5.2.5 Solved Problems

Problem 1

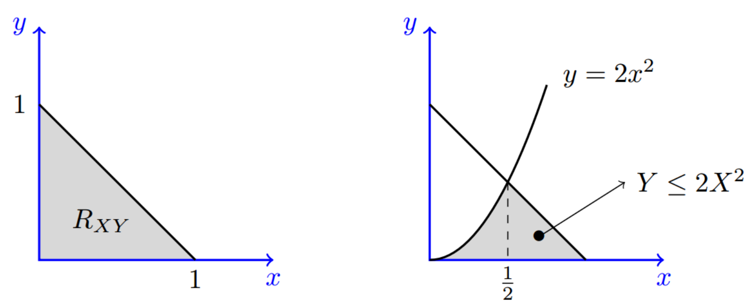
Let X and Y be jointly continuous random variables with joint PDF

$$f_{X,Y}(x,y) = \begin{cases} cx+1 & x,y \geq 0, x+y<1 \\ 0 & \text{otherwise} \end{cases}$$

1. Show the range of (X,Y), R_{XY}, in the $x-y$ plane.
2. Find the constant c.
3. Find the marginal PDFs $f_X(x)$ and $f_Y(y)$.
4. Find $P(Y<2X^2)$.

Solution

1. Figure 5.8(a) shows R_{XY} in the $x-y$ plane.



The figure shows (a) R_{XY} as well as (b) the integration region for finding $P(Y<2X^2)$ for Solved Problem 1.

2. To find the constant c, we write

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)dxdy \\ &= \int_0^1 \int_0^{1-x} cx+1 \; dydx \\ &= \int_0^1 (cx+1)(1-x) \; dx \\ &= \frac{1}{2}+\frac{1}{6}c. \end{aligned}$$

Thus, we conclude $c = 3$.

3. We first note that $R_X = R_Y = [0, 1]$.

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{XY}(x,y)dy \\ &= \int_0^{1-x} 3x + 1 \ \ dy \\ &= (3x+1)(1-x), \quad \text{ for } x \in [0,1]. \end{aligned}$$

Thus, we have

$$f_X(x) = \begin{cases} (3x+1)(1-x) & 0 \leq x \leq 1 \\ & \\ 0 & \text{otherwise} \end{cases}$$

Similarly, we obtain

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{XY}(x,y)dx \\ &= \int_0^{1-y} 3x + 1 \ \ dx \\ &= \frac{1}{2}(1-y)(5-3y), \quad \text{ for } y \in [0,1]. \end{aligned}$$

Thus, we have

$$f_Y(y) = \begin{cases} \frac{1}{2}(1-y)(5-3y) & 0 \leq y \leq 1 \\ & \\ 0 & \text{otherwise} \end{cases}$$

4. To find $P(Y < 2X^2)$, we need to integrate $f_{XY}(x,y)$ over the region shown in Figure 5.8(b). We have

$$\begin{aligned} P(Y < 2X^2) &= \int_{-\infty}^{\infty} \int_{-\infty}^{2x^2} f_{XY}(x,y)dydx \\ &= \int_0^1 \int_0^{\min(2x^2, 1-x)} 3x + 1 \ \ dydx \\ &= \int_0^1 (3x+1)\min(2x^2, 1-x) \ \ dx \\ &= \int_0^{\frac{1}{2}} 2x^2(3x+1) \ \ dx + \int_{\frac{1}{2}}^1 (3x+1)(1-x) \ \ dx \\ &= \frac{53}{96}. \end{aligned}$$

Problem 2

Let X and Y be jointly continuous random variables with joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 6e^{-(2x+3y)} & x, y \geq 0 \\ \\ 0 & \text{otherwise} \end{cases}$$

1. Are X and Y independent?
2. Find $E[Y|X > 2]$.
3. Find $P(X > Y)$.

Solution

1. We can write

$$f_{X,Y}(x,y) = f_X(x)f_Y(y),$$

where

$$f_X(x) = 2e^{-2x}u(x), \qquad f_Y(y) = 3e^{-3y}u(y).$$

Thus, X and Y are independent.

