1.4.5 Solved Problems: Conditional Probability

In die and coin problems, unless stated otherwise, it is assumed coins and dice are fair and repeated trials are independent.

Problem 1

You purchase a certain product. The manual states that the lifetime T of the product, defined as the amount of time (in years) the product works properly until it breaks down, satisfies

$$P(T \geq t) = e^{-\frac{t}{5}}, \text{ for all } t \geq 0.$$

For example, the probability that the product lasts more than (or equal to) 2 years is $P(T \geq 2) = e^{-\frac{2}{5}} = 0.6703$. I purchase the product and use it for two years without any problems. What is the probability that it breaks down in the third year?

Solution

Let A be the event that a purchased product breaks down in the third year. Also, let B be the event that a purchased product does not break down in the first two years. We are interested in $P(A|B)$. We have

$$\begin{aligned} P(B) &= P(T \geq 2) \\ &= e^{-\frac{2}{5}}. \end{aligned}$$

We also have

$$\begin{aligned} P(A) &= P(2 \leq T \leq 3) \\ &= P(T \geq 2) - P(T \geq 3) \\ &= e^{-\frac{2}{5}} - e^{-\frac{3}{5}}. \end{aligned}$$

Finally, since $A \subset B$, we have $A \cap B = A$. Therefore,

$$P(A|B)$$

$$
\begin{aligned}
&= \frac{P(A \cap B)}{P(B)} \\
&= \frac{P(A)}{P(B)} \\
&= \frac{e^{-\frac{2}{5}} - e^{-\frac{3}{5}}}{e^{-\frac{2}{5}}} \\
&= 0.1813
\end{aligned}
$$

Problem 2

You toss a fair coin three times:

a. What is the probability of three heads, HHH?
b. What is the probability that you observe exactly one heads?
c. Given that you have observed *at least* one heads, what is the probability that you observe at least two heads?

Solution

We assume that the coin tosses are independent.

a. $P(HHH) = P(H) \cdot P(H) \cdot P(H) = 0.5^3 = \frac{1}{8}$.
b. To find the probability of exactly one heads, we can write

$$
\begin{aligned}
P(\text{One heads}) &= P(HTT \cup THT \cup TTH) \\
&= P(HTT) + P(THT) + P(TTH) \\
&= \frac{1}{8} + \frac{1}{8} + \frac{1}{8} \\
&= \frac{3}{8}.
\end{aligned}
$$

c. Given that you have observed *at least* one heads, what is the probability that you observe at least two heads? Let A_1 be the event that you observe at least one heads, and A_2 be the event that you observe at least two heads. Then

$$A_1 = S - \{TTT\}, \text{ and } P(A_1) = \frac{7}{8};$$

$$A_2 = \{HHT, HTH, THH, HHH\}, \text{ and } P(A_2) = \frac{4}{8}.$$

Thus, we can write

$$
\begin{aligned}
P(A_2|A_1) &= \frac{P(A_2 \cap A_1)}{P(A_1)} \\
&= \frac{P(A_2)}{P(A_1)} \\
&= \frac{4}{8} \cdot \frac{8}{7} = \frac{4}{7}.
\end{aligned}
$$

Problem 3

For three events A, B, and C, we know that

- A and C are independent,
- B and C are independent,
- A and B are disjoint,
- $P(A \cup C) = \frac{2}{3}, P(B \cup C) = \frac{3}{4}, P(A \cup B \cup C) = \frac{11}{12}$

Find $P(A), P(B)$, and $P(C)$.

Solution

We can use the Venn diagram in Figure 1.26 to better visualize the events in this problem. We assume $P(A) = a, P(B) = b$, and $P(C) = c$. Note that the assumptions about independence and disjointness of sets are already included in the figure.

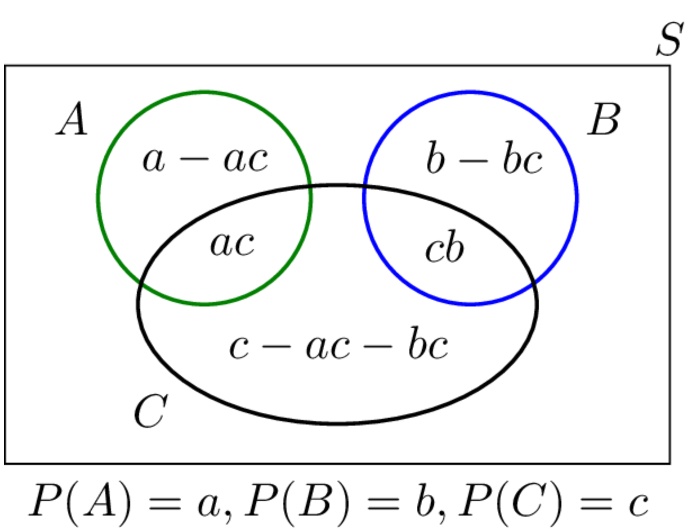

Fig.1.26 - Venn diagram for Problem 3.

