
6.3.0 Chapter Problems

Problem 1

Let X, Y and Z be three jointly continuous random variables with joint PDF

$$f_{XYZ}(x,y,z) = \begin{cases} x+y & 0 \leq x,y,z \leq 1 \\ & \\ 0 & \text{otherwise} \end{cases}$$

1. Find the joint PDF of X and Y.
2. Find the marginal PDF of X.
3. Find the conditional PDF of $f_{XY|Z}(x,y|z)$ using

$$f_{XY|Z}(x,y|z) = \frac{f_{XYZ}(x,y,z)}{f_Z(z)}.$$

4. Are X and Y independent of Z?

Problem 2

Suppose that X, Y, and Z are three independent random variables. If $X, Y \sim N(0,1)$ and $Z \sim Exponential(1)$, find

1. $E[XY|Z=1]$,
2. $E[X^2Y^2Z^2|Z=1]$.

Problem 3

Let X, Y, and Z be three independent $N(1,1)$ random variables. Find $E[XY|Y+Z=1]$.

Problem 4

Let $X_1, X_2, \cdots, X_n$ be i.i.d. random variables, where $X_i \sim Bernoulli(p)$. Define

$$\begin{aligned} Y_1 &= X_1 X_2, \\ Y_2 &= X_2 X_3, \\ &\vdots \\ Y_{n-1} &= X_{n-1} X_n, \\ Y_n &= X_n X_1. \end{aligned}$$

If $Y = Y_1 + Y_2 + \cdots + Y_n$, find

1. $E[Y]$,
2. $\mathrm{Var}(Y)$.

Problem 5

In this problem, our goal is to find the variance of the hypergeometric distribution. Let's remember the random experiment behind the hypergeometric distribution. You have a bag that contains b blue marbles and r red marbles. You choose $k \leq b + r$ marbles at random (without replacement) and let X be the number of blue marbles in your sample. Then $X \sim Hypergeometric(b, r, k)$. Now let us define the indicator random variables X_i as follows.

$$X_i = \begin{cases} 1 & \text{if the } i\text{th chosen marble is blue} \\ 0 & \text{otherwise} \end{cases}$$

Then, we can write

$$X = X_1 + X_2 + \cdots + X_k.$$

Using the above equation, show

1. $EX = \frac{kb}{b+r}$,
2. $\mathrm{Var}(X) = \frac{kbr}{(b+r)^2} \frac{b+r-k}{b+r-1}$.

Problem 6

(MGF of the geometric distribution) If $X \sim Geometric(p)$, find the MGF of X.

Problem 7

If $M_X(s) = \frac{1}{4} + \frac{1}{2}e^s + \frac{1}{4}e^{2s}$, find EX and $\mathrm{Var}(X)$.

Problem 8

Using MGFs show that if $X \sim N(\mu_X, \sigma_X^2)$ and $Y \sim N(\mu_Y, \sigma_Y^2)$ are independent, then

$$X + Y \sim N\left(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2\right).$$

Problem 9
(MGF of the Laplace distribution) Let X be a continuous random variable with the following PDF

$$f_X(x) = \frac{\lambda}{2} e^{-\lambda |x|}.$$

Find the MGF of X, $M_X(s)$.

Problem 10
(MGF of Gamma distribution) Remember that a continuous random variable X is said to have a *Gamma* distribution with parameters $\alpha > 0$ and $\lambda > 0$, shown as $X \sim Gamma(\alpha, \lambda)$, if its PDF is given by

$$f_X(x) = \begin{cases} \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

If $X \sim Gamma(\alpha, \lambda)$, find the MGF of X. *Hint:* Remember that $\int_0^\infty x^{\alpha-1} e^{-\lambda x} \mathrm{d}x = \frac{\Gamma(\alpha)}{\lambda^\alpha}, \text{for } \alpha, \lambda > 0.$

Problem 11
Using the MGFs show that if $Y = X_1 + X_2 + \cdots + X_n$, where the X_i's are independent $Exponential(\lambda)$ random variables, then $Y \sim Gamma(n, \lambda)$.

Problem 12
Let X be a random variable with characteristic function $\phi_X(\omega)$. If $Y = aX + b$, show that

$$\phi_Y(\omega) = e^{j\omega b} \phi_X(a\omega).$$

Problem 13
Let X and Y be two jointly continuous random variables with joint PDF

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{2}(3x+y) & 0 \leq x, y \leq 1 \\ \\ 0 & \text{otherwise} \end{cases}$$

and let the random vector **U** be defined as

$$U = \begin{bmatrix} X \\ Y \end{bmatrix}.$$

1. Find the mean vector of **U**, $E\mathbf{U}$.
2. Find the correlation matrix of **U**, $\mathbf{R_U}$.
3. Find the covariance matrix of **U**, $\mathbf{C_U}$.

