7.2.5 Convergence in Probability

Convergence in probability is stronger than convergence in distribution. In particular, for a sequence X_1, X_2, X_3, $\cdots$ to converge to a random variable X, we must have that $P(|X_n - X| \geq \epsilon)$ goes to 0 as $n \to \infty$, for any $\epsilon > 0$. To say that X_n converges in probability to X, we write

$$X_n \xrightarrow{p} X.$$

Here is the formal definition of convergence in probability:

Convergence in Probability

A sequence of random variables X_1, X_2, X_3, $\cdots$ converges **in probability** to a random variable X, shown by $X_n \xrightarrow{p} X$, if

$$\lim_{n \to \infty} P\big(|X_n - X| \geq \epsilon\big) = 0, \qquad \text{for all } \epsilon > 0.$$

Example 7.8

Let $X_n \sim Exponential(n)$, show that $X_n \xrightarrow{p} 0$. That is, the sequence X_1, X_2, X_3, $\cdots$ converges in probability to the zero random variable X.

Solution

We have

$$\begin{aligned}
\lim_{n \to \infty} P\big(|X_n - 0| \geq \epsilon\big) &= \lim_{n \to \infty} P\big(X_n \geq \epsilon\big) && (\text{ since } X_n \geq 0 \text{ }) \\
&= \lim_{n \to \infty} e^{-n\epsilon} && (\text{ since } X_n \sim Exponential(n) \text{ }) \\
&= 0, \qquad \text{for all } \epsilon > 0.
\end{aligned}$$

Example 7.9

Let X be a random variable, and $X_n = X + Y_n$, where

$$EY_n = \frac{1}{n}, \qquad \mathrm{Var}(Y_n) = \frac{\sigma^2}{n},$$

where $\sigma > 0$ is a constant. Show that $X_n \xrightarrow{p} X$.

Solution

First note that by the triangle inequality, for all $a, b \in \mathbb{R}$, we have $|a+b| \leq |a| + |b|$. Choosing $a = Y_n - EY_n$ and $b = EY_n$, we obtain

$$|Y_n| \leq |Y_n - EY_n| + \frac{1}{n}.$$

Now, for any $\epsilon > 0$, we have

$$\begin{aligned}
P\big(|X_n - X| \geq \epsilon\big) &= P\big(|Y_n| \geq \epsilon\big) \\
&\leq P\left(|Y_n - EY_n| + \frac{1}{n} \geq \epsilon\right) \\
&= P\left(|Y_n - EY_n| \geq \epsilon - \frac{1}{n}\right) \\
&\leq \frac{\mathrm{Var}(Y_n)}{\left(\epsilon - \frac{1}{n}\right)^2} && \text{(by Chebyshev's inequality)} \\
&= \frac{\sigma^2}{n\left(\epsilon - \frac{1}{n}\right)^2} \rightarrow 0 \qquad \text{as } n \rightarrow \infty.
\end{aligned}$$

Therefore, we conclude $X_n \xrightarrow{p} X$.

As we mentioned previously, convergence in probability is stronger than convergence in distribution. That is, if $X_n \xrightarrow{p} X$, then $X_n \xrightarrow{d} X$. The converse is not necessarily true. For example, let $X_1, X_2, X_3, \cdots$ be a sequence of i.i.d. $Bernoulli\left(\frac{1}{2}\right)$ random variables. Let also $X \sim Bernoulli\left(\frac{1}{2}\right)$ be independent from the X_i's. Then, $X_n \xrightarrow{d} X$. However, X_n does not converge in probability to X, since $|X_n - X|$ is in fact also a $Bernoulli\left(\frac{1}{2}\right)$ random variable and

$$P\big(|X_n - X| \geq \epsilon\big) = \frac{1}{2}, \qquad \text{for } 0 < \epsilon < 1.$$

A special case in which the converse is true is when $X_n \ \xrightarrow{d}\ c$, where c is a constant. In this case, convergence in distribution implies convergence in probability. We can state the following theorem:

Theorem 7.2 If $X_n \ \xrightarrow{d}\ c$, where c is a constant, then $X_n \ \xrightarrow{p}\ c$.

Proof

Since $X_n \ \xrightarrow{d}\ c$, we conclude that for any $\epsilon > 0$, we have

$$\lim_{n\rightarrow\infty} F_{X_n}(c-\epsilon) = 0,$$
$$\lim_{n\rightarrow\infty} F_{X_n}(c+\frac{\epsilon}{2}) = 1.$$

We can write for any $\epsilon > 0$,

$$\begin{aligned}
\lim_{n\rightarrow\infty} P\big(|X_n - c| \geq \epsilon\big) &= \lim_{n\rightarrow\infty} \bigg[P\big(X_n \leq c-\epsilon\big) + P\big(X_n \geq c+\epsilon\big)\bigg] \\
&= \lim_{n\rightarrow\infty} P\big(X_n \leq c-\epsilon\big) + \lim_{n\rightarrow\infty} P\big(X_n \geq c+\epsilon\big) \\
&= \lim_{n\rightarrow\infty} F_{X_n}(c-\epsilon) + \lim_{n\rightarrow\infty} P\big(X_n \geq c+\epsilon\big) \\
&= 0 + \lim_{n\rightarrow\infty} P\big(X_n \geq c+\epsilon\big) \qquad\qquad (\text{since } \lim_{n\rightarrow\infty} F_{X_n}(c-\epsilon) = 0) \\
&\leq \lim_{n\rightarrow\infty} P\big(X_n > c+\frac{\epsilon}{2}\big) \\
&= 1 - \lim_{n\rightarrow\infty} F_{X_n}(c+\frac{\epsilon}{2}) \\
&= 0 \qquad\qquad (\text{since } \lim_{n\rightarrow\infty} F_{X_n}(c+\frac{\epsilon}{2}) = 1).
\end{aligned}$$

Since $\lim_{n\rightarrow\infty} P\big(|X_n - c| \geq \epsilon\big) \geq 0$, we conclude that

$$\lim_{n\rightarrow\infty} P\big(|X_n - c| \geq \epsilon\big) = 0, \qquad \text{for all } \epsilon > 0,$$

which means $X_n \ \xrightarrow{p}\ c$.

The most famous example of convergence in probability is the weak law of large numbers (WLLN). We proved WLLN in Section 7.1.1. The WLLN states that if X_1, X_2,

$X_3, \cdots$ are i.i.d. random variables with mean $EX_i = \mu < \infty$, then the average sequence defined by

$$\overline{X}_n = \frac{X_1 + X_2 + \ldots + X_n}{n}$$

converges in probability to μ. It is called the "weak" law because it refers to convergence in probability. There is another version of the law of large numbers that is called the strong law of large numbers (SLLN). We will discuss SLLN in Section 7.2.7.