9.2.0 End of Chapter Problems

Problem 1

Let X be a continuous random variable with the following PDF

$$f_X(x) = \begin{cases} 6x(1-x) & \text{if } 0 \leq x \leq 1 \\ \\ 0 & \text{otherwise} \end{cases}$$

Suppose that we know

$$Y \mid X = x \quad \sim \quad Geometric(x).$$

Find the posterior density of X given $Y = 2$, $f_{X|Y}(x|2)$.

Problem 2

Let X be a continuous random variable with the following PDF

$$f_X(x) = \begin{cases} 3x^2 & \text{if } 0 \leq x \leq 1 \\ \\ 0 & \text{otherwise} \end{cases}$$

Also, suppose that

$$Y \mid X = x \quad \sim \quad Geometric(x).$$

Find the MAP estimate of X given $Y = 5$.

Problem 3

Let X and Y be two jointly continuous random variables with joint PDF

$$f_{XY}(x,y) = \begin{cases} x + \frac{3}{2}y^2 & 0 \leq x, y \leq 1 \\ \\ 0 & \text{otherwise.} \end{cases}$$

Find the MAP and the ML estimates of X given $Y = y$.

Problem 4

Let X be a continuous random variable with the following PDF

$$f_X(x) = \begin{cases} 2x^2 + \frac{1}{3} & \text{if } 0 \leq x \leq 1 \\ \\ 0 & \text{otherwise} \end{cases}$$

We also know that

$$f_{Y|X}(y|x) = \begin{cases} xy - \frac{x}{2} + 1 & \text{if } 0 \leq y \leq 1 \\ \\ 0 & \text{otherwise} \end{cases}$$

Find the MMSE estimate of X, given $Y = y$ is observed.

Problem 5

Let $X \sim N(0,1)$ and

$$Y = 2X + W,$$

where $W \sim N(0,1)$ is independent of X.

a. Find the MMSE estimator of X given Y, ($\hat{X}_M$).
b. Find the MSE of this estimator, using $MSE = E[(X - \hat{X_M})^2]$.
c. Check that $E[X^2] = E[\hat{X}_M^2] + E[\tilde{X}^2]$.

Problem 6

Suppose $X \sim Uniform(0,1)$, and given $X = x$, $Y \sim Exponential(\lambda = \frac{1}{2x})$.

a. Find the linear MMSE estimate of X given Y.
b. Find the MSE of this estimator.
c. Check that $E[\tilde{X}Y] = 0$.

Problem 7
Suppose that the signal $X \sim N(0, \sigma_X^2)$ is transmitted over a communication channel. Assume that the received signal is given by

$$Y = X + W,$$

where $W \sim N(0, \sigma_W^2)$ is independent of X.

a. Find the MMSE estimator of X given Y, ($\hat{X}_M$).
b. Find the MSE of this estimator.

Problem 8
Let X be an unobserved random variable with $EX = 0$, $\mathrm{Var}(X) = 5$. Assume that we have observed Y_1 and Y_2 given by

$$Y_1 = 2X + W_1,$$
$$Y_2 = X + W_2,$$

where $EW_1 = EW_2 = 0$, $\mathrm{Var}(W_1) = 2$, and $\mathrm{Var}(W_2) = 5$. Assume that W_1, W_2 , and X are independent random variables. Find the linear MMSE estimator of X, given Y_1 and Y_2.

Problem 9
Consider again Problem 8, in which X is an unobserved random variable with $EX = 0$, $\mathrm{Var}(X) = 5$. Assume that we have observed Y_1 and Y_2 given by

$$Y_1 = 2X + W_1,$$
$$Y_2 = X + W_2,$$

where $EW_1 = EW_2 = 0$, $\mathrm{Var}(W_1) = 2$, and $\mathrm{Var}(W_2) = 5$. Assume that W_1, W_2 , and X are independent random variables. Find the linear MMSE estimator of X, given Y_1 and Y_2, using the vector formula

$$\hat{\mathbf{X}}_L = \mathbf{C_{XY}}\mathbf{C_Y}^{-1}(\mathbf{Y} - E[\mathbf{Y}]) + E[\mathbf{X}].$$

Problem 10

Let X be an unobserved random variable with $EX = 0$, $\text{Var}(X) = 5$. Assume that we have observed Y_1, Y_2, and Y_3 given by

$$Y_1 = 2X + W_1,$$
$$Y_2 = X + W_2,$$
$$Y_3 = X + 2W_3,$$

where $EW_1 = EW_2 = EW_3 = 0$, $\text{Var}(W_1) = 2$, $\text{Var}(W_2) = 5$, and $\text{Var}(W_3) = 3$. Assume that W_1, W_2, W_3, and X are independent random variables. Find the linear MMSE estimator of X, given Y_1, Y_2, and Y_3.

Problem 11

Consider two random variables X and Y with the joint PMF given by the table below.

	$Y = 0$	$Y = 1$
$X = 0$	$\frac{1}{7}$	$\frac{3}{7}$
$X = 1$	$\frac{3}{7}$	0

a. Find the linear MMSE estimator of X given Y, ($\hat{X}_L$).
b. Find the MMSE estimator of X given Y, ($\hat{X}_M$).
c. Find the MSE of $\hat{X}_M$.

Problem 12

Consider two random variables X and Y with the joint PMF given by the table below.

