Countably Infinite Markov Chains:

When a Markov chain has an infinite (but countable) number of states, we need to distinguish between two types of recurrent states: *positive* recurrent and *null* recurrent states.

Remember that if state i is recurrent, then that state will be visited an infinite number of times (any time that we visit that state, we will return to it with probability one in the future). We previously defined r_i as the expected number of transitions between visits to state i. Consider a recurrent state i. If $r_i < \infty$, then state i is a *positive* recurrent state. Otherwise, it is called *null* recurrent.

Let i be a recurrent state. Assuming $X_0 = i$, let R_i be the number of transitions needed to return to state i, i.e.,

$$R_i = \min\{n \geq 1 : X_n = i\}.$$

If $r_i = E[R_i|X_0 = i] < \infty$, then i is said to be **positive recurrent**. If $E[R_i|X_0 = i] = \infty$, then i is said to be **null recurrent**.

Theorem 11.2
Consider an infinite Markov chain $\{X_n, n = 0, 1, 2, \dots\}$ where $X_n \in S = \{0, 1, 2, \cdots\}$. Assume that the chain is irreducible and aperiodic. Then, one of the following cases can occur:

1. All states are transient, and

$$\lim_{n\to\infty} P(X_n = j | X_0 = i) = 0, \text{ for all } i, j.$$

2. All states are null recurrent, and

$$\lim_{n\to\infty} P(X_n = j | X_0 = i) = 0, \text{ for all } i, j.$$

3. All states are positive recurrent. In this case, there exists a limiting distribution, $\pi = [\pi_0, \pi_1, \cdots]$, where

$$\pi_j = \lim_{n\to\infty} P(X_n = j | X_0 = i) > 0,$$

for all $i, j \in S$. The limiting distribution is the unique solution to the equations

$$\pi_j = \sum_{k=0}^{\infty} \pi_k P_{kj}, \quad \text{for } j = 0, 1, 2, \cdots,$$
$$\sum_{j=0}^{\infty} \pi_j = 1.$$

We also have

$$r_j = \frac{1}{\pi_j}, \quad \text{for all } j = 0, 1, 2, \cdots,$$

where r_j is the mean return time to state j.

How do we use the above theorem? Consider an infinite Markov chain $\{X_n, n = 0, 1, 2, \dots\}$, where $X_n \in S = \{0, 1, 2, \cdots\}$. Assume that the chain is irreducible and aperiodic. We first try to find a stationary distribution π by solving the equations

$$\pi_j = \sum_{k=0}^{\infty} \pi_k P_{kj}, \quad \text{for } j = 0, 1, 2, \cdots,$$
$$\sum_{j=0}^{\infty} \pi_j = 1.$$

If the above equations have a unique solution, we conclude that the chain is positive recurrent and the stationary distribution is the limiting distribution of this chain. On the other hand, if no stationary solution exists, we conclude that the chain is either transient or null recurrent, so

$$\lim_{n \to \infty} P(X_n = j | X_0 = i) = 0, \text{ for all } i, j.$$

Example 11.16

Consider the Markov chain shown in Figure 11.15. Assume that $0 < p < \frac{1}{2}$. Does this chain have a limiting distribution?

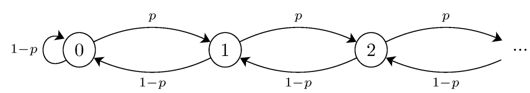

Figure 11.15 - A state transition diagram.

Solution

This chain is irreducible since all states communicate with each other. It is also aperiodic since it includes a self-transition, $P_{00} > 0$. Let's write the equations for a stationary distribution. For state 0, we can write

$$\pi_0 = (1-p)\pi_0 + (1-p)\pi_1,$$

which results in

$$\pi_1 = \frac{p}{1-p}\pi_0.$$

For state 1, we can write

$$\begin{aligned}\pi_1 &= p\pi_0 + (1-p)\pi_2 \\ &= (1-p)\pi_1 + (1-p)\pi_2,\end{aligned}$$

which results in

$$\pi_2 = \frac{p}{1-p}\pi_1.$$

Similarly, for any $j \in \{1, 2, \cdots\}$, we obtain

$$\pi_j = \alpha \pi_{j-1},$$

where $\alpha = \frac{p}{1-p}$. Note that since $0 < p < \frac{1}{2}$, we conclude that $0 < \alpha < 1$. We obtain

$$\pi_j = \alpha^j \pi_0, \quad \text{for } j = 1, 2, \cdots.$$

Finally, we must have

$$\begin{aligned} 1 &= \sum_{j=0}^{\infty} \pi_j \\ &= \sum_{j=0}^{\infty} \alpha^j \pi_0, \qquad &&(\text{where } 0 < \alpha < 1) \\ &= \frac{1}{1-\alpha}\pi_0 \qquad &&(\text{geometric series}). \end{aligned}$$

Thus, $\pi_0 = 1 - \alpha$. Therefore, the stationary distribution is given by

$$\pi_j = (1-\alpha)\alpha^j, \quad \text{for } j = 0, 1, 2, \cdots.$$

Since this chain is irreducible and aperiodic and we have found a stationary distribution, we conclude that all states are positive recurrent and $\pi = [\pi_0, \pi_1, \cdots]$ is the limiting distribution.
