11.2.6 Stationary and Limiting Distributions

Here, we would like to discuss long-term behavior of Markov chains. In particular, we would like to know the fraction of times that the Markov chain spends in each state as n becomes large. More specifically, we would like to study the distributions

$$\pi^{(n)} = [\, P(X_n = 0) \quad P(X_n = 1) \quad \cdots \quad]$$

as $n \to \infty$. To better understand the subject, we will first look at an example and then provide a general analysis.

Example 11.12

Consider a Markov chain with two possible states, $S = \{0, 1\}$. In particular, suppose that the transition matrix is given by

$$P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix},$$

where a and b are two real numbers in the interval $[0, 1]$ such that $0 < a + b < 2$. Suppose that the system is in state 0 at time $n = 0$ with probability α, i.e.,

$$\pi^{(0)} = [\, P(X_0 = 0) \quad P(X_0 = 1)\,] = [\, \alpha \quad 1-\alpha\,],$$

where $\alpha \in [0, 1]$.

a. Using induction (or any other method), show that

$$P^n = \frac{1}{a+b}\begin{bmatrix} b & a \\ b & a \end{bmatrix} + \frac{(1-a-b)^n}{a+b}\begin{bmatrix} a & -a \\ -b & b \end{bmatrix}.$$

b. Show that

$$\lim_{n\to\infty} P^n = \frac{1}{a+b}\begin{bmatrix} b & a \\ b & a \end{bmatrix}.$$

c. Show that

$$\lim_{n\to\infty} \pi^{(n)} = \begin{bmatrix} \frac{b}{a+b} & \frac{a}{a+b} \end{bmatrix}.$$

Solution

a. For $n = 1$, we have

$$\begin{aligned} P^1 &= \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix} \\ &= \frac{1}{a+b}\begin{bmatrix} b & a \\ b & a \end{bmatrix} + \frac{1-a-b}{a+b}\begin{bmatrix} a & -a \\ -b & b \end{bmatrix}. \end{aligned}$$

Assuming that the statement of the problem is true for n, we can write P^{n+1} as

$$\begin{aligned} P^{n+1} = P^n P &= \frac{1}{a+b}\left(\begin{bmatrix} b & a \\ b & a \end{bmatrix} + (1-a-b)^n \begin{bmatrix} a & -a \\ -b & b \end{bmatrix}\right) \cdot \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix} \\ &= \frac{1}{a+b}\begin{bmatrix} b & a \\ b & a \end{bmatrix} + \frac{(1-a-b)^{n+1}}{a+b}\begin{bmatrix} a & -a \\ -b & b \end{bmatrix}, \end{aligned}$$

which completes the proof.

b. By assumption $0 < a+b < 2$, which implies $-1 < 1-a-b < 1$. Thus,

$$\lim_{n\to\infty} (1-a-b)^n = 0.$$

Therefore,

$$\lim_{n\to\infty} P^n = \frac{1}{a+b}\begin{bmatrix} b & a \\ b & a \end{bmatrix}.$$

c. We have

$$\begin{aligned} \lim_{n\to\infty} \pi^{(n)} &= \lim_{n\to\infty} \left[\pi^{(0)} P^n\right] \\ &= \pi^{(0)} \lim_{n\to\infty} P^n \\ &= \begin{bmatrix} \alpha & 1-\alpha \end{bmatrix} \cdot \frac{1}{a+b}\begin{bmatrix} b & a \\ b & a \end{bmatrix} \\ &= \begin{bmatrix} \frac{b}{a+b} & \frac{a}{a+b} \end{bmatrix}. \end{aligned}$$

In the above example, the vector

$$\lim_{n\to\infty} \pi^{(n)} = \begin{bmatrix} \frac{b}{a+b} & \frac{a}{a+b} \end{bmatrix}$$

is called the *limiting distribution* of the Markov chain. Note that the limiting distribution does not depend on the initial probabilities α and $1-\alpha$. In other words, the initial state (X_0) does not matter as n becomes large. Thus, for $i = 1, 2$, we can write

$$\lim_{n \rightarrow \infty} P(X_n = 0|X_0 = i) = \frac{b}{a+b},$$
$$\lim_{n \rightarrow \infty} P(X_n = 1|X_0 = i) = \frac{a}{a+b}.$$

Remember that we show $P(X_n = j|X_0 = i)$ by $P_{ij}^{(n)}$, which is the entry in the ith row and jth column of P^n.

Limiting Distributions

The probability distribution $\pi = [\pi_0, \pi_1, \pi_2, \cdots]$ is called the **limiting distribution** of the Markov chain X_n if

$$\pi_j = \lim_{n \rightarrow \infty} P(X_n = j|X_0 = i)$$

for all $i, j \in S$, and we have

$$\sum_{j \in S} \pi_j = 1.$$

By the above definition, when a limiting distribution exists, it does not depend on the initial state ($X_0 = i$), so we can write

$$\pi_j = \lim_{n \rightarrow \infty} P(X_n = j), \text{ for all } j \in S.$$

So far we have shown that the Markov chain in Example 11.12 has the following limiting distribution:

$$\pi = \begin{bmatrix} \pi_0 & \pi_1 \end{bmatrix} = \begin{bmatrix} \frac{b}{a+b} & \frac{a}{a+b} \end{bmatrix}.$$

Let's now look at mean return times for this Markov chain.

Example 11.13

Consider a Markov chain in Example 11.12: a Markov chain with two possible states, $S = \{0, 1\}$, and the transition matrix

$$P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix},$$

where a and b are two real numbers in the interval $[0, 1]$ such that $0 < a + b < 2$. Find the mean return times, r_0 and r_1, for this Markov chain.

Solution

We can use the method of the law of total probability that we explained before to find the mean return times (Example 11.11). We can also find r_0 and r_1 directly as follows:

Let R be the first return time to state 0, i.e., $r_0 = E[R|X_0 = 0]$. If $X_0 = 0$, then $X_1 = 0$ with probability $1 - a$, and $X_1 = 1$ with probability a. Thus, using the law of total probability, and assuming $X_0 = 0$, we can write

$$\begin{aligned} r_0 &= E[R|X_1 = 0]P(X_1 = 0) + E[R|X_1 = 1]P(X_1 = 1) \\ &= E[R|X_1 = 0] \cdot (1 - a) + E[R|X_1 = 1] \cdot a. \end{aligned}$$

If $X_1 = 0$, then $R = 1$, so

$$E[R|X_1 = 0] = 1.$$

If $X_1 = 1$, then $R \sim 1 + Geometric(b)$, so

$$\begin{aligned} E[R|X_1 = 1] &= 1 + E[Geometric(b)] \\ &= 1 + \frac{1}{b}. \end{aligned}$$

We conclude

$$\begin{aligned} r_0 &= E[R|X_1 = 0]P(X_1 = 0) + E[R|X_1 = 1]P(X_1 = 1) \\ &= 1 \cdot (1 - a) + \left(1 + \frac{1}{b}\right) \cdot a \\ &= \frac{a + b}{b}. \end{aligned}$$

Similarly, we can obtain the mean return time to state 1:

$$r_1 = \frac{a + b}{a}.$$