10.2.1 Power Spectral Density

So far, we have studied random processes in the time domain. It is often very useful to study random processes in the frequency domain as well. To do this, we need to use the **Fourier** transform. Here, we will assume that you are familiar with the Fourier transform. A brief review of the Fourier transform and its properties is given in the appendix.

Consider a WSS random process $X(t)$ with autocorrelation function $R_X(\tau)$. We define the *Power Spectral Density (PSD)* of $X(t)$ as the Fourier transform of $R_X(\tau)$. We show the PSD of $X(t)$, by $S_X(f)$. More specifically, we can write

$$S_X(f) = \mathcal{F}\{R_X(\tau)\} = \int_{-\infty}^{\infty} R_X(\tau) e^{-2j\pi f\tau} \, d\tau,$$

where $j = \sqrt{-1}$.

Power Spectral Density (PSD)

$$S_X(f) = \mathcal{F}\{R_X(\tau)\} = \int_{-\infty}^{\infty} R_X(\tau) e^{-2j\pi f\tau} \, d\tau, \quad \text{where } j = \sqrt{-1}.$$

From this definition, we can conclude that $R_X(\tau)$ can be obtained by the inverse Fourier transform of $S_X(f)$. That is

$$R_X(\tau) = \mathcal{F}^{-1}\{S_X(f)\} = \int_{-\infty}^{\infty} S_X(f) e^{2j\pi f\tau} \, df.$$

As we have seen before, if $X(t)$ is a real-valued random process, then $R_X(\tau)$ is an even, real-valued function of τ. From the properties of the Fourier transform, we conclude that $S_X(f)$ is also real-valued and an even function of f. Also, from what we will discuss later on, we can conclude that $S_X(f)$ is non-negative for all f.

1. $S_X(-f) = S_X(f)$, for all f;
2. $S_X(f) \geq 0$, for all f.

Before going any further, let's try to understand the idea behind the PSD. To do so, let's choose $\tau = 0$. We know that expected power in $X(t)$ is given by

$$
\begin{aligned}
E\big[X(t)^2\big] = R_X(0) &= \int_{-\infty}^{\infty} S_X(f) e^{2j\pi f \cdot 0}\, df \\
&= \int_{-\infty}^{\infty} S_X(f)\, df.
\end{aligned}
$$

We conclude that the expected power in $X(t)$ can be obtained by integrating the PSD of $X(t)$. This fact helps us to understand why $S_X(f)$ is called the power spectral density. In fact, as we will see shortly, we can find the expected power of $X(t)$ in a specific frequency range by integrating the PSD over that specific range.

The expected power in $X(t)$ can be obtained as

$$E\big[X(t)^2\big] = R_X(0) = \int_{-\infty}^{\infty} S_X(f)\, df.$$

Example 10.13
Consider a WSS random process $X(t)$ with

$$R_X(\tau) = e^{-a|\tau|},$$

where a is a positive real number. Find the PSD of $X(t)$.