11.4.1 Brownian Motion as the Limit of a Symmetric Random Walk

Here, we introduce a construction of Brownian motion from a symmetric random walk. Divide the half-line $[0, \infty)$ to tiny subintervals of length δ as shown in Figure 11.30.

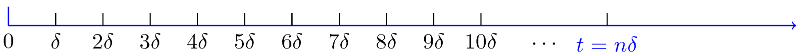

Figure 11.30 - Dividing the half-line $[0, \infty)$ to tiny subintervals of length δ.

Each subinterval corresponds to a time slot of length δ. Thus, the intervals are $(0, \delta]$, $(\delta, 2\delta]$, $(2\delta, 3\delta]$, $\cdots$. More generally, the kth interval is $\big((k-1)\delta, k\delta\big]$. We assume that in each time slot, we toss a fair coin. We define the random variables X_i as follows. $X_i = \sqrt{\delta}$ if the kth coin toss results in heads, and $X_i = -\sqrt{\delta}$ if the kth coin toss results in tails. Thus,

$$X_i = \begin{cases} \sqrt{\delta} & \text{with probability } \frac{1}{2} \\ & \\ -\sqrt{\delta} & \text{with probability } \frac{1}{2} \end{cases}$$

Moreover, the X_i's are independent. Note that

$$\begin{aligned} E[X_i] &= 0, \\ \mathrm{Var}(X_i) &= \delta. \end{aligned}$$

Now, we would like to define the process $W(t)$ as follows. We let $W(0) = 0$. At time $t = n\delta$, the value of $W(t)$ is given by

$$W(t) = W(n\delta) = \sum_{i=1}^{n} X_i.$$

Since $W(t)$ is the sum of n i.i.d. random variables, we know how to find $E[W(t)]$ and $\mathrm{Var}(W(t))$. In particular,

$$
\begin{aligned}
E[W(t)] &= \sum_{i=1}^{n} E[X_i] \\
&= 0, \\
\mathrm{Var}(W(t)) &= \sum_{i=1}^{n} \mathrm{Var}(X_i) \\
&= n\mathrm{Var}(X_1) \\
&= n\delta \\
&= t.
\end{aligned}
$$

For any $t \in (0, \infty)$, as n goes to ∞, δ goes to 0. By the central limit theorem, $W(t)$ will become a normal random variable,

$$W(t) \sim N(0, t).$$

Since the coin tosses are independent, we conclude that $W(t)$ has *independent increments*. That is, for all $0 \le t_1 < t_2 < t_3 \cdots < t_n$, the random variables

$$W(t_2) - W(t_1),\ W(t_3) - W(t_2),\ \cdots,\ W(t_n) - W(t_{n-1})$$

are independent. Remember that we say that a random process $X(t)$ has *stationary increments* if, for all $t_2 > t_1 \ge 0$, and all $r > 0$, the two random variables $X(t_2) - X(t_1)$ and $X(t_2 + r) - X(t_1 + r)$ have the same distributions. In other words, the distribution of the difference depends only on the length of the interval $(t_1, t_2]$, and not on the exact location of the interval on the real line. We now claim that the random process $W(t)$, defined above, has stationary increments. To see this, we argue as follows. For $0 \le t_1 < t_2$, if we have $t_1 = n_1\delta$ and $t_2 = n_2\delta$, we obtain

$$
\begin{aligned}
W(t_1) &= W(n_1\delta) = \sum_{i=1}^{n_1} X_i, \\
W(t_2) &= W(n_2\delta) = \sum_{i=1}^{n_2} X_i.
\end{aligned}
$$

Then, we can write

$$W(t_2) - W(t_1) = \sum_{i=n_1+1}^{n_2} X_i.$$

Therefore, we conclude

$$
\begin{aligned}
E[W(t_2) - W(t_1)] &= \sum_{i=n_1+1}^{n_2} E[X_i] \\
&= 0, \\
\mathrm{Var}(W(t_2) - W(t_1)) &= \sum_{i=n_1+1}^{n_2} \mathrm{Var}(X_i) \\
&= (n_2 - n_1)\mathrm{Var}(X_1) \\
&= (n_2 - n_1)\delta \\
&= t_2 - t_1.
\end{aligned}
$$

Therefore, for any $0 \leq t_1 < t_2$, the distribution of $W(t_2) - W(t_1)$ only depends on the lengths of the interval $[t_1, t_2]$, i.e., how many coin tosses are in that interval. In particular, for any $0 \leq t_1 < t_2$, the distribution of $W(t_2) - W(t_1)$ converges to $N(0, t_2 - t_1)$. Therefore, we conclude that $W(t)$ has *stationary increments*.

The above construction can be made more rigorous. The random process $W(t)$ is called the standard *Brownian motion* or the standard *Wiener process*. Brownian motion has continuous sample paths, i.e., $W(t)$ is a continuous function of t (See Figure 11.29). However, it can be shown that it is nowhere differentiable.