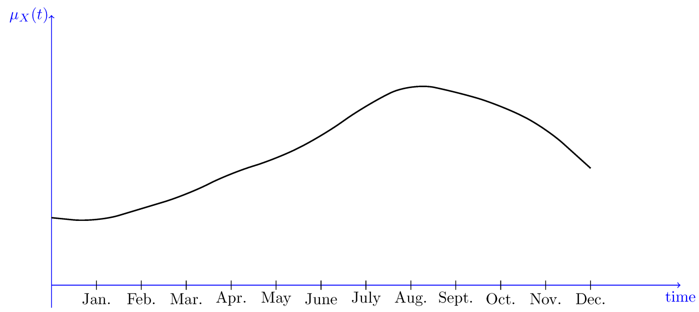

Figure 10.3 - The mean function, $\mu_X(t)$, for the temperature in a certain city.

Example 10.4

Find the mean functions for the random processes given in Examples 10.1 and 10.2.

Solution

For $\{X_n, n = 0, 1, 2, \cdots\}$ given in Example 10.1, we have

$$\begin{aligned}
\mu_X(n) &= E[X_n] \\
&= 1000E[Y^n] \quad \big(\text{where } Y = 1 + R \quad \sim \quad Uniform(1.04, 1.05)\big) \\
&= 1000 \int_{1.04}^{1.05} 100y^n \quad dy \quad (\text{by LOTUS}) \\
&= \frac{10^5}{n+1} \bigg[y^{n+1}\bigg]_{1.04}^{1.05} \\
&= \frac{10^5}{n+1} \bigg[(1.05)^{n+1} - (1.04)^{n+1}\bigg], \qquad \text{for all } n \in \{0, 1, 2, \cdots\}.
\end{aligned}$$

For $\{X(t), t \in [0, \infty)\}$ given in Example 10.2, we have

$$\begin{aligned}
\mu_X(t) &= E[X(t)] \\
&= E[A + Bt] \\
&= E[A] + E[B]t \\
&= 1 + t, \qquad \text{for all } t \in [0, \infty).
\end{aligned}$$

Autocorrelation and Autocovariance:

The mean function $\mu_X(t)$ gives us the expected value of $X(t)$ at time t, but it does not give us any information about how $X(t_1)$ and $X(t_2)$ are related. To get some insight on the relation between $X(t_1)$ and $X(t_2)$, we define correlation and covariance functions.

For a random process $\big\{X(t), t \in J\big\}$, the **autocorrelation function** or, simply, the **correlation function**, $R_X(t_1, t_2)$, is defined by

$$R_X(t_1,t_2) = E[X(t_1)X(t_2)], \quad \text{for } t_1, t_2 \in J.$$

For a random process $\big\{X(t), t \in J\big\}$, the **autocovariance function** or, simply, the **covariance function**, $C_X(t_1, t_2)$, is defined by

$$\begin{aligned} C_X(t_1,t_2) &= \mathrm{Cov}\big(X(t_1), X(t_2)\big) \\ &= R_X(t_1,t_2) - \mu_X(t_1)\mu_X(t_2), \quad \text{for } t_1, t_2 \in J. \end{aligned}$$

Note that if we let $t_1 = t_2 = t$, we obtain

$$\begin{aligned} R_X(t,t) &= E[X(t)X(t)] \\ &= E[X(t)^2], \quad \text{for } t \in J; \end{aligned}$$

$$\begin{aligned} C_X(t,t) &= \mathrm{Cov}\big(X(t), X(t)\big) \\ &= \mathrm{Var}\big(X(t)\big), \quad \text{for } t \in J. \end{aligned}$$

If $t_1 \neq t_2$, then the covariance function $C_X(t_1, t_2)$ gives us some information about how $X(t_1)$ and $X(t_2)$ are statistically related. In particular, note that

$$C_X(t_1,t_2) = E\Bigg[\bigg(X(t_1) - E\big[X(t_1)\big]\bigg)\bigg(X(t_2) - E\big[X(t_2)\big]\bigg)\Bigg].$$

Intuitively, $C_X(t_1, t_2)$ shows how $X(t_1)$ and $X(t_2)$ move relative to each other. If large values of $X(t_1)$ tend to imply large values of $X(t_2)$, then $\big(X(t_1) - E\big[X(t_1)\big]\big)\big(X(t_2) - E\big[X(t_2)\big]\big)$ is positive on average. In this case, $C_X(t_1, t_2)$ is positive, and we say $X(t_1)$ and $X(t_2)$ are positively correlated. On the other hand, if large values of $X(t_1)$ imply small values of $X(t_2)$, then $\big(X(t_1) - E\big[X(t_1)\big]\big)\big(X(t_2) - E\big[X(t_2)\big]\big)$ is negative on average, and we say $X(t_1)$ and $X(t_2)$ are negatively correlated. If $C_X(t_1, t_2) = 0$, then $X(t_1)$ and $X(t_2)$ are uncorrelated.

Example 10.5

Find the correlation functions and covariance functions for the random processes given in Examples 10.1 and 10.2.

Solution

For $\{X_n, n = 0, 1, 2, \cdots\}$ given in Example 10.1, we have

$$\begin{aligned} R_X(m,n) &= E[X_m X_n] \\ &= 10^6 E[Y^m Y^n] \quad \big(\text{where } Y = 1+R \quad \sim \quad Uniform(1.04, 1.05)\big) \\ &= 10^6 \int_{1.04}^{1.05} 100 y^{(m+n)} \quad dy \quad (\text{by LOTUS}) \\ &= \frac{10^8}{m+n+1}\Big[y^{m+n+1}\Big]_{1.04}^{1.05} \\ &= \frac{10^8}{m+n+1}\Big[(1.05)^{m+n+1} - (1.04)^{m+n+1}\Big], \qquad \text{for all } m,n \in \{0,1,2,\cdots\}. \end{aligned}$$

To find the covariance function, we write

$$\begin{aligned} C_X(m,n) &= R_X(m,n) - E[X_m]E[X_n] \\ &= \frac{10^8}{m+n+1}\Big[(1.05)^{m+n+1} - (1.04)^{m+n+1}\Big] \\ &\quad - \frac{10^{10}}{(m+1)(n+1)}\Big[(1.05)^{m+1} - (1.04)^{m+1}\Big]\Big[(1.05)^{n+1} - (1.04)^{n+1}\Big]. \end{aligned}$$

For $\{X(t), t \in [0,\infty)\}$ given in Example 10.2, we have

$$\begin{aligned} R_X(t_1,t_2) &= E[X(t_1)X(t_2)] \\ &= E[(A+Bt_1)(A+Bt_2)] \\ &= E[A^2] + E[AB](t_1+t_2) + E[B^2]t_1 t_2 \\ &= 2 + E[A]E[B](t_1+t_2) + 2t_1t_2 \quad (\text{since } A \text{ and } B \text{ are independent}) \\ &= 2 + t_1 + t_2 + 2t_1t_2, \qquad \text{for all } t_1, t_2 \in [0,\infty). \end{aligned}$$

Finally, to find the covariance function for $X(t)$, we can write

$$\begin{aligned} C_X(t_1,t_2) &= R_X(t_1,t_2) - E[X(t_1)]E[X(t_2)] \\ &= 2 + t_1 + t_2 + 2t_1t_2 - (1+t_1)(1+t_2) \\ &= 1 + t_1t_2, \qquad \text{for all } t_1, t_2 \in [0,\infty). \end{aligned}$$
