## 1.2.2 Set Operations

The **union** of two sets is a set containing all elements that are in $A$ <u>or</u> in $B$ (possibly both). For example, $\{1, 2\} \cup \{2, 3\} = \{1, 2, 3\}$. Thus, we can write $x \in (A \cup B)$ if and only if $(x \in A)$ or $(x \in B)$. Note that $A \cup B = B \cup A$. In Figure 1.4, the union of sets $A$ and $B$ is shown by the shaded area in the Venn diagram.

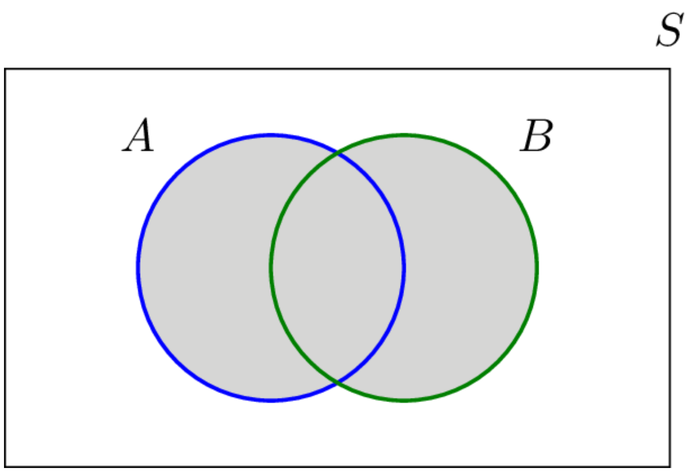


Fig.1.4 - The shaded area shows the set $B \cup A$.

Similarly we can define the union of three or more sets. In particular, if $A_1, A_2, A_3, \cdots, A_n$ are $n$ sets, their union $A_1 \cup A_2 \cup A_3 \cdots \cup A_n$ is a set containing all elements that are in at least one of the sets. We can write this union more compactly by

$$\bigcup_{i=1}^{n} A_i.$$

For example, if $A_1 = \{a, b, c\}, A_2 = \{c, h\}, A_3 = \{a, d\}$, then $\bigcup_i A_i = A_1 \cup A_2 \cup A_3 = \{a, b, c, h, d\}$. We can similarly define the union of infinitely many sets $A_1 \cup A_2 \cup A_3 \cup \cdots$.

The **intersection** of two sets $A$ and $B$, denoted by $A \cap B$, consists of all elements that are both in $A$ $\underline{\text{and}}$ $B$. For example, $\{1, 2\} \cap \{2, 3\} = \{2\}$. In Figure 1.5, the intersection of sets $A$ and $B$ is shown by the shaded area using a Venn diagram.

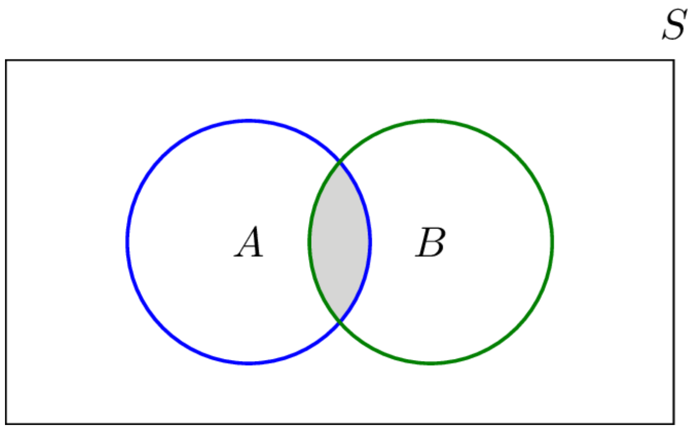


Fig.1.5 - The shaded area shows the set $B \cap A$.

More generally, for sets $A_1, A_2, A_3, \cdots$, their intersection $\bigcap_i A_i$ is defined as the set consisting of the elements that are in all $A_i$'s. Figure 1.6 shows the intersection of three sets.

