Chapter 14

Recursive Methods

14.1 Using Recursion

Some problems in combinatorics and probability can be solved using recursive methods. Here is the basic idea: Suppose we are interested in computing a sequence a_n, for $n = 0, 1, 2, \dots$. The value a_n could be the number of elements in a set or the probability of a certain event. We may be able to find a **recursive** relation that relates a_n to other a_i's where $i < n$. For example, suppose that for a certain problem, we can show that

$$a_n = a_{n-1} + 2a_{n-3}, \text{ for } n = 3, 4, 5, \dots$$

Now, if we know a_0, a_1, and a_2, we can use the above recursive equation to find a_n for all n. Let's look at an example.

Example 1. Find the total number of sequences of length n (using H and T) such that no two Hs are next to each other. For example, for $n = 2$, we have 3 possible sequences: HT, TH, TT.

Let a_n be the number of such sequences. Let's call these sequences "NO-HH sequences." We have $a_1 = 2, a_2 = 3$. But how do we find, say a_{1000}? To do this, we will show that

$$a_n = a_{n-1} + a_{n-2} \text{ for } n = 3, 4, 5, \dots \tag{14.1}$$

To show this, consider a NO-HH sequence of length n. This sequence either starts with a T or an H.

- If it starts with a T, then the rest of the sequence is simply a NO-HH sequence of length $n-1$. Conversely, by adding a T in front of any NO-HH sequence of length $n-1$, we can obtain a NO-HH sequence of length n.

- If it starts with an H, then the second element in the sequence must be a T. In this case, the rest of the sequence is simply a NO-HH sequence of length $n-2$. Conversely, by adding HT in front of any NO-HH sequence of length $n-2$, we can obtain a NO-HH sequence of length n.

Thus, we conclude that $a_n = a_{n-1} + a_{n-2}$. Since we already know that $a_1 = 2$ and $a_2 = 3$, we can use this recursive equation to find

$$a_3 = 5, a_4 = 8, a_5 = 13, ...$$

Using a computer program we can compute a_n for the larger values of n. However, there is also a straight-forward method to solving Equation 14.1 in order to obtain a simple formula for a_n that does not involve previous values in the sequence. We will discuss how to solve these equations in general shortly. Here, we solve Equation 14.1 to find a formula for the number of No-HH sequences of length n. The trick, as we will see, is to let $a_k = x^k$ and find non-zero values of x that satisfy the recursive equation. In particular, letting $a_k = x^k$ in Equation 14.1, we obtain

$$x^2 = x + 1,$$

which gives

$$x_1 = \frac{1+\sqrt{5}}{2}, x_2 = \frac{1-\sqrt{5}}{2}.$$

Then the general solution can be written in the form of

$$a_n = \alpha_1 x_1^n + \alpha_2 x_2^n,$$

where α_1 and α_2 are constants to be determined from the known values of a_n. For example, here we know $a_1 = 2, a_2 = 3$. Using these two values we can find α_1 and α_2. It is a little bit easier to use a_0 and a_1. That is, since $a_2 = a_1 + a_0$, we obtain $a_0 = 1$. Thus we have

$$\begin{aligned} a_0 = 1 = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^0 + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^0 \\ a_1 = 2 = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^1 + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^1. \end{aligned}$$

Thus, we obtain

$$\begin{cases} \alpha_1 + \alpha_2 = 1 \\ \alpha_1(\frac{1+\sqrt{5}}{2}) + \alpha_2(\frac{1-\sqrt{5}}{2}) = 2 \end{cases}$$

By solving these equations, we obtain $\alpha_1 = \frac{5+3\sqrt{5}}{10}$, and $\alpha_2 = \frac{5-3\sqrt{5}}{10}$. Finally,

$$a_n = \frac{5+3\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{5-3\sqrt{5}}{10}\left(\frac{1-\sqrt{5}}{2}\right)^n \tag{14.2}$$

This might seem somewhat strange as it does not look like an integer. However, you can evaluate the above expression for small values of n to see that, in fact, square roots always cancel out and the resulting values of a_n are always integers. If the above calculation seems confusing, don't worry. We will now discuss in general how to solve recursive equations such as the one given in Equation 14.1.

