4.4 End of Chapter Problems

Problem 1

Choose a real number uniformly at random in the interval $[2, 6]$ and call it X.

a. Find the CDF of X, $F_X(x)$.
b. Find EX.

Problem 2

Let X be a continuous random variable with the following PDF

$$f_X(x) = \begin{cases} ce^{-4x} & x \geq 0 \\ 0 & otherwise \end{cases}$$

where c is a positive constant.

a. Find c.
b. Find the CDF of X, $F_X(x)$.
c. Find $P(2 < X < 5)$.
d. Find EX.

Problem 3

Let X be a continuous random variable with PDF

$$f_X(x) = \begin{cases} x^2 + \frac{2}{3} & 0 \leq x \leq 1 \\ 0 & otherwise \end{cases}$$

a. Find $E(X^n)$, for $n = 1, 2, 3, \cdots$.
b. Find the variance of X.

Problem 4

Let X be a $uniform(0, 1)$ random variable, and let $Y = e^{-X}$.

a. Find the CDF of Y.

b. Find the PDF of Y.
c. Find EY.

Problem 5
Let X be a continuous random variable with PDF

$$f_X(x) = \begin{cases} \frac{5}{32}x^4 & 0 < x \leq 2 \\ 0 & otherwise \end{cases}$$

and let $Y = X^2$.

a. Find the CDF of Y.
b. Find the PDF of Y.
c. Find EY.

Problem 6
Let $X \sim Exponential(\lambda)$, and $Y = aX$, where a is a positive real number. Show that

$$Y \sim Exponential\left(\frac{\lambda}{a}\right).$$

Problem 7
Let $X \sim Exponential(\lambda)$. Show that

a. $EX^n = \frac{n}{\lambda}EX^{n-1}$, for $n = 1, 2, 3, \cdots$;
b. $EX^n = \frac{n!}{\lambda^n}$, for $n = 1, 2, 3, \cdots$.

Problem 8
Let $X \sim N(3, 9)$.

a. Find $P(X > 0)$.
b. Find $P(-3 < X < 8)$.
c. Find $P(X > 5 | X > 3)$.

Problem 9

Let $X \sim N(3,9)$ and $Y = 5 - X$.

a. Find $P(X > 2)$.
b. Find $P(-1 < Y < 3)$.
c. Find $P(X > 4 | Y < 2)$.

Problem 10

Let X be a continuous random variable with PDF

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \qquad \text{for all } x \in \mathbb{R}.$$

and let $Y = \sqrt{|X|}$. Find $f_Y(y)$.

Problem 11

Let $X \sim Exponential(2)$ and $Y = 2 + 3X$.

a. Find $P(X > 2)$.
b. Find EY and $Var(Y)$.
c. Find $P(X > 2 | Y < 11)$.

Problem 12

The **median** of a continuous random variable X can be defined as the unique real number *m* that satisfies

$$P(X \geq m) = P(X < m) = \frac{1}{2}.$$

Find the median of the following random variables

a. $X \sim Uniform(a, b)$.
b. $Y \sim Exponential(\lambda)$.
c. $W \sim N(\mu, \sigma^2)$.

Problem 13

Let X be a random variable with the following CDF

$$F_X(x) = \begin{cases} 0 & \text{for } x < 0 \\ x & \text{for } 0 \leq x < \frac{1}{4} \\ x + \frac{1}{2} & \text{for } \frac{1}{4} \leq x < \frac{1}{2} \\ 1 & \text{for } x \geq \frac{1}{2} \end{cases}$$

a. Plot $F_X(x)$ and explain why X is a mixed random variable.
b. Find $P(X \leq \frac{1}{3})$.
c. Find $P(X \geq \frac{1}{4})$.
d. Write CDF of X in the form of

$$F_X(x) = C(x) + D(x),$$

where $C(x)$ is a continuous function and $D(x)$ is in the form of a staircase function, i.e.,

$$D(x) = \sum_k a_k u(x - x_k).$$

e. Find $c(x) = \frac{d}{dx} C(x)$.
f. Find EX using $EX = \int_{-\infty}^{\infty} xc(x)dx + \sum_k x_k a_k$

Problem 14

Let X be a random variable with the following CDF

$$F_X(x) = \begin{cases} 0 & \text{for } x < 0 \\ x & \text{for } 0 \leq x < \frac{1}{4} \\ x + \frac{1}{2} & \text{for } \frac{1}{4} \leq x < \frac{1}{2} \\ 1 & \text{for } x \geq \frac{1}{2} \end{cases}$$

a. Find the generalized PDF of $X, f_X(x)$.
b. Find EX using $f_X(x)$.
c. Find $Var(X)$ using $f_X(x)$.

