Note that $|X_n| = \frac{1}{n}$. Thus, $|X_n| > \epsilon$ if and only if $n < \frac{1}{\epsilon}$. Thus, we conclude

$$\sum_{n=1}^{\infty} P\big(|X_n| > \epsilon\big) \leq \sum_{n=1}^{\lfloor \frac{1}{\epsilon} \rfloor} P\big(|X_n| > \epsilon\big)$$
$$= \lfloor \frac{1}{\epsilon} \rfloor < \infty.$$

Theorem 7.5 provides only a sufficient condition for almost sure convergence. In particular, if we obtain

$$\sum_{n=1}^{\infty} P\big(|X_n - X| > \epsilon\big) = \infty,$$

then we still don't know whether the X_n's converge to X almost surely or not. Here, we provide a condition that is both necessary and sufficient.

Theorem 7.6

Consider the sequence $X_1, X_2, X_3, \cdots$. For any $\epsilon > 0$, define the set of events

$$A_m = \{|X_n - X| < \epsilon, \qquad \text{for all } n \geq m\}.$$

Then $X_n \xrightarrow{a.s.} X$ if and only if for any $\epsilon > 0$, we have

$$\lim_{m \rightarrow \infty} P(A_m) = 1.$$

Example 7.15

Let $X_1, X_2, X_3, \cdots$ be independent random variables, where $X_n \sim Bernoulli\left(\frac{1}{n}\right)$ for $n = 2, 3, \cdots$. The goal here is to check whether $X_n \xrightarrow{a.s.} 0$.

1. Check that $\sum_{n=1}^{\infty} P\big(|X_n| > \epsilon\big) = \infty$.
2. Show that the sequence $X_1, X_2, \ldots$ does not converge to 0 almost surely using Theorem 7.6

   .

Solution

1. We first note that for $0 < \epsilon < 1$, we have

$$\begin{aligned} \sum_{n=1}^{\infty} P\big(|X_n| > \epsilon\big) &= \sum_{n=1}^{\infty} P(X_n = 1) \\ &= \sum_{n=1}^{\infty} \frac{1}{n} = \infty. \end{aligned}$$

2. To use Theorem 7.6, we define

$$A_m = \{|X_n| < \epsilon, \qquad \text{for all } n \geq m\}.$$

Note that for $0 < \epsilon < 1$, we have

$$A_m = \{X_n = 0, \qquad \text{for all } n \geq m\}.$$

According to Theorem 7.6 , it suffices to show that

$$\lim_{m \to \infty} P(A_m) < 1.$$

We can in fact show that $\lim_{m \to \infty} P(A_m) = 0$. To show this, we will prove $P(A_m) = 0$, for every $m \geq 2$. For $0 < \epsilon < 1$, we have

$$\begin{aligned} P(A_m) &= P\big(\{X_n = 0, \qquad \text{for all } n \geq m\}\big) \\ &\leq P\big(\{X_n = 0, \qquad \text{for } n = m, m+1, \cdots, N\}\big) \quad (\text{for every positive integer } N \geq m) \\ &= P(X_m = 0) P(X_{m+1} = 0) \cdots P(X_N = 0) \qquad (\text{since the } X_i\text{'s are independent}) \\ &= \frac{m-1}{m} \cdot \frac{m}{m+1} \cdots \frac{N-1}{N} \\ &= \frac{m-1}{N}. \end{aligned}$$

Thus, by choosing N large enough, we can show that $P(A_m)$ is less than any positive number. Therefore, $P(A_m) = 0$ for all $m \geq 2$. We conclude that $\lim_{m \to \infty} P(A_m) = 0$. Thus, according to Theorem 7.6, the sequence X_1, X_2, $\ldots$ does not converge to 0 almost surely.

An important example for almost sure convergence is the **strong law of large numbers (SLLN)**. Here, we state the SLLN without proof. The interested reader can find a proof of SLLN in [19]. A simpler proof can be obtained if we assume the finiteness of the fourth moment. (See [20] for example.)

The strong law of large numbers (SLLN)

Let X_1,X_2,...,X_n be i.i.d. random variables with a finite expected value $EX_i = \mu < \infty$. Let also

$$M_n = \frac{X_1 + X_2 + \ldots + X_n}{n}.$$

Then $M_n \xrightarrow{a.s.} \mu$.

We end this section by stating a version of the **continuous mapping theorem**. This theorem is sometimes useful when proving the convergence of random variables.

Theorem 7.7 Let $X_1, X_2, X_3, \cdots$ be a sequence of random variables. Let also $h : \mathbb{R} \mapsto \mathbb{R}$ be a <u>continuous</u> function. Then, the following statements are true:

1. If $X_n \xrightarrow{d} X$, then $h(X_n) \xrightarrow{d} h(X)$.
2. If $X_n \xrightarrow{p} X$, then $h(X_n) \xrightarrow{p} h(X)$.
3. If $X_n \xrightarrow{a.s.} X$, then $h(X_n) \xrightarrow{a.s.} h(X)$.