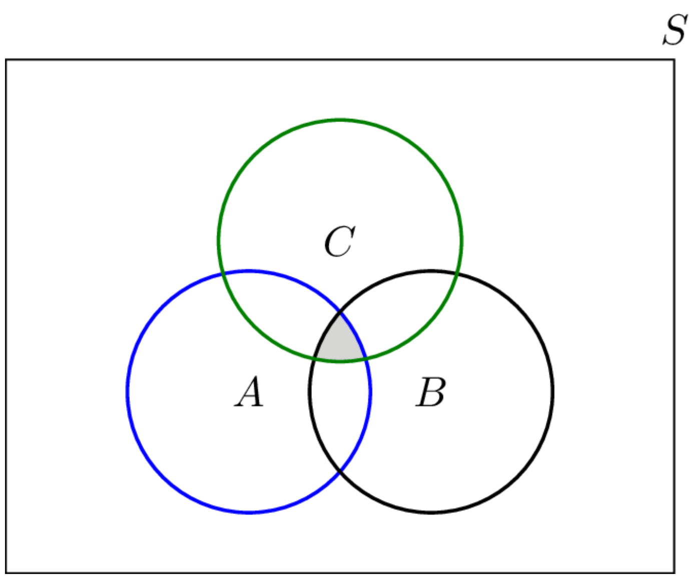


Fig.1.6 - The shaded area shows the set $A \cap B \cap C$.

The **complement** of a set $A$, denoted by $A^c$ or $\bar{A}$, is the set of all elements that are in the universal set $S$ but are not in $A$. In Figure 1.7, $\bar{A}$ is shown by the shaded area using a Venn diagram.

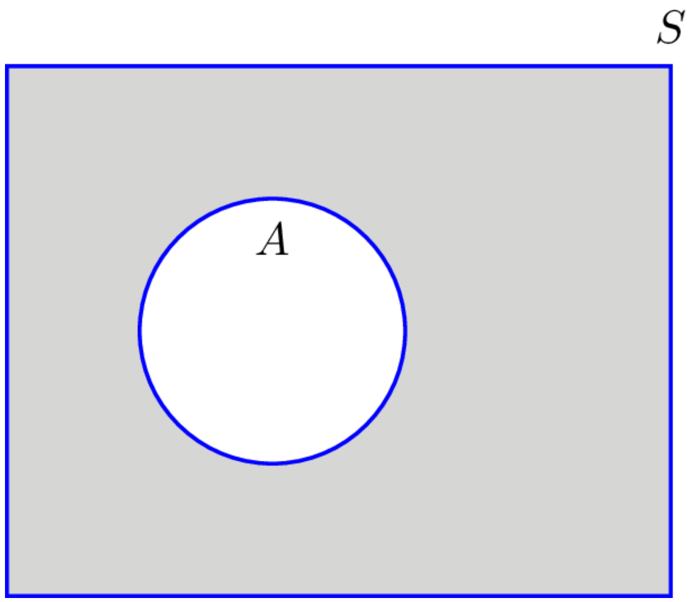


Fig.1.7 - The shaded area shows the set $\bar{A} = A^c$.

The **difference (subtraction)** is defined as follows. The set $A - B$ consists of elements that are in $A$ but not in $B$. For example if $A = \{1, 2, 3\}$ and $B = \{3, 5\}$, then $A - B = \{1, 2\}$. In Figure 1.8, $A - B$ is shown by the shaded area using a Venn diagram. Note that $A - B = A \cap B^c$.

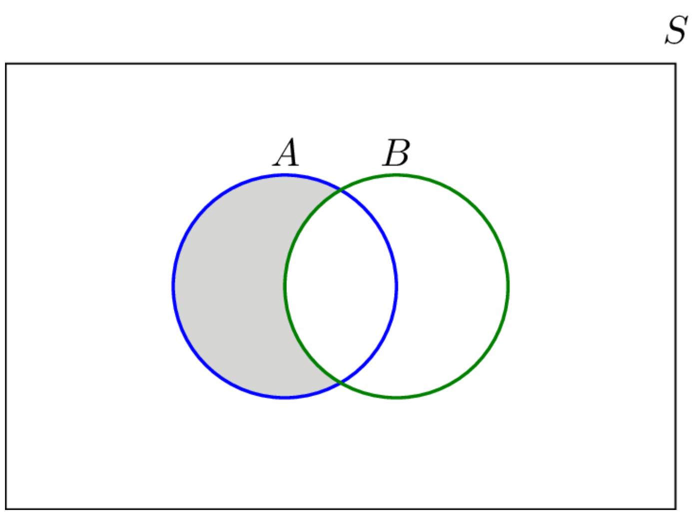


Fig.1.8 - The shaded area shows the set $A - B$.