14.1.1 Solving Linear Homogeneous Recurrence Equations with Constant Coefficients

Suppose that we have the following recursive equation:

$$a_n + c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + ... + c_d a_{n-d} = 0 \tag{14.3}$$

where the c_i's are known constants. Also suppose that we already know the values of a_i for d different values of i. For example, we might know the values of $a_1, a_2, ..., a_d$. To solve this recursive equation, we first solve the following *characteristic* equation

$$x^d + c_1 x^{d-1} + c_2 x^{d-2} + c_3 x^{n-3} + ... + c_d = 0 \tag{14.4}$$

This equation is obtained by replacing a_i by x^i in the recursive Equation 14.3. Let $x_1, x_2, ..., x_d$ be d distinct roots of the characteristic polynomial (we will discuss the case of repeated roots shortly). Then the general format for solutions to the recursive Equation 14.3 is given by

$$a_n = \alpha_1 x_1^n + \alpha_2 x_2^n + \alpha_3 x_3^n + ... + \alpha_d x_d^n \tag{14.5}$$

The values of $\alpha_1, \alpha_2, ..., \alpha_d$ can be obtained from the known values of a_i. If a root is repeated r times, we need to include r terms for that root, each scaled by a power of n. For example, if x_1 is a repeated root of multiplicity r, then we write

$$a_n = \alpha_{11} x_1^n + \alpha_{12} n x_1^n + \alpha_{13} n^2 x_1^n + ... + \alpha_{1r} n^{r-1} x_1^n + \alpha_2 x_2^n + \alpha_3 x_3^n + ... + \alpha_d x_d^n \tag{14.6}$$

To better understand all this, let's look at some examples.

Example 2. Solve the following recurrence equations:

(a) $a_n = 3a_{n-1} - 2a_{n-2}$, where $a_0 = 2$, $a_1 = 3$;

(b) $a_n = 4a_{n-1} - 5a_{n-2} + 2a_{n-3}$, where $a_0 = 0$, $a_1 = 2$, and $a_2 = 5$.

Solution

(a) The characteristic polynomial for $a_n = 3a_{n-1} - 2a_{n-2}$ is $x^2 - 3x + 2$. It has roots $x_1 = 2$ and $x_2 = 1$. Thus, the general solution is of the form

$$a_n = \alpha 2^n + \beta.$$

Since $a_0 = 2$, $a_1 = 3$, we obtain $\alpha = 1$, $\beta = 1$. Therefore, a_n is given by

$$a_n = 2^n + 1, \text{ for } n = 0, 1, 2, ...$$

(b) The characteristic polynomial for $a_n = 4a_{n-1} - 5a_{n-2} + 2a_{n-3}$ is $x^3 - 4x^2 + 5x - 2$. We can factor this polynomial as

$$x^3 - 4x^2 + 5x - 2 = (x-1)^2(x-2).$$

Thus we have two roots, $x_1 = 1$ with multiplicity 2, and $x_2 = 2$. The general formula for x_n can be written as

$$a_n = \alpha_1 + \alpha_2 n + \alpha_3 2^n.$$

Using $a_0 = 0$, $a_1 = 2$, and $a_2 = 5$, we obtain

$$a_n = 2^n + n - 1.$$

Note that recurrences could be much more complicated than the form given in Equation 14.3, and sometimes we may not be able to find simple closed form solutions for them. Nevertheless, we can usually use computer programs to compute them for at least moderate values of n. In general, if the recursion is not in the form of Equation 14.3, a good start would be to compute a_n for small n and see if we can identify a pattern and guess a general formula for a_n. If we are lucky, and we can guess a general formula, then we usually can prove it mathematically using induction.

