$$
\begin{aligned}
E[Y(t)^2] &= \int_{-\infty}^{\infty} S_Y(f)\, df \\
&= \int_{-f_2}^{-f_1} S_X(f)\, df + \int_{f_1}^{f_2} S_X(f)\, df \\
&= 2\int_{f_1}^{f_2} S_X(f)\, df\ b\big(\text{since } S_X(-f) = S_X(f)\big)
\end{aligned}
$$

Therefore, we conclude that, if we integrate $S_X(f)$ over the frequency range $f_1 < |f| < f_2$, we will obtain the expected power in $X(t)$ in that frequency range. That is why $S_X(f)$ is called the *power spectral density* of $X(t)$.

Gaussian Processes through LTI Systems:

Let $X(t)$ be a stationary Gaussian random process that goes through an LTI system with impulse response $h(t)$. Then, the output process is given by

$$
\begin{aligned}
Y(t) &= h(t) * X(t) \\
&= \int_{-\infty}^{\infty} h(\alpha) X(t-\alpha)\, d\alpha.
\end{aligned}
$$

For each t, you can think of the above integral as a limit of a sum. Now, since the different sums of jointly normal random variables are also jointly normal, you can argue that $Y(t)$ is also a Gaussian random process. Indeed, we can conclude that $X(t)$ and $Y(t)$ are jointly normal. Note that, for Gaussian processes, stationarity and wide-sense stationarity are equal.

> Let $X(t)$ be a stationary Gaussian process. If $X(t)$ is the input to an LTI system, then the output random process, $Y(t)$, is also a stationary Gaussian process. Moreover, $X(t)$ and $Y(t)$ are jointly Gaussian.

Example 10.15

Let $X(t)$ be a zero-mean Gaussian random process with $R_X(\tau) = 8\,\mathrm{sinc}(4\tau)$. Suppose that $X(t)$ is input to an LTI system with transfer function

$$H(f) = \begin{cases} \frac{1}{2} & |f| < 1 \\ \\ 0 & \text{otherwise} \end{cases}$$

If $Y(t)$ is the output, find $P(Y(2) < 1 | Y(1) = 1)$.

Solution

Since $X(t)$ is a WSS Gaussian process, $Y(t)$ is also a WSS Gaussian process. Thus, it suffices to find μ_Y and $R_Y(\tau)$. Since $\mu_X = 0$, we have

$$\mu_Y = \mu_X H(0) = 0.$$

Also, note that

$$\begin{aligned} S_X(f) &= \mathcal{F}\{R_X(\tau)\} \\ &= \begin{cases} 2 & |f| < 2 \\ \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

We can then find $S_Y(f)$ as

$$\begin{aligned} S_Y(f) &= S_X(f)|H(f)|^2 \\ &= \begin{cases} \frac{1}{2} & |f| < 1 \\ \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Thus, $R_Y(\tau)$ is given by

$$\begin{aligned} R_Y(\tau) &= \mathcal{F}^{-1}\{S_X(f)\} \\ &= \text{sinc}(2\tau). \end{aligned}$$

Therefore,

$$E[Y(t)^2] = R_Y(0) = 1.$$

We conclude that $Y(t) \sim N(0, 1)$, for all t. Since $Y(1)$ and $Y(2)$ are jointly Gaussian, to determine their joint PDF, it only remains to find their covariance. We have

$$\begin{aligned} E[Y(1)Y(2)] &= R_Y(-1) \\ &= \text{sinc}(-2) \\ &= \frac{\sin(-2\pi)}{-2\pi} \\ &= 0. \end{aligned}$$

Since $E[Y(1)] = E[Y(2)] = 0$, we conclude that $Y(1)$ and $Y(2)$ are uncorrelated. Since $Y(1)$ and $Y(2)$ are jointly normal, we conclude that they are independent, so

$$\begin{aligned} P(Y(2) < 1|Y(1) = 1) &= P(Y(2) < 1) \\ &= \Phi(1) \approx 0.84 \end{aligned}$$