2. Since X and Y are independent, we have $E[Y|X > 2] = E[Y]$. Note that $Y \sim Exponential(3)$, thus $EY = \frac{1}{3}$.
3. We have

$$\begin{aligned} P(X > Y) &= \int_0^\infty \int_y^\infty 6e^{-(2x+3y)} dx dy \\ &= \int_0^\infty 3e^{-5y} dy \\ &= \frac{3}{5}. \end{aligned}$$

Problem 3

Let X be a continuous random variable with PDF

$$f_X(x) = \begin{cases} 2x & 0 \leq x \leq 1 \\ \\ 0 & \text{otherwise} \end{cases}$$

We know that given $X=x$, the random variable Y is uniformly distributed on $[-x,x]$.

1. Find the joint PDF $f_{XY}(x,y)$.
2. Find $f_Y(y)$.
3. Find $P(|Y|<X^3)$.

Solution

1. First note that, by the assumption

$$f_{Y|X}(y|x)=\begin{cases}\frac{1}{2x} & -x\leq y\leq x\\ \\ 0 & \text{otherwise}\end{cases}$$

   Thus, we have

$$f_{XY}(x,y)=f_{Y|X}(y|x)f_X(x)=\begin{cases}1 & 0\leq x\leq 1, -x\leq y\leq x\\ \\ 0 & \text{otherwise}\end{cases}$$

   Thus,

$$f_{XY}(x,y)=\begin{cases}1 & |y|\leq x\leq 1\\ \\ 0 & \text{otherwise}\end{cases}$$

2. First, note that $R_Y=[-1,1]$. To find $f_Y(y)$, we can write

$$\begin{aligned}f_Y(y)&=\int_{-\infty}^{\infty}f_{XY}(x,y)dx\\&=\int_{|y|}^{1}1dx\\&=1-|y|.\end{aligned}$$

   Thus,

$$f_Y(y)=\begin{cases}1-|y| & |y|\leq 1\\ \\ 0 & \text{otherwise}\end{cases}$$

3. To find $P(|Y|<X^3)$, we can use the law of total probability (Equation 5.16):

$$\begin{aligned} P(|Y| < X^3) &= \int_0^1 P(|Y| < X^3 | X = x) f_X(x) dx \\ &= \int_0^1 P(|Y| < x^3 | X = x) 2x dx \\ &= \int_0^1 \left(\frac{2x^3}{2x} \right) 2x dx \qquad \text{since } Y|X = x \quad \sim \quad Uniform(-x, x) \\ &= \frac{1}{2}. \end{aligned}$$

Problem 4

Let X and Y be two jointly continuous random variables with joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 6xy & 0 \le x \le 1, 0 \le y \le \sqrt{x} \\ \\ 0 & \text{otherwise} \end{cases}$$

1. Show R_{XY} in the $x - y$ plane.
2. Find $f_X(x)$ and $f_Y(y)$.
3. Are X and Y independent?
4. Find the conditional PDF of X given $Y = y$, $f_{X|Y}(x|y)$.
5. Find $E[X|Y = y]$, for $0 \le y \le 1$.
6. Find $\text{Var}(X|Y = y)$, for $0 \le y \le 1$.

