## 8.5.5 Solved Problems

**Problem 1**

Consider the following observed values of $(x_i, y_i)$:

$$(-1,6), \quad (0,3), \quad (1,2), \quad (2,-1)$$

a. Find the estimated regression line

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x,$$

based on the observed data.

b. For each $x_i$, compute the fitted value of $y_i$ using

$$\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i.$$

c. Compute the residuals, $e_i = y_i - \hat{y}_i$.

d. Find $R$-squared (the coefficient of determination).

**Solution**

a. We have

$$\begin{aligned}
\overline{x} &= \frac{-1+0+1+2}{4} = 0.5, \\
\overline{y} &= \frac{6+3+2+(-1)}{4} = 2.5, \\
s_{xx} &= (-1-0.5)^2 + (0-0.5)^2 + (1-0.5)^2 + (2-0.5)^2 = 5, \\
s_{xy} &= (-1-0.5)(6-2.5) + (0-0.5)(3-2.5) \\
&\quad + (1-0.5)(2-2.5) + (2-0.5)(-1-2.5) = -11.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
\hat{\beta}_1 &= \frac{s_{xy}}{s_{xx}} = \frac{-11}{5} = -2.2, \\
\hat{\beta}_0 &= 2.5 - (-2.2)(0.5) = 3.6
\end{aligned}$$

The following MATLAB code can be used to obtain the estimated regression line

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x=[-1;0;1;2];
x0=ones(size(x));
y=[6;3;2;-1];
beta = regress(y,[x0,x]);
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b. The fitted values are given by

$$\hat{y}_i = 3.6 - 2.2x_i,$$

so we obtain

$$\hat{y}_1 = 5.8, \quad \hat{y}_2 = 3.6, \quad \hat{y}_3 = 1.4, \quad \hat{y}_4 = -0.8$$

c. We have

$$\begin{aligned}
e_1 &= y_1 - \hat{y}_1 = 6 - 5.8 = 0.2, \\
e_2 &= y_2 - \hat{y}_2 = 3 - 3.6 = -0.6, \\
e_3 &= y_3 - \hat{y}_3 = 2 - 1.4 = 0.6, \\
e_4 &= y_4 - \hat{y}_4 = -1 - (-0.8) = -0.2
\end{aligned}$$

d. We have

$$s_{yy} = (6 - 2.5)^2 + (3 - 2.5)^2 + (2 - 2.5)^2 + (-1 - 2.5)^2 = 25.$$

We conclude

$$r^2 = \frac{(-11)^2}{5 \times 25} \approx 0.968$$

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**Problem 2**
Consider the model

$$Y = \beta_0 + \beta_1 X + \epsilon,$$

where $\epsilon$ is a $N(0, \sigma^2)$ random variable independent of $X$. Let also

$$\hat{Y} = \beta_0 + \beta_1 X.$$

Show that

$$E[(Y - EY)^2] = E[(\hat{Y} - EY)^2] + E[(Y - \hat{Y})^2].$$

**Solution**

Since $X$ and $\epsilon$ are independent, we can write

$$\mathrm{Var}(Y) = \beta_1^2 \mathrm{Var}(X) + \mathrm{Var}(\epsilon) \qquad (8.10)$$

Note that,

$$\begin{aligned}\hat{Y} - EY &= (\beta_0 + \beta_1 X) - (\beta_0 + \beta_1 EX) \\ &= \beta_1 (X - EX).\end{aligned}$$

Therefore,

$$E[(\hat{Y} - EY)^2] = \beta_1^2 \mathrm{Var}(X).$$

Also,

$$E[(Y - EY)^2] = \mathrm{Var}(Y), \quad E[(Y - \hat{Y})^2] = \mathrm{Var}(\epsilon).$$

Combining with Equation 8.10, we conclude

$$E[(Y - EY)^2] = E[(\hat{Y} - EY)^2] + E[(Y - \hat{Y})^2].$$

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**Problem 3**

Show that, in a simple linear regression, the estimated coefficients $\hat{\beta}_0$ and $\hat{\beta}_1$ (least squares estimates of $\beta_0$ and $\beta_1$) satisfy the following equations

$$\sum_{i=1}^{n} e_i = 0, \quad \sum_{i=1}^{n} e_i x_i = 0, \quad \sum_{i=1}^{n} e_i \hat{y}_i = 0,$$

where $e_i = y_i - \hat{y}_i = y_i - \hat{\beta}_0 - \hat{\beta}_1 x$. *Hint:* $\hat{\beta}_0$ and $\hat{\beta}_1$ satisfy Equation 8.8 and Equation 8.9. By cancelling the $(-2)$ factor, you can write

$$\sum_{i=1}^{n} (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) = 0,$$
$$\sum_{i=1}^{n} (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) x_i = 0.$$

Use the above equations to show the desired equations.

