$$\left[\overline{X}-t_{\frac{\alpha}{2},n-1}\frac{S}{\sqrt{n}},\overline{X}+t_{\frac{\alpha}{2},n-1}\frac{S}{\sqrt{n}}\right]=$$

$$=\left[9.26-2.26\cdot\frac{\sqrt{3.96}}{\sqrt{10}},9.26+2.26\cdot\frac{\sqrt{3.96}}{\sqrt{10}}\right]$$

$$=[7.84,10.68].$$

Therefore, $[7.84, 10.68]$ is a 95% confidence interval for μ.

Confidence Intervals for the Variance of Normal Random Variables

Now, suppose that we would like to estimate the variance of a normal distribution. More specifically, assume that X_1, X_2, X_3, $\dots$, X_n is a random sample from a normal distribution $N(\mu, \sigma^2)$, and our goal is to find an interval estimator for σ^2. We assume that μ is also unknown. Again, n could be any positive integer. By Theorem 8.3, the random variable Y defined as

$$Q=\frac{(n-1)S^2}{\sigma^2}=\frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i-\overline{X})^2$$

has a chi-squared distribution with $n-1$ degrees of freedom, i.e., $Q \sim \chi^2(n-1)$. In particular, Q is a pivotal quantity since it is a function of the X_i's and σ^2, and its distribution does not depend on σ^2 or any other unknown parameters. Using the definition of $\chi^2_{p,n}$, a $(1-\alpha)$ interval for Q can be stated as

$$P\left(\chi^2_{1-\frac{\alpha}{2},n-1}\leq Q\leq\chi^2_{\frac{\alpha}{2},n-1}\right)=1-\alpha.$$

Therefore,

$$P\left(\chi^2_{1-\frac{\alpha}{2},n-1}\leq\frac{(n-1)S^2}{\sigma^2}\leq\chi^2_{\frac{\alpha}{2},n-1}\right)=1-\alpha.$$

which is equivalent to

$$P\left(\frac{(n-1)S^2}{\chi^2_{\frac{\alpha}{2},n-1}}\leq\sigma^2\leq\frac{(n-1)S^2}{\chi^2_{1-\frac{\alpha}{2},n-1}}\right)=1-\alpha.$$

We conclude that $\left[\frac{(n-1)S^2}{\chi^2_{\frac{\alpha}{2},n-1}},\frac{(n-1)S^2}{\chi^2_{1-\frac{\alpha}{2},n-1}}\right]$ is a $(1-\alpha)100\%$ confidence interval for σ^2.

Assumptions: A random sample X_1, X_2, X_3, ..., X_n is given from a $N(\mu, \sigma^2)$ distribution, where $\mu = EX_i$ and $\text{Var}(X_i) = \sigma^2$ are unknown.

Parameter to be Estimated: $\text{Var}(X_i) = \sigma^2$.

Confidence Interval: $\left[\frac{(n-1)S^2}{\chi^2_{\frac{\alpha}{2},n-1}}, \frac{(n-1)S^2}{\chi^2_{1-\frac{\alpha}{2},n-1}}\right]$ is a $(1-\alpha)100\%$ confidence interval for σ^2.

Example 8.21

For the data given in Example 8.20, find a 95% confidence interval for σ^2. Again, assume that the weight is normally distributed with mean μ and and variance σ, where μ and σ are unknown.

Solution

As before, using the data we obtain

$$\overline{X} = 9.26,$$
$$S^2 = 3.96$$

Here, $n = 10$, $\alpha = 0.05$, so we need

$$\chi^2_{0.025,9} = 19.02, \quad \chi^2_{0.975,9} = 2.70$$

The above values can obtained in MATLAB using the commands `chi2inv(0.975,9)` and `chi2inv(0.025,9)`, respectively. Thus, we can obtain a 95% confidence interval for σ^2 as

$$\left[\frac{(n-1)S^2}{\chi^2_{\frac{\alpha}{2},n-1}}, \frac{(n-1)S^2}{\chi^2_{1-\frac{\alpha}{2},n-1}}\right] = \left[\frac{9 \times 3.96}{19.02}, \frac{9 \times 3.96}{2.70}\right]$$
$$= [1.87, 13.20].$$

Therefore, $[1.87, 13.20]$ is a 95% confidence interval for σ^2.
