7.2.0 Convergence of Random Variables

In some situations, we would like to see if a sequence of random variables X_1, X_2, X_3, $\cdots$ "converges" to a random variable X. That is, we would like to see if X_n gets closer and closer to X in some sense as n increases. For example, suppose that we are interested in knowing the value of a random variable X, but we are not able to observe X directly. Instead, you can do some measurements and come up with an estimate of X: call it X_1. You then perform more measurements and update your estimate of X and call it X_2. You continue this process to obtain X_1, X_2, X_3, $\cdots$. Your hope is that as n increases, your estimate gets better and better. That is, you hope that as n increases, X_n gets closer and closer to X. In other words, you hope that X_n converges to X.

In fact, we have already seen the concept of convergence in Section 7.1.0 when we discussed limit theorems (the weak law of large numbers (WLLN) and the central limit theorem (CLT)). The WLLN states that the average of a large number of i.i.d. random variables converges in probability to the expected value. The CLT states that the normalized average of a sequence of i.i.d. random variables converges in distribution to a standard normal distribution. In this section, we will develop the theoretical background to study the convergence of a sequence of random variables in more detail. In particular, we will define different types of convergence. When we say that the sequence X_n converges to X, it means that X_n's are getting "closer and closer" to X. Different types of convergence refer to different ways of defining what "closer" means. We also discuss how different types of convergence are related.