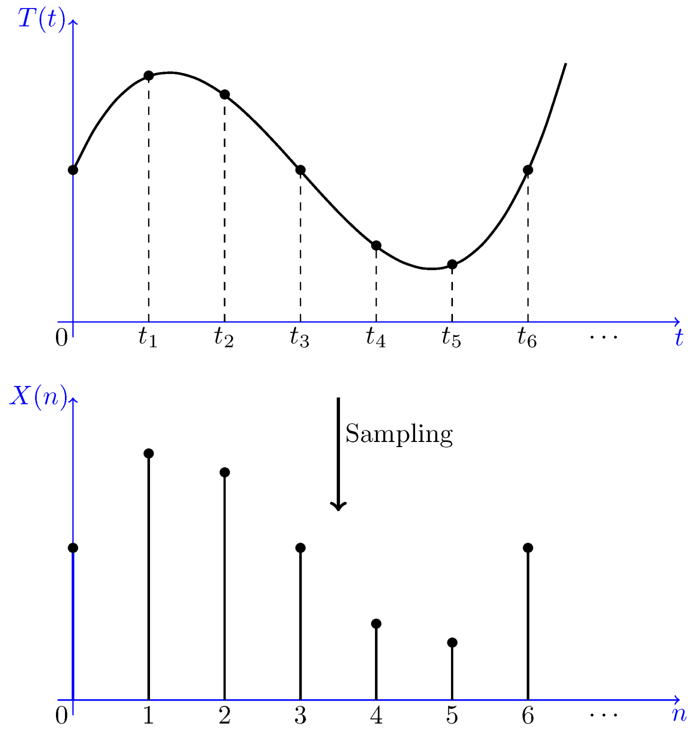

Figure 10.2 - Possible realization of the random process $\{X_n, n = 1, 2, 3, \cdots\}$ where X_n shows the temperature in New York City at noon on day n.

A **continuous-time** random process is a random process $\{X(t), t \in J\}$, where J is an interval on the real line such as $[-1, 1]$, $[0, \infty)$, $(-\infty, \infty)$, etc.

A **discrete-time** random process (or a **random sequence**) is a random process $\{X(n) = X_n, n \in J\}$, where J is a countable set such as $\mathbb{N}$ or $\mathbb{Z}$.

Random Processes as Random Functions:

Consider a random process $\{X(t), t \in J\}$. This random process is resulted from a random experiment, e.g., observing the stock prices of a company over a period of time. Remember that any random experiment is defined on a sample space S. After observing the values of $X(t)$, we obtain a function of time such as the one showed in Figure 10.1. The function shown in this figure is just one of the many possible outcomes of this random experiment. We call each of these possible functions of $X(t)$ a **sample function** or **sample path**. It is also called a **realization** of $X(t)$.
From this point of view, a random process can be thought of as a random function of time. You are familiar with the concept of functions. The difference here is that $\{X(t), t \in J\}$ will be equal to one of many possible sample functions after we are done with our random experiment. In engineering applications, random processes are often referred to as ***random signals***.

> A random process is a random function of time.

Example 10.1
You have 1000 dollars to put in an account with interest rate R, compounded annually. That is, if X_n is the value of the account at year n, then

$$X_n = 1000(1+R)^n, \quad \text{for } n = 0, 1, 2, \cdots.$$

The value of R is a random variable that is determined when you put the money in the bank, but it does not not change after that. In particular, assume that $R \sim Uniform(0.04, 0.05)$.

a. Find all possible sample functions for the random process $\{X_n, n = 0, 1, 2, \ldots\}$.
b. Find the expected value of your account at year three. That is, find $E[X_3]$.

Solution

a. Here, the randomness in X_n comes from the random variable R. As soon as you know R, you know the entire sequence X_n for $n = 0, 1, 2, \cdots$. In particular, if $R = r$, then

$$X_n = 1000(1+r)^n, \quad \text{for all } n \in \{0, 1, 2, \cdots\}.$$

Thus, here sample functions are of the form $f(n) = 1000(1+r)^n$, $n = 0, 1, 2, \cdots$, where $r \in [0.04, 0.05]$. For any $r \in [0.04, 0.05]$, you obtain a sample function for the random process X_n.

b. The random variable X_3 is given by

$$X_3 = 1000(1+R)^3.$$

If you let $Y = 1 + R$, then $Y \sim Uniform(1.04, 1.05)$, so

$$f_Y(y) = \begin{cases} 100 & 1.04 \leq y \leq 1.05 \\ 0 & \text{otherwise} \end{cases}$$

To obtain $E[X_3]$, we can write

$$\begin{aligned} E[X_3] &= 1000E[Y^3] \\ &= 1000 \int_{1.04}^{1.05} 100y^3 \quad dy \quad \text{(by LOTUS)} \\ &= \frac{10^5}{4} \bigg[y^4\bigg]_{1.04}^{1.05} \\ &= \frac{10^5}{4} \bigg[(1.05)^4 - (1.04)^4\bigg] \\ &\approx 1,141.2 \end{aligned}$$

Example 10.2

Let $\{X(t), t \in [0, \infty)\}$ be defined as

$$X(t) = A + Bt, \qquad \text{for all } t \in [0, \infty),$$

where A and B are independent normal $N(1, 1)$ random variables.

a. Find all possible sample functions for this random process.
b. Define the random variable $Y = X(1)$. Find the PDF of Y.
c. Let also $Z = X(2)$. Find $E[YZ]$.

Solution

a. Here, we note that the randomness in $X(t)$ comes from the two random variables A and B. The random variable A can take any real value $a \in \mathbb{R}$. The random variable B can also take any real value $b \in \mathbb{R}$. As soon as we know the values of A and B, the entire process $X(t)$ is known. In particular, if $A = a$ and $B = b$, then

$$X(t) = a + bt, \qquad \text{for all } t \in [0, \infty).$$

Thus, here, sample functions are of the form $f(t) = a + bt$, $t \geq 0$, where $a, b \in \mathbb{R}$. For any $a, b \in \mathbb{R}$ you obtain a sample function for the random process $X(t)$.

b. We have

$$Y = X(1) = A + B.$$

Since A and B are independent $N(1,1)$ random variables, $Y = A + B$ is also normal with

$$\begin{aligned} EY &= E[A + B] \\ &= E[A] + E[B] \\ &= 1 + 1 \\ &= 2, \end{aligned}$$

$$\begin{aligned} \mathrm{Var}(Y) &= \mathrm{Var}(A + B) \\ &= \mathrm{Var}(A) + \mathrm{Var}(B) \quad (\text{since } A \text{ and } B \text{ are independent}) \\ &= 1 + 1 \\ &= 2. \end{aligned}$$

Thus, we conclude that $Y \sim N(2, 2)$:

$$f_Y(y) = \frac{1}{\sqrt{4\pi}} e^{-\frac{(y-2)^2}{4}}.$$

c. We have

$$\begin{aligned} E[YZ] &= E[(A + B)(A + 2B)] \\ &= E[A^2 + 3AB + 2B^2] \\ &= E[A^2] + 3E[AB] + 2E[B^2] \\ &= 2 + 3E[A]E[B] + 2 \cdot 2 \quad (\text{since } A \text{ and } B \text{ are independent}) \\ &= 9. \end{aligned}$$

The random processes in the above examples were relatively simple in the sense that the randomness in the process originated from one or two random variables. We will see more complicated examples later on.