10.3.0 End of Chapter Problems

Problem 1

Let $\{X_n, n \in \mathbb{Z}\}$ be a discrete-time random process, defined as

$$X_n = 2\cos\left(\frac{\pi n}{8} + \Phi\right),$$

where $\Phi \sim Uniform(0, 2\pi)$.

a. Find the mean function, $\mu_X(n)$.
b. Find the correlation function $R_X(m, n)$.
c. Is X_n a WSS process?

Problem 2

Let $\{X(t), t \in \mathbb{R}\}$ be a continuous-time random process, defined as

$$X(t) = A\cos(2t + \Phi),$$

where $A \sim U(0, 1)$ and $\Phi \sim U(0, 2\pi)$ are two independent random variables.

a. Find the mean function $\mu_X(t)$.
b. Find the correlation function $R_X(t_1, t_2)$.
c. Is $X(t)$ a WSS process?

Problem 3

Let $\{X(n), n \in \mathbb{Z}\}$ be a WSS discrete-time random process with $\mu_X(n) = 1$ and $R_X(m, n) = e^{-(m-n)^2}$. Define the random process $Z(n)$ as

$$Z(n) = X(n) + X(n-1), \quad \text{for all } n \in \mathbb{Z}.$$

a. Find the mean function of $Z(n)$, $\mu_Z(n)$.
b. Find the autocorrelation function of $Z(n)$, $R_Z(m, n)$.
c. Is $Z(n)$ a WSS random process?

Problem 4

Let $g : \mathbb{R} \mapsto \mathbb{R}$ be a periodic function with period T, i.e.,

$$g(t+T) = g(t), \quad \text{for all } t \in \mathbb{R}.$$

Define the random process $\{X(t), t \in \mathbb{R}\}$ as

$$X(t) = g(t+U), \quad \text{for all } t \in \mathbb{R},$$

where $U \sim Uniform(0, T)$. Show that $X(t)$ is a WSS random process.

Problem 5

Let $\{X(t), t \in \mathbb{R}\}$ and $\{Y(t), t \in \mathbb{R}\}$ be two independent random processes. Let $Z(t)$ be defined as

$$Z(t) = X(t)Y(t), \quad \text{for all } t \in \mathbb{R}.$$

Prove the following statements:

a. $\mu_Z(t) = \mu_X(t)\mu_Y(t)$, for all $t \in \mathbb{R}$.
b. $R_Z(t_1, t_2) = R_X(t_1, t_2)R_Y(t_1, t_2)$, for all $t \in \mathbb{R}$.
c. If $X(t)$ and $Y(t)$ are WSS, then they are jointly WSS.
d. If $X(t)$ and $Y(t)$ are WSS, then $Z(t)$ is also WSS.
e. If $X(t)$ and $Y(t)$ are WSS, then $X(t)$ and $Z(t)$ are jointly WSS.

Problem 6

Let $X(t)$ be a Gaussian process such that for all $t > s \geq 0$ we have

$$X(t) - X(s) \sim N\left(0, t-s\right).$$

Show that $X(t)$ is mean-square continuous at any time $t \geq 0$.

Problem 7

Let $X(t)$ be a WSS Gaussian random process with $\mu_X(t) = 1$ and $R_X(\tau) = 1 + 4\text{sinc}(\tau)$
.

a. Find $P(1 < X(1) < 2)$.
b. Find $P(1 < X(1) < 2, X(2) < 3)$.

Problem 8
Let $X(t)$ be a Gaussian random process with $\mu_X(t) = 0$ and $R_X(t_1, t_2) = \min(t_1, t_2)$.
Find $P(X(4) < 3|X(1) = 1)$.

Problem 9
Let $\{X(t), t \in \mathbb{R}\}$ be a continuous-time random process, defined as

$$X(t) = \sum_{k=0}^{n} A_k t^k,$$

where A_0, A_1, $\cdots$, A_n are i.i.d. $N(0, 1)$ random variables and n is a fixed positive integer.

a. Find the mean function $\mu_X(t)$.
b. Find the correlation function $R_X(t_1, t_2)$.
c. Is $X(t)$ a WSS process?
d. Find $P(X(1) < 1)$. Assume $n = 10$.
e. Is $X(t)$ a Gaussian process?

Problem 10
(Complex Random Processes) In some applications, we need to work with complex-valued random processes. More specifically, a complex random process $X(t)$ can be written as

$$X(t) = X_r(t) + jX_i(t),$$

where $X_r(t)$ and $X_i(t)$ are two real-valued random processes and $j = \sqrt{-1}$. We define the mean function and the autocorrelation function as

$$
\begin{aligned}
\mu_X(t) &= E[X(t)] \\
&= E[X_r(t)] + jE[X_i(t)] \\
&= \mu_{X_r}(t) + j\mu_{X_i}(t);
\end{aligned}
$$

$$
\begin{aligned}
R_X(t_1,t_2) &= E[X(t_1)X^*(t_2)] \\
&= E\left[\big(X_r(t_1) + jX_i(t_1)\big)\big(X_r(t_2) - jX_i(t_2)\big)\right].
\end{aligned}
$$

Let $X(t)$ be a complex-valued random process defined as

$$
X(t) = Ae^{j(\omega t + \Phi)},
$$

where $\Phi \sim Uniform(0, 2\pi)$, and A is a random variable independent of Φ with $EA = \mu$ and $\mathrm{Var}(A) = \sigma^2$.

a. Find the mean function of $X(t)$, $\mu_X(t)$.
b. Find the autocorrelation function of $X(t)$, $R_X(t_1, t_2)$.

