8.4.6 Solved Problems

Problem 1

Let $X \sim Geometric(\theta)$. We observe X and we need to decide between

H_0: $\theta = \theta_0 = 0.5$,

H_1: $\theta = \theta_1 = 0.1$

a. Design a level 0.05 test $(\alpha = 0.05)$ to decide between H_0 and H_1.
b. Find the probability of type-II error β.

Solution

a. We choose a threshold $c \in \mathbb{N}$ and compare the observed value of $X = x$ to c. We accept H_0 if $x \leq c$ and reject it if $x > c$. The probability of type I error is given by

$$\begin{aligned} P(\text{type I error}) &= P(\text{Reject } H_0 \mid H_0) \\ &= P(\text{Reject } H_0 \mid \theta = 0.5) \\ &= P(X > c \mid \theta = 0.5) \\ &= \sum_{k=c+1}^{\infty} P(X = k) \quad \big(\text{where } X \sim Geometric(\theta_0 = 0.5)\big) \\ &= \sum_{k=c+1}^{\infty} (1-\theta_0)^{k-1}\theta_0 \\ &= (1-\theta_0)^c \theta_0 \sum_{l=0}^{\infty} (1-\theta_0)^l \\ &= (1-\theta_0)^c. \end{aligned}$$

To have $\alpha = 0.05$, we need to choose c such that $(1-\theta_0)^c \leq \alpha = 0.05$, so we obtain

$$
\begin{aligned}
c &\geq \frac{\ln \alpha}{\ln(1-\theta_0)} \\
&= \frac{\ln(0.05)}{\ln(.5)} \\
&= 4.32
\end{aligned}
$$

Since we would like $c \in \mathbb{N}$, we can let $c = 5$. To summarize, we have the following decision rule: Accept H_0 if the observed value of X is in the set $A = \{1, 2, 3, 4, 5\}$, and reject H_0 otherwise.

b. Since the alternative hypothesis H_1 is a simple hypothesis ($\theta = \theta_1$), there is only one value for β,

$$
\begin{aligned}
\beta &= P(\text{type II error}) = P(\text{accept } H_0 \mid H_1) \\
&= P(X \leq c \mid H_1) \\
&= 1 - (1-\theta_1)^c \\
&= 1 - (0.9)^5 \\
&= 0.41
\end{aligned}
$$

Problem 2

Let X_1,X_2,X_3, X_4 be a random sample from a $N(\mu, 1)$ distribution, where μ is unknown. Suppose that we have observed the following values

$$2.82 \quad 2.71 \quad 3.22 \quad 2.67$$

We would like to decide between

H_0: $\mu = \mu_0 = 2$,

H_1: $\mu \neq 2$.

a. Assuming $\alpha = 0.1$, Do you accept H_0 or H_1?
b. If we require significance level α, find β as a function of μ and α.

Solution

a. We have a sample from a normal distribution with known variance, so using the first row in Table 8.2, we define the test statistic as

$$W = \frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}}.$$

We have $\overline{X} = 2.85$, $\mu_0 = 2$, $\sigma = 1$, and $n = 4$. So, we obtain

$$W = \frac{2.85 - 2}{1/2}$$
$$= 1.7$$

Here, $\alpha = 0.1$, so $z_{\frac{\alpha}{2}} = z_{0.05} = 1.645$. Since

$$|W| > z_{\frac{\alpha}{2}},$$

we reject H_0 and accept H_1.

b. Here, the test statistic W is

$$W \sim = 2(\overline{X} - 2).$$

If $X \sim (\mu, 1)$, then

$$\overline{X} \sim N\left(\mu, \frac{1}{4}\right),$$

and

$$W \sim N(2(\mu - 2), 1).$$

Thus, we have

$$\begin{aligned}\beta &= P(\text{type II error}) = P(\text{accept } H_0 \mid \mu) \\ &= P(|W| < z_{\frac{\alpha}{2}} \mid \mu) \\ &= P(|W| < z_{\frac{\alpha}{2}}) \quad \big(\text{when } W \sim N(2(\mu-2), 1)\big) \\ &= \Phi\left(z_{\frac{\alpha}{2}} - 2\mu + 4\right) - \Phi\left(-z_{\frac{\alpha}{2}} - 2\mu + 4\right).\end{aligned}$$