14.1.2 Using Recursion with Conditioning

As we have seen so far, conditioning is a powerful method for solving probability problems. In some problems, after conditioning we get a recursive relation that can help us solve the problem. As an easy example, let's start with a problem closely related to Example 1.

Example 3. I toss a fair coin n times and record the sequence of heads and tails. What is the probability that I do not observe two consecutive heads in the sequence?

Solution: Let p_n be the probability of not observing two consecutive heads in n coin tosses. One way to solve this problem is to use our answer to Example 1. In that example, we found the total number of sequences of length n with no HH to be

$$a_n = \frac{5+3\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{5-3\sqrt{5}}{10}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Now, since the total number of sequences of length n is 2^n, and all sequences are equally likely, we obtain

$$\begin{aligned} p_n &= \frac{a_n}{2^n} = \\ &= \frac{5+3\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{4}\right)^n + \frac{5-3\sqrt{5}}{10}\left(\frac{1-\sqrt{5}}{4}\right)^n. \end{aligned} \tag{14.7}$$

Here we will solve this problem directly using conditioning. Let A_n be the event that we observe no consecutive heads in n coin tosses, i.e., $p_n = P(A_n)$. The idea is to condition on the result of the first coin toss. There are two possibilities. Using the law of total probability and by conditioning on the result of the first coin toss, we can write

$$\begin{aligned} p_n &= P(A_n) = P(A_n|H)P(H) + P(A_n|T)P(T) \\ &= \frac{1}{2}P(A_n|H) + \frac{1}{2}P(A_n|T) \end{aligned} \tag{14.8}$$

Now, to find $P(A_n|T)$ note that if the first coin toss is a T, then in order to not observe an HH in the entire sequence, we must not observe an HH in the remaining $n-1$ coin tosses. Thus, we have

$$P(A_n|T) = P(A_{n-1}) = p_{n-1}.$$

Similarly, if the first coin toss results in an H, the second one must be a T and we must not observe an HH in the remaining $n-2$ coin tosses, thus we have

$$P(A_n|H) = \frac{1}{2} \cdot P(A_{n-2}) = \frac{1}{2}p_{n-2}.$$

Plugging back into Equation 14.8 we obtain

$$p_n = \frac{1}{2}p_{n-1} + \frac{1}{4}p_{n-2}.$$

Note that we also know that $p_1 = 1$ and $p_2 = \frac{3}{4}$. Using the recursion, we also obtain $p_0 = 1$. Thus we can solve this recursion. We obtain the following characteristic equation

$$x^2 - \frac{1}{2}x - \frac{1}{4} = 0.$$

The characteristic equation has roots $x_1 = \frac{1+\sqrt{5}}{4}$ and $x_2 = \frac{1-\sqrt{5}}{4}$, so the general solution is given by

$$p_n = \alpha\left(\frac{1+\sqrt{5}}{4}\right)^n + \beta\left(\frac{1-\sqrt{5}}{4}\right)^n.$$

Using $p_0 = 1$ and $p_1 = 1$, we obtain

$$p_n = \frac{5+3\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{4}\right)^n + \frac{5-3\sqrt{5}}{10}\left(\frac{1-\sqrt{5}}{4}\right)^n.$$

Which, as we expect, is the same as Equation 14.7.

Gambler's Ruin Problem:

Here we discuss a famous problem called the Gambler's Ruin. It refers to a simple gambling game in which two gamblers play repeatedly until one of them runs out of money. This is also an example of a *random walk*.

Example 4. Two gamblers, call them Gambler A and Gambler B, play repeatedly. In each round, A wins 1 dollar with probability p or loses 1 dollar with probability $q = 1 - p$ (thus, equivalently, in each round B wins 1 dollar with probability $q = 1 - p$ and loses 1 dollar with probability p). We assume different rounds are independent. Suppose that initially A has i dollars and B has $N - i$ dollars. The game ends when one of the gamblers runs out of money (in which case the other gambler will have N dollars). Find p_i, the probability that A wins the game given that he has initially i dollars.

