11.1.1 Counting Processes

In some problems, we count the occurrences of some types of events. In such scenarios, we are dealing with a *counting process*. For example, you might have a random process $N(t)$ that shows the number of customers who arrive at a supermarket by time t starting from time 0. For such a processes, we usually assume $N(0)=0$, so as time passes and customers arrive, $N(t)$ takes positive integer values.

Definition 11.1

A random process $\{N(t), t \in [0, \infty)\}$ is said to be a **counting process** if $N(t)$ is the number of events occurred from time 0 up to and including time t. For a counting process, we assume

1. $N(0)=0$;
2. $N(t) \in \{0, 1, 2, \cdots\}$, for all $t \in [0, \infty)$;
3. for $0 \leq s < t$, $N(t)-N(s)$ shows the number of events that occur in the interval $(s, t]$.

Since counting processes have been used to model arrivals (such as the supermarket example above), we usually refer to the occurrence of each event as an "arrival". For example, if $N(t)$ is the number of accidents in a city up to time t, we still refer to each accident as an arrival. Figure 11.1 shows a possible realization and the corresponding sample function of a counting process.

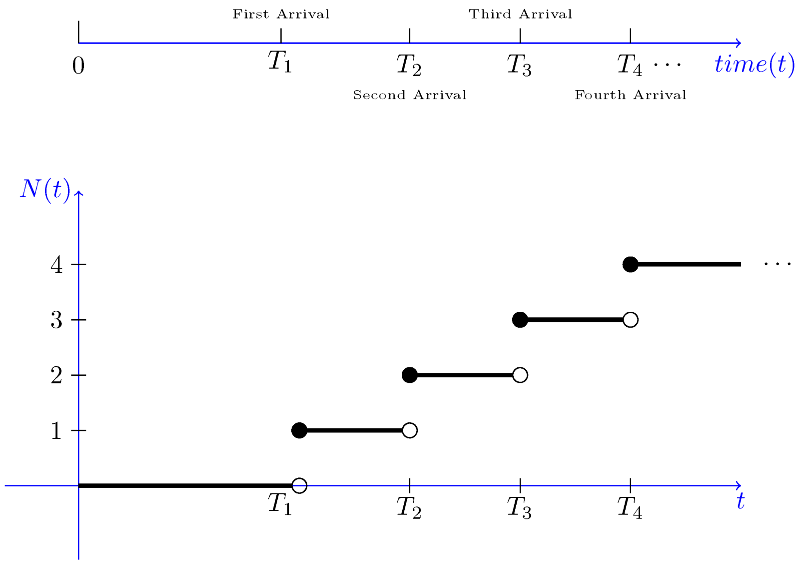

Figure 11.1 - A possible realization and the corresponding sample path of a counting process..

By the above definition, the only sources of randomness are the arrival times T_i. Before introducing the Poisson process, we would like to provide two definitions.

Definition 11.2

Let $\{X(t), t \in [0, \infty)\}$ be a continuous-time random process. We say that $X(t)$ has **independent increments** if, for all $0 \leq t_1 < t_2 < t_3 \cdots < t_n$, the random variables

$$X(t_2) - X(t_1),\ X(t_3) - X(t_2),\ \cdots,\ X(t_n) - X(t_{n-1})$$

are independent.

Note that for a counting process, $N(t_i) - N(t_{i-1})$ is the number of arrivals in the interval $(t_{i-1}, t_i]$. Thus, a counting process has independent increments if the numbers of arrivals in non-overlapping (disjoint) intervals

$$(t_1, t_2], (t_2, t_3],\ \cdots,\ (t_{n-1}, t_n]$$

are independent. Having independent increments simplifies analysis of a counting process. For example, suppose that we would like to find the probability of having 2 arrivals in the interval $(1, 2]$, and 3 arrivals in the interval $(3, 5]$. Since the two intervals $(1, 2]$ and $(3, 5]$ are disjoint, we can write

$$P\bigg(2 \text{ arrivals in } (1,2] \quad \text{and} \quad 3 \text{ arrivals in } (3,5]\bigg) = P\bigg(2 \text{ arrivals in } (1,2]\bigg) \cdot P\bigg(3 \text{ arrivals in } (3,5]\bigg).$$

Here is another useful definition.

Definition 11.3

Let $\{X(t), t \in [0, \infty)\}$ be a continuous-time random process. We say that $X(t)$ has **stationary increments** if, for all $t_2 > t_1 \geq 0$, and all $r > 0$, the two random variables $X(t_2) - X(t_1)$ and $X(t_2 + r) - X(t_1 + r)$ have the same distributions. In other words, the distribution of the difference depends only on the length of the interval $(t_1, t_2]$, and not on the exact location of the interval on the real line.

Note that for a counting process $N(t)$, $N(t_2) - N(t_1)$ is the number of arrivals in the interval $(t_1, t_2]$. We also assume $N(0) = 0$. Therefore, a counting process has stationary increments if for all $t_2 > t_1 \geq 0$, $N(t_2) - N(t_1)$ has the same distribution as $N(t_2 - t_1)$. This means that the distribution of the number of arrivals in any interval depends only on the length of the interval, and not on the exact location of the interval on the real line.

A counting process has **independent increments** if the numbers of arrivals in non-overlapping (disjoint) intervals are independent.

A counting process has **stationary increments** if, for all $t_2 > t_1 \geq 0$, $N(t_2) - N(t_1)$ has the same distribution as $N(t_2 - t_1)$.