Now we can write

$$P(A \cup C) = a + c - ac = \frac{2}{3};$$

$$P(B \cup C) = b + c - bc = \frac{3}{4};$$

$$P(A \cup B \cup C) = a + b + c - ac - bc = \frac{11}{12}.$$

By subtracting the third equation from the sum of the first and second equations, we immediately obtain $c = \frac{1}{2}$, which then gives $a = \frac{1}{3}$ and $b = \frac{1}{2}$.

Problem 4

Let $C_1, C_2, \cdots, C_M$ be a partition of the sample space S, and A and B be two events. Suppose we know that

- A and B are conditionally independent given C_i, for all $i \in \{1, 2, \cdots, M\}$;
- B is independent of all C_i's.

Prove that A and B are independent.

Solution

Since the C_i's form a partition of the sample space, we can apply the law of total probability for $A \cap B$:

$$\begin{aligned} P(A \cap B) &= \sum_{i=1}^{M} P(A \cap B|C_i)P(C_i) \\ &= \sum_{i=1}^{M} P(A|C_i)P(B|C_i)P(C_i) \\ &= \sum_{i=1}^{M} P(A|C_i)P(B)P(C_i) \\ &= P(B)\sum_{i=1}^{M} P(A|C_i)P(C_i) \\ &= P(B)P(A) \qquad \text{(law of total probability).} \end{aligned}$$

Problem 5

In my town, it's rainy one third of the days. Given that it is rainy, there will be heavy traffic with probability $\frac{1}{2}$, and given that it is not rainy, there will be heavy traffic with probability $\frac{1}{4}$. If it's rainy and there is heavy traffic, I arrive late for work with probability $\frac{1}{2}$. On the other hand, the probability of being late is reduced to $\frac{1}{8}$ if it is not rainy and there is no heavy traffic. In other situations (rainy and no traffic, not rainy and traffic) the probability of being late is 0.25. You pick a random day.

a. What is the probability that it's not raining and there is heavy traffic and I am not late?
b. What is the probability that I am late?
c. Given that I arrived late at work, what is the probability that it rained that day?

Solution

Let R be the event that it's rainy, T be the event that there is heavy traffic, and L be the event that I am late for work. As it is seen from the problem statement, we are given conditional probabilities in a chain format. Thus, it is useful to draw a tree diagram. Figure 1.27 shows a tree diagram for this problem. In this figure, each leaf in the tree corresponds to a single outcome in the sample space. We can calculate the probabilities of each outcome in the sample space by multiplying the probabilities on the edges of the tree that lead to the corresponding outcome.

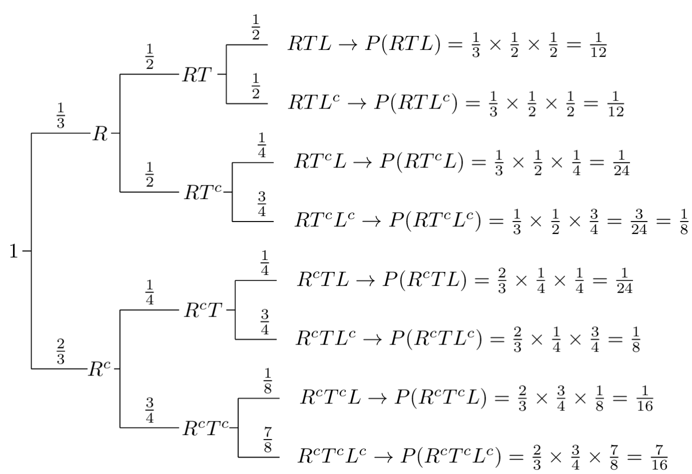

Fig.1.27 - Tree diagram for Problem 5.

a. The probability that it's not raining and there is heavy traffic and I am not late can be found using the tree diagram which is in fact applying the chain rule:

$$
\begin{aligned}
P(R^c \cap T \cap L^c) &= P(R^c)P(T|R^c)P(L^c|R^c \cap T) \\
&= \frac{2}{3} \cdot \frac{1}{4} \cdot \frac{3}{4} \\
&= \frac{1}{8}.
\end{aligned}
$$

b. The probability that I am late can be found from the tree. All we need to do is sum the probabilities of the outcomes that correspond to me being late. In fact, we are using the law of total probability here.

