## 6.1.6 Solved Problems

**Problem 1**

Let $X, Y$ and $Z$ be three jointly continuous random variables with joint PDF

$$f_{XYZ}(x,y,z) = \begin{cases} \frac{1}{3}(x+2y+3z) & 0 \leq x,y,z \leq 1 \\ \\ 0 & \text{otherwise} \end{cases}$$

Find the joint PDF of $X$ and $Y$, $f_{XY}(x,y)$.

**Solution**

$$\begin{aligned} f_{XY}(x,y) &= \int_{-\infty}^{\infty} f_{XYZ}(x,y,z)dz \\ &= \int_{0}^{1} \frac{1}{3}(x+2y+3z)dz \\ &= \frac{1}{3}\left[(x+2y)z + \frac{3}{2}z^2\right]_0^1 \\ &= \frac{1}{3}\left(x+2y+\frac{3}{2}\right), \qquad \text{for} \quad 0 \leq x,y \leq 1. \end{aligned}$$

Thus,

$$f_{XY}(x,y) = \begin{cases} \frac{1}{3}\left(x+2y+\frac{3}{2}\right) & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ \\ 0 & \text{otherwise} \end{cases}$$

**Problem 2**

Let $X, Y$ and $Z$ be three independent random variables with $X \sim N(\mu, \sigma^2)$, and $Y, Z \sim Uniform(0,2)$. We also know that

$$\begin{aligned} E[X^2Y + XYZ] &= 13, \\ E[XY^2 + ZX^2] &= 14. \end{aligned}$$

Find $\mu$ and $\sigma$.

**Solution**

$$X, Y, \text{ and } Z\, are \text{ independent } \Rightarrow \begin{cases} EX^2 \cdot EY + EX \cdot EY \cdot EZ = 13 \\ EX \cdot EY^2 + EZ \cdot EX^2 = 14 \end{cases}$$

Since $Y, Z \sim Uniform(0,2)$, we conclude

$$EY = EZ = 1; \ \mathrm{Var}(Y) = \mathrm{Var}(Z) = \frac{(2-0)^2}{12} = \frac{1}{3}.$$

Therefore,

$$EY^2 = \frac{1}{3} + 1 = \frac{4}{3}.$$

Thus,

$$\begin{cases} EX^2 + EX = 13 \\ \frac{4}{3}EX + EX^2 = 14 \end{cases}$$

We conclude $EX = 3$, $EX^2 = 10$. Therefore,

$$\begin{cases} \mu = 3 \\ \mu^2 + \sigma^2 = 10 \end{cases}$$

So, we obtain $\mu = 3$,$\sigma = 1$.

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**Problem 3**

Let $X_1$, $X_2$, and $X_3$ be three i.i.d $Bernoulli(p)$ random variables and

$$\begin{aligned} Y_1 &= \max(X_1, X_2), \\ Y_2 &= \max(X_1, X_3), \\ Y_3 &= \max(X_2, X_3), \\ Y &= Y_1 + Y_2 + Y_3. \end{aligned}$$

Find $EY$ and $\mathrm{Var}(Y)$.

**Solution**

We have

$$EY = EY_1 + EY_2 + EY_3 = 3EY_1, \quad \text{by symmetry.}$$

Also,

$$\begin{aligned}\mathrm{Var}(Y) &= \mathrm{Var}(Y_1)+\mathrm{Var}(Y_2)+\mathrm{Var}(Y_3)+2\mathrm{Cov}(Y_1,Y_2)+2\mathrm{Cov}(Y_1,Y_3)+2\mathrm{Cov}(Y_2,Y_3)\\ &= 3\mathrm{Var}(Y_1)+6\mathrm{Cov}(Y_1,Y_2), \quad \text{by symmetry.}\end{aligned}$$

