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### 1.3.6 Solved Problems: Random Experiments and Probabilities

#### Problem 1

Consider a sample space  $S$  and three events  $A$ ,  $B$ , and  $C$ . For each of the following events draw a Venn diagram representation as well as a set expression.

- Among  $A$ ,  $B$ , and  $C$ , only  $A$  occurs.
- At least one of the events  $A$ ,  $B$ , or  $C$  occurs.
- $A$  or  $C$  occurs, but not  $B$ .
- At most two of the events  $A$ ,  $B$ , or  $C$  occur.

#### Solution

- Among  $A$ ,  $B$ , and  $C$ , only  $A$  occurs:  $A - B - C = A - (B \cup C)$ .
- At least one of the events  $A$ ,  $B$ , or  $C$  occurs:  $A \cup B \cup C$ .
- $A$  or  $C$  occurs, but not  $B$ :  $(A \cup C) - B$ .
- At most two of the events  $A$ ,  $B$ , or  $C$  occur:  $(A \cap B \cap C)^c = A^c \cup B^c \cup C^c$ .

The Venn diagrams are shown in Figure 1.19.

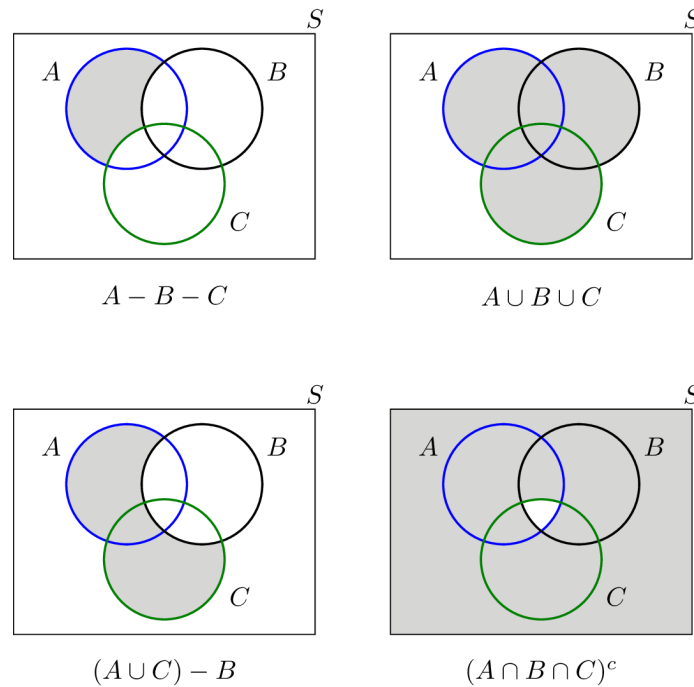


Fig.1.19 - Venn diagrams for solved problem 1.

## Problem 2

Write the sample space  $S$  for the following random experiments.

- We toss a coin until we see two consecutive tails. We record the total number of coin tosses.
- A bag contains 4 balls: one is red, one is blue, one is white, and one is green. We choose two distinct balls and record their color in order.
- A customer arrives at a bank and waits in the line. We observe  $T$ , which is the total time (in hours) that the customer waits in the line. The bank has a strict policy that no customer waits more than 20 minutes under any circumstances.

## Solution

Remember that the sample space is the set of all possible outcomes. Usually, when you have a random experiment, there are different ways to define the sample space  $S$  depending on what you observe as the outcome. In this problem, for each experiment

it is stated what outcomes we observe in order to help you write down the sample space  $S$ .

- a. We toss a coin until we see two consecutive tails. We record the total number of coin tosses: Here, the total number of coin tosses is a natural number larger than or equal to 2. The sample space is

$$S = \{2, 3, 4, \dots\}.$$

- b. A bag contains 4 balls: one is red, one is blue, one is white, and one is green. We choose two distinct balls and record their color in order: The sample space can be written as

$$S = \{(R, B), (B, R), (R, W), (W, R), (R, G), (G, R), \\ (B, W), (W, B), (B, G), (G, B), (W, G), (G, W)\}.$$

- c. A customer arrives at a bank and waits in the line. We observe  $T$ ...: In theory  $T$  can be any real number between 0 and  $\frac{1}{3} = 20$  minutes. Thus,

$$S = \left[0, \frac{1}{3}\right] = \left\{x \in \mathbb{R} \mid 0 \leq x \leq \frac{1}{3}\right\}.$$

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### Problem 3

Let  $A$ ,  $B$ , and  $C$  be three events in the sample space  $S$ . Suppose we know

- $A \cup B \cup C = S$ ,
- $P(A) = \frac{1}{2}$ ,
- $P(B) = \frac{2}{3}$ ,
- $P(A \cup B) = \frac{5}{6}$ .

