Note that states 0 and 3 have the following property: once you enter those states, you never leave them. For this reason, we call them **absorbing** states. For our example here, there are two absorbing states. The process will eventually get absorbed in one of them. The first question that we would like to address deals with finding absorption probabilities.

Absorption Probabilities:

Consider the Markov chain in Figure 11.12. Let's define a_i as the absorption probability in state 0 if we start from state i. More specifically,

$$\begin{aligned} a_0 &= P(\text{absorption in } 0 | X_0 = 0), \\ a_1 &= P(\text{absorption in } 0 | X_0 = 1), \\ a_2 &= P(\text{absorption in } 0 | X_0 = 2), \\ a_3 &= P(\text{absorption in } 0 | X_0 = 3). \end{aligned}$$

By the above definition, we have $a_0 = 1$ and $a_3 = 0$. To find the values of a_1 and a_2, we apply the law of total probability with recursion. The main idea is the following: if $X_n = i$, then the next state will be $X_{n+1} = k$ with probability p_{ik}. Thus, we can write

$$a_i = \sum_k a_k p_{ik}, \quad \text{for } i = 0, 1, 2, 3 \qquad (11.6)$$

Solving the above equations will give us the values of a_1 and a_2. More specifically, using Equation 11.6, we obtain

$$\begin{aligned} a_0 &= a_0, \\ a_1 &= \frac{1}{3}a_0 + \frac{2}{3}a_2, \\ a_2 &= \frac{1}{2}a_1 + \frac{1}{2}a_3, \\ a_3 &= a_3. \end{aligned}$$

We also know $a_0 = 1$ and $a_3 = 0$. Solving for a_1 and a_2, we obtain

$$\begin{aligned} a_1 &= \frac{1}{2}, \\ a_2 &= \frac{1}{4}. \end{aligned}$$

Let's now define b_i as the absorption probability in state 3 if we start from state i. Since $a_i + b_i = 1$, we conclude

$$b_0 = 0, \quad b_1 = \frac{1}{2}, \quad b_2 = \frac{3}{4}, \quad b_3 = 1.$$

Nevertheless, for practice, let's find the b_i's directly.

Example 11.10

Consider the Markov chain in Figure 11.12. Let's define b_i as the absorption probability in state 3 if we start from state i. Use the above procedure to obtain b_i for $i = 0, 1, 2, 3$.

Solution

From the definition of b_i and the Markov chain graph, we have $b_0 = 0$ and $b_3 = 1$. Writing Equation 11.6 for $i = 1, 2$, we obtain

$$\begin{aligned} b_1 &= \frac{1}{3}b_0 + \frac{2}{3}b_2 \\ &= \frac{2}{3}b_2, \\ b_2 &= \frac{1}{2}b_1 + \frac{1}{2}b_3 \\ &= \frac{1}{2}b_1 + \frac{1}{2}. \end{aligned}$$

Solving the above equations, we obtain

$$\begin{aligned} b_1 &= \frac{1}{2}, \\ b_2 &= \frac{3}{4}. \end{aligned}$$