Note that $Y_i$'s are also Bernoulli random variables (but they are not independent). In particular, we have

$$\begin{aligned}P(Y_1=1) &= P\big((X_1=1) \text{ or } (X_2=1)\big)\\ &= P(X_1=1)+P(X_2=1)-P(X_1=1,X_2=1) \qquad \text{(comma means "and")}\\ &= 2p-p^2.\end{aligned}$$

Thus, $Y_1 \sim Bernoulli(2p-p^2)$, and we obtain

$$\begin{aligned}EY_1 &= 2p-p^2 = p(2-p),\\ \mathrm{Var}(Y_1) &= (2p-p^2)(1-2p+p^2) = p(2-p)(1-p)^2.\end{aligned}$$

It remains to find $\mathrm{Cov}(Y_1,Y_2)$. We can write

$$\begin{aligned}\mathrm{Cov}(Y_1,Y_2) &= E[Y_1Y_2]-E[Y_1]E[Y_2]\\ &= E[Y_1Y_2]-p^2(2-p)^2.\end{aligned}$$

Note that $Y_1Y_2$ is also a Bernoulli random variable. We have

$$\begin{aligned}E[Y_1Y_2] &= P\big(Y_1=1,Y_2=1\big)\\ &= P\big((X_1=1) \text{ or } \big(X_2=1,X_3=1\big)\big)\\ &= P(X_1=1)+P\big(X_2=1,X_3=1\big)-P\big(X_1=1,X_2=1,X_3=1\big)\\ &= p+p^2-p^3.\end{aligned}$$

Thus, we obtain

$$\begin{aligned}\mathrm{Cov}(Y_1,Y_2) &= E[Y_1Y_2]-p^2(2-p)^2\\ &= p+p^2-p^3-p^2(2-p)^2.\end{aligned}$$

Finally, we obtain

$$EY = 3EY_1 = 3p(2-p).$$

Also,

$$\begin{aligned}\mathrm{Var}(Y) &= 3\mathrm{Var}(Y_1)+6\mathrm{Cov}(Y_1,Y_2)\\ &= 3p(2-p)(1-p)^2+6(p+p^2-p^3-p^2(2-p)^2).\end{aligned}$$

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**Problem 4**
Let $M_X(s)$ be finite for $s \in [-c, c]$, where $c > 0$. Show that MGF of $Y = aX + b$ is given by

$$M_Y(s) = e^{sb} M_X(as),$$

and it is finite in $\left[-\frac{c}{|a|}, \frac{c}{|a|}\right]$.

**Solution**

We have

$$\begin{aligned} M_Y(s) &= E[e^{sY}] \\ &= E[e^{saX} e^{sb}] \\ &= e^{sb} E[e^{(sa)X}] \\ &= e^{sb} M_X(as). \end{aligned}$$

Also, since $M_X(s)$ is finite for $s \in [-c, c]$, $M_X(as)$ is finite for $s \in \left[-\frac{c}{|a|}, \frac{c}{|a|}\right]$.

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**Problem 5**
Let $Z \sim N(0, 1)$ Find the MGF of $Z$. Extend your result to $X \sim N(\mu, \sigma)$.

**Solution**

We have

$$\begin{aligned} M_Z(s) &= E[e^{sZ}] \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{sx} e^{-\frac{x^2}{2}} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{s^2}{2}} e^{-\frac{(x-s)^2}{2}} dx \\ &= e^{\frac{s^2}{2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{2}} dx \\ &= e^{\frac{s^2}{2}} \quad \text{(PDF of normal integrates to 1).} \end{aligned}$$

Using Problem 4, we obtain

$$M_X(s) = e^{s\mu + \frac{\sigma^2 s^2}{2}}, \quad \text{for all} \quad s \in \mathbb{R}.$$

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**Problem 6**
Let $Y = X_1 + X_2 + X_3 + ... + X_n$, where $X_i$'s are independent and $X_i \sim Poisson(\lambda_i)$. Find the distribution of $Y$.

