11.2.7 Solved Problems

Problem 1

Consider the Markov chain with three states, $S = \{1, 2, 3\}$, that has the following transition matrix

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}.$$

a. Draw the state transition diagram for this chain.
b. If we know $P(X_1 = 1) = P(X_1 = 2) = \frac{1}{4}$, find $P(X_1 = 3, X_2 = 2, X_3 = 1)$.

Solution

a. The state transition diagram is shown in Figure 11.6

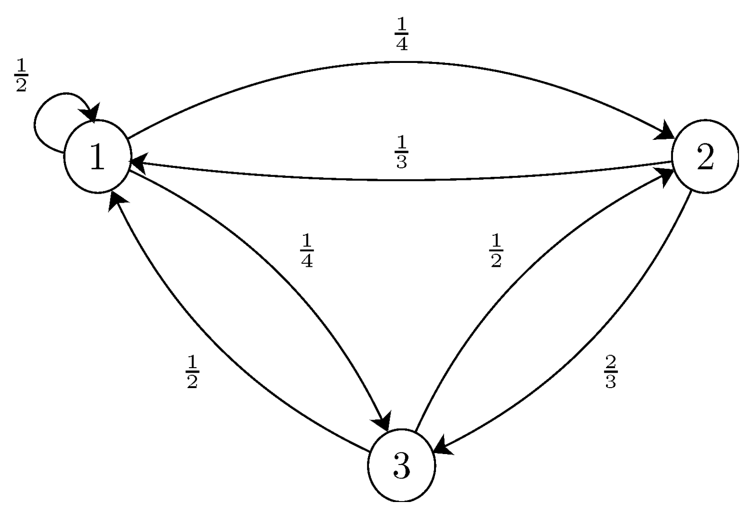

Figure 11.6 - A state transition diagram.

b. First, we obtain

$$\begin{aligned} P(X_1 = 3) &= 1 - P(X_1 = 1) - P(X_1 = 2) \\ &= 1 - \frac{1}{4} - \frac{1}{4} \\ &= \frac{1}{2}. \end{aligned}$$

We can now write

$$\begin{aligned} P(X_1 = 3, X_2 = 2, X_3 = 1) &= P(X_1 = 3) \cdot p_{32} \cdot p_{21} \\ &= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{3} \\ &= \frac{1}{12}. \end{aligned}$$

Problem 2

Consider the Markov chain in Figure 11.17. There are two recurrent classes, $R_1 = \{1, 2\}$, and $R_2 = \{5, 6, 7\}$. Assuming $X_0 = 3$, find the probability that the chain gets absorbed in R_1.

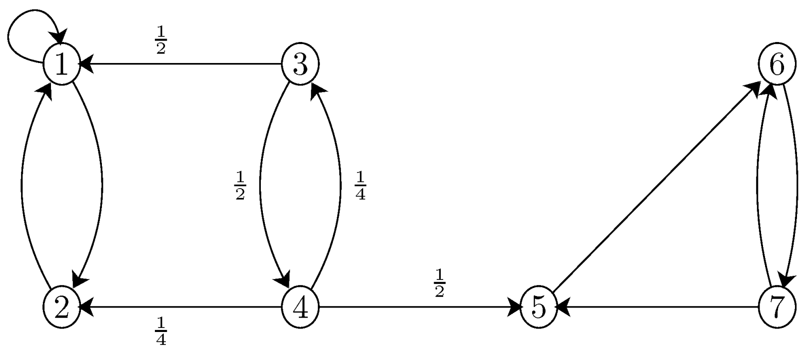

Figure 11.17 - A state transition diagram.

