5.3.3 Solved Problems

Problem 1

Let X and Y be two jointly continuous random variables with joint PDF

$$f_{XY}(x,y) = \begin{cases} 2 & y+x \leq 1, x>0, y>0 \\ \\ 0 & \text{otherwise} \end{cases}$$

Find $\text{Cov}(X,Y)$ and $\rho(X,Y)$.

Solution

For $0 \leq x \leq 1$, we have

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{XY}(x,y)dy \\ &= \int_0^{1-x} 2dy \\ &= 2(1-x). \end{aligned}$$

Thus,

$$f_X(x) = \begin{cases} 2(1-x) & 0 \leq x \leq 1 \\ \\ 0 & \text{otherwise} \end{cases}$$

Similarly, we obtain

$$f_Y(y) = \begin{cases} 2(1-y) & 0 \leq y \leq 1 \\ \\ 0 & \text{otherwise} \end{cases}$$

Thus, we have

$$\begin{aligned} EX &= \int_0^1 2x(1-x)dx \\ &= \frac{1}{3} = EY, \end{aligned}$$

$$
\begin{aligned}
EX^2 &= \int_0^1 2x^2(1-x)dx \\
&= \frac{1}{6} = EY^2.
\end{aligned}
$$

Thus,

$$
Var(X) = Var(Y) = \frac{1}{18}.
$$

We also have

$$
\begin{aligned}
EXY &= \int_0^1 \int_0^{1-x} 2xydydx \\
&= \int_0^1 x(1-x)^2 dx \\
&= \frac{1}{12}.
\end{aligned}
$$

Now, we can find $\mathrm{Cov}(X, Y)$ and $\rho(X, Y)$:

$$
\begin{aligned}
\mathrm{Cov}(X, Y) &= EXY - EXEY \\
&= \frac{1}{12} - \left(\frac{1}{3}\right)^2 \\
&= -\frac{1}{36},
\end{aligned}
$$

$$
\begin{aligned}
\rho(X, Y) &= \frac{\mathrm{Cov}(X, Y)}{\sqrt{Var(X)Var(Y)}} \\
&= -\frac{1}{2}.
\end{aligned}
$$

Problem 2

I roll a fair die n times. Let X be the number of 1's that I observe and let Y be the number of 2's that I observe. Find $\mathrm{Cov}(X, Y)$ and $\rho(X, Y)$. *Hint:* One way to solve this problem is to look at $Var(X + Y)$.

Solution

Note that you can look at this as a binomial experiment. In particular, we can say that X and Y are $Binomial(n, \frac{1}{6})$. Also, $X + Y$ is $Binomial(n, \frac{2}{6})$. Remember the variance of a $Binomial(n, p)$ random variable is $np(1-p)$. Thus, we can write

$$
\begin{aligned}
n\frac{2}{6}.\frac{4}{6} &= Var(X+Y) \\
&= Var(X) + Var(Y) + 2\text{Cov}(X,Y) \\
&= n\frac{1}{6}.\frac{5}{6} + n\frac{1}{6}.\frac{5}{6} + 2\text{Cov}(X,Y).
\end{aligned}
$$

Thus,

$$\text{Cov}(X,Y) = -\frac{n}{36}.$$

And,

$$\rho(X,Y) = \frac{\text{Cov}(X,Y)}{\sqrt{Var(X)Var(Y)}} = -\frac{1}{5}.$$

Problem 3

In this problem, you will provide another proof for the fact that $|\rho(X,Y)| \leq 1$. By definition $\rho_{XY} = \text{Cov}(U,V)$, where U and V are the normalized versions of X and Y as defined in Equation 5.22:

$$U = \frac{X - EX}{\sigma_X}, \quad V = \frac{Y - EY}{\sigma_Y}.$$

Use the fact that $\text{Var}(U+V) \geq 0$ to show that $|\rho(X,Y)| \leq 1$.