Problem 15

Let X be a mixed random variable with the following generalized PDF

$$f_X(x) = \frac{1}{3}\delta(x+2) + \frac{1}{6}\delta(x-1) + \frac{1}{2}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}.$$

a. Find $P(X = 1)$ and $P(X = -2)$.
b. Find $P(X \geq 1)$.
c. Find $P(X = 1|X \geq 1)$.
d. Find EX and $Var(X)$.

Problem 16

A company makes a certain device. We are interested in the lifetime of the device. It is estimated that around 2% of the devices are defective from the start so they have a lifetime of 0 years. If a device is not defective, then the lifetime of the device is exponentially distributed with a parameter $\lambda = 2$ years. Let X be the lifetime of a randomly chosen device.

a. Find the generalized PDF of X.
b. Find $P(X \geq 1)$.
c. Find $P(X > 2|X \geq 1)$.
d. Find EX and $Var(X)$.

Problem 17

A continuous random variable is said to have a $Laplace(\mu, b)$ distribution [14] if its PDF is given by

$$f_X(x) = \frac{1}{2b} \exp\left(-\frac{|x-\mu|}{b}\right)$$
$$= \begin{cases} \frac{1}{2b} \exp\left(\frac{x-\mu}{b}\right) & \text{if } x < \mu \\ \frac{1}{2b} \exp\left(-\frac{x-\mu}{b}\right) & \text{if } x \geq \mu \end{cases}$$

where $\mu \in \mathbb{R}$ and $b > 0$.

a. If $X \sim Laplace(0, 1)$, find EX and $Var(X)$.
b. If $X \sim Laplace(0, 1)$ and $Y = bX + \mu$, show that $Y \sim Laplace(\mu, b)$.
c. Let $Y \sim Laplace(\mu, b)$, where $\mu \in \mathbb{R}$ and $b > 0$. Find EY and $Var(Y)$.

Problem 18

Let $X \sim Laplace(0, b)$, i.e.,

$$f_X(x) = \frac{1}{2b} \exp\left(-\frac{|x|}{b}\right),$$

where $b > 0$. Define $Y = |X|$. Show that $Y \sim Exponential\left(\frac{1}{b}\right)$.

Problem 19
A continuous random variable is said to have the **standard Cauchy** distribution if its PDF is given by

$$f_X(x) = \frac{1}{\pi(1+x^2)}.$$

If X has a standard Cauchy distribution, show that EX is not well-defined. Also, show $EX^2 = \infty$.

Problem 20
A continuous random variable is said to have a **Rayleigh distribution** with parameter σ if its PDF is given by

$$f_X(x) = \frac{x}{\sigma^2} e^{-x^2/2\sigma^2} u(x)$$
$$= \begin{cases} \frac{x}{\sigma^2} e^{-x^2/2\sigma^2} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

where $\sigma > 0$.

a. If $X \sim Rayleigh(\sigma)$, find EX.
b. If $X \sim Rayleigh(\sigma)$, find the CDF of $X, F_X(x)$.
c. If $X \sim Exponential(1)$ and $Y = \sqrt{2\sigma^2 X}$, show that $Y \sim Rayleigh(\sigma)$.

Problem 21
A continuous random variable is said to have a $Pareto(x_m, \alpha)$ distribution [15] if its PDF is given by

$$f_X(x) = \begin{cases} \alpha \frac{x_m^\alpha}{x^{\alpha+1}} & \text{for } x \geq x_m \\ 0 & \text{for } x < x_m \end{cases}$$

where $x_m, \alpha > 0$. Let $X \sim Pareto(x_m, \alpha)$.

a. Find the CDF of X, $F_X(x)$.
b. Find $P(X > 3x_m | X > 2x_m)$.
c. If $\alpha > 2$, find EX and $Var(X)$.

Problem 22

Let $Z \sim N(0,1)$. If we define $X = e^{\sigma Z + \mu}$, then we say that X has a log-normal distribution with parameters μ and σ, and we write $X \sim LogNormal(\mu, \sigma)$.

a. If $X \sim LogNormal(\mu, \sigma)$, find the CDF of X in terms of the Φ function.
b. Find EX and $Var(X)$.

Problem 23

Let X_1, X_2, $\cdots$, X_n be independent random variables with $X_i \sim Exponential(\lambda)$. Define

$$Y = X_1 + X_2 + \cdots + X_n.$$

As we will see later, Y has a **Gamma** distribution with parameters n and λ, i.e., $Y \sim Gamma(n, \lambda)$. Using this, show that if $Y \sim Gamma(n, \lambda)$, then $EY = \frac{n}{\lambda}$ and $Var(Y) = \frac{n}{\lambda^2}$.
