## 9.1.4 Conditional Expectation (MMSE)

Remember that the posterior distribution, $f_{X|Y}(x|y)$, contains all the knowledge that we have about the unknown quantity $X$. Therefore, to find a point estimate of $X$, we can just choose a summary statistic of the posterior such as its mean, median, or mode. If we choose the mode (the value of $x$ that maximizes $f_{X|Y}(x|y)$), we obtain the MAP estimate of $X$. Another option would be to choose the posterior mean, i.e.,

$$\hat{x} = E[X|Y = y].$$

We will show that $E[X|Y = y]$ will give us the best estimate of $X$ in terms of the *mean squared error*. For this reason, the conditional expectation is called the *minimum mean squared error (MMSE) estimate* of $X$. It is also called the *least mean squares (LMS) estimate* or simply the *Bayes' estimate* of $X$.

Minimum Mean Squared Error (MMSE) Estimation

The **minimum mean squared error (MMSE)** estimate of the random variable $X$, given that we have observed $Y = y$, is given by

$$\hat{x}_M = E[X|Y = y].$$

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**Example 9.6**

Let $X$ be a continuous random variable with the following PDF

$$f_X(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq 1 \\ \\ 0 & \text{otherwise} \end{cases}$$

We also know that

$$f_{Y|X}(y|x) = \begin{cases} 2xy - x + 1 & \text{if } 0 \leq y \leq 1 \\ \\ 0 & \text{otherwise} \end{cases}$$

Find the MMSE estimate of $X$, given $Y=y$ is observed.

**Solution**

First we need to find the posterior density, $f_{X|Y}(x|y)$. We have

$$f_{X|Y}(x|y)=\frac{f_{Y|X}(y|x)f_X(x)}{f_Y(y)}.$$

We can find $f_Y(y)$ as

$$\begin{aligned} f_Y(y) &= \int_0^1 f_{Y|X}(y|x)f_X(x)dx \\ &= \int_0^1 (2xy-x+1)2xdx \\ &= \frac{4}{3}y+\frac{1}{3}, \quad \text{for } 0 \leq y \leq 1. \end{aligned}$$

We conclude

$$f_{X|Y}(x|y)=\frac{6x(2xy-x+1)}{4y+1}, \quad \text{for } 0 \leq x \leq 1.$$

The MMSE estimate of $X$ given $Y=y$ is then given by

$$\begin{aligned} \hat{x}_M &= E[X|Y=y] \\ &= \int_0^1 x f_{X|Y}(x|y)dx \\ &= \frac{1}{4y+1}\int_0^1 6x^2(2xy-x+1)dx \\ &= \frac{3y+\frac{1}{2}}{4y+1}. \end{aligned}$$

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