heads in n coin flips. We conclude that $N(t) \sim Binomial(n, p)$. Note that here $p = \lambda\delta$, so

$$\begin{aligned} np &= n\lambda\delta \\ &= \frac{t}{\delta} \cdot \lambda\delta \\ &= \lambda t. \end{aligned}$$

Thus, by Theorem 11.1, as $\delta \to 0$, the PMF of $N(t)$ converges to a Poisson distribution with rate λt. More generally, we can argue that the number of arrivals in any interval of length τ follows a $Poisson(\lambda\tau)$ distribution as $\delta \to 0$.

Consider several non-overlapping intervals. The number of arrivals in each interval is determined by the results of the coin flips for that interval. Since different coin flips are independent, we conclude that the above counting process has independent increments.

Definition of the Poisson Process:

The above construction can be made mathematically rigorous. The resulting random process is called a Poisson process with rate (or intensity) λ. Here is a formal definition of the Poisson process.

The Poisson Process

Let $\lambda > 0$ be fixed. The counting process $\{N(t), t \in [0, \infty)\}$ is called a **Poisson process** with **rate**s λ if all the following conditions hold:

1. $N(0) = 0$;
2. $N(t)$ has independent increments;
3. the number of arrivals in any interval of length $\tau > 0$ has $Poisson(\lambda\tau)$ distribution.

Note that from the above definition, we conclude that in a Poisson process, the distribution of the number of arrivals in any interval depends only on the length of the interval, and not on the exact location of the interval on the real line. Therefore the *Poisson process has stationary increments*.