Two sets $A$ and $B$ are **mutually exclusive** or **disjoint** if they do not have any shared elements; i.e., their intersection is the empty set, $A \cap B = \emptyset$. More generally, several sets are called disjoint if they are pairwise disjoint, i.e., no two of them share a common elements. Figure 1.9 shows three disjoint sets.

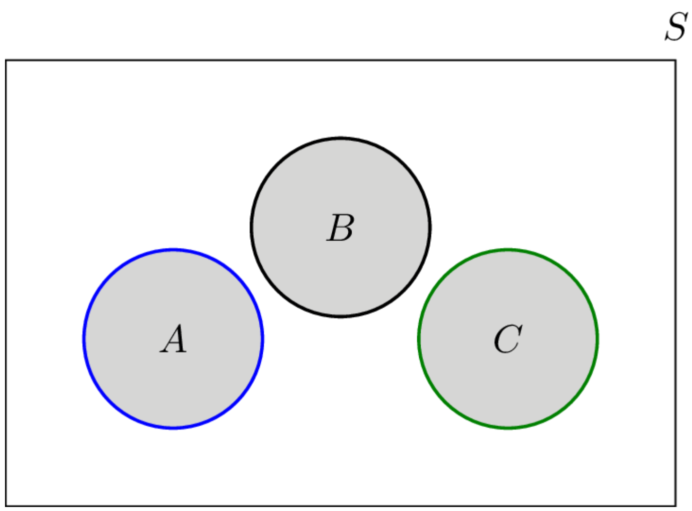


Fig.1.9 - Sets $A, B$, and $C$ are disjoint.

If the earth's surface is our sample space, we might want to partition it to the different continents. Similarly, a country can be partitioned to different provinces. In general, a collection of nonempty sets $A_1, A_2, \cdots$ is a **partition** of a set $A$ if they are disjoint and their union is $A$. In Figure 1.10, the sets $A_1, A_2, A_3$ and $A_4$ form a partition of the universal set $S$.

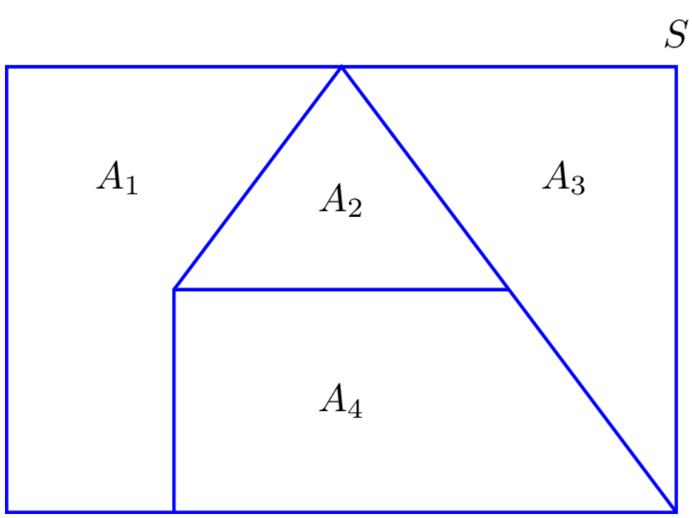


Fig.1.10 - The collection of sets $A_1, A_2, A_3$ and $A_4$ is a partition of $S$.

Here are some rules that are often useful when working with sets. We will see examples of their usage shortly.

