5.2.2 Joint Cumulative Distribution Function (CDF)

We have already seen the joint CDF for discrete random variables. The joint CDF has the same definition for continuous random variables. It also satisfies the same properties.

The **joint cumulative function** of two random variables X and Y is defined as

$$F_{XY}(x,y) = P(X \leq x, Y \leq y).$$

The joint CDF satisfies the following properties:

1. $F_X(x) = F_{XY}(x,\infty)$, for any x (marginal CDF of X);
2. $F_Y(y) = F_{XY}(\infty,y)$, for any y (marginal CDF of Y);
3. $F_{XY}(\infty,\infty) = 1$;
4. $F_{XY}(-\infty,y) = F_{XY}(x,-\infty) = 0$;
5. $P(x_1 < X \leq x_2, \;\; y_1 < Y \leq y_2) =$
   $$F_{XY}(x_2,y_2) - F_{XY}(x_1,y_2) - F_{XY}(x_2,y_1) + F_{XY}(x_1,y_1)$$
   ;
6. if X and Y are independent, then $F_{XY}(x,y) = F_X(x)F_Y(y)$.

Example 5.18

Let X and Y be two independent $Uniform(0,1)$ random variables. Find $F_{XY}(x,y)$.

Solution

Since $X, Y \sim Uniform(0,1)$, we have

$$F_X(x) = \begin{cases} 0 & \text{for } x < 0 \\ x & \text{for } 0 \leq x \leq 1 \\ 1 & \text{for } x > 1 \end{cases}$$

$$F_Y(y) = \begin{cases} 0 & \text{for } y < 0 \\ y & \text{for } 0 \le y \le 1 \\ 1 & \text{for } y > 1 \end{cases}$$

Since X and Y are independent, we obtain

$$F_{XY}(x,y) = F_X(x)F_Y(y) = \begin{cases} 0 & \text{for } y < 0 \text{ or } x < 0 \\ xy & \text{for } 0 \le x \le 1, 0 \le y \le 1 \\ y & \text{for } x > 1, 0 \le y \le 1 \\ x & \text{for } y > 1, 0 \le x \le 1 \\ 1 & \text{for } x > 1, y > 1 \end{cases}$$

Figure 5.7 shows the values of $F_{XY}(x,y)$ in the $x-y$ plane. Note that $F_{XY}(x,y)$ is a continuous function in both arguments. This is always true for jointly continuous random variables. This fact sometimes simplifies finding $F_{XY}(x,y)$. The next example (Example 5.19) shows how we can use this fact.

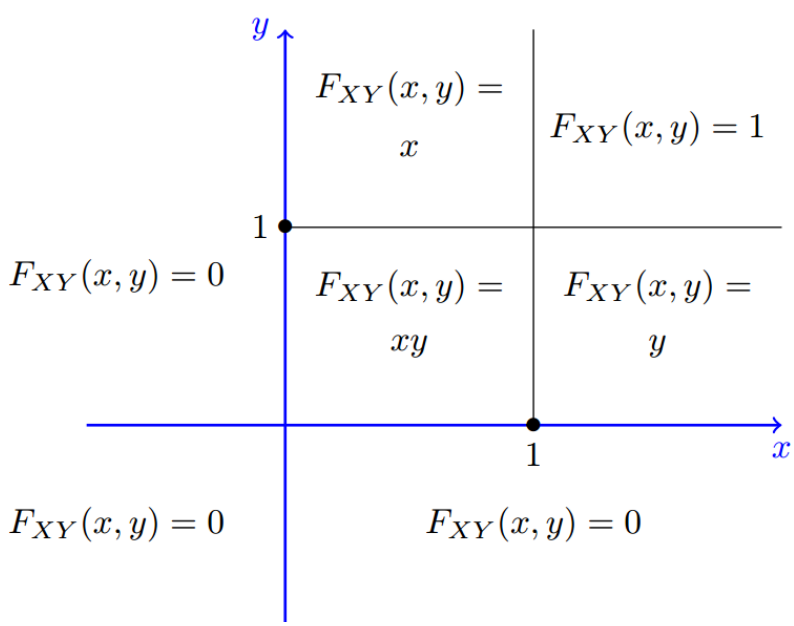

Figure 5.7: The joint CDF of two independent $Uniform(0,1)$ random variables X and Y.

Remember that, for a single random variable, we have the following relationship between the PDF and CDF:

$$\begin{aligned} F_X(x) &= \int_{-\infty}^{x} f_X(u)du, \\ f_X(x) &= \frac{dF_X(x)}{dx}. \end{aligned}$$

Similar formulas hold for jointly continuous random variables. In particular, we have the following:

$$F_{XY}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{XY}(u,v)dudv,$$

$$f_{XY}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x,y)$$

Example 5.19

Find the joint CDF for X and Y in Example 5.15

Solution

In Example 5.15, we found

$$f_{XY}(x,y) = \begin{cases} x + \frac{3}{2}y^2 & 0 \leq x, y \leq 1 \\ \\ 0 & \text{otherwise} \end{cases}$$

First, note that since $R_{XY} = \{(x,y) | 0 \leq x, y \leq 1\}$, we find that

$$\begin{aligned} F_{XY}(x,y) &= 0, \text{ for } x < 0 \text{ or } y < 0, \\ F_{XY}(x,y) &= 1, \text{ for } x \geq 1 \text{ and } y \geq 1. \end{aligned}$$

To find the joint CDF for $x > 0$ and $y > 0$, we need to integrate the joint PDF:

$$
\begin{aligned}
F_{XY}(x,y) &= \int_{-\infty}^{y}\int_{-\infty}^{x} f_{XY}(u,v)dudv \\
&= \int_{0}^{y}\int_{0}^{x} f_{XY}(u,v)dudv \\
&= \int_{0}^{\min(y,1)}\int_{0}^{\min(x,1)} \left(u+\frac{3}{2}v^2\right) dudv.
\end{aligned}
$$

For $0 \leq x, y \leq 1$, we obtain

$$
\begin{aligned}
F_{XY}(x,y) &= \int_{0}^{y}\int_{0}^{x} \left(u+\frac{3}{2}v^2\right) dudv \\
&= \int_{0}^{y} \left[\frac{1}{2}u^2+\frac{3}{2}v^2u\right]_0^x dv \\
&= \int_{0}^{y} \left(\frac{1}{2}x^2+\frac{3}{2}xv^2\right) dv \\
&= \frac{1}{2}x^2y+\frac{1}{2}xy^3.
\end{aligned}
$$

For $0 \leq x \leq 1$ and $y \geq 1$, we use the fact that F_{XY} is continuous to obtain

$$
\begin{aligned}
F_{XY}(x,y) &= F_{XY}(x,1) \\
&= \frac{1}{2}x^2+\frac{1}{2}x.
\end{aligned}
$$

Similarly, for $0 \leq y \leq 1$ and $x \geq 1$, we obtain

$$
\begin{aligned}
F_{XY}(x,y) &= F_{XY}(1,y) \\
&= \frac{1}{2}y+\frac{1}{2}y^3.
\end{aligned}
$$