6.2.3 Chernoff Bounds

If X is a random variable, then for any $a \in \mathbb{R}$, we can write

$$P(X \geq a) = P(e^{sX} \geq e^{sa}), \qquad \text{for } s > 0,$$
$$P(X \leq a) = P(e^{sX} \geq e^{sa}), \qquad \text{for } s < 0.$$

Now, note that e^{sX} is always a positive random variable for all $s \in \mathbb{R}$. Thus, we can apply Markov's inequality. So for $s > 0$, we can write

$$\begin{aligned} P(X \geq a) &= P(e^{sX} \geq e^{sa}) \\ &\leq \frac{E[e^{sX}]}{e^{sa}}, \qquad \text{by Markov's inequality.} \end{aligned}$$

Similarly, for $s < 0$, we can write

$$\begin{aligned} P(X \leq a) &= P(e^{sX} \geq e^{sa}) \\ &\leq \frac{E[e^{sX}]}{e^{sa}}. \end{aligned}$$

Note that $E[e^{sX}]$ is in fact the moment generating function, $M_X(s)$. Thus, we conclude

Chernoff Bounds:

$$P(X \geq a) \leq e^{-sa} M_X(s), \qquad \text{for all } s > 0,$$
$$P(X \leq a) \leq e^{-sa} M_X(s), \qquad \text{for all } s < 0$$

Since Chernoff bounds are valid for all values of $s > 0$ and $s < 0$, we can choose s in a way to obtain the best bound, that is we can write

$$P(X \geq a) \leq \min_{s>0} e^{-sa} M_X(s),$$
$$P(X \leq a) \leq \min_{s<0} e^{-sa} M_X(s).$$

Let us look at an example to see how we can use Chernoff bounds.

Example 6.22

Let $X \sim Binomial(n, p)$. Using Chernoff bounds, find an upper bound on $P(X \geq \alpha n)$, where $p < \alpha < 1$. Evaluate the bound for $p = \frac{1}{2}$ and $\alpha = \frac{3}{4}$.

Solution

For $X \sim Binomial(n, p)$, we have

$$M_X(s) = (pe^s + q)^n, \qquad \text{where } q = 1 - p.$$

Thus, the Chernoff bound for $P(X \geq a)$ can be written as

$$\begin{aligned} P(X \geq \alpha n) &\leq \min_{s>0} e^{-sa} M_X(s) \\ &= \min_{s>0} e^{-sa} (pe^s + q)^n. \end{aligned}$$

To find the minimizing value of s, we can write

$$\frac{d}{ds} e^{-sa} (pe^s + q)^n = 0,$$

which results in

$$e^s = \frac{aq}{np(1-\alpha)}.$$

By using this value of s in Equation 6.3 and some algebra, we obtain

$$P(X \geq \alpha n) \leq \left(\frac{1-p}{1-\alpha}\right)^{(1-\alpha)n} \left(\frac{p}{\alpha}\right)^{\alpha n}.$$

For $p = \frac{1}{2}$ and $\alpha = \frac{3}{4}$, we obtain

$$P(X \geq \frac{3}{4} n) \leq \left(\frac{16}{27}\right)^{\frac{n}{4}}.$$

Comparison between Markov, Chebyshev, and Chernoff Bounds:

Above, we found upper bounds on $P(X \geq \alpha n)$ for $X \sim Binomial(n, p)$. It is interesting to compare them. Here are the results that we obtain for $p = \frac{1}{4}$ and $\alpha = \frac{3}{4}$:

$$P(X \geq \frac{3n}{4}) \leq \frac{2}{3} \qquad \text{Markov,}$$
$$P(X \geq \frac{3n}{4}) \leq \frac{4}{n} \qquad \text{Chebyshev,}$$
$$P(X \geq \frac{3n}{4}) \leq \left(\frac{16}{27}\right)^{\frac{n}{4}} \qquad \text{Chernoff.}$$

The bound given by Markov is the "weakest" one. It is constant and does not change as n increases. The bound given by Chebyshev's inequality is "stronger" than the one given by Markov's inequality. In particular, note that $\frac{4}{n}$ goes to zero as n goes to infinity. The strongest bound is the Chernoff bound. It goes to zero exponentially fast.