Solution

We have

$$
\begin{aligned}
\text{Var}(U+V) &= \text{Var}(U) + \text{Var}(V) + 2\text{Cov}(U,V) \\
&= 1 + 1 + 2\rho_{XY}.
\end{aligned}
$$

Since $\text{Var}(U+V) \geq 0$, we conclude $\rho(X,Y) \geq -1$. Also, from this we conclude that

$$\rho(-X,Y) \geq -1.$$

But $\rho(-X,Y) = -\rho(X,Y)$, so we conclude $\rho(X,Y) \leq 1$.

Problem 4

Let X and Y be two independent $Uniform(0,1)$ random variables. Let also $Z = \max(X,Y)$ and $W = \min(X,Y)$. Find $\textrm{Cov}(Z,W)$.

Solution

It is useful to find the distributions of Z and W. To find the CDF of Z, we can write

$$\begin{aligned}
F_Z(z) &= P(Z \leq z) \\
&= P(\max(X,Y) \leq z) \\
&= P\Big((X \leq z) \textrm{ and } (Y \leq z) \Big) \\
&= P(X \leq z)P(Y \leq z) && \textrm{(since } X \textrm{ and } Y \textrm{ are independent)} \\
&= F_X(z)F_Y(z).
\end{aligned}$$

Thus, we conclude

$$F_Z(z) = \begin{cases} 0 & z < 0 \\ z^2 & 0 \leq z \leq 1 \\ 1 & z > 1 \end{cases}$$

Therefore,

$$f_Z(z) = \begin{cases} 2z & 0 \leq z \leq 1 \\ 0 & \textrm{otherwise} \end{cases}$$

From this we obtain $EZ = \frac{2}{3}$. Note that we can find EW as follows

$$\begin{aligned}
1 = E[X+Y] &= E[Z+W] \\
&= EZ + EW \\
&= \frac{2}{3} + EW.
\end{aligned}$$

Thus, $EW = \frac{1}{3}$. Nevertheless, it is a good exercise to find the CDF and PDF of W, too. To find the CDF of W, we can write

$$\begin{aligned}
F_W(w) &= P(W \leq w) \\
&= P(\min(X,Y) \leq w) \\
&= 1 - P(\min(X,Y) > w) \\
&= 1 - P\Big((X > w) \textrm{ and } (Y > w) \Big) \\
&= 1 - P(X > w)P(Y > w) && \textrm{(since } X \textrm{ and } Y \textrm{ are independent)} \\
&= 1 - (1 - F_X(w))(1 - F_Y(w)) \\
&= F_X(w) + F_Y(w) - F_X(w)F_Y(w).
\end{aligned}$$

Thus,

$$F_W(w) = \begin{cases} 0 & w < 0 \\ 2w - w^2 & 0 \leq w \leq 1 \\ 1 & w > 1 \end{cases}$$

Therefore,

$$f_W(w) = \begin{cases} 2 - 2w & 0 \leq w \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

From the above PDF we can verify that $EW = \frac{1}{3}$. Now, to find $\text{Cov}(Z, W)$, we can write

$$\begin{aligned} \text{Cov}(Z, W) &= E[ZW] - EZEW \\ &= E[XY] - EZEW \\ &= E[X]E[Y] - E[Z]E[W] \qquad \text{(since } X \text{ and } Y \text{ are independent)} \\ &= \frac{1}{2} \cdot \frac{1}{2} - \frac{2}{3} \cdot \frac{1}{3} \\ &= \frac{1}{36}. \end{aligned}$$

Note that $\text{Cov}(Z, W) > 0$ as we expect intuitively.

Problem 5

Let X and Y be jointly (bivariate) normal, with $Var(X) = Var(Y)$. Show that the two random variables $X + Y$ and $X - Y$ are independent.

