8.4.5 Likelihood Ratio Tests

So far we have focused on specific examples of hypothesis testing problems. Here, we would like to introduce a relatively general hypothesis testing procedure called the *likelihood ratio test*. Before doing so, let us quickly review the definition of the likelihood function, which was previously discussed in Section 8.2.3.

Review of the Likelihood Function:

Let X_1, X_2, X_3, $\ldots$, X_n be a random sample from a distribution with a parameter θ. Suppose that we have observed $X_1 = x_1$, $X_2 = x_2$, $\cdots$, $X_n = x_n$.

- If the X_i's are discrete, then the **likelihood function** is defined as

$$L(x_1, x_2, \cdots, x_n; \theta) = P_{X_1 X_2 \cdots X_n}(x_1, x_2, \cdots, x_n; \theta).$$

- If the X_i's are jointly continuous, then the likelihood function is defined as

$$L(x_1, x_2, \cdots, x_n; \theta) = f_{X_1 X_2 \cdots X_n}(x_1, x_2, \cdots, x_n; \theta).$$

Likelihood Ratio Tests:

Consider a hypothesis testing problem in which both the null and the alternative hypotheses are simple. That is

H_0: $\theta = \theta_0$,

H_1: $\theta = \theta_1$.

Now, let X_1, X_2, X_3, $\ldots$, X_n be a random sample from a distribution with a parameter θ. Suppose that we have observed $X_1 = x_1$, $X_2 = x_2$, $\cdots$, $X_n = x_n$. One way to decide between H_0 and H_1 is to compare the corresponding likelihood functions:

$$l_0 = L(x_1, x_2, \cdots, x_n; \theta_0), \qquad l_1 = L(x_1, x_2, \cdots, x_n; \theta_1).$$

More specifically, if l_0 is much larger than l_1, we should accept H_0. On the other hand if l_1 is much larger, we tend to reject H_0. Therefore, we can look at the ratio $\frac{l_0}{l_1}$ to decide between H_0 and H_1. This is the idea behind *likelihood ratio tests*.

Likelihood Ratio Test for Simple Hypotheses

Let X_1, X_2, X_3, ..., X_n be a random sample from a distribution with a parameter θ. Suppose that we have observed $X_1 = x_1$, $X_2 = x_2$, $\cdots$, $X_n = x_n$. To decide between two simple hypotheses

H_0: $\theta = \theta_0$,

H_1: $\theta = \theta_1$,

we define

$$\lambda(x_1, x_2, \cdots, x_n) = \frac{L(x_1, x_2, \cdots, x_n; \theta_0)}{L(x_1, x_2, \cdots, x_n; \theta_1)}.$$

To perform a **likelihood ratio test (LRT)**, we choose a constant c. We reject H_0 if $\lambda < c$ and accept it if $\lambda \geq c$. The value of c can be chosen based on the desired α.

Let's look at an example to see how we can perform a likelihood ratio test.

Example 8.30

Here, we look again at the radar problem (Example 8.23). More specifically, we observe the random variable X:

$$X = \theta + W,$$

where $W \sim N(0, \sigma^2 = \frac{1}{9})$. We need to decide between

H_0: $\theta = \theta_0 = 0$,

H_1: $\theta = \theta_1 = 1$.

Let $X = x$. Design a level 0.05 test ($\alpha = 0.05$) to decide between H_0 and H_1.