**Solution**

We have

$$M_{X_i}(s) = e^{\lambda_i(e^s - 1)}, \text{ for all } s \in \mathbb{R}.$$

Thus,

$$\begin{aligned} M_Y(s) &= \prod_{i=1}^{n} e^{\lambda_i(e^s-1)} \\ &= e^{(\sum_{i=1}^{n} \lambda_i)(e^s-1)}, \text{ for all } s \in \mathbb{R}. \end{aligned}$$

which is the MGF of a Poisson random variable with parameter $\lambda = \sum_{i=1}^{n} \lambda_i$, thus

$$Y \sim Poisson(\sum_{i=1}^{n} \lambda_i).$$

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**Problem 7**
**Probability Generating Functions (PGFs):** For many important discrete random variables, the range is a subset of $\{0, 1, 2, ...\}$. For these random variables it is usually more useful to work with *probability generating functions (PGF)s* defined as

$$G_X(z) = E[Z^X] = \sum_{n=0}^{\infty} P(X = n) Z^n,$$

for all $Z \in \mathbb{R}$ that $G_X(Z)$ is finite.

1. Show that $G_X(Z)$ is always finite for $|Z| \leq 1$.
2. Show that if $X$ and $Y$ are independent, then

$$G_{X+Y}(Z) = G_X(Z) G_Y(Z).$$

3. Show that

$$\frac{1}{k!}\frac{d^k G_X(z)}{dz^k}|_{z=0} = P(X = k).$$

4. Show that

$$\frac{d^k G_X(z)}{dz^k}|_{z=1} = E[X(X-1)(X-2)\dots(X-k+1)].$$

**Solution**

1. If $|Z| \le 1$, then $Z^n \le |Z| \le 1$, so we have

$$\begin{aligned} G_X(z) &= \sum_{n=0}^{\infty} P(X=n)Z^n \\ &\le \sum_{n=0}^{\infty} P(X=n) = 1. \end{aligned}$$

2. If $X$ and $Y$ are independent, then

$$\begin{aligned} G_{X+Y}(Z) &= E[Z^{X+Y}] \\ &= E[Z^X Z^Y] \\ &= E[Z^X]E[Z^Y] \quad \text{(since } X \text{ and } Y \text{ are independent)} \\ &= G_X(Z)G_Y(Z). \end{aligned}$$

3. By differentiation we obtain

$$\frac{d^k G_X(z)}{dz^k} = \sum_{n=k}^{\infty} n(n-1)(n-2)\dots(n-k+1)P(X=n)Z^{n-k}.$$

Thus,

$$\frac{d^k G_X(z)}{dz^k} = k!P(X=k) + \sum_{n=k+1}^{\infty} n(n-1)(n-2)\dots(n-k+1)P(X=n)Z^{n-k}.$$

Thus,

$$\frac{1}{k!}\frac{d^k G_X(z)}{dz^k}|_{z=0} = P(X = k).$$

4. By letting $Z = 1$ in

$$\frac{d^k G_X(z)}{dz^k} = \sum_{n=k}^{\infty} n(n-1)(n-2)\dots(n-k+1)P(X=n)Z^{n-k},$$

we obtain

$$\frac{d^k G_X(z)}{dz^k}|_{z=1} = \sum_{n=k}^{\infty} n(n-1)(n-2)\ldots(n-k+1)P(X=n),$$

which by LOTUS is equal to $E[X(X-1)(X-2)\ldots(X-k+1)]$.

