5.2.1 Joint Probability Density Function (PDF)

Here, we will define jointly continuous random variables. Basically, two random variables are jointly continuous if they have a joint probability density function as defined below.

Definition 5.1
Two random variables X and Y are **jointly continuous** if there exists a nonnegative function $f_{XY} : \mathbb{R}^2 \rightarrow \mathbb{R}$, such that, for any set $A \in \mathbb{R}^2$, we have

$$P\big((X,Y) \in A\big) = \iint\limits_A f_{XY}(x,y)dxdy \qquad (5.15)$$

The function $f_{XY}(x,y)$ is called the **joint probability density function (PDF)** of X and Y.

In the above definition, the domain of $f_{XY}(x,y)$ is the entire $\mathbb{R}^2$. We may define the range of (X,Y) as

$$R_{XY} = \{(x,y) | f_{X,Y}(x,y) > 0\}.$$

The above double integral (Equation 5.15) exists for all sets A of practical interest. If we choose $A = \mathbb{R}^2$, then the probability of $(X,Y) \in A$ must be one, so we must have

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x,y)dxdy = 1$$

The intuition behind the joint density $f_{XY}(x,y)$ is similar to that of the PDF of a single random variable. In particular, remember that for a random variable X and small positive δ, we have

$$P(x < X \leq x+\delta) \approx f_X(x)\delta.$$

Similarly, for small positive δ_x and δ_y, we can write

$$P(x < X \leq x+\delta_x, y \leq Y \leq y+\delta_y) \approx f_{XY}(x,y)\delta_x\delta_y.$$

Example 5.15

Let X and Y be two jointly continuous random variables with joint PDF

$$f_{XY}(x,y) = \begin{cases} x+cy^2 & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ \\ 0 & \text{otherwise} \end{cases}$$

a. Find the constant c.
b. Find $P(0 \leq X \leq \frac{1}{2}, 0 \leq Y \leq \frac{1}{2})$.

Solution

a. To find c, we use

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)dxdy = 1.$$

Thus, we have

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)dxdy \\ &= \int_{0}^{1}\int_{0}^{1} x+cy^2 \; dxdy \\ &= \int_{0}^{1} \left[\frac{1}{2}x^2+cy^2x\right]_{x=0}^{x=1} dy \\ &= \int_{0}^{1} \frac{1}{2}+cy^2 \; dy \\ &= \left[\frac{1}{2}y+\frac{1}{3}cy^3\right]_{y=0}^{y=1} \\ &= \frac{1}{2}+\frac{1}{3}c. \end{aligned}$$

Therefore, we obtain $c = \frac{3}{2}$.

b. To find $P(0 \leq X \leq \frac{1}{2}, 0 \leq Y \leq \frac{1}{2})$, we can write

$$P\big((X,Y) \in A\big) = \iint_A f_{XY}(x,y)dxdy, \quad \text{for } A = \{(x,y)|0 \leq x, y \leq 1\}.$$

Thus,

$$\begin{aligned} P(0 \leq X \leq \frac{1}{2}, 0 \leq Y \leq \frac{1}{2}) &= \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \left(x + \frac{3}{2}y^2\right) dxdy \\ &= \int_0^{\frac{1}{2}} \left[\frac{1}{2}x^2 + \frac{3}{2}y^2 x\right]_0^{\frac{1}{2}} dy \\ &= \int_0^{\frac{1}{2}} \left(\frac{1}{8} + \frac{3}{4}y^2\right) dy \\ &= \frac{3}{32}. \end{aligned}$$

We can find marginal PDFs of X and Y from their joint PDF. This is exactly analogous to what we saw in the discrete case. In particular, by integrating over all y's, we obtain $f_X(x)$. We have

Marginal PDFs

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{XY}(x,y)dy, \quad \text{for all } x, \\ f_Y(y) &= \int_{-\infty}^{\infty} f_{XY}(x,y)dx, \quad \text{for all } y. \end{aligned}$$

Example 5.16

In Example 5.15 find the marginal PDFs $f_X(x)$ and $f_Y(y)$.