Solution

1. Figure 5.9 shows R_{XY} in the $x - y$ plane.

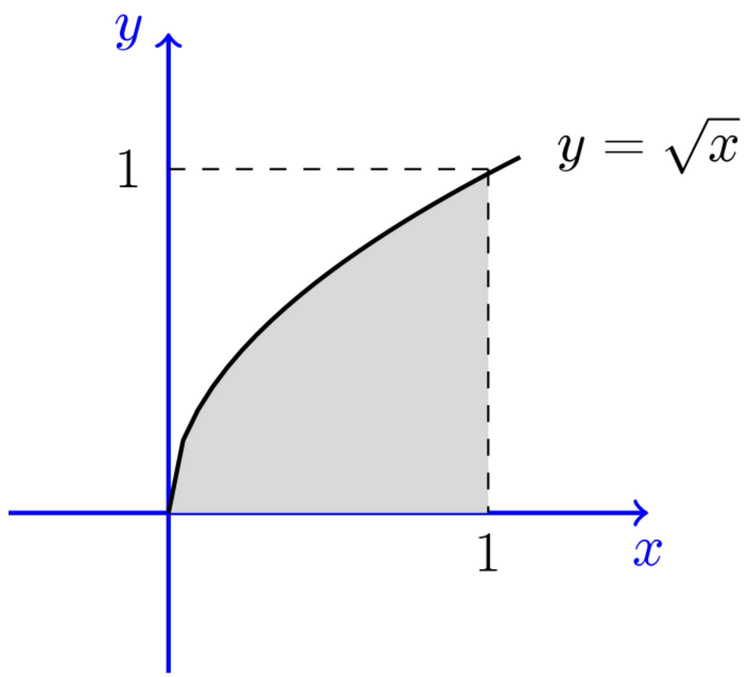


Figure 5.9: The figure shows R_{XY} for Solved Problem 4.

2. First, note that $R_X = R_Y = [0, 1]$. To find $f_X(x)$ for $0 \leq x \leq 1$, we can write

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{XY}(x, y) \; dy \\ &= \int_{0}^{\sqrt{x}} 6xy \; dy \\ &= 3x^2. \end{aligned}$$

Thus,

$$f_X(x) = \begin{cases} 3x^2 & 0 \leq x \leq 1 \\ & \\ 0 & \text{otherwise} \end{cases}$$

To find $f_Y(y)$ for $0 \leq y \leq 1$, we can write

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{XY}(x, y) \; dx \\ &= \int_{y^2}^{1} 6xy \; dx \\ &= 3y(1 - y^4). \end{aligned}$$

$$f_Y(y) = \begin{cases} 3y(1 - y^4) & 0 \leq y \leq 1 \\ & \\ 0 & \text{otherwise} \end{cases}$$

3. X and Y are not independent, since $f_{XY}(x,y) \neq f_x(x)f_Y(y)$.
4. We have

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$
$$= \begin{cases} \frac{2x}{1-y^4} & y^2 \leq x \leq 1 \\ \\ 0 & \text{otherwise} \end{cases}$$

5. We have

$$E[X|Y=y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y) \; dx$$
$$= \int_{y^2}^{1} x \frac{2x}{1-y^4} \; dx$$
$$= \frac{2(1-y^6)}{3(1-y^4)}.$$

6. We have

$$E[X^2|Y=y] = \int_{-\infty}^{\infty} x^2 f_{X|Y}(x|y) \; dx$$
$$= \int_{y^2}^{1} x^2 \frac{2x}{1-y^4} \; dx$$
$$= \frac{1-y^8}{2(1-y^4)}.$$

Thus,

$$\text{Var}(X|Y=y) = E[X^2|Y=y] - (E[X|Y=y])^2$$
$$= \frac{1-y^8}{2(1-y^4)} - \left(\frac{2(1-y^6)}{3(1-y^4)}\right)^2.$$

Problem 5

Consider the unit disc

$$D = \{(x,y)|x^2 + y^2 \leq 1\}.$$

Suppose that we choose a point (X,Y) uniformly at random in D. That is, the joint PDF of X and Y is given by

$$f_{XY}(x,y) = \begin{cases} \frac{1}{\pi} & (x,y) \in D \\ \\ 0 & \text{otherwise} \end{cases}$$

Let (R, Θ) be the corresponding polar coordinates as shown in Figure 5.10. The inverse transformation is given by

$$\begin{cases} X = R\cos\Theta \\ Y = R\sin\Theta \end{cases}$$

where $R \geq 0$ and $-\pi < \Theta \leq \pi$. Find the joint PDF of R and Θ.