Solution

Note that since X and Y are jointly normal, we conclude that the random variables $X + Y$ and $X - Y$ are also jointly normal. We have

$$\begin{aligned} \text{Cov}(X + Y, X - Y) &= \text{Cov}(X, X) - \text{Cov}(X, Y) + \text{Cov}(Y, X) - \text{Cov}(Y, Y) \\ &= Var(X) - Var(Y) \\ &= 0. \end{aligned}$$

Since $X + Y$ and $X - Y$ are jointly normal and uncorrelated, they are independent.

Problem 6

Let X and Y be jointly normal random variables with parameters $\mu_X = 0$, $\sigma_X^2 = 1$, $\mu_Y = -1$, $\sigma_Y^2 = 4$, and $\rho = -\frac{1}{2}$.

1. Find $P(X + Y > 0)$.
2. Find the constant a if we know $aX + Y$ and $X + 2Y$ are independent.
3. Find $P(X + Y > 0 | 2X - Y = 0)$.

Solution

1. Since X and Y are jointly normal, the random variable $U = X + Y$ is normal. We have

$$EU = EX + EY = -1,$$

$$\begin{aligned} Var(U) &= Var(X) + Var(Y) + 2\text{Cov}(X, Y) \\ &= 1 + 4 + 2\sigma_X\sigma_Y\rho(X, Y) \\ &= 5 - 2 \times 1 \times 2 \times \frac{1}{2} \\ &= 3. \end{aligned}$$

Thus, $U \sim N(-1, 3)$. Therefore,

$$P(U > 0) = 1 - \Phi\left(\frac{0 - (-1)}{\sqrt{3}}\right) = 1 - \Phi\left(\frac{1}{\sqrt{3}}\right) = 0.2819$$

2. Note that $aX + Y$ and $X + 2Y$ are jointly normal. Thus, for them, independence is equivalent to having $\text{Cov}(aX + Y, X + 2Y) = 0$. Also, note that $\text{Cov}(X, Y) = \sigma_X\sigma_Y\rho(X, Y) = -1$. We have

$$\begin{aligned} \text{Cov}(aX + Y, X + 2Y) &= a\text{Cov}(X, X) + 2a\text{Cov}(X, Y) + \text{Cov}(Y, X) + 2\text{Cov}(Y, Y) \\ &= a - (2a + 1) + 8 \\ &= -a + 7. \end{aligned}$$

Thus, $a = 7$.

3. If we define $U = X + Y$ and $V = 2X - Y$, then note that U and V are jointly normal. We have

$$\begin{aligned} EU &= -1, Var(U) = 3, \\ EV &= 1, Var(V) = 12, \end{aligned}$$

and

$$\begin{aligned}
\mathrm{Cov}(U,V) &= \mathrm{Cov}(X+Y, 2X-Y) \\
&= 2\mathrm{Cov}(X,X) - \mathrm{Cov}(X,Y) + 2\mathrm{Cov}(Y,X) - \mathrm{Cov}(Y,Y) \\
&= 2Var(X) + \mathrm{Cov}(X,Y) - Var(Y) \\
&= 2 - 1 - 4 \\
&= -3.
\end{aligned}$$

Thus,

$$\begin{aligned}
\rho(U,V) &= \frac{\mathrm{Cov}(U,V)}{\sqrt{Var(U)Var(V)}} \\
&= -\frac{1}{2}.
\end{aligned}$$

Using Theorem 5.4, we conclude that given $V = 0$, U is normally distributed with

$$\begin{aligned}
&E[U|V=0] = \mu_U + \rho(U,V)\sigma_U \frac{0-\mu_V}{\sigma_V} = -\frac{3}{4}, \\
&Var(U|V=0) = (1-\rho_{UV}^2)\sigma_U^2 = \frac{9}{4}.
\end{aligned}$$

Thus

$$\begin{aligned}
P(X+Y>0|2X-Y=0) &= P(U>0|V=0) \\
&= 1 - \Phi\left(\frac{0-(-\frac{3}{4})}{\frac{3}{2}}\right) \\
&= 1 - \Phi\left(\frac{1}{2}\right) = 0.3085.
\end{aligned}$$