4.1.2 Expected Value and Variance

As we mentioned earlier, the theory of continuous random variables is very similar to the theory of discrete random variables. In particular, usually summations are replaced by integrals and PMFs are replaced by PDFs. The proofs and ideas are very analogous to the discrete case, so sometimes we state the results without mathematical derivations for the purpose of brevity.

Remember that the expected value of a discrete random variable can be obtained as

$$EX = \sum_{x_k \in R_X} x_k P_X(x_k).$$

Now, by replacing the sum by an integral and PMF by PDF, we can write the definition of expected value of a continuous random variable as

$$\boxed{EX = \int_{-\infty}^{\infty} x f_X(x) dx}$$

Example 4.3
Let $X \sim Uniform(a, b)$. Find EX.

Solution

As we saw, the PDF of X is given by

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & x < a \text{ or } x > b \end{cases}$$

so to find its expected value, we can write

$$\begin{aligned} EX &= \int_{-\infty}^{\infty} x f_X(x) dx \\ &= \int_a^b x \left(\frac{1}{b-a}\right) dx \end{aligned}$$

$$= \frac{1}{b-a}\left[\frac{1}{2}x^2\right]_a^b dx$$
$$= \frac{a+b}{2}.$$

This result is intuitively reasonable: since X is uniformly distributed over the interval $[a, b]$, we expect its mean to be the middle point, i.e., $EX = \frac{a+b}{2}$.

Example 4.4
Let X be a continuous random variable with PDF

$$f_X(x) = \begin{cases} 2x & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the expected value of X.

Solution

We have

$$\begin{aligned} EX &= \int_{-\infty}^{\infty} x f_X(x) dx \\ &= \int_0^1 x(2x) dx \\ &= \int_0^1 2x^2 dx \\ &= \frac{2}{3}. \end{aligned}$$

Expected Value of a Function of a Continuous Random Variable
Remember the law of the unconscious statistician (LOTUS) for discrete random variables:

$$E[g(X)] = \sum_{x_k \in R_X} g(x_k) P_X(x_k) \tag{4.2}$$

Now, by changing the sum to integral and changing the PMF to PDF we will obtain the similar formula for continuous random variables.

Law of the unconscious statistician (LOTUS) for continuous random variables:

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx \tag{4.3}$$

As we have seen before, expectation is a linear operation, thus we always have

- $E[aX + b] = aEX + b$, for all $a, b \in \mathbb{R}$, and
- $E[X_1 + X_2 + ... + X_n] = EX_1 + EX_2 + ... + EX_n$, for any set of random variables $X_1, X_2, \ldots, X_n$.

Example 4.5

Let X be a continuous random variable with PDF

$$f_X(x) = \begin{cases} x + \frac{1}{2} & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find $E(X^n)$, where $n \in \mathbb{N}$.

Solution

Using LOTUS we have

$$\begin{aligned} E[X^n] &= \int_{-\infty}^{\infty} x^n f_X(x) dx \\ &= \int_0^1 x^n (x + \tfrac{1}{2}) dx \\ &= \left[\frac{1}{n+2} x^{n+2} + \frac{1}{2(n+1)} x^{n+1} \right]_0^1 \\ &= \frac{3n+4}{2(n+1)(n+2)}. \end{aligned}$$

Variance

Remember that the variance of any random variable is defined as

$$\mathrm{Var}(X) = E\big[(X - \mu_X)^2\big] = EX^2 - (EX)^2.$$

So for a continuous random variable, we can write

$$\mathrm{Var}(X) = E\big[(X-\mu_X)^2\big] = \int_{-\infty}^{\infty} (x-\mu_X)^2 f_X(x)dx$$

$$= EX^2 - (EX)^2 = \int_{-\infty}^{\infty} x^2 f_X(x)dx - \mu_X^2$$

Also remember that for $a, b \in \mathbb{R}$, we always have

$$\mathrm{Var}(aX+b) = a^2\mathrm{Var}(X). \tag{4.4}$$

Example 4.6

Let X be a continuous random variable with PDF

$$f_X(x) = \begin{cases} \frac{3}{x^4} & x \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the mean and variance of X.

Solution

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} x f_X(x)dx \\ &= \int_{1}^{\infty} \frac{3}{x^3} dx \\ &= \left[-\frac{3}{2}x^{-2}\right]_1^{\infty} \\ &= \frac{3}{2}. \end{aligned}$$

Next, we find EX^2 using LOTUS,

$$\begin{aligned} E[X^2] &= \int_{-\infty}^{\infty} x^2 f_X(x)dx \\ &= \int_{1}^{\infty} \frac{3}{x^2} dx \\ &= \left[-3x^{-1}\right]_1^{\infty} \\ &= 3. \end{aligned}$$

Thus, we have

$$\mathrm{Var}(X) = EX^2 - (EX)^2 = 3 - \frac{9}{4} = \frac{3}{4}.$$