4.2.6 Solved Problems: Special Continuous Distributions

Problem 1

Suppose the number of customers arriving at a store obeys a Poisson distribution with an average of λ customers per unit time. That is, if Y is the number of customers arriving in an interval of length t, then $Y \sim Poisson(\lambda t)$. Suppose that the store opens at time $t = 0$. Let X be the arrival time of the first customer. Show that $X \sim Exponential(\lambda)$.

Solution

We first find $P(X > t)$:

$$\begin{aligned} P(X > t) &= P(\text{No arrival in } [0, t]) \\ &= e^{-\lambda t} \frac{(\lambda t)^0}{0!} \\ &= e^{-\lambda t}. \end{aligned}$$

Thus, the CDF of X for $x > 0$ is given by

$$F_X(x) = 1 - P(X > x) = 1 - e^{-\lambda x},$$

which is the CDF of $Exponential(\lambda)$. Note that by the same argument, the time between the first and second customer also has $Exponential(\lambda)$ distribution. In general, the time between the k'th and $k+1$'th customer is $Exponential(\lambda)$.

Problem 2 (Exponential as the limit of Geometric)

Let $Y \sim Geometric(p)$, where $p = \lambda\Delta$. Define $X = Y\Delta$, where $\lambda, \Delta > 0$. Prove that for any $x \in (0, \infty)$, we have

$$\lim_{\Delta \to 0} F_X(x) = 1 - e^{-\lambda x}.$$

Solution

If $Y \sim Geometric(p)$ and $q = 1 - p$, then

$$P(Y \leq n) = \sum_{k=1}^{n} pq^{k-1}$$
$$= p. \frac{1-q^n}{1-q} = 1 - (1-p)^n.$$

Then for any $y \in (0, \infty)$, we can write

$$P(Y \leq y) = 1 - (1-p)^{\lfloor y \rfloor},$$

where $\lfloor y \rfloor$ is the largest integer less than or equal to y. Now, since $X = Y\Delta$, we have

$$F_X(x) = P(X \leq x)$$
$$= P\left(Y \leq \frac{x}{\Delta}\right)$$
$$= 1 - (1-p)^{\lfloor \frac{x}{\Delta} \rfloor} = 1 - (1 - \lambda\Delta)^{\lfloor \frac{x}{\Delta} \rfloor}.$$

Now, we have

$$\lim_{\Delta \to 0} F_X(x) = \lim_{\Delta \to 0} 1 - (1 - \lambda\Delta)^{\lfloor \frac{x}{\Delta} \rfloor}$$
$$= 1 - \lim_{\Delta \to 0} (1 - \lambda\Delta)^{\lfloor \frac{x}{\Delta} \rfloor}$$
$$= 1 - e^{-\lambda x}.$$

The last equality holds because $\frac{x}{\Delta} - 1 \leq \lfloor \frac{x}{\Delta} \rfloor \leq \frac{x}{\Delta}$, and we know

$$\lim_{\Delta \to 0^+} (1 - \lambda\Delta)^{\frac{1}{\Delta}} = e^{-\lambda}.$$

Problem 3

Let $U \sim Uniform(0, 1)$ and $X = -\ln(1 - U)$. Show that $X \sim Exponential(1)$.

Solution

First note that since $R_U = (0, 1)$, $R_X = (0, \infty)$. We will find the CDF of X. For $x \in (0, \infty)$, we have

$$F_X(x) = P(X \leq x)$$
$$= P(-\ln(1 - U) \leq x)$$
$$= P\left(\frac{1}{1-U} \leq e^x\right)$$
$$= P(U \leq 1 - e^{-x}) = 1 - e^{-x},$$

which is the CDF of an $Exponential(1)$ random variable.

Problem 4

Let $X \sim N(2, 4)$ and $Y = 3 - 2X$.

a. Find $P(X > 1)$.
b. Find $P(-2 < Y < 1)$.
c. Find $P(X > 2|Y < 1)$.

Solution

a. Find $P(X > 1)$: We have $\mu_X = 2$ and $\sigma_X = 2$. Thus,

$$\begin{aligned} P(X > 1) &= 1 - \Phi\left(\frac{1-2}{2}\right) \\ &= 1 - \Phi(-0.5) = \Phi(0.5) = 0.6915 \end{aligned}$$

b. Find $P(-2 < Y < 1)$: Since $Y = 3 - 2X$, using Theorem 4.3, we have $Y \sim N(-1, 16)$. Therefore,

$$\begin{aligned} P(-2 < Y < 1) &= \Phi\left(\frac{1-(-1)}{4}\right) - \Phi\left(\frac{(-2)-(-1)}{4}\right) \\ &= \Phi(0.5) - \Phi(-0.25) = 0.29 \end{aligned}$$

c. Find $P(X > 2|Y < 1)$:

$$\begin{aligned} P(X > 2|Y < 1) &= P(X > 2|3 - 2X < 1) \\ &= P(X > 2|X > 1) \\ &= \frac{P(X>2, X>1)}{P(X>1)} \\ &= \frac{P(X>2)}{P(X>1)} \\ &= \frac{1-\Phi(\frac{2-2}{2})}{1-\Phi(\frac{1-2}{2})} \\ &= \frac{1-\Phi(0)}{1-\Phi(-0.5)} \\ &\approx 0.72 \end{aligned}$$

Problem 5

Let $X \sim N(0, \sigma^2)$. Find $E|X|$

Solution

We can write $X = \sigma Z$, where $Z \sim N(0,1)$. Thus, $E|X| = \sigma E|Z|$. We have

$$\begin{aligned} E|Z| &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |t| e^{-\frac{t^2}{2}} dt \\ &= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} |t| e^{-\frac{t^2}{2}} dt \qquad \text{(integral of an even function)} \\ &= \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} t e^{-\frac{t^2}{2}} dt \\ &= \sqrt{\frac{2}{\pi}} \left[-e^{-\frac{t^2}{2}} \right]_0^{\infty} = \sqrt{\frac{2}{\pi}} \end{aligned}$$

Thus, we conclude $E|X| = \sigma E|Z| = \sigma \sqrt{\frac{2}{\pi}}$.

