$Binomial$	$pbinom$	$qbinom$	$dbinom$	$rbinom$
$Geometric$	$pgeom$	$qgeom$	$dgeom$	$rgeom$
$NegativeBinomial$	$pnbinom$	$qnbinom$	$dnbinom$	$rnbinom$
$Poisson$	$ppois$	$qpois$	$dpois$	$rpois$
$Beta$	$pbeta$	$qbeta$	$dbeta$	$rbeta$
$Beta$	$pbeta$	$qbeta$	$dbeta$	$rbeta$
$Exponential$	$pexp$	$qexp$	$dexp$	$rexp$
$Gamma$	$pgamma$	$qgamma$	$dgamma$	$rgamma$
$Studentt$	pt	qt	dt	rt
$Uniform$	$punif$	$qunif$	$dunif$	$runif$

13.5 Exercises

1. Write R programs to generate Geometric(p) and Negative Binomial(i,p) random variables.

   Solution: To generate a Geometric random variable, we run a loop of Bernoulli trials until the first success occurs. K counts the number of failures plus one success, which is equal to the total number of trials.

$$
\boxed{\begin{array}{l}
K = 1; \\
p = 0.2; \\
while(runif(1) > p) \\
K = K + 1; \\
K
\end{array}}
$$

   Now, we can generate Geometric random variable i times to obtain a Negative Binomial(i, p) variable as a sum of i independent Geometric (p) random variables.

$K = 1;$
$p = 0.2;$
$r = 2;$
$success = 0;$
$while(success < r)$
$\{if \quad (runif(1) > p)$
$\{K = K + 1;$
$print = 0 \quad \#Failure$
$\}else$
$\{success = success + 1;$
$print = 1\}\} \quad \#Success$
$K + r - 1 \quad \#Number \quad of \quad trials \quad needed \quad to \quad obtain \quad r \quad successes$

2. (Poisson) Use the algorithm for generating discrete random variables to obtain a Poisson random variable with parameter $\lambda = 2$.

   Solution: We know a Poisson random variable takes all nonnegative integer values with probabilities

$$p_i = P(X = x_i) = e^{-\lambda}\frac{\lambda^i}{i!} \quad \text{for} \quad i = 0, 1, 2, \cdots$$

   To generate a $Poisson(\lambda)$, first we generate a random number U. Next, we divide the interval $[0, 1]$ into subintervals such that the jth subinterval has length p_j (Figure 13.2). Assume

$$X = \begin{cases} x_0 & \text{if} \quad (U < p_0) \\ x_1 & \text{if} \quad (p_0 \leq U < p_0 + p_1) \\ \vdots & \\ x_j & \text{if} \quad \left(\sum_{k=0}^{j-1} p_k \leq U < \sum_{k=0}^{j} p_k\right) \\ \vdots & \end{cases}$$

   Here $x_i = i - 1$, so

$$X = i \quad \text{if} \quad p_0 + \cdots + p_{i-1} \leq U < p_0 + \cdots + p_{i-1} + p_i$$
$$F(i-1) \leq U < F(i) \quad \text{F is CDF}$$

$$
\begin{aligned}
&lambda = 2;\\
&i = 0;\\
&U = runif(1);\\
&cdf = exp(-lambda);\\
&while(U >= cdf)\\
&\{i = i + 1;\\
&cdf = cdf + exp(-lambda) * lambda\^{}i/gamma(i + 1);\}\\
&X = i;
\end{aligned}
$$

3. Explain how to generate a random variable with the density

$$f(x) = 2.5x\sqrt{x} \quad \text{for} \quad 0 < x < 1$$

if your random number generator produces a Standard Uniform random variable U. Hint: use the inverse transformation method.

Solution:

$$
\begin{aligned}
F_X(X) = X^{\frac{5}{2}} &= U \quad (0 < x < 1)\\
X &= U^{\frac{2}{5}}
\end{aligned}
$$

$$
\begin{aligned}
&U = runif(1);\\
&X = U^{\frac{2}{5}};
\end{aligned}
$$

We have the desired distribution.

