If we do not know the variance of the X_i's, we use

$$W(X_1, X_2, \cdots, X_n) = \frac{\overline{X} - \mu_0}{S/\sqrt{n}},$$

where S is the sample standard deviation,

$$S = \sqrt{\frac{1}{n-1} \sum_{k=1}^{n} (X_k - \overline{X})^2} = \sqrt{\frac{1}{n-1} \left(\sum_{k=1}^{n} X_k^2 - n\overline{X}^2 \right)}.$$

In any case, we will be able to find the distribution of W, and thus we can design our tests by calculating error probabilities. Let us start with the first case.

Two-sided Tests for the Mean:

Here, we are given a random sample X_1,X_2,...,X_n from a distribution. Let $\mu = EX_i$. Our goal is to decide between

H_0: $\mu = \mu_0$,

H_1: $\mu \neq \mu_0$.

Example 8.22, which we saw previously is an instance of this case. If H_0 is true, we expect $\overline{X}$ to be close to μ_0, and so we expect $W(X_1, X_2, \cdots, X_n)$ to be close to 0 (see the definition of W above).
Therefore, we can suggest the following test. Choose a threshold, and call it c. If $|W| \leq c$, accept H_0, and if $|W| > c$, accept H_1. How do we choose c? If α is the required significance level, we must have

$$\begin{aligned} P(\text{type I error}) &= P(\text{Reject } H_0 \mid H_0) \\ &= P(|W| > c \mid H_0) \leq \alpha. \end{aligned}$$

Thus, we can choose c such that $P(|W| > c \mid H_0) = \alpha$. Let us look at an example.

Example 8.24
Let X_1,X_2,...,X_n be a random sample from a $N(\mu, \sigma^2)$ distribution, where μ is unknown but σ is known. Design a level α test to choose between

H_0: $\mu = \mu_0$,

H_1: $\mu \neq \mu_0$.

Solution

As discussed above, we let

$$W(X_1, X_2, \cdots, X_n) = \frac{\overline{X} - \mu_0}{\sigma / \sqrt{n}}.$$

Note that, assuming H_0, $W \sim N(0,1)$. We will choose a threshold, c. If $|W| \leq c$, we accept H_0, and if $|W| > c$, accept H_1. To choose c, we let

$$P(|W| > c \mid H_0) = \alpha.$$

Since the standard normal PDF is symmetric around 0, we have

$$P(|W| > c \mid H_0) = 2P(W > c|\ H_0).$$

Thus, we conclude $P(W > c|\ H_0) = \frac{\alpha}{2}$. Therefore,

$$c = z_{\frac{\alpha}{2}}.$$

Therefore, we accept H_0 if

$$\left| \frac{\overline{X} - \mu_0}{\sigma / \sqrt{n}} \right| \leq z_{\frac{\alpha}{2}},$$

and reject it otherwise.

Relation to Confidence Intervals: It is interesting to examine the above acceptance region. Here, we accept H_0 if

$$\left| \frac{\overline{X} - \mu_0}{\sigma / \sqrt{n}} \right| \leq z_{\frac{\alpha}{2}}.$$

We can rewrite the above condition as

$$\mu_0 \in \left[\overline{X} - z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}, \overline{X} + z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} \right].$$

The above interval should look familiar to you. It is the $(1-\alpha)100\%$ confidence interval for μ_0. This is not a coincidence as there is a general relationship between confidence interval problems and hypothesis testing problems.

Example 8.25
For the above example (Example 8.24), find β, the probability of type II error, as a function of μ.

