7.1.1 Law of Large Numbers

The **law of large numbers** has a very central role in probability and statistics. It states that if you repeat an experiment independently a large number of times and average the result, what you obtain should be close to the expected value. There are two main versions of the law of large numbers. They are called the **weak** and **strong** laws of the large numbers. The difference between them is mostly theoretical. In this section, we state and prove the weak law of large numbers (WLLN). The strong law of large numbers is discussed in Section 7.2. Before discussing the WLLN, let us define the *sample mean*.

Definition 7.1. For i.i.d. random variables $X_1, X_2, \ldots, X_n$, the **sample mean**, denoted by $\overline{X}$, is defined as

$$\overline{X} = \frac{X_1 + X_2 + \ldots + X_n}{n}.$$

Another common notation for the sample mean is M_n. If the X_i's have CDF $F_X(x)$, we might show the sample mean by $M_n(X)$ to indicate the distribution of the X_i's.

Note that since the X_i's are random variables, the sample mean, $\overline{X} = M_n(X)$, is also a random variable. In particular, we have

$$\begin{aligned} E[\overline{X}] &= \frac{EX_1 + EX_2 + \ldots + EX_n}{n} && \text{(by linearity of expectation)} \\ &= \frac{nEX}{n} && \text{(since } EX_i = EX\text{)} \\ &= EX. \end{aligned}$$

Also, the variance of $\overline{X}$ is given by

$$\begin{aligned}
\mathrm{Var}(\overline{X}) &= \frac{\mathrm{Var}(X_1 + X_2 + \ldots + X_n)}{n^2} && \text{(since } \mathrm{Var}(aX) = a^2\mathrm{Var}(X)\text{)} \\
&= \frac{\mathrm{Var}(X_1) + \mathrm{Var}(X_2) + \ldots + \mathrm{Var}(X_n)}{n^2} && \text{(since the } X_i\text{'s are independent)} \\
&= \frac{n\mathrm{Var}(X)}{n^2} && \text{(since } \mathrm{Var}(X_i) = \mathrm{Var}(X)\text{)} \\
&= \frac{\mathrm{Var}(X)}{n}.
\end{aligned}$$

Now let us state and prove the **weak law of large numbers (WLLN).**

The weak law of large numbers (WLLN)

Let X_1, X_2 , ... , X_n be i.i.d. random variables with a finite expected value $EX_i = \mu < \infty$. Then, for any $\epsilon > 0$,

$$\lim_{n \to \infty} P(|\overline{X} - \mu| \geq \epsilon) = 0.$$

Proof

The proof of the weak law of large number is easier if we assume $\mathrm{Var}(X) = \sigma^2$ is finite. In this case we can use Chebyshev's inequality to write

$$\begin{aligned}
P(|\overline{X} - \mu| \geq \epsilon) &\leq \frac{\mathrm{Var}(\overline{X})}{\epsilon^2} \\
&= \frac{\mathrm{Var}(X)}{n\epsilon^2},
\end{aligned}$$

which goes to zero as $n \to \infty$.