$$
\begin{aligned}
P(L) &= P(R,T,L) + P(R,T^c,L) + P(R^c,T,L) + P(R^c,T^c,L) \\
&= \frac{1}{12} + \frac{1}{24} + \frac{1}{24} + \frac{1}{16} \\
&= \frac{11}{48}.
\end{aligned}
$$

c. We can find $P(R|L)$ using $P(R|L) = \frac{P(R \cap L)}{P(L)}$. We have already found $P(L) = \frac{11}{48}$, and we can find $P(R \cap L)$ similarly by adding the probabilities of the outcomes that belong to $R \cap L$. In particular,

$$
\begin{aligned}
P(R \cap L) &= P(R,T,L) + P(R,T^c,L) \\
&= \frac{1}{12} + \frac{1}{24} \\
&= \frac{1}{8}.
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
P(R|L) &= \frac{P(R \cap L)}{P(L)} \\
&= \frac{1}{8} \cdot \frac{48}{11} \\
&= \frac{6}{11}.
\end{aligned}
$$

Problem 6

A box contains three coins: two regular coins and one fake two-headed coin ($P(H) = 1$),

- You pick a coin at random and toss it. What is the probability that it lands heads up?
- You pick a coin at random and toss it, and get heads. What is the probability that it is the two-headed coin?

Solution

This is another typical problem for which the law of total probability is useful. Let C_1 be the event that you choose a regular coin, and let C_2 be the event that you choose the two-headed coin. Note that C_1 and C_2 form a partition of the sample space. We already know that

$$P(H|C_1) = 0.5,$$
$$P(H|C_2) = 1.$$

a. Thus, we can use the law of total probability to write

$$\begin{aligned} P(H) &= P(H|C_1)P(C_1) + P(H|C_2)P(C_2) \\ &= \tfrac{1}{2}.\tfrac{2}{3} + 1.\tfrac{1}{3} \\ &= \tfrac{2}{3}. \end{aligned}$$

b. Now, for the second part of the problem, we are interested in $P(C_2|H)$. We use Bayes' rule

$$\begin{aligned} P(C_2|H) &= \frac{P(H|C_2)P(C_2)}{P(H)} \\ &= \frac{1.\frac{1}{3}}{\frac{2}{3}} \\ &= \tfrac{1}{2}. \end{aligned}$$

Problem 7

Here is another variation of the family-with-two-children problem [1] [7]. A family has two children. We ask the father, "Do you have at least one daughter named Lilia?" He replies, "Yes!" What is the probability that both children are girls? In other words, we want to find the probability that both children are girls, given that the family has at least one daughter named Lilia. Here you can assume that if a child is a girl, her name will be Lilia with probability $\alpha \ll 1$ independently from other children's names. If the child is a boy, his name will not be Lilia. Compare your result with the second part of Example 1.18.

Solution

Here we have four possibilities, $GG = (\text{girl, girl}), GB, BG, BB$, and $P(GG) = P(GB) = P(BG) = P(BB) = \frac{1}{4}$. Let also L be the event that the family has at least one child named Lilia. We have

$$P(L|BB) = 0,$$

$$P(L|BG) = P(L|GB) = \alpha,$$

$$P(L|GG) = \alpha(1-\alpha) + (1-\alpha)\alpha + \alpha^2 = 2\alpha - \alpha^2.$$

We can use Bayes' rule to find $P(GG|L)$:

$$\begin{aligned}
P(GG|L) &= \frac{P(L|GG)P(GG)}{P(L)} \\
&= \frac{P(L|GG)P(GG)}{P(L|GG)P(GG)+P(L|GB)P(GB)+P(L|BG)P(BG)+P(L|BB)P(BB)} \\
&= \frac{(2\alpha-\alpha^2)\frac{1}{4}}{(2\alpha-\alpha^2)\frac{1}{4}+\alpha\frac{1}{4}+\alpha\frac{1}{4}+0.\frac{1}{4}} \\
&= \frac{2-\alpha}{4-\alpha} \approx \frac{1}{2}.
\end{aligned}$$

Let's compare the result with part (b) of Example 1.18. Amazingly, we notice that the extra information about the name of the child increases the conditional probability of GG from $\frac{1}{3}$ to about $\frac{1}{2}$. How can we explain this intuitively? Here is one way to look at the problem. In part (b) of Example 1.18, we know that the family has at least one girl. Thus, the sample space reduces to three equally likely outcomes: GG, GB, BG, thus the conditional probability of GG is one third in this case. On the other hand, in this problem, the available information is that the event L has occurred. The conditional sample space here still is GG, GB, BG, but these events are not equally likely anymore. A family with two girls is more likely to name at least one of them Lilia than a family who has only one girl ($P(L|BG) = P(L|GB) = \alpha$, $P(L|GG) = 2\alpha - \alpha^2$), thus in this case the conditional probability of GG is higher. We would like to mention here that these problems are confusing and counterintuitive to most people. So, do not be disappointed if they seem confusing to you. We seek several goals by including such problems.