Problem 6

Show that the constant in the normal distribution must be $\frac{1}{\sqrt{2\pi}}$. That is, show that

$$I = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = \sqrt{2\pi}.$$

Hint: Write I^2 as a double integral in polar coordinates.

Solution

Let $I = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx$. We show that $I^2 = 2\pi$. To see this, note

$$\begin{aligned} I^2 &= \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2}} dx dy. \end{aligned}$$

To evaluate this double integral we can switch to polar coordinates. This can be done by change of variables $x = r\cos\theta, y = r\sin\theta$, and $dxdy = rdrd\theta$. In particular, we have

$$I^2$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2}} dxdy$$
$$= \int_0^{\infty} \int_0^{2\pi} e^{-\frac{r^2}{2}} rd\theta dr$$
$$= 2\pi \int_0^{\infty} re^{-\frac{r^2}{2}} dr$$
$$= 2\pi \Big[-e^{-\frac{r^2}{2}} \Big]_0^{\infty} = 2\pi.$$

Problem 7

Let $Z \sim N(0,1)$. Prove for all $x \geq 0$,

$$\frac{1}{\sqrt{2\pi}} \frac{x}{x^2+1} e^{-\frac{x^2}{2}} \leq P(Z \geq x) \leq \frac{1}{\sqrt{2\pi}} \frac{1}{x} e^{-\frac{x^2}{2}}.$$

Solution

To show the upper bound, we can write

$$P(Z \geq x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-\frac{u^2}{2}} du$$
$$= \frac{1}{\sqrt{2\pi}} \frac{1}{x} \Big[-e^{-\frac{u^2}{2}} \Big]_x^{\infty}$$
$$= \frac{1}{\sqrt{2\pi}} \frac{1}{x} e^{-\frac{x^2}{2}}.$$

To show the lower bound, let $Q(x) = P(Z \geq x)$, and

$$h(x) = Q(x) - \frac{1}{\sqrt{2\pi}} \frac{x}{x^2+1} e^{-\frac{x^2}{2}}, \quad \text{for all } x \geq 0.$$

It suffices to show that

$$h(x) \geq 0, \quad \text{for all } x \geq 0.$$

To see this, note that the function h has the following properties

1. $h(0) = \frac{1}{2}$;
2. $\lim_{x \to \infty} h(x) = 0$;

3. $h'(x) = -\frac{2}{\sqrt{2\pi}}\left(\frac{e^{-\frac{x^2}{2}}}{(x^2+1)}\right) < 0$, for all $x \geq 0$.

Therefore, $h(x)$ is a strictly decreasing function that starts at $h(0) = \frac{1}{2}$ and decreases as x increases. It approaches 0 as x goes to infinity. We conclude that $h(x) \geq 0$, for all $x \geq 0$.

Problem 8

Let $X \sim Gamma(\alpha, \lambda)$, where $\alpha, \lambda > 0$. Find EX, and $Var(X)$.

Solution

To find EX we can write

$$\begin{aligned}
EX &= \int_0^\infty x f_X(x) dx \\
&= \int_0^\infty x \cdot \frac{\lambda^\alpha}{\Gamma\alpha} x^{\alpha-1} e^{-\lambda x} \mathrm{d}x \\
&= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty x \cdot x^{\alpha-1} e^{-\lambda x} \mathrm{d}x \\
&= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty x^\alpha e^{-\lambda x} \mathrm{d}x \\
&= \frac{\lambda^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha+1)}{\lambda^{\alpha+1}} & \text{(using Property 2 of the gamma function)} \\
&= \frac{\alpha\Gamma(\alpha)}{\lambda\Gamma(\alpha)} & \text{(using Property 3 of the gamma function)} \\
&= \frac{\alpha}{\lambda}.
\end{aligned}$$

Similarly, we can find EX^2:

$$
\begin{aligned}
EX^2 &= \int_0^\infty x^2 \mathrm{d}x \\
&= \int_0^\infty x^2 \cdot \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} \mathrm{d}x \\
&= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty x^2 \cdot x^{\alpha-1} e^{-\lambda x} \mathrm{d}x \\
&= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha+1} e^{-\lambda x} \mathrm{d}x \\
&= \frac{\lambda^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha+2)}{\lambda^{\alpha+2}} && \text{(using Property 2 of the gamma function)} \\
&= \frac{(\alpha+1)\Gamma(\alpha+1)}{\lambda^2 \Gamma(\alpha)} && \text{(using Property 3 of the gamma function)} \\
&= \frac{(\alpha+1)\alpha\Gamma(\alpha)}{\lambda^2 \Gamma(\alpha)} && \text{(using Property 3 of the gamma function)} \\
&= \frac{\alpha(\alpha+1)}{\lambda^2}.
\end{aligned}
$$

So, we conclude

$$
\begin{aligned}
Var(X) &= EX^2 - (EX)^2 \\
&= \frac{\alpha(\alpha+1)}{\lambda^2} - \frac{\alpha^2}{\lambda^2} \\
&= \frac{\alpha}{\lambda^2}.
\end{aligned}
$$