11.2.3 Probability Distributions

State Probability Distributions:

Consider a Markov chain $\{X_n, n = 0, 1, 2, \ldots\}$, where $X_n \in S = \{1, 2, \cdots, r\}$. Suppose that we know the probability distribution of X_0. More specifically, define the row vector $\pi^{(0)}$ as

$$\pi^{(0)} = [\, P(X_0 = 1) \quad P(X_0 = 2) \quad \cdots \quad P(X_0 = r) \,].$$

How can we obtain the probability distribution of X_1, X_2, $\cdots$? We can use the law of total probability. More specifically, for any $j \in S$, we can write

$$\begin{aligned} P(X_1 = j) &= \sum_{k=1}^{r} P(X_1 = j | X_0 = k) P(X_0 = k) \\ &= \sum_{k=1}^{r} p_{kj} P(X_0 = k). \end{aligned}$$

If we generally define

$$\pi^{(n)} = [\, P(X_n = 1) \quad P(X_n = 2) \quad \cdots \quad P(X_n = r) \,],$$

we can rewrite the above result in the form of matrix multiplication

$$\pi^{(1)} = \pi^{(0)} P,$$

where P is the state transition matrix. Similarly, we can write

$$\pi^{(2)} = \pi^{(1)} P = \pi^{(0)} P^2.$$

More generally, we can write

$$\boxed{\begin{aligned} &\pi^{(n+1)} = \pi^{(n)} P, \quad \text{for } n = 0, 1, 2, \cdots; \\ &\pi^{(n)} = \pi^{(0)} P^n, \quad \text{for } n = 0, 1, 2, \cdots. \end{aligned}}$$

Example 11.5

Consider a system that can be in one of two possible states, $S = \{0, 1\}$. In particular, suppose that the transition matrix is given by

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}.$$

Suppose that the system is in state 0 at time $n = 0$, i.e., $X_0 = 0$.

a. Draw the state transition diagram.
b. Find the probability that the system is in state 1 at time $n = 3$.

Solution

a. The state transition diagram is shown in Figure 11.8.

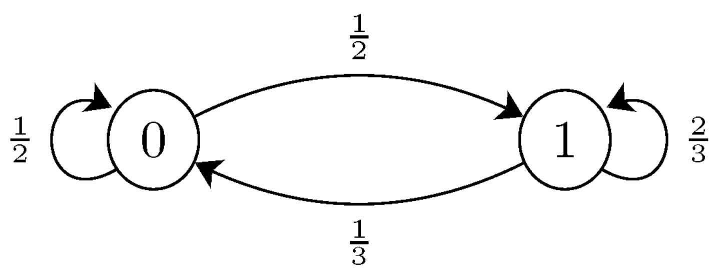

Figure 11.8 - A state transition diagram.

b. Here, we know

$$\begin{aligned} \pi^{(0)} &= \begin{bmatrix} P(X_0 = 0) & P(X_0 = 1) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix}. \end{aligned}$$

Thus,

$$\begin{aligned} \pi^{(3)} &= \pi^{(0)} P^3 \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}^3 \\ &= \begin{bmatrix} \frac{29}{72} & \frac{43}{72} \end{bmatrix}. \end{aligned}$$

Thus, the probability that the system is in state 1 at time $n = 3$ is $\frac{43}{72}$.