Similarly, a **random matrix** is a matrix whose elements are random variables. In particular, we can have an m by n random matrix **M** as

$$\mathbf{M} = \begin{bmatrix} X_{11} & X_{12} & \dots & X_{1n} \\ X_{21} & X_{22} & \dots & X_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ X_{m1} & X_{m2} & \dots & X_{mn} \end{bmatrix}.$$

We sometimes write this as **M**$= [X_{ij}]$, which means that X_{ij} is the element in the ith row and jth column of **M**. The mean matrix of **M** is given by

$$E\mathbf{M} = \begin{bmatrix} EX_{11} & EX_{12} & \dots & EX_{1n} \\ EX_{21} & EX_{22} & \dots & EX_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ EX_{m1} & EX_{m2} & \dots & EX_{mn} \end{bmatrix}.$$

Linearity of expectation is also valid for random vectors and matrices. In particular, let **X** be an n-dimensional random vector and the random vector **Y** be defined as

$$\mathbf{Y} = \mathbf{AX} + \mathbf{b},$$

where **A** is a fixed (non-random) m by n matrix and **b** is a fixed m-dimensional vector. Then we have

$$E\mathbf{Y} = \mathbf{AEX} + \mathbf{b}.$$

Also, if $\mathbf{X_1, X_2, \cdots, X_k}$ are n-dimensional random vectors, then we have

$$\mathbf{E[X_1 + X_2 + \cdots + X_k] = EX_1 + EX_2 + \cdots + EX_k}.$$

Correlation and Covariance Matrix

For a random vector $\mathbf{X}$, we define the **correlation matrix**, $\mathbf{R_X}$, as

$$\mathbf{R_X} = \mathbf{E[XX^T]} = E \begin{bmatrix} X_1^2 & X_1X_2 & \dots & X_1X_n \\ X_2X_1 & X_2^2 & \dots & X_2X_n \\ \vdots & \vdots & \vdots & \vdots \\ X_nX_1 & X_nX_2 & \dots & X_n^2 \end{bmatrix} = \begin{bmatrix} EX_1^2 & E[X_1X_2] & \dots & E[X_1X_n] \\ EX_2X_1 & E[X_2^2] & \dots & E[X_2X_n] \\ \vdots & \vdots & \vdots & \vdots \\ E[X_nX_1] & E[X_nX_2] & \dots & E[X_n^2] \end{bmatrix},$$

where ^{T} shows matrix transposition.

The **covariance matrix**, C_X, is defined as

$$\mathbf{C_X} = \mathbf{E}[(\mathbf{X} - \mathbf{EX})(\mathbf{X} - \mathbf{EX})^{\mathbf{T}}]$$

$$= E \begin{bmatrix} (X_1 - EX_1)^2 & (X_1 - EX_1)(X_2 - EX_2) & \dots & (X_1 - EX_1)(X_n - EX_n) \\ (X_2 - EX_2)(X_1 - EX_1) & (X_2 - EX_2)^2 & \dots & (X_2 - EX_2)(X_n - EX_n) \\ \vdots & \vdots & \vdots & \vdots \\ (X_n - EX_n)(X_1 - EX_1) & (X_n - EX_n)(X_2 - EX_2) & \dots & (X_n - EX_n)^2 \end{bmatrix}$$

$$= \begin{bmatrix} \mathrm{Var}(X_1) & \mathrm{Cov}(X_1, X_2) & \dots & \mathrm{Cov}(X_1, X_n) \\ \mathrm{Cov}(X_2, X_1) & \mathrm{Var}(X_2) & \dots & \mathrm{Cov}(X_2, X_n) \\ \vdots & \vdots & \vdots & \vdots \\ \mathrm{Cov}(X_n, X_1) & \mathrm{Cov}(X_n X_2) & \dots & \mathrm{Var}(X_n) \end{bmatrix}.$$

The covariance matrix is a generalization of the variance of a random variable. Remember that for a random variable, we have $\mathrm{Var}(X) = EX^2 - (EX)^2$. The following example extends this formula to random vectors.

