Example 11.1

The number of customers arriving at a grocery store can be modeled by a Poisson process with intensity $\lambda = 10$ customers per hour.

1. Find the probability that there are 2 customers between 10:00 and 10:20.
2. Find the probability that there are 3 customers between 10:00 and 10:20 and 7 customers between 10:20 and 11.

Solution

1. Here, $\lambda = 10$ and the interval between 10:00 and 10:20 has length $\tau = \frac{1}{3}$ hours. Thus, if X is the number of arrivals in that interval, we can write $X \sim Poisson(10/3)$. Therefore,

$$P(X=2) = \frac{e^{-\frac{10}{3}}\left(\frac{10}{3}\right)^2}{2!} \approx 0.2$$

2. Here, we have two non-overlapping intervals $I_1 =$(10:00 a.m., 10:20 a.m.] and $I_2 =$ (10:20 a.m., 11 a.m.]. Thus, we can write

$$P\Big(3 \text{ arrivals in } I_1 \quad \text{and} \quad 7 \text{ arrivals in } I_2\Big) = P\Big(3 \text{ arrivals in } I_1\Big) \cdot P\Big(7 \text{ arrivals in } I_2\Big).$$

Since the lengths of the intervals are $\tau_1 = 1/3$ and $\tau_2 = 2/3$ respectively, we obtain $\lambda\tau_1 = 10/3$ and $\lambda\tau_2 = 20/3$. Thus, we have

$$P\Big(3 \text{ arrivals in } I_1 \quad \text{and} \quad 7 \text{ arrivals in } I_2\Big) = \frac{e^{-\frac{10}{3}}\left(\frac{10}{3}\right)^3}{3!} \cdot \frac{e^{-\frac{20}{3}}\left(\frac{20}{3}\right)^7}{7!} \approx 0.0325$$

Second Definition of the Poisson Process:

Let $N(t)$ be a Poisson process with rate λ. Consider a very short interval of length Δ. Then, the number of arrivals in this interval has the same distribution as $N(\Delta)$. In particular, we can write

$$\begin{aligned} P(N(\Delta)=0) &= e^{-\lambda\Delta} \\ &= 1-\lambda\Delta+\frac{\lambda^2}{2}\Delta^2-\cdots \text{ (Taylor Series).} \end{aligned}$$

Note that if Δ is small, the terms that include second or higher powers of Δ are negligible compared to Δ. We write this as

$$P(N(\Delta)=0)=1-\lambda\Delta+o(\Delta) \qquad (11.1)$$

Here $o(\Delta)$ shows a function that is negligible compared to Δ, as $\Delta \to 0$. More precisely, $g(\Delta)=o(\Delta)$ means that

$$\lim_{\Delta\to 0}\frac{g(\Delta)}{\Delta}=0.$$

Now, let us look at the probability of having one arrival in an interval of length Δ.

$$\begin{aligned} P(N(\Delta)=1) &= e^{-\lambda\Delta}\lambda\Delta \\ &= \lambda\Delta\left(1-\lambda\Delta+\frac{\lambda^2}{2}\Delta^2-\cdots\right) \text{ (Taylor Series)} \\ &= \lambda\Delta+\left(-\lambda^2\Delta^2+\frac{\lambda^3}{2}\Delta^3\cdots\right) \\ &= \lambda\Delta+o(\Delta). \end{aligned}$$

We conclude that

$$P(N(\Delta)=1)=\lambda\Delta+o(\Delta) \qquad (11.2)$$

Similarly, we can show that

$$P(N(\Delta)\geq 2)=o(\Delta) \qquad (11.3)$$

In fact, equations 11.1, 11.2, and 11.3 give us another way to define a Poisson process.