**Solution**

We have

$$\sum_{i=1}^{n}(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) = 0,$$
$$\sum_{i=1}^{n}(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)x_i = 0.$$

Since $e_i = y_i - \hat{\beta}_0 - \hat{\beta}_1 x$, we conclude

$$\sum_{i=1}^{n} e_i = 0,$$
$$\sum_{i=1}^{n} e_i x_i = 0.$$

Moreover,

$$\begin{aligned}\sum_{i=1}^{n} e_i \hat{y}_i &= \sum_{i=1}^{n} e_i(\hat{\beta}_0 + \hat{\beta}_1 x_i) \\ &= \hat{\beta}_0 \sum_{i=1}^{n} e_i + \hat{\beta}_1 \sum_{i=1}^{n} e_i x_i \\ &= 0 + 0 = 0.\end{aligned}$$

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**Problem 4**

Show that the coefficient of determination can also be obtained as

$$r^2 = \frac{\sum_{i=1}^{n}(\hat{y}_i - \overline{y})^2}{\sum_{i=1}^{n}(y_i - \overline{y})^2}.$$

**Solution**

We know

$$\hat{y}_i = \beta_0 + \beta_1 x_i,$$
$$\overline{y} = \beta_0 + \beta_1 \overline{x}.$$

Therefore,

$$\begin{aligned}\sum_{i=1}^{n}(\hat{y}_i-\overline{y})^2 &= \sum_{i=1}^{n}(\beta_1 x_i-\beta_1\overline{x})^2\\ &= \beta_1^2\sum_{i=1}^{n}(x_i-\overline{x})^2\\ &= \beta_1^2 s_{xx}.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{\sum_{i=1}^{n}(\hat{y}_i-\overline{y})^2}{\sum_{i=1}^{n}(y_i-\overline{y})^2} &= \frac{\beta_1^2 s_{xx}}{s_{yy}}\\ &= \frac{s_{xy}^2}{s_{xx}s_{yy}} \quad (\text{since } \beta_1=\frac{s_{xy}}{s_{xx}})\\ &= r^2.\end{aligned}$$

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**Problem 5**

(**The Method of Maximum Likelihood**) This problem assumes that you are familiar with the maximum likelihood method discussed in Section 8.2.3. Consider the model

$$Y_i=\beta_0+\beta_1 x_i+\epsilon_i,$$

where $\epsilon_i$'s are independent $N(0,\sigma^2)$ random variables. Our goal is to estimate $\beta_0$ and $\beta_1$. We have the observed data pairs $(x_1,y_1)$, $(x_2,y_2)$, $\cdots$, $(x_n,y_n)$.

a. Argue that, for given values of $\beta_0$, $\beta_1$, and $x_i$, $Y_i$ is a normal random variable with mean $\beta_0+\beta_1 x_i$ and variance $\sigma^2$. Moreover, show that the $Y_i$'s are independent.
b. Find the likelihood function

$$L(y_1,y_2,\cdots,y_n;\beta_0,\beta_1)=f_{Y_1Y_2\cdots Y_n}(y_1,y_2,\cdots,y_n;\beta_0,\beta_1).$$

c. Show that the maximum likelihood estimates of $\beta_0$ and $\beta_1$ are the same as the ones we obtained using the least squares method.

**Solution**

a. Given values of $\beta_0$, $\beta_1$, and $x_i$, $c=\beta_0+\beta_1 x_i$ is a constant. Therefore, $Y_i=c+\epsilon_i$ is a normal random variable with mean $c$ and variance $\sigma^2$. Also, since the $\epsilon_i$'s are independent, we conclude that $Y_i$'s are also independent random variables.
b. By the previous part, for given values of $\beta_0$, $\beta_1$, and $x_i$,

$$f_{Y_i}(y;\beta_0,\beta_1) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{1}{2}(y-\beta_0-\beta_1 x_i)^2\right\}.$$

Therefore, the likelihood function is given by

$$\begin{aligned} L(y_1,y_2,\cdots,y_n;\beta_0,\beta_1) &= f_{Y_1Y_2\cdots Y_n}(y_1,y_2,\cdots,y_n;\beta_0,\beta_1) \\ &= f_{Y_1}(y_1;\beta_0,\beta_1)f_{Y_2}(y_2;\beta_0,\beta_1)\cdots f_{Y_n}(y_n;\beta_0,\beta_1) \\ &= \frac{1}{(2\pi\sigma^2)^{\frac{n}{2}}}\exp\left\{-\frac{1}{2}\sum_{i=1}^{n}(y-\beta_0-\beta_1 x_i)^2\right\}. \end{aligned}$$

c. To find the maximum likelihood estimates (MLE) of $\beta_0$ and $\beta_1$, we need to find $\hat{\beta}_0$ and $\hat{\beta}_1$ such that the likelihood function

$$L(y_1,y_2,\cdots,y_n;\beta_0,\beta_1) = \frac{1}{(2\pi\sigma^2)^{\frac{n}{2}}}\exp\left\{-\frac{1}{2}\sum_{i=1}^{n}(y-\beta_0-\beta_1 x_i)^2\right\}$$

is maximized. This is equivalent to minimizing

$$\sum_{i=1}^{n}(y-\beta_0-\beta_1 x_i)^2.$$

The above expression is the sum of the squared errors, $g(\beta_0,\beta_1)$ (Equation 8.7). Therefore, the maximum likelihood estimation for this model is the same as the least squares method.