10.1.0 Basic Concepts

In real-life applications, we are often interested in multiple observations of random values over a period of time. For example, suppose that you are observing the stock price of a company over the next few months. In particular, let $S(t)$ be the stock price at time $t \in [0, \infty)$. Here, we assume $t = 0$ refers to current time. Figure 10.1 shows a possible outcome of this random experiment from time $t = 0$ to time $t = 1$.

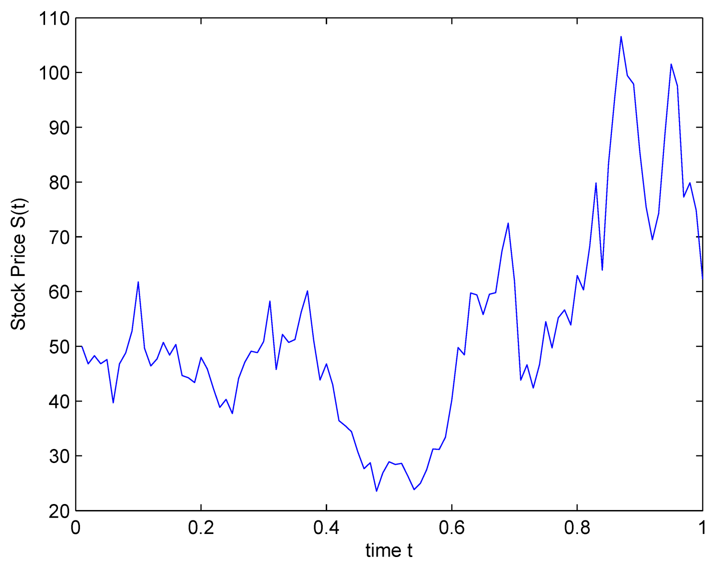

Figure 10.1 - A possible realization of values of a stock observed as a function of time. Here, $S(t)$ is an example of a random process.

Note that at any fixed time $t_1 \in [0, \infty)$, $S(t_1)$ is a random variable. Based on your knowledge of finance and the historical data, you might be able to provide a PDF for $S(t_1)$. If you choose another time $t_2 \in [0, \infty)$, you obtain another random variable $S(t_2)$ that could potentially have a different PDF. When we consider the values of $S(t)$ for $t \in [0, \infty)$ collectively, we say $S(t)$ is a **random process** or a **stochastic process**. We may show this process by

$$\{S(t), t \in [0, \infty)\}.$$

Therefore, a random process is a collection of random variables usually indexed by time (or sometimes by space).

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The process $S(t)$ mentioned here is an example of a **continuous-time** random process. In general, when we have a random process $X(t)$ where t can take real values in an interval on the real line, then $X(t)$ is a continuous-time random process. Here are a few more examples of continuous-time random processes:

– Let $N(t)$ be the number of customers who have visited a bank from $t = 9$ (when the bank opens at 9:00 am) until time t, on a given day, for $t \in [9, 16]$. Here, we measure t in hours, but t can take any real value between 9 and 16. We assume that $N(9) = 0$, and $N(t) \in \{0, 1, 2, \ldots\}$ for all $t \in [9, 16]$. Note that for any time t_1, the random variable $N(t_1)$ is a discrete random variable. Thus, $N(t)$ is a *discrete-valued* random process. However, since t can take any real value between 9 and 16, $N(t)$ is a continuous-time random process.

– Let $W(t)$ be the thermal noise voltage generated across a resistor in an electric circuit at time t, for $t \in [0, \infty)$. Here, $W(t)$ can take real values.

– Let $T(t)$ be the temperature in New York City at time $t \in [0, \infty)$. We can assume here that t is measured in hours and $t = 0$ refers to the time we start measuring the temperature.

In all of these examples, we are dealing with an uncountable number of random variables. For example, for any given $t_1 \in [9, 16]$, $N(t_1)$ is a random variable. Thus, the random process $N(t)$ consists of an uncountable number of random variables. A random process can be defined on the entire real line, i.e., $t \in (-\infty, \infty)$. In fact, it is sometimes convenient to assume that the process starts at $t = -\infty$ even if we are interested in $X(t)$ only on a finite interval. For example, we can assume that the $T(t)$ defined above is a random process defined for all $t \in \mathbb{R}$ although we get to observe only a finite portion of it.
On the other hand, you can have a **discrete-time** random process. A discrete-time random process is a process

$$\big\{X(t), t \in J\big\},$$

where J is a countable set. Since J is countable, we can write $J = \{t_1, t_2, \cdots\}$. We usually define $X(t_n) = X(n)$ or $X(t_n) = X_n$, for $n = 1, 2, \cdots$, (the index values n could be from any countable set such as $\mathbb{N}$ or $\mathbb{Z}$). Therefore, a discrete-time random process is just a sequence of random variables. For this reason, discrete-time random processes are sometimes referred to as **random sequences**. We can denote such a discrete-time process as

$$\big\{X(n), n = 0, 1, 2, \dots\big\} \quad \text{or} \quad \big\{X_n, n = 0, 1, 2, \dots\big\}.$$

Or, if the process is defined for all integers, then we may show the process by

$$\big\{X(n), n \in \mathbb{Z}\big\} \quad \text{or} \quad \big\{X_n, n \in \mathbb{Z}\big\}.$$

Here is an example of a discrete-time random process. Suppose that we are observing customers who visit a bank starting at a given time. Let X_n for $n \in \mathbb{N}$ be the amount of time the ith customer spends at the bank. This process consists of a countable number of random variables

$$X_1, X_2, X_3, \dots$$

Thus, we say that the process $\big\{X_n, n = 1, 2, 3..\big\}$ is a discrete-time random process. Discrete-time processes are sometimes obtained from continuous-time processes by discretizing time (sampling at specific times). For example, if you only record the temperature in New York City once a day (let's say at noon), then you can define a process

$$\begin{aligned}
X_1 &= T(12) & \text{(temperature at noon on day } 1, t = 12) \\
X_2 &= T(36) & \text{(temperature at noon on day } 2, t = 12 + 24) \\
X_3 &= T(60) & \text{(temperature at noon on day } 3, t = 12 + 24 + 24) \\
&\dots
\end{aligned}$$

And, in general, $X_n = T(t_n)$ where $t_n = 24(n-1) + 12$ for $n \in \mathbb{N}$. Here, X_n is a discrete-time random process. Figure 10.2 shows a possible realization of this random process.