Solution: At first it might not be clear that this problem can be solved using recursive methods. The main idea is very simple. Condition on the result of the first round. After the first round, A will have either $i - 1$ dollars (if he loses) or will have $i + 1$ dollars (if he wins). This way we can relate p_i to p_{i-1} and p_{i+1}. In particular, applying the law of total probability, we obtain

$$\begin{aligned}
p_i = {} & P(\text{A wins the game}|\text{A wins the first round})P(\text{A wins the first round})+ \\
& P(\text{A wins the game}|\text{A loses the first round})P(\text{A loses the first round}) \\
& = p_{i+1}p + p_{i-1}(1-p).
\end{aligned}$$

Thus we obtain the recursive equation

$$p_i = p_{i+1}p + p_{i-1}(1-p).$$

We can rewrite the equation as

$$pp_{i+1} = p_i - (1-p)p_{i-1}.$$

To solve this recursion we need to know the value of p_i for two different values of i. We already know that $p_0 = 0$. If A starts with 0 dollars he is automatically the loser. Similarly, if B starts with 0 dollars (i.e., A starts with N dollars), then A is automatically the winner. Thus, we have $p_N = 1$. The characteristic equation is

$$px^2 - x + (1-p) = 0.$$

Solving this equation, we obtain two roots, $x_1 = 1$ and $x_2 = \frac{1-p}{p} = \frac{q}{p}$. The roots are different if $p \neq q$. Thus, we need to consider two cases: if $p \neq q$ (i.e., when $p \neq \frac{1}{2}$) we can write the general solution as

$$p_i = \alpha + \beta \left(\frac{q}{p}\right)^i.$$

Using $p_0 = 0$ and $p_N = 1$, we obtain

$$p_i = \frac{1-(\frac{q}{p})^i}{1-(\frac{q}{p})^N}.$$

If $p = q = \frac{1}{2}$, the characteristic equation has a repeated root of $x_1 = 1$, so the general solution can be written as

$$p_i = \alpha' + \beta' i.$$

Using $p_0 = 0$ and $p_N = 1$, we obtain

$$p_i = \frac{i}{N}.$$

To summarize, for $i = 0, 1, 2, ..., N$, we have

$$p_i = \begin{cases} \frac{1-(\frac{q}{p})^i}{1-(\frac{q}{p})^N} \text{ if } p \neq \frac{1}{2} \\ \frac{i}{N} \text{ if } p = \frac{1}{2} \end{cases}$$

Discussion: Using the above answer, we can draw some conclusions. The setup of this problem in some sense can be a model for someone who goes to the casino and gambles repeatedly. We can look at this problem from two points of view. First, let us be somewhat optimistic and assume that the casino games are fair. In that case, you can win or lose with probability $p = q = \frac{1}{2}$ each time. But the casino has usually much more money than an individual gambler, that is $i << N$. This means that your chance of winning, $\frac{i}{N}$ is very small. Thus, if you gamble repeatedly you are most likely to lose all your money. What if you are very rich? Assume that you have the same amount of money as the casino. Even in that case, you are in no luck. The reason is that the games are usually unfair (casino has some advantage), so $p < \frac{1}{2}$. Now, if you and the casino both have a large sum of money $\frac{N}{2}$ then your chance of winning is

$$\begin{aligned}
p_i &= \frac{1-(\frac{q}{p})^i}{1-(\frac{q}{p})^N} \\
&= \frac{1-(\frac{q}{p})^{\frac{N}{2}}}{1-(\frac{q}{p})^N} && \text{since } i = \frac{N}{2} \\
&\approx \frac{-(\frac{q}{p})^{\frac{N}{2}}}{-(\frac{q}{p})^N} && \text{since } N \text{ is large and } q > p \\
&= \frac{1}{(\frac{q}{p})^{\frac{N}{2}}} \rightarrow 0 \text{ as } N \text{ becomes large.}
\end{aligned}$$

Thus, even if you have the same amount of money as the casino, you will most likely lose all your money if you gamble repeatedly.