Solution

Here, we can replace each recurrent class with one absorbing state. The resulting state diagram is shown in Figure 11.18

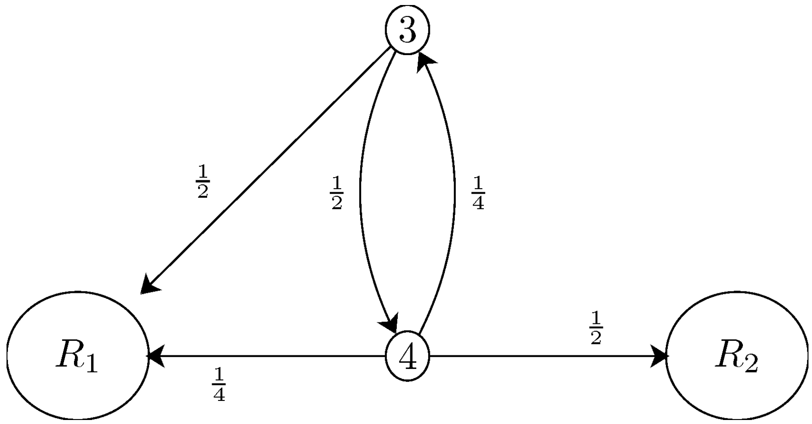

Figure 11.18 - The state transition diagram in which we have replaced each recurrent class with one absorbing state.

Now we can apply our standard methodology to find probability of absorption in state R_1. In particular, define

$$a_i = P(\text{absorption in } R_1 | X_0 = i), \quad \text{for all } i \in S.$$

By the above definition, we have $a_{R_1} = 1$, and $a_{R_2} = 0$. To find the unknown values of a_i's, we can use the following equations

$$a_i = \sum_k a_k p_{ik}, \quad \text{for } i \in S.$$

We obtain

$$\begin{aligned} a_3 &= \frac{1}{2} a_{R_1} + \frac{1}{2} a_4 \\ &= \frac{1}{2} + \frac{1}{2} a_4, \end{aligned}$$

$$\begin{aligned} a_4 &= \frac{1}{4} a_{R_1} + \frac{1}{4} a_3 + \frac{1}{2} a_{R_2} \\ &= \frac{1}{4} + \frac{1}{4} a_3. \end{aligned}$$

Solving the above equations, we obtain

$$a_3 = \frac{5}{7}, \quad a_4 = \frac{3}{7}.$$

Therefore, if $X_0 = 3$, the chain will end up in class R_1 with probability $a_3 = \frac{5}{7}$.

Problem 3

Consider the Markov chain of Example 2. Again assume $X_0 = 3$. We would like to find the expected time (number of steps) until the chain gets absorbed in R_1 or R_2. More specifically, let T be the absorption time, i.e., the first time the chain visits a state in R_1 or R_2. We would like to find $E[T|X_0 = 3]$.

Solution

Here we follow our standard procedure for finding mean hitting times. Consider Figure 11.18. Let T be the first time the chain visits R_1 or R_2. For all $i \in S$, define

$$t_i = E[T|X_0 = i].$$

By the above definition, we have $t_{R_1} = t_{R_2} = 0$. To find t_3 and t_4, we can use the following equations

$$t_i = 1 + \sum_k t_k p_{ik}, \quad \text{ for } i = 3, 4.$$

Specifically, we obtain

$$\begin{aligned} t_3 &= 1 + \frac{1}{2} t_{R_1} + \frac{1}{2} t_4 \\ &= 1 + \frac{1}{2} t_4, \end{aligned}$$

$$\begin{aligned} t_4 &= 1 + \frac{1}{4} t_{R_1} + \frac{1}{4} t_3 + \frac{1}{2} t_{R_2} \\ &= 1 + \frac{1}{4} t_3. \end{aligned}$$

Solving the above equations, we obtain

$$t_3 = \frac{12}{7}, \quad t_4 = \frac{10}{7}.$$

Therefore, if $X_0 = 3$, it will take on average $\frac{12}{7}$ steps until the chain gets absorbed in R_1 or R_2.

Problem 4

Consider the Markov chain shown in Figure 11.19. Assume $X_0 = 1$, and let R be the first time that the chain returns to state 1, i.e.,

$$R = \min\{n \geq 1 : X_n = 1\}.$$

Find $E[R|X_0 = 1]$.

