Since $Y|H_1 \sim N(-1, \sigma^2)$,

$$\begin{aligned} P(\text{choose } H_0|H_1) &= P(Y \geq c|H_1) \\ &= 1 - \Phi\left(\frac{c+1}{\sigma}\right) \\ &= 1 - \Phi\left(\frac{\sigma}{2}\ln\left(\frac{1-p}{p}\right) + \frac{1}{\sigma}\right). \end{aligned}$$

Figure 9.4 shows the two error probabilities for this example. Therefore, the average error probability is given by

$$\begin{aligned} P_e &= P(\text{choose } H_1|H_0)P(H_0) + P(\text{choose } H_0|H_1)P(H_1) \\ &= p \cdot \Phi\left(\frac{\sigma}{2}\ln\left(\frac{1-p}{p}\right) - \frac{1}{\sigma}\right) + (1-p) \cdot \left[1 - \Phi\left(\frac{\sigma}{2}\ln\left(\frac{1-p}{p}\right) + \frac{1}{\sigma}\right)\right]. \end{aligned}$$

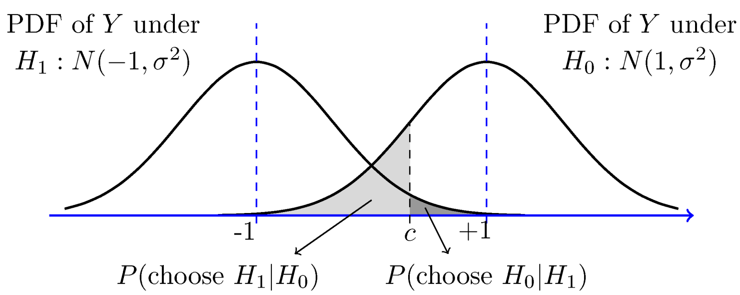

Figure 9.4 - Error probabilities for Example 9.10 and Example 9.11

Minimum Cost Hypothesis Test:

Suppose that you are building a sensor network to detect fires in a forest. Based on the information collected by the sensors, the system needs to decide between two opposing hypotheses:

H_0: There is no fire,

H_1: There is a fire.

There are two possible types of errors that we can make: We might accept H_0 while H_1 is true, or we might accept H_1 while H_0 is true. Note that the cost associated with these two errors are not the same. In other words, if there is a fire and we miss it, we will be making a costlier error. To address situations like this, we associate a cost to each error type:

C_{10}: The cost of choosing H_1, given that H_0 is true.

C_{01}: The cost of choosing H_0, given that H_1 is true.

Then, the average cost can be written as

$$C = C_{10} P(\text{choose } H_1 | H_0) P(H_0) + C_{01} P(\text{choose } H_0 | H_1) P(H_1).$$

The goal of *minimum cost hypothesis testing* is to minimise the above expression. Luckily, this can be done easily. Note that we can rewrite the average cost as

$$C = P(\text{choose } H_1 | H_0) \cdot [P(H_0) C_{10}] + P(\text{choose } H_0 | H_1) \cdot [P(H_1) C_{01}].$$

The above expression is very similar to the average error probability of the MAP test (Equation 9.6). The only difference is that we have $p(H_0)C_{10}$ instead of $P(H_0)$, and we have $p(H_1)C_{01}$ instead of $P(H_1)$. Therefore, we can use a decision rule similar to the MAP decision rule. More specifically, we choose H_0 if and only if

$$f_Y(y|H_0) P(H_0) C_{10} \geq f_Y(y|H_1) P(H_1) C_{01} \qquad (9.7)$$

Here is another way to interpret the above decision rule. If we divide both sides of Equation 9.7 by $f_Y(y)$ and apply Bayes' rule, we conclude the following: We choose H_0 if and only if

$$P(H_0|y) C_{10} \geq P(H_1|y) C_{01}.$$

Note that $P(H_0|y)C_{10}$ is the expected cost of accepting H_1. We call this the **posterior risk** of accepting H_1. Similarly, $P(H_1|y)C_{01}$ is the posterior risk (expected cost) of accepting H_0. Therefore, we can summarize the minimum cost test as follows: We accept the hypothesis with the lowest posterior risk.

Minimum Cost Hypothesis Test

Assuming the following costs

C_{10}: The cost of choosing H_1, given that H_0 is true.

C_{01}: The cost of choosing H_0, given that H_1 is true.

We choose H_0 if and only if

$$\frac{f_Y(y|H_0)}{f_Y(y|H_1)} \geq \frac{P(H_1)C_{01}}{P(H_0)C_{10}}.$$

Equivalently, we choose H_0 if and only if

$$P(H_0|y)C_{10} \geq P(H_1|y)C_{01}.$$

Example 9.12

A surveillance system is in charge of detecting intruders to a facility. There are two hypotheses to choose from:

H_0: No intruder is present.

H_1: There is an intruder.

The system sends an alarm message if it accepts H_1. Suppose that after processing the data, we obtain $P(H_1|y) = 0.05$. Also, assume that the cost of missing an intruder is 10 times the cost of a false alarm. Should the system send an alarm message (accept H_1)?

Solution

First note that

$$P(H_0|y) = 1 - P(H_1|y) = 0.95$$

The posterior risk of accepting H_1 is

$$P(H_0|y)C_{10} = 0.95C_{10}.$$

We have $C_{01} = 10C_{10}$, so the posterior risk of accepting H_0 is

$$\begin{aligned} P(H_1|y)C_{01} &= (0.05)(10C_{10}) \\ &= 0.5C_{10}. \end{aligned}$$

Since $P(H_0|y)C_{10} \geq P(H_1|y)C_{01}$, we accept H_0, so no alarm message needs to be sent.