	$Y = 0$	$Y = 1$	$Y = 2$
$X = 0$	$\frac{1}{6}$	$\frac{1}{3}$	0
$X = 1$	$\frac{1}{3}$	0	$\frac{1}{6}$

a. Find the linear MMSE estimator of X given Y, ($\hat{X}_L$).
b. Find the MSE of $\hat{X}_L$.
c. Find the MMSE estimator of X given Y, ($\hat{X}_M$).
d. Find the MSE of $\hat{X}_M$.

Problem 13
Suppose that the random variable X is transmitted over a communication channel. Assume that the received signal is given by

$$Y = 2X + W,$$

where $W \sim N(0, \sigma^2)$ is independent of X. Suppose that $X = 1$ with probability p, and $X = -1$ with probability $1 - p$. The goal is to decide between $X = -1$ and $X = 1$ by observing the random variable Y. Find the MAP test for this problem.

Problem 14
Find the average error probability in Problem 13.

Problem 15
A monitoring system is in charge of detecting malfunctioning machinery in a facility. There are two hypotheses to choose from:

H_0: There is not a malfunction,

H_1: There is a malfunction.

The system notifies a maintenance team if it accepts H_1. Suppose that, after processing the data, we obtain $P(H_1|y) = 0.10$. Also, assume that the cost of missing a malfunction is 30 times the cost of a false alarm. Should the system alert a maintenance team (accept H_1)?

Problem 16

Let X and Y be jointly normal and $X \sim N(2, 1)$, $Y \sim N(1, 5)$, and $\rho(X, Y) = \frac{1}{4}$. Find a 90% credible interval for X, given $Y = 1$ is observed.

Problem 17

When the choice of a prior distribution is subjective, it is often advantageous to choose a prior distribution that will result in a posterior distribution of the same distributional family. When the prior and posterior distributions share the same distributional family, they are called *conjugate distributions*, and the prior is called a *conjugate prior*. Conjugate priors are used out of ease because they always result in a closed form posterior distribution. One example of this is to use a gamma prior for Poisson distributed data. Assume our data Y given X is distributed $Y \mid X = x \sim Poisson(\lambda = x)$ and we chose the prior to be $X \sim Gamma(\alpha, \beta)$. Then the PMF for our data is

$$P_{Y|X}(y|x) = \frac{e^{-x}x^y}{y!}, \quad \text{for } x > 0, y \in \{0, 1, 2, \ldots\},$$

and the PDF of the prior is given by

$$f_X(x) = \frac{\beta^\alpha x^{\alpha-1}e^{-\beta x}}{\Gamma(\alpha)}, \quad \text{for } x > 0,\ \alpha, \beta > 0.$$

a. Show that the posterior distribution is $Gamma(\alpha + y, \beta + 1)$.
(*Hint: Remove all the terms not containing x by putting them into some normalizing constant, c, and noting that* $f_{X|Y}(x|y) \propto P_{Y|X}(y|x)f_X(x)$.)
b. Write out the PDF for the posterior distribution, $f_{X|Y}(x|y)$.
c. Find mean and variance of the posterior distribution, $E[X|Y]$ and $\mathrm{Var}(X|Y)$.

Problem 18

Assume our data Y given X is distributed $Y \mid X = x \sim Binomial(n, p = x)$ and we chose the prior to be $X \sim Beta(\alpha, \beta)$. Then the PMF for our data is

$$P_{Y|X}(y|x) = \binom{n}{y} x^y (1-x)^{n-y}, \quad \text{for } x \in [0,1], y \in \{0, 1, \dots, n\},$$

and the PDF of the prior is given by

$$f_X(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \quad \text{for } 0 \le x \le 1, \alpha > 0, \beta > 0.$$

Note that, $EX = \frac{\alpha}{\alpha+\beta}$ and $\mathrm{Var}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$.

a. Show that the posterior distribution is $Beta(\alpha + y, \beta + n - y)$.
b. Write out the PDF for the posterior distribution, $f_{X|Y}(x|y)$.
c. Find mean and variance of the posterior distribution, $E[X|Y]$ and $\mathrm{Var}(X|Y)$.

Problem 19

Assume our data Y given X is distributed $Y \mid X = x \sim Geometric(p = x)$ and we chose the prior to be $X \sim Beta(\alpha, \beta)$. Refer to Problem 18 for the PDF and moments of the $Beta$ distribution.

a. Show that the posterior distribution is $Beta(\alpha + 1, \beta + y - 1)$.
b. Write out the PDF for the posterior distribution, $f_{X|Y}(x|y)$.
c. Find mean and variance of the posterior distribution, $E[X|Y]$ and $\mathrm{Var}(X|Y)$.

Problem 20

Assume our data $\mathbf{Y} = (y_1, y_2, \dots, y_n)^T$ given X is independently identically distributed, $\mathbf{Y} \mid X = x \overset{i.i.d.}{\sim} Exponential(\lambda = x)$, and we chose the prior to be $X \sim Gamma(\alpha, \beta)$.

a. Find the likelihood of the function, $L(\mathbf{Y}; X) = f_{Y_1, Y_2, \dots, Y_n|X}(y_1, y_2, \dots, y_n|x)$.
b. Using the likelihood function of the data, show that the posterior distribution is $Gamma(\alpha + n, \beta + \sum_{i=1}^n y_i)$.
c. Write out the PDF for the posterior distribution, $f_{X|\mathbf{Y}}(x|\mathbf{y})$.
d. Find mean and variance of the posterior distribution, $E[X|\mathbf{Y}]$ and $\mathrm{Var}(X|\mathbf{Y})$.