Answer the following questions:

- Find  $P(A \cap B)$ .
- Do  $A$ ,  $B$ , and  $C$  form a partition of  $S$ ?
- Find  $P(C - (A \cup B))$ .
- If  $P(C \cap (A \cup B)) = \frac{5}{12}$ , find  $P(C)$ .

**Solution**

As before, it is always useful to draw a Venn diagram; however, here we provide the solution without using a Venn diagram.

a. Using the inclusion-exclusion principle, we have

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Thus,

$$\begin{aligned} P(A \cap B) &= P(A) + P(B) - P(A \cup B) \\ &= \frac{1}{2} + \frac{2}{3} - \frac{5}{6} \\ &= \frac{1}{3}. \end{aligned}$$

b. No, since  $A \cap B \neq \emptyset$ .

c. We can write

$$\begin{aligned} C - (A \cup B) &= \left( C \cup (A \cup B) \right) - (A \cup B) \\ &= S - (A \cup B) \\ &= (A \cup B)^c. \end{aligned}$$

Thus

$$\begin{aligned} P(C - (A \cup B)) &= P((A \cup B)^c) \\ &= 1 - P(A \cup B) \\ &= \frac{1}{6}. \end{aligned}$$

d. We have

$$P(C) = P(C \cap (A \cup B)) + P(C - (A \cup B)) = \frac{5}{12} + \frac{1}{6} = \frac{7}{12}.$$

#### Problem 4

I roll a fair die twice and obtain two numbers  $X_1$  = result of the first roll, and  $X_2$  = result of the second roll. Find the probability of the following events:

- $A$  defined as " $X_1 < X_2$ ";
- $B$  defined as "You observe a 6 at least once".

### Solution

As we saw before, the sample space  $S$  has 36 elements.

a. We have

$$A = \{(1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 3), (2, 4), (2, 5), \\ (2, 6), (3, 4), (3, 5), (3, 6), (4, 5), (4, 6), (5, 6)\}.$$

Then, we obtain

$$P(A) = \frac{|A|}{|S|} = \frac{15}{36} = \frac{5}{12}.$$

b. We have

$$B = \{(6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6), (1, 6), (2, 6), (3, 6), (4, 6), (5, 6)\}.$$

We obtain

$$P(B) = \frac{|B|}{|S|} = \frac{11}{36}.$$

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### Problem 5

You purchase a certain product. The manual states that the lifetime  $T$  of the product, defined as the amount of time (in years) the product works properly until it breaks down, satisfies

$$P(T \geq t) = e^{-\frac{t}{5}} \text{ for all } t \geq 0.$$

For example, the probability that the product lasts more than (or equal to) 2 years is  $P(T \geq 2) = e^{-\frac{2}{5}} = 0.6703$ .

- This is an example of a continuous probability model. Write down the sample space  $S$ .
- Check that the statement in the manual makes sense by finding  $P(T \geq 0)$  and  $\lim_{t \rightarrow \infty} P(T \geq t)$ .
- Also check that if  $t_1 < t_2$ , then  $P(T \geq t_1) \geq P(T \geq t_2)$ . Why does this need to be true?

- d. Find the probability that the product breaks down within three years of the purchase time.
- e. Find the probability that the product breaks down in the second year, i.e., find  $P(1 \leq T < 2)$ .

### Solution

- a. The sample space  $S$  is the set of all possible outcomes. Here, the possible outcomes are the possible values for  $T$  which can be any real number larger than or equal to zero. Thus

$$S = [0, \infty).$$

- b. We have

$$P(T \geq 0) = e^{-\frac{0}{5}} = 1,$$

$$\lim_{t \rightarrow \infty} P(T \geq t) = e^{-\infty} = 0,$$

which is what we expect. In particular,  $T$  is always larger than or equal to zero, thus we expect  $P(T \geq 0) = 1$ . Also, since the product will eventually fail at some point, we expect that  $P(T \geq t)$  approaches zero as  $t$  goes to infinity.

