7.2.6 Convergence in Mean

One way of interpreting the convergence of a sequence X_n to X is to say that the "distance" between X and X_n is getting smaller and smaller. For example, if we define the distance between X_n and X as $P\big(|X_n - X| \geq \epsilon\big)$, we have convergence in probability. One way to define the distance between X_n and X is

$$E\left(|X_n - X|^r\right),$$

where $r \geq 1$ is a fixed number. This refers to **convergence in mean**. (*Note: for convergence in mean, it is usually required that $E|X_n^r| < \infty$.*) The most common choice is $r = 2$, in which case it is called the **mean-square convergence**. (*Note: Some authors refer to the case $r = 1$ as convergence in mean.*)

Convergence in Mean

Let $r \geq 1$ be a fixed number. A sequence of random variables X_1, X_2, X_3, $\cdots$ converges **in the rth mean** or **in the L^r norm** to a random variable X, shown by $X_n \ \xrightarrow{L^r}\ X$, if

$$\lim_{n \rightarrow \infty} E\left(|X_n - X|^r\right) = 0.$$

If $r = 2$, it is called the **mean-square convergence**, and it is shown by $X_n \ \xrightarrow{m.s.}\ X$.

Example 7.10

Let $X_n \sim Uniform\left(0, \frac{1}{n}\right)$. Show that $X_n \ \xrightarrow{L^r}\ 0$, for any $r \geq 1$.

Solution

The PDF of X_n is given by

$$f_{X_n}(x) = \begin{cases} n & 0 \leq x \leq \frac{1}{n} \\ & \\ 0 & \text{otherwise} \end{cases}$$

We have

$$\begin{aligned} E\left(|X_n - 0|^r\right) &= \int_0^{\frac{1}{n}} x^r n \quad dx \\ &= \frac{1}{(r+1)n^r} \rightarrow 0, \qquad \text{for all } r \geq 1. \end{aligned}$$

Theorem 7.3

Let $1 \leq r \leq s$. If $X_n \ \xrightarrow{L^s}\ X$, then $X_n \ \xrightarrow{L^r}\ X$.

Proof

We can use Hölder's inequality, which was proved in Section . Hölder's Inequality states that

$$E|XY| \leq \left(E|X|^p\right)^{\frac{1}{p}} \left(E|Y|^q\right)^{\frac{1}{q}},$$

where $1 < p$, $q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. In Hölder's inequality, choose

$$\begin{aligned} X &= |X_n - X|^r, \\ Y &= 1, \\ p &= \frac{s}{r} > 1. \end{aligned}$$

We obtain

$$E|X_n - X|^r \leq \left(E|X_n - X|^s\right)^{\frac{1}{p}}.$$

Now, by assumption $X_n \ \xrightarrow{L^s}\ X$, which means

$$\lim_{n \rightarrow \infty} E\left(|X_n - X|^s\right) = 0.$$

We conclude

$$\begin{aligned}\lim_{n \to \infty} E\left(|X_n - X|^r\right) &\leq \lim_{n \to \infty} \left(E|X_n - X|^s\right)^{\frac{1}{p}} \\ &= 0.\end{aligned}$$

Therefore, $X_n \ \xrightarrow{L^r}\ X$.

As we mentioned before, convergence in mean is stronger than convergence in probability. We can prove this using Markov's inequality.

Theorem 7.4

If $X_n \ \xrightarrow{L^r}\ X$ for some $r \geq 1$, then $X_n \ \xrightarrow{p}\ X$.

Proof

For any $\epsilon > 0$, we have

$$\begin{aligned}P\big(|X_n - X| \geq \epsilon\big) &= P\big(|X_n - X|^r \geq \epsilon^r\big) && \text{(since } r \geq 1) \\ &\leq \frac{E|X_n - X|^r}{\epsilon^r} && \text{(by Markov's inequality).}\end{aligned}$$

Since by assumption $\lim_{n \to \infty} E\left(|X_n - X|^r\right) = 0$, we conclude

$$\lim_{n \to \infty} P\big(|X_n - X| \geq \epsilon\big) = 0, \qquad \text{for all } \epsilon > 0.$$

The converse of Theorem 7.4 is not true in general. That is, there are sequences that converge in probability but not in mean. Let us look at an example.

Example 7.11

Consider a sequence $\{X_n, n = 1, 2, 3, \cdots\}$ such that

$$X_n = \begin{cases} n^2 & \text{with probability } \frac{1}{n} \\ & \\ 0 & \text{with probability } 1 - \frac{1}{n} \end{cases}$$

Show that

a. $X_n \xrightarrow{p} 0$.
b. X_n does not converge in the rth mean for any $r \geq 1$.

Solution

a. To show $X_n \xrightarrow{p} 0$, we can write, for any $\epsilon > 0$

$$\begin{aligned}\lim_{n\to\infty} P\big(|X_n| \geq \epsilon\big) &= \lim_{n\to\infty} P(X_n = n^2)\\ &= \lim_{n\to\infty} \frac{1}{n}\\ &= 0.\end{aligned}$$

We conclude that $X_n \xrightarrow{p} 0$.

b. For any $r \geq 1$, we can write

$$\begin{aligned}\lim_{n\to\infty} E\left(|X_n|^r\right) &= \lim_{n\to\infty}\left(n^{2r}\cdot\frac{1}{n} + 0\cdot\left(1-\frac{1}{n}\right)\right)\\ &= \lim_{n\to\infty} n^{2r-1}\\ &= \infty \qquad (\text{since } r \geq 1).\end{aligned}$$

Therefore, X_n does not converge in the rth mean for any $r \geq 1$. In particular, it is interesting to note that, although $X_n \xrightarrow{p} 0$, the expected value of X_n does not converge to 0.
