Thus,

$$f_X(x) = \begin{cases} \frac{2}{\pi}\sqrt{1-x^2} & -1 \leq x \leq 1 \\ & \\ 0 & \text{otherwise} \end{cases}$$

Similarly,

$$f_Y(y) = \begin{cases} \frac{2}{\pi}\sqrt{1-y^2} & -1 \leq y \leq 1 \\ & \\ 0 & \text{otherwise} \end{cases}$$

c. We have

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f_{XY}(x,y)}{f_Y(y)} \\ &= \begin{cases} \frac{1}{2\sqrt{1-y^2}} & -\sqrt{1-y^2} \leq x \leq \sqrt{1-y^2} \\ & \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Note that the above equation indicates that, given $Y=y$, X is uniformly distributed on $[-\sqrt{1-y^2}, \sqrt{1-y^2}]$. We write

$$X|Y=y \;\; \sim \;\; Uniform(-\sqrt{1-y^2}, \sqrt{1-y^2}).$$

d. Are X and Y independent? No, because $f_{XY}(x,y) \neq f_X(x)f_Y(y)$.

Law of Total Probability:

Now, we'll discuss the law of total probability for continuous random variables. This is completely analogous to the discrete case. In particular, the law of total probability, the law of total expectation (law of iterated expectations), and the law of total variance can be stated as follows:

Law of Total Probability:

$$P(A) = \int_{-\infty}^{\infty} P(A|X=x) f_X(x) \ dx \qquad (5.16)$$

Law of Total Expectation:

$$\begin{aligned} E[Y] &= \int_{-\infty}^{\infty} E[Y|X=x] f_X(x) \ dx \qquad (5.17) \\ &= E[E[Y|X]] \end{aligned}$$

Law of Total Variance:

$$\mathrm{Var}(Y) = E[\mathrm{Var}(Y|X)] + \mathrm{Var}(E[Y|X]) \qquad (5.18)$$

Let's look at some examples.

Example 5.25

Let X and Y be two independent $Uniform(0,1)$ random variables. Find $P(X^3 + Y > 1)$
.

Solution

Using the law of total probability (Equation 5.16), we can write

$$
\begin{aligned}
P(X^3+Y>1) &= \int_{-\infty}^{\infty} P(X^3+Y>1|X=x)f_X(x)\ \ dx \\
&= \int_0^1 P(x^3+Y>1|X=x)\ \ dx \\
&= \int_0^1 P(Y>1-x^3)\ \ dx & \text{(since } X \text{ and } Y \text{ are independent)} \\
&= \int_0^1 x^3\ \ dx & \text{(since } Y \sim Uniform(0,1)\text{)} \\
&= \frac{1}{4}.
\end{aligned}
$$

Example 5.26

Suppose $X \sim Uniform(1,2)$ and given $X=x$, Y is an exponential random variable with parameter $\lambda = x$, so we can write

$$Y|X=x \quad \sim \quad Exponential(x).$$

We sometimes write this as

$$Y|X \quad \sim \quad Exponential(X).$$

a. Find EY.
b. Find $Var(Y)$.

Solution

a. We use the law of total expectation (Equation 5.17) to find EY. Remember that if $Y \sim Exponential(\lambda)$, then $EY = \frac{1}{\lambda}$. Thus we conclude

$$E[Y|X=x] = \frac{1}{x}.$$

Using the law of total expectation, we have

$$
\begin{aligned}
EY &= \int_{-\infty}^{\infty} E[Y|X=x]f_X(x)dx \\
&= \int_1^2 E[Y|X=x] \cdot 1dx \\
&= \int_1^2 \frac{1}{x}dx \\
&= \ln 2.
\end{aligned}
$$

Another way to write the above calculation is

$$\begin{aligned} EY &= E[E[Y|X]] && \text{(law of total expectation)} \\ &= E\left[\frac{1}{X}\right] && (\text{since } E[Y|X] = \frac{1}{X}) \\ &= \int_1^2 \frac{1}{x} dx \\ &= \ln 2. \end{aligned}$$

b. To find $Var(Y)$, we can write

$$\begin{aligned} Var(Y) &= E[Y^2] - (E[Y])^2 \\ &= E[Y^2] - (\ln 2)^2 \\ &= E\big[E[Y^2|X]\big] - (\ln 2)^2 && \text{(law of total expectation)} \\ &= E\left[\frac{2}{X^2}\right] - (\ln 2)^2 && \big(\text{since } Y|X \sim Exponential(X)\big) \\ &= \int_1^2 \frac{2}{x^2} dx - (\ln 2)^2 \\ &= 1 - (\ln 2)^2. \end{aligned}$$

Another way to find $Var(Y)$ is to apply the law of total variance:

$$\mathrm{Var}(Y) = E[\mathrm{Var}(Y|X)] + \mathrm{Var}(E[Y|X]).$$

Since $Y|X \sim Exponential(X)$, we conclude

$$\begin{aligned} E[Y|X] &= \frac{1}{X}, \\ Var(Y|X) &= \frac{1}{X^2}. \end{aligned}$$

Therefore

$$\begin{aligned} \mathrm{Var}(Y) &= E\left[\frac{1}{X^2}\right] + \mathrm{Var}\left(\frac{1}{X}\right) \\ &= E\left[\frac{1}{X^2}\right] + E\left[\frac{1}{X^2}\right] - \left(E\left[\frac{1}{X}\right]\right)^2 \\ &= E\left[\frac{2}{X^2}\right] - (\ln 2)^2 \\ &= 1 - (\ln 2)^2. \end{aligned}$$