11.1.5 Solved Problems

Problem 1

Let $\{N(t), t \in [0, \infty)\}$ be a Poisson process with rate $\lambda = 0.5$.

a. Find the probability of no arrivals in $(3, 5]$.
b. Find the probability that there is exactly one arrival in each of the following intervals: $(0, 1]$, $(1, 2]$, $(2, 3]$, and $(3, 4]$.

Solution

a. If Y is the number arrivals in $(3, 5]$, then $Y \sim Poisson(\mu = 0.5 \times 2)$. Therefore,

$$\begin{aligned} P(Y = 0) &= e^{-1} \\ &= 0.37 \end{aligned}$$

b. Let Y_1, Y_2, Y_3 and Y_4 be the numbers of arrivals in the intervals $(0, 1]$, $(1, 2]$, $(2, 3]$, and $(3, 4]$. Then $Y_i \sim Poisson(0.5)$ and Y_i's are independent, so

$$\begin{aligned} P(Y_1 = 1, Y_2 = 1, Y_3 = 1, Y_4 = 1) &= P(Y_1 = 1) \cdot P(Y_2 = 1) \cdot P(Y_3 = 1) \cdot P(Y_4 = 1) \\ &= \left[0.5e^{-0.5}\right]^4 \\ &\approx 8.5 \times 10^{-3}. \end{aligned}$$

Problem 2

Let $\{N(t), t \in [0, \infty)\}$ be a Poisson process with rate λ. Find the probability that there are two arrivals in $(0, 2]$ and three arrivals in $(1, 4]$.

Solution

Note that the two intervals $(0, 2]$ and $(1, 4]$ are not disjoint. Thus, we cannot multiply the probabilities for each interval to obtain the desired probability. In particular,

$$(0, 2] \cap (1, 4] = (1, 2].$$

Let X, Y, and Z be the numbers of arrivals in $(0,1]$, $(1,2]$, and $(2,4]$ respectively. Then X, Y, and Z are independent, and

$$\begin{aligned} X &\sim Poisson(\lambda \cdot 1), \\ Y &\sim Poisson(\lambda \cdot 1), \\ Z &\sim Poisson(\lambda \cdot 2). \end{aligned}$$

Let A be the event that there are two arrivals in $(0,2]$ and three arrivals in $(1,4]$. We can use the law of total probability to obtain $P(A)$. In particular,

$$\begin{aligned} P(A) &= P(X+Y=2 \text{ and } Y+Z=3) \\ &= \sum_{k=0}^{\infty} P\big(X+Y=2 \text{ and } Y+Z=3|Y=k\big)P(Y=k) \\ &= P\big(X=2, Z=3|Y=0\big)P(Y=0) + P(X=1, Z=2|Y=1)P(Y=1)+ \\ &\qquad + P(X=0, Z=1|Y=2)P(Y=2) \\ &= P\big(X=2, Z=3\big)P(Y=0) + P(X=1, Z=2)P(Y=1)+ \\ &\qquad P(X=0, Z=1)P(Y=2) \\ &= P(X=2)P(Z=3)P(Y=0) + P(X=1)P(Z=2)P(Y=1)+ \\ &\qquad P(X=0)P(Z=1)P(Y=2) \\ &= \left(\frac{e^{-\lambda}\lambda^2}{2}\right) \cdot \left(\frac{e^{-2\lambda}(2\lambda)^3}{6}\right) \cdot \left(e^{-\lambda}\right) + \left(\lambda e^{-\lambda}\right) \cdot \left(\frac{e^{-2\lambda}(2\lambda)^2}{2}\right) \cdot \left(\lambda e^{-\lambda}\right) + \\ &\qquad \left(e^{-\lambda}\right) \cdot \left(e^{-2\lambda}(2\lambda)\right) \cdot \left(\frac{e^{-\lambda}\lambda^2}{2}\right). \end{aligned}$$

Problem 3

Let $\{N(t), t \in [0, \infty)\}$ be a Poisson Process with rate λ. Find its covariance function

$$C_N(t_1, t_2) = \text{Cov}\big(N(t_1), N(t_2)\big), \quad \text{for } t_1, t_2 \in [0, \infty)$$

Solution

Let's assume $t_1 \geq t_2 \geq 0$. Then, by the independent increment property of the Poisson process, the two random variables $N(t_1) - N(t_2)$ and $N(t_2)$ are independent. We can write

$$
\begin{aligned}
C_N(t_1, t_2) &= \mathrm{Cov}\big(N(t_1), N(t_2)\big) \\
&= \mathrm{Cov}\big(N(t_1) - N(t_2) + N(t_2), N(t_2)\big) \\
&= \mathrm{Cov}\big(N(t_1) - N(t_2), N(t_2)\big) + \mathrm{Cov}\big(N(t_2), N(t_2)\big) \\
&= \mathrm{Cov}\big(N(t_2), N(t_2)\big) \\
&= \mathrm{Var}\big(N(t_2)\big) \\
&= \lambda t_2, \quad \text{since } N(t_2) \sim Poisson(\lambda t_2).
\end{aligned}
$$