14.1.3 Solved Problems

1. Solve the following recursive equations. That is, find a closed form formula for a_n.

   (a) $a_n = a_{n-1} + n$, with $a_0 = 0$.

   (b) $a_n = na_{n-1}$, with $a_0 = 1$.

   (c) $a_n = 5a_{n-1} - 6a_{n-2}$, with $a_0 = 3, a_1 = 8$.

   (d) $a_n = 3a_{n-1} - 4a_{n-3}$, with $a_0 = 3, a_1 = 5$, $a_2 = 17$.

   (e) $a_n = 2a_{n-1} - 2a_{n-2}$, with $a_0 = a_1 = 2$.

   Solution:

   (a) Note that this equation is NOT of the form of Equation 14.3, so we cannot use our general methodology to solve this problem. In these situations, it is always a good idea to compute the first few terms in the sequence and try to guess a general formula, and then prove it (possibly using mathematical induction). For this problem we quickly observe that

   $$a_n = 1 + 2 + 3 + \cdots + n.$$

   This is the sum of the numbers from 1 to n, and it is given by $a_n = \frac{n(n+1)}{2}$. To obtain this formula, you can write

$$\begin{aligned} a_n + a_n &= \Big(1 + 2 + 3 + \cdots + n\Big) \\ &\quad + \Big(n + (n-1) + \cdots + 2 + 1\Big) \\ &= (1+n) + (2 + (n-1)) + \cdots + (n+1) \\ &= (n+1)n. \end{aligned}$$

(b) Again, this is not in the form of Equation 14.3. By writing the first few a_i's we observe that

$$a_n = 1 \cdot 2 \cdot 3 \cdots n.$$

Thus, we conclude that

$$a_n = n!$$

(c) This recurrence equation is in the form of Equation 14.3, so we can use our general method. In particular, the characteristic polynomial is

$$x^2 - 5x + 6.$$

This polynomial has two roots, $x_1 = 2$ and $x_2 = 3$, so the solution will be in the form

$$a_n = \alpha 2^n + \beta 3^n.$$

Using $a_0 = 3$ and $a_1 = 8$, we obtain

$$a_n = 2^n + 2 \cdot 3^n.$$

(d) This recurrence equation is in the form of Equation 14.3, so we can use our general method. In particular, the characteristic polynomial is

$$x^3 - 3x^2 + 4 = (x-2)^2(x+1).$$

Thus, the solution will be in the form

$$a_n = \alpha_1 2^n + \alpha_2 n 2^n + \alpha_3 (-1)^n.$$

Using $a_0 = 3, a_1 = 5$, $a_2 = 17$, we obtain

$$a_n = (2+n)2^n + (-1)^n.$$

(e) This recurrence equation is in the form of Equation 14.3, so we can use our general method. In particular, the characteristic polynomial is

$$x^2 - 2x + 2,$$

which has two complex roots, $x_1 = 1 + i$, $x_2 = 1 - i$ (where $i = \sqrt{-1}$). Thus, the solution will be in the form

$$a_n = \alpha(1 + i)^n + \beta(1 - i)^n.$$

Using $a_0 = a_1 = 2$, we obtain

$$a_n = (1 + i)^n + (1 - i)^n.$$

2. Let a_n be the total number of sequences of length n (using H and T) that do not include three consecutive Hs. For example, we know that $a_1 = 2, a_2 = 4$ and $a_3 = 7$. Find a recurrence equation for a_n.

   Solution: We can solve this problem with a similar approach to the method we used in Example 1. In particular, we will show that

   $$a_n = a_{n-1} + a_{n-2} + a_{n-3}.$$

   Let's call these sequences NO-HHH sequences. Consider a NO-HHH sequence of length n. This sequence either starts with an T or an H.

