5.1.6 Solved Problems

Problem 1

Consider two random variables X and Y with joint PMF given in Table 5.3.

a. Find $P(X \leq 2, Y \leq 4)$.
b. Find the marginal PMFs of X and Y.
c. Find $P(Y = 2|X = 1)$.
d. Are X and Y independent?

	$Y = 2$	$Y = 4$	$Y = 5$
$X = 1$	$\frac{1}{12}$	$\frac{1}{24}$	$\frac{1}{24}$
$X = 2$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{8}$
$X = 3$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{12}$

Solution

a. To find $P(X \leq 2, Y \leq 4)$, we can write

$$\begin{aligned} P(X \leq 2, Y \leq 4) &= P_{XY}(1,2) + P_{XY}(1,4) + P_{XY}(2,2) + P_{XY}(2,4) \\ &= \frac{1}{12} + \frac{1}{24} + \frac{1}{6} + \frac{1}{12} = \frac{3}{8}. \end{aligned}$$

b. Note from the table that

$$R_X = \{1, 2, 3\} \text{ and } R_Y = \{2, 4, 5\}.$$

Now we can use Equation 5.1 to find the marginal PMFs:

$$P_X(x) = \begin{cases} \frac{1}{6} & x = 1 \\ \frac{3}{8} & x = 2 \\ \frac{11}{24} & x = 3 \\ 0 & \text{otherwise} \end{cases}$$

$$P_Y(y) = \begin{cases} \frac{1}{2} & y = 2 \\ \frac{1}{4} & y = 4 \\ \frac{1}{4} & y = 5 \\ 0 & \text{otherwise} \end{cases}$$

c. Using the formula for conditional probability, we have

$$\begin{aligned} P(Y = 2|X = 1) &= \frac{P(X = 1, Y = 2)}{P(X = 1)} \\ &= \frac{P_{XY}(1, 2)}{P_X(1)} \\ &= \frac{\frac{1}{12}}{\frac{1}{6}} = \frac{1}{2}. \end{aligned}$$

d. Are X and Y independent? To check whether X and Y are independent, we need to check that $P(X = x_i, Y = y_j) = P(X = x_i)P(Y = y_j)$, for all $x_i \in R_X$ and all $y_j \in R_Y$. Looking at the table and the results from previous parts, we find

$$P(X = 2, Y = 2) = \frac{1}{6} \neq P(X = 2)P(Y = 2) = \frac{3}{16}.$$

Thus, we conclude that X and Y are not independent.

Problem 2

I have a bag containing 40 blue marbles and 60 red marbles. I choose 10 marbles (without replacement) at random. Let X be the number of blue marbles and y be the number of red marbles. Find the joint PMF of X and Y.

Solution

This is, in fact, a hypergeometric distribution. First, note that we must have $X+Y=10$, so

$$\begin{aligned} R_{XY} &= \{(i,j) | i+j=10, i,j \in \mathbb{Z}, i,j \geq 0\} \\ &= \{(0,10),(1,9),(2,8),\ldots,(10,0)\}. \end{aligned}$$

Then, we can write

$$P_{XY}(i,j) = \begin{cases} \frac{\binom{40}{i}\binom{60}{j}}{\binom{100}{10}} & i+j=10, i,j \in \mathbb{Z}, i,j \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Problem 3

Let X and Y be two independent discrete random variables with the same CDFs F_X and F_Y. Define

$$\begin{aligned} Z &= \max(X,Y), \\ W &= \min(X,Y). \end{aligned}$$

Find the CDFs of Z and W.

Solution

To find the CDF of Z, we can write

$$\begin{aligned} F_Z(z) &= P(Z \leq z) \\ &= P(\max(X,Y) \leq z) \\ &= P\Big((X \leq z) \text{ and } (Y \leq z)\Big) \\ &= P(X \leq z)P(Y \leq z) && (\text{ since } X \text{ and } Y \text{ are independent}) \\ &= F_X(z)F_Y(z). \end{aligned}$$

To find the CDF of W, we can write

$$
\begin{aligned}
F_W(w) &= P(W \leq w) \\
&= P(\min(X,Y) \leq w) \\
&= 1 - P(\min(X,Y) > w) \\
&= 1 - P\Big((X > w) \text{ and } (Y > w)\Big) \\
&= 1 - P(X > w)P(Y > w) && (\text{ since } X \text{ and } Y \text{ are independent}) \\
&= 1 - (1 - F_X(w))(1 - F_Y(w)) \\
&= F_X(w) + F_Y(w) - F_X(w)F_Y(w).
\end{aligned}
$$

