4.3.3 Solved Problems: Mixed Random Variables

Problem 1

Here is one way to think about a mixed random variable. Suppose that we have a discrete random variable X_d with (generalized) PDF and CDF $f_d(x)$ and $F_d(x)$, and a continuous random variable X_c with PDF and CDF $f_c(x)$ and $F_c(x)$. Now we create a new random variable X in the following way. We have a coin with $P(H) = p$. We toss the coin once. If it lands heads, then the value of X is determined according to the probability distribution of X_d. If the coin lands tails, the value of X is determined according to the probability distribution of X_c.

a. Find the CDF of $X, F_X(x)$.
b. Find the PDF of $X, f_X(x)$.
c. Find EX.
d. Find Var(X).

Solution

a. Find the CDF of $X, F_X(x)$: We can write

$$\begin{aligned} F_X(x) &= P(X \leq x) \\ &= P(X \leq x|H)P(H) + P(X \leq x|T)P(T) (\text{law of total probability}) \\ &= pP(X_d \leq x) + (1-p)P(X_c \leq x) \\ &= pF_d(x) + (1-p)F_c(x). \end{aligned}$$

b. Find the PDF of $X, f_X(x)$: By differentiating $F_X(x)$, we obtain

$$\begin{aligned} f_X(x) &= \frac{dF_X(x)}{dx} \\ &= pf_d(x) + (1-p)f_c(x). \end{aligned}$$

c. Find EX: We have

$$\begin{aligned} EX &= \int_{-\infty}^{\infty} xf_X(x)dx \\ &= p\int_{-\infty}^{\infty} xf_d(x)dx + (1-p)\int_{-\infty}^{\infty} xf_c(x)dx \\ &= pEX_d + (1-p)EX_c. \end{aligned}$$

d. Find Var(X):

$$\begin{aligned} EX^2 &= \int_{-\infty}^{\infty} x^2 f_X(x)dx \\ &= p\int_{-\infty}^{\infty} x^2 f_d(x)dx + (1-p)\int_{-\infty}^{\infty} x^2 f_c(x)dx \\ &= pEX_d^2 + (1-p)EX_c^2. \end{aligned}$$

Thus,

$$\begin{aligned} \mathrm{Var}(X) &= EX^2 - (EX)^2 \\ &= pEX_d^2 + (1-p)EX_c^2 - (pEX_d + (1-p)EX_c)^2 \\ &= pEX_d^2 + (1-p)EX_c^2 - p^2(EX_d)^2 - (1-p)^2(EX_c)^2 - 2p(1-p)EX_dEX_c \\ &= p(EX_d^2 - (EX_d)^2) + (1-p)(EX_c^2 - (EX_c)^2) + p(1-p)(EX_d - EX_c)^2 \\ &= p\mathrm{Var}(X_d) + (1-p)\mathrm{Var}(X_c) + p(1-p)\big(EX_d - EX_c\big)^2. \end{aligned}$$

Problem 2

Let X be a random variable with CDF

$$F_X(x) = \begin{cases} 1 & x \geq 1 \\ \frac{1}{2} + \frac{x}{2} & 0 \leq x < 1 \\ 0 & x < 0 \end{cases}$$

a. What kind of random variable is X: discrete, continuous, or mixed?
b. Find the PDF of X, $f_X(x)$.
c. Find $E(e^X)$.
d. Find $P(X=0|X \leq 0.5)$.

Solution

a. What kind of random variable is X: discrete, continuous, or mixed? We note that the CDF has a discontinuity at $x=0$, and it is continuous at other points. Since $F_X(x)$ is not flat in other locations, we conclude X is a mixed random variable. Indeed, we can write

$$F_X(x) = \frac{1}{2}u(x) + \frac{1}{2}F_Y(x),$$

where Y is a $Uniform(0,1)$ random variable. If we use the interpretation of Problem 1, we can say the following. We toss a fair coin. If it lands heads then $X=0$, otherwise X is obtained according the a $Uniform(0,1)$ distribution.

b. Find the PDF of X, $f_X(x)$: By differentiating the CDF, we obtain

$$f_X(x) = \frac{1}{2}\delta(x) + \frac{1}{2}f_Y(x),$$

where $f_Y(x)$ is the PDF of $Uniform(0,1)$, i.e.,

$$f_Y(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

c. Find $E(e^X)$: We can use LOTUS to write

$$\begin{aligned} E(e^X) &= \int_{-\infty}^{\infty} e^x f_X(x)dx \\ &= \tfrac{1}{2}\int_{-\infty}^{\infty} e^x \delta(x)dx + \tfrac{1}{2}\int_{-\infty}^{\infty} e^x f_Y(x)dx \\ &= \tfrac{1}{2}e^0 + \tfrac{1}{2}\int_0^1 e^x dx \\ &= \tfrac{1}{2} + \tfrac{1}{2}(e-1) \\ &= \tfrac{1}{2}e. \end{aligned}$$

Here is another way to think about this part: similar to part (c) of Problem 1, we can write

$$\begin{aligned} E(e^X) &= \tfrac{1}{2} \times e^0 + \tfrac{1}{2}E[e^Y] \\ &= \tfrac{1}{2} + \tfrac{1}{2}\int_0^1 e^y dy \\ &= \tfrac{1}{2}e. \end{aligned}$$

d. Find $P(X = 0|X \leq 0.5)$: We have

$$\begin{aligned} P(X=0|X \leq 0.5) &= \frac{P(X=0, X \leq 0.5)}{P(X \leq 0.5)} \\ &= \frac{P(X=0)}{P(X \leq 0.5)} \\ &= \frac{0.5}{\int_0^{0.5} f_X(x)dx} \\ &= \frac{0.5}{0.75} = \frac{2}{3}. \end{aligned}$$

Problem 3

Let X be a $Uniform(-2,2)$ continuous random variable. We define $Y = g(X)$, where the function $g(x)$ is defined as

$$g(x) = \begin{cases} 1 & x > 1 \\ x & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the CDF and PDF of Y.

Solution

Note that $R_Y = [0, 1]$. Therefore,

$$F_Y(y) = 0, \quad \text{for} \quad y < 0,$$

$$F_Y(y) = 1, \quad \text{for} \quad y \geq 1.$$

We also note that

$$P(Y = 0) = P(X < 0) = \frac{1}{2},$$

$$P(Y = 1) = P(X > 1) = \frac{1}{4}.$$

Also for $0 < y < 1$,

$$F_Y(y) = P(Y \leq y) = P(X \leq y) = F_X(y) = \frac{y + 2}{4}.$$

Thus, the CDF of Y is given by

$$F_Y(y) = \begin{cases} 1 & y \geq 1 \\ \frac{y+2}{4} & 0 \leq y < 1 \\ 0 & \text{otherwise} \end{cases}$$

In particular, we note that there are two jumps in the CDF, one at $y = 0$ and another at $y = 1$. We can find the generalized PDF of Y by differentiating $F_Y(y)$:

$$f_Y(y) = \frac{1}{2}\delta(y) + \frac{1}{4}\delta(y - 1) + \frac{1}{4}(u(y) - u(y - 1)).$$