

---

## 10.2.4 White Noise

A very commonly-used random process is *white noise*. White noise is often used to model the thermal noise in electronic systems. By definition, the random process  $X(t)$  is called white noise if  $S_X(f)$  is constant for all frequencies. By convention, the constant is usually denoted by  $\frac{N_0}{2}$ .

The random process  $X(t)$  is called a **white noise** process if

$$S_X(f) = \frac{N_0}{2}, \quad \text{for all } f.$$

Before going any further, let's calculate the expected power in  $X(t)$ . We have

$$\begin{aligned} E[X(t)^2] &= \int_{-\infty}^{\infty} S_X(f) df \\ &= \int_{-\infty}^{\infty} \frac{N_0}{2} df = \infty. \end{aligned}$$

Thus, white noise, as defined above, has infinite power! In reality, white noise is in fact an approximation to the noise that is observed in real systems. To better understand the idea, consider the PSDs shown in Figure 10.8.

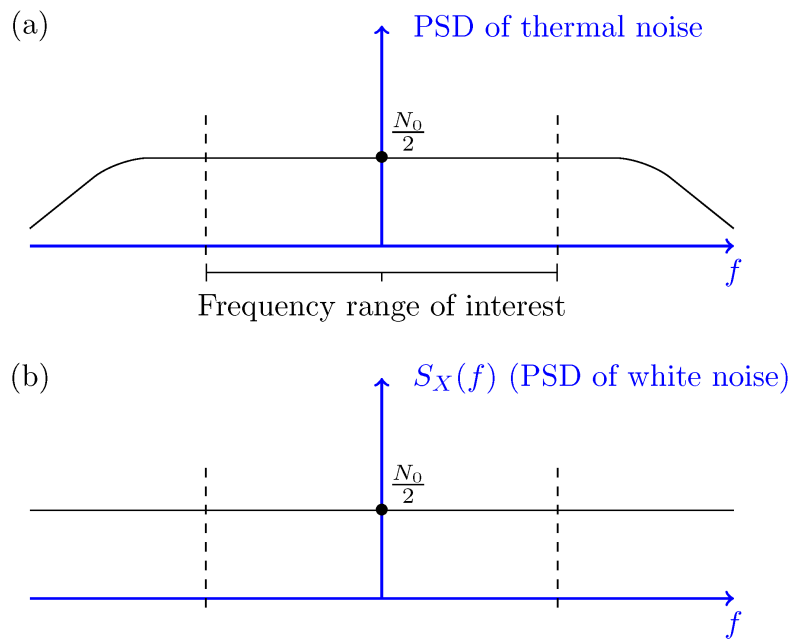


Figure 10.8 - Part (a): PSD of thermal noise; Part (b) PSD of white noise.

Part (a) in the figure shows what the real PSD of a thermal noise might look like. As we see, the PSD is not constant for all frequencies; however, it is approximately constant over the frequency range that we are interested in. In other words, real systems are bandlimited and work on a limited range of frequencies. For the frequency range that we are interested in, the two PSDs (the PSD in Part (a) and the PSD of the white noise, shown in Part (b)) are approximately the same.

The thermal noise in electronic systems is usually modeled as a white Gaussian noise process. It is usually assumed that it has zero mean  $\mu_X = 0$  and is Gaussian.

The random process  $X(t)$  is called a **white Gaussian noise** process if  $X(t)$  is a stationary Gaussian random process with zero mean,  $\mu_X = 0$ , and flat power spectral density,

$$S_X(f) = \frac{N_0}{2}, \quad \text{for all } f.$$

Since the PSD of a white noise process is given by  $S_X(f) = \frac{N_0}{2}$ , its autocorrelation function is given by

$$\begin{aligned}
 R_Y(\tau) &= \mathcal{F}^{-1} \left\{ \frac{N_0}{2} \right\} \\
 &= \frac{N_0}{2} \delta(\tau),
 \end{aligned}$$

where  $\delta(\tau)$  is the dirac delta function

$$\delta(x) = \begin{cases} \infty & x = 0 \\ 0 & \text{otherwise} \end{cases}$$

This again confirms that white noise has infinite power,  $E[X(t)^2] = R_X(0)$ . We also note that  $R_X(\tau) = 0$  for any  $\tau \neq 0$ . This means that  $X(t_1)$  and  $X(t_2)$  are uncorrelated for any  $t_1 \neq t_2$ . Therefore, for a white Gaussian noise,  $X(t_1)$  and  $X(t_2)$  are independent for any  $t_1 \neq t_2$ . White Gaussian noise can be described as the "derivative" of *Brownian motion*. Brownian motion is an important random process that will be discussed in the next chapter.

---

### Example 10.16

Let  $X(t)$  be a white Gaussian noise process that is input to an LTI system with transfer function

$$|H(f)| = \begin{cases} 2 & 1 < |f| < 2 \\ 0 & \text{otherwise} \end{cases}$$

If  $Y(t)$  is the output, find  $P(Y(1) < \sqrt{N_0})$ .

**Solution**

Since  $X(t)$  is a zero-mean Gaussian process,  $Y(t)$  is also a zero-mean Gaussian process.  $S_Y(f)$  is given by

$$\frac{N_0}{2} |H(f)|^2 = \begin{cases} 2N_0 & 1 < |f| < 2 \\ 0 & \text{otherwise} \end{cases}$$

Therefore,

$$\begin{aligned}
 E[Y(t)^2] &= \int_{-\infty}^{\infty} S_Y(f) df \\
 &= 4N_0.
 \end{aligned}$$

Thus,

$$Y(t) \sim N(0, 4N_0).$$

To find  $P(Y(1) < \sqrt{N_0})$ , we can write

$$\begin{aligned} P(Y(1) < \sqrt{N_0}) &= \Phi\left(\frac{\sqrt{N_0}}{\sqrt{4N_0}}\right) \\ &= \Phi\left(\frac{1}{2}\right) \approx 0.69 \end{aligned}$$


---