9.1.1 Prior and Posterior

Let X be the random variable whose value we try to estimate. Let Y be the observed random variable. That is, we have observed $Y = y$, and we would like to estimate X. Assuming both X and Y are discrete, we can write

$$\begin{aligned} P(X = x|Y = y) &= \frac{P(X = x, Y = y)}{P(Y = y)} \\ &= \frac{P(Y = y|X = x)P(X = x)}{P(Y = y)}. \end{aligned}$$

Using our notation for PMF and conditional PMF, the above equation can be rewritten as

$$P_{X|Y}(x|y) = \frac{P_{Y|X}(y|x)P_X(x)}{P_Y(y)}.$$

The above equation, as we have seen before, is just one way of writing Bayes' rule. If either X or Y are continuous random variables, we can replace the corresponding PMF with PDF in the above formula. For example, if X is a continuous random variable, while Y is discrete we can write

$$f_{X|Y}(x|y) = \frac{P_{Y|X}(y|x)f_X(x)}{P_Y(y)}.$$

To find the denominator ($P_Y(y)$ or $f_Y(y)$), we often use the law of total probability. Let's look at an example.

Example 9.3

Let $X \sim Uniform(0, 1)$. Suppose that we know

$$Y \mid X = x \quad \sim \quad Geometric(x).$$

Find the posterior density of X given $Y = 2$, $f_{X|Y}(x|2)$.

Solution

Using Bayes' rule we have

$$f_{X|Y}(x|2) = \frac{P_{Y|X}(2|x) f_X(x)}{P_Y(2)}.$$

We know $Y \mid X = x \quad \sim \quad Geometric(x)$, so

$$P_{Y|X}(y|x) = x(1-x)^{y-1}, \quad \text{for } y = 1, 2, \cdots.$$

Therefore,

$$P_{Y|X}(2|x) = x(1-x).$$

To find $P_Y(2)$, we can use the law of total probability

$$\begin{aligned} P_Y(2) &= \int_{-\infty}^{\infty} P_{Y|X}(2|x) f_X(x) \quad \mathrm{d}x \\ &= \int_0^1 x(1-x) \cdot 1 \quad \mathrm{d}x \\ &= \frac{1}{6}. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} f_{X|Y}(x|2) &= \frac{x(1-x) \cdot 1}{\frac{1}{6}} \\ &= 6x(1-x), \quad \text{for } 0 \leq x \leq 1. \end{aligned}$$

For the remainder of this chapter, for simplicity, we often write the posterior PDF as

$$f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x) f_X(x)}{f_Y(y)},$$

which implies that both X and Y are continuous. Nevertheless, we understand that if either X or Y is discrete, we need to replace the PDF by the corresponding PMF.