country.

What if the sample is not from a normal distribution? In the case that n is large, we can say that

$$W(X_1, X_2, \cdots, X_n) = \frac{\overline{X} - \mu_0}{S/\sqrt{n}}$$

is approximately standard normal. Therefore, we accept $H_0: \mu = \mu_0$ if

$$\left| \frac{\overline{X} - \mu_0}{S/\sqrt{n}} \right| \leq z_{\frac{\alpha}{2}},$$

and reject it otherwise (i.e., accept $H_1: \mu \neq \mu_0$).

Let us summarize what we have obtained for the two-sided test for the mean.

Table 8.2: Two-sided hypothesis testing for the mean: $H_0: \mu = \mu_0$, H_1: $\mu \neq \mu_0$.

Case	Test Statistic	Acceptance Region
$X_i \sim N(\mu, \sigma^2)$, σ known	$W = \frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}}$	$\lvert W\rvert \leq z_{\frac{\alpha}{2}}$
n large, X_i non-normal	$W = \frac{\overline{X} - \mu_0}{S/\sqrt{n}}$	$\lvert W\rvert \leq z_{\frac{\alpha}{2}}$
$X_i \sim N(\mu, \sigma^2)$, σ unknown	$W = \frac{\overline{X} - \mu_0}{S/\sqrt{n}}$	$\lvert W\rvert \leq t_{\frac{\alpha}{2}, n-1}$

One-sided Tests for the Mean:

We can provide a similar analysis when we have a one-sided test. Let's show this by an example.

Example 8.28

Let X_1,X_2,...,X_n be a random sample from a $N(\mu, \sigma^2)$ distribution, where μ is unknown and σ is known. Design a level α test to choose between

H_0: $\mu \leq \mu_0$,

H_1: $\mu > \mu_0$.

Solution

As before, we define the test statistic as

$$W(X_1, X_2, \cdots, X_n) = \frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}}.$$

If H_0 is true (i.e., $\mu \leq \mu_0$), we expect $\overline{X}$ (and thus W) to be relatively small, while if H_1 is true, we expect $\overline{X}$ (and thus W) to be larger. This suggests the following test: Choose a threshold, and call it c. If $W \leq c$, accept H_0, and if $W > c$, accept H_1. How do we choose c? If α is the required significance level, we must have

$$\begin{aligned} P(\text{type I error}) &= P(\text{Reject } H_0 \mid H_0) \\ &= P(W > c \mid \mu \leq \mu_0) \leq \alpha. \end{aligned}$$

Here, the probability of type I error depends on μ. More specifically, for any $\mu_0 \leq \mu$, we can write

$$\begin{aligned} P(\text{type I error} \mid \mu) &= P(\text{Reject } H_0 \mid \mu) \\ &= P(W > c \mid \mu) \\ &= P\left(\frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}} > c \mid \mu\right) \\ &= P\left(\frac{\overline{X} - \mu}{\sigma/\sqrt{n}} + \frac{\mu - \mu_0}{\sigma/\sqrt{n}} > c \mid \mu\right) \\ &= P\left(\frac{\overline{X} - \mu}{\sigma/\sqrt{n}} > c + \frac{\mu_0 - \mu}{\sigma/\sqrt{n}} \mid \mu\right) \\ &\leq P\left(\frac{\overline{X} - \mu}{\sigma/\sqrt{n}} > c \mid \mu\right) \quad (\text{ since } \mu \leq \mu_0) \\ &= 1 - \Phi(c) \quad \left(\text{since given } \mu, \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)\right). \end{aligned}$$

Thus, we can choose $\alpha = 1 - \Phi(c)$, which results in

$$c = z_\alpha.$$

Therefore, we accept H_0 if

$$\frac{\overline{X}-\mu_0}{\sigma/\sqrt{n}} \leq z_\alpha,$$

and reject it otherwise.

The above analysis can be repeated for other cases. More generally, suppose that we are given a random sample X_1,X_2,...,X_n from a distribution. Let $\mu = EX_i$. Our goal is to decide between

H_0: $\mu \leq \mu_0$,

H_1: $\mu > \mu_0$.

We define the test statistic as before, i.e., we define W as

$$W(X_1, X_2, \cdots, X_n) = \frac{\overline{X}-\mu_0}{\sigma/\sqrt{n}},$$

if $\sigma = \sqrt{\mathrm{Var}(X_i)}$ is known, and as

$$W(X_1, X_2, \cdots, X_n) = \frac{\overline{X}-\mu_0}{S/\sqrt{n}},$$

if σ is unknown. If H_0 is true (i.e., $\mu \leq \mu_0$), we expect that $\overline{X}$ (and thus W) to be relatively small, while if H_1 is true, we expect $\overline{X}$ (and thus W) to be larger. This suggests the following test: Choose a threshold c. If $W \leq c$, accept H_0, and if $W > c$, accept H_1. To choose c, note that

$$\begin{aligned} P(\text{type I error}) &= P(\text{Reject } H_0 \mid H_0) \\ &= P(W > c \mid \mu \leq \mu_0) \\ &\leq P(W > c \mid \mu = \mu_0). \end{aligned}$$

Note that the last inequality resulted because if we make μ larger, the probability of $W > c$ can only increase. In other words, we assumed the worst case scenario, i.e, $\mu = \mu_0$ for the probability of error. Thus, we can choose c such that $P(W > c \mid \mu = \mu_0) = c$. By doing this procedure, we obtain the acceptance regions reflected in Table 8.3.

Table 8.3: One-sided hypothesis testing for the mean: $H_0 : \mu \leq \mu_0$, H_1: $\mu > \mu_0$.

Case	**Test Statistic**	**Acceptance Region**

$X_i \sim N(\mu, \sigma^2)$, σ known	$W = \frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}}$	$W \leq z_\alpha$
n large, X_i non-normal	$W = \frac{\overline{X} - \mu_0}{S/\sqrt{n}}$	$W \leq z_\alpha$
$X_i \sim N(\mu, \sigma^2)$, σ unknown	$W = \frac{\overline{X} - \mu_0}{S/\sqrt{n}}$	$W \leq t_{\alpha, n-1}$

Note that the tests mentioned in Table 8.3 remain valid if we replace the null hypothesis by $\mu = \mu_0$. The reason for this is that in choosing the threshold c, we assumed the worst case scenario, i.e, $\mu = \mu_0$. Finally, if we need to decide between

H_0: $\mu \geq \mu_0$,

H_1: $\mu < \mu_0$,

we can again repeat the above analysis and we obtain the acceptance regions reflected in Table 8.4.

Table 8.4: One-sided hypothesis testing for the mean: $H_0 : \mu \geq \mu_0$, H_1: $\mu < \mu_0$.

Case	**Test Statistic**	**Acceptance Region**
$X_i \sim N(\mu, \sigma^2)$, σ known	$W = \frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}}$	$W \geq -z_\alpha$
n large, X_i non-normal	$W = \frac{\overline{X} - \mu_0}{S/\sqrt{n}}$	$W \geq -z_\alpha$
$X_i \sim N(\mu, \sigma^2)$, σ unknown	$W = \frac{\overline{X} - \mu_0}{S/\sqrt{n}}$	$W \geq -t_{\alpha, n-1}$