Problem 4

Let X and Y be two discrete random variables, with range

$$R_{XY} = \{(i,j) \in \mathbb{Z}^2 | i,j \geq 0, |i-j| \leq 1\},$$

and joint PMF given by

$$P_{XY}(i,j) = \frac{1}{6 \cdot 2^{\min(i,j)}}, \quad \text{for } (i,j) \in R_{XY}.$$

a. Pictorially show R_{XY} in the $x-y$ plane.
b. Find the marginal PMFs $P_X(i)$, $P_Y(j)$.
c. Find $P(X = Y | X < 2)$.
d. Find $P(1 \leq X^2 + Y^2 \leq 5)$.
e. Find $P(X = Y)$.
f. Find $E[X|Y = 2]$.
g. Find Var$(X|Y = 2)$.

Solution

a. Figure 5.5 shows the R_{XY} in the $x-y$ plane.

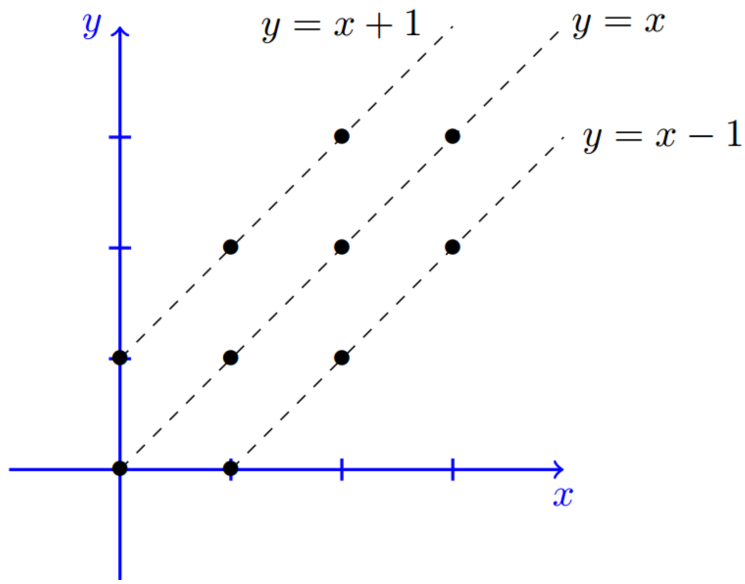

Figure 5.5: Figure shows R_{XY} for X and Y in problem 4.

b. First, by symmetry we note that X and Y have the same PMF. Next, we can write

$$
\begin{aligned}
P_X(0) &= P_{XY}(0,0) + P_{XY}(0,1) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}, \\
P_X(1) &= P_{XY}(1,0) + P_{XY}(1,1) + P_{XY}(1,2) = \frac{1}{6}\left(1 + \frac{1}{2} + \frac{1}{2}\right) = \frac{1}{3}, \\
P_X(2) &= P_{XY}(2,1) + P_{XY}(2,2) + P_{XY}(2,3) = \frac{1}{6}\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{4}\right) = \frac{1}{6}, \\
P_X(3) &= P_{XY}(3,2) + P_{XY}(3,3) + P_{XY}(3,4) = \frac{1}{6}\left(\frac{1}{4} + \frac{1}{8} + \frac{1}{8}\right) = \frac{1}{12}.
\end{aligned}
$$

In general, we obtain

$$
P_X(k) = P_Y(k) = \begin{cases} \frac{1}{3} & k = 0 \\ \frac{1}{3 \cdot 2^{k-1}} & k = 1, 2, 3, \ldots \\ 0 & \text{otherwise} \end{cases}
$$

c. Find $P(X = Y | X < 2)$: We have

$$
\begin{aligned}
P(X=Y|X<2) &= \frac{P(X=Y, X<2)}{P(X<2)} \\
&= \frac{P_{XY}(0,0)+P_{XY}(1,1)}{P_X(0)+P_X(1)} \\
&= \frac{\frac{1}{6}+\frac{1}{12}}{\frac{1}{3}+\frac{1}{3}} \\
&= \frac{3}{8}.
\end{aligned}
$$

d. Find $P(1 \le X^2+Y^2 \le 5)$: We have

$$
\begin{aligned}
P(1 \le X^2+Y^2 \le 5) &= P_{XY}(0,1)+P_{XY}(1,0)+P_{XY}(1,1)+P_{XY}(1,2)+P_{XY}(2,1) \\
&= \frac{1}{6}+\frac{1}{6}+\frac{1}{12}+\frac{1}{12}+\frac{1}{12} \\
&= \frac{7}{12}.
\end{aligned}
$$

e. By symmetry, we can argue that $P(X=Y)=\frac{1}{3}$. The reason is that R_{XY} consists of three lines with points with the same probabilities. We can also find $P(X=Y)$ by