Solution

We have

$$\begin{aligned}\beta(\mu) &= P(\text{type II error}) = P(\text{accept } H_0 \mid \mu)\\ &= P\left(\left|\frac{\overline{X}-\mu_0}{\sigma/\sqrt{n}}\right| < z_{\frac{\alpha}{2}} \mid \mu\right).\end{aligned}$$

If $X_i \sim N(\mu, \sigma^2)$, then $\overline{X} \sim N(\mu, \frac{\sigma^2}{n})$. Thus,

$$\begin{aligned}\beta(\mu) &= P\left(\left|\frac{\overline{X}-\mu_0}{\sigma/\sqrt{n}}\right| < z_{\frac{\alpha}{2}} \mid \mu\right)\\ &= P\left(\mu_0 - z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}} \leq \overline{X} \leq \mu_0 + z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\right)\\ &= \Phi\left(z_{\frac{\alpha}{2}} + \frac{\mu_0-\mu}{\sigma/\sqrt{n}}\right) - \Phi\left(-z_{\frac{\alpha}{2}} + \frac{\mu_0-\mu}{\sigma/\sqrt{n}}\right).\end{aligned}$$

Unknown variance: The above results (Example 8.25) can be extended to the case when we do not know the variance using the T distribution. More specifically, consider the following example.

Example 8.26
Let X_1,X_2,...,X_n be a random sample from a $N(\mu, \sigma^2)$ distribution, where μ and σ are unknown. Design a level α test to choose between

H_0: $\mu = \mu_0$,

H_1: $\mu \neq \mu_0$.

Solution

Let S^2 be the standard variance for this random sample. Then, the random variable W defined as

$$W(X_1, X_2, \cdots, X_n) = \frac{\overline{X} - \mu_0}{S/\sqrt{n}}$$

has a t-distribution with $n-1$ degrees of freedom, i.e., $W \sim T(n-1)$. Thus, we can repeat the analysis of Example 8.24 here. The only difference is that we need to replace σ by S and $z_{\frac{\alpha}{2}}$ by $t_{\frac{\alpha}{2},n-1}$. Therefore, we accept H_0 if

$$|W| \leq t_{\frac{\alpha}{2},n-1},$$

and reject it otherwise. Let us look at a numerical example of this case.

Example 8.27

The average adult male height in a certain country is 170 cm. We suspect that the men in a certain city in that country might have a different average height due to some environmental factors. We pick a random sample of size 9 from the adult males in the city and obtain the following values for their heights (in cm):

$$176.2 \quad 157.9 \quad 160.1 \quad 180.9 \quad 165.1 \quad 167.2 \quad 162.9 \quad 155.7 \quad 166.2$$

Assume that the height distribution in this population is normally distributed. Here, we need to decide between

H_0: $\mu = 170$,

H_1: $\mu \neq 170$.

Based on the observed data, is there enough evidence to reject H_0 at significance level $\alpha = 0.05$?

Solution

Let's first compute the sample mean and the sample standard deviation. The sample mean is

$$\begin{aligned}\overline{X} &= \frac{X_1+X_2+X_3+X_4+X_5+X_6+X_7+X_8+X_9}{9} \\ &= 165.8\end{aligned}$$

The sample variance is given by

$$S^2 = \frac{1}{9-1}\sum_{k=1}^{9}(X_k - \overline{X})^2 = 68.01$$

The sample standard deviation is given by

$$S = \sqrt{S^2} = 8.25$$

The following MATLAB code can be used to obtain these values:

```
x=[176.2,157.9,160.1,180.9,165.1,167.2,162.9,155.7,166.2];
m=mean(x);
v=var(x);
s=std(x);
```

Now, our test statistic is

$$\begin{aligned}W(X_1, X_2, \cdots, X_9) &= \frac{\overline{X}-\mu_0}{S/\sqrt{n}} \\ &= \frac{165.8-170}{8.25/3} = -1.52\end{aligned}$$

Thus, $|W| = 1.52$. Also, we have

$$t_{\frac{\alpha}{2},n-1} = t_{0.025,8} \approx 2.31$$

The above value can be obtained in MATLAB using the command `tinv(0.975,8)`. Thus, we conclude

$$|W| \leq t_{\frac{\alpha}{2},n-1}.$$

Therefore, we accept H_0. In other words, we do not have enough evidence to conclude that the average height in the city is different from the average height in the