4. Use the inverse transformation method to generate a random variable having distribution function

$$F(x) = \frac{x^2 + x}{2}, \quad 0 \le x \le 1$$

Solution:

$$
\begin{aligned}
\frac{X^2 + X}{2} &= U\\
(X + \frac{1}{2})^2 - \frac{1}{4} &= 2U\\
X + \frac{1}{2} &= \sqrt{2U + \frac{1}{4}}\\
X &= \sqrt{2U + \frac{1}{4}} - \frac{1}{2} \quad (X, U \in [0, 1])
\end{aligned}
$$

By generating a random number, U, we have the desired distribution.

$$U = runif(1);$$
$$X = sqrt\left(2U + \frac{1}{4}\right) - \frac{1}{2};$$

5. Let X have a standard Cauchy distribution.

$$F_X(x) = \frac{1}{\pi}\arctan(x) + \frac{1}{2}$$

Assuming you have $U \sim Uniform(0,1)$, explain how to generate X. Then, use this result to produce 1000 samples of X and compute the sample mean. Repeat the experiment 100 times. What do you observe and why?

Solution: Using Inverse Transformation Method:

$$\begin{aligned} U - \frac{1}{2} &= \frac{1}{\pi}\arctan(X) \\ \pi\left(U - \frac{1}{2}\right) &= \arctan(X) \\ X &= \tan\left(\pi(U - \frac{1}{2})\right) \end{aligned}$$

Next, here is the R code:

$$\begin{aligned} &U = numeric(1000); \\ &n = 100; \\ &average = numeric(n); \\ &for \quad (i \quad in \quad 1:n) \\ &\{U = runif(1000); \\ &X = tan(pi * (U - 0.5)); \\ &average[i] = mean(X);\} \\ &plot(1:n, average[1:n], type = "l", lwd = 2, col = "blue") \end{aligned}$$

Cauchy distribution has no mean (Figure 13.6), or higher moments defined.

6. (The Rejection Method) When we use the Inverse Transformation Method, we need a simple form of the cdf $F(x)$ that allows direct computation of $X = F^{-1}(U)$. When $F(x)$ doesn't have a simple form but the pdf $f(x)$ is available, random variables with density $f(x)$ can be generated by the **rejection method**. Suppose you have a method

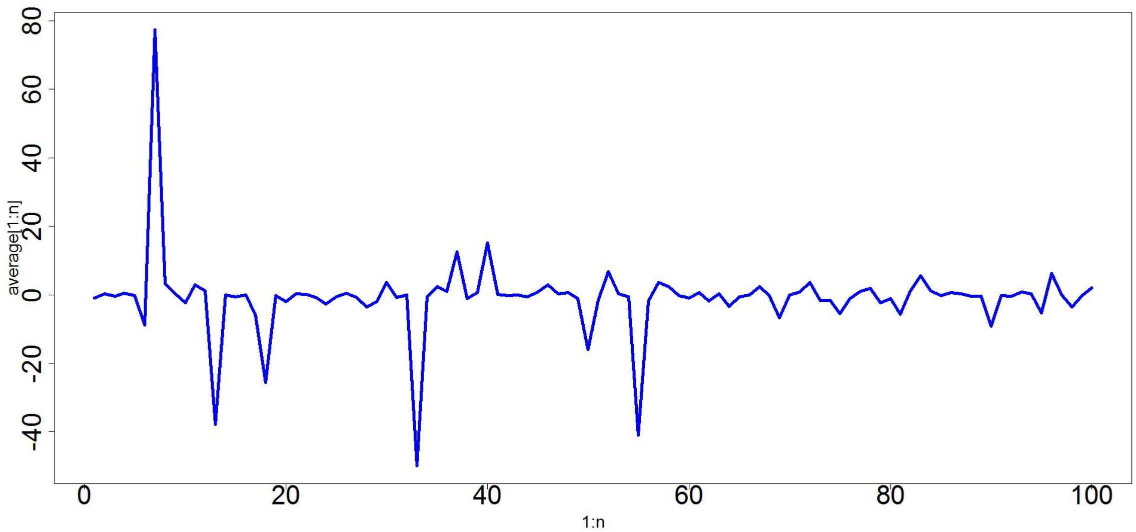

Figure 13.6: Cauchy Mean

for generating a random variable having density function $g(x)$. Now, assume you want to generate a random variable having density function $f(x)$. Let c be a constant such that

$$\frac{f(y)}{g(y)} \leq c \quad \text{(for all y)}$$

Show that the following method generates a random variable with density function $f(x)$.

