5.4.0 End of Chapter Problems

Problem 1

Consider two random variables X and Y with joint PMF given in Table 5.4

Joint PMF of X and Y in Problem 1

	$Y=1$	$Y=2$
$X=1$	$\frac{1}{3}$	$\frac{1}{12}$
$X=2$	$\frac{1}{6}$	0
$X=4$	$\frac{1}{12}$	$\frac{1}{3}$

a. Find $P(X \leq 2, Y > 1)$.
b. Find the marginal PMFs of X and Y.
c. Find $P(Y=2|X=1)$.
d. Are X and Y independent?

Problem 2

Let X and Y be as defined in Problem 1. I define a new random variable $Z = X - 2Y$.

a. Find the PMF of Z.
b. Find $P(X=2|Z=0)$.

Problem 3

A box contains two coins: a regular coin and a biased coin with $P(H) = \frac{2}{3}$. I choose a coin at random and toss it once. I define the random variable X as a Bernoulli random variable associated with this coin toss, i.e., $X=1$ if the result of the coin toss is heads and $X=0$ otherwise. Then I take the remaining coin in the box and toss it once. I define the random variable Y as a Bernoulli random variable associated with the second coin toss. Find the joint PMF of X and Y. Are X and Y independent?

Problem 4

Consider two random variables X and Y with joint PMF given by

$$P_{XY}(k,l) = \frac{1}{2^{k+l}}, \text{for } k,l = 1,2,3,\ldots$$

a. Show that X and Y are independent and find the marginal PMFs of X and Y.
b. Find $P(X^2+Y^2 \leq 10)$.

Problem 5

Let X and Y be as defined in Problem 1. Also, suppose that we are given that $Y=1$.

a. Find the conditional PMF of X given $Y=1$. That is, find $P_{X|Y}(x|1)$.
b. Find $E[X|Y=1]$.
c. Find $Var(X|Y=1)$.

Problem 6

The number of customers visiting a store in one hour has a Poisson distribution with mean $\lambda=10$. Each customer is a female with probability $p=\frac{3}{4}$ independent of other customers. Let X be the total number of customers in a one-hour interval and Y be the total number of female customers in the same interval. Find the joint PMF of X and Y.

Problem 7

Let $X \sim Geometric(p)$. Find $Var(X)$ as follows: Find EX and EX^2 by conditioning on the result of the first "coin toss", and use $Var(X) = EX^2 - (EX)^2$.

Problem 8

Let X and Y be two independent $Geometric(p)$ random variables. Find $E\left[\frac{X^2+Y^2}{XY}\right]$.

Problem 9

Consider the set of points in the set C:

$$C = \{(x,y) | x,y \in \mathbb{Z}, x^2+|y| \leq 2\}.$$

Suppose that we pick a point (X, Y) from this set completely at random. Thus, each point has a probability of $\frac{1}{11}$ of being chosen.

a. Find the joint and marginal PMFs of X and Y.
b. Find the conditional PMF of X given $Y = 1$.
c. Are X and Y independent?
d. Find $E[XY^2]$.

Problem 10

Consider the set of points in the set C:

$$C = \{(x, y) | x, y \in \mathbb{Z}, x^2 + |y| \leq 2\}.$$

Suppose that we pick a point (X, Y) from this set completely at random. Thus, each point has a probability of $\frac{1}{11}$ of being chosen.

a. Find $E[X|Y = 1]$.
b. Find $Var(X|Y = 1)$.
c. Find $E[X||Y| \leq 1]$.
d. Find $E[X^2||Y| \leq 1]$.

Problem 11

The number of cars being repaired at a small repair shop has the following PMF:

$$P_N(n) = \begin{cases} \frac{1}{8} & \text{for } n = 0 \\ \frac{1}{8} & \text{for } n = 1 \\ \frac{1}{4} & \text{for } n = 2 \\ \frac{1}{2} & \text{for } n = 3 \\ 0 & \text{otherwise} \end{cases}$$

Each car that is being repaired is a four-door car with probability $\frac{3}{4}$ and a two-door car with probability $\frac{1}{4}$, independently from other cars and independently from the number of cars being repaired. Let X be the number of four-door cars and Y be the number of two-door cars currently being repaired.

a. Find the marginal PMFs of X and Y.
b. Find the joint PMF of X and Y.
c. Are X and Y independent?

