Functions of Random Vectors: The Method of Transformations

A function of a random vector is a random vector. Thus, the methods that we discussed regarding functions of two random variables can be used to find distributions of functions of random vectors. For example, we can state a more general form of Theorem 5.1 (method of transformations). Let us first explain the method and then see some examples on how to use it. Let $\mathbf{X}$ be an n-dimensional random vector with joint PDF $f_{\mathbf{X}}(\mathbf{x})$. Let $G : \mathbb{R}^n \mapsto \mathbb{R}^n$ be a continuous and invertible function with continuous partial derivatives and let $H = G^{-1}$. Suppose that the random vector $\mathbf{Y}$ is given by $\mathbf{Y} = G(\mathbf{X})$ and thus $\mathbf{X} = G^{-1}(\mathbf{Y}) = H(\mathbf{Y})$. That is,

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \\ . \\ . \\ . \\ X_n \end{bmatrix} = \begin{bmatrix} H_1(Y_1, Y_2, \ldots, Y_n) \\ H_2(Y_1, Y_2, \ldots, Y_n) \\ . \\ . \\ . \\ H_n(Y_1, Y_2, \ldots, Y_n) \end{bmatrix}.$$

Then, the PDF of Y, $f_{Y_1,Y_2,\ldots,Y_n}(y_1, y_2, \ldots, y_n)$, is given by

$$\boxed{f_{\mathbf{Y}}(\mathbf{y}) = f_{\mathbf{X}}\big(H(\mathbf{y})\big)|J|}$$

where J is the Jacobian of H defined by

$$J = \det \begin{bmatrix} \frac{\partial H_1}{\partial y_1} & \frac{\partial H_1}{\partial y_2} & \ldots & \frac{\partial H_1}{\partial y_n} \\ \frac{\partial H_2}{\partial y_1} & \frac{\partial H_2}{\partial y_2} & \ldots & \frac{\partial H_2}{\partial y_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial H_n}{\partial y_1} & \frac{\partial H_n}{\partial y_2} & \ldots & \frac{\partial H_n}{\partial y_n} \end{bmatrix},$$

and evaluated at $(y_1, y_2, \ldots, y_n)$.

Example 6.15

Let **X** be an n-dimensional random vector. Let **A** be a fixed (non-random) invertible n by n matrix, and **b** be a fixed n-dimensional vector. Define the random vector **Y** as