- Generate Y having density g.
- Generate a random number U from Uniform $(0, 1)$.
- If $U \leq \frac{f(Y)}{cg(Y)}$, set $X = Y$. Otherwise, return to step 1.

Solution: The number of times N that the first two steps of the algorithm need to be called is itself a random variable and has a geometric distribution with "success" probability

$$p = P\left(U \leq \frac{f(Y)}{cg(Y)}\right)$$

Thus, $E(N) = \frac{1}{p}$. Also, we can compute p:

$$\begin{aligned}
P\left(U \le \frac{f(Y)}{cg(Y)}|Y = y\right) &= \frac{f(y)}{cg(y)} \\
p &= \int_{-\infty}^{\infty} \frac{f(y)}{cg(y)} g(y)dy \\
&= \frac{1}{c}\int_{-\infty}^{\infty} f(y)dy \\
&= \frac{1}{c} \\
\text{Therefore,}\quad E(N) &= c
\end{aligned}$$

Let F be the desired CDF (CDF of X). Now, we must show that the conditional distribution of Y given that $U \le \frac{f(Y)}{cg(Y)}$ is indeed F, i.e. $P(Y \le y|U \le \frac{f(Y)}{cg(Y)}) = F(y)$. Assume $M = \{U \le \frac{f(Y)}{cg(Y)}\}$, $K = \{Y \le y\}$. We know $P(M) = p = \frac{1}{c}$. Also, we can compute

$$\begin{aligned}
P(U \le \frac{f(Y)}{cg(Y)}|Y \le y) &= \frac{P(U \le \frac{f(Y)}{cg(Y)}, Y \le y)}{G(y)} \\
&= \int_{-\infty}^{y} \frac{P(U \le \frac{f(y)}{cg(y)}|Y = v \le y)}{G(y)} g(v)dv \\
&= \frac{1}{G(y)} \int_{-\infty}^{y} \frac{f(v)}{cg(v)} g(v)dv \\
&= \frac{1}{cG(y)} \int_{-\infty}^{y} f(v)dv \\
&= \frac{F(y)}{cG(y)}
\end{aligned}$$

Thus,

$$\begin{aligned}
P(K|M) &= P(M|K)P(K)/P(M) \\
&= P(U \le \frac{f(Y)}{cg(Y)}|Y \le y) \times \frac{G(y)}{\frac{1}{c}} \\
&= \frac{F(y)}{cG(y)} \times \frac{G(y)}{\frac{1}{c}} \\
&= F(y)
\end{aligned}$$

7. Use the rejection method to generate a random variable having density function Beta$(2, 4)$. Hint: Assume $g(x) = 1$ for $0 < x < 1$.

   Solution:

$$\begin{aligned}
f(x) &= 20x(1-x)^3 \quad 0 < x < 1 \\
g(x) &= 1 \quad 0 < x < 1 \\
\frac{f(x)}{g(x)} &= 20x(1-x)^3
\end{aligned}$$

We need to find the smallest constant c such that $f(x)/g(x) \leq c$. Differentiation of this quantity yields

$$\frac{d\left(\frac{f(x)}{g(x)}\right)}{dx} = 0$$
$$\text{Thus,} \quad x = \frac{1}{4}$$
$$\text{Therefore,} \quad \frac{f(x)}{g(x)} \leq \frac{135}{64}$$
$$\text{Hence,} \quad \frac{f(x)}{cg(x)} = \frac{256}{27} x(1-x)^3$$

```
n = 1;
while   (n == 1)
{U1 = runif(1);
U2 = runif(1);
if   (U2 <= 256/27 * U1 * (1 - U1)^3)
{X = U1;
n = 0; }}
```

8. Use the rejection method to generate a random variable having the $Gamma(\frac{5}{2}, 1)$ density function. Hint: Assume $g(x)$ is the pdf of the $Gamma\left(\alpha = \frac{5}{2}, \lambda = 1\right)$.

