b. We have

$$\begin{aligned}
E[X|X>1] &= \int_1^\infty x f_{X|X>1}(x)dx \\
&= \int_1^\infty x e^{-x+1} dx \\
&= e\int_1^\infty x e^{-x} dx \\
&= e\bigg[-e^{-x} - xe^{-x}\bigg]_1^\infty \\
&= e\frac{2}{e} \\
&= 2.
\end{aligned}$$

c. We have

$$\begin{aligned}
E[X^2|X>1] &= \int_1^\infty x^2 f_{X|X>1}(x)dx \\
&= \int_1^\infty x^2 e^{-x+1} dx \\
&= e\int_1^\infty x^2 e^{-x} dx \\
&= e\bigg[-2e^{-x} - 2xe^{-x} - x^2e^{-x}\bigg]_1^\infty \\
&= e\frac{5}{e} \\
&= 5.
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathrm{Var}(X|X>1) &= E[X^2|X>1] - (E[X|X>1])^2 \\
&= 5 - 4 = 1.
\end{aligned}$$

Conditioning by Another Random Variable:

If X and Y are two jointly continuous random variables, and we obtain some information regarding Y, we should update the PDF and CDF of X based on the new information. In particular, if we get to observe the value of the random variable Y, then how do we need to update the PDF and CDF of X? Remember for the discrete case, the conditional PMF of X given $Y = y$ is given by

$$P_{X|Y}(x_i|y_j) = \frac{P_{XY}(x_i, y_j)}{P_Y(y_j)}.$$

Now, if X and Y are jointly continuous, the conditional PDF of X given Y is given by

$$f_{X|Y}(x|y) = \frac{f_{XY}(x, y)}{f_Y(y)}.$$

This means that if we get to observe $Y = y$, then we need to use the above conditional density for the random variable X. To get an intuition about the formula, note that by definition, for small Δ_x and Δ_y we should have

$$\begin{aligned}
f_{X|Y}(x|y) &\approx \frac{P(x \leq X \leq x + \Delta_x | y \leq Y \leq y + \Delta_y)}{\Delta_x} && \text{(definition of PDF)} \\
&= \frac{P(x \leq X \leq x + \Delta_x, y \leq Y \leq y + \Delta_y)}{P(y \leq Y \leq y + \Delta_y)\Delta_x} \\
&\approx \frac{f_{XY}(x, y)\Delta_x\Delta_y}{f_Y(y)\Delta_y\Delta_x} \\
&= \frac{f_{XY}(x, y)}{f_Y(y)}.
\end{aligned}$$

Similarly, we can write the conditional PDF of Y, given $X = x$, as

$$f_{Y|X}(y|x) = \frac{f_{XY}(x, y)}{f_X(x)}.$$

For two jointly continuous random variables X and Y, we can define the following conditional concepts:

1. The conditional PDF of X given $Y = y$:

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

2. The conditional probability that $X \in A$ given $Y = y$:

$$P(X \in A|Y = y) = \int_A f_{X|Y}(x|y)dx$$

3. The conditional CDF of X given $Y = y$:

$$F_{X|Y}(x|y) = P(X \leq x|Y = y) = \int_{-\infty}^{x} f_{X|Y}(x|y)dx$$

Example 5.21 Let X and Y be two jointly continuous random variables with joint PDF

$$f_{XY}(x,y) = \begin{cases} \frac{x^2}{4} + \frac{y^2}{4} + \frac{xy}{6} & 0 \leq x \leq 1, 0 \leq y \leq 2 \\ \\ 0 & \text{otherwise} \end{cases}$$

For $0 \leq y \leq 2$, find

a. the conditional PDF of X given $Y = y$;
b. $P(X < \frac{1}{2}|Y = y)$.

Solution

a. Let us first find the marginal PDF of Y. We have

$$\begin{aligned} f_Y(y) &= \int_0^1 \frac{x^2}{4} + \frac{y^2}{4} + \frac{xy}{6} \ dx \\ &= \frac{3y^2 + y + 1}{12}, \qquad \text{for } 0 \leq y \leq 2. \end{aligned}$$

Thus, for $0 \leq y \leq 2$, we obtain

$$
\begin{aligned}
f_{X|Y}(x|y) &= \frac{f_{XY}(x,y)}{f_Y(y)} \\
&= \frac{3x^2+3y^2+2xy}{3y^2+y+1}, \qquad \text{for } 0 \leq x \leq 1.
\end{aligned}
$$

Thus, for $0 \leq y \leq 2$, we have

$$
f_{X|Y}(x|y) = \begin{cases} \frac{3x^2+3y^2+2xy}{3y^2+y+1} & 0 \leq x \leq 1 \\ \\ 0 & \text{otherwise} \end{cases}
$$

b. We have

$$
\begin{aligned}
P\left(X < \frac{1}{2} | Y = y\right) &= \int_0^{\frac{1}{2}} \frac{3x^2+3y^2+2xy}{3y^2+y+1} \; dx \\
&= \frac{1}{3y^2+y+1} \left[x^3 + yx^2 + 3y^2x \right]_0^{\frac{1}{2}} \\
&= \frac{\frac{3}{2}y^2 + \frac{y}{4} + \frac{1}{8}}{3y^2+y+1}.
\end{aligned}
$$

Note that, as we expect, $P\left(X < \frac{1}{2}|Y=y\right)$ depends on y.

Conditional expectation and variance are similarly defined. Given $Y=y$, we need to replace $f_X(x)$ by $f_{X|Y}(x|y)$ in the formulas for expectation:

For two jointly continuous random variables X and Y, we have:

1. Expected value of X given $Y = y$:

$$E[X|Y = y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y) dx$$

2. Conditional LOTUS:

$$E[g(X)|Y = y] = \int_{-\infty}^{\infty} g(x) f_{X|Y}(x|y) dx$$

3. Conditional variance of X given $Y = y$:

$$Var(X|Y = y) = E[X^2|Y = y] - (E[X|Y = y])^2$$

Example 5.22

Let X and Y be as in Example 5.21. Find $E[X|Y = 1]$ and Var$(X|Y = 1)$.

Solution

$$\begin{aligned} E[X|Y = 1] &= \int_{-\infty}^{\infty} x f_{X|Y}(x|1) dx \\ &= \int_0^1 x \frac{3x^2 + 3y^2 + 2xy}{3y^2 + y + 1}\Big|_{y=1} \ dx \\ &= \int_0^1 x \frac{3x^2 + 3 + 2x}{3 + 1 + 1} \ dx \qquad\qquad (y = 1) \\ &= \frac{1}{5} \int_0^1 3x^3 + 2x^2 + 3x \ dx \\ &= \frac{7}{12}, \end{aligned}$$

$$\begin{aligned} E[X^2|Y = 1] &= \int_{-\infty}^{\infty} x^2 f_{X|Y}(x|1) dx \\ &= \frac{1}{5} \int_0^1 3x^4 + 2x^3 + 3x^2 \ dx \\ &= \frac{21}{50}. \end{aligned}$$

So we have