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**Problem 8**

Let $M_X(s)$ be finite for $s \in [-c, c]$ where $c > 0$. Prove

$$\lim_{n\to\infty} \left[M_X(\frac{s}{n})\right]^n = e^{sEX}.$$

**Solution**

Equivalently, we show

$$\lim_{n\to\infty} n \ln\Big(M_X(\frac{s}{n})\Big) = sEX.$$

We have

$$\begin{aligned}\lim_{n\to\infty} n \ln\Big(M_X(\frac{s}{n})\Big) &= \lim_{n\to\infty} \frac{\ln\big(M_X(\frac{s}{n})\big)}{\frac{1}{n}} \\ &= \frac{0}{0}.\end{aligned}$$

So, we can use L'Hôpital's rule

$$\begin{aligned}\lim_{n\to\infty} \frac{\ln\big(M_X(\frac{s}{n})\big)}{\frac{1}{n}} &= \lim_{t\to 0} \frac{\ln(M_X(ts))}{t} \quad \left(\text{let} \quad t = \frac{1}{n}\right) \\ &= \lim_{t\to 0} \frac{\frac{sM_X'(ts)}{M_X(ts)}}{1} \quad (\text{by L'Hôpital's rule}) \\ &= \frac{sM_X'(0)}{M_X(0)} \\ &= s\mu \quad (\text{since} \quad M_X'(0) = \mu, M_X(0) = 1).\end{aligned}$$

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**Problem 9**

Let $M_X(s)$ be finite for $s \in [-c, c]$, where $c > 0$. Assume $EX = 0$, and $\text{Var}(X) = 1$. Prove

$$\lim_{n \to \infty} \left[ M_X\left(\frac{s}{\sqrt{n}}\right) \right]^n = e^{\frac{s^2}{2}}.$$

*Note:* From this, we can prove the Central Limit Theorem (CLT) which is discussed in Section 7.1.

**Solution**

Equivalently, we show

$$\lim_{n \to \infty} n \ln\left( M_X(\frac{s}{\sqrt{n}}) \right) = \frac{s^2}{2}.$$

We have

$$\begin{aligned}
\lim_{n \to \infty} n \ln\left( M_X(\frac{s}{\sqrt{n}}) \right) &= \lim_{n \to \infty} \frac{\ln\left( M_X(\frac{s}{\sqrt{n}}) \right)}{\frac{1}{n}} \quad (\text{let} \quad t = \frac{1}{\sqrt{n}}) \\
&= \lim_{t \to 0} \frac{\ln(M_X(ts))}{t^2} \\
&= \lim_{t \to 0} \frac{\frac{sM_X'(ts)}{M_X(ts)}}{2t} \quad (\text{by L'Hôpital's rule}) \\
&= \lim_{t \to 0} \frac{sM_X'(ts)}{2t} \quad (\text{again} \quad \frac{0}{0},) \\
&= \lim_{t \to 0} \frac{s^2 M_X''(ts)}{2} \quad (\text{by L'Hôpital's rule}) \\
&= \frac{s^2}{2} \quad (\text{since} \quad M_X''(0) = EX^2 = 1).
\end{aligned}$$

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**Problem 10**

We can define MGF for jointly distributed random variables as well. For example, for two random variables $(X, Y)$, the MGF is defined by

$$M_{XY}(s, t) = E[e^{sX + tY}].$$

Similar to the MGF of a single random variable, the MGF of the joint distributions uniquely determines the joint distribution. Let $X$ and $Y$ be two jointly normal random variables with $EX = \mu_X$, $EY = \mu_Y$, $\mathrm{Var}(X) = \sigma_X^2$, $\mathrm{Var}(Y) = \sigma_Y^2$, $\rho(X, Y) = \rho$ . Find $M_{XY}(s, t)$.

**Solution**

Note that $U = sX + tY$ is a linear combination of $X$ and $Y$ and thus it is a normal random variable. We have

$$\begin{aligned} EU &= sEX + tEY = s\mu_X + t\mu_Y, \\ \mathrm{Var}(U) &= s^2\mathrm{Var}(X) + t^2\mathrm{Var}(Y) + 2st\rho(X, Y)\sigma_X\sigma_Y \\ &= s^2\sigma_X^2 + t^2\sigma_Y^2 + 2st\rho\sigma_X\sigma_Y. \end{aligned}$$