Solution

If $\theta = \theta_0 = 0$, then $X \sim N(0, \sigma^2 = \frac{1}{9})$. Therefore,

$$L(x; \theta_0) = f_X(x; \theta_0) = \frac{3}{\sqrt{2\pi}} e^{-\frac{9x^2}{2}}.$$

On the other hand, if $\theta = \theta_1 = 1$, then $X \sim N(1, \sigma^2 = \frac{1}{9})$. Therefore,

$$L(x; \theta_1) = f_X(x; \theta_1) = \frac{3}{\sqrt{2\pi}} e^{-\frac{9(x-1)^2}{2}}.$$

Therefore,

$$\begin{aligned}\lambda(x) = \frac{L(x; \theta_0)}{L(x; \theta_1)} &= \exp\left\{-\frac{9x^2}{2} + \frac{9(x-1)^2}{2}\right\} \\ &= \exp\left\{\frac{9(1-2x)}{2}\right\}.\end{aligned}$$

Thus, we accept H_0 if

$$\exp\left\{\frac{9(1-2x)}{2}\right\} \geq c,$$

where c is the threshold. Equivalently, we accept H_0 if

$$x \leq \frac{1}{2}\left(1 - \frac{2}{9}\ln c\right).$$

Let us define $c' = \frac{1}{2}\left(1 - \frac{2}{9}\ln c\right)$, where c' is a new threshold. Remember that x is the observed value of the random variable X. Thus, we can summarize the decision rule as follows. We accept H_0 if

$$X \leq c'.$$

How to do we choose c'? We use the required α.

$$\begin{aligned}P(\text{type I error}) &= P(\text{Reject } H_0 \mid H_0) \\ &= P(X > c' \mid H_0) \\ &= P(X > c') \quad \left(\text{where } X \sim N\left(0, \frac{1}{3}\right)\right) \\ &= 1 - \Phi(3c').\end{aligned}$$

Letting $P(\text{type I error}) = \alpha$, we obtain

$$c' = \frac{1}{3}\Phi^{-1}(1-\alpha).$$

Letting $\alpha = 0.05$, we obtain

$$c' = \frac{1}{3}\Phi^{-1}(.95) = 0.548$$

As we see, in this case, the likelihood ratio test is exactly the same test that we obtained in Example 8.23.

How do we perform the likelihood ratio test if the hypotheses are not simple? Suppose that θ is an unknown parameter. Let S be the set of possible values for θ and suppose that we can partition S into two disjoint sets S_0 and S_1. Consider the following hypotheses:

H_0: $\theta \in S_0$,

H_1: $\theta \in S_1$.

The idea behind the general likelihood ratio test can be explained as follows: We first find the likelihoods corresponding to the most likely values of θ in S_0 and S_1 respectively. That is, we find

$$l_0 = \max\{L(x_1, x_2, \cdots, x_n; \theta) : \theta \in S_0\},$$
$$l = \max\{L(x_1, x_2, \cdots, x_n; \theta) : \theta \in S\}.$$

(To be more accurate, we need to replace $\max$ by $\sup$.) Let us consider two extreme cases. First, if $l_0 = l$, then we can say that the most likely value of θ belongs to S_0. This indicates that we should not reject H_0. On the other hand, if $\frac{l_0}{l_1}$ is much smaller than 1, we should probably reject H_0 in favor of H_1. To conduct a likelihood ratio test, we choose a threshold $0 \leq c \leq 1$ and compare $\frac{l_0}{l}$ to c. If $\frac{l_0}{l} \geq c$, we accept H_0. If $\frac{l_0}{l} < c$, we reject H_0. The value of c can be chosen based on the desired α.

Likelihood Ratio Tests

Let X_1, X_2, X_3, ..., X_n be a random sample from a distribution with a parameter θ. Suppose that we have observed $X_1 = x_1$, $X_2 = x_2$, $\cdots$, $X_n = x_n$. Define

$$\lambda(x_1, x_2, \cdots, x_n) = \frac{\sup\{L(x_1, x_2, \cdots, x_n; \theta) : \theta \in S_0\}}{\sup\{L(x_1, x_2, \cdots, x_n; \theta) : \theta \in S\}}.$$

To perform a **likelihood ratio test (LRT)**, we choose a constant c in $[0, 1]$. We reject H_0 if $\lambda < c$ and accept it if $\lambda \geq c$. The value of c can be chosen based on the desired α.