Problem 3

Let X_1,X_2,...,X_{100} be a random sample from an unknown distribution. After observing this sample, the sample mean and the sample variance are calculated to be

$$\overline{X} = 21.32, \quad S^2 = 27.6$$

Design a level 0.05 test to choose between

H_0: $\mu = 20$,

H_1: $\mu > 20$.

Do you accept or reject H_0?

Solution

Here, we have a non-normal sample, where $n = 100$ is large. As we have discussed previously, to test for the above hypotheses, we can use the results of Table 8.3. More specifically, using the second row of Table 8.3, we define the test statistic as

$$\begin{aligned} W &= \frac{\overline{X} - \mu_0}{S/\sqrt{n}} \\ &= \frac{21.32 - 20}{\sqrt{27.6}/\sqrt{100}} \\ &= 2.51 \end{aligned}$$

Here, $\alpha = 0.05$, so $z_\alpha = z_{0.05} = 1.645$. Since

$$W > z_\alpha,$$

we reject H_0 and accept H_1.

Problem 4 Let X_1,X_2,X_3, X_4 be a random sample from a $N(\mu, \sigma^2)$ distribution, where μ and σ are unknown. Suppose that we have observed the following values

$$3.58 \quad 10.03 \quad 4.77 \quad 14.66$$

We would like to decide between

H_0: $\mu \geq 10$,

H_1: $\mu < 10$.

Assuming $\alpha = 0.05$, Do you accept H_0 or H_1?

Solution

Here, we have a sample from a normal distribution with unknown mean and unknown variance. Thus, using the third row in Table 8.4, we define the test statistic as

$$W = \frac{\overline{X} - \mu_0}{S/\sqrt{n}}.$$

Using the data we obtain

$$\overline{X} = 8.26, \quad S = 5.10$$

Therefore, we obtain

$$\begin{aligned} W &= \frac{8.26 - 10}{5.10/2} \\ &= -0.68 \end{aligned}$$

Here, $\alpha = 0.05$, so $n = 4$, $t_{\alpha,n-1} = t_{0.05,3} = 2.35$. Since

$$W > -t_{\alpha,n-1},$$

we fail to reject H_0, so we accept H_0.

Problem 5 Let X_1,X_2,...,X_{81} be a random sample from an unknown distribution. After observing this sample, the sample mean and the sample variance are calculated to be

$$\overline{X} = 8.25, \quad S^2 = 14.6$$

Design a test to decide between

H_0: $\mu = 9$,

H_1: $\mu < 9$,

and calculate the P-value for the observed data.

Solution

Here, we have a non-normal sample, where $n = 81$ is large. As we have discussed previously, to test for the above hypotheses, we can use the results of Table 8.4. More specifically, using the second row of Table 8.4, we define the test statistic as

$$\begin{aligned} W &= \frac{\overline{X} - \mu_0}{S/\sqrt{n}} \\ &= \frac{8.25 - 9}{\sqrt{14.6}/\sqrt{81}} \\ &= -1.767 \end{aligned}$$

The P-value is $P(\text{type I error})$ when the test threshold c is chosen to be $c = -1.767$. Since the threshold for this test (as indicated by Table 8.4) is $-z_\alpha$, we obtain

$$-z_\alpha = -1.767$$

Noting that by definition $z_\alpha = \Phi^{-1}(1-\alpha)$, we obtain $P(\text{type I error})$ as

$$\alpha = 1 - \Phi(1.767) \approx 0.0386$$

Therefore,

$$P - \text{value} \approx 0.0386$$
