10.1.6 Solved Problems

Problem 1

Let Y_1, Y_2, Y_3, $\cdots$ be a sequence of i.i.d. random variables with mean $EY_i = 0$ and $\mathrm{Var}(Y_i) = 4$. Define the discrete-time random process $\{X(n), n \in \mathbb{N}\}$ as

$$X(n) = Y_1 + Y_2 + \cdots + Y_n, \quad \text{for all } n \in \mathbb{N}.$$

Find $\mu_X(n)$ and $R_X(m,n)$, for all $n, m \in \mathbb{N}$.

Solution

We have

$$\begin{aligned}
\mu_X(n) &= E[X(n)] \\
&= E[Y_1 + Y_2 + \cdots + Y_n] \\
&= E[Y_1] + E[Y_2] + \cdots + E[Y_n] \\
&= 0.
\end{aligned}$$

Let $m \leq n$, then

$$\begin{aligned}
R_X(m,n) &= E[X(m)X(n)] \\
&= E\left[X(m)\big(X(m) + Y_{m+1} + Y_{m+2} + \cdots + Y_n\big)\right] \\
&= E[X(m)^2] + E[X(m)]E[Y_{m+1} + Y_{m+2} + \cdots + Y_n] \\
&= E[X(m)^2] + 0 \\
&= \mathrm{Var}\big(X(m)\big) \\
&= \mathrm{Var}\big(Y_1\big) + \mathrm{Var}\big(Y_2\big) + \cdots + \mathrm{Var}\big(Y_m\big) \\
&= 4m.
\end{aligned}$$

Similarly, for $m \geq n$, we have

$$\begin{aligned}
R_X(m,n) &= E[X(m)X(n)] \\
&= 4n.
\end{aligned}$$

We conclude

$$R_X(m,n) = 4\min(m,n).$$

Problem 2

For any $k \in \mathbb{Z}$, define the function $g_k(t)$ as

$$g_k(t) = \begin{cases} 1 & k < t \leq k+1 \\ \\ 0 & \text{otherwise} \end{cases}$$

Now, consider the continuous-time random process $\{X(t), t \in \mathbb{R}\}$ defined as

$$X(t) = \sum_{k=-\infty}^{+\infty} A_k g_k(t),$$

where A_1, A_2, $\cdots$ are i.i.d. random variables with $EA_k = 1$ and $\mathrm{Var}(A_k) = 1$. Find $\mu_X(t)$, $R_X(s,t)$, and $C_X(s,t)$ for all $s, t \in \mathbb{R}$.

Solution

Note that, for any $k \in \mathbb{Z}$, $g(t) = 0$ outside of the interval $(k, k+1]$. Thus, if $k < t \leq k+1$, we can write

$$X(t) = A_k.$$

Thus,

$$\begin{aligned} \mu_X(t) &= E[X(t)] \\ &= E[A_k] = 1. \end{aligned}$$

So, $\mu_X(t) = 1$ for all $t \in \mathbb{R}$. Now consider two real numbers s and t. If for some $k \in \mathbb{Z}$, we have

$$k < s, t \leq k+1,$$

then

$$\begin{aligned} R_X(s,t) &= E[X(s)X(t)] \\ &= E[A_k^2] = 1 + 1 = 2. \end{aligned}$$

On the other hand, if s and t are in two different subintervals of $\mathbb{R}$, that is if

$$k < s \leq k+1, \quad \text{and} \quad l < t \leq l+1,$$

where k and l are two different integers, then

$$
\begin{aligned}
R_X(s,t) &= E[X(s)X(t)] \\
&= E[A_k A_l] = E[A_k]E[A_l] = 1.
\end{aligned}
$$

To find $C_X(s,t)$, note that if \

$$k < s, t \leq k+1,$$

then

$$
\begin{aligned}
C_X(s,t) &= R_X(s,t) - E[X(s)]E[X(t)] \\
&= 2 - 1 \cdot 1 = 1.
\end{aligned}
$$

On the other hand, if

$$k < s \leq k+1, \quad \text{and} \quad l < t \leq l+1,$$

where k and l are two different integers, then

$$
\begin{aligned}
C_X(s,t) &= R_X(s,t) - E[X(s)]E[X(t)] \\
&= 1 - 1 \cdot 1 = 0.
\end{aligned}
$$

Problem 3

Let $X(t)$ be a continuous-time WSS process with mean $\mu_X = 1$ and

$$
R_X(\tau) = \begin{cases} 3 - |\tau| & -2 \leq \tau \leq 2 \\ \\ 1 & \text{otherwise} \end{cases}
$$

a. Find the expected power in $X(t)$.

b. Find $E\left[\left(X(1) + X(2) + X(3)\right)^2\right]$.

