10.2.5 Solved Problems

Problem 1

Consider a WSS random process $X(t)$ with

$$R_X(\tau) = \begin{cases} 1 - |\tau| & \quad -1 \leq \tau \leq 1 \\ & \\ 0 & \quad \text{otherwise} \end{cases}$$

Find the PSD of $X(t)$, and $E[X(t)^2]$.

Solution

First, we have

$$E[X(t)^2] = R_X(0) = 1.$$

We can write triangular function, $R_X(\tau) = \Lambda(\tau)$, as

$$R_X(\tau) = \Pi(\tau) * \Pi(\tau),$$

where

$$\Pi(\tau) = \begin{cases} 1 & \quad -\frac{1}{2} \leq \tau \leq \frac{1}{2} \\ & \\ 0 & \quad \text{otherwise} \end{cases}$$

Thus, we conclude

$$\begin{aligned} S_X(f) &= \mathcal{F}\{R_X(\tau)\} \\ &= \mathcal{F}\{\Pi(\tau) * \Pi(\tau)\} \\ &= \mathcal{F}\{\Pi(\tau)\} \cdot \mathcal{F}\{\Pi(\tau)\} \\ &= \big[\text{sinc}(f)\big]^2. \end{aligned}$$

Problem 2

Let $X(t)$ be a random process with mean function $\mu_X(t)$ and autocorrelation function $R_X(s,t)$ ($X(t)$ is not necessarily a WSS process). Let $Y(t)$ be given by

$$Y(t) = h(t) * X(t),$$

where $h(t)$ is the impulse response of the system. Show that

a. $\mu_Y(t) = \mu_X(t) * h(t)$.
b. $R_{XY}(t_1, t_2) = h(t_2) * R_X(t_1, t_2) = \int_{-\infty}^{\infty} h(\alpha) R_X(t_1, t_2 - \alpha)\, d\alpha$.

Solution

a. We have

$$\begin{aligned} \mu_Y(t) = E[Y(t)] &= E\left[\int_{-\infty}^{\infty} h(\alpha) X(t-\alpha)\, d\alpha\right] \\ &= \int_{-\infty}^{\infty} h(\alpha) E[X(t-\alpha)]\, d\alpha \\ &= \int_{-\infty}^{\infty} h(\alpha) \mu_X(t-\alpha)\, d\alpha \\ &= \mu_X(t) * h(t). \end{aligned}$$

b. We have

$$\begin{aligned} R_{XY}(t_1, t_2) = E[X(t_1) Y(t_2)] &= E\left[X(t_1) \int_{-\infty}^{\infty} h(\alpha) X(t_2-\alpha)\, d\alpha\right] \\ &= E\left[\int_{-\infty}^{\infty} h(\alpha) X(t_1) X(t_2-\alpha)\, d\alpha\right] \\ &= \int_{-\infty}^{\infty} h(\alpha) E[X(t_1) X(t_2-\alpha)]\, d\alpha \\ &= \int_{-\infty}^{\infty} h(\alpha) R_X(t_1, t_2-\alpha)\, d\alpha. \end{aligned}$$

Problem 3

Prove the third part of Theorem 10.2: Let $X(t)$ be a WSS random process and $Y(t)$ be given by

$$Y(t) = h(t) * X(t),$$

where $h(t)$ is the impulse response of the system. Show that

$$R_Y(s, t) = R_Y(s-t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\alpha) h(\beta) R_X(s - t - \alpha + \beta)\, d\alpha d\beta.$$

Also, show that we can rewrite the above integral as $R_Y(\tau) = h(\tau) * h(-\tau) * R_X(\tau)$.

Solution

$$\begin{aligned}
R_Y(s,t) &= E[X(s)Y(t)] \\
&= E\left[\int_{-\infty}^{\infty} h(\alpha)X(s-\alpha)\, d\alpha \int_{-\infty}^{\infty} h(\beta)X(s-\beta)\, d\beta\right] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\alpha)h(\beta)E[X(s-\alpha)X(t-\beta)]\, d\alpha\, d\beta \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\alpha)h(\beta)R_X(s-t-\alpha+\beta)\, d\alpha\, d\beta.
\end{aligned}$$

We now compute $h(\tau) * h(-\tau) * R_X(\tau)$. First, let $g(\tau) = h(\tau) * h(-\tau)$. Note that

$$\begin{aligned}
g(\tau) &= h(\tau) * h(-\tau) \\
&= \int_{-\infty}^{\infty} h(\alpha)h(\alpha-\tau)\, d\alpha.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
g(\tau) * R_X(\tau) &= \int_{-\infty}^{\infty} g(\theta)R_X(\theta-\tau)\, d\theta \\
&= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} h(\alpha)h(\alpha-\theta)\, d\alpha\right] R_X(\theta-\tau)\, d\theta \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\alpha)h(\alpha-\theta)R_X(\theta-\tau)\, d\alpha\, d\theta \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\alpha)h(\beta)R_X(\alpha-\beta-\tau)\, d\alpha\, d\beta \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\alpha)h(\beta)R_X(\tau-\alpha+\beta)\, d\alpha\, d\beta \qquad \big(\text{since } R_X(-\tau) = R_X(\tau)\big).
\end{aligned}$$

Problem 4
Let $X(t)$ be a WSS random process. Assuming that $S_X(f)$ is continuous at f_1, show that $S_X(f_1) \geq 0$.

Solution

Let $f_1 \in \mathbb{R}$. Suppose that $X(t)$ goes through an LTI system with the following transfer function

$$H(f) = \begin{cases} 1 & f_1 < |f| < f_1 + \Delta \\ \\ 0 & \text{otherwise} \end{cases}$$

where Δ is chosen to be very small. The PSD of $Y(t)$ is given by

$$S_Y(f) = S_X(f)|H(f)|^2 = \begin{cases} S_X(f) & f_1 < |f| < f_1 + \Delta \\ \\ 0 & \text{otherwise} \end{cases}$$

Thus, the power in $Y(t)$ is

$$\begin{aligned} E[Y(t)^2] &= \int_{-\infty}^{\infty} S_Y(f)\, df \\ &= 2\int_{f_1}^{f_1+\Delta} S_X(f)\, df \\ &\approx 2\Delta S_X(f_1). \end{aligned}$$

Since $E[Y(t)^2] \geq 0$, we conclude that $S_X(f_1) \geq 0$.

Problem 5

Let $X(t)$ be a white Gaussian noise with $S_X(f) = \frac{N_0}{2}$. Assume that $X(t)$ is input to an LTI system with

$$h(t) = e^{-t}u(t).$$

Let $Y(t)$ be the output.

a. Find $S_Y(f)$.
b. Find $R_Y(\tau)$.
c. Find $E[Y(t)^2]$.

Solution

First, note that

$$\begin{aligned} H(f) &= \mathcal{F}\{h(t)\} \\ &= \frac{1}{1 + j2\pi f}. \end{aligned}$$

a. To find $S_Y(f)$, we can write

$$\begin{aligned} S_Y(f) &= S_X(f)|H(f)|^2 \\ &= \frac{N_0/2}{1+(2\pi f)^2}. \end{aligned}$$

b. To find $R_Y(\tau)$, we can write

$$\begin{aligned} R_Y(\tau) &= \mathcal{F}^{-1}\{S_Y(f)\} \\ &= \frac{N_0}{4}e^{-|\tau|}. \end{aligned}$$

c. We have

$$\begin{aligned} E[Y(t)^2] &= R_Y(0) \\ &= \frac{N_0}{4}. \end{aligned}$$