Solution

For $0 \leq x \leq 1$, we have

$$
\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f_{XY}(x,y)dy \\
&= \int_0^1 \left(x + \frac{3}{2}y^2\right) dy \\
&= \left[xy + \frac{1}{2}y^3\right]_0^1 \\
&= x + \frac{1}{2}.
\end{aligned}
$$

Thus,

$$
f_X(x) = \begin{cases} x + \frac{1}{2} & 0 \leq x \leq 1 \\ \\ 0 & \text{otherwise} \end{cases}
$$

Similarly, for $0 \leq y \leq 1$, we have

$$
\begin{aligned}
f_Y(y) &= \int_{-\infty}^{\infty} f_{XY}(x,y)dx \\
&= \int_0^1 \left(x + \frac{3}{2}y^2\right) dx \\
&= \left[\frac{1}{2}x^2 + \frac{3}{2}y^2 x\right]_0^1 \\
&= \frac{3}{2}y^2 + \frac{1}{2}.
\end{aligned}
$$

Thus,

$$
f_Y(y) = \begin{cases} \frac{3}{2}y^2 + \frac{1}{2} & 0 \leq y \leq 1 \\ \\ 0 & \text{otherwise} \end{cases}
$$

Example 5.17

Let X and Y be two jointly continuous random variables with joint PDF

$$
f_{XY}(x,y) = \begin{cases} cx^2y & 0 \leq y \leq x \leq 1 \\ \\ 0 & \text{otherwise} \end{cases}
$$

a. Find R_{XY} and show it in the $x - y$ plane.
b. Find the constant c.
c. Find marginal PDFs, $f_X(x)$ and $f_Y(y)$.

d. Find $P(Y \leq \frac{X}{2})$.
e. Find $P(Y \leq \frac{X}{4} | Y \leq \frac{X}{2})$.

Solution

a. From the joint PDF, we find that

$$R_{XY} = \{(x, y) \in \mathbb{R}^2 | 0 \leq y \leq x \leq 1\}.$$

Figure 5.6 shows R_{XY} in the $x - y$ plane.

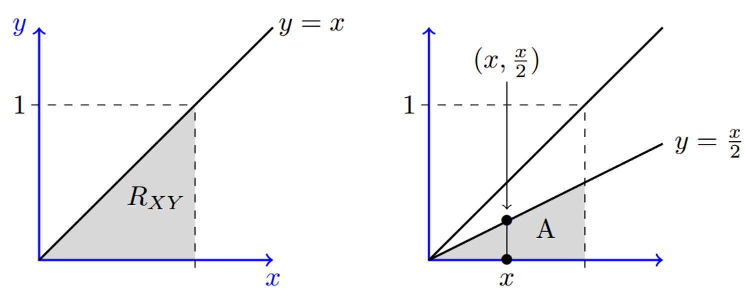

Figure 5.6: Figure shows R_{XY} as well as integration region for finding $P(Y \leq \frac{X}{2})$.

b. To find the constant c, we can write

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x, y) dx dy \\ &= \int_{0}^{1} \int_{0}^{x} cx^2 y \; dy dx \\ &= \int_{0}^{1} \frac{c}{2} x^4 dx \\ &= \frac{c}{10}. \end{aligned}$$

Thus, $c = 10$.

c. To find the marginal PDFs, first note that $R_X = R_Y = [0, 1]$. For $0 \leq x \leq 1$, we can write

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{XY}(x, y) dy \\ &= \int_{0}^{x} 10x^2 y dy \\ &= 5x^4. \end{aligned}$$

Thus,

$$f_X(x) = \begin{cases} 5x^4 & 0 \leq x \leq 1 \\ \\ 0 & \text{otherwise} \end{cases}$$

For $0 \leq y \leq 1$, we can write

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{XY}(x,y)dx \\ &= \int_{y}^{1} 10x^2 y dx \\ &= \frac{10}{3} y(1-y^3). \end{aligned}$$

Thus,

$$f_Y(y) = \begin{cases} \frac{10}{3} y(1-y^3) & 0 \leq y \leq 1 \\ \\ 0 & \text{otherwise} \end{cases}$$

d. To find $P(Y \leq \frac{X}{2})$, we need to integrate $f_{XY}(x,y)$ over region A shown in Figure 5.6. In particular, we have

$$\begin{aligned} P\left(Y \leq \frac{X}{2}\right) &= \int_{-\infty}^{\infty} \int_{0}^{\frac{x}{2}} f_{XY}(x,y)dydx \\ &= \int_{0}^{1} \int_{0}^{\frac{x}{2}} 10x^2 y \;\; dydx \\ &= \int_{0}^{1} \frac{5}{4} x^4 dx \\ &= \frac{1}{4}. \end{aligned}$$

e. To find $P(Y \leq \frac{X}{4} | Y \leq \frac{X}{2})$, we have

$$\begin{aligned} P\left(Y \leq \frac{X}{4} | Y \leq \frac{X}{2}\right) &= \frac{P\left(Y \leq \frac{X}{4}, Y \leq \frac{X}{2}\right)}{P\left(Y \leq \frac{X}{2}\right)} \\ &= 4P\left(Y \leq \frac{X}{4}\right) \\ &= 4\int_{0}^{1} \int_{0}^{\frac{x}{4}} 10x^2 y \;\; dydx \\ &= 4\int_{0}^{1} \frac{5}{16} x^4 dx \\ &= \frac{1}{4}. \end{aligned}$$