11.2.4 Classification of States

To better understand Markov chains, we need to introduce some definitions. The first definition concerns the accessibility of states from each other: If it is possible to go from state i to state j, we say that state j is *accessible* from state i. In particular, we can provide the following definitions.

> We say that state j is **accessible** from state i, written as $i \to j$, if $p_{ij}^{(n)} > 0$ for some n. We assume every state is accessible from itself since $p_{ii}^{(0)} = 1$.

> Two states i and j are said to **communicate**, written as $i \leftrightarrow j$, if they are **accessible** from each other. In other words,
>
> $$i \leftrightarrow j \;\text{ means }\; i \to j \text{ and } j \to i.$$

Communication is an *equivalence* relation. That means that

$-$every state communicates with itself, $i \leftrightarrow i$;
$-$if $i \leftrightarrow j$, then $j \leftrightarrow i$;
$-$if $i \leftrightarrow j$ and $j \leftrightarrow k$, then $i \leftrightarrow k$.

Therefore, the states of a Markov chain can be partitioned into communicating *classes* such that only members of the same class communicate with each other. That is, two states i and j belong to the same class if and only if $i \leftrightarrow j$.

Example11.6
Consider the Markov chain shown in Figure 11.9. It is assumed that when there is an arrow from state i to state j, then $p_{ij} > 0$. Find the equivalence classes for this Markov chain.

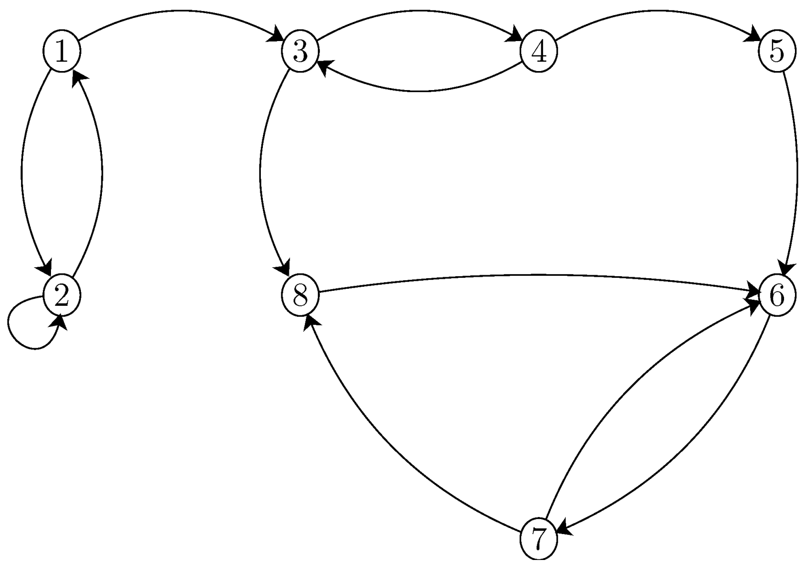

Figure 11.9 - A state transition diagram.

Solution

There are four communicating classes in this Markov chain. Looking at Figure 11.10, we notice that states 1 and 2 communicate with each other, but they do not communicate with any other nodes in the graph. Similarly, nodes 3 and 4 communicate with each other, but they do not communicate with any other nodes in the graph. State 5 does not communicate with any other states, so it by itself is a class. Finally, states 6 , 7, and 8 construct another class. Thus, here are the classes:

$$\begin{aligned}
&\text{Class } 1 = \{\text{state } 1, \text{state } 2\}, \\
&\text{Class } 2 = \{\text{state } 3, \text{state } 4\}, \\
&\text{Class } 3 = \{\text{state } 5\}, \\
&\text{Class } 4 = \{\text{state } 6, \text{state } 7, \text{state } 8\}.
\end{aligned}$$

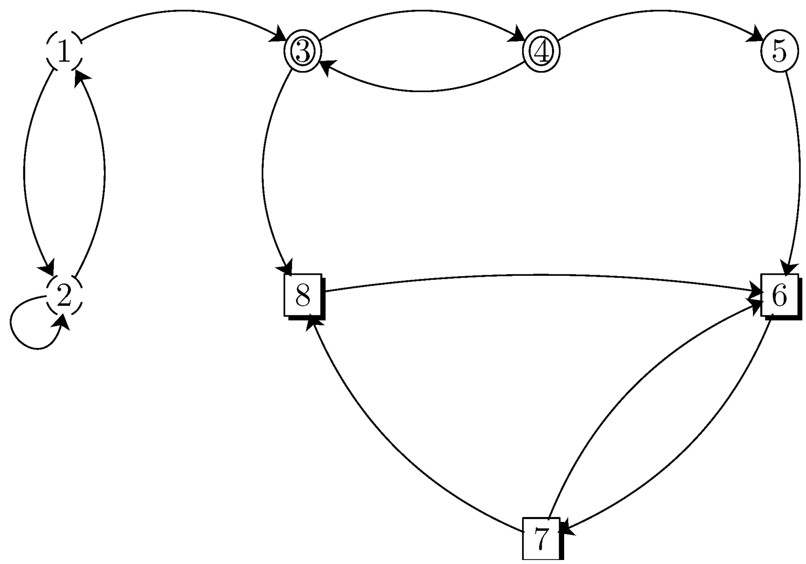

Figure 11.10 - Equivalence classes.

A Markov chain is said to be *irreducible* if it has only one communicating class. As we will see shortly, irreducibility is a desirable property in the sense that it can simplify analysis of the limiting behavior.

> A Markov chain is said to be **irreducible** if all states communicate with each other.

Looking at Figure 11.10, we notice that there are two kinds of classes. In particular, if at any time the Markov chain enters Class 4, it will always stay in that class. On the other hand, for other classes this is not true. For example, if $X_0 = 1$, then the Markov chain might stay in Class 1 for a while, but at some point, it will leave that class and it will never return to that class again. The states in Class 4 are called *recurrent* states, while the other states in this chain are called *transient*.

In general, a state is said to be recurrent if, any time that we leave that state, we will return to that state in the future with probability one. On the other hand, if the probability of returning is less than one, the state is called transient. Here, we provide a formal definition:

For any state i, we define

$$f_{ii} = P(X_n = i, \text{ for some } n \geq 1 | X_0 = i).$$

State i is **recurrent** if $f_{ii} = 1$, and it is **transient** if $f_{ii} < 1$.

It is relatively easy to show that if two states are in the same class, either both of them are recurrent, or both of them are transient. Thus, we can extend the above definitions to classes. A class is said to be recurrent if the states in that class are recurrent. If, on the other hand, the states are transient, the class is called transient. In general, a Markov chain might consist of several transient classes as well as several recurrent classes.

Consider a Markov chain and assume $X_0 = i$. If i is a recurrent state, then the chain will return to state i any time it leaves that state. Therefore, the chain will visit state i an infinite number of times. On the other hand, if i is a transient state, the chain will return to state i with probability $f_{ii} < 1$. Thus, in that case, the total number of visits to state i will be a Geometric random variable with parameter $1 - f_{ii}$.

Consider a discrete-time Markov chain. Let V be the total number of visits to state i.

a. If i is a recurrent state, then

$$P(V = \infty | X_0 = i) = 1.$$

b. If i is a transient state, then

$$V | X_0 = i \ \sim \ Geometric(1 - f_{ii}).$$

Example 11.7

Show that in a finite Markov chain, there is at least one recurrent class.

Solution