7.2.8 Solved Problems

Problem 1

Let X_1, X_2, X_3, $\cdots$ be a sequence of random variables such that

$$X_n \sim Geometric\left(\frac{\lambda}{n}\right), \qquad \text{for } n = 1, 2, 3, \cdots,$$

where $\lambda > 0$ is a constant. Define a new sequence Y_n as

$$Y_n = \frac{1}{n} X_n, \qquad \text{for } n = 1, 2, 3, \cdots.$$

Show that Y_n converges in distribution to $Exponential(\lambda)$.

Solution

Note that if $W \sim Geometric(p)$, then for any positive integer l, we have

$$\begin{aligned} P(W \leq l) &= \sum_{k=1}^{l} (1-p)^{k-1} p \\ &= p \sum_{k=1}^{l} (1-p)^{k-1} \\ &= p \cdot \frac{1-(1-p)^l}{1-(1-p)} \\ &= 1-(1-p)^l. \end{aligned}$$

Now, since $Y_n = \frac{1}{n} X_n$, for any positive real number, we can write

$$\begin{aligned} P(Y_n \leq y) &= P(X_n \leq ny) \\ &= 1 - \left(1 - \frac{\lambda}{n}\right)^{\lfloor ny \rfloor}, \end{aligned}$$

where $\lfloor ny \rfloor$ is the largest integer less than or equal to ny. We then write

$$\begin{aligned}\lim_{n\to\infty} F_{Y_n}(y) &= \lim_{n\to\infty} 1-\left(1-\frac{\lambda}{n}\right)^{\lfloor ny\rfloor}\\ &= 1-\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{\lfloor ny\rfloor}\\ &= 1-e^{-\lambda y}.\end{aligned}$$

The last equality holds because $ny-1 \leq \lfloor ny \rfloor \leq ny$, and

$$\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{ny} = e^{-\lambda y}.$$

Problem 2

Let X_1, X_2, X_3, $\cdots$ be a sequence of i.i.d. $Uniform(0,1)$ random variables. Define the sequence Y_n as

$$Y_n = \min(X_1, X_2, \cdots, X_n).$$

Prove the following convergence results independently (i.e, do not conclude the weaker convergence modes from the stronger ones).

a. $Y_n \xrightarrow{d} 0$.
b. $Y_n \xrightarrow{p} 0$.
c. $Y_n \xrightarrow{L^r} 0$, for all $r \geq 1$.
d. $Y_n \xrightarrow{a.s} 0$.

Solution

a. $Y_n \xrightarrow{d} 0$: Note that

$$F_{X_n}(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$

Also, note that $R_{Y_n} = [0,1]$. For $0 \leq y \leq 1$, we can write

$$
\begin{aligned}
F_{Y_n}(y) &= P(Y_n \leq y) \\
&= 1 - P(Y_n > y) \\
&= 1 - P(X_1 > y, X_2 > y, \cdots, X_n > y) \\
&= 1 - P(X_1 > y)P(X_2 > y) \cdots P(X_n > y) \qquad \text{(since } X_i\text{'s are independent)} \\
&= 1 - (1 - F_{X_1}(y))(1 - F_{X_2}(y)) \cdots (1 - F_{X_n}(y)) \\
&= 1 - (1 - y)^n.
\end{aligned}
$$

Therefore, we conclude

$$
\lim_{n\to\infty} F_{Y_n}(y) = \begin{cases} 0 & y \leq 0 \\ 1 & y > 0 \end{cases}
$$

Therefore, $Y_n \xrightarrow{d} 0$.

b. $Y_n \xrightarrow{p} 0$: Note that as we found in part (a)

$$
F_{Y_n}(y) = \begin{cases} 0 & y < 0 \\ 1 - (1 - y)^n & 0 \leq y \leq 1 \\ 1 & y > 1 \end{cases}
$$

In particular, note that Y_n is a continuous random variable. To show $Y_n \xrightarrow{p} 0$, we need to show that

$$
\lim_{n\to\infty} P\big(|Y_n| \geq \epsilon\big) = 0, \qquad \text{for all } \epsilon > 0.
$$

Since $Y_n \geq 0$, it suffices to show that

$$
\lim_{n\to\infty} P\big(Y_n \geq \epsilon\big) = 0, \qquad \text{for all } \epsilon > 0.
$$

