7.2.7 Almost Sure Convergence

Consider a sequence of random variables X_1, X_2, X_3, $\cdots$ that is defined on an underlying sample space S. For simplicity, let us assume that S is a finite set, so we can write

$$S = \{s_1, s_2, \cdots, s_k\}.$$

Remember that each X_n is a function from S to the set of real numbers. Thus, we may write

$$X_n(s_i) = x_{ni}, \qquad \text{for } i = 1, 2, \cdots, k.$$

After this random experiment is performed, one of the s_i's will be the outcome of the experiment, and the values of the X_n's are known. If s_j is the outcome of the experiment, we observe the following sequence:

$$x_{1j}, x_{2j}, x_{3j}, \cdots.$$

Since this is a sequence of real numbers, we can talk about its convergence. Does it converge? If yes, what does it converge to? **Almost sure** convergence is defined based on the convergence of such sequences. Before introducing almost sure convergence let us look at an example.

Example 7.12
Consider the following random experiment: A fair coin is tossed once. Here, the sample space has only two elements $S = \{H, T\}$. We define a sequence of random variables X_1, X_2, X_3, $\cdots$ on this sample space as follows:

$$X_n(s) = \begin{cases} \frac{n}{n+1} & \text{if } s = H \\ \\ (-1)^n & \text{if } s = T \end{cases}$$

a. For each of the possible outcomes (H or T), determine whether the resulting sequence of real numbers converges or not.
b. Find

$$P\left(\left\{s_i \in S : \lim_{n \to \infty} X_n(s_i) = 1\right\}\right).$$

Solution

a. If the outcome is H, then we have $X_n(H) = \frac{n}{n+1}$, so we obtain the following sequence

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \cdots.$$

This sequence converges to 1 as n goes to infinity. If the outcome is T, then we have $X_n(T) = (-1)^n$, so we obtain the following sequence

$$-1, 1, -1, 1, -1, \cdots.$$

This sequence does not converge as it oscillates between -1 and 1 forever.

b. By part (a), the event $\{s_i \in S : \lim_{n\rightarrow\infty} X_n(s_i) = 1\}$ happens if and only if the outcome is H, so

$$\begin{aligned} P\left(\left\{s_i \in S : \lim_{n\rightarrow\infty} X_n(s_i) = 1\right\}\right) &= P(H) \\ &= \frac{1}{2}. \end{aligned}$$

In the above example, we saw that the sequence $X_n(s)$ converged when $s = H$ and did not converge when $s = T$. In general, if the probability that the sequence $X_n(s)$ converges to $X(s)$ is equal to 1, we say that X_n converges to X **almost surely** and write

$$X_n \xrightarrow{a.s.} X.$$

Almost Sure Convergence

A sequence of random variables X_1, X_2, X_3, $\cdots$ converges **almost surely** to a random variable X, shown by $X_n \xrightarrow{a.s.} X$, if

$$P\left(\left\{s \in S : \lim_{n\rightarrow\infty} X_n(s) = X(s)\right\}\right) = 1.$$

Example 7.13

Consider the sample space $S = [0, 1]$ with a probability measure that is uniform on this space, i.e.,

$$P([a, b]) = b - a, \qquad \text{for all } 0 \leq a \leq b \leq 1.$$

Define the sequence $\{X_n, n = 1, 2, \cdots\}$ as follows:

$$X_n(s) = \begin{cases} 1 & 0 \leq s < \frac{n+1}{2n} \\ 0 & \text{otherwise} \end{cases}$$

Also, define the random variable X on this sample space as follows:

$$X(s) = \begin{cases} 1 & 0 \leq s < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

Show that $X_n \xrightarrow{a.s.} X$.

Solution

Define the set A as follows:

$$A = \left\{ s \in S : \lim_{n \to \infty} X_n(s) = X(s) \right\}.$$

We need to prove that $P(A) = 1$. Let's first find A. Note that $\frac{n+1}{2n} > \frac{1}{2}$, so for any $s \in [0, \frac{1}{2})$, we have

$$X_n(s) = X(s) = 1.$$

Therefore, we conclude that $[0, 0.5) \subset A$. Now if $s > \frac{1}{2}$, then

$$X(s) = 0.$$

Also, since $2s - 1 > 0$, we can write

$$X_n(s) = 0, \qquad \text{for all } n > \frac{1}{2s - 1}.$$

Therefore,

$$\lim_{n \to \infty} X_n(s) = 0 = X(s), \qquad \text{for all } s > \frac{1}{2}.$$

We conclude $(\frac{1}{2}, 1] \subset S$. You can check that $s = \frac{1}{2} \notin A$, since

$$X_n\left(\frac{1}{2}\right) = 1, \qquad \text{for all } n,$$

while $X\left(\frac{1}{2}\right) = 0$. We conclude

$$A = \left[0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right] = S - \left\{\frac{1}{2}\right\}.$$

Since $P(A) = 1$, we conclude $X_n \xrightarrow{a.s.} X$.

In some problems, proving almost sure convergence directly can be difficult. Thus, it is desirable to know some sufficient conditions for almost sure convergence. Here is a result that is sometimes useful when we would like to prove almost sure convergence.

Theorem 7.5
Consider the sequence $X_1, X_2, X_3, \cdots$. If for all $\epsilon > 0$, we have

$$\sum_{n=1}^{\infty} P\big(|X_n - X| > \epsilon\big) < \infty,$$

then $X_n \xrightarrow{a.s.} X$.

Example 7.14
Consider a sequence $\{X_n, n = 1, 2, 3, \cdots\}$ such that

$$X_n = \begin{cases} -\frac{1}{n} & \text{with probability } \frac{1}{2} \\ \\ \frac{1}{n} & \text{with probability } \frac{1}{2} \end{cases}$$

Show that $X_n \xrightarrow{a.s.} 0$.

Solution

By the Theorem above, it suffices to show that

$$\sum_{n=1}^{\infty} P\big(|X_n| > \epsilon\big) < \infty.$$