10.1.4 Stationary Processes

We can classify random processes based on many different criteria. One of the important questions that we can ask about a random process is whether it is a **stationary** process. Intuitively, a random process $\{X(t), t \in J\}$ is stationary if its statistical properties do not change by time. For example, for a stationary process, $X(t)$ and $X(t+\Delta)$ have the same probability distributions. In particular, we have

$$F_{X(t)}(x) = F_{X(t+\Delta)}(x), \quad \text{ for all } t, t+\Delta \in J.$$

More generally, for a stationary process, the joint distribution of $X(t_1)$ and $X(t_2)$ is the same as the joint distribution of $X(t_1+\Delta)$ and $X(t_2+\Delta)$. For example, if you have a stationary process $X(t)$, then

$$P\Bigg(\Big(X(t_1), X(t_2)\Big) \in A\Bigg) = P\Bigg(\Big(X(t_1+\Delta), X(t_2+\Delta)\Big) \in A\Bigg),$$

for any set $A \in \mathbb{R}^2$. In sum, a random process is stationary if *a time shift does not change its statistical properties*. Here is a formal definition of stationarity of continuous-time processes.

> A continuous-time random process $\{X(t), t \in \mathbb{R}\}$ is **strict-sense stationary** or simply **stationary** if, for all $t_1, t_2, \cdots, t_r \in \mathbb{R}$ and all $\Delta \in \mathbb{R}$, the joint CDF of
>
> $$X(t_1), X(t_2), \cdots, X(t_r)$$
>
> is the same as the joint CDF of
>
> $$X(t_1+\Delta), X(t_2+\Delta), \cdots, X(t_r+\Delta).$$
>
> That is, for all real numbers $x_1, x_2, \cdots, x_r$, we have
>
> $$F_{X(t_1)X(t_2)\cdots X(t_r)}(x_1, x_2, \cdots, x_r) = F_{X(t_1+\Delta)X(t_2+\Delta)\cdots X(t_r+\Delta)}(x_1, x_2, \cdots, x_r).$$

We can provide similar definition for discrete-time processes.

strict-sense stationary or simply **stationary**, if for all $n_1, n_2, \cdots, n_r \in \mathbb{Z}$ and all $D \in \mathbb{Z}$, the joint CDF of

$$X(n_1), X(n_2), \cdots, X(n_r)$$

is the same as the joint CDF of

$$X(n_1 + D), X(n_2 + D), \cdots, X(n_r + D).$$

That is, for all real numbers $x_1, x_2, \cdots, x_r$, we have

$$F_{X(n_1)X(n_2)\cdots X(n_r)}(x_1, x_2, \cdots, x_n) = F_{X(n_1+D)X(n_2+D)\cdots X(n_r+D)}(x_1, x_2, \cdots, x_r).$$