Find $R_{XY}(t_1,t_2)$ and $C_{XY}(t_1,t_2)$, for $t_1,t_2 \in [0,\infty)$.

Solution

First, note that

$$\begin{aligned}\mu_X(t) &= E[X(t)] \\ &= EA + EB \cdot t \\ &= 1+t, \qquad \text{for all } t \in [0,\infty).\end{aligned}$$

Similarly,

$$\begin{aligned}\mu_Y(t) &= E[Y(t)] \\ &= EA + EC \cdot t \\ &= 1+t, \qquad \text{for all } t \in [0,\infty).\end{aligned}$$

To find $R_{XY}(t_1,t_2)$ for $t_1,t_2 \in [0,\infty)$, we write

$$\begin{aligned}R_{XY}(t_1,t_2) &= E[X(t_1)Y(t_2)] \\ &= E\big[(A+Bt_1)(A+Ct_2)\big] \\ &= E\big[A^2 + ACt_2 + BAt_1 + BCt_1t_2\big] \\ &= E[A^2] + E[AC]t_2 + E[BA]t_1 + E[BC]t_1t_2 \\ &= E[A^2] + E[A]E[C]t_2 + E[B]E[A]t_1 + E[B]E[C]t_1t_2, \quad \text{(by independence)} \\ &= 2 + t_1 + t_2 + t_1t_2.\end{aligned}$$

To find $C_{XY}(t_1,t_2)$ for $t_1,t_2 \in [0,\infty)$, we write

$$\begin{aligned}C_{XY}(t_1,t_2) &= R_{XY}(t_1,t_2) - \mu_X(t_1)\mu_Y(t_2) \\ &= \big(2 + t_1 + t_2 + t_1t_2\big) - \big(1+t_1\big)\big(1+t_2\big) \\ &= 1.\end{aligned}$$

Independent Random Processes:

We have seen independence for random variables. In particular, remember that random variables X_1, X_2,...,X_n are independent if, for all $(x_1,x_2,\ldots,x_n) \in \mathbb{R}^n$, we have

$$F_{X_1,X_2,\ldots,X_n}(x_1,x_2,\ldots,x_n) = F_{X_1}(x_1)F_{X_2}(x_2)\ldots F_{X_n}(x_n).$$

Now, note that a random process is a collection of random variables. Thus, we can define the concept of independence for random processes, too. In particular, if for two random processes $X(t)$ and $Y(t)$, the random variables $X(t_i)$ are independent from the random variables $Y(t_j)$, we say that the two random processes are independent. More precisely, we have the following definition:

Two random processes $\{X(t), t \in J\}$ and $\{Y(t), t \in J'\}$ are said to be **independent** if, for all

$$t_1, t_2, \ldots, t_m \in J$$
$$\text{and}$$
$$t'_1, t'_2, \ldots, t'_n \in J',$$

the set of random variables

$$X(t_1), X(t_2), \cdots, X(t_m)$$

are independent of the set of random variables

$$Y(t'_1), Y(t'_2), \cdots, Y(t'_n).$$

The above definition implies that for all real numbers $x_1, x_2, \cdots, x_m$ and $y_1, y_2, \cdots, y_n$, we have

$$\begin{aligned} &F_{X(t_1),X(t_2),\cdots,X(t_m),Y(t'_1),Y(t'_2),\cdots,Y(t'_n)}(x_1, x_2, \cdots, x_m, y_1, y_2, \cdots, y_n) \\ &\quad = F_{X(t_1),X(t_2),\cdots,X(t_m)}(x_1, x_2, \cdots, x_m) \cdot F_{Y(t'_1),Y(t'_2),\cdots,Y(t'_n)}(y_1, y_2, \cdots, y_n). \end{aligned}$$

The above equation might seem complicated; however, in many real-life applications we can often argue that two random processes are independent by looking at the problem structure. For example, in engineering we can reasonably assume that the thermal noise processes in two separate systems are independent. Note that if two random processes $X(t)$ and $Y(t)$ are independent, then their covariance function, $C_{XY}(t_1, t_2)$, for all t_1 and t_2 is given by

$$\begin{aligned} C_{XY}(t_1, t_2) &= \text{Cov}\big(X(t_1), Y(t_2)\big) \\ &= 0 \quad (\text{since } X(t_1) \text{ and } Y(t_2) \text{ are independent}). \end{aligned}$$