4.1.4 Solved Problems: Continuous Random Variables

Problem 1

Let X be a random variable with PDF given by

$$f_X(x) = \begin{cases} cx^2 & |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

a. Find the constant c.
b. Find EX and Var(X).
c. Find $P(X \geq \frac{1}{2})$.

Solution

a. To find c, we can use $\int_{-\infty}^{\infty} f_X(u)du = 1$:

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} f_X(u)du \\ &= \int_{-1}^{1} cu^2 du \\ &= \frac{2}{3}c. \end{aligned}$$

Thus, we must have $c = \frac{3}{2}$.

b. To find EX, we can write

$$\begin{aligned} EX &= \int_{-1}^{1} u f_X(u)du \\ &= \frac{3}{2}\int_{-1}^{1} u^3 du \\ &= 0. \end{aligned}$$

In fact, we could have guessed $EX = 0$ because the PDF is symmetric around $x = 0$. To find Var(X), we have

$$\begin{aligned} \mathrm{Var}(X) &= EX^2 - (EX)^2 = EX^2 \\ &= \int_{-1}^{1} u^2 f_X(u)du \\ &= \frac{3}{2}\int_{-1}^{1} u^4 du \\ &= \frac{3}{5}. \end{aligned}$$

c. To find $P(X \geq \frac{1}{2})$, we can write

$$P(X \geq \frac{1}{2}) = \frac{3}{2}\int_{\frac{1}{2}}^{1} x^2 dx = \frac{7}{16}.$$

Problem 2

Let X be a continuous random variable with PDF given by

$$f_X(x) = \frac{1}{2}e^{-|x|}, \qquad \text{for all } x \in \mathbb{R}.$$

If $Y = X^2$, find the CDF of Y.

Solution

First, we note that $R_Y = [0, \infty)$. For $y \in [0, \infty)$, we have

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(X^2 \leq y) \\ &= P(-\sqrt{y} \leq X \leq \sqrt{y}) \\ &= \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{2}e^{-|x|} dx \\ &= \int_0^{\sqrt{y}} e^{-x} dx \\ &= 1 - e^{-\sqrt{y}}. \end{aligned}$$

Thus,

$$F_Y(y) = \begin{cases} 1 - e^{-\sqrt{y}} & y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Problem 3

Let X be a continuous random variable with PDF

$$f_X(x) = \begin{cases} 4x^3 & 0 < x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find $P(X \leq \frac{2}{3} | X > \frac{1}{3})$.

Solution

We have

$$\begin{aligned} P(X \leq \tfrac{2}{3} | X > \tfrac{1}{3}) &= \frac{P(\frac{1}{3} < X \leq \frac{2}{3})}{P(X > \frac{1}{3})} \\ &= \frac{\int_{\frac{1}{3}}^{\frac{2}{3}} 4x^3 dx}{\int_{\frac{1}{3}}^{1} 4x^3 dx} \\ &= \frac{3}{16}. \end{aligned}$$

Problem 4

Let X be a continuous random variable with PDF

$$f_X(x) = \begin{cases} x^2\left(2x + \frac{3}{2}\right) & \quad 0 < x \leq 1 \\ 0 & \quad \text{otherwise} \end{cases}$$

If $Y = \frac{2}{X} + 3$, find Var(Y).

Solution

First, note that

$$\text{Var}(Y) = \text{Var}\left(\frac{2}{X} + 3\right) = 4\text{Var}\left(\frac{1}{X}\right), \quad \text{using Equation 4.4}$$

Thus, it suffices to find Var$(\frac{1}{X}) = E[\frac{1}{X^2}] - (E[\frac{1}{X}])^2$. Using LOTUS, we have

$$E\left[\frac{1}{X}\right] = \int_0^1 x\left(2x + \frac{3}{2}\right) dx = \frac{17}{12}$$

$$E\left[\frac{1}{X^2}\right] = \int_0^1 \left(2x + \frac{3}{2}\right) dx = \frac{5}{2}.$$

Thus, Var$\left(\frac{1}{X}\right) = E[\frac{1}{X^2}] - (E[\frac{1}{X}])^2 = \frac{71}{144}$. So, we obtain

$$\text{Var}(Y) = 4\text{Var}\left(\frac{1}{X}\right) = \frac{71}{36}.$$

Problem 5

Let X be a <u>positive</u> continuous random variable. Prove that $EX = \int_0^\infty P(X \geq x)dx$.

