The Method of Transformations:

When we have functions of two or more jointly continuous random variables, we may be able to use a method similar to Theorems 4.1 and 4.2 to find the resulting PDFs. In particular, we can state the following theorem. While the statement of the theorem might look a little confusing, its application is quite straightforward and we will see a few examples to illustrate the methodology.

Theorem 5.1
Let X and Y be two jointly continuous random variables. Let $(Z,W) = g(X,Y) = (g_1(X,Y), g_2(X,Y))$, where $g : \mathbb{R}^2 \mapsto \mathbb{R}^2$ is a continuous one-to-one (invertible) function with continuous partial derivatives. Let $h = g^{-1}$, i.e., $(X,Y) = h(Z,W) = (h_1(Z,W), h_2(Z,W))$. Then Z and W are jointly continuous and their joint PDF, $f_{ZW}(z,w)$, for $(z,w) \in R_{ZW}$ is given by

$$f_{ZW}(z,w) = f_{XY}(h_1(z,w), h_2(z,w))|J|,$$

where J is the Jacobian of h defined by

$$J = \det \begin{bmatrix} \frac{\partial h_1}{\partial z} & \frac{\partial h_1}{\partial w} \\ \\ \frac{\partial h_2}{\partial z} & \frac{\partial h_2}{\partial w} \end{bmatrix} = \frac{\partial h_1}{\partial z}.\frac{\partial h_2}{\partial w} - \frac{\partial h_2}{\partial z}\frac{\partial h_1}{\partial w}.$$

The following examples show how to apply the above theorem.

Example 5.29
Let X and Y be two independent standard normal random variables. Let also

$$\begin{cases} Z = 2X - Y \\ W = -X + Y \end{cases}$$

Find $f_{ZW}(z,w)$.

Solution

X and Y are jointly continuous and their joint PDF is given by

$$f_{XY}(x,y) = f_X(x)f_Y(y) = \frac{1}{2\pi}\exp\left\{-\frac{x^2+y^2}{2}\right\}, \qquad \text{for all } x,y \in \mathbb{R}.$$

Here, the function g is defined by $(z,w) = g(x,y) = (g_1(x,y), g_2(x,y)) = (2x-y, -x+y)$. Solving for x and y, we obtain the inverse function h:

$$\begin{cases} x = z + w = h_1(z,w) \\ y = z + 2w = h_2(z,w) \end{cases}$$

We have

$$\begin{aligned} f_{ZW}(z,w) &= f_{XY}(h_1(z,w), h_2(z,w))|J| \\ &= f_{XY}(z+w, z+2w)|J|, \end{aligned}$$

where

$$J = \det \begin{bmatrix} \frac{\partial h_1}{\partial z} & \frac{\partial h_1}{\partial w} \\ \\ \frac{\partial h_2}{\partial z} & \frac{\partial h_2}{\partial w} \end{bmatrix} = \det \begin{bmatrix} 1 & 1 \\ \\ 1 & 2 \end{bmatrix} = 1.$$

Thus, we conclude that

$$\begin{aligned} f_{ZW}(z,w) &= f_{XY}(z+w, z+2w)|J| \\ &= \frac{1}{2\pi}\exp\left\{-\frac{(z+w)^2+(z+2w)^2}{2}\right\} \\ &= \frac{1}{2\pi}\exp\left\{-\frac{2z^2+5w^2+6zw}{2}\right\}. \end{aligned}$$

Example 5.30

Let X and Y be two random variables with joint PDF $f_{XY}(x,y)$. Let $Z = X + Y$. Find $f_Z(z)$.

Solution

To apply Theorem 5.1, we need two random variables Z and W. We can simply define $W = X$. Thus, the function g is given by

$$\begin{cases} z = x + y \\ w = x \end{cases}$$

Then, we can find the inverse transform:

$$\begin{cases} x = w \\ y = z - w \end{cases}$$

Then, we have

$$|J| = |\det \begin{bmatrix} 0 & 1 \\ \\ 1 & -1 \end{bmatrix}| = |-1| = 1.$$

Thus,

$$f_{ZW}(z, w) = f_{XY}(w, z - w).$$

But since we are interested in the marginal PDF, $f_Z(z)$, we have

$$f_Z(z) = \int_{-\infty}^{\infty} f_{XY}(w, z - w) dw.$$

Note that, if X and Y are independent, then $f_{XY}(x, y) = f_X(x) f_Y(y)$ and we conclude that

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(w) f_Y(z - w) dw.$$

The above integral is called the *convolution* of f_X and f_Y, and we write

$$\begin{aligned} f_Z(z) &= f_X(z) * f_Y(z) \\ &= \int_{-\infty}^{\infty} f_X(w) f_Y(z - w) dw = \int_{-\infty}^{\infty} f_Y(w) f_X(z - w) dw. \end{aligned}$$

If X and Y are two jointly continuous random variables and $Z = X + Y$, then

$$f_Z(z) = \int_{-\infty}^{\infty} f_{XY}(w, z-w)dw = \int_{-\infty}^{\infty} f_{XY}(z-w, w)dw.$$

If X and Y are also independent, then

$$\begin{aligned} f_Z(z) &= f_X(z) * f_Y(z) \\ &= \int_{-\infty}^{\infty} f_X(w) f_Y(z-w)dw = \int_{-\infty}^{\infty} f_Y(w) f_X(z-w)dw. \end{aligned}$$

Example 5.31

Let X and Y be two independent standard normal random variables, and let $Z = X + Y$. Find the PDF of Z.

Solution

We have

$$\begin{aligned} f_Z(z) &= f_X(z) * f_Y(z) \\ &= \int_{-\infty}^{\infty} f_X(w) f_Y(z-w)dw \\ &= \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{w^2}{2}} e^{-\frac{(z-w)^2}{2}} dw \\ &= \frac{1}{\sqrt{4\pi}} e^{\frac{-z^2}{4}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-(w-\frac{z}{2})^2} dw \\ &= \frac{1}{\sqrt{4\pi}} e^{\frac{-z^2}{4}}, \end{aligned}$$

where $\int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-(w-\frac{z}{2})^2} dw = 1$ because it is the integral of the PDF of a normal random variable with mean $\frac{z}{2}$ and variance $\frac{1}{2}$. Thus, we conclude that $Z \sim N(0, 2)$. In fact, this is one of the interesting properties of the normal distribution: the sum of two independent normal random variables is also normal. In particular, similar to our calculation above, we can show the following:

Theorem 5.2

If $X \sim N(\mu_X, \sigma_X^2)$ and $Y \sim N(\mu_Y, \sigma_Y^2)$ are independent, then

$$X + Y \;\sim\; N\left(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2\right).$$

We will see an easier proof of Theorem 5.2 when we discuss *moment generating functions*.