6.1.2 Sums of Random Variables

In many applications, we need to work with a sum of several random variables. In particular, we might need to study a random variable Y given by

$$Y = X_1 + X_2 + \cdots + X_n.$$

The linearity of expectation tells us that

$$EY = EX_1 + EX_2 + \cdots + EX_n.$$

We can also find the variance of Y based on our discussion in Section 5.3. In particular, we saw that the variance of a sum of two random variables is

$$\mathrm{Var}(X_1 + X_2) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + 2\mathrm{Cov}(X_1, X_2).$$

For $Y = X_1 + X_2 + \cdots + X_n$, we can obtain a more general version of the above equation. We can write

$$\begin{aligned}
\mathrm{Var}(Y) &= \mathrm{Cov}\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{n} X_j\right) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \mathrm{Cov}(X_i, X_j) && \text{(using part 7 of Lemma 5.3)} \\
&= \sum_{i=1}^{n} \mathrm{Var}(X_i) + 2 \sum_{i<j} \mathrm{Cov}(X_i, X_j).
\end{aligned}$$

$$\boxed{\mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mathrm{Var}(X_i) + 2 \sum_{i<j} \mathrm{Cov}(X_i, X_j)}$$

If the X_i's are independent, then $\mathrm{Cov}(X_i, X_j) = 0$ for $i \neq j$. In this case, we can write

$$\boxed{\text{If } X_1, X_2, \ldots, X_n \text{ are independent, } \mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mathrm{Var}(X_i).}$$

Example 6.2

N people sit around a round table, where $N > 5$. Each person tosses a coin. Anyone whose outcome is different from his/her two neighbors will receive a present. Let X be the number of people who receive presents. Find EX and $\textrm{Var}(X)$.

Solution

Number the N people from 1 to N. Let X_i be the indicator random variable for the ith person, that is, $X_i=1$ if the ith person receives a present and zero otherwise. Then

$$X=X_1+X_2+\ldots+X_N.$$

First note that $P(X_i=1)=\frac{1}{4}$. This is the probability that the person to the right has a different outcome times the probability that the person to the left has a different outcome. In other words, if we define H_i and T_i be the events that the ith person's outcome is heads and tails respectively, then we can write

$$\begin{aligned}EX_i&=P(X_i=1)\\&=P(H_{i-1},T_i,H_{i+1})+P(T_{i-1},H_i,T_{i+1})\\&=\frac{1}{8}+\frac{1}{8}=\frac{1}{4}.\end{aligned}$$

Thus, we find

$$EX=EX_1+EX_2+\ldots+EX_N=\frac{N}{4}.$$

Next, we can write

$$\textrm{Var}(X)=\sum_{i=1}^{N}\textrm{Var}(X_i)+\sum_{i=1}^{N}\sum_{j\neq i}\textrm{Cov}(X_i,X_j).$$

Since $X_i \sim Bernoulli(\frac{1}{4})$, we have

$$\textrm{Var}(X_i)=\frac{1}{4}.\frac{3}{4}=\frac{3}{16}.$$

It remains to find $\textrm{Cov}(X_i,X_j)$. First note that X_i and X_j are independent if there are at least two people between the ith person and the jth person. In other words, if $2<|i-j|<N-2$, then X_i and X_j are independent, so

$$\textrm{Cov}(X_i,X_j)=0, \textrm{ for } 2<|i-j|<N-2.$$

Also, note that there is a lot of symmetry in the problem:

$$\begin{aligned}&\textrm{Cov}(X_1,X_2)=\textrm{Cov}(X_2,X_3)=\textrm{Cov}(X_3,X_4)=\ldots=\textrm{Cov}(X_{N-1},X_N)=\textrm{Cov}(X_N,X_1),\\&\textrm{Cov}(X_1,X_3)=\textrm{Cov}(X_2,X_4)=\textrm{Cov}(X_3,X_5)=\ldots=\textrm{Cov}(X_{N-1},X_1)=\textrm{Cov}(X_N,X_2).\end{aligned}$$

Thus, we can write

$$\begin{aligned}\textrm{Var}(X)&=N\textrm{Var}(X_1)+2N\textrm{Cov}(X_1,X_2)+2N\textrm{Cov}(X_1,X_3)\\&=\frac{3N}{16}+2N\textrm{Cov}(X_1,X_2)+2N\textrm{Cov}(X_1,X_3).\end{aligned}$$

So we need to find $\mathrm{Cov}(X_1, X_2)$ and $\mathrm{Cov}(X_1, X_3)$. We have

$$\begin{aligned}E[X_1X_2] &= P(X_1 = 1, X_2 = 1)\\ &= P(H_N, T_1, H_2, T_3) + P(T_N, H_1, T_2, H_3)\\ &= \frac{1}{16} + \frac{1}{16} = \frac{1}{8}.\end{aligned}$$

Thus,

$$\begin{aligned}\mathrm{Cov}(X_1, X_2) &= E[X_1X_2] - E[X_1]E[X_2]\\ &= \frac{1}{8} - \frac{1}{16} = \frac{1}{16},\end{aligned}$$

$$\begin{aligned}E[X_1X_3] &= P(X_1 = 1, X_3 = 1)\\ &= P(H_N, T_1, H_2, T_3, H_4) + P(T_N, H_1, T_2, H_3, T_4)\\ &= \frac{1}{32} + \frac{1}{32} = \frac{1}{16}.\end{aligned}$$

Thus,

$$\begin{aligned}\mathrm{Cov}(X_1, X_3) &= E[X_1X_3] - E[X_1]E[X_3]\\ &= \frac{1}{16} - \frac{1}{16} = 0.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathrm{Var}(X) &= \frac{3N}{16} + 2N\mathrm{cov}(X_1, X_2) + 2N\mathrm{cov}(X_1, X_3)\\ &= \frac{3N}{16} + \frac{2N}{16}\\ &= \frac{5N}{16}.\end{aligned}$$

We now know how to find the mean and variance of a sum of n random variables, but we might need to go beyond that. Specifically, what if we need to know the PDF of $Y = X_1 + X_2 + ... + X_n$? In fact we have addressed that problem for the case where $Y = X_1 + X_2$ and X_1 and X_2 are independent (Example 5.30 in Section 5.2.4). For this case, we found out the PDF is given by convolving the PDF of X_1 and X_2, that is

$$f_Y(y) = f_{X_1}(y) * f_{X_2}(y) = \int_{-\infty}^{\infty} f_{X_1}(x) f_{X_2}(y - x)dx.$$

For $Y = X_1 + X_2 + ... + X_n$, we can use the above formula repeatedly to obtain the PDF of Y:

$$f_Y(y) = f_{X_1}(y) * f_{X_2}(y) * \ldots * f_{X_n}(y).$$

Nevertheless, this quickly becomes computationally difficult. Thus, we often resort to other methods if we can. One method that is often useful is using moment generating functions, as we discuss in the next section.