Example 6.11

For a random vector **X**, show

$$\mathbf{C_X} = \mathbf{R_X} - \mathbf{EXEX^T}.$$

Solution

We have

$$\begin{aligned} \mathbf{C_X} &= \mathbf{E}[(\mathbf{X} - \mathbf{EX})(\mathbf{X} - \mathbf{EX})^{\mathbf{T}}] \\ &= \mathbf{E}[(\mathbf{X} - \mathbf{EX})(\mathbf{X}^{\mathbf{T}} - \mathbf{EX}^{\mathbf{T}})] \\ &= \mathbf{E}[\mathbf{XX^T}] - \mathbf{EXEX^T} - \mathbf{EXEX^T} + \mathbf{EXEX^T} \quad \text{(by linearity of expectation)} \\ &= \mathbf{R_X} - \mathbf{EXEX^T}. \end{aligned}$$

Correlation matrix of X:

$$\mathbf{R_X} = \mathbf{E}[\mathbf{X}\mathbf{X}^\mathbf{T}]$$

Covariance matrix of X:

$$\mathbf{C_X} = \mathbf{E}[(\mathbf{X} - \mathbf{EX})(\mathbf{X} - \mathbf{EX})^\mathbf{T}] = \mathbf{R_X} - \mathbf{EXEX^T}$$

Example 6.12

Let **X** be an n-dimensional random vector and the random vector **Y** be defined as

$$\mathbf{Y} = \mathbf{AX} + \mathbf{b},$$

where **A** is a fixed m by n matrix and **b** is a fixed m-dimensional vector. Show that

$$\mathbf{C_Y} = \mathbf{AC_XA^T}.$$

Solution

Note that by linearity of expectation, we have

$$\mathbf{EY} = \mathbf{AEX} + \mathbf{b}.$$

By definition, we have

$$\begin{aligned}
\mathbf{C_Y} &= \mathbf{E}[(\mathbf{Y} - \mathbf{EY})(\mathbf{Y} - \mathbf{EY})^\mathbf{T}] \\
&= \mathbf{E}[(\mathbf{AX} + \mathbf{b} - \mathbf{AEX} - \mathbf{b})(\mathbf{AX} + \mathbf{b} - \mathbf{AEX} - \mathbf{b})^\mathbf{T}] \\
&= E[\mathbf{A}(\mathbf{X} - \mathbf{EX})(\mathbf{X} - \mathbf{EX})^\mathbf{T}\mathbf{A}^\mathbf{T}] \\
&= \mathbf{AE}[(\mathbf{X} - \mathbf{EX})(\mathbf{X} - \mathbf{EX})^\mathbf{T}]\mathbf{A}^\mathbf{T} && \text{(by linearity of expectation)} \\
&= \mathbf{AC_XA^T}.
\end{aligned}$$

Example 6.13

Let X and Y be two jointly continuous random variables with joint PDF

$$f_{X,Y}(x,y) = \begin{cases} \frac{3}{2}x^2 + y & 0 < x, y < 1 \\ & \\ 0 & \text{otherwise} \end{cases}$$

and let the random vector **U** be defined as

$$\mathbf{U} = \begin{bmatrix} X \\ Y \end{bmatrix}.$$

Find the correlation and covariance matrices of **U**.

Solution

We first obtain the marginal PDFs of X and Y. Note that $R_X = R_Y = (0,1)$. We have for $x \in R_X$

$$\begin{aligned} f_X(x) &= \int_0^1 \frac{3}{2}x^2 + y \;\; dy \\ &= \frac{3}{2}x^2 + \frac{1}{2}, \quad \text{for } 0 < x < 1. \end{aligned}$$

Similarly, for $y \in R_Y$, we have

$$\begin{aligned} f_Y(y) &= \int_0^1 \frac{3}{2}x^2 + y \;\; dx \\ &= y + \frac{1}{2}, \quad \text{for } 0 < y < 1. \end{aligned}$$

From these, we obtain $EX = \frac{5}{8}$, $EX^2 = \frac{7}{15}$, $EY = \frac{7}{12}$, and $EY^2 = \frac{5}{12}$. We also need EXY. By LOTUS, we can write

$$\begin{aligned} EXY &= \int_0^1 \int_0^1 xy\left(\frac{3}{2}x^2 + y\right) dxdy \\ &= \int_0^1 \frac{3}{8}y + \frac{1}{2}y^2 dy \\ &= \frac{17}{48}. \end{aligned}$$

From this, we also obtain

$$\begin{aligned} \text{Cov}(X,Y) &= EXY - EXEY \\ &= \frac{17}{48} - \frac{5}{8} \cdot \frac{7}{12} \\ &= -\frac{1}{96}. \end{aligned}$$

The correlation matrix R_U is given by

$$\mathbf{R_U} = \mathbf{E}[\mathbf{U}\mathbf{U}^\mathbf{T}] = \begin{bmatrix} EX^2 & EXY \\ EYX & EY^2 \end{bmatrix} = \begin{bmatrix} \frac{7}{15} & \frac{17}{48} \\ \frac{17}{48} & \frac{5}{12} \end{bmatrix}.$$