3.3 End of Chapter Problems

Problem 1

Let X be a discrete random variable with the following PMF:

$$P_X(x) = \begin{cases} \frac{1}{2} & \text{for } x = 0 \\ \frac{1}{3} & \text{for } x = 1 \\ \frac{1}{6} & \text{for } x = 2 \\ 0 & \text{otherwise} \end{cases}$$

a. Find R_X, the range of the random variable X.
b. Find $P(X \geq 1.5)$.
c. Find $P(0 < X < 2)$.
d. Find $P(X = 0|X < 2)$

Problem 2

Let X be the number of the cars being repaired at a repair shop. We have the following information:

- At any time, there are at most 3 cars being repaired.
- The probability of having 2 cars at the shop is the same as the probability of having one car.
- The probability of having no car at the shop is the same as the probability of having 3 cars.
- The probability of having 1 or 2 cars is half of the probability of having 0 or 3 cars.

Find the PMF of X.

Problem 3

I roll two dice and observe two numbers X and Y. If $Z = X - Y$, find the range and PMF of Z.

Problem 4

Let X and Y be two independent discrete random variables with the following PMFs:

$$P_X(k) = \begin{cases} \frac{1}{4} & \text{for } k = 1 \\ \frac{1}{8} & \text{for } k = 2 \\ \frac{1}{8} & \text{for } k = 3 \\ \frac{1}{2} & \text{for } k = 4 \\ 0 & \text{otherwise} \end{cases}$$

and

$$P_Y(k) = \begin{cases} \frac{1}{6} & \text{for } k = 1 \\ \frac{1}{6} & \text{for } k = 2 \\ \frac{1}{3} & \text{for } k = 3 \\ \frac{1}{3} & \text{for } k = 4 \\ 0 & \text{otherwise} \end{cases}$$

a. Find $P(X \leq 2 \text{ and } Y \leq 2)$.
b. Find $P(X > 2 \text{ or } Y > 2)$.
c. Find $P(X > 2|Y > 2)$.
d. Find $P(X < Y)$.

Problem 5
50 students live in a dormitory. The parking lot has the capacity for 30 cars. If each student has a car with probability $\frac{1}{2}$ (independently from other students), what is the probability that there won't be enough parking spaces for all the cars?

Problem 6 (The Matching Problem)
N guests arrive at a party. Each person is wearing a hat. We collect all the hats and then randomly redistribute the hats, giving each person one of the N hats randomly. Let X_N be the number of people who receive their own hats. Find the PMF of X_N.

Hint: We previously found that (Problem 7 in Section 2.1.5)

$$P(X_N = 0) = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots (-1)^N \frac{1}{N!}$$

, for $N = 1, 2, \cdots$. Using this, find $P(X_N = k)$ for all $k \in \{0, 1, \cdots N\}$.

Problem 7

For each of the following random variables, find $P(X > 5)$, $P(2 < X \leq 6)$ and $P(X > 5|X < 8)$.

a. $X \sim Geometric(\frac{1}{5})$
b. $X \sim Binomial(10, \frac{1}{3})$
c. $X \sim Pascal(3, \frac{1}{2})$
d. $X \sim Hypergeometric(10, 10, 12)$
e. $X \sim Poisson(5)$

Problem 8

Suppose you take a pass-fail test repeatedly. Let S_k be the event that you are successful in your k^{th} try, and F_k be the event that you fail the test in your k^{th} try. On your first try, you have a 50 percent chance of passing the test.

$$P(S_1) = 1 - P(F_1) = \frac{1}{2}.$$

Assume that as you take the test more often, your chance of failing the test goes down. In particular,

$$P(F_k) = \frac{1}{2} \cdot P(F_{k-1}), \qquad \text{for } k = 2, 3, 4, \cdots$$

However, the result of different exams are independent. Suppose you take the test repeatedly until you pass the test for the first time. Let X be the total number of tests you take, so $Range(X) = \{1, 2, 3, \cdots\}$.

a. Find $P(X = 1), P(X = 2), P(X = 3)$.
b. Find a general formula for $P(X = k)$ for $k = 1, 2, \cdots$.
c. Find the probability that you take the test more than 2 times.
d. Given that you take the test more than once, find the probability that you take the test exactly twice.

Problem 9

In this problem, we would like to show that the geometric random variable is **memoryless**. Let $X \sim Geometric(p)$. Show that

$$P(X > m + l|X > m) = P(X > l), \text{ for } m, l \in \{1, 2, 3, \cdots\}.$$

We can interpret this in the following way: Remember that a geometric random variable can be obtained by tossing a coin repeatedly until observing the first heads. If we toss the coin several times, and do not observe a heads, from now on it is like we start all over again. In other words, the failed coin tosses do not impact the distribution of waiting time from this point forward. The reason for this is that the coin tosses are independent.

Problem 10

An urn consists of 20 red balls and 30 green balls. We choose 10 balls at random from the urn. The sampling is done **without** replacement (repetition not allowed).

a. What is the probability that there will be exactly 4 red balls among the chosen balls?
b. Given that there are at least 3 red balls among the chosen balls, what is the probability that there are exactly 4 red balls?

Problem 11

The number of emails that I get in a weekday (Monday through Friday) can be modeled by a Poisson distribution with an average of $\frac{1}{6}$ emails per minute. The number of emails that I receive on weekends (Saturday and Sunday) can be modeled by a Poisson distribution with an average of $\frac{1}{30}$ emails per minute.

a. What is the probability that I get no emails in an interval of length 4 hours on a Sunday?
b. A random day is chosen (all days of the week are equally likely to be selected), and a random interval of length one hour is selected on the chosen day. It is observed that I did not receive any emails in that interval. What is the probability that the chosen day is a weekday?

