From the recursion, we obtain $a_0 = 1$, so we have the following equation:

$$a_n = (1 - 2p)a_{n-1} + p, \qquad \text{with } a_0 = 1.$$

This recursion is in the form given in problem 6 ($\alpha = 1 - 2p$, $\beta = p$), so we obtain

$$\begin{aligned} a_n &= (1 - 2p)^n + p\left(\frac{1 - (1 - 2p)^n}{2p}\right) \\ &= \frac{1 + (1 - 2p)^n}{2}. \end{aligned}$$

14.2 End of Chapter Problems

1. Solve the following recurrence equations, that is, find a closed form formula for a_n.

   (a) $a_n = 2a_{n-1} - \frac{3}{4}a_{n-2}$, with $a_0 = 0, a_1 = -1$.

   (b) $a_n = 4a_{n-1} - 4a_{n-2}$, with $a_0 = 2, a_1 = 6$.

2. I toss a biased coin n times. Let $P(H) = p$ and let $a_{n,k}$ be the probability that I observe k heads.

   (a) By conditioning, on the last coin toss, show that $a_{n+1,k+1} = p \cdot a_{n,k} + (1 - p) \cdot a_{n,k+1}$.

   (b) Using part (a), prove that for $0 \leq k < n$, we have $\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k}$.

3. * You toss a biased coin repeatedly. If $P(H) = p$, what is the probability that two consecutive Hs are observed before we observe two consecutive Ts? For example, this event happens if the observed sequence is $THT\underline{HH}THTT\cdots$.

4. I toss a biased coin n times and record the sequence of heads and tails. Assume $P(H) = p$ (where $0 < p < 1$). Let a_n be the probability that the number of heads is divisible by 3. Write a set of recursive equations to compute a_n.