Problem 14

Let $X \sim Uniform(0,1)$. Suppose that given $X = x$, Y and Z are independent and $Y|X = x \sim Uniform(0,x)$ and $Z|X = x \sim Uniform(0,2x)$. Define the random vector **U** as

$$U = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}.$$

1. Find the PDFs of Y and Z.
2. Find the PDF of **U**, $f_\mathbf{U}(\mathbf{u})$, by using

$$\begin{aligned} f_\mathbf{U}(\mathbf{u}) &= f_{XYZ}(x,y,z) \\ &= f_X(x) f_{Y|X}(y|x) f_{Z|X,Y}(z|x,y). \end{aligned}$$

Problem 15

Let

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

be a normal random vector with the following mean and covariance matrices

$$m = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, C = \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix}.$$

Let also

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 1 \\ 1 & 3 \end{bmatrix},$$

$$b = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},$$

$$Y = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = AX + b.$$

1. Find $P(X_2 > 0)$.
2. Find expected value vector of $\mathbf{Y}$, $\mathbf{m_Y} = E\mathbf{Y}$.
3. Find the covariance matrix of $\mathbf{Y}$, $\mathbf{C_Y}$.
4. Find $P(Y_2 \leq 2)$.

Problem 16

Let $X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$ be a normal random vector with the following mean and covariance

$$m = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 9 & 1 & -1 \\ 1 & 4 & 2 \\ -1 & 2 & 4 \end{bmatrix}.$$

Find the MGF of $\mathbf{X}$ defined as

$$M_{\mathbf{X}}(s,t,r) = E\left[e^{sX_1+tX_2+rX_3}\right].$$

Problem 17

A system consists of 4 components in a series, so the system works properly if all of the components are functional. In other words, the system fails if and only if at least one of its components fails. Suppose the probability that the component i fails is less than or equal to $p_f = \frac{1}{100}$, for $i = 1, 2, 3, 4$. Find an upper bound on the probability that the system fails.

Problem 18

A sensor network consists of n sensors that are distributed randomly on the unit square. Each node's location is uniform over the unit square and is independent of the locations of the other node. A node is isolated if there are no nodes that are within distance r of that node, where $0 < r < 1$.

1. Show that the probability that a given node is isolated is less than or equal to $p_d = (1 - \frac{\pi r^2}{4})^{(n-1)}$.
2. Using the union bound, find an upper bound on the probability that the sensor network contains at least one isolated node.

Problem 19

Let $X \sim Geometric(p)$. Using Markov's inequality find an upper bound for $P(X \geq a)$, for a positive integer a. Compare the upper bound with the real value of $P(X \geq a)$.

Problem 20

in $Geometric(p)$. Using Chebyshev's inequality find an upper bound for $P(|X - EX| \geq b)$.

Problem 21

(Cantelli's inequality [16]) Let X be a random variable with $EX = 0$ and $\mathrm{Var}(X) = \sigma^2$. We would like to prove that for any $a > 0$, we have

$$P(X \geq a) \leq \frac{\sigma^2}{\sigma^2 + a^2}.$$

This inequality is sometimes called the one-sided Chebyshev inequality. *Hint:* One way to show this is to use $P(X \geq a) = P(X + c \geq a + c)$ for any constant $c \in \mathbb{R}$.

Problem 22

The number of customers visiting a store during a day is a random variable with mean $EX = 100$ and variance $Var(X) = 225$.

1. Using Chebyshev's inequality, find an upper bound for having more than 120 or less than 80 customers in a day. That is, find an upper bound on

$$P(X \leq 80 \text{ or } X \geq 120).$$

2. Using the one-sided Chebyshev inequality (Problem 21), find an upper bound for having more than 120 customers in a day.

Problem 23

Let X_i be i.i.d. and $X_i \sim Exponential(\lambda)$. Using Chernoff bounds find an upper bound for $P(X_1 + X_2 + \cdots + X_n \geq a)$, where $a > \frac{n}{\lambda}$. Show that the bound goes to zero exponentially fast as a function of n.

Problem 24

(Minkowski's inequality [17]) Prove for two random variables X and Y with finite moments, and $1 \leq p < \infty$, we have

$$E[|X+Y|^p]^{\frac{1}{p}} \leq E[|X|^p]^{\frac{1}{p}} + E[|Y|^p]^{\frac{1}{p}}.$$

Hint: Note that

$$\begin{aligned} |X+Y|^p &= |X+Y|^{p-1}|X+Y| \\ &\leq |X+Y|^{p-1}(|X|+|Y|) \\ &\leq |X+Y|^{p-1}|X| + |X+Y|^{p-1}|Y|. \end{aligned}$$

Therefore

$$E|X+Y|^p \leq E[|X+Y|^{p-1}|X|] + E[|X+Y|^{p-1}|Y|].$$

Now, apply Hölder's inequality.

Problem 25

Let X be a positive random variable with $EX = 10$. What can you say about the following quantities?

1. $E[X - X^3]$
2. $E[X \ln \sqrt{X}]$
3. $E\left[|2 - X|\right]$

Problem 26

Let X be a random variable with $EX = 1$ and $R_X = (0, 2)$. If $Y = X^3 - 6X^2$, show that $EY \leq -5$.