Problem 12

Let X and Y be two independent random variables with PMFs

$$P_X(k) = P_Y(k) = \begin{cases} \frac{1}{5} & \text{for } x = 1, 2, 3, 4, 5 \\ 0 & \text{otherwise} \end{cases}$$

Define $Z = X - Y$. Find the PMF of Z.

Problem 13

Consider two random variables X and Y with joint PMF given in Table 5.5

Table 5.5: Joint PMF of X and Y in Problem 13

	$Y=0$	$Y=1$	$Y=2$
$X=0$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{8}$
$X=1$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{4}$

Define the random variable Z as $Z = E[X|Y]$.

a. Find the Marginal PMFs of X and Y.
b. Find the conditional PMF of X, given $Y = 0$ and $Y = 1$, i.e., find $P_{X|Y}(x|0)$ and $P_{X|Y}(x|1)$.
c. Find the PMF of Z.
d. Find EZ, and check that $EZ = EX$.
e. Find Var(Z).

Problem 14

Let X, Y, and $Z = E[X|Y]$ be as in Problem 13. Define the random variable V as $V = Var(X|Y)$.

a. Find the PMF of V.
b. Find EV.
c. Check that $Var(X) = EV + Var(Z)$.

Problem 15

Let N be the number of phone calls made by the customers of a phone company in a given hour. Suppose that $N \sim Poisson(\beta)$, where $\beta > 0$ is known. Let X_i be the length of the i'th phone call, for $i = 1, 2, \ldots, N$. We assume X_i's are independent of each other and also independent of N. We further assume

$$X_i \sim Exponential(\lambda),$$

where $\lambda > 0$ is known. Let Y be the sum of the lengths of the phone calls, i.e.,

$$Y = \sum_{i=1}^{N} X_i.$$

Find EY and Var(Y).

Problem 16

Let X and Y be two jointly continuous random variables with joint PDF

$$f_{XY}(x,y) = \begin{cases} \frac{1}{2}e^{-x} + \frac{cy}{(1+x)^2} & 0 \le x, 0 \le y \le 1 \\ \\ 0 & \text{otherwise} \end{cases}$$

a. Find the constant c.
b. Find $P(0 \le X \le 1, 0 \le Y \le \frac{1}{2})$.
c. Find $P(0 \le X \le 1)$.

Problem 17

Let X and Y be two jointly continuous random variables with joint PDF

$$f_{XY}(x,y) = \begin{cases} e^{-xy} & 1 \le x \le e, y > 0 \\ \\ 0 & \text{otherwise} \end{cases}$$

a. Find the marginal PDFs, $f_X(x)$ and $f_Y(y)$.
b. Write an integral to compute $P(0 \le Y \le 1, 1 \le X \le \sqrt{e})$.

Problem 18

Let X and Y be two jointly continuous random variables with joint PDF

$$f_{XY}(x,y) = \begin{cases} \frac{1}{4}x^2 + \frac{1}{6}y & -1 \leq x \leq 1, 0 \leq y \leq 2 \\ \\ 0 & \text{otherwise} \end{cases}$$

a. Find the marginal PDFs, $f_X(x)$ and $f_Y(y)$.
b. Find $P(X > 0, Y < 1)$.
c. Find $P(X > 0 \text{ or } Y < 1)$.
d. Find $P(X > 0 | Y < 1)$.
e. Find $P(X + Y > 0)$.

Problem 19

Let X and Y be two jointly continuous random variables with joint CDF

$$F_{XY}(x,y) = \begin{cases} 1 - e^{-x} - e^{-2y} + e^{-(x+2y)} & x, y > 0 \\ \\ 0 & \text{otherwise} \end{cases}$$

a. Find the joint PDF, $f_{XY}(x,y)$.
b. Find $P(X < 2Y)$.
c. Are X and Y independent?

