We can use the joint PMF to find $P\big((X,Y) \in A\big)$ for any set $A \subset \mathbb{R}^2$. Specifically, we have

$$\boxed{P\big((X,Y) \in A\big) = \sum_{(x_i,y_j) \in (A \cap R_{XY})} P_{XY}(x_i,y_j)}$$

Note that the event $X = x$ can be written as $\{(x_i,y_j) : x_i = x, y_j \in R_Y\}$. Also, the event $Y = y$ can be written as $\{(x_i,y_j) : x_i \in R_X, y_j = y\}$. Thus, we can write

$$\begin{aligned} P_{XY}(x,y) &= P(X = x, Y = y) \\ &= P\big((X = x) \cap (Y = y)\big). \end{aligned}$$

Marginal PMFs

The joint PMF contains all the information regarding the distributions of X and Y. This means that, for example, we can obtain PMF of X from its joint PMF with Y. Indeed, we can write

$$\begin{aligned} P_X(x) &= P(X = x) \\ &= \sum_{y_j \in R_Y} P(X = x, Y = y_j) \qquad \text{law of total probablity} \\ &= \sum_{y_j \in R_Y} P_{XY}(x,y_j). \end{aligned}$$

Here, we call $P_X(x)$ the **marginal PMF** of X. Similarly, we can find the marginal PMF of Y as

$$P_Y(Y) = \sum_{x_i \in R_X} P_{XY}(x_i,y).$$

Marginal PMFs of X and Y:

$$P_X(x) = \sum_{y_j \in R_Y} P_{XY}(x, y_j), \qquad \text{for any } x \in R_X$$
$$P_Y(y) = \sum_{x_i \in R_X} P_{XY}(x_i, y), \qquad \text{for any } y \in R_Y \qquad (5.1)$$

Let's practice these concepts by looking at an example.

Example 5.1

Consider two random variables X and Y with joint PMF given in Table 5.1.

Table 5.1 Joint PMF of X and Y in Example 5.1

	$Y=0$	$Y=1$	$Y=2$
$X=0$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$
$X=1$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$

Figure 5.1 shows $P_{XY}(x, y)$.

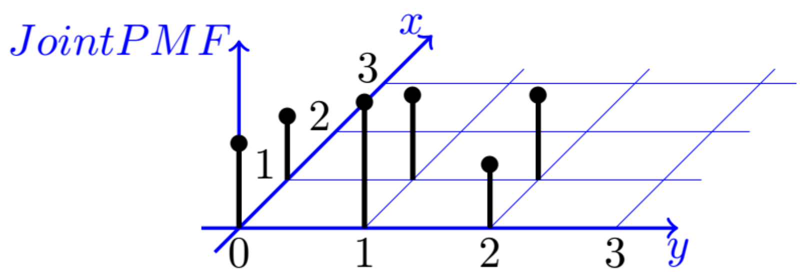

Figure 5.1: Joint PMF of X and Y (Example 5.1).

a. Find $P(X=0, Y \leq 1)$.
b. Find the marginal PMFs of X and Y.
c. Find $P(Y=1|X=0)$.
d. Are X and Y independent?

Solution

a. To find $P(X=0, Y \leq 1)$, we can write

$$P(X=0, Y \leq 1) = P_{XY}(0,0) + P_{XY}(0,1) = \frac{1}{6} + \frac{1}{4} = \frac{5}{12}.$$

b. Note that from the table,

$$R_X = \{0,1\} \qquad \text{and} \qquad R_Y = \{0,1,2\}.$$

Now we can use Equation 5.1 to find the marginal PMFs. For example, to find $P_X(0)$, we can write

$$\begin{aligned} P_X(0) &= P_{XY}(0,0) + P_{XY}(0,1) + P_{XY}(0,2) \\ &= \frac{1}{6} + \frac{1}{4} + \frac{1}{8} \\ &= \frac{13}{24}. \end{aligned}$$

We obtain

$$P_X(x) = \begin{cases} \frac{13}{24} & x = 0 \\ \frac{11}{24} & x = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$P_Y(y) = \begin{cases} \frac{7}{24} & y = 0 \\ \frac{5}{12} & y = 1 \\ \frac{7}{24} & y = 2 \\ 0 & \text{otherwise} \end{cases}$$

c. Find $P(Y=1|X=0)$: Using the formula for conditional probability, we have

$$
\begin{aligned}
P(Y=1|X=0) &= \frac{P(X=0, Y=1)}{P(X=0)} \\
&= \frac{P_{XY}(0,1)}{P_X(0)} \\
&= \frac{\frac{1}{4}}{\frac{13}{24}} = \frac{6}{13}.
\end{aligned}
$$

d. Are X and Y independent? X and Y are not independent, because as we just found out

$$P(Y=1|X=0) = \frac{6}{13} \neq P(Y=1) = \frac{5}{12}.$$

Caution: If we want to show that X and Y are independent, we need to check that $P(X=x_i, Y=y_j) = P(X=x_i)P(Y=y_j)$, for all $x_i \in R_X$ and all $y_j \in R_Y$. Thus, even if in the above calculation we had found $P(Y=1|X=0) = P(Y=1)$, we would not yet have been able to conclude that X and Y are independent. For that, we would need to check the independence condition for all $x_i \in R_X$ and all $y_j \in R_Y$.