5.1.5 Conditional Expectation (Revisited) and Conditional Variance

In Section 5.1.3, we briefly discussed conditional expectation. Here, we will discuss the properties of conditional expectation in more detail as they are quite useful in practice. We will also discuss conditional variance. An important concept here is that we interpret the conditional expectation as a random variable.

Conditional Expectation as a Function of a Random Variable:

Remember that the conditional expectation of X given that $Y=y$ is given by

$$E[X|Y=y]=\sum_{x_i \in R_X} x_i P_{X|Y}(x_i|y).$$

Note that $E[X|Y=y]$ depends on the value of y. In other words, by changing y, $E[X|Y=y]$ can also change. Thus, we can say $E[X|Y=y]$ is a function of y, so let's write

$$g(y)=E[X|Y=y].$$

Thus, we can think of $g(y)=E[X|Y=y]$ as a function of the value of random variable Y. We then write

$$g(Y)=E[X|Y].$$

We use this notation to indicate that $E[X|Y]$ is a random variable whose value equals $g(y)=E[X|Y=y]$ when $Y=y$. Thus, if Y is a random variable with range $R_Y=\{y_1,y_2,\cdots\}$, then $E[X|Y]$ is also a random variable with

$$E[X|Y]=\begin{cases} E[X|Y=y_1] & \quad \text{with probability } P(Y=y_1) \\ E[X|Y=y_2] & \quad \text{with probability } P(Y=y_2) \\ \quad\quad . & \quad\quad\quad . \\ \quad\quad . & \quad\quad\quad . \\ \quad\quad . & \quad\quad\quad . \end{cases}$$

Let's look at an example.

Example 5.10 Let $X = aY + b$. Then $E[X|Y = y] = E[aY + b|Y = y] = ay + b$. Here, we have $g(y) = ay + b$, and therefore,

$$E[X|Y] = aY + b,$$

which is a function of the random variable Y.

Since $E[X|Y]$ is a random variable, we can find its PMF, CDF, variance, etc. Let's look at an example to better understand $E[X|Y]$.

Example 5.11 Consider two random variables X and Y with joint PMF given in Table 5.2. Let $Z = E[X|Y]$.

a. Find the Marginal PMFs of X and Y.
b. Find the conditional PMF of X given $Y = 0$ and $Y = 1$, i.e., find $P_{X|Y}(x|0)$ and $P_{X|Y}(x|1)$.
c. Find the PMF of Z.
d. Find EZ, and check that $EZ = EX$.
e. Find Var(Z).

Table 5.2: Joint PMF of X and Y in example 5.11

	$Y=0$	$Y=1$
$X=0$	$\frac{1}{5}$	$\frac{2}{5}$
$X=1$	$\frac{2}{5}$	0

Solution

a. Using the table we find out

$$P_X(0) = \frac{1}{5} + \frac{2}{5} = \frac{3}{5},$$
$$P_X(1) = \frac{2}{5} + 0 = \frac{2}{5},$$
$$P_Y(0) = \frac{1}{5} + \frac{2}{5} = \frac{3}{5},$$
$$P_Y(1) = \frac{2}{5} + 0 = \frac{2}{5}.$$

Thus, the marginal distributions of X and Y are both $Bernoulli(\frac{2}{5})$. However, note that X and Y are not independent.

b. We have

$$P_{X|Y}(0|0) = \frac{P_{XY}(0,0)}{P_Y(0)}$$
$$= \frac{\frac{1}{5}}{\frac{3}{5}} = \frac{1}{3}.$$

Thus,

$$P_{X|Y}(1|0) = 1 - \frac{1}{3} = \frac{2}{3}.$$

We conclude

$$X|Y = 0 \quad \sim \quad Bernoulli\left(\frac{2}{3}\right).$$

Similarly, we find

$$P_{X|Y}(0|1) = 1,$$
$$P_{X|Y}(1|1) = 0.$$

Thus, given $Y = 1$, we have always $X = 0$.

c. We note that the random variable Y can take two values: 0 and 1. Thus, the random variable $Z = E[X|Y]$ can take two values as it is a function of Y. Specifically,

$$Z = E[X|Y] = \begin{cases} E[X|Y=0] & \text{if } Y = 0 \\ \\ E[X|Y=1] & \text{if } Y = 1 \end{cases}$$

Now, using the previous part, we have

$$E[X|Y = 0] = \frac{2}{3}, \qquad E[X|Y = 1] = 0,$$

and since $P(y=0)=\frac{3}{5}$, and $P(y=1)=\frac{2}{5}$, we conclude that

$$Z=E[X|Y]=\begin{cases}\frac{2}{3} & \text{with probability } \frac{3}{5}\\ \\ 0 & \text{with probability } \frac{2}{5}\end{cases}$$

So we can write

$$P_Z(z)=\begin{cases}\frac{3}{5} & \text{if } z=\frac{2}{3}\\ \\ \frac{2}{5} & \text{if } z=0\\ \\ 0 & \text{otherwise}\end{cases}$$

d. Now that we have found the PMF of Z, we can find its mean and variance. Specifically,

$$E[Z]=\frac{2}{3}\cdot\frac{3}{5}+0\cdot\frac{2}{5}=\frac{2}{5}.$$

We also note that $EX=\frac{2}{5}$. Thus, here we have

$$E[X]=E[Z]=E[E[X|Y]].$$

In fact, as we will prove shortly, the above equality always holds. It is called the law of iterated expectations.

e. To find Var(Z), we write

$$\begin{aligned}\text{Var}(Z)&=E[Z^2]-(EZ)^2\\&=E[Z^2]-\frac{4}{25},\end{aligned}$$

where

$$E[Z^2]=\frac{4}{9}\cdot\frac{3}{5}+0\cdot\frac{2}{5}=\frac{4}{15}.$$

Thus,

$$\begin{aligned}\text{Var}(Z)&=\frac{4}{15}-\frac{4}{25}\\&=\frac{8}{75}.\end{aligned}$$