Problem 20

Let $X \sim N(0,1)$.

a. Find the conditional PDF and CDF of X given $X > 0$.
b. Find $E[X|X > 0]$.
c. Find Var$(X|X > 0)$.

Problem 21

Let X and Y be two jointly continuous random variables with joint PDF

$$f_{XY}(x,y) = \begin{cases} x^2 + \frac{1}{3}y & -1 \leq x \leq 1, 0 \leq y \leq 1 \\ \\ 0 & \text{otherwise} \end{cases}$$

For $0 \leq y \leq 1$, find the following:

a. The conditional PDF of X given $Y = y$.

b. $P(X > 0|Y = y)$. Does this value depend on y?
c. Are X and Y independent?

Problem 22
Let X and Y be two jointly continuous random variables with joint PDF

$$f_{XY}(x,y) = \begin{cases} \frac{1}{2}x^2 + \frac{2}{3}y & -1 \leq x \leq 1, 0 \leq y \leq 1 \\ \\ 0 & \text{otherwise} \end{cases}$$

Find $E[Y|X = 0]$ and Var$(Y|X = 0)$.

Problem 23
Consider the set

$$E = \{(x,y)||x| + |y| \leq 1\}.$$

Suppose that we choose a point (X, Y) uniformly at random in E. That is, the joint PDF of X and Y is given by

$$f_{XY}(x,y) = \begin{cases} c & (x,y) \in E \\ \\ 0 & \text{otherwise} \end{cases}$$

a. Find the constant c.
b. Find the marginal PDFs $f_X(x)$ and $f_Y(y)$.
c. Find the conditional PDF of X given $Y = y$, where $-1 \leq y \leq 1$.
d. Are X and Y independent?

Problem 24
Let X and Y be two independent $Uniform(0, 2)$ random variables. Find $P(XY < 1)$.

Problem 25
Suppose $X \sim Exponential(1)$ and given $X = x$, Y is a uniform random variable in $[0, x]$, i.e.,

$$Y|X = x \quad \sim \quad Uniform(0, x),$$

or equivalently

$$Y|X \quad \sim \quad Uniform(0, X).$$

a. Find EY.
b. Find $Var(Y)$.

Problem 26

Let X and Y be two independent $Uniform(0,1)$ random variables. Find

a. $E[XY]$
b. $E[e^{X+Y}]$
c. $E[X^2+Y^2+XY]$
d. $E[Ye^{XY}]$

Problem 27

Let X and Y be two independent $Uniform(0,1)$ random variables, and $Z=\frac{X}{Y}$. Find the CDF and PDF of Z.

Problem 28

Let X and Y be two independent $N(0,1)$ random variables, and $U=X+Y$.