   - If it starts with a T, then the rest of the sequence is simply a NO-HHH sequence of length $n - 1$. Conversely, by adding a T in front of any NO-HHH sequence of length $n - 1$, we can obtain a NO-HHH sequence of length n.
   - If it starts with an H, then the second element in the sequence is either an H or a T:
     – If the second element is a T, then the rest of the sequence is simply a NO-HHH sequence of length $n - 2$. Conversely, by adding HT in front of any NO-HHH sequence of length $n - 2$, we can obtain a NO-HHH sequence of length n.
     – If the second element is also an H, then the third element must be a T, and thus the rest of the sequence is a NO-HHH sequence of length $n - 3$. Conversely, by adding HHT in front of any NO-HHH sequence of length $n - 3$, we will obtain a NO-HHH sequence of length n.

   Thus, we conclude that $a_n = a_{n-1} + a_{n-2} + a_{n-3}$.

3. Let k be a fixed given integer larger than 1. Let $f(n, k)$ be the total number of sequences of length n (using H and T) that do not include k consecutive Hs. Find a recurrence equation for $f(n, k)$.

Solution: Similar to Example 1 and Problem 2, we can argue that

$$f(n,k) = f(n-1,k) + f(n-2,k) + ... + f(n-k,k), \text{ for } n > k.$$

And $f(n,k) = 2^n$ for $n = 1, 2, 3, ..., k-1$, and $f(k,k) = 2^k - 1$. Using the above recursion we can define $f(0,k) = 1$ so that the above recursion also holds for $n = k$.

4. Let $f(n,k,l)$ be the number of binary sequences of length n with exactly k ones and at least l consecutive zeros. Find a recursive equation along with initial values to compute $f(n,k,l)$. Assume that $n \geq k + l$.

   Solution: Let also $g(n,k,l)$ be the number of binary sequences of length n with exactly k ones and NO sequences of l consecutive zeros. We have

   $$f(n,k,l) = \binom{n}{k} - g(n,k,l).$$

   We provide a recursive formula for $f(n,k,l)$. First note that

   $$\begin{aligned} f(n,k,l) &= n - l + 1, \text{ for } k + l = n \\ f(n,0,l) &= 1, \text{ for } l \geq 1. \end{aligned}$$

   Now we prove for $n \geq 1, l \geq 1, k \geq 0$, and $k + l \leq n$ we have

   $$f(n,k,l) = f(n-1,k-1,l) + f(n-1,k,l) + g(n-l-1,k-1,l).$$

   Thus we have the following recursion for $f(n,k,l)$:

   $$\begin{aligned} f(n,k,l) = f(n-1,k-1,l) + f(n-1,k,l) + \\ \binom{n-l-1}{k-1} - f(n-l-1,k-1,l). \end{aligned}$$

   The above equation can be used to compute $f(n,k,l)$ for moderate values of n, k, and l.
   Proof: Consider a sequence of length n with exactly k ones and at least l consecutive zeros. We consider two cases:

   (a) The first $n-1$ bits (positions) include at least l consecutive zeros. In this case the last bit is either 0 or a 1 which results in $f(n-1,k-1,l) + f(n-1,k,l)$ distinct sequences.

   (b) The first $n-1$ bits do NOT include l consecutive zeros. In this case the last l bits must be zeros and the $(n-l)$th bit must be a one. Thus the first $n-l-1$ bits must have exactly $k-1$ ones and no consecutive l zeros. This results in $g(n-l-1,k-1,l)$ distinct sequences.

5. I toss a biased coin n times and record the sequence of heads and tails. If $P(H) = p$ (where $0 < p < 1$), what is the probability that I do not observe two consecutive heads in the sequence?