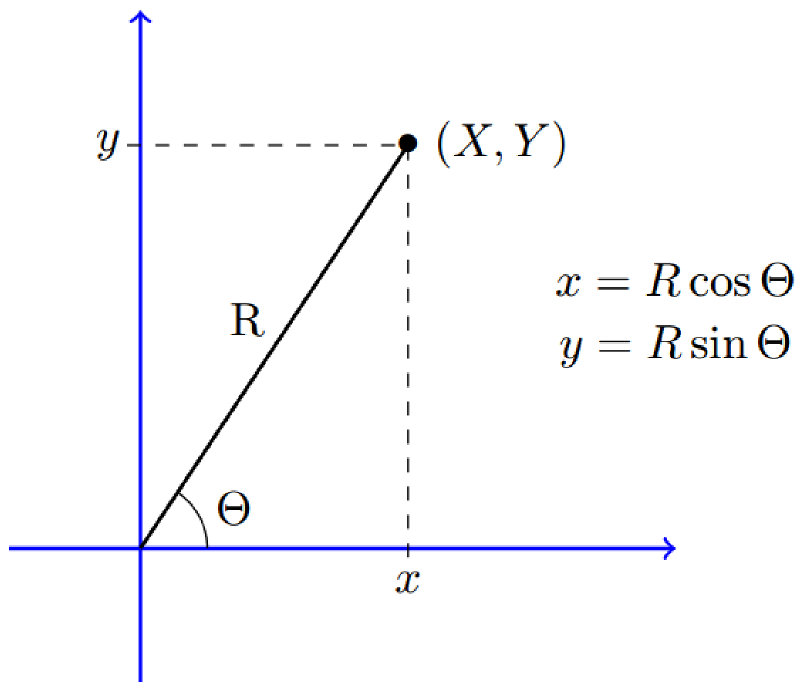


Figure 5.10: Polar Coordinates

Solution

Here (X,Y) are jointly continuous and are related to (R, Θ) by a one-to-one relationship. We use the method of transformations (Theorem 5.1). The function $h(r,\theta)$ is given by

$$\begin{cases} x = h_1(r,\theta) = r\cos\theta \\ y = h_2(r,\theta) = r\sin\theta \end{cases}$$

Thus, we have

$$\begin{aligned} f_{R\Theta}(r,\theta) &= f_{XY}(h_1(r,\theta), h_2(r,\theta))|J| \\ &= f_{XY}(r\cos\theta, r\sin\theta)|J|. \end{aligned}$$

where

$$J = \det \begin{bmatrix} \frac{\partial h_1}{\partial r} & \frac{\partial h_1}{\partial \theta} \\ & \\ \frac{\partial h_2}{\partial r} & \frac{\partial h_2}{\partial \theta} \end{bmatrix} = \det \begin{bmatrix} \cos\theta & -r\sin\theta \\ & \\ \sin\theta & r\cos\theta \end{bmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

We conclude that

$$\begin{aligned} f_{R\Theta}(r,\theta) &= f_{XY}(r\cos\theta, r\sin\theta)|J| \\ &= \begin{cases} \frac{r}{\pi} & r \in [0,1], \theta \in (-\pi,\pi] \\ & \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Note that from above we can write

$$f_{R\Theta}(r,\theta) = f_R(r) f_\Theta(\theta),$$

where

$$f_R(r) = \begin{cases} 2r & r \in [0,1] \\ & \\ 0 & \text{otherwise} \end{cases}$$

$$f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi} & \theta \in (-\pi,\pi] \\ & \\ 0 & \text{otherwise} \end{cases}$$

Thus, we conclude that R and Θ are independent.