7.3.0 End of Chapter Problems

Problem 1

Let X_i be i.i.d. $Uniform(0,1)$. We define the sample mean as

$$M_n = \frac{X_1 + X_2 + \ldots + X_n}{n}.$$

a. Find $E[M_n]$ and $\mathrm{Var}(M_n)$ as a function of n.
b. Using Chebyshev's inequality, find an upper bound on

$$P\left(\left|M_n - \frac{1}{2}\right| \geq \frac{1}{100}\right).$$

c. Using your bound, show that

$$\lim_{n\to\infty} P\left(\left|M_n - \frac{1}{2}\right| \geq \frac{1}{100}\right) = 0.$$

Problem 2

The number of accidents in a certain city is modeled by a Poisson random variable with an average rate of 10 accidents per day. Suppose that the number of accidents on different days are independent. Use the central limit theorem to find the probability that there will be more than 3800 accidents in a certain year. Assume that there are 365 days in a year.

Problem 3

In a communication system, each codeword consists of 1000 bits. Due to the noise, each bit may be received in error with probability 0.1. It is assumed bit errors occur independently. Since error correcting codes are used in this system, each codeword can be decoded reliably if there are less than or equal to 125 errors in the received codeword, otherwise the decoding fails. Using the CLT, find the probability of decoding failure.

Problem 4

50 students live in a dormitory. The parking lot has the capacity for 30 cars. Each student has a car with probability $\frac{1}{2}$, independently from other students. Use the CLT (with continuity correction) to find the probability that there won't be enough parking spaces for all the cars.

Problem 5

The amount of time needed for a certain machine to process a job is a random variable with mean $EX_i = 10$ minutes and $\mathrm{Var}(X_i) = 2$ minutes^2. The times needed for different jobs are independent from each other. Find the probability that the machine processes less than or equal to 40 jobs in 7 hours.

Problem 6

You have a fair coin. You toss the coin n times. Let X be the portion of times that you observe heads. How large n has to be so that you are 95% sure that $0.45 \leq X \leq 0.55$? In other words, how large n has to be so that

$$P\big(0.45 \leq X \leq 0.55\big) \geq .95 \qquad ?$$

Problem 7

An engineer is measuring a quantity q. It is assumed that there is a random error in each measurement, so the engineer will take n measurements and reports the average of the measurements as the estimated value of q. Specifically, if Y_i is the value that is obtained in the i'th measurement, we assume that

$$Y_i = q + X_i,$$

where X_i is the error in the ith measurement. We assume that X_i's are i.i.d. with $EX_i = 0$ and $\mathrm{Var}(X_i) = 4$ units. The engineer reports the average of measurements

$$M_n = \frac{Y_1 + Y_2 + \ldots + Y_n}{n}.$$

How many measurements does the engineer need to make until he is 95% sure that the final error is less than 0.1 units? In other words, what should the value of n be such that

$$P\big(q - 0.1 \leq M_n \leq q + 0.1\big) \geq 0.95 \qquad ?$$

Problem 8

Let $X_2, X_3, X_4, \cdots$ be a sequence of random variables such that

$$F_{X_n}(x) = \begin{cases} \frac{e^{n(x-1)}}{1+e^{n(x-1)}} & x > 0 \\ \\ 0 & \text{otherwise} \end{cases}$$

Show that X_n converges in distribution to $X = 1$.

Problem 9

Let $X_2, X_3, X_4, \cdots$ be a sequence of random variables such that

$$F_{X_n}(x) = \begin{cases} \frac{e^{nx}+xe^{n}}{e^{nx}+\left(\frac{n+1}{n}\right)e^{n}} & 0 \leq x \leq 1 \\ \\ \frac{e^{nx}+e^{n}}{e^{nx}+\left(\frac{n+1}{n}\right)e^{n}} & x > 1 \end{cases}$$

Show that X_n converges in distribution to $Uniform(0, 1)$.

Problem 10

Consider a sequence $\{X_n, n = 1, 2, 3, \cdots\}$ such that

$$X_n = \begin{cases} n & \text{with probability } \frac{1}{n^2} \\ \\ 0 & \text{with probability } 1 - \frac{1}{n^2} \end{cases}$$

Show that

a. $X_n \xrightarrow{p} 0$.
b. $X_n \xrightarrow{L^r} 0$, for $r < 2$.
c. X_n does not converge to 0 in the rth mean for any $r \geq 2$.
d. $X_n \xrightarrow{a.s.} 0$.