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**Theorem 1.1: De Morgan's law**
For any sets $A_1$, $A_2$, $\cdots$, $A_n$, we have

- $(A_1 \cup A_2 \cup A_3 \cup \cdots A_n)^c = A_1^c \cap A_2^c \cap A_3^c \cdots \cap A_n^c$;
- $(A_1 \cap A_2 \cap A_3 \cap \cdots A_n)^c = A_1^c \cup A_2^c \cup A_3^c \cdots \cup A_n^c$.

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**Theorem 1.2: Distributive law**
For any sets $A$, $B$, and $C$ we have

- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$;
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

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**Example 1.4**
If the universal set is given by $S = \{1, 2, 3, 4, 5, 6\}$, and $A = \{1, 2\}$, $B = \{2, 4, 5\}, C = \{1, 5, 6\}$ are three sets, find the following sets:

a. $A \cup B$
b. $A \cap B$
c. $\overline{A}$
d. $\overline{B}$
e. Check De Morgan's law by finding $(A \cup B)^c$ and $A^c \cap B^c$.
f. Check the distributive law by finding $A \cap (B \cup C)$ and $(A \cap B) \cup (A \cap C)$.

**Solution**

a. $A \cup B = \{1, 2, 4, 5\}$.
b. $A \cap B = \{2\}$.
c. $\overline{A} = \{3, 4, 5, 6\}$ ($\overline{A}$ consists of elements that are in $S$ but not in $A$).
d. $\overline{B} = \{1, 3, 6\}$.
e. We have

$$(A \cup B)^c = \{1, 2, 4, 5\}^c = \{3, 6\},$$

which is the same as

$$A^c \cap B^c = \{3,4,5,6\} \cap \{1,3,6\} = \{3,6\}.$$

f. We have

$$A \cap (B \cup C) = \{1,2\} \cap \{1,2,4,5,6\} = \{1,2\},$$

which is the same as

$$(A \cap B) \cup (A \cap C) = \{2\} \cup \{1\} = \{1,2\}.$$

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A **Cartesian product** of two sets $A$ and $B$, written as $A \times B$, is the set containing **ordered** pairs from $A$ and $B$. That is, if $C = A \times B$, then each element of $C$ is of the form $(x, y)$, where $x \in A$ and $y \in B$:

$$A \times B = \{(x,y) | x \in A \text{ and } y \in B\}.$$

For example, if $A = \{1,2,3\}$ and $B = \{H,T\}$, then

$$A \times B = \{(1,H),(1,T),(2,H),(2,T),(3,H),(3,T)\}.$$

Note that here the pairs are ordered, so for example, $(1,H) \neq (H,1)$. Thus $A \times B$ is **not** the same as $B \times A$.

If you have two finite sets $A$ and $B$, where $A$ has $M$ elements and $B$ has $N$ elements, then $A \times B$ has $M \times N$ elements. This rule is called the **multiplication principle** and is very useful in counting the numbers of elements in sets. The number of elements in a set is denoted by $|A|$, so here we write $|A| = M, |B| = N$, and $|A \times B| = MN$. In the above example, $|A| = 3, |B| = 2$, thus $|A \times B| = 3 \times 2 = 6$. We can similarly define the Cartesian product of $n$ sets $A_1, A_2, \cdots, A_n$ as

$$A_1 \times A_2 \times A_3 \times \cdots \times A_n = \{(x_1, x_2, \cdots, x_n) | x_1 \in A_1 \text{ and } x_2 \in A_2 \text{ and } \cdots x_n \in A_n\}.$$

The multiplication principle states that for finite sets $A_1, A_2, \cdots, A_n$, if

$$|A_1| = M_1, |A_2| = M_2, \cdots, |A_n| = M_n,$$

then

$$|A_1 \times A_2 \times A_3 \times \cdots \times A_n| = M_1 \times M_2 \times M_3 \times \cdots \times M_n.$$

An important example of sets obtained using a Cartesian product is $\mathbb{R}^n$, where $n$ is a natural number. For $n = 2$, we have

$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$$

$$= \{(x, y) | x \in \mathbb{R}, y \in \mathbb{R}\}.$$

Thus, $\mathbb{R}^2$ is the set consisting of all points in the two-dimensional plane. Similarly, $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ and so on.

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