9.1.8 Bayesian Hypothesis Testing

Suppose that we need to decide between two hypotheses H_0 and H_1. In the Bayesian setting, we assume that we know prior probabilities of H_0 and H_1. That is, we know $P(H_0) = p_0$ and $P(H_1) = p_1$, where $p_0 + p_1 = 1$. We observe the random variable (or the random vector) Y. We know the distribution of Y under the two hypotheses, i.e, we know

$$f_Y(y|H_0), \quad \text{and} \quad f_Y(y|H_1).$$

Using Bayes' rule, we can obtain the posterior probabilities of H_0 and H_1:

$$P(H_0|Y=y) = \frac{f_Y(y|H_0)P(H_0)}{f_Y(y)},$$
$$P(H_1|Y=y) = \frac{f_Y(y|H_1)P(H_1)}{f_Y(y)}.$$

One way to decide between H_0 and H_1 is to compare $P(H_0|Y=y)$ and $P(H_1|Y=y)$, and accept the hypothesis with the higher posterior probability. This is the idea behind the *maximum a posteriori (MAP) test*. Here, since we are choosing the hypothesis with the highest probability, it is relatively easy to show that the error probability is minimized.
To be more specific, according to the MAP test, we choose H_0 if and only if

$$P(H_0|Y=y) \geq P(H_1|Y=y).$$

In other words, we choose H_0 if and only if

$$f_Y(y|H_0)P(H_0) \geq f_Y(y|H_1)P(H_1).$$

Note that as always, we use the PMF instead of the PDF if Y is a discrete random variable. We can generalize the MAP test to the case where you have more than two hypotheses. In that case, again we choose the hypothesis with the highest posterior probability.

MAP Hypothesis Test

Choose the hypothesis with the highest posterior probability, $P(H_i|Y=y)$. Equivalently, choose hypothesis H_i with the highest $f_Y(y|H_i)P(H_i)$.

Example 9.10

Suppose that the random variable X is transmitted over a communication channel. Assume that the received signal is given by

$$Y = X + W,$$

where $W \sim N(0, \sigma^2)$ is independent of X. Suppose that $X = 1$ with probability p, and $X = -1$ with probability $1-p$. The goal is to decide between $X = 1$ and $X = -1$ by observing the random variable Y. Find the MAP test for this problem.

Solution

Here, we have two hypotheses:

H_0: $X = 1$,

H_1: $X = -1$.

Under H_0, $Y = 1 + W$, so $Y|H_0 \sim N(1, \sigma^2)$. Therefore,

$$f_Y(y|H_0) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y-1)^2}{2\sigma^2}}.$$

Under H_1, $Y = -1 + W$, so $Y|H_1 \sim N(-1, \sigma^2)$. Therefore,

$$f_Y(y|H_1) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y+1)^2}{2\sigma^2}}.$$

Thus, we choose H_0 if and only if

$$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y-1)^2}{2\sigma^2}} P(H_0) \geq \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y+1)^2}{2\sigma^2}} P(H_1).$$

We have $P(H_0) = p$, and $P(H_1) = 1 - p$. Therefore, we choose H_0 if and only if

$$\exp\left(\frac{2y}{\sigma^2}\right) \geq \frac{1-p}{p}.$$

Equivalently, we choose H_0 if and only if

$$y \geq \frac{\sigma^2}{2} \ln\left(\frac{1-p}{p}\right).$$

Note that the average error probability for a hypothesis test can be written as

$$P_e = P(\text{choose } H_1|H_0)P(H_0) + P(\text{choose } H_0|H_1)P(H_1). \qquad (9.6)$$

As we mentioned earlier, the MAP test achieves the minimum possible average error probability.

Example 9.11

Find the average error probability in Example 9.10

Solution

in Example 9.10, we arrived at the following decision rule: We choose H_0 if and only if

$$y \geq c,$$

where

$$c = \frac{\sigma^2}{2} \ln\left(\frac{1-p}{p}\right).$$

Since $Y|H_0 \ \sim \ N(1, \sigma^2)$,

$$\begin{aligned} P(\text{choose } H_1|H_0) &= P(Y < c|H_0) \\ &= \Phi\left(\frac{c-1}{\sigma}\right) \\ &= \Phi\left(\frac{\sigma}{2}\ln\left(\frac{1-p}{p}\right) - \frac{1}{\sigma}\right). \end{aligned}$$