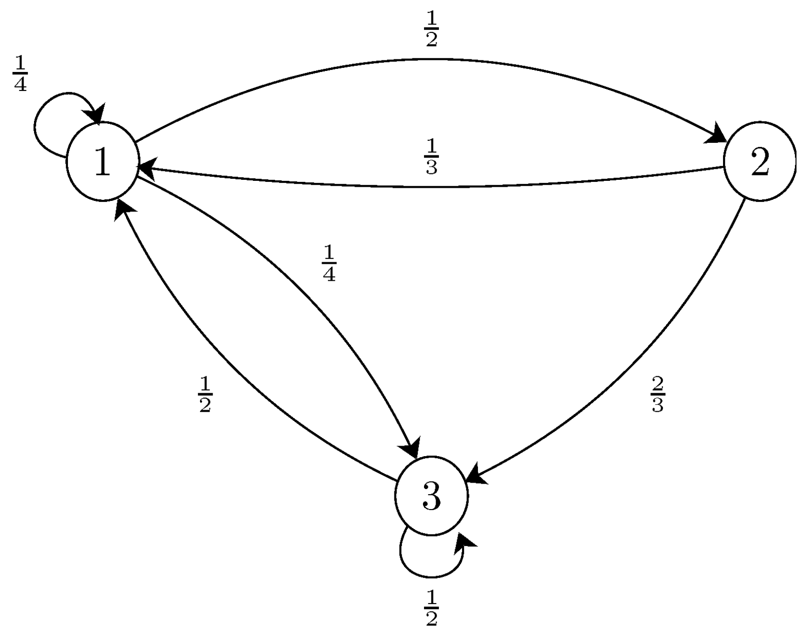

Figure 11.19 - A state transition diagram.

Solution

In this question, we are asked to find the mean return time to state 1. Let r_1 be the mean return time to state 1, i.e., $r_1 = E[R|X_0 = 1]$. Then

$$r_1 = 1 + \sum_k t_k p_{1k},$$

where t_k is the expected time until the chain hits state 1 given $X_0 = k$. Specifically,

$$\begin{aligned} t_1 &= 0, \\ t_k &= 1 + \sum_j t_j p_{kj}, \quad \text{for } k \neq 1. \end{aligned}$$

So, let's first find t_k's. We obtain

$$\begin{aligned} t_2 &= 1 + \frac{1}{3} t_1 + \frac{2}{3} t_3 \\ &= 1 + \frac{2}{3} t_3, \end{aligned}$$

$$\begin{aligned} t_3 &= 1 + \frac{1}{2} t_3 + \frac{1}{2} t_1 \\ &= 1 + \frac{1}{2} t_3. \end{aligned}$$

Solving the above equations, we obtain

$$t_3 = 2, \quad t_2 = \frac{7}{3}.$$

Now, we can write

$$\begin{aligned} r_1 &= 1 + \frac{1}{4}t_1 + \frac{1}{2}t_2 + \frac{1}{4}t_3 \\ &= 1 + \frac{1}{4} \cdot 0 + \frac{1}{2} \cdot \frac{7}{3} + \frac{1}{4} \cdot 2 \\ &= \frac{8}{3}. \end{aligned}$$

Problem 5

Consider the Markov chain shown in Figure 11.20.

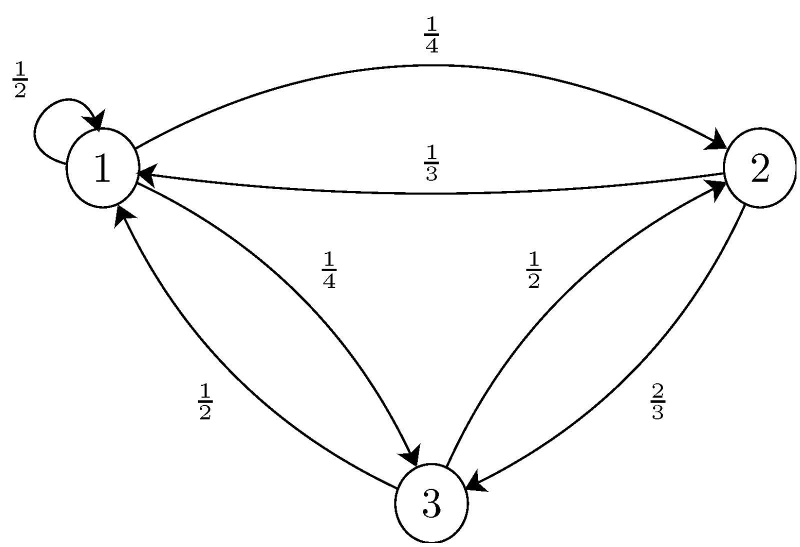

Figure 11.20 - A state transition diagram.

a. Is this chain irreducible?
b. Is this chain aperiodic?
c. Find the stationary distribution for this chain.
d. Is the stationary distribution a limiting distribution for the chain?

