6.2.6 Solved Problems

Problem 1

Your friend tells you that he had four job interviews last week. He says that based on how the interviews went, he thinks he has a 20% chance of receiving an offer from each of the companies he interviewed with. Nevertheless, since he interviewed with four companies, he is 90% sure that he will receive at least one offer. Is he right?

Solution

Let A_i be the event that your friend receives an offer from the ith company, i=1,2,3,4. Then, by the union bound:

$$\begin{aligned} P\left(\bigcup_{i=1}^{4} A_i\right) &\leq \sum P(A_i) \\ &= 0.2 + 0.2 + 0.2 + 0.2 \\ &= 0.8 \end{aligned}$$

Thus the probability of receiving at least one offer is less than or equal to 80%.

Problem 2

An isolated edge in a network is an edge that connects two nodes in the network such that neither of the two nodes is connected to any other nodes in the network. Let C_n be the event that a graph randomly generated according to $G(n, p)$ model has at least one isolated edge.

a. Show that

$$P(C_n) \leq \binom{n}{2} p(1-p)^{2(n-2)}$$

b. Show that, for any constant $b > \frac{1}{2}$, if $p = p_n = b\frac{\ln(n)}{n}$ then

$$\lim_{n\to\infty} P(C_n) = 0.$$

Solution

There are $\binom{n}{2}$ possible edges in the graph. Let E_i be the event that the ith edge is an isolated edge, then

$$P(E_i) = p(1-p)^{2(n-2)},$$

where p in the above equation is the probability that the ith edge is present and $(1-p)^{2(n-2)}$ is the probability that no other nodes are connected to this edge. By the union bound, we have

$$\begin{aligned} P(C_n) &= P\left(\bigcup E_i\right) \\ &\leq \sum_i P(E_i) \\ &= \binom{n}{2} p(1-p)^{2(n-2)}, \end{aligned}$$

which is the desired result. Now, let $p = b\frac{\ln n}{n}$, where $b > \frac{1}{2}$.

Here, it is convenient to use the following inequality:

$$1 - x \leq e^{-x}, \quad \text{for all} \quad x \in \mathbb{R}.$$

You can prove it by differentiating $f(x) = e^{-x} + x - 1$, and showing that the minimum occurs at $x = 0$.

Now, we can write

$$\begin{aligned} P(C_n) &= \binom{n}{2} p(1-p)^{2(n-2)} \\ &= \frac{n(n-1)}{2} \frac{b \ln n}{n} (1-p)^{2(n-2)} \\ &\leq \frac{(n-1)b}{2} e^{-2p(n-2)} \quad (\text{using} \quad 1 - x \leq e^{-x}) \\ &= \frac{(n-1)}{2} b e^{-2\frac{b \ln n}{n}(n-2)}. \end{aligned}$$

Thus,

$$\begin{aligned} \lim_{n \to \infty} P(C_n) &\leq \lim_{n \to \infty} \frac{(n-1)}{2} b e^{-2\frac{b \ln n}{n}(n-2)} \\ &= \lim_{n \to \infty} \frac{(n-1)}{2} b n^{-2b} \\ &= \frac{b}{2} \lim_{n \to \infty} (n^{1-2b}) \\ &= 0 \quad (\text{since} \quad b > \tfrac{1}{2}). \end{aligned}$$

Problem 3

Let $X \sim Exponential(\lambda)$. Using Markov's inequality find an upper bound for $P(X \geq a)$. Compare the upper bound with the actual value of $P(X \geq a)$.

Solution

If $X \sim Exponential(\lambda)$, then $EX = \frac{1}{\lambda}$, using Markov's inequality

$$P(X \geq a) \leq \frac{EX}{a} = \frac{1}{\lambda a}.$$

The actual value of $P(X \geq a)$ is $e^{-\lambda a}$, and we always have $\frac{1}{\lambda a} \geq e^{-\lambda a}$.

Problem 4

Let $X \sim Exponential(\lambda)$. Using Chebyshev's inequality find an upper bound for $P(|X - EX| \geq b)$.

Solution

a. We have $EX = \frac{1}{\lambda}$ and $VarX = \frac{1}{\lambda^2}$. Using Chebyshev's inequality, we have

$$\begin{aligned} P(|X - EX| \geq b) &\leq \frac{Var(X)}{b^2} \\ &= \frac{1}{\lambda^2 b^2}. \end{aligned}$$

Problem 5

Let $X \sim Exponential(\lambda)$. Using Chernoff bounds find an upper bound for $P(X \geq a)$, where $a > EX$. Compare the upper bound with the actual value of $P(X \geq a)$.

Solution

If $X \sim Exponential(\lambda)$, then

$$M_X(s) = \frac{\lambda}{\lambda - s}, \quad \text{for} \quad s < \lambda.$$

Using Chernoff bounds, we have

$$\begin{aligned} P(X \geq a) &\leq \min_{s>0} \left[e^{-sa} M_X(s)\right] \\ &= \min_{s>0} \left[e^{-sa} \frac{\lambda}{\lambda - s}\right]. \end{aligned}$$

If $f(s) = e^{-sa} \frac{\lambda}{\lambda - s}$, to find $\min_{s>0} f(s)$ we write

$$\frac{d}{ds} f(s) = 0.$$

Therefore,

$$s^* = \lambda - \frac{1}{a}.$$

Note since $a > EX = \frac{1}{\lambda}$, then $\lambda - \frac{1}{a} > 0$. Thus,

$$P(X \geq a) \leq e^{-s^* a} \frac{\lambda}{\lambda - s^*} = a\lambda e^{1-\lambda a}.$$

The real value of $P(X \geq a)$ is $e^{-\lambda a}$ and we have $e^{-\lambda a} \leq a\lambda e^{1-\lambda a}$, or equivalently, $a\lambda e \geq 1$, which is true since $a > \frac{1}{\lambda}$.

