Example 5.12

Let X and Y be two random variables and g and h be two functions. Show that

$$E[g(X)h(Y)|X] = g(X)E[h(Y)|X].$$

Solution

Note that $E[g(X)h(Y)|X]$ is a random variable that is a function of X. In particular, if $X = x$, then $E[g(X)h(Y)|X] = E[g(X)h(Y)|X = x]$. Now, we can write

$$\begin{aligned} E[g(X)h(Y)|X = x] &= E[g(x)h(Y)|X = x] \\ &= g(x)E[h(Y)|X = x] \qquad \text{(since } g(x) \text{ is a constant).} \end{aligned}$$

Thinking of this as a function of the random variable X, it can be rewritten as $E[g(X)h(Y)|X] = g(X)E[h(Y)|X]$. This rule is sometimes called "taking out what is known." The idea is that, given X, $g(X)$ is a known quantity, so it can be taken out of the conditional expectation.

$$\boxed{E[g(X)h(Y)|X] = g(X)E[h(Y)|X] \qquad (5.6)}$$

Iterated Expectations:

Let us look again at the law of total probability for expectation. Assuming $g(Y) = E[X|Y]$, we have

$$\begin{aligned} E[X] &= \sum_{y_j \in R_Y} E[X|Y = y_j]P_Y(y_j) \\ &= \sum_{y_j \in R_Y} g(y_j)P_Y(y_j) \\ &= E[g(Y)] \qquad \text{by LOTUS (Equation 5.2)} \\ &= E[E[X|Y]]. \end{aligned}$$

Thus, we conclude

$$E[X] = E[E[X|Y]]. \qquad (5.7)$$