$$
\begin{aligned}
P(X=Y) &= \sum_{i=0}^{\infty} P_{XY}(i,i) \\
&= \sum_{i=0}^{\infty} \frac{1}{6.2^i} \\
&= \frac{1}{3}.
\end{aligned}
$$

f. To find $E[X|Y=2]$, we first need the conditional PMF of X given $Y=2$. We have

$$
\begin{aligned}
P_{X|Y}(k|2) &= \frac{P_{XY}(k,2)}{P(Y=2)} \\
&= 6P_{XY}(k,2),
\end{aligned}
$$

so we obtain

$$
P_{X|Y}(k|2) = \begin{cases} \frac{1}{2} & k=1 \\ \frac{1}{4} & k=2,3 \\ 0 & \text{otherwise} \end{cases}
$$

Thus,

$$\begin{aligned} E[X|Y=2] &= 1\cdot\frac{1}{2}+2\cdot\frac{1}{4}+3\cdot\frac{1}{4} \\ &= \frac{7}{4}. \end{aligned}$$

g. Find Var$(X|Y=2)$: we have

$$\begin{aligned} E[X^2|Y=2] &= 1\cdot\frac{1}{2}+4\cdot\frac{1}{4}+9\cdot\frac{1}{4} \\ &= \frac{15}{4}. \end{aligned}$$

Thus,

$$\begin{aligned} \mathrm{Var}(X) &= E[X^2|Y=2]-\big(E[X|Y=2]\big)^2 \\ &= \frac{15}{4}-\frac{49}{16} \\ &= \frac{11}{16}. \end{aligned}$$

Problem 5

Suppose that the number of customers visiting a fast food restaurant in a given day is $N \sim Poisson(\lambda)$. Assume that each customer purchases a drink with probability p, independently from other customers, and independently from the value of N. Let X be the number of customers who purchase drinks. Let Y be the number of customers that do not purchase drinks; so $X+Y=N$.

a. Find the marginal PMFs of X and Y.
b. Find the joint PMF of X and Y.
c. Are X and Y independent?
d. Find $E[X^2Y^2]$.

Solution

a. First note that $R_X=R_Y=\{0,1,2,\ldots\}$. Also, given $N=n$, X is a sum of n independent $Bernoulli(p)$ random variables. Thus, given $N=n$, X has a binomial distribution with parameters n and p, so

$$\begin{aligned} X|N=n &\sim Binomial(n,p), \\ Y|N=n &\sim Binomial(n,q=1-p). \end{aligned}$$

We have

$$
\begin{aligned}
P_X(k) &= \sum_{n=0}^{\infty} P(X=k|N=n)P_N(n) && \text{(law of total probability)}\\
&= \sum_{n=k}^{\infty} \binom{n}{k} p^k q^{n-k} exp(-\lambda)\frac{\lambda^n}{n!}\\
&= \sum_{n=k}^{\infty} \frac{p^k q^{n-k} exp(-\lambda)\lambda^n}{k!(n-k)!}\\
&= \frac{exp(-\lambda)(\lambda p)^k}{k!} \sum_{n=k}^{\infty} \frac{(\lambda q)^{n-k}}{(n-k)!}\\
&= \frac{exp(-\lambda)(\lambda p)^k}{k!} exp(\lambda q) && \text{(Taylor series for } e^x\text{)}\\
&= \frac{exp(-\lambda p)(\lambda p)^k}{k!}, \qquad \text{for } k = 0,1,2,\ldots
\end{aligned}
$$

Thus, we conclude that

$$
\begin{aligned}
X &\sim Poisson(\lambda p),\\
Y &\sim Poisson(\lambda q).
\end{aligned}
$$

b. To find the joint PMF of X and Y, we can also use the law of total probability:

$$
P_{XY}(i,j) = \sum_{n=0}^{\infty} P(X=i, Y=j|N=n)P_N(n) \qquad \text{(law of total probability).}
$$

