Similarly, we can write

$$
\begin{aligned}
t_2 &= 1 + \frac{1}{2}t_1 + \frac{1}{2}t_3 \\
&= 1 + \frac{1}{2}t_1.
\end{aligned}
$$

Solving the above equations, we obtain

$$
t_1 = \frac{5}{2}, \quad t_2 = \frac{9}{4}.
$$

Generally, let $A \subset S$ be a set of states. The above procedure can be used to find the expected time until the chain first hits one of the states in the set $A$.

Mean Hitting Times

Consider a finite Markov chain $\{X_n, n = 0, 1, 2, \cdots\}$ with state space $S = \{0, 1, 2, \cdots, r\}$. Let $A \subset S$ be a set of states. Let $T$ be the first time the chain visits a state in $A$. For all $i \in S$, define

$$
t_i = E[T|X_0 = i].
$$

By the above definition, we have $t_j = 0$, for all $j \in A$. To find the unknown values of $t_i$'s, we can use the following equations

$$
t_i = 1 + \sum_k t_k p_{ik}, \quad \text{for } i \in S - A.
$$

## Mean Return Times:

Another interesting random variable is the *first return time*. In particular, assuming the chain is in state $l$, we consider the expected time (number of steps) needed until the chain returns to state $l$. For example, consider a Markov chain for which $X_0 = 2$. If the chain gets the values

$$
X_0 = 2,\ X_1 = 1,\ X_2 = 4,\ X_3 = 3,\ X_4 = 2,\ X_5 = 3,\ X_6 = 2,\ X_7 = 3,\ \cdots,
$$

then the first return to state $2$ occurs at time $n = 4$. Thus, the first return time to state $2$ is equal to $4$ for this example. Here, we are interested in the expected value of the first return time. In particular, assuming $X_0 = l$, let's define $r_l$ as the expected number of steps needed until the chain returns to state $l$. To make the definition more precise, let's define

$$R_l = \min\{n \geq 1 : X_n = l\}.$$

Then,

$$r_l = E[R_l | X_0 = l].$$

Note that by definition, $R_l \geq 1$, so we conclude $r_l \geq 1$. In fact, $r_l = 1$ if and only if $l$ is an absorbing state (i.e., $p_{ll} = 1$). As before, we can apply the law of total probability to obtain $r_l$. Again, let's define $t_k$ as the expected time until the chain hits state $l$ for the first time, given that $X_0 = k$. We have already seen how to find $t_k$'s (mean hitting times). Using the law of total probability, we can write

$$r_l = 1 + \sum_k p_{lk} t_k.$$

Let's look at an example to see how we can find the mean return time.

---

**Example 11.11**

Consider the Markov chain shown in Figure 11.13. Let $t_k$ be the expected number of steps until the chain hits state $1$ for the first time, given that $X_0 = k$. Clearly, $t_1 = 0$. Also, let $r_1$ be the mean return time to state $1$.

1. Find $t_2$ and $t_3$.
2. Find $r_1$.

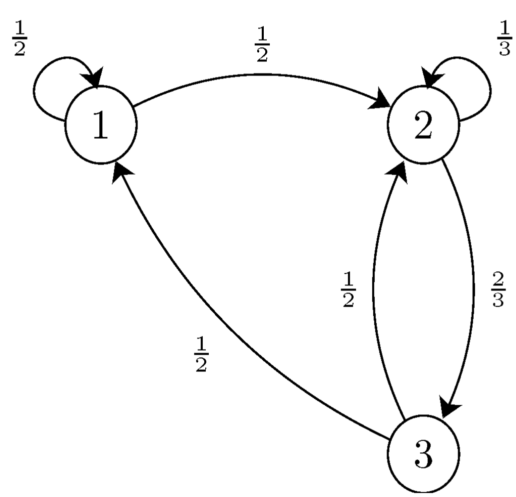


Figure 11.13 - A state transition diagram.

**Solution**

1. To find $t_2$ and $t_3$, we use the law of total probability with recursion as before. For example, if $X_0 = 2$, then after one step, we have $X_1 = 2$ or $X_1 = 3$. Thus, we can write

$$t_2 = 1 + \frac{1}{3}t_2 + \frac{2}{3}t_3.$$

   Similarly, we can write

$$\begin{aligned} t_3 &= 1 + \frac{1}{2}t_1 + \frac{1}{2}t_2 \\ &= 1 + \frac{1}{2}t_2. \end{aligned}$$

   Solving the above equations, we obtain

$$t_2 = 5, \quad t_3 = \frac{7}{2}.$$

2. To find $r_1$, we note that if $X_0 = 1$, then $X_1 = 1$ or $X_1 = 2$. We can write

$$\begin{aligned} r_1 &= 1 + \frac{1}{2} \cdot t_1 + \frac{1}{2}t_2 \\ &= 1 + \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 5 \\ &= \frac{7}{2}. \end{aligned}$$

---

Here, we summarize the formulas for finding the mean return times. As we mentioned before, there is no need to memorize these formulas once you understand how they are derived.

## Mean Return Times

Consider a finite irreducible Markov chain $\{X_n, n = 0, 1, 2, \cdots\}$ with state space $S = \{0, 1, 2, \cdots, r\}$. Let $l \in S$ be a state. Let $r_l$ be the **mean return time** to state $l$. Then

$$r_l = 1 + \sum_k t_k p_{lk},$$

where $t_k$ is the expected time until the chain hits state $l$ given $X_0 = k$. Specifically,

$$\begin{aligned} t_l &= 0, \\ t_k &= 1 + \sum_j t_j p_{kj}, \quad \text{for } k \neq l. \end{aligned}$$