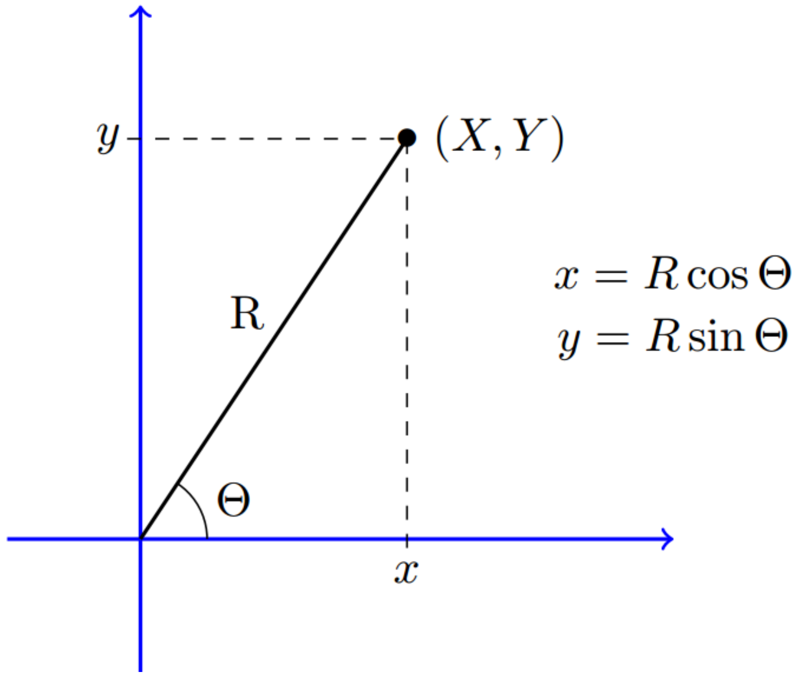
a. Find the conditional PDF of U given $X=x$, $f_{U|X}(u|x)$.
b. Find the PDF of U, $f_U(u)$.
c. Find the conditional PDF of X given $U=u$, $f_{X|U}(x|u)$.
d. Find $E[X|U=u]$, and $Var(X|U=u)$.

Problem 29

Let X and Y be two independent standard normal random variables. Consider the point (X,Y) in the $x-y$ plane. Let (R,Θ) be the corresponding polar coordinates as shown in Figure 5.11. The inverse transformation is given by

$$\begin{cases} X=R\cos\Theta \\ Y=R\sin\Theta \end{cases}$$

where, $R\geq 0$ and $-\pi<\Theta\leq\pi$. Find the joint PDF of R and Θ. Show that R and Θ are independent.



Problem 30

In Problem 29, suppose that X and Y are independent $Uniform(0,1)$ random variables. Find the joint PDF of R and Θ. Are R and Θ independent?

Problem 31

Consider two random variables X and Y with joint PMF given in Table 5.6.

Table 5.6: Joint PMF of X and Y in Problem 31

	$Y = 0$	$Y = 1$	$Y = 2$
$X = 0$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$
$X = 1$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$

Find $Cov(X, Y)$ and $\rho(X, Y)$.

Problem 32

Let X and Y be two independent $N(0,1)$ random variable and

$$Z = 11 - X + X^2 Y,$$
$$W = 3 - Y.$$

Find Cov(Z, W).

Problem 33

Let X and Y be two random variables. Suppose that $\sigma_X^2 = 4$, and $\sigma_Y^2 = 9$. If we know that the two random variables $Z = 2X - Y$ and $W = X + Y$ are independent, find $Cov(X, Y)$ and $\rho(X, Y)$.

Problem 34

Let $X \sim Uniform(1, 3)$ and $Y|X \sim Exponential(X)$. Find $Cov(X, Y)$.

Problem 35

Let X and Y be two independent $N(0, 1)$ random variable and

$$Z = 7 + X + Y,$$
$$W = 1 + Y.$$

Find $\rho(Z, W)$.

Problem 36

Let X and Y be jointly normal random variables with parameters $\mu_X = -1$, $\sigma_X^2 = 4$, $\mu_Y = 1$, $\sigma_Y^2 = 1$, and $\rho = -\frac{1}{2}$.

a. Find $P(X + 2Y \leq 3)$.
b. Find $Cov(X - Y, X + 2Y)$.

Problem 37

Let X and Y be jointly normal random variables with parameters $\mu_X = 1$, $\sigma_X^2 = 4$, $\mu_Y = 1$, $\sigma_Y^2 = 1$, and $\rho = 0$.

a. Find $P(X + 2Y > 4)$.
b. Find $E[X^2 Y^2]$.

Problem 38

Let X and Y be jointly normal random variables with parameters $\mu_X = 2$, $\sigma_X^2 = 4$, $\mu_Y = 1$, $\sigma_Y^2 = 9$, and $\rho = -\frac{1}{2}$.

a. Find $E[Y|X = 3]$.

b. Find $Var(Y|X=2)$.

c. Find $P(X+2Y \leq 5|X+Y=3)$.