11.4.3 Solved Problems

Problem 1

Let $W(t)$ be a standard Brownian motion. Find $P(W(1)+W(2)>2)$.

Solution

Let $X=W(1)+W(2)$. Since $W(t)$ is a Gaussian process, X is a normal random variable.

$$EX = E[W(1)] + E[W(2)] = 0,$$

$$\begin{aligned}\mathrm{Var}(X) &= \mathrm{Var}\big(W(1)\big) + \mathrm{Var}\big(W(2)\big) + 2\mathrm{Cov}\big(W(1), W(2)\big) \\ &= 1+2+2\cdot 1 \\ &= 5.\end{aligned}$$

We conclude

$$X \sim N(0,5).$$

Thus,

$$\begin{aligned}P(X>2) &= 1-\Phi\left(\frac{2-0}{\sqrt{5}}\right) \\ &\approx 0.186\end{aligned}$$

Problem 2

Let $W(t)$ be a standard Brownian motion, and $0 \leq s < t$. Find the conditional PDF of $W(s)$ given $W(t)=a$.

Solution

It is useful to remember the following result from the previous chapters: Suppose X and Y are jointly normal random variables with parameters μ_X, σ^2_X, μ_Y, σ^2_Y, and ρ. Then, given $X=x$, Y is normally distributed with

$$
\begin{aligned}
E[Y|X=x] &= \mu_Y + \rho\sigma_Y \frac{x-\mu_X}{\sigma_X}, \\
\mathrm{Var}(Y|X=x) &= (1-\rho^2)\sigma_Y^2.
\end{aligned}
$$

Now, if we let $X = W(t)$ and $Y = W(s)$, we have $X \sim N(0,t)$ and $Y \sim N(0,s)$ and

$$
\begin{aligned}
\rho &= \frac{\mathrm{Cov}(X,Y)}{\sigma_x \sigma_Y} \\
&= \frac{\min(s,t)}{\sqrt{t}\sqrt{s}} \\
&= \frac{s}{\sqrt{t}\sqrt{s}} \\
&= \sqrt{\frac{s}{t}}.
\end{aligned}
$$

We conclude that

$$
\begin{aligned}
E[Y|X=a] &= \frac{s}{t}a, \\
\mathrm{Var}(Y|X=a) &= s\left(1-\frac{s}{t}\right).
\end{aligned}
$$

Therefore,

$$
W(s)|W(t) = a \ \sim \ N\left(\frac{s}{t}a, s\left(1-\frac{s}{t}\right)\right).
$$

Problem 3

(**Geometric Brownian Motion**) Let $W(t)$ be a standard Brownian motion. Define

$$
X(t) = \exp\{W(t)\}, \quad \text{for all t } \in [0,\infty).
$$

a. Find $E[X(t)]$, for all $t \in [0,\infty)$.
b. Find $\mathrm{Var}(X(t))$, for all $t \in [0,\infty)$.
c. Let $0 \leq s \leq t$. Find $\mathrm{Cov}(X(s), X(t))$.

Solution

It is useful to remember the MGF of the normal distribution. In particular, if $X \sim N(\mu,\sigma)$, then

$$
M_X(s) = E[e^{sX}] = \exp\left\{s\mu + \frac{\sigma^2 s^2}{2}\right\}, \qquad \text{for all} \quad s \in \mathbb{R}.
$$

a. We have

$$\begin{aligned} E[X(t)] &= E[e^{W(t)}], \qquad\qquad (\text{where } W(t) \sim N(0,t)) \\ &= \exp\left\{\frac{t}{2}\right\}. \end{aligned}$$

b. We have

$$\begin{aligned} E[X^2(t)] &= E[e^{2W(t)}], \qquad\qquad (\text{where } W(t) \sim N(0,t)) \\ &= \exp\{2t\}. \end{aligned}$$

Thus,

$$\begin{aligned} \mathrm{Var}(X(t)) &= E[X^2(t)] - E[X(t)]^2 \\ &= \exp\{2t\} - \exp\{t\}. \end{aligned}$$

c. Let $0 \le s \le t$. Then, we have

$$\begin{aligned} \mathrm{Cov}(X(s), X(t)) &= E[X(s)X(t)] - E[X(s)]E[X(t)] \\ &= E[X(s)X(t)] - \exp\left\{\frac{s+t}{2}\right\}. \end{aligned}$$

To find $E[X(s)X(t)]$, we can write

$$\begin{aligned} E[X(s)X(t)] &= E\bigg[\exp\{W(s)\}\exp\{W(t)\}\bigg] \\ &= E\bigg[\exp\{W(s)\}\exp\{W(s) + W(t) - W(s)\}\bigg] \\ &= E\bigg[\exp\{2W(s)\}\exp\{W(t) - W(s)\}\bigg] \\ &= E\bigg[\exp\{2W(s)\}\bigg]E\bigg[\exp\{W(t) - W(s)\}\bigg] \\ &= \exp\{2s\}\exp\left\{\frac{t-s}{2}\right\} \\ &= \exp\left\{\frac{3s+t}{2}\right\}. \end{aligned}$$

We conclude, for $0 \le s \le t$,

$$\mathrm{Cov}(X(s), X(t)) = \exp\left\{\frac{3s+t}{2}\right\} - \exp\left\{\frac{s+t}{2}\right\}.$$
