8.2.5 Solved Problems

Problem 1

Let X be the height of a randomly chosen individual from a population. In order to estimate the mean and variance of X, we observe a random sample $X_1, X_2, \cdots, X_7$. Thus, X_i's are i.i.d. and have the same distribution as X. We obtain the following values (in centimeters):

$$166.8, 171.4, 169.1, 178.5, 168.0, 157.9, 170.1$$

Find the values of the sample mean, the sample variance, and the sample standard deviation for the observed sample.

Solution

$$\begin{aligned}\overline{X} &= \frac{X_1 + X_2 + X_3 + X_4 + X_5 + X_6 + X_7}{7} \\ &= \frac{166.8 + 171.4 + 169.1 + 178.5 + 168.0 + 157.9 + 170.1}{7} \\ &= 168.8\end{aligned}$$

The sample variance is given by

$$S^2 = \frac{1}{7-1} \sum_{k=1}^{7} (X_k - 168.8)^2 = 37.7$$

Finally, the sample standard deviation is given by

$$= \sqrt{S^2} = 6.1$$

The following MATLAB code can be used to obtain these values:

```
x=[166.8, 171.4, 169.1, 178.5, 168.0, 157.9,
170.1];
m=mean(x);
v=var(x);
s=std(x);
```

Problem 2

Prove the following:

a. If $\hat{\Theta}_1$ is an unbiased estimator for θ, and W is a zero mean random variable, then

$$\hat{\Theta}_2 = \hat{\Theta}_1 + W$$

is also an unbiased estimator for θ.

b. If $\hat{\Theta}_1$ is an estimator for θ such that $E[\hat{\Theta}_1] = a\theta + b$, where $a \neq 0$, show that

$$\hat{\Theta}_2 = \frac{\hat{\Theta}_1 - b}{a}$$

is an unbiased estimator for θ.

Solution

a. We have

$$\begin{aligned} E[\hat{\Theta}_2] &= E[\hat{\Theta}_1] + E[W] && \text{(by linearity of expectation)} \\ &= \theta + 0 && \text{(since } \hat{\Theta}_1 \text{ is unbiased and } EW = 0) \\ &= \theta. \end{aligned}$$

Thus, $\hat{\Theta}_2$ is an unbiased estimator for θ.

b. We have

$$\begin{aligned} E[\hat{\Theta}_2] &= \frac{E[\hat{\Theta}_1] - b}{a} \text{(by linearity of expectation)} \\ &= \frac{a\theta + b - b}{a} \\ &= \theta. \end{aligned}$$

Thus, $\hat{\Theta}_2$ is an unbiased estimator for θ.

Problem 3

Let X_1, X_2, X_3, ..., X_n be a random sample from a $Uniform(0, \theta)$ distribution, where θ is unknown. Define the estimator

$$\hat{\Theta}_n = \max\{X_1, X_2, \cdots, X_n\}.$$

a. Find the bias of $\hat{\Theta}_n$, $B(\hat{\Theta}_n)$.
b. Find the MSE of $\hat{\Theta}_n$, $MSE(\hat{\Theta}_n)$.
c. Is $\hat{\Theta}_n$ a consistent estimator of θ?

Solution

If $X \sim Uniform(0, \theta)$, then the PDF and CDF of X are given by

$$f_X(x) = \begin{cases} \frac{1}{\theta} & 0 \leq x \leq \theta \\ \\ 0 & \text{otherwise} \end{cases}$$

and

$$F_X(x) = \begin{cases} \frac{x}{\theta} & 0 \leq x \leq \theta \\ \\ 0 & \text{otherwise} \end{cases}$$

By Theorem 8.1, the PDF of $\hat{\Theta}_n$ is given by

$$\begin{aligned} f_{\hat{\Theta}_n}(y) &= n f_X(x) \big[F_X(x)\big]^{n-1} \\ &= \begin{cases} \frac{ny^{n-1}}{\theta^n} & 0 \leq y \leq \theta \\ \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

a. To find the bias of $\hat{\Theta}_n$, we have

$$\begin{aligned} E[\hat{\Theta}_n] &= \int_0^\theta y \cdot \frac{ny^{n-1}}{\theta^n} dy \\ &= \frac{n}{n+1}\theta. \end{aligned}$$

Thus, the bias is given by

$$\begin{aligned} B(\hat{\Theta}_n) &= E[\hat{\Theta}_n] - \theta \\ &= \frac{n}{n+1}\theta - \theta \\ &= -\frac{\theta}{n+1}. \end{aligned}$$

b. To find $MSE(\hat{\Theta}_n)$, we can write

$$\begin{aligned} MSE(\hat{\Theta}_n) &= \mathrm{Var}(\hat{\Theta}_n) + B(\hat{\Theta}_n)^2 \\ &= \mathrm{Var}(\hat{\Theta}_n) + \frac{\theta^2}{(n+1)^2}. \end{aligned}$$

