5.2.3 Conditioning and Independence

Here, we will discuss conditioning for continuous random variables. In particular, we will discuss the conditional PDF, conditional CDF, and conditional expectation. We have discussed conditional probability for discrete random variables before. The ideas behind conditional probability for continuous random variables are very similar to the discrete case. The difference lies in the fact that we need to work with probability density in the case of continuous random variables. Nevertheless, we would like to emphasize again that there is only one main formula regarding conditional probability which is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \text{ when } P(B) > 0.$$

Any other formula regarding conditional probability can be derived from the above formula. In fact, for some problems we only need to apply the above formula. You have already used this in Example 5.17. As another example, if you have two random variables X and Y, you can write

$$P(X \in C|Y \in D) = \frac{P(X \in C, Y \in D)}{P(Y \in D)}, \text{ where } C, D \subset \mathbb{R}.$$

However, sometimes we need to use the concepts of conditional PDFs and CDFs. The formulas for conditional PDFs and CDFs of continuous random variables are very similar to those of discrete random variables. Since there are no new fundamental ideas in this section, we usually provide the main formulas and guidelines, and then work on examples. Specifically, we do not spend much time deriving formulas. Nevertheless, to give you the basic idea of how to derive these formulas, we start by deriving a formula for the conditional CDF and PDF of a random variable X given that $X \in I = [a, b]$. Consider a continuous random variable X. Suppose that we know that the event $X \in I = [a, b]$ has occurred. Call this event A. The conditional CDF of X given A, denoted by $F_{X|A}(x)$ or $F_{X|a \leq X \leq b}(x)$, is

$$\begin{aligned} F_{X|A}(x) &= P(X \leq x|A) \\ &= P(X \leq x|a \leq X \leq b) \\ &= \frac{P(X \leq x, a \leq X \leq b)}{P(A)}. \end{aligned}$$

Now if $x < a$, then $F_{X|A}(x) = 0$. On the other hand, if $a \leq x \leq b$, we have

$$\begin{aligned} F_{X|A}(x) &= \frac{P(X \leq x, a \leq X \leq b)}{P(A)} \\ &= \frac{P(a \leq X \leq x)}{P(A)} \\ &= \frac{F_X(x) - F_X(a)}{F_X(b) - F_X(a)}. \end{aligned}$$

Finally, if $x > b$, then $F_{X|A}(x) = 1$. Thus, we obtain

$$F_{X|A}(x) = \begin{cases} 1 & x > b \\ \\ \frac{F_X(x) - F_X(a)}{F_X(b) - F_X(a)} & a \leq x < b \\ \\ 0 & \text{otherwise} \end{cases}$$

Note that since X is a continuous random variable, we do not need to be careful about end points, i.e., changing $x > b$ to $x \geq b$ does not make a difference in the above formula. To obtain the conditional PDF of X, denoted by $f_{X|A}(x)$, we can differentiate $F_{X|A}(x)$. We obtain

$$f_{X|A}(x) = \begin{cases} \frac{f_X(x)}{P(A)} & a \leq x < b \\ \\ 0 & \text{otherwise} \end{cases}$$

It is insightful if we derive the above formula for $f_{X|A}(x)$ directly from the definition of the PDF for continuous random variables. Recall that the PDF of X can be defined as

$$f_X(x) = \lim_{\Delta \rightarrow 0^+} \frac{P(x < X \leq x + \Delta)}{\Delta}.$$

Now, the conditional PDF of X given A, denoted by $f_{X|A}(x)$, is

$$\begin{aligned} f_{X|A}(x) &= \lim_{\Delta \rightarrow 0^+} \frac{P(x < X \leq x + \Delta | A)}{\Delta} \\ &= \lim_{\Delta \rightarrow 0^+} \frac{P(x < X \leq x + \Delta, A)}{\Delta P(A)} \\ &= \lim_{\Delta \rightarrow 0^+} \frac{P(x < X \leq x + \Delta, a \leq X \leq b)}{\Delta P(A)}. \end{aligned}$$

Now consider two cases. If $a \leq x < b$, then

$$
\begin{aligned}
f_{X|A}(x) &= \lim_{\Delta \to 0^+} \frac{P(x < X \leq x + \Delta, a \leq X \leq b)}{\Delta P(A)} \\
&= \frac{1}{P(A)} \lim_{\Delta \to 0^+} \frac{P(x < X \leq x + \Delta)}{\Delta} \\
&= \frac{f_X(x)}{P(A)}.
\end{aligned}
$$

On the other hand, if $x < a$ or $x \geq b$, then

$$
\begin{aligned}
f_{X|A}(x) &= \lim_{\Delta \to 0^+} \frac{P(x < X \leq x + \Delta, a \leq X \leq b)}{\Delta P(A)} \\
&= 0.
\end{aligned}
$$

If X is a continuous random variable, and A is the event that $a < X < b$ (where possibly $b = \infty$ or $a = -\infty$), then

$$
F_{X|A}(x) = \begin{cases} 1 & x > b \\ \frac{F_X(x) - F_X(a)}{F_X(b) - F_X(a)} & a \leq x < b \\ 0 & x < a \end{cases}
$$

$$
f_{X|A}(x) = \begin{cases} \frac{f_X(x)}{P(A)} & a \leq x < b \\ 0 & \text{otherwise} \end{cases}
$$

The conditional expectation and variance are defined by replacing the PDF by conditional PDF in the definitions of expectation and variance. In general, for a random variable X and an event A, we have the following:

$$E[X|A] = \int_{-\infty}^{\infty} x f_{X|A}(x)dx,$$
$$E[g(X)|A] = \int_{-\infty}^{\infty} g(x) f_{X|A}(x)dx,$$
$$\text{Var}(X|A) = E[X^2|A] - (E[X|A])^2$$

Example 5.20

Let $X \sim Exponential(1)$.

a. Find the conditional PDF and CDF of X given $X > 1$.
b. Find $E[X|X > 1]$.
c. Find Var$(X|X > 1)$.

Solution

a. Let A be the event that $X > 1$. Then

$$P(A) = \int_{1}^{\infty} e^{-x}dx = \frac{1}{e}.$$

Thus,

$$f_{X|X>1}(x) = \begin{cases} e^{-x+1} & x > 1 \\ 0 & \text{otherwise} \end{cases}$$

For $x > 1$, we have

$$F_{X|A}(A) = \frac{F_X(x) - F_X(1)}{P(A)} = 1 - e^{-x+1}.$$

Thus,

$$F_{X|A}(x) = \begin{cases} 1 - e^{-x+1} & x > 1 \\ 0 & \text{otherwise} \end{cases}$$