Problem 12

Let X be a discrete random variable with the following PMF:

$$P_X(x) = \begin{cases} 0.2 & \text{for } x = -2 \\ 0.3 & \text{for } x = -1 \\ 0.2 & \text{for } x = 0 \\ 0.2 & \text{for } x = 1 \\ 0.1 & \text{for } x = 2 \\ 0 & \text{otherwise} \end{cases}$$

Find and plot the CDF of X.

Problem 13

Let X be a discrete random variable with the following CDF:

$$F_X(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{1}{6} & \text{for } 0 \leq x < 1 \\ \frac{1}{2} & \text{for } 1 \leq x < 2 \\ \frac{3}{4} & \text{for } 2 \leq x < 3 \\ 1 & \text{for } x \geq 3 \end{cases}$$

Find the range and PMF of X.

Problem 14

Let X be a discrete random variable with the following PMF

$$P_X(k) = \begin{cases} 0.5 & \text{for } k = 1 \\ 0.3 & \text{for } k = 2 \\ 0.2 & \text{for } k = 3 \\ 0 & \text{otherwise} \end{cases}$$

a. Find EX.
b. Find Var(X) and $SD(X)$.
c. If $Y = \frac{2}{X}$, find EY.

Problem 15

Let $X \sim Geometric(\frac{1}{3})$, and let $Y = |X - 5|$. Find the range and PMF of Y.

Problem 16

Let X be a discrete random variable with the following PMF

$$P_X(k) = \begin{cases} \frac{1}{21} & \text{for } k \in \{-10, -9, \cdots, -1, 0, 1, \cdots, 9, 10\} \\ \\ 0 & \text{otherwise} \end{cases}$$

The random variable $Y = g(X)$ is defined as

$$Y = g(X) = \begin{cases} 0 & \text{if } X \leq 0 \\ X & \text{if } 0 < X \leq 5 \\ 5 & \text{otherwise} \end{cases}$$

Find the PMF of Y.

Problem 17

Let $X \sim Geometric(p)$. Find $Var(X)$.

Problem 18

Let $X \sim Pascal(m, p)$. Find $Var(X)$.

Problem 19

Suppose that $Y = -2X + 3$. If we know $EY = 1$ and $EY^2 = 9$, find EX and $Var(X)$.

Problem 20

There are 1000 households in a town. Specifically, there are 100 households with one member, 200 households with 2 members, 300 households with 3 members, 200 households with 4 members, 100 households with 5 members, and 100 households with 6 members. Thus, the total number of people living in the town is

$$N = 100 \cdot 1 + 200 \cdot 2 + 300 \cdot 3 + 200 \cdot 4 + 100 \cdot 5 + 100 \cdot 6 = 3300.$$

a. We pick a household at random, and define the random variable X as the number of people in the chosen household. Find the PMF and the expected value of X.
b. We pick a person in the town at random, and define the random variable Y as the number of people in the household where the chosen person lives. Find the PMF and the expected value of Y.

Problem 21(Coupon collector's problem [8])

Suppose that there are N different types of coupons. Each time you get a coupon, it is equally likely to be any of the N possible types. Let X be the number of coupons you will need to get before having observed each coupon at least once.

a. Show that you can write $X = X_0 + X_1 + \cdots + X_{N-1}$, where $X_i \sim Geometric(\frac{N-i}{N})$.
b. Find EX.

Problem 22 (St. Petersburg Paradox [9])

Here is a famous problem called the St. Petersburg Paradox. Wikipedia states the problem as follows: "A casino offers a game of chance for a single player in which a fair coin is tossed at each stage. The pot starts at 1 dollar and is doubled every time a head appears. The first time a tail appears, the game ends and the player wins whatever is in the pot. Thus the player wins 1 dollar if a tail appears on the first toss, 2 dollars if a head appears on the first toss and a tail on the second, 4 dollars if a head appears on the first two tosses and a tail on the third, 8 dollars if a head appears on the first three tosses and a tail on the fourth, and so on. In short, the player wins 2^{k-1} dollars if the coin is tossed k times until the first tail appears. What would be a fair price to pay the casino for entering the game?"

a. Let X be the amount of money (in dollars) that the player wins. Find EX.
b. What is the probability that the player wins more than 65 dollars?
c. Now suppose that the casino only has a finite amount of money. Specifically, suppose that the maximum amount of the money that the casion will pay you is 2^{30} dollars (around 1.07 billion dollars). That is, if you win more than 2^{30} dollars, the casino is going to pay you only 2^{30} dollars. Let Y be the money that the player wins in this case. Find EY.

Problem 23

Let X be a random variable with mean $EX = \mu$. Define the function $f(\alpha)$ as

$$f(\alpha) = E[(X - \alpha)^2].$$

Find the value of α that minimizes f.

Problem 24

You are offered to play the following game. You roll a fair die once and observe the result which is shown by the random variable X. At this point, you can stop the game and win X dollars. You can also choose to roll the die for the second time to observe the value Y. In this case, you will win Y dollars. Let W be the value that you win in this game. What strategy do you use to maximize EW? What is the maximum EW you can achieve using your strategy?

Problem 25

The **median** of a random variable X is defined as any number m that satisfies both of the following conditions:

$$P(X \geq m) \geq \frac{1}{2} \quad \text{and} \quad P(X \leq m) \geq \frac{1}{2}$$

Note that the median of X is not necessarily unique. Find the median of X if

a. The PMF of X is given by

$$P_X(k) = \begin{cases} 0.4 & \text{for } k = 1 \\ 0.3 & \text{for } k = 2 \\ 0.3 & \text{for } k = 3 \\ 0 & \text{otherwise} \end{cases}$$

b. X is the result of a rolling of a fair die.

c. $X \sim Geometric(p)$, where $0 < p < 1$.
