11.1.4 Nonhomogeneous Poisson Processes

Let $N(t)$ be the number of customers arriving at a fast food restaurant by time t. We think that the customers arrive somewhat randomly, so we might want to model $N(t)$ as a Poisson process. However, we notice that this process does not have stationary increments. For example, we note that the arrival rate of customers is larger during lunch time compared to, say, 4 p.m. In such scenarios, we might model $N(t)$ as a *nonhomogeneous Poisson process*. Such a process has all the properties of a Poisson process, except for the fact that its rate is a function of time, i.e., $\lambda = \lambda(t)$.

Nonhomogeneous Poisson Process

Let $\lambda(t) : [0, \infty) \mapsto [0, \infty)$ be an integrable function. The counting process $\{N(t), t \in [0, \infty)\}$ is called a **nonhomogeneous Poisson process** with **rate** $\lambda(t)$ if all the following conditions hold.

1. $N(0) = 0$;
2. $N(t)$ has independent increments;
3. for any $t \in [0, \infty)$, we have

$$
\begin{aligned}
&P(N(t+\Delta) - N(t) = 0) = 1 - \lambda(t)\Delta + o(\Delta), \\
&P(N(t+\Delta) - N(t) = 1) = \lambda(t)\Delta + o(\Delta), \\
&P(N(t+\Delta) - N(t) \geq 2) = o(\Delta).
\end{aligned}
$$

For a nonhomogeneous Poisson process with rate $\lambda(t)$, the number of arrivals in any interval is a Poisson random variable; however, its parameter can depend on the location of the interval. More specifically, we can write

$$\boxed{N(t+s) - N(t) \;\sim\; Poisson\left(\int_t^{t+s} \lambda(\alpha) d\alpha\right).}$$