First, we would like to emphasize that we should not rely too much on our intuition when solving probability problems. Intuition is useful, but at the end, we must use laws of probability to solve problems. Second, after obtaining counterintuitive results, you are encouraged to think deeply about them to explain your confusion. This thinking process can be very helpful to improve our understanding of probability. Finally, I personally think these paradoxical-looking problems make probability more interesting.

Problem 8

If you are not yet confused, let's look at another family-with-two-children problem! I know that a family has two children. I see one of the children in the mall and notice that she is a girl. What is the probability that both children are girls? Again compare your result with the second part of Example 1.18. Note: Let's agree on what precisely the problem statement means. Here is a more precise statement of the problem: "A family has two children. We choose one of them at random and find out that she is a girl. What is the probability that both children are girls?"

Solution

Here again, we have four possibilities, $GG = (\text{girl}, \text{girl}), GB, BG, BB$, and $P(GG) = P(GB) = P(BG) = P(BB) = \frac{1}{4}$. Now, let G_r be the event that a randomly chosen child is a girl. Then we have

$$P(G_r|GG) = 1,$$

$$P(G_r|GB) = P(G_r|BG) = \frac{1}{2},$$

$$P(G_r|BB) = 0.$$

We can use Bayes' rule to find $P(GG|G_r)$:

$$\begin{aligned} P(GG|G_r) &= \frac{P(G_r|GG)P(GG)}{P(G_r)} \\ &= \frac{P(G_r|GG)P(GG)}{P(G_r|GG)P(GG)+P(G_r|GB)P(GB)+P(G_r|BG)P(BG)+P(G_r|BB)P(BB)} \\ &= \frac{1.\frac{1}{4}}{1.\frac{1}{4}+\frac{1}{2}\frac{1}{4}+\frac{1}{2}\frac{1}{4}+0.\frac{1}{4}} \\ &= \frac{1}{2}. \end{aligned}$$

So the answer again is different from the second part of Example 1.18. This is surprising to most people. The two problem statements look very similar but the answers are completely different. This is again similar to the previous problem (please read the explanation there). The conditional sample space here still is GG, GB, BG, but the point here is that these are not equally likely as in Example 1.18. The probability that a randomly chosen child from a family with two girls is a girl is one, while this probability for a family who has only one girl is $\frac{1}{2}$. Thus, intuitively, the conditional probability of the outcome GG in this case is higher than GB and BG, and thus this conditional probability must be larger than one third.

Problem 9

Okay, another family-with-two-children problem. Just kidding! This problem has nothing to do with the two previous problems. I toss a coin repeatedly. The coin is unfair and $P(H) = p$. The game ends the first time that two consecutive heads (HH) or two consecutive tails (TT) are observed. I win if HH is observed and lose if TT is observed. For example if the outcome is $HTH\underline{TT}$, I lose. On the other hand, if the outcome is $THTHT\underline{HH}$, I win. Find the probability that I win.

Solution

Let W be the event that I win. We can write down the set W by listing all the different sequences that result in my winning. It is cleaner if we divide W into two parts depending on the result of the first coin toss,

$$W = \{HH, HTHH, HTHTHH, \cdots\} \cup \{THH, THTHH, THTHTHH, \cdots\}.$$

Let $q = 1 - p$. Then

$$\begin{aligned}
W &= P(\{HH, HTHH, HTHTHH, \cdots\}) + P(\{THH, THTHH, THTHTHH, \cdots\}) \\
&= p^2 + p^3q + p^4q^2 + \cdots + p^2q + p^3q^2 + p^4q^3 + \cdots \\
&= p^2(1 + pq + (pq)^2 + (pq)^3 + \cdots) + p^2q(1 + pq + (pq)^2 + (pq)^3 + \cdots) \\
&= p^2(1 + q)(1 + pq + (pq)^2 + (pq)^3 + \cdots) \\
&= \frac{p^2(1+q)}{1-pq}, \qquad \text{Using the geometric series formula} \\
&= \frac{p^2(2-p)}{1-p+p^2}.
\end{aligned}$$