For $\epsilon \in (0, 1)$, we have

$$
\begin{aligned}
P\big(Y_n \geq \epsilon\big) &= 1 - P(Y_n < \epsilon) \\
&= 1 - P(Y_n \leq \epsilon) \qquad \text{(since } Y_n \text{ is a continuous random variable)} \\
&= 1 - F_{Y_n}(\epsilon) \\
&= (1 - \epsilon)^n.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\lim_{n\to\infty} P\big(|Y_n| \geq \epsilon\big) &= \lim_{n\to\infty} (1 - \epsilon)^n \\
&= 0, \qquad \text{for all } \epsilon \in (0, 1].
\end{aligned}
$$

c. $Y_n \xrightarrow{L^r} 0$, for all $r \geq 1$: By differentiating $F_{Y_n}(y)$, we obtain

$$f_{Y_n}(y) = \begin{cases} n(1-y)^{n-1} & 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Thus, for $r \geq 1$, we can write

$$\begin{aligned} E|Y_n|^r &= \int_0^1 n y^r (1-y)^{n-1} dy \\ &\leq \int_0^1 n y (1-y)^{n-1} dy \qquad \text{(since } r \geq 1\text{)} \\ &= \Big[-y(1-y)^n \Big]_0^1 + \int_0^1 (1-y)^n dy \qquad \text{(integration by parts)} \\ &= \frac{1}{n+1}. \end{aligned}$$

Therefore

$$\lim_{n \to \infty} E\left(|Y_n|^r\right) = 0.$$

d. $Y_n \xrightarrow{a.s} 0$: We will prove

$$\sum_{n=1}^{\infty} P\big(|Y_n| > \epsilon\big) < \infty,$$

which implies $Y_n \xrightarrow{a.s} 0$. By our discussion in part (b),

$$\begin{aligned} \sum_{n=1}^{\infty} P\big(|Y_n| > \epsilon\big) &= \sum_{n=1}^{\infty} (1-\epsilon)^n \\ &= \frac{1-\epsilon}{\epsilon} < \infty \qquad \text{(geometric series).} \end{aligned}$$

Problem 3

Let $X_n \sim N(0, \frac{1}{n})$. Show that $X_n \xrightarrow{a.s.} 0$. *Hint:* You may decide to use the inequality given in Equation 4.7 , which is

$$1 - \Phi(x) \leq \frac{1}{\sqrt{2\pi}} \frac{1}{x} e^{-\frac{x^2}{2}}.$$

Solution

We will prove

$$\sum_{n=1}^{\infty} P\big(|X_n| > \epsilon\big) < \infty,$$

which implies $X_n \xrightarrow{a.s} 0$. In particular,

$$\begin{aligned} P\big(|X_n| > \epsilon\big) &= 2\big(1 - \Phi(\epsilon n)\big) \qquad (\text{since } X_n \sim N(0, \frac{1}{n})) \\ &\leq \frac{1}{\sqrt{2\pi}} \frac{2}{\epsilon n} e^{-\frac{\epsilon^2 n^2}{2}} \\ &\leq \frac{1}{\sqrt{2\pi}} \frac{2}{\epsilon} e^{-\frac{\epsilon^2 n^2}{2}} \\ &\leq \frac{1}{\sqrt{2\pi}} \frac{2}{\epsilon} e^{-\frac{\epsilon^2 n}{2}}. \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{n=1}^{\infty} P\big(|X_n| > \epsilon\big) &\leq \sum_{n=1}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{2}{\epsilon} e^{-\frac{\epsilon^2 n}{2}} \\ &= \frac{1}{\sqrt{2\pi}} \frac{2}{\epsilon} \sum_{n=1}^{\infty} e^{-\frac{\epsilon^2 n}{2}} \\ &= \frac{1}{\sqrt{2\pi}} \frac{2}{\epsilon} \frac{e^{-\frac{\epsilon^2}{2}}}{1 - e^{-\frac{\epsilon^2}{2}}} < \infty \qquad (\text{geometric series}). \end{aligned}$$

Problem 4

Consider the sample space $S = [0, 1]$ with uniform probability distribution, i.e.,

$$P([a, b]) = b - a, \qquad \text{for all } 0 \leq a \leq b \leq 1.$$

Define the sequence $\{X_n, n = 1, 2, \cdots\}$ as $X_n(s) = \frac{n}{n+1}s + (1 - s)^n$. Also, define the random variable X on this sample space as $X(s) = s$. Show that $X_n \xrightarrow{a.s.} X$.

Solution

For any $s \in (0, 1]$, we have

$$\begin{aligned} \lim_{n\to\infty} X_n(s) &= \lim_{n\to\infty} \left[\frac{n}{n+1}s + (1 - s)^n\right] \\ &= s = X(s). \end{aligned}$$

However, if $s = 0$, then

$$\begin{aligned}\lim_{n\to\infty} X_n(0) &= \lim_{n\to\infty}\left[\frac{n}{n+1}\cdot 0 + (1-0)^n\right]\\ &= 1.\end{aligned}$$

Thus, we conclude

$$\lim_{n\to\infty} X_n(s) = X(s), \qquad \text{for all } s \in (0,1].$$

Since $P\big((0,1]\big) = 1$, we conclude $X_n \xrightarrow{a.s.} X$.