Problem 11

(Time Averages) Let $\{X(t), t \in \mathbb{R}\}$ be a continuous-time random process. The time average mean of $X(t)$ is defined as (assuming that the limit exists in mean-square sense)

$$
\langle X(t) \rangle = \lim_{T \to \infty} \left[\frac{1}{2T} \int_{-T}^{T} X(t) dt \right].
$$

Consider the random process $\big\{X(t), t \in \mathbb{R}\big\}$ defined as

$$
X(t) = \cos(t + U),
$$

where $U \sim Uniform(0, 2\pi)$. Find $\langle X(t) \rangle$.

Problem 12

(Ergodicity) Let $X(t)$ be a WSS process. We say that $X(t)$ is *mean ergodic* if $\langle X(t) \rangle$ (defined above) is equal to μ_X. Let A_0, A_1, A_{-1}, A_2, A_{-2}, $\cdots$ be a sequence of i.i.d. random variables with mean $EA_i = \mu < \infty$. Define the random process $\{X(t), t \in \mathbb{R}\}$ as

$$X(t) = \sum_{k=-\infty}^{\infty} A_k g(t-k),$$

where, $g(t)$ is given by

$$g(t) = \begin{cases} 1 & 0 \le t < 1 \\ \\ 0 & \text{otherwise} \end{cases}$$

Show that $X(t)$ is mean ergodic.

Problem 13

Let $\{X(t), t \in \mathbb{R}\}$ be a WSS random process. Show that for any $\alpha > 0$, we have

$$P\Big(|X(t+\tau) - X(t)| > \alpha\Big) \le \frac{2R_X(0) - 2R_X(\tau)}{\alpha^2}.$$

Problem 14

Let $\{X(t), t \in \mathbb{R}\}$ be a WSS random process. Suppose that $R_X(\tau) = R_X(0)$ for some $\tau > 0$. Show that, for any t, we have

$$X(t+\tau) = X(t), \quad \text{with probability one.}$$

Problem 15

Let $X(t)$ be a real-valued WSS random process with autocorrelation function $R_X(\tau)$. Show that the Power Spectral Density (PSD) of $X(t)$ is given by

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau) \cos(2\pi f \tau) \, d\tau.$$

Problem 16

Let $X(t)$ and $Y(t)$ be real-valued jointly WSS random processes. Show that

$$S_{YX}(f) = S^*_{XY}(f),$$

where, * shows the complex conjugate.

Problem 17

Let $X(t)$ be a WSS process with autocorrelation function

$$R_X(\tau) = \frac{1}{1+\pi^2\tau^2}.$$

Assume that $X(t)$ is input to a low-pass filter with frequency response

$$H(f) = \begin{cases} 3 & |f| < 2 \\ \\ 0 & \text{otherwise} \end{cases}$$

Let $Y(t)$ be the output.

a. Find $S_X(f)$.
b. Find $S_{XY}(f)$.
c. Find $S_Y(f)$.
d. Find $E[Y(t)^2]$.

Problem 18

Let $X(t)$ be a WSS process with autocorrelation function

$$R_X(\tau) = 1 + \delta(\tau).$$

Assume that $X(t)$ is input to an LTI system with impulse response

$$h(t) = e^{-t}u(t).$$

Let $Y(t)$ be the output.

a. Find $S_X(f)$.
b. Find $S_{XY}(f)$.
c. Find $R_{XY}(\tau)$.
d. Find $S_Y(f)$.
e. Find $R_Y(\tau)$.

f. Find $E[Y(t)^2]$.

Problem 19

Let $X(t)$ be a zero-mean WSS Gaussian random process with $R_X(\tau) = e^{-\pi\tau^2}$. Suppose that $X(t)$ is input to an LTI system with transfer function

$$|H(f)| = e^{-\frac{3}{2}\pi f^2}.$$

Let $Y(t)$ be the output.

a. Find μ_Y.
b. Find $R_Y(\tau)$ and $\mathrm{Var}(Y(t))$.
c. Find $E[Y(3)|Y(1) = -1]$.
d. Find $\mathrm{Var}(Y(3)|Y(1) = -1)$.
e. Find $P(Y(3) < 0|Y(1) = -1)$.

Problem 20

Let $X(t)$ be a white Gaussian noise with $S_X(f) = \frac{N_0}{2}$. Assume that $X(t)$ is input to a bandpass filter with frequency response

$$H(f) = \begin{cases} 2 & 1 < |f| < 3 \\ 0 & \text{otherwise} \end{cases}$$

Let $Y(t)$ be the output.

a. Find $S_Y(f)$.
b. Find $R_Y(\tau)$.
c. Find $E[Y(t)^2]$.