4.2.4 Gamma Distribution

The gamma distribution is another widely used distribution. Its importance is largely due to its relation to exponential and normal distributions. Here, we will provide an introduction to the gamma distribution. In Chapters 6 and 11, we will discuss more properties of the gamma random variables. Before introducing the gamma random variable, we need to introduce the gamma function.

Gamma function: The gamma function [10], shown by $\Gamma(x)$, is an extension of the factorial function to real (and complex) numbers. Specifically, if $n \in \{1, 2, 3, \ldots\}$, then

$$\Gamma(n) = (n-1)!$$

More generally, for any positive real number α, $\Gamma(\alpha)$ is defined as

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} \mathrm{d}x, \qquad \text{for } \alpha > 0.$$

Figure 4.9 shows the gamma function for positive real values.

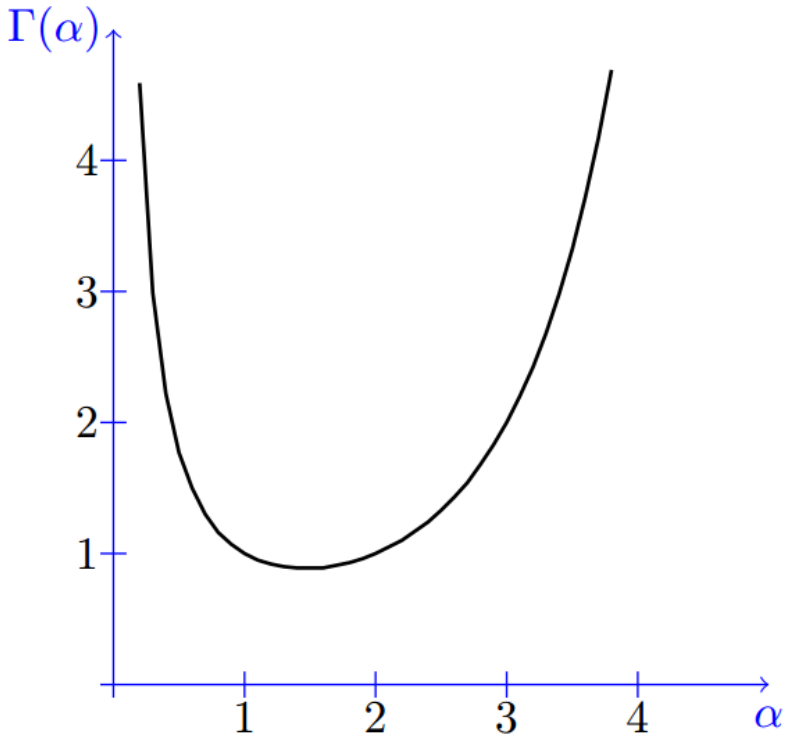

Figure 4.9: The Gamma function for some real values of α.

Note that for $\alpha = 1$, we can write

$$\begin{aligned}\Gamma(1) &= \int_0^\infty e^{-x} dx \\ &= 1.\end{aligned}$$

Using the change of variable $x = \lambda y$, we can show the following equation that is often useful when working with the gamma distribution:

$$\Gamma(\alpha) = \lambda^\alpha \int_0^\infty y^{\alpha-1} e^{-\lambda y} dy \qquad \text{for } \alpha, \lambda > 0.$$

Also, using integration by parts it can be shown that

$$\Gamma(\alpha + 1) = \alpha\Gamma(\alpha), \qquad \text{for } \alpha > 0.$$

Note that if $\alpha = n$, where n is a positive integer, the above equation reduces to

$$n! = n \cdot (n-1)!$$