditional expectation of X. Specifically, the conditional expectation of X given that $Y = y$ is

$$E[X|Y = y] = \sum_{x_i \in R_X} x_i P_{X|Y}(x_i|y).$$

Conditional Expectation of X:

$$E[X|A] = \sum_{x_i \in R_X} x_i P_{X|A}(x_i),$$
$$E[X|Y = y_j] = \sum_{x_i \in R_X} x_i P_{X|Y}(x_i|y_j)$$

Example 5.5 Let X and Y be the same as in Example 5.4.

a. Find $E[X|Y = 1|$.
b. Find $E[X| -1 < Y < 2]$.
c. Find $E[|X|| -1 < Y < 2]$.

Solution

a. To find $E[X|Y = 1]$, we have

$$E[X|Y = 1] = \sum_{x_i \in R_X} x_i P_{X|Y}(x_i|1).$$

We found in Example 5.4 that given $Y = 1$, X is uniformly distributed over the set $\{-1, 0, 1\}$. Thus, we conclude that

$$E[X|Y = 1] = \frac{1}{3}(-1 + 0 + 1) = 0.$$

b. To find $E[X| -1 < Y < 2]$, let A be the event that $-1 < Y < 2$, i.e., $Y \in \{0, 1\}$. To find $E[X|A]$, we need to find the conditional PMF, $P_{X|A}(k)$, for $k = -2, 1, 0, 1, 2$. First, note that

$$P(A) = P_Y(0) + P_Y(1) = \frac{5}{13} + \frac{3}{13} = \frac{8}{13}.$$

Thus, for $k = -2, 1, 0, 1, 2$, we have

$$P_{X|A}(k) = \frac{13}{8} P(X = k, A).$$

So, we can write