Problem 5

Let $\{X_n, n = 1, 2, \cdots\}$ and $\{Y_n, n = 1, 2, \cdots\}$ be two sequences of random variables, defined on the sample space S. Suppose that we know

$$\begin{aligned}X_n &\xrightarrow{a.s.} X,\\ Y_n &\xrightarrow{a.s.} Y.\end{aligned}$$

Prove that $X_n + Y_n \xrightarrow{a.s.} X + Y$.

Solution

Define the sets A and B as follows:

$$\begin{aligned}A &= \left\{s \in S : \lim_{n\to\infty} X_n(s) = X(s)\right\},\\ B &= \left\{s \in S : \lim_{n\to\infty} Y_n(s) = Y(s)\right\}.\end{aligned}$$

By definition of almost sure convergence, we conclude $P(A) = P(B) = 1$. Therefore, $P(A^c) = P(B^c) = 0$. We conclude

$$\begin{aligned}P(A \cap B) &= 1 - P(A^c \cup B^c)\\ &\geq 1 - P(A^c) - P(B^c)\\ &= 1.\end{aligned}$$

Thus, $P(A \cap B) = 1$. Now, consider the sequence $\{Z_n, n = 1, 2, \cdots\}$, where $Z_n = X_n + Y_n$, and define the set C as

$$C = \left\{s \in S : \lim_{n\to\infty} Z_n(s) = X(s) + Y(s)\right\}.$$

We claim $A \cap B \subset C$. Specifically, if $s \in A \cap B$, then we have

$$\lim_{n \to \infty} X_n(s) = X(s), \qquad \lim_{n \to \infty} Y_n(s) = Y(s).$$

Therefore,

$$\begin{aligned} \lim_{n \to \infty} Z_n(s) &= \lim_{n \to \infty} \Big[X_n(s) + Y_n(s)\Big] \\ &= \lim_{n \to \infty} X_n(s) + \lim_{n \to \infty} Y_n(s) \\ &= X(s) + Y(s). \end{aligned}$$

Thus, $s \in C$. We conclude $A \cap B \subset C$. Thus,

$$P(C) \geq P(A \cap B) = 1,$$

which implies $P(C) = 1$. This means that $Z_n \xrightarrow{a.s.} X + Y$.

Problem 6

Let $\{X_n, n = 1, 2, \cdots\}$ and $\{Y_n, n = 1, 2, \cdots\}$ be two sequences of random variables, defined on the sample space S. Suppose that we know

$$\begin{aligned} X_n &\xrightarrow{p} X, \\ Y_n &\xrightarrow{p} Y. \end{aligned}$$

Prove that $X_n + Y_n \xrightarrow{p} X + Y$.

Solution

For $n \in \mathbb{N}$, define the following events

$$\begin{aligned} A_n &= \left\{ |X_n - X| < \frac{\epsilon}{2} \right\}, \\ B_n &= \left\{ |Y_n - Y| < \frac{\epsilon}{2} \right\}. \end{aligned}$$

Since $X_n \xrightarrow{p} X$ and $Y_n \xrightarrow{p} Y$, we have for all $\epsilon > 0$

$$\begin{aligned} \lim_{n \to \infty} P\big(A_n\big) &= 1, \\ \lim_{n \to \infty} P\big(B_n\big) &= 1. \end{aligned}$$

We can also write

$$
\begin{aligned}
P(A_n \cap B_n) &= P(A_n) + P(B_n) - P(A_n \cup B_n) \\
&\geq P(A_n) + P(B_n) - 1.
\end{aligned}
$$

Therefore,

$$
\lim_{n \to \infty} P\big(A_n \cap B_n) = 1.
$$

Now, let us define the events C_n and D_n as follows:

$$
\begin{aligned}
C_n &= \bigg\{|X_n - X| + |Y_n - Y| < \epsilon\bigg\}, \\
D_n &= \bigg\{|X_n + Y_n - X - Y| < \epsilon\bigg\}.
\end{aligned}
$$

Now, note that $(A_n \cap B_n) \subset C_n$, thus $P(A_n \cap B_n) \leq P(C_n)$. Also, by the triangle inequality for absolute values, we have

$$
|(X_n - X) + (Y_n - Y)| \leq |X_n - X| + |Y_n - Y|.
$$

Therefore, $C_n \subset D_n$, which implies

$$
P(C_n) \leq P(D_n).
$$

We conclude

$$
P(A_n \cap B_n) \leq P(C_n) \leq P(D_n).
$$

Since $\lim_{n \to \infty} P(A_n \cap B_n) = 1$, we conclude $\lim_{n \to \infty} P(D_n) = 1$. This by definition means that $X_n + Y_n \ \xrightarrow{p}\ X + Y$.