11.4.2 Definition and Some Properties

Here, we provide a more formal definition for Brownian Motion.

Standard Brownian Motion

A Gaussian random process $\{W(t), t \in [0, \infty)\}$ is called a (standard) **Brownian motion** or a (standard) **Wiener process** if

1. W(0)=0;
2. for all $0 \leq t_1 < t_2$, $W(t_2) - W(t_1) \sim N(0, t_2 - t_1)$;
3. W(t) has independent increments. That is, for all $0 \leq t_1 < t_2 < t_3 \cdots < t_n$, the random variables

$$W(t_2) - W(t_1),\ W(t_3) - W(t_2),\ \cdots,\ W(t_n) - W(t_{n-1})$$

   are independent;
4. W(t) has continuous sample paths.

A more general process is obtained if we define $X(t) = \mu + \sigma W(t)$. In this case, $X(t)$ is a Brownian motion with

$$E[X(t)] = \mu, \quad \text{Var}(X(t)) = \sigma^2 t.$$

Nevertheless, since $X(t)$ is obtained by simply shifting and scaling $W(t)$, it suffices to study properties of the standard Brownian motion, $W(t)$.

Example 11.23
Let $W(t)$ be a standard Brownian motion. For all $s, t \in [0, \infty)$, find

$$C_W(s,t) = \text{Cov}(W(s), W(t)).$$

Solution

Let's assume $s \le t$. Then, we have

$$\begin{aligned}\mathrm{Cov}\big(W(s), W(t)\big) &= \mathrm{Cov}\big(W(s), W(s) + W(t) - W(s)\big) \\ &= \mathrm{Cov}\big(W(s), W(s)\big) + \mathrm{Cov}\big(W(s), W(t) - W(s)\big) \\ &= \mathrm{Var}\big(W(s)\big) + \mathrm{Cov}\big(W(s), W(t) - W(s)\big) \\ &= s + \mathrm{Cov}\big(W(s), W(t) - W(s)\big).\end{aligned}$$

Brownian motion has independent increments, so the two random variables $W(s) = W(s) - W(0)$ and $W(t) - W(s)$ are independent. Therefore, $\mathrm{Cov}\big(W(s), W(t) - W(s)\big) = 0$. We conclude

$$\mathrm{Cov}\big(W(s), W(t)\big) = s.$$

Similarly, if $t \le s$, we obtain

$$\mathrm{Cov}\big(W(s), W(t)\big) = t.$$

We conclude

$$\mathrm{Cov}\big(W(s), W(t)\big) = \min(s, t), \quad \text{for all } s, t.$$

If $W(t)$ is a standard Brownian motion, we have

$$\mathrm{Cov}(W(s), W(t)) = \min(s, t), \quad \text{for all } s, t.$$

Example 11.24
Let $W(t)$ be a standard Brownian motion.

a. Find $P(1 < W(1) < 2)$.
b. Find $P(W(2) < 3 | W(1) = 1)$.

Solution

a. We have $W(1) \sim N(0, 1)$. Thus,

$$\begin{aligned} P(1 < W(1) < 2) &= \Phi(2) - \Phi(1) \\ &\approx 0.136 \end{aligned}$$

b. Note that $W(2) = W(1) + W(2) - W(1)$. Also, note that $W(1)$ and $W(2) - W(1)$ are independent, and

$$W(2) - W(1) \sim N(0, 1).$$

We conclude that

$$W(2)|W(1) = 1 \ \sim \ N(1, 1).$$

Thus,

$$\begin{aligned} P(W(2) < 3|W(1) = 1) &= \Phi\left(\frac{3-1}{1}\right) \\ &= \Phi(2) \approx 0.98 \end{aligned}$$
