b. Y is a $Uniform(0,1)$ random variable.

Solution

a. For X, we have

$$M_X(s) = E\left[e^{sX}\right] = \frac{1}{3}e^s + \frac{2}{3}e^{2s}.$$

which is well-defined for all $s \in \mathbb{R}$.

b. For Y, we can write

$$M_Y(s) = E\left[e^{sY}\right] = \int_0^1 e^{sy}dy = \frac{e^s - 1}{s}.$$

Note that we always have $M_Y(0) = E[e^{0 \cdot Y}] = 1$, thus $M_Y(s)$ is also well-defined for all $s \in \mathbb{R}$.

Why is the MGF useful? There are basically two reasons for this. First, the MGF of X gives us all moments of X. That is why it is called the moment generating function. Second, the MGF (if it exists) uniquely determines the distribution. That is, if two random variables have the same MGF, then they must have the same distribution. Thus, if you find the MGF of a random variable, you have indeed determined its distribution. We will see that this method is very useful when we work on sums of several independent random variables. Let's discuss these in detail.

Finding Moments from MGF:

Remember the Taylor series for e^x: for all $x \in \mathbb{R}$, we have

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

Now, we can write

$$e^{sX} = \sum_{k=0}^{\infty} \frac{(sX)^k}{k!} = \sum_{k=0}^{\infty} \frac{X^k s^k}{k!}.$$

Thus, we have

$$M_X(s) = E[e^{sX}] = \sum_{k=0}^{\infty} E[X^k] \frac{s^k}{k!}.$$

We conclude that the kth moment of X is the coefficient of $\frac{s^k}{k!}$ in the Taylor series of $M_X(s)$. Thus, if we have the Taylor series of $M_X(s)$, we can obtain all moments of X.

Example 6.4

If $Y \sim Uniform(0,1)$, find $E[Y^k]$ using $M_Y(s)$.

Solution

We found $M_Y(s)$ in Example 6.3, so we have

$$\begin{aligned} M_Y(s) &= \frac{e^s - 1}{s} \\ &= \frac{1}{s}\left(\sum_{k=0}^{\infty} \frac{s^k}{k!} - 1\right) \\ &= \frac{1}{s} \sum_{k=1}^{\infty} \frac{s^k}{k!} \\ &= \sum_{k=1}^{\infty} \frac{s^{k-1}}{k!} \\ &= \sum_{k=0}^{\infty} \frac{1}{k+1} \frac{s^k}{k!}. \end{aligned}$$

Thus, the coefficient of $\frac{s^k}{k!}$ in the Taylor series for $M_Y(s)$ is $\frac{1}{k+1}$, so

$$E[X^k] = \frac{1}{k+1}.$$

We remember from calculus that the coefficient of $\frac{s^k}{k!}$ in the Taylor series of $M_X(s)$ is obtained by taking the kth derivative of $M_X(s)$ and evaluating it at $s = 0$. Thus, we can write

$$E[X^k] = \frac{d^k}{ds^k} M_X(s)|_{s=0}.$$

We can obtain all moments of X^k from its MGF:

$$M_X(s) = \sum_{k=0}^{\infty} E[X^k] \frac{s^k}{k!},$$

$$E[X^k] = \frac{d^k}{ds^k} M_X(s)|_{s=0}.$$

Example 6.5

Let $X \sim Exponential(\lambda)$. Find the MGF of X, $M_X(s)$, and all of its moments, $E[X^k]$.

Solution

Recall that the PDF of X is

$$f_X(x) = \lambda e^{-\lambda x} u(x),$$

where $u(x)$ is the unit step function. We conclude

$$\begin{aligned} M_X(s) &= E[e^{sX}] \\ &= \int_0^{\infty} \lambda e^{-\lambda x} e^{sx} dx \\ &= \left[-\frac{\lambda}{\lambda - s} e^{-(\lambda - s)x} \right]_0^{\infty}, \quad \text{for } s < \lambda \\ &= \frac{\lambda}{\lambda - s}, \quad \text{for } s < \lambda. \end{aligned}$$

Therefore, $M_X(s)$ exists for all $s < \lambda$. To find the moments of X, we can write

$$
\begin{aligned}
M_X(s) &= \frac{\lambda}{\lambda - s} \\
&= \frac{1}{1 - \frac{s}{\lambda}} \\
&= \sum_{k=0}^{\infty} \left(\frac{s}{\lambda}\right)^k, \quad \text{for } \left|\frac{s}{\lambda}\right| < 1 \\
&= \sum_{k=0}^{\infty} \frac{k!}{\lambda^k} \frac{s^k}{k!}.
\end{aligned}
$$

We conclude that

$$
E[X^k] = \frac{k!}{\lambda^k}, \quad \text{for } k = 0, 1, 2, \ldots
$$

Example 6.6

Let $X \sim Poisson(\lambda)$. Find the MGF of X, $M_X(s)$.

Solution

We have

$$
P_X(k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad \text{for } k = 0, 1, 2, \ldots
$$

Thus,

$$
\begin{aligned}
M_X(s) &= E[e^{sX}] \\
&= \sum_{k=0}^{\infty} e^{sk} e^{-\lambda} \frac{\lambda^k}{k!} \\
&= e^{-\lambda} \sum_{k=0}^{\infty} e^{sk} \frac{\lambda^k}{k!} \\
&= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^s)^k}{k!} \\
&= e^{-\lambda} e^{\lambda e^s} \quad \text{(Taylor series for } e^x\text{)} \\
&= e^{\lambda(e^s - 1)}, \quad \text{for all } s \in \mathbb{R}.
\end{aligned}
$$

As we discussed previously, the MGF uniquely determines the distribution. This is a