Solution

a. The chain is irreducible since we can go from any state to any other states in a finite number of steps.
b. The chain is aperiodic since there is a self-transition, i.e., $p_{11} > 0$.
c. To find the stationary distribution, we need to solve

$$\begin{aligned}\pi_1 &= \frac{1}{2}\pi_1 + \frac{1}{3}\pi_2 + \frac{1}{2}\pi_3, \\ \pi_2 &= \frac{1}{4}\pi_1 + \frac{1}{2}\pi_3, \\ \pi_3 &= \frac{1}{4}\pi_1 + \frac{2}{3}\pi_2, \\ \pi_1 + \pi_2 + \pi_3 &= 1.\end{aligned}$$

We find

$$\pi_1 \approx 0.457, \ \pi_2 \approx 0.257, \ \pi_3 \approx 0.286$$

d. The above stationary distribution is a limiting distribution for the chain because the chain is irreducible and aperiodic.

Problem 6

Consider the Markov chain shown in Figure 11.21. Assume that $\frac{1}{2} < p < 1$. Does this chain have a limiting distribution? For all $i, j \in \{0, 1, 2, \cdots\}$, find

$$\lim_{n \to \infty} P(X_n = j | X_0 = i).$$

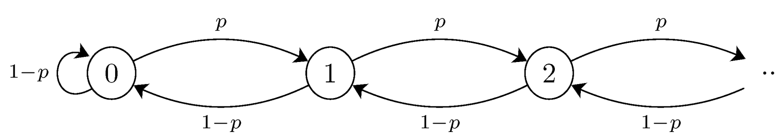

Figure 11.21 - A state transition diagram.

Solution

This chain is irreducible since all states communicate with each other. It is also aperiodic since it includes a self-transition, $P_{00} > 0$. Let's write the equations for a stationary distribution. For state 0, we can write

$$\pi_0 = (1-p)\pi_0 + (1-p)\pi_1,$$

which results in

$$\pi_1 = \frac{p}{1-p}\pi_0.$$

For state 1, we can write

$$\begin{aligned}\pi_1 &= p\pi_0 + (1-p)\pi_2 \\ &= (1-p)\pi_1 + (1-p)\pi_2,\end{aligned}$$

which results in

$$\pi_2 = \frac{p}{1-p}\pi_1.$$

Similarly, for any $j \in \{1, 2, \cdots\}$, we obtain

$$\pi_j = \alpha \pi_{j-1},$$

where $\alpha = \frac{p}{1-p}$. Note that since $\frac{1}{2} < p < 1$, we conclude that $\alpha > 1$. We obtain

$$\pi_j = \alpha^j \pi_0, \quad \text{ for } j = 1, 2, \cdots.$$

Finally, we must have

$$\begin{aligned}1 &= \sum_{j=0}^{\infty} \pi_j \\ &= \sum_{j=0}^{\infty} \alpha^j \pi_0, \qquad (\text{where } \alpha > 1) \\ &= \infty \pi_0.\end{aligned}$$

Therefore, the above equation cannot be satisfied if $\pi_0 > 0$. If $\pi_0 = 0$, then all π_j's must be zero, so they cannot sum to 1. We conclude that there is no stationary distribution. This means that either all states are transient, or all states are null recurrent. In either case, we have

$$\lim_{n \rightarrow \infty} P(X_n = j | X_0 = i) = 0, \text{ for all } i, j.$$

We will see how to figure out if the states are transient or null recurrent in the End of Chapter Problems (see Problem 15 in Section 11.5).