   Solution: Let A_n be the event that we observe no consecutive heads in n coin tosses, i.e., $p_n = P(A_n)$. The idea is to condition on the result of the first coin toss. There are two possibilities. Using the law of total probability and by conditioning on the result of the first coin toss, we can write

$$\begin{aligned} p_n = P(A_n) &= P(A_n|H)P(H) + P(A_n|T)P(T) \\ &= p \cdot P(A_n|H) + (1-p) \cdot P(A_n|T) \qquad (14.9) \end{aligned}$$

   Now, to find $P(A_n|T)$ note that if the first coin toss is a T, then in order to not observe an HH in the entire sequence, we must not observe an HH in the remaining $n-1$ coin tosses. Thus, we have

$$P(A_n|T) = P(A_{n-1}) = p_{n-1}.$$

   Similarly, if the first coin toss results in an H, the second one must be a T and we must not observe an HH in the remaining $n-2$ coin tosses, thus we have

$$P(A_n|H) = (1-p) \cdot P(A_{n-2}) = (1-p)p_{n-2}.$$

   Plugging back into Equation 14.9 we obtain

$$p_n = (1-p)p_{n-1} + p(1-p)p_{n-2}.$$

   Note that we also know that $p_1 = 1$ and $p_2 = 1 - p^2$. Using the recursion, we also obtain $p_0 = 1$. Thus we can solve this recursion. We obtain the following characteristic equation:

$$x^2 - (1-p)x - p(1-p) = 0.$$

   The characteristic equation has roots $x_1 = \frac{1-p+\sqrt{(1-p)(1+3p)}}{2}$ and $x_2 = \frac{1-p-\sqrt{(1-p)(1+3p)}}{2}$, so the general solution is given by

$$p_n = \alpha\left(\frac{1-p+\sqrt{(1-p)(1+3p)}}{2}\right)^n + \beta\left(\frac{1-p-\sqrt{(1-p)(1+3p)}}{2}\right)^n.$$

   Using $p_0 = 1$ and $p_1 = 1$, we obtain

$$\begin{aligned} \alpha &= \frac{1+p+\sqrt{(1-p)(1+3p)}}{2\sqrt{(1-p)(1+3p)}} \\ \beta = 1-\alpha &= \frac{\sqrt{(1-p)(1+3p)}-1-p}{2\sqrt{(1-p)(1+3p)}}. \end{aligned}$$

6. Solve the following recurrence equation, that is find a closed form formula for a_n.

$$a_n = \alpha a_{n-1} + \beta, \qquad \text{with } a_0 = 1,$$

where $\alpha \neq 0$ and β are known constants.

Solution: This equation is NOT exactly of the form of Equation 14.3 (because of the constant β), so we cannot directly use our general methodology to solve this problem. However, by computing a_n for a few values of n we can identify the solution:

$$\begin{aligned}
a_1 &= \alpha + \beta \\
a_2 &= \alpha^2 + \beta(1 + \alpha) \\
a_3 &= \alpha^3 + \beta(1 + \alpha + \alpha^2) \\
a_3 &= \alpha^4 + \beta(1 + \alpha + \alpha^2 + \alpha^3).
\end{aligned}$$

In general, we obtain

$$\begin{aligned}
a_n &= \alpha^n + \beta(1 + \alpha + \alpha^2 + \cdot + \alpha^{n-1}) \\
&= \alpha^n + \beta\left(\frac{1 - \alpha^n}{1 - \alpha}\right).
\end{aligned}$$

7. This problem has applications in coding theory: I toss a biased coin n times and record the sequence of heads and tails. If $P(H) = p$ (where $0 < p < 1$), what is the probability that I observe an even number of Hs?

Solution: Let A_n be the event that I observe an even number of heads for $n = 1, 2, ...$ Let $a_n = P(A_n)$. Conditioning on the last coin toss, we can write for $n \geq 2$:

$$\begin{aligned}
a_n &= P(A_n|H)P(H) + P(A_n|T)P(T) \\
&= p \cdot P(A_n|H) + (1 - p) \cdot P(A_n|T) \\
&= p \cdot P(A^c_{n-1}) + (1 - p) \cdot P(A_{n-1}) \\
&= p(1 - a_{n-1}) + (1 - p)a_{n-1}.
\end{aligned}$$

Thus, we obtain the following recursive equation:

$$a_n = (1 - 2p)a_{n-1} + p, \qquad \text{with } a_1 = 1 - p.$$