Thus

$$U \sim N(s\mu_X + t\mu_Y, s^2\sigma_X^2 + t^2\sigma_Y^2 + 2st\rho\sigma_X\sigma_Y).$$

Note that for a normal random variable with mean $\mu$ and variance $\sigma^2$ the MGF is given by $e^{s\mu + \frac{\sigma^2 s^2}{2}}$. Thus

$$\begin{aligned} M_{XY}(s, t) &= E[e^U] = M_U(1) \\ &= e^{\mu_U + \frac{\sigma_U^2}{2}} \\ &= e^{s\mu_X + t\mu_Y + \frac{1}{2}(s^2\sigma_X^2 + t^2\sigma_Y^2 + 2st\rho\sigma_X\sigma_Y)}. \end{aligned}$$

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**Problem 11**

Let $\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ be a normal random vector with the following mean vector and covariance matrix

$$\mathbf{m} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}.$$

Let also

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \mathbf{Y} = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \mathbf{AX} + \mathbf{b}.$$

a. Find $P(0 \le X_2 \le 1)$.

b. Find the expected value vector of $\mathbf{Y}$, $\mathbf{m_Y} = E\mathbf{Y}$.

c. Find the covariance matrix of $\mathbf{Y}$, $\mathbf{C_Y}$.

d. Find $P(Y_3 \leq 4)$.

**Solution**

(a) From $m$ and $c$ we have $X_2 \sim N(1, 2)$. Thus

$$\begin{aligned} P(0 \leq X_2 \leq 1) &= \Phi\left(\frac{1-1}{\sqrt{2}}\right) - \Phi\left(\frac{0-1}{\sqrt{2}}\right) \\ &= \Phi(0) - \Phi\left(\frac{-1}{\sqrt{2}}\right) = 0.2602 \end{aligned}$$

(b)

$$\begin{aligned} m_Y &= EY = AEX + b \\ &= \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}. \end{aligned}$$

(c)

$$\begin{aligned} C_Y &= AC_XA^T \\ &= \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 5 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}. \end{aligned}$$

(d) From $m_Y$ and $c_Y$ we have $Y_3 \sim N(3, 1)$, thus

$$P(Y_3 \leq 4) = \Phi\left(\frac{4-3}{1}\right) = \Phi(1) = 0.8413$$

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**Problem 12**

(Whitening/decorrelating transformation) Let $\mathbf{X}$ be an $n$-dimensional zero-mean random vector. Since $C_X$ is a real symmetric matrix, we conclude that it can be diagonalized. That is, there exists an $n$ by $n$ matrix **Q** such that

$$
\begin{aligned}
&QQ^T = I \quad (I \text{ is the identity matrix}),\\
&C_X = QDQ^T,
\end{aligned}
$$

where D is a diagonal matrix

$$
D = \begin{bmatrix} d_{11} & 0 & \dots & 0 \\ 0 & d_{22} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & d_{nn} \end{bmatrix}.
$$

Now suppose we define a new random vector $\mathbf{Y}$ as $Y = Q^T X$, thus

$$
X = QY.
$$

Show that $\mathbf{Y}$ has a diagonal covariance matrix, and conclude that components of $\mathbf{Y}$ are uncorrelated, i.e., $\textrm{Cov}(Y_i, Y_j) = 0$ if $i \neq j$.

**Solution**

$$
\begin{aligned}
C_Y &= E[(Y - EY)(Y - EY)^T]\\
&= E[(Q^T X - EQ^T X)(Q^T X - EQ^T X)^T]\\
&= E[Q^T (X - EX)(X - EX)^T]Q]\\
&= Q^T C_X Q\\
&= Q^T QDQ^T Q\\
&= D \quad (\text{since} \quad Q^T Q = I).
\end{aligned}
$$

Therefore, $\mathbf{Y}$ has a diagonal covariance matrix, and $\textrm{Cov}(Y_i, Y_j) = 0$ if $i \neq j$.

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