   Solution:

$$f(x) = \frac{4}{3\sqrt{\pi}} x^{\frac{3}{2}} e^{-x}, x > 0$$
$$g(x) = \frac{2}{5} e^{-\frac{2x}{5}} \quad x > 0$$
$$\frac{f(x)}{g(x)} = \frac{10}{3\sqrt{\pi}} x^{\frac{3}{2}} e^{-\frac{3x}{5}}$$
$$\frac{d\left(\frac{f(x)}{g(x)}\right)}{dx} = 0$$
$$\text{Hence,} \quad x = \frac{5}{2}$$
$$c = \frac{10}{3\sqrt{\pi}} \left(\frac{5}{2}\right)^{\frac{3}{2}} e^{\frac{-3}{2}}$$
$$\frac{f(x)}{cg(x)} = \frac{x^{\frac{3}{2}} e^{\frac{-3x}{5}}}{\left(\frac{5}{2}\right)^{\frac{3}{2}} e^{\frac{-3}{2}}}$$

We know how to generate an Exponential random variable.

- Generate a random number U_1 and set $Y = -\frac{5}{2}\log U_1$.
- Generate a random number U_2.
- If $U_2 < \frac{Y^{\frac{3}{2}} e^{\frac{-3Y}{5}}}{(\frac{5}{2})^{\frac{3}{2}} e^{\frac{-3}{2}}}$, set $X = Y$. Otherwise, execute the step 1.

9. Use the rejection method to generate a standard normal random variable. Hint: Assume $g(x)$ is the pdf of the exponential distribution with $\lambda = 1$.

Solution:

$$f(x) = \frac{2}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad 0 < x < \infty$$

$$g(x) = e^{-x} \quad 0 < x < \infty \quad \text{(Exponential density function with mean 1)}$$

$$\text{Thus,} \quad \frac{f(x)}{g(x)} = \sqrt{\frac{2}{\pi}} e^{x - \frac{x^2}{2}}$$

$$\text{Thus,} \quad x = 1 \quad \text{maximizes} \quad \frac{f(x)}{g(x)}$$

$$\text{Thus,} \quad c = \sqrt{\frac{2e}{\pi}}$$

$$\frac{f(x)}{cg(x)} = e^{-\frac{(x-1)^2}{2}}$$

- Generate Y, an exponential random variable with mean 1.
- Generate a random number U.
- If $U \le e^{\frac{-(Y-1)^2}{2}}$ set $X = Y$. Otherwise, return to step 1.

10. Use the rejection method to generate a $Gamma(2, 1)$ random variable conditional on its value being greater than 5, that is

$$\begin{aligned} f(x) &= \frac{xe^{-x}}{\int_5^\infty xe^{-x}dx} \\ &= \frac{xe^{-x}}{6e^{-5}} \quad (x \ge 5) \end{aligned}$$

Hint: Assume $g(x)$ be the density function of exponential distribution.

Solution: Since Gamma(2,1) random variable has expected value 2, we use an exponential distribution with mean 2 that is conditioned to be greater than 5.

$$\begin{aligned} f(x) &= \frac{xe^{(-x)}}{\int_5^\infty xe^{(-x)}dx} \\ &= \frac{xe^{(-x)}}{6e^{(-5)}} \quad x \ge 5 \\ g(x) &= \frac{\frac{1}{2}e^{(-\frac{x}{2})}}{e^{\frac{-5}{2}}} \quad x \ge 5 \\ \frac{f(x)}{g(x)} &= \frac{x}{3} e^{-(\frac{x-5}{2})} \end{aligned}$$

We obtain the maximum in $x = 5$ since $\frac{f(x)}{g(x)}$ is decreasing. Therefore,

$$c = \frac{f(5)}{g(5)} = \frac{5}{3}$$

- Generate a random number V.
- $Y = 5 - 2\log(V)$.
- Generate a random number U.
- If $U < \frac{Y}{5} e^{-(\frac{Y-5}{2})}$, set $X = Y$; otherwise return to step 1.