9.1.3 Comparison to ML Estimation

We discussed maximum likelihood estimation in the previous chapter. Assuming that we have observed $Y = y$, the maximum likelihood (ML) estimate of X is the value of x that maximizes

$$f_{Y|X}(y|x) \qquad (9.1)$$

We show the ML estimate of X by $\hat{x}_{ML}$. On the other hand, the MAP estimate of X is the value of x that maximizes

$$f_{Y|X}(y|x)f_X(x) \qquad (9.2)$$

The two expressions in Equations 9.1 and 9.2 are somewhat similar. The difference is that Equation 9.2 has an extra term, $f_X(x)$. For example, if X is uniformly distributed over a finite interval, then the ML and the MAP estimate will be the same.

Example 9.5
Suppose that the signal $X \sim N(0, \sigma_X^2)$ is transmitted over a communication channel. Assume that the received signal is given by

$$Y = X + W,$$

where $W \sim N(0, \sigma_W^2)$ is independent of X.

1. Find the ML estimate of X, given $Y = y$ is observed.
2. Find the MAP estimate of X, given $Y = y$ is observed.

Solution

Here, we have

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_X} e^{-\frac{x^2}{2\sigma_X^2}}.$$

We also have, $Y|X = x \quad \sim \quad N(x, \sigma_W^2)$, so

$$f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi}\sigma_W} e^{-\frac{(y-x)^2}{2\sigma_W^2}}.$$

1. The ML estimate of X, given $Y = y$, is the value of x that maximizes

$$f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi}\sigma_W} e^{-\frac{(y-x)^2}{2\sigma_W^2}}.$$

To maximize the above function, we should minimize $(y-x)^2$. Therefore, we conclude

$$\hat{x}_{ML} = y.$$

2. The MAP estimate of X, given $Y = y$, is the value of x that maximizes

$$f_{Y|X}(y|x)f_X(x) = c\exp\left\{-\left[\frac{(y-x)^2}{2\sigma_W^2} + \frac{x^2}{2\sigma_X^2}\right]\right\},$$

where c is a constant. To maximize the above function, we should minimize

$$\frac{(y-x)^2}{2\sigma_W^2} + \frac{x^2}{2\sigma_X^2}.$$

By differentiation, we obtain the MAP estimate of x as

$$\hat{x}_{MAP} = \frac{\sigma_X^2}{\sigma_X^2 + \sigma_W^2} y.$$
