9.1.9 Bayesian Interval Estimation

Interval estimation has a very natural interpretation in Bayesian inference. Suppose that we would like to estimate the value of an unobserved random variable X, given that we have observed $Y = y$. After calculating the posterior density $f_{X|Y}(x|y)$, we can simply find an interval $[a, b]$ for which we have

$$P(a \leq X \leq b | Y = y) = 1 - \alpha.$$

Such an interval is said to be a $(1 - \alpha)100\%$ *credible* interval for X.

Bayesian Credible Intervals

Given the observation $Y = y$, the interval $[a, b]$ is said to be a $(1 - \alpha)100\%$ **credible interval** for X, if the posterior probability of X being in $[a, b]$ is equal to $1 - \alpha$. In other words,

$$P(a \leq X \leq b | Y = y) = 1 - \alpha.$$

Example 9.13
Let X and Y be jointly normal and $X \sim N(0, 1)$, $Y \sim N(1, 4)$, and $\rho(X, Y) = \frac{1}{2}$. Find a 95% credible interval for X, given $Y = 2$ is observed.

Solution

As we have seen before, if X and Y are jointly normal random variables with parameters μ_X, σ_X^2, μ_Y, σ_Y^2, and ρ, then, given $Y = y$, X is normally distributed with

$$E[X|Y = y] = \mu_X + \rho \sigma_X \frac{y - \mu_Y}{\sigma_Y},$$
$$\mathrm{Var}(X|Y = y) = (1 - \rho^2)\sigma_X^2.$$

Therefore, $X|Y = 2$ is normal with

$$E[X|Y=y] = 0 + \frac{1}{2} \cdot \frac{2-1}{2} = \frac{1}{4},$$

$$\mathrm{Var}(X|Y=y) = \left(1 - \frac{1}{4}\right) \cdot 1 = \frac{3}{4}.$$

Here $\alpha = 0.05$, so we need an interval $[a, b]$ for which

$$P(a \leq X \leq b|Y=2) = 0.95$$

We usually choose a symmetric interval around the expected value $E[X|Y=y] = \frac{1}{4}$. That is, we choose the interval in the form of

$$\left[\frac{1}{4} - c, \frac{1}{4} + c\right].$$

Thus, we need to have

$$\begin{aligned} P\left(\frac{1}{4} - c \leq X \leq \frac{1}{4} + c|Y=2\right) &= \Phi\left(\frac{c}{\sqrt{3/4}}\right) - \Phi\left(\frac{-c}{\sqrt{3/4}}\right) \\ &= 2\Phi\left(\frac{c}{\sqrt{3/4}}\right) - 1 = 0.95 \end{aligned}$$

Solving for c, we obtain

$$c = \sqrt{3/4}\Phi^{-1}(0.975) \approx 1.70$$

Therefore, the 95% credible interval for X is

$$\left[\frac{1}{4} - c, \frac{1}{4} + c\right] \approx [-1.45, 1.95].$$