Similarly, if $t_2 \geq t_1 \geq 0$, we conclude

$$C_N(t_1, t_2) = \lambda t_1.$$

Therefore, we can write

$$C_N(t_1, t_2) = \lambda \min(t_1, t_2), \quad \text{for } t_1, t_2 \in [0, \infty).$$

Problem 4

Let $\{N(t), t \in [0, \infty)\}$ be a Poisson process with rate λ, and X_1 be its first arrival time. Show that given $N(t) = 1$, then X_1 is uniformly distributed in $(0, t]$. That is, show that

$$P(X_1 \leq x | N(t) = 1) = \frac{x}{t}, \quad \text{for } 0 \leq x \leq t.$$

Solution

For $0 \leq x \leq t$, we can write

$$P(X_1 \leq x | N(t) = 1) = \frac{P(X_1 \leq x, N(t) = 1)}{P\big(N(t) = 1\big)}.$$

We know that

$$P\big(N(t) = 1\big) = \lambda t e^{-\lambda t},$$

and

$$
\begin{aligned}
P(X_1 \leq x, N(t) = 1) &= P\Big(\text{one arrival in } (0, x] \quad \text{and} \quad \text{no arrivals in } (x, t]\Big) \\
&= \big[\lambda x e^{-\lambda x}\big] \cdot \Big[e^{-\lambda (t - x)}\Big] \\
&= \lambda x e^{-\lambda t}.
\end{aligned}
$$

Thus,

$$P(X_1 \leq x | N(t) = 1) = \frac{x}{t}, \quad \text{for } 0 \leq x \leq t.$$

Note: The above result can be generalized for n arrivals. That is, given that $N(t) = n$, the n arrival times have the same joint CDF as the order statistics of n independent $Uniform(0, t)$ random variables. This fact is discussed more in detail in the End of Chapter Problems.

Problem 5

Let $N_1(t)$ and $N_2(t)$ be two independent Poisson processes with rates $\lambda_1 = 1$ and $\lambda_2 = 2$, respectively. Let $N(t)$ be the merged process $N(t) = N_1(t) + N_2(t)$.

a. Find the probability that $N(1) = 2$ and $N(2) = 5$.
b. Given that $N(1) = 2$, find the probability that $N_1(1) = 1$.

Solution

$N(t)$ is a Poisson process with rate $\lambda = 1 + 2 = 3$.

a. We have

$$\begin{aligned} P(N(1) = 2, N(2) = 5) &= P\Big(\underline{two} \text{ arrivals in } (0, 1] \text{ and } \underline{three} \text{ arrivals in } (1, 2]\Big) \\ &= \left[\frac{e^{-3}3^2}{2!}\right] \cdot \left[\frac{e^{-3}3^3}{3!}\right] \\ &\approx .05 \end{aligned}$$

b.

$$\begin{aligned} P(N_1(1) = 1 | N(1) = 2) &= \frac{P\big(N_1(1) = 1, N(1) = 2\big)}{P(N(1) = 2)} \\ &= \frac{P\big(N_1(1) = 1, N_2(1) = 1\big)}{P(N(1) = 2)} \\ &= \frac{P\big(N_1(1) = 1\big) \cdot P\big(N_2(1) = 1\big)}{P(N(1) = 2)} \\ &= \left[e^{-1} \cdot 2e^{-2}\right] / \left[\frac{e^{-3}3^2}{2!}\right] \\ &= \frac{4}{9}. \end{aligned}$$

Problem 6

Let $N_1(t)$ and $N_2(t)$ be two independent Poisson processes with rates $\lambda_1 = 1$ and $\lambda_2 = 2$, respectively. Find the probability that the second arrival in $N_1(t)$ occurs before the third arrival in $N_2(t)$. *Hint:* One way to solve this problem is to think of $N_1(t)$ and $N_2(t)$ as two processes obtained from splitting a Poisson process.

Solution

Let $N(t)$ be a Poisson process with rate $\lambda = 1 + 2 = 3$. We split $N(t)$ into two processes $N_1(t)$ and $N_2(t)$ in the following way. For each arrival, a coin with $P(H) = \frac{1}{3}$ is tossed. If the coin lands heads up, the arrival is sent to the first process ($N_1(t)$), otherwise it is sent to the second process. The coin tosses are independent of each other and are independent of $N(t)$. Then

a. $N_1(t)$ is a Poisson process with rate $\lambda p = 1$;
b. $N_2(t)$ is a Poisson process with rate $\lambda(1 - p) = 2$;
c. $N_1(t)$ and $N_2(t)$ are independent.

Thus, $N_1(t)$ and $N_2(t)$ have the same probabilistic properties as the ones stated in the problem. We can now restate the probability that the second arrival in $N_1(t)$ occurs before the third arrival in $N_2(t)$ as the probability of observing at least two heads in four coin tosses, which is

$$\sum_{k=2}^{4} \binom{4}{k} \left(\frac{1}{3}\right)^k \left(\frac{2}{3}\right)^{4-k}.$$