Thus, we need to find $\mathrm{Var}(\hat{\Theta})$. We have

$$\begin{aligned} E\left[\hat{\Theta}_n^2\right] &= \int_0^\theta y^2 \cdot \frac{ny^{n-1}}{\theta^n} dy \\ &= \frac{n}{n+2}\theta^2. \end{aligned}$$

Thus,

$$\begin{aligned} \mathrm{Var}(\hat{\Theta}_n) &= E\left[\hat{\Theta}_n^2\right] - \left(E[\hat{\Theta}_n]\right)^2 \\ &= \frac{n}{(n+2)(n+1)^2}\theta^2. \end{aligned}$$

Therefore,

$$\begin{aligned} MSE(\hat{\Theta}_n) &= \frac{n}{(n+2)(n+1)^2}\theta^2 + \frac{\theta^2}{(n+1)^2} \\ &= \frac{2\theta^2}{(n+2)(n+1)}. \end{aligned}$$

c. Note that

$$\lim_{n\to\infty} MSE(\hat{\Theta}_n) = \lim_{n\to\infty} \frac{2\theta^2}{(n+2)(n+1)} = 0.$$

Thus, by Theorem 8.2, $\hat{\Theta}_n$ is a consistent estimator of θ.

Problem 4

Let X_1, X_2, X_3, ..., X_n be a random sample from a $Geometric(\theta)$ distribution, where θ is unknown. Find the maximum likelihood estimator (MLE) of θ based on this random sample.

Solution

If $X_i \sim Geometric(\theta)$, then

$$P_{X_i}(x;\theta) = (1-\theta)^{x-1}\theta.$$

Thus, the likelihood function is given by

$$\begin{aligned} L(x_1, x_2, \cdots, x_n; \theta) &= P_{X_1 X_2 \cdots X_n}(x_1, x_2, \cdots, x_n; \theta) \\ &= P_{X_1}(x_1;\theta) P_{X_2}(x_2;\theta) \cdots P_{X_n}(x_n;\theta) \\ &= (1-\theta)^{\left[\sum_{i=1}^n x_i - n\right]} \theta^n. \end{aligned}$$

Then, the log likelihood function is given by

$$\ln L(x_1, x_2, \cdots, x_n; \theta) = \left(\sum_{i=1}^n x_i - n\right) \ln(1-\theta) + n \ln \theta.$$

Thus,

$$\frac{d \ln L(x_1, x_2, \cdots, x_n; \theta)}{d\theta} = \left(\sum_{i=1}^n x_i - n\right) \cdot \frac{-1}{1-\theta} + \frac{n}{\theta}.$$

By setting the derivative to zero, we can check that the maximizing value of θ is given by

$$\hat{\theta}_{ML} = \frac{n}{\sum_{i=1}^n x_i}.$$

Thus, the MLE can be written as

$$\hat{\Theta}_{ML} = \frac{n}{\sum_{i=1}^n X_i}.$$

Problem 5
Let X_1, X_2, X_3, ..., X_n be a random sample from a $Uniform(0, \theta)$ distribution, where θ is unknown. Find the maximum likelihood estimator (MLE) of θ based on this random sample.

Solution

If $X_i \sim Uniform(0, \theta)$, then

$$f_X(x) = \begin{cases} \frac{1}{\theta} & 0 \leq x \leq \theta \\ & \\ 0 & \text{otherwise} \end{cases}$$

The likelihood function is given by

$$\begin{aligned} L(x_1, x_2, \cdots, x_n; \theta) &= f_{X_1 X_2 \cdots X_n}(x_1, x_2, \cdots, x_n; \theta) \\ &= f_{X_1}(x_1; \theta) f_{X_2}(x_2; \theta) \cdots f_{X_n}(x_n; \theta) \\ &= \begin{cases} \frac{1}{\theta^n} & 0 \leq x_1, x_2, \cdots, x_n \leq \theta \\ & \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Note that $\frac{1}{\theta^n}$ is a decreasing function of θ. Thus, to minimize it, we need to choose the smallest possible value for θ. For $i = 1, 2, \ldots, n$, we need to have $\theta \geq x_i$. Thus, the smallest possible value for θ is

$$\hat{\theta}_{ML} = \max(x_1, x_2, \cdots, x_n).$$

Therefore, the MLE can be written as

$$\hat{\Theta}_{ML} = \max(X_1, X_2, \cdots, X_n).$$

Note that this is one of those cases wherein $\hat{\theta}_{ML}$ cannot be obtained by setting the derivative of the likelihood function to zero. Here, the maximum is achieved at an endpoint of the acceptable interval.
