$$
\begin{aligned}
R_{XY}(t_1,t_2) = E[X(t_1)Y(t_2)] &= E\left[X(t_1)\int_{-\infty}^{\infty} h(\alpha)X(t_2-\alpha)\ d\alpha\right] \\
&= E\left[\int_{-\infty}^{\infty} h(\alpha)X(t_1)X(t_2-\alpha)\ d\alpha\right] \\
&= \int_{-\infty}^{\infty} h(\alpha)E[X(t_1)X(t_2-\alpha)]\ d\alpha \\
&= \int_{-\infty}^{\infty} h(\alpha)R_X(t_1,t_2-\alpha)\ d\alpha \\
&= \int_{-\infty}^{\infty} h(\alpha)R_X(t_1-t_2+\alpha)\ d\alpha \qquad\qquad \text{(since } X(t) \text{ is WSS).}
\end{aligned}
$$

We note that $R_{XY}(t_1,t_2)$ is only a function of $\tau=t_1-t_2$, so we may write

$$
\begin{aligned}
R_{XY}(\tau) &= \int_{-\infty}^{\infty} h(\alpha)R_X(\tau+\alpha)\ d\alpha \\
&= h(\tau) * R_X(-\tau) = h(-\tau) * R_X(\tau).
\end{aligned}
$$

Similarly, you can show that

$$R_Y(\tau) = h(\tau) * h(-\tau) * R_X(\tau).$$

This has been shown in the Solved Problems section. From the above results we conclude that $X(t)$ and $Y(t)$ are jointly WSS. The following theorem summarizes the results.

Theorem 10.2
Let $X(t)$ be a WSS random process and $Y(t)$ be given by

$$Y(t) = h(t) * X(t),$$

where $h(t)$ is the impulse response of the system. Then $X(t)$ and $Y(t)$ are jointly WSS. Moreover,

1. $\mu_Y(t) = \mu_Y = \mu_X \int_{-\infty}^{\infty} h(\alpha)\ d\alpha$;
2. $R_{XY}(\tau) = h(-\tau) * R_X(\tau) = \int_{-\infty}^{\infty} h(-\alpha)R_X(t-\alpha)\ d\alpha$;
3. $R_Y(\tau) = h(\tau) * h(-\tau) * R_X(\tau)$.

Frequency Domain Analysis:

Let's now rewrite the statement of Theorem 10.2 in the frequency domain. Let $H(f)$ be the Fourier transform of $h(t)$,

$$H(f) = \mathcal{F}\{h(t)\} = \int_{-\infty}^{\infty} h(t)e^{-2j\pi ft}\, dt.$$

$H(f)$ is called the **transfer function** of the system. We can rewrite

$$\mu_Y = \mu_X \int_{-\infty}^{\infty} h(\alpha)\, d\alpha$$

as

$$\boxed{\mu_Y = \mu_X H(0)}$$

Since $h(t)$ is assumed to be a real signal, we have

$$\mathcal{F}\{h(-t)\} = H(-f) = H^*(f),$$

where * shows the complex conjugate. By taking the Fourier transform from both sides of $R_{XY}(\tau) = R_X(\tau) * h(-\tau)$, we conclude

$$\boxed{S_{XY}(f) = S_X(f)H(-f) = S_X(f)H^*(f).}$$

Finally, by taking the Fourier transform from both sides of $R_Y(\tau) = h(\tau) * h(-\tau) * R_X(\tau)$, we conclude

$$\begin{aligned} S_Y(f) &= S_X(f)H^*(f)H(f) \\ &= S_X(f)|H(f)|^2. \end{aligned}$$

$$\boxed{S_Y(f) = S_X(f)|H(f)|^2}$$

Example 10.14

Let $X(t)$ be a zero-mean WSS process with $R_X(\tau) = e^{-|\tau|}$. $X(t)$ is input to an LTI system with

$$|H(f)| = \begin{cases} \sqrt{1+4\pi^2 f^2} & |f| < 2 \\ & \\ 0 & \text{otherwise} \end{cases}$$

Let $Y(t)$ be the output.

a. Find $\mu_Y(t) = E[Y(t)]$.
b. Find $R_Y(\tau)$.
c. Find $E[Y(t)^2]$.

Solution

Note that since $X(t)$ is WSS, $X(t)$ and $Y(t)$ are jointly WSS, and therefore $Y(t)$ is WSS.

a. To find $\mu_Y(t)$, we can write

$$\begin{aligned} \mu_Y &= \mu_X H(0) \\ &= 0 \cdot 1 = 0. \end{aligned}$$

b. To find $R_Y(\tau)$, we first find $S_Y(f)$.

$$S_Y(f) = S_X(f)|H(f)|^2.$$

From Fourier transform tables, we can see that

$$\begin{aligned} S_X(f) &= \mathcal{F}\{e^{-|\tau|}\} \\ &= \frac{2}{1+(2\pi f)^2}. \end{aligned}$$

Then, we can find $S_Y(f)$ as

$$\begin{aligned} S_Y(f) &= S_X(f)|H(f)|^2 \\ &= \begin{cases} 2 & |f| < 2 \\ & \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

We can now find $R_Y(\tau)$ by taking the inverse Fourier transform of $S_Y(f)$.

$$R_Y(\tau) = 8\text{sinc}(4\tau),$$

where

$$\mathrm{sinc}(f) = \frac{\sin(\pi f)}{\pi f}.$$

c. We have

$$E[Y(t)^2] = R_Y(0) = 8.$$