Merging Independent Poisson Processes

Let $N_1(t)$, $N_2(t)$, $\cdots$, $N_m(t)$ be m independent Poisson processes with rates λ_1, λ_2, $\cdots$, λ_m. Let also

$$N(t) = N_1(t) + N_2(t) + \cdots + N_m(t), \quad \text{for all } t \in [0, \infty).$$

Then, $N(t)$ is a Poisson process with rate $\lambda_1 + \lambda_2 + \cdots + \lambda_m$.

Splitting (Thinning) of Poisson Processes:

Here, we will talk about splitting a Poisson process into two independent Poisson processes. The idea will be better understood if we look at a concrete example.

Example 11.3

Suppose that the number of customers visiting a fast food restaurant in a given time interval I is $N \sim Poisson(\mu)$. Assume that each customer purchases a drink with probability p, independently from other customers, and independently from the value of N. Let X be the number of customers who purchase drinks in that time interval. Also, let Y be the number of customers that do not purchase drinks; so $X + Y = N$.

a. Find the marginal PMFs of X and Y.
b. Find the joint PMF of X and Y.
c. Are X and Y independent?

Solution

a. First, note that $R_X = R_Y = \{0, 1, 2, \dots\}$. Also, given $N = n$, X is a sum of n independent $Bernoulli(p)$ random variables. Thus, given $N = n$, X has a binomial distribution with parameters n and p, so

$$\begin{aligned} X|N = n &\sim Binomial(n, p), \\ Y|N = n &\sim Binomial(n, q = 1 - p). \end{aligned}$$

We have

$$
\begin{aligned}
P_X(k) &= \sum_{n=0}^{\infty} P(X=k|N=n)P_N(n) && \text{(law of total probability)}\\
&= \sum_{n=k}^{\infty} \binom{n}{k} p^k q^{n-k} e^{-\mu} \frac{\mu^n}{n!}\\
&= \sum_{n=k}^{\infty} \frac{p^k q^{n-k} e^{-\mu} \mu^n}{k!(n-k)!}\\
&= \frac{e^{-\mu}(\mu p)^k}{k!} \sum_{n=k}^{\infty} \frac{(\mu q)^{n-k}}{(n-k)!}\\
&= \frac{e^{-\mu}(\mu p)^k}{k!} e^{\mu q} && \text{(Taylor series for } e^x\text{)}\\
&= \frac{e^{-\mu p}(\mu p)^k}{k!}, \qquad \text{for } k = 0,1,2,\dots
\end{aligned}
$$

Thus, we conclude that

$$X \sim Poisson(\mu p).$$

Similarly, we obtain

$$Y \sim Poisson(\mu q).$$

b. To find the joint PMF of X and Y, we can also use the law of total probability:

$$P_{XY}(i,j) = \sum_{n=0}^{\infty} P(X=i, Y=j|N=n)P_N(n) \qquad \text{(law of total probability).}$$

However, note that $P(X=i, Y=j|N=n) = 0$ if $N \neq i+j$, thus

$$
\begin{aligned}
P_{XY}(i,j) &= P(X=i, Y=j|N=i+j)P_N(i+j)\\
&= P(X=i|N=i+j)P_N(i+j)\\
&= \binom{i+j}{i} p^i q^j e^{-\mu} \frac{\mu^{i+j}}{(i+j)!}\\
&= \frac{e^{-\mu}(\mu p)^i(\mu q)^j}{i!j!}\\
&= \frac{e^{-\mu p}(\mu p)^i}{i!} \cdot \frac{e^{-\mu q}(\mu q)^j}{j!}\\
&= P_X(i)P_Y(j).
\end{aligned}
$$

c. X and Y are independent since, as we saw above,

$$P_{XY}(i,j) = P_X(i)P_Y(j).$$

The above example was given for a specific interval I, in which a Poisson random variable N was split to two independent Poisson random variables X and Y. However, the argument can be used to show the same result for splitting a Poisson process to two independent Poisson processes. More specifically, we have the following result.

Splitting a Poisson Processes

Let $N(t)$ be a Poisson process with rate λ. Here, we divide $N(t)$ to two processes $N_1(t)$ and $N_2(t)$ in the following way (Figure 11.6). For each arrival, a coin with $P(H) = p$ is tossed. If the coin lands heads up, the arrival is sent to the first process ($N_1(t)$), otherwise it is sent to the second process. The coin tosses are independent of each other and are independent of $N(t)$. Then,

1. $N_1(t)$ is a Poisson process with rate λp;
2. $N_2(t)$ is a Poisson process with rate $\lambda(1-p)$;
3. $N_1(t)$ and $N_2(t)$ are independent.

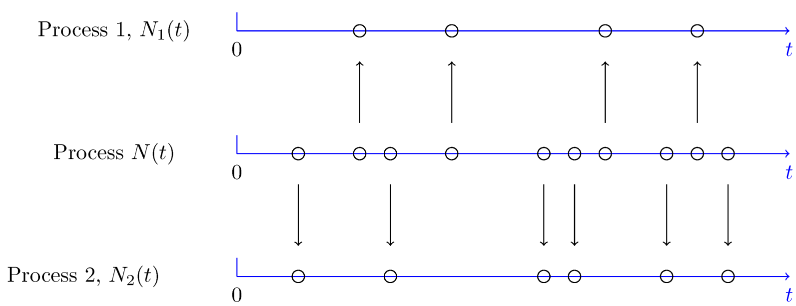

Figure 11.6 - Splitting a Poisson process to two independent Poisson processes.