But note that $P(X=i, Y=j|N=n) = 0$ if $N \neq i+j$, thus

$$
\begin{aligned}
P_{XY}(i,j) &= P(X=i, Y=j|N=i+j)P_N(i+j)\\
&= P(X=i|N=i+j)P_N(i+j)\\
&= \binom{i+j}{i} p^i q^j exp(-\lambda)\frac{\lambda^{i+j}}{(i+j)!}\\
&= \frac{exp(-\lambda)(\lambda p)^i(\lambda q)^j}{i!j!}\\
&= \frac{exp(-\lambda p)(\lambda p)^i}{i!} \cdot \frac{exp(-\lambda q)(\lambda q)^j}{j!}\\
&= P_X(i)P_Y(j).
\end{aligned}
$$

c. X and Y are independent, since as we saw above

$$
P_{XY}(i,j) = P_X(i)P_Y(j).
$$

d. Since X and Y are independent, we have

$$
E[X^2Y^2] = E[X^2]E[Y^2].
$$

Also, note that for a Poisson random variable W with parameter λ,

$$E[W^2] = \mathrm{Var}(W) + (EW)^2 = \lambda + \lambda^2.$$

Thus,

$$\begin{aligned} E[X^2Y^2] &= E[X^2]E[Y^2] \\ &= (\lambda p + \lambda^2 p^2)(\lambda q + \lambda^2 q^2) \\ &= \lambda^2 pq(\lambda^2 pq + \lambda + 1). \end{aligned}$$

Problem 6

I have a coin with $P(H) = p$. I toss the coin repeatedly until I observe two consecutive heads. Let X be the total number of coin tosses. Find EX.

Solution

We solve this problem using a similar approach as in Example 5.6. Let $\mu = EX$. We first condition on the result of the first coin toss. Specifically,

$$\begin{aligned} \mu = EX &= E[X|H]P(H) + E[X|T]P(T) \\ &= E[X|H]p + (1 + \mu)(1 - p). \end{aligned}$$

In this equation, $E[X|T] = 1 + EX$, because the tosses are independent, so if the first toss is tails, it is like starting over on the second toss. Thus,

$$p\mu = pE[X|H] + (1 - p) \tag{5.14}$$

We still need to find $E[X|H]$ so we condition on the second coin toss

$$\begin{aligned} E[X|H] &= E[X|HH]P + E[X|HT](1 - p) \\ &= 2p + (2 + \mu)(1 - p) \\ &= 2 + (1 - p)\mu. \end{aligned}$$

Here, $E[X|HT] = 2 + EX$ because, if the first two tosses are HT, we have wasted two coin tosses and we start over at the third toss. By letting $E[X|H] = 2 + (1 - p)\mu$ in Equation 5.14, we obtain

$$\mu = EX = \frac{1 + p}{p^2}.$$

Problem 7

Let $X, Y \sim Geometric(p)$ be independent, and let $Z = \frac{X}{Y}$.

a. Find the range of Z.
b. Find the PMF of Z.
c. Find EZ.

Solution

a. The range of Z is given by

$$R_Z = \left\{ \frac{m}{n} | m, n \in \mathbb{N} \right\},$$

which is the set of all positive rational numbers.

b. To find PMF of Z, let $m, n \in \mathbb{N}$ such that $(m, n) = 1$, where (m, n) is the largest divisor of m and n. Then

$$\begin{aligned}
P_Z\left(\frac{m}{n}\right) &= \sum_{k=1}^{\infty} P(X = mk, Y = nk) \\
&= \sum_{k=1}^{\infty} P(X = mk)P(Y = nk) \qquad \text{(since X and $Y$are independent)} \\
&= \sum_{k=1}^{\infty} pq^{mk-1}pq^{nk-1} \qquad \text{(where $q = 1 - p$)} \\
&= p^2 q^{-2} \sum_{k=1}^{\infty} q^{(m+n)k} \\
&= \frac{p^2 q^{m+n-2}}{1 - q^{m+n}} \\
&= \frac{p^2 (1-p)^{m+n-2}}{1 - (1-p)^{m+n}}.
\end{aligned}$$

c. Find EZ: We can use LOTUS to find EZ. Let us first remember the following useful identities:

$$\sum_{k=1}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}, \qquad \text{for } |x| < 1,$$

$$-\ln(1-x) = \sum_{k=1}^{\infty} \frac{x^k}{k}, \qquad \text{for } |x| < 1.$$

The first one is obtained by taking derivative of the geometric sum formula, and the second one is a Taylor series. Now, let's apply LOTUS.

$$
\begin{aligned}
E\left[\frac{X}{Y}\right] &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{m}{n}P(X=m,Y=n) \\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{m}{n}p^2q^{m-1}q^{n-1} \\
&= \sum_{n=1}^{\infty}\frac{1}{n}p^2q^{n-1}\sum_{m=1}^{\infty}mq^{m-1} \\
&= \sum_{n=1}^{\infty}\frac{1}{n}p^2q^{n-1}\frac{1}{(1-q)^2} \\
&= \sum_{n=1}^{\infty}\frac{1}{n}q^{n-1} \\
&= \frac{1}{q}\sum_{n=1}^{\infty}\frac{q^n}{n} \\
&= \frac{1}{1-p}\ln\frac{1}{p}.
\end{aligned}
$$