Solution

We have

$$P(X \geq x) = \int_x^\infty f_X(t)dt.$$

Thus, we need to show that

$$\int_0^\infty \int_x^\infty f_X(t)dtdx = EX.$$

The left hand side is a double integral. In particular, it is the integral of $f_X(t)$ over the shaded region in Figure 4.4.

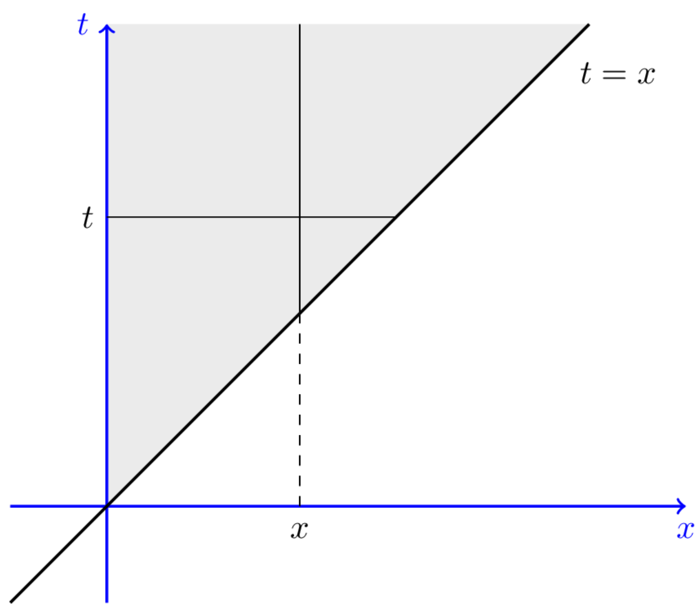

Fig.4.4 - The shaded area shows the region of the double integral of Problem 5.

We can take the integral with respect to x or t. Thus, we can write

$$\begin{aligned}\int_0^\infty \int_x^\infty f_X(t)dtdx &= \int_0^\infty \int_0^t f_X(t)dxdt \\ &= \int_0^\infty f_X(t)\left(\int_0^t 1dx\right)dt\end{aligned}$$

Problem 6

Let $X \sim Uniform(-\frac{\pi}{2}, \pi)$ and $Y = \sin(X)$. Find $f_Y(y)$.

Solution

Here $Y = g(X)$, where g is a differentiable function. Although g is not monotone, it can be divided to a finite number of regions in which it is monotone. Thus, we can use Equation 4.6. We note that since $R_X = [-\frac{\pi}{2}, \pi]$, $R_Y = [-1, 1]$. By looking at the plot of $g(x) = \sin(x)$ over $[-\frac{\pi}{2}, \pi]$, we notice that for $y \in (0, 1)$ there are two solutions to $y = g(x)$, while for $y \in (-1, 0)$, there is only one solution. In particular, if $y \in (0, 1)$, we have two solutions: $x_1 = \arcsin(y)$, and $x_2 = \pi - \arcsin(y)$. If $y \in (-1, 0)$ we have one solution, $x_1 = \arcsin(y)$. Thus, for $y \in (-1, 0)$, we have

$$\begin{aligned} f_Y(y) &= \frac{f_X(x_1)}{|g'(x_1)|} \\ &= \frac{f_X(\arcsin(y))}{|\cos(\arcsin(y))|} \\ &= \frac{\frac{2}{3\pi}}{\sqrt{1-y^2}}. \end{aligned}$$

For $y \in (0, 1)$, we have

$$\begin{aligned} f_Y(y) &= \frac{f_X(x_1)}{|g'(x_1)|} + \frac{f_X(x_2)}{|g'(x_2)|} \\ &= \frac{f_X(\arcsin(y))}{|\cos(\arcsin(y))|} + \frac{f_X(\pi - \arcsin(y))}{|\cos(\pi - \arcsin(y))|} \\ &= \frac{\frac{2}{3\pi}}{\sqrt{1-y^2}} + \frac{\frac{2}{3\pi}}{\sqrt{1-y^2}} \\ &= \frac{4}{3\pi\sqrt{1-y^2}}. \end{aligned}$$

To summarize, we can write

$$f_Y(y) = \begin{cases} \frac{2}{3\pi\sqrt{1-y^2}} & -1 < y < 0 \\ \frac{4}{3\pi\sqrt{1-y^2}} & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$