Properties of the gamma function

For any positive real number α:

1. $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$;
2. $\int_0^\infty x^{\alpha-1} e^{-\lambda x} dx = \frac{\Gamma(\alpha)}{\lambda^\alpha}, \qquad \text{for } \lambda > 0;$
3. $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$;
4. $\Gamma(n) = (n-1)!, \text{ for } n = 1, 2, 3, \cdots;$
5. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Example 4.12

Answer the following questions:

1. Find $\Gamma(\frac{7}{2})$.
2. Find the value of the following integral:

$$I = \int_0^\infty x^6 e^{-5x} dx.$$

Solution

1. To find $\Gamma(\frac{7}{2})$, we can write

$$\begin{aligned}\Gamma(\frac{7}{2}) &= \frac{5}{2}\cdot\Gamma(\frac{5}{2}) \qquad \text{(using Property 3)}\\ &= \frac{5}{2}\cdot\frac{3}{2}\cdot\Gamma(\frac{3}{2}) \qquad \text{(using Property 3)}\\ &= \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma(\frac{1}{2})\text{(using Property 3)}\\ &= \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi} \qquad \text{(using Property 5)}\\ &= \frac{15}{8}\sqrt{\pi}.\end{aligned}$$

2. Using Property 2 with $\alpha = 7$ and $\lambda = 5$, we obtain

$$\begin{aligned}I &= \int_0^\infty x^6 e^{-5x} dx\\ &= \frac{\Gamma(7)}{5^7}\\ &= \frac{6!}{5^7} \qquad \text{(using Property 4)}\\ &\approx 0.0092\end{aligned}$$

Gamma Distribution:

We now define the gamma distribution by providing its PDF:

A continuous random variable X is said to have a *gamma* distribution with parameters $\alpha > 0$ and $\lambda > 0$, shown as $X \sim Gamma(\alpha, \lambda)$, if its PDF is given by

$$f_X(x) = \begin{cases} \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

If we let $\alpha = 1$, we obtain

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Thus, we conclude $Gamma(1, \lambda) = Exponential(\lambda)$. More generally, if you sum n independent $Exponential(\lambda)$ random variables, then you will get a $Gamma(n, \lambda)$ random variable. We will prove this later on using the moment generating function. The gamma distribution is also related to the normal distribution as will be discussed later. Figure 4.10 shows the PDF of the gamma distribution for several values of α.

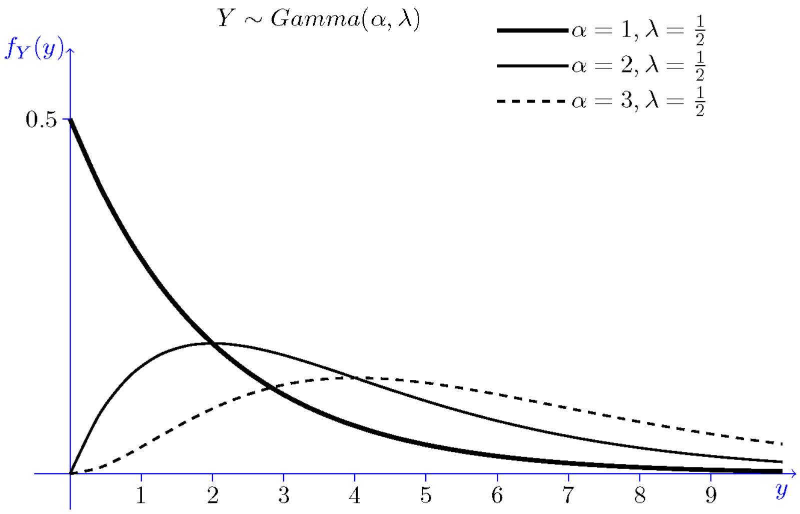

Figure 4.10: PDF of the gamma distribution for some values of α and λ.

Example 4.13
Using the properties of the gamma function, show that the gamma PDF integrates to 1, i.e., show that for $\alpha, \lambda > 0$, we have

$$\int_0^\infty \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} dx = 1.$$

Solution

We can write

$$\begin{aligned} \int_0^\infty \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} dx &= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{-\lambda x} dx \\ &= \frac{\lambda^\alpha}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha)}{\lambda^\alpha} \qquad \text{(using Property 2 of the gamma function)} \\ &= 1. \end{aligned}$$

In the Solved Problems section, we calculate the mean and variance for the gamma distribution. In particular, we find out that if $X \sim Gamma(\alpha, \lambda)$, then

$$EX = \frac{\alpha}{\lambda}, \qquad Var(X) = \frac{\alpha}{\lambda^2}.$$