Problem 6

Let X and Y be two random variables with $EX = 1, Var(X) = 4$, and $EY = 2, Var(Y) = 1$. Find the maximum possible value for $E[XY]$.

Solution

Using $\rho(X, Y) \leq 1$ and $\rho(X, Y) = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$, we conclude

$$\frac{EXY - EXEY}{\sigma_X \sigma_Y} \leq 1.$$

Thus

$$\begin{aligned} EXY &\leq \sigma_X \sigma_Y + EXEY \\ &= 2 \times 1 + 2 \times 1 \\ &= 4. \end{aligned}$$

In fact, we can achieve $EXY = 4$, if we choose $Y = aX + b$.

$$Y = aX + b \quad \Rightarrow \quad \begin{cases} 2 = a + b \\ 1 = (a^2)(4) \end{cases}$$

Solving for a and b, we obtain

$$a = \frac{1}{2}, \quad b = \frac{3}{2}.$$

Note that if you use the Cauchy-Schwarz inequality directly, you obtain:

$$\begin{aligned} |EXY|^2 &\leq EX^2 \cdot EY^2 \\ &= 5 \times 5. \end{aligned}$$

Thus

$$EXY \leq 5.$$

But $EXY = 5$ cannot be achieved because equality in the Cauchy-Schwarz is obtained only when $Y = \alpha X$. But here this is not possible.

Problem 7
(Hölder's Inequality) Prove

$$E[|XY|] \leq E\left[|X|^p\right]^{\frac{1}{p}} E\left[|Y|^q\right]^{\frac{1}{q}},$$

where $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Note that, for $p = q = \frac{1}{2}$, Hölder's ineqality becomes the Cauchy-Schwarz inequality. *Hint:* You can use Young's inequality [4] which states that for nonnegative real numbers α and β and integers p and q such that $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\alpha\beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q},$$

with equality only if $\alpha^p = \beta^q$.

Solution

Using Young's inequality, we conclude that for random variables U and V we have

$$E|UV| \leq \frac{E|U|^p}{p} + \frac{E|V|^q}{q}.$$

Choose $U = \frac{|X|}{(E|X|^p)^{\frac{1}{p}}}$ and $V = \frac{|Y|}{(E|Y|^q)^{\frac{1}{q}}}$. We obtain

$$\begin{aligned} \frac{E|XY|}{(E|X|^p)^{\frac{1}{p}}(E|Y|^q)^{\frac{1}{q}}} &\leq \frac{E|X|^p}{pE|X|^p} + \frac{E|Y|^q}{qE|Y|^q} \\ &= \frac{1}{p} + \frac{1}{q} \\ &= 1. \end{aligned}$$

Problem 8

Show that if $h : \mathbb{R} \mapsto \mathbb{R}$ is convex and non-decreasing, and $g : \mathbb{R} \mapsto \mathbb{R}$ is convex, then $h(g(x))$ is a convex function.

Solution

Since g is convex, we have

$$g(\alpha x + (1-\alpha)y) \leq \alpha g(x) + (1-\alpha)g(y), \quad \text{for all } \alpha \in [0,1].$$

Therefore, we have

$$\begin{aligned} h(g(\alpha x + (1-\alpha)y)) &\leq h(\alpha g(x) + (1-\alpha)g(y)) \quad \text{(h is non-decreasing)} \\ &\leq \alpha h(g(x)) + (1-\alpha)h(g(y)) \quad \text{(h is convex).} \end{aligned}$$

Problem 9

Let X be a positive random variable with $EX = 10$. What can you say about the following quantities?

a. $E\left[\frac{1}{X+1}\right]$

b. $E\left[e^{\frac{1}{X+1}}\right]$

c. $E[\ln \sqrt{X}]$

Solution

a.

$$\begin{aligned} g(x) &= \frac{1}{x+1}, \\ g''(x) &= \frac{2}{(1+x)^3} > 0, \quad \text{for } x > 0. \end{aligned}$$

Thus g is convex on $(0,\infty)$

$$\begin{aligned} E\left[\frac{1}{X+1}\right] &\geq \frac{1}{1+EX} \quad \text{(Jensen's inequality)} \\ &= \frac{1}{1+10} \\ &= \frac{1}{11}. \end{aligned}$$

b. If we let $h(x)=e^x, g(x)=\frac{1}{1+x}$ then h is convex and non-decreasing and g is convex thus by problem 8, $e^{\frac{1}{x+1}}$ is a convex function, thus

$$\begin{aligned} E\left[e^{\frac{1}{1+X}}\right] &\geq e^{\frac{1}{1+EX}} \quad \text{(by Jensen's inequality)} \\ &= e^{\frac{1}{11}}. \end{aligned}$$

c. If $g(x)=\ln\sqrt{x}=\frac{1}{2}\ln x$, then $g^{'}(x)=\frac{1}{2x}$ for $x>0$ and $g^{''}(x)=-\frac{1}{2x^2}$. Thus g is concave on $(0,\infty)$. We conclude

$$\begin{aligned} E\left[\ln\sqrt{X}\right] &= E\left[\frac{1}{2}\ln X\right] \\ &\leq \frac{1}{2}\ln EX \quad \text{(by Jensen's inequality)} \\ &= \frac{1}{2}\ln 10. \end{aligned}$$