7.2.4 Convergence in Distribution

Convergence in distribution is in some sense the weakest type of convergence. All it says is that the CDF of X_n's converges to the CDF of X as n goes to infinity. It does not require any dependence between the X_n's and X. We saw this type of convergence before when we discussed the central limit theorem. To say that X_n converges in distribution to X, we write

$$X_n \xrightarrow{d} X.$$

Here is a formal definition of convergence in distribution:

Convergence in Distribution

A sequence of random variables $X_1, X_2, X_3, \cdots$ converges **in distribution** to a random variable X, shown by $X_n \xrightarrow{d} X$, if

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x),$$

for all x at which $F_X(x)$ is continuous.

Example 7.5

If $X_1, X_2, X_3, \cdots$ is a sequence of i.i.d. random variables with CDF $F_X(x)$, then $X_n \xrightarrow{d} X$. This is because

$$F_{X_n}(x) = F_X(x), \qquad \text{for all } x.$$

Therefore,

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x), \qquad \text{for all } x.$$

Example 7.6
Let X_2, X_3, X_4, $\cdots$ be a sequence of random variable such that

$$F_{X_n}(x) = \begin{cases} 1-\left(1-\frac{1}{n}\right)^{nx} & x>0 \\ & \\ 0 & \text{otherwise} \end{cases}$$

Show that X_n converges in distribution to $Exponential(1)$.

Solution

Let $X \sim Exponential(1)$. For $x \leq 0$, we have

$$F_{X_n}(x) = F_X(x) = 0, \qquad \text{for } n = 2, 3, 4, \cdots.$$

For $x \geq 0$, we have

$$\begin{aligned} \lim_{n\to\infty} F_{X_n}(x) &= \lim_{n\to\infty} \left(1-\left(1-\frac{1}{n}\right)^{nx}\right) \\ &= 1 - \lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{nx} \\ &= 1-e^{-x} \\ &= F_X(x), \qquad \text{for all } x. \end{aligned}$$

Thus, we conclude that $X_n \xrightarrow{d} X$.

When working with integer-valued random variables, the following theorem is often useful.

Theorem 7.1 Consider the sequence X_1, X_2, X_3, $\cdots$ and the random variable X. Assume that X and X_n (for all n) are non-negative and integer-valued, i.e.,

$$\begin{aligned} &R_X \subset \{0, 1, 2, \cdots\}, \\ &R_{X_n} \subset \{0, 1, 2, \cdots\}, \qquad \text{for } n = 1, 2, 3, \cdots. \end{aligned}$$

Then $X_n \xrightarrow{d} X$ if and only if

$$\lim_{n\to\infty} P_{X_n}(k) = P_X(k), \qquad \text{for } k = 0, 1, 2, \cdots.$$

Proof

Since X is integer-valued, its CDF, $F_X(x)$, is continuous at all $x \in \mathbb{R} - \{0, 1, 2, \ldots\}$. If $X_n \xrightarrow{d} X$, then

$$\lim_{n \to \infty} F_{X_n}(x) = F_X(x), \qquad \text{for all } x \in \mathbb{R} - \{0, 1, 2, \ldots\}.$$

Thus, for $k = 0, 1, 2, \cdots$, we have

$$\begin{aligned}
\lim_{n \to \infty} P_{X_n}(k) &= \lim_{n \to \infty} \left[F_{X_n}\left(k + \frac{1}{2}\right) - F_{X_n}\left(k - \frac{1}{2}\right) \right] \qquad (X_n\text{'s are integer-valued}) \\
&= \lim_{n \to \infty} F_{X_n}\left(k + \frac{1}{2}\right) - \lim_{n \to \infty} F_{X_n}\left(k - \frac{1}{2}\right) \\
&= F_X\left(k + \frac{1}{2}\right) - F_X\left(k - \frac{1}{2}\right) \qquad (\text{since } X_n \xrightarrow{d} X) \\
&= P_X(k) \qquad (\text{since } X \text{ is integer-valued}).
\end{aligned}$$

To prove the converse, assume that we know

$$\lim_{n \to \infty} P_{X_n}(k) = P_X(k), \qquad \text{for } k = 0, 1, 2, \cdots.$$

Then, for all $x \in \mathbb{R}$, we have

$$\begin{aligned}
\lim_{n \to \infty} F_{X_n}(x) &= \lim_{n \to \infty} P(X_n \leq x) \\
&= \lim_{n \to \infty} \sum_{k=0}^{\lfloor x \rfloor} P_{X_n}(k),
\end{aligned}$$

where $\lfloor x \rfloor$ shows the largest integer less than or equal to x. Since for any fixed x, the set $\{0, 1, \cdots, \lfloor x \rfloor\}$ is a finite set, we can change the order of the limit and the sum, so we obtain

$$\begin{aligned}
\lim_{n \to \infty} F_{X_n}(x) &= \sum_{k=0}^{\lfloor x \rfloor} \lim_{n \to \infty} P_{X_n}(k) \\
&= \sum_{k=0}^{\lfloor x \rfloor} P_X(k) \qquad (\text{by assumption}) \\
&= P(X \leq x) = F_X(x).
\end{aligned}$$

Example 7.7

Let X_1, X_2, X_3, $\cdots$ be a sequence of random variable such that

$$X_n \sim Binomial\left(n, \frac{\lambda}{n}\right), \qquad \text{for } n \in \mathbb{N}, n > \lambda,$$

where $\lambda > 0$ is a constant. Show that X_n converges in distribution to $Poisson(\lambda)$.

Solution

By **Theorem 7.1**, it suffices to show that

$$\lim_{n\to\infty} P_{X_n}(k) = P_X(k), \qquad \text{for all } k = 0, 1, 2, \cdots.$$

We have

$$\begin{aligned}
\lim_{n\to\infty} P_{X_n}(k) &= \lim_{n\to\infty} \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} \\
&= \lambda^k \lim_{n\to\infty} \frac{n!}{k!(n-k)!} \left(\frac{1}{n^k}\right) \left(1 - \frac{\lambda}{n}\right)^{n-k} \\
&= \frac{\lambda^k}{k!} . \lim_{n\to\infty} \left(\left[\frac{n(n-1)(n-2)\ldots(n-k+1)}{n^k}\right] \left[\left(1-\frac{\lambda}{n}\right)^n\right] \left[\left(1-\frac{\lambda}{n}\right)^{-k}\right] \right).
\end{aligned}$$

Note that for a fixed k, we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{n(n-1)(n-2)\ldots(n-k+1)}{n^k} &= 1, \\
\lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^{-k} &= 1, \\
\lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^{n} &= e^{-\lambda}.
\end{aligned}$$

Thus, we conclude

$$\lim_{n\to\infty} P_{X_n}(k) = \frac{e^{-\lambda}\lambda^k}{k!}.$$

We end this section by reminding you that the most famous example of convergence in distribution is the central limit theorem (CLT). The CLT states that the normalized average of i.i.d. random variables $X_1, X_2, X_3, \cdots$ converges in distribution to a standard normal random variable.