Problem 11

We perform the following random experiment. We put $n \geq 10$ blue balls and n red balls in a bag. We pick 10 balls at random (without replacement) from the bag. Let X_n be the number of blue balls. We perform this experiment for $n = 10, 11, 12, \cdots$. Prove that $X_n \xrightarrow{d} Binomial\left(10, \frac{1}{2}\right)$.

Problem 12

Find two sequences of random variables $\{X_n, n = 1, 2, \cdots\}$ and $\{Y_n, n = 1, 2, \cdots\}$ such that

$$X_n \xrightarrow{d} X,$$
$$\text{and}$$
$$Y_n \xrightarrow{d} Y,$$

but $X_n + Y_n$ does not converge in distribution to $X + Y$.

Problem 13

Let X_1, X_2, X_3, $\cdots$ be a sequence of continuous random variable such that

$$f_{X_n}(x) = \frac{n}{2} e^{-n|x|}.$$

Show that X_n converges in probability to 0.

Problem 14

Let X_1, X_2, X_3, $\cdots$ be a sequence of continuous random variable such that

$$f_{X_n}(x) = \begin{cases} \frac{1}{nx^2} & x > \frac{1}{n} \\ \\ 0 & \text{otherwise} \end{cases}$$

Show that X_n converges in probability to 0.

Problem 15

Let Y_1, Y_2, Y_3, $\cdots$ be a sequence of i.i.d. random variables with mean $EY_i = \mu$ and finite variance $\mathrm{Var}(Y_i) = \sigma^2$. Define the sequence $\{X_n, n = 2, 3, \ldots\}$ as

$$X_n = \frac{Y_1Y_2 + Y_2Y_3 + \cdots Y_{n-1}Y_n + Y_nY_1}{n}, \qquad \text{for } n = 2, 3, \cdots.$$

Show that $X_n \xrightarrow{p} \mu^2$.

Problem 16

Let Y_1, Y_2, Y_3, $\cdots$ be a sequence of positive i.i.d. random variables with $0 < E[\ln Y_i] = \gamma < \infty$. Define the sequence $\{X_n, n = 1, 2, 3, \ldots\}$ as

$$X_n = (Y_1Y_2Y_3 \cdots Y_{n-1}Y_n)^{\frac{1}{n}}, \qquad \text{for } n = 1, 2, 3, \cdots.$$

Show that $X_n \xrightarrow{p} e^{\gamma}$.

Problem 17 Let X_1, X_2, X_3, $\cdots$ be a sequence of random variable such that

$$X_n \sim Poisson(n\lambda), \qquad \text{for } n = 1, 2, 3, \cdots,$$

where $\lambda > 0$ is a constant. Define a new sequence Y_n as

$$Y_n = \frac{1}{n}X_n, \qquad \text{for } n = 1, 2, 3, \cdots.$$

Show that Y_n converges in mean square to λ, i.e., $Y_n \xrightarrow{m.s.} \lambda$.

Problem 18

Let $\{X_n, n = 1, 2, \cdots\}$ and $\{Y_n, n = 1, 2, \cdots\}$ be two sequences of random variables, defined on the sample space S. Suppose that we know

$$X_n \xrightarrow{L^r} X,$$
$$Y_n \xrightarrow{L^r} Y.$$

Prove that $X_n + Y_n \xrightarrow{L^r} X + Y$. *Hint:* You may want to use Minkowski's inequality which states that for two random variables X and Y with finite moments, and $1 \leq p < \infty$, we have

$$E\Big[|X+Y|^p\Big] \leq E\big[|X|^p\big]^{\frac{1}{p}} + E\big[|Y|^p\big]^{\frac{1}{p}}.$$

Problem 19

Let X_1, X_2, X_3, $\cdots$ be a sequence of random variable such that $X_n \sim Rayleigh(\frac{1}{n})$, i.e.,

$$f_{X_n}(x) = \begin{cases} n^2 x \exp\left\{-\frac{n^2x^2}{2}\right\} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Show that $X_n \xrightarrow{a.s.} 0$.

Problem 20

Let Y_1,Y_2, $\cdots$ be independent random variables, where $Y_n \sim Bernoulli\left(\frac{n}{n+1}\right)$ for $n = 1, 2, 3, \cdots$. We define the sequence $\{X_n, n = 2, 3, 4, \cdots\}$ as

$$X_{n+1} = Y_1 Y_2 Y_3 \cdots Y_n, \qquad \text{for } n = 1, 2, 3, \cdots.$$

Show that $X_n \xrightarrow{a.s.} 0$.