$$\begin{aligned} P_{X|Y}(x_i|y_j) &= P(X=x_i|Y=y_j) \\ &= \frac{P(X=x_i, Y=y_j)}{P(Y=y_j)} \\ &= \frac{P_{XY}(x_i,y_j)}{P_Y(y_j)}. \end{aligned}$$

Similarly, we can define the conditional probability of Y given X:

$$\begin{aligned} P_{Y|X}(y_j|x_i) &= P(Y=y_j|X=x_i) \\ &= \frac{P_{XY}(x_i,y_j)}{P_X(x_i)}. \end{aligned}$$

For discrete random variables X and Y, the **conditional PMFs** of X given Y and vice versa are defined as

$$\begin{aligned} P_{X|Y}(x_i|y_j) &= \frac{P_{XY}(x_i,y_j)}{P_Y(y_j)}, \\ P_{Y|X}(y_j|x_i) &= \frac{P_{XY}(x_i,y_j)}{P_X(x_i)} \end{aligned}$$

for any $x_i \in R_X$ and $y_j \in R_Y$.

Independent Random Variables:

We have defined independent random variables previously. Now that we have seen joint PMFs and CDFs, we can restate the independence definition.

Two discrete random variables X and Y are independent if

$$P_{XY}(x,y) = P_X(x)P_Y(y), \quad \text{for all } x, y.$$

Equivalently, X and Y are independent if

$$F_{XY}(x,y) = F_X(x)F_Y(y), \quad \text{for all } x, y.$$

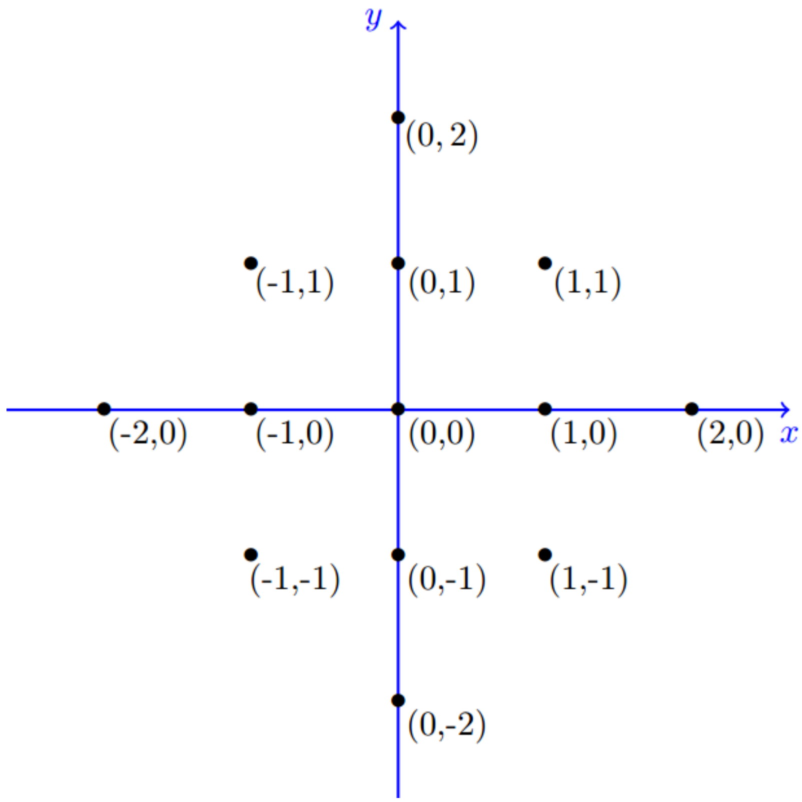

Figure 5.4: Grid for example 5.4

So, if X and Y are independent, we have

$$\begin{aligned} P_{X|Y}(x_i|y_j) &= P(X = x_i|Y = y_j) \\ &= \frac{P_{XY}(x_i, y_j)}{P_Y(y_j)} \\ &= \frac{P_X(x_i)P_Y(y_j)}{P_Y(y_j)} \\ &= P_X(x_i). \end{aligned}$$

As we expect, for independent random variables, the conditional PMF is equal to the marginal PMF. In other words, knowing the value of Y does not provide any information about X.

Example 5.4

Consider the set of points in the grid shown in Figure 5.4. These are the points in set G defined as

$$G = \{(x, y)|x, y \in \mathbb{Z}, |x| + |y| \leq 2\}.$$

Suppose that we pick a point (X, Y) from this grid completely at random. Thus, each point has a probability of $\frac{1}{13}$ of being chosen.

a. Find the joint and marginal PMFs of X and Y.
b. Find the conditional PMF of X given $Y=1$.
c. Are X and Y independent?

Solution

a. Here, note that

$$R_{XY}=G=\{(x,y)|x,y\in\mathbb{Z},|x|+|y|\leq 2\}.$$

Thus, the joint PMF is given by

$$P_{XY}(x,y)=\begin{cases}\frac{1}{13} & (x,y)\in G\\ 0 & \text{otherwise}\end{cases}$$

To find the marginal PMF of X, $P_X(i)$, we use Equation 5.1. Thus,

$$\begin{aligned}P_X(-2)&=P_{XY}(-2,0)=\frac{1}{13},\\ P_X(-1)&=P_{XY}(-1,-1)+P_{XY}(-1,0)+P_{XY}(-1,1)=\frac{3}{13},\\ P_X(0)&=P_{XY}(0,-2)+P_{XY}(0,-1)+P_{XY}(0,0)\\ &\quad+P_{XY}(0,1)+P_{XY}(0,2)=\frac{5}{13},\\ P_X(1)&=P_{XY}(1,-1)+P_{XY}(1,0)+P_{XY}(1,-1)=\frac{3}{13},\\ P_X(2)&=P_{XY}(2,0)=\frac{1}{13}.\end{aligned}$$

Similarly, we can find

$$P_Y(j)=\begin{cases}\frac{1}{13} & \text{for } j=2,-2\\ \frac{3}{13} & \text{for } j=-1,1\\ \frac{5}{13} & \text{for } j=0\\ 0 & \text{otherwise}\end{cases}$$

We can write this in a more compact form as

$$P_X(k)=P_Y(k)=\frac{5-2|k|}{13},\quad \text{for } k=-2,-1,0,1,2.$$

b. For $i=-1,0,1$, we can write