11.2.2 State Transition Matrix and Diagram

We often list the transition probabilities in a matrix. The matrix is called the **state transition matrix** or **transition probability matrix** and is usually shown by P. Assuming the states are $1, 2, \cdots, r$, then the state transition matrix is given by

$$P = \begin{bmatrix} p_{11} & p_{12} & \dots & p_{1r} \\ p_{21} & p_{22} & \dots & p_{2r} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ p_{r1} & p_{r2} & \dots & p_{rr} \end{bmatrix}.$$

Note that $p_{ij} \geq 0$, and for all i, we have

$$\begin{aligned} \sum_{k=1}^{r} p_{ik} &= \sum_{k=1}^{r} P(X_{m+1} = k | X_m = i) \\ &= 1. \end{aligned}$$

This is because, given that we are in state i, the next state must be one of the possible states. Thus, when we sum over all the possible values of k, we should get one. That is, the rows of any state transition matrix must sum to one.

State Transition Diagram:

A Markov chain is usually shown by a **state transition diagram**. Consider a Markov chain with three possible states 1, 2, and 3 and the following transition probabilities

$$P = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}.$$

Figure 11.7 shows the state transition diagram for the above Markov chain. In this diagram, there are three possible states 1, 2, and 3, and the arrows from each state to other states show the transition probabilities p_{ij}. When there is no arrow from state i to state j, it means that $p_{ij} = 0$.

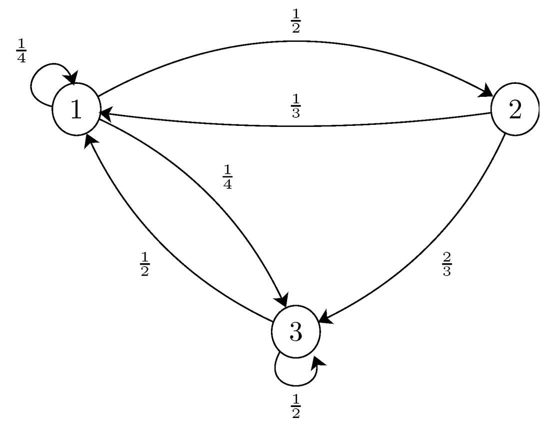

Figure 11.7 - A state transition diagram.

Example 11.4

Consider the Markov chain shown in Figure 11.7.

a. Find $P(X_4 = 3|X_3 = 2)$.
b. Find $P(X_3 = 1|X_2 = 1)$.
c. If we know $P(X_0 = 1) = \frac{1}{3}$, find $P(X_0 = 1, X_1 = 2)$.
d. If we know $P(X_0 = 1) = \frac{1}{3}$, find $P(X_0 = 1, X_1 = 2, X_2 = 3)$.

Solution

a. By definition

$$P(X_4 = 3|X_3 = 2) = p_{23} = \frac{2}{3}.$$

b. By definition

$$P(X_3 = 1|X_2 = 1) = p_{11} = \frac{1}{4}.$$

c. We can write

$$\begin{aligned} P(X_0 = 1, X_1 = 2) &= P(X_0 = 1)P(X_1 = 2|X_0 = 1) \\ &= \frac{1}{3} \cdot p_{12} \\ &= \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}. \end{aligned}$$

d. We can write

$$
\begin{aligned}
&P(X_0 = 1, X_1 = 2, X_2 = 3) \\
&\quad = P(X_0 = 1)P(X_1 = 2 | X_0 = 1)P(X_2 = 3 | X_1 = 2, X_0 = 1) \\
&\quad = P(X_0 = 1)P(X_1 = 2 | X_0 = 1)P(X_2 = 3 | X_1 = 2) \quad \text{(by Markov property)} \\
&\quad = \frac{1}{3} \cdot p_{12} \cdot p_{23} \\
&\quad = \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{2}{3} \\
&\quad = \frac{1}{9}.
\end{aligned}
$$