- c. First note that if  $t_1 < t_2$ , then  $P(T \geq t_1) = e^{-\frac{t_1}{5}} > e^{-\frac{t_2}{5}} = P(T \geq t_2)$  (since  $f(x) = e^{(x)}$  is an increasing function). Here we have two events,  $A$  is the event that  $T \geq t_1$  and  $B$  is the event that  $T \geq t_2$ . That is,

$$A = [t_1, \infty), B = [t_2, \infty).$$

Since  $B$  is a subset of  $A$ ,  $B \subset A$ , we must have  $P(B) \leq P(A)$ , thus

$$P(A) = P(T \geq t_1) \geq P(T \geq t_2) = P(B).$$

- d. The probability that the product breaks down within three years of the purchase time is

$$P(T < 3) = 1 - P(T \geq 3) = 1 - e^{-\frac{3}{5}} \approx 0.4512$$

- e. Note that if  $A \subset B$ , then

$$P(B - A) = P(B) - P(B \cap A)$$

$$= P(B) - P(A) \quad (\text{since } A \subset B).$$

Choosing  $A = [1, \infty)$  and  $B = [2, \infty)$ , we can write

$$\begin{aligned}P(1 \leq T < 2) &= P(T \geq 1) - P(T \geq 2) \\&= e^{-\frac{1}{5}} - e^{-\frac{2}{5}} \\&= 0.1484\end{aligned}$$

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### Problem 6

I first saw this question in a math contest many years ago: You get a stick and break it randomly into three pieces. What is the probability that you can make a triangle using the three pieces? You can assume the break points are chosen completely at random, i.e. if the length of the original stick is 1 unit, and  $x, y, z$  are the lengths of the three pieces, then  $(x, y, z)$  are uniformly chosen from the set

$$\{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 1, x, y, z \geq 0\}.$$

#### Solution

This is again a problem on a continuous probability space. The basic idea is pretty simple. First, we need to identify the sample space  $S$ . In this case the sample space is going to be a two-dimensional set. Second, we need to identify the set  $A$  that contains the favorable outcomes (the set of  $(x, y, z)$  in  $S$  that form a triangle). And finally, since the space is uniform, we will divide area of set  $A$  by the area of  $S$  to obtain  $P(A)$ .

First, we need to find the sets  $S$  and  $A$ . This is basically a geometry problem. The two sets,  $S$  and  $A$ , are shown in Figure 1.20.

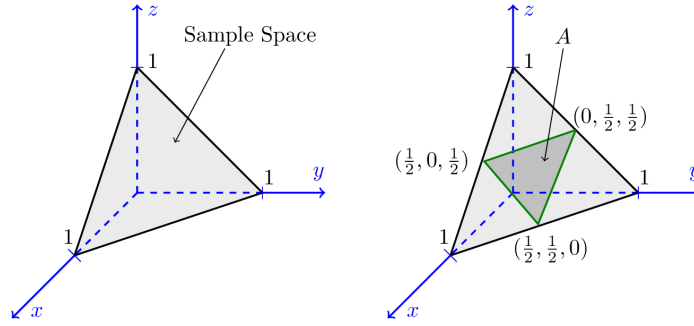


Fig.1.20 - The sample space and set  $A$  for Problem 6.

Note that in  $\mathbb{R}^3$ ,  $x + y + z = 1$  represents a plane that goes through the points  $(1, 0, 0)$ ,  $(0, 1, 0)$ ,  $(0, 0, 1)$ . To find the sample space  $S$ , note that  $S = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 1, x, y, z \geq 0\}$ , thus  $S$  is the part of the plane that is shown in Figure 1.20.

To find the set  $A$ , note that we need  $(x, y, z)$  to satisfy the triangle inequality

$$x + y > z,$$

$$y + z > x,$$

$$x + z > y.$$

Note that since  $x + y + z = 1$ , we can equivalently write the three equations as

$$x < \frac{1}{2},$$

$$y < \frac{1}{2},$$

$$z < \frac{1}{2}.$$

Thus, we conclude that the set  $A$  is the area shown in Figure 20. In particular, we note that the set  $S$  consists of four triangles with equal areas. Therefore, its area is four times the area of  $A$ , and we have

$$P(A) = \frac{\text{Area of } A}{\text{Area of } S} = \frac{1}{4}.$$