Consider a finite Markov chain with r states, $S = \{1, 2, \cdots, r\}$. Suppose that all states are transient. Then, starting from time 0, the chain might visit state 1 several times, but at some point the chain will leave state 1 and will never return to it. That is, there exists an integer $M_1 > 0$ such that $X_n \neq 1$, for all $n \geq M_1$. Similarly, there exists an integer $M_2 > 0$ such that $X_n \neq 2$, for all $n \geq M_2$, and so on. Now, if you choose

$$n \geq \max\{M_1, M_2, \cdots, M_r\},$$

then X_n cannot be equal to any of the states $1, 2, \cdots, r$. This is a contradiction, so we conclude that there must be at least one recurrent state, which means that there must be at least one recurrent class.

Periodicity:

Consider the Markov chain shown in Figure 11.11. There is a periodic pattern in this chain. Starting from state 0, we only return to 0 at times $n = 3, 6, \cdots$. In other words, $p_{00}^{(n)} = 0$, if n is not divisible by 3. Such a state is called a *periodic* state with period $d(0) = 3$.

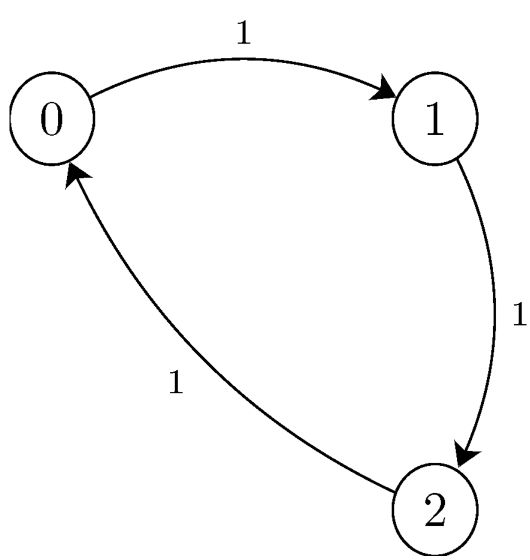

Figure 11.11 - A state transition diagram.

The **period** of a state i is the largest integer d satisfying the following property: $p_{ii}^{(n)} = 0$, whenever n is not divisible by d. The period of i is shown by $d(i)$. If $p_{ii}^{(n)} = 0$, for all $n > 0$, then we let $d(i) = \infty$.

$-$If $d(i) > 1$, we say that state i is **periodic**.
$-$If $d(i) = 1$, we say that state i is **aperiodic**.

You can show that all states in the same communicating class have the same period. A class is said to be periodic if its states are periodic. Similarly, a class is said to be aperiodic if its states are aperiodic. Finally, a Markov chain is said to be aperiodic if all of its states are aperiodic.

If $i \leftrightarrow j$, then $d(i) = d(j)$.

Why is periodicity important? As we will see shortly, it plays a roll when we discuss limiting distributions. It turns out that in a typical problem, we are given an irreducible Markov chain, and we need to check if it is aperiodic.

How do we check that a Markov chain is aperiodic? Here is a useful method. Remember that two numbers m and l are said to be *co-prime* if their greatest common divisor (gcd) is 1, i.e., $\gcd(l, m) = 1$. Now, suppose that we can find two co-prime numbers l and m such that $p_{ii}^{(l)} > 0$ and $p_{ii}^{(m)} > 0$. That is, we can go from state i to itself in l steps, and also in m steps. Then, we can conclude state i is aperiodic. If we have an irreducible Markov chain, this means that the chain is aperiodic. Since the number 1 is co-prime to every integer, any state with a self-transition is aperiodic.

Consider a finite irreducible Markov chain X_n:

a. If there is a self-transition in the chain ($p_{ii} > 0$ for some i), then the chain is aperiodic.
b. Suppose that you can go from state i to state i in l steps, i.e., $p_{ii}^{(l)} > 0$. Also suppose that $p_{ii}^{(m)} > 0$. If $\gcd(l, m) = 1$, then state i is aperiodic.
c. The chain is aperiodic if and only if there exists a positive integer n such that all elements of the matrix P^n are strictly positive, i.e.,

$$p_{ij}^{(n)} > 0, \quad \text{for all } i, j \in S.$$

Example 11.8

Consider the Markov chain in Example 11.6.

a. Is $\text{Class } 1 = \{\text{state } 1, \text{state } 2\}$ aperiodic?
b. Is $\text{Class } 2 = \{\text{state } 3, \text{state } 4\}$ aperiodic?
c. Is $\text{Class } 4 = \{\text{state } 6, \text{state } 7, \text{state } 8\}$ aperiodic?

Solution

a. $\text{Class } 1 = \{\text{state } 1, \text{state } 2\}$ is aperiodic since it has a self-transition, $p_{22} > 0$.
b. $\text{Class } 2 = \{\text{state } 3, \text{state } 4\}$ is periodic with period 2.
c. $\text{Class } 4 = \{\text{state } 6, \text{state } 7, \text{state } 8\}$ is aperiodic. For example, note that we can go from state 6 to state 6 in two steps ($6 - 7 - 6$) and in three steps ($6 - 7 - 8 - 6$). Since $\gcd(2, 3) = 1$, we conclude state 6 and its class are aperiodic.