Solution

a. The expected power in $X(t)$ at time t is $E[X(t)^2]$, which is given by

$$R_X(0) = 3.$$

b. We have

$$
\begin{aligned}
E\left[\left(X(1)+X(2)+X(3)\right)^2\right] &= E\Big[X(1)^2+X(2)^2+X(3)^2 \\
&\quad + 2X(1)X(2)+2X(1)X(3)+2X(2)X(3)\Big] \\
&= 3R_X(0)+2R_X(-1)+2R_X(-2)+2R_X(-1) \\
&= 3\cdot 3+2\cdot 2+2\cdot 1+2\cdot 2 \\
&= 19.
\end{aligned}
$$

Problem 4

Let $X(t)$ be a continuous-time WSS process with mean $\mu_X = 0$ and

$$R_X(\tau) = \delta(\tau),$$

where $\delta(\tau)$ is the Dirac delta function. We define the random process $Y(t)$ as

$$Y(t) = \int_{t-2}^{t} X(u)du.$$

a. Find $\mu_Y(t) = E[Y(t)]$.

b. Find $R_{XY}(t_1, t_2)$.

Solution

a. We have

$$
\begin{aligned}
\mu_Y(t) &= E\left[\int_{t-2}^{t} X(u)du\right] \\
&= \int_{t-2}^{t} E[X(u)]\ du \\
&= \int_{t-2}^{t} 0\ du \\
&= 0.
\end{aligned}
$$

b. We have

$$\begin{aligned} R_{XY}(t_1,t_2) &= E\left[X(t_1)\int_{t_2-2}^{t_2} X(u)du\right] \\ &= E\left[\int_{t_2-2}^{t_2} X(t_1)X(u)du\right] \\ &= \int_{t_2-2}^{t_2} R_X(t_1-u)\,du \\ &= \int_{t_2-2}^{t_2} \delta(t_1-u)\,du \\ &= \begin{cases} 1 & t_2-2<t_1<t_2 \\ & \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Problem 5

Let $X(t)$ be a Gaussian process with $\mu_X(t)=t$, and $R_X(t_1,t_2)=1+2t_1t_2$, for all $t,t_1,t_2\in\mathbb{R}$. Find $P\big(2X(1)+X(2)<3\big)$.

Solution

Let $Y=2X(1)+X(2)$. Then, Y is a normal random variable. We have

$$\begin{aligned} EY &= 2E[X(1)]+E[X(2)] \\ &= 2\cdot 1+2=4. \end{aligned}$$

$$\mathrm{Var}(Y)=4\mathrm{Var}\big(X(1)\big)+\mathrm{Var}\big(X(2)\big)+4\mathrm{Cov}\big(X(1),X(2)\big).$$

Note that

$$\begin{aligned} \mathrm{Var}\big(X(1)\big) &= E[X(1)^2]-E[X(1)]^2 \\ &= R_X(1,1)-\mu_X(1)^2 \\ &= 1+2\cdot 1\cdot 1-1=2. \end{aligned}$$

$$\begin{aligned} \mathrm{Var}\big(X(2)\big) &= E[X(2)^2]-E[X(2)]^2 \\ &= R_X(2,2)-\mu_X(2)^2 \\ &= 1+2\cdot 2\cdot 2-4=5. \end{aligned}$$

$$\begin{aligned} \mathrm{Cov}\big(X(1),X(2)\big) &= E[X(1)X(2)]-E[X(1)]E[X(2)] \\ &= R_X(1,2)-\mu_X(1)\mu_X(2) \\ &= 1+2\cdot 1\cdot 2-1\cdot 2=3. \end{aligned}$$

Therefore,

$$\mathrm{Var}(Y) = 4 \cdot 2 + 5 + 4 \cdot 3 = 25.$$

We conclude $Y \sim N(4, 25)$. Thus,

$$\begin{aligned} P(Y < 3) &= \Phi\left(\frac{3-4}{5}\right) \\ &= \Phi(-0.2) \approx 0.42 \end{aligned}$$
