



1.4.2 Law of Total Probability

Let us start this section by asking a very simple question: In a certain country there are three provinces, call them B_1 , B_2 , and B_3 (i.e., the country is partitioned into three disjoint sets B_1 , B_2 , and B_3). We are interested in the total forest area in the country. Suppose that we know that the forest area in B_1 , B_2 , and B_3 are $100km^2$, $50km^2$, and $150km^2$, respectively. What is the total forest area in the country? If your answer is

$$100km^2 + 50km^2 + 150km^2 = 300km^2,$$

you are right. That is, you can simply add forest areas in each province (partition) to obtain the forest area in the whole country. This is the idea behind the law of total probability, in which the *area of forest* is replaced by *probability of an event A* . In particular, if you want to find $P(A)$, you can look at a partition of S , and add the amount of probability of A that falls in each partition. We have already seen the special case where the partition is B and B^c : we saw that for any two events A and B ,

$$P(A) = P(A \cap B) + P(A \cap B^c)$$

and using the definition of conditional probability, $P(A \cap B) = P(A|B)P(B)$, we can write

$$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c).$$

We can state a more general version of this formula which applies to a general partition of the sample space S .

Law of Total Probability:

If B_1, B_2, B_3, \dots is a partition of the sample space S , then for any event A we have

$$P(A) = \sum_i P(A \cap B_i) = \sum_i P(A|B_i)P(B_i).$$

Using a Venn diagram, we can pictorially see the idea behind the law of total probability. In Figure 1.24, we have

$$A_1 = A \cap B_1,$$

$$A_2 = A \cap B_2,$$

$$A_3 = A \cap B_3.$$

As it can be seen from the figure, A_1 , A_2 , and A_3 form a partition of the set A , and thus by the third axiom of probability

$$P(A) = P(A_1) + P(A_2) + P(A_3).$$

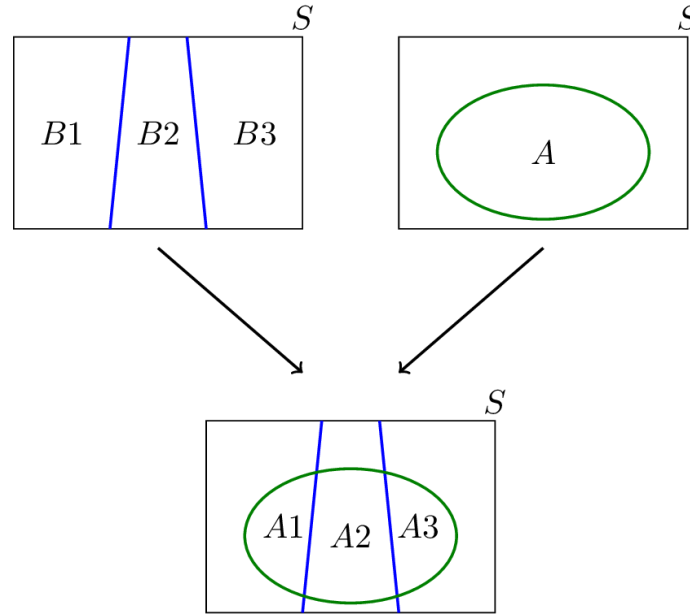


Fig.1.24 - Law of total probability.

Here is a proof of the law of total probability using probability axioms:

Proof

Since B_1, B_2, B_3, \dots is a partition of the sample space S , we can write

$$\begin{aligned} S &= \bigcup_i B_i \\ A &= A \cap S \\ &= A \cap \left(\bigcup_i B_i \right) \\ &= \bigcup_i (A \cap B_i) \quad \text{by the distributive law (Theorem 1.2).} \end{aligned}$$

Now note that the sets $A \cap B_i$ are disjoint (since the B_i 's are disjoint). Thus, by the third probability axiom,

$$P(A) = P\left(\bigcup_i (A \cap B_i)\right) = \sum_i P(A \cap B_i) = \sum_i P(A|B_i)P(B_i).$$

Here is a typical scenario in which we use the law of total probability. We are interested in finding the probability of an event A , but we don't know how to find $P(A)$ directly. Instead, we know the conditional probability of A given some events B_i , where the B_i 's form a partition of the sample space. Thus, we will be able to find $P(A)$ using the law of total probability, $P(A) = \sum_i P(A|B_i)P(B_i)$.

Example 1.24

I have three bags that each contain 100 marbles:

- Bag 1 has 75 red and 25 blue marbles;
- Bag 2 has 60 red and 40 blue marbles;
- Bag 3 has 45 red and 55 blue marbles.

I choose one of the bags at random and then pick a marble from the chosen bag, also at random. What is the probability that the chosen marble is red?

Solution

Let R be the event that the chosen marble is red. Let B_i be the event that I choose Bag i . We already know that

$$P(R|B_1) = 0.75,$$

$$P(R|B_2) = 0.60,$$

$$P(R|B_3) = 0.45$$

We choose our partition as B_1, B_2, B_3 . Note that this is a valid partition because, firstly, the B_i 's are disjoint (only one of them can happen), and secondly, because their union is the entire sample space as one the bags will be chosen for sure, i.e., $P(B_1 \cup B_2 \cup B_3) = 1$. Using the law of total probability, we can write

$$\begin{aligned} P(R) &= P(R|B_1)P(B_1) + P(R|B_2)P(B_2) + P(R|B_3)P(B_3) \\ &= (0.75)\frac{1}{3} + (0.60)\frac{1}{3} + (0.45)\frac{1}{3} \\ &= 0.60 \end{aligned}$$
