5.2.4 Functions of Two Continuous Random Variables

So far, we have seen several examples involving functions of random variables. When we have two continuous random variables $g(X,Y)$, the ideas are still the same. First, if we are just interested in $E[g(X,Y)]$, we can use LOTUS:

LOTUS for two continuous random variables:

$$E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{XY}(x,y) \; dxdy \qquad (5.19)$$

Example 5.27

Let X and Y be two jointly continuous random variables with joint PDF

$$f_{XY}(x,y) = \begin{cases} x+y & 0 \leq x,y \leq 1 \\ \\ 0 & \text{otherwise} \end{cases}$$

Find $E[XY^2]$.

Solution

We have

$$\begin{aligned} E[XY^2] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (xy^2) f_{XY}(x,y) \; dxdy \\ &= \int_{0}^{1}\int_{0}^{1} xy^2(x+y) \; dxdy \\ &= \int_{0}^{1}\int_{0}^{1} x^2y^2 + xy^3 \; dxdy \\ &= \int_{0}^{1} \left(\frac{1}{3}y^2 + \frac{1}{2}y^3\right) dy \\ &= \frac{17}{72}. \end{aligned}$$

If $Z=g(X,Y)$ and we are interested in its distribution, we can start by writing

$$\begin{aligned} F_Z(z) &= P(Z \leq z) \\ &= P(g(X,Y) \leq z) \\ &= \iint\limits_D f_{XY}(x,y) \ \ dxdy, \end{aligned}$$

where $D=\{(x,y)|g(x,y)<z\}$. To find the PDF of Z, we differentiate $F_Z(z)$.

Example 5.28

Let X and Y be two independent $Uniform(0,1)$ random variables, and $Z=XY$. Find the CDF and PDF of Z.

Solution

First note that $R_Z=[0,1]$. Thus,

$$\begin{aligned} F_Z(z) &= 0, \qquad & \textrm{for } z \leq 0, \\ F_Z(z) &= 1, \qquad & \textrm{for } z \geq 1. \end{aligned}$$

For $0<z<1$, we have

$$\begin{aligned} F_Z(z) &= P(Z \leq z) \\ &= P(XY \leq z) \\ &= P\left(X \leq \frac{z}{Y}\right). \end{aligned}$$

Just to get some practice, we will show you two ways to calculate $P(X \leq \frac{z}{Y})$ for $0<z<1$. The first way is just integrating $f_{XY}(x,y)$ in the region $x \leq \frac{z}{y}$. We have

$$\begin{aligned} P\left(X \leq \frac{z}{Y}\right) &= \int_0^1 \int_0^{\frac{z}{y}} f_{XY}(x,y) \ \ dxdy \\ &= \int_0^1 \int_0^{\min(1,\frac{z}{y})} 1 \ \ dxdy \\ &= \int_0^1 \min\left(1,\frac{z}{y}\right) \ \ dy. \end{aligned}$$

Note that if we let $g(y)=\min\left(1,\frac{z}{y}\right)$, then

$$g(y) = \begin{cases} 1 & \text{for } 0 < y < z \\ \\ \frac{z}{y} & \text{for } z \leq y \leq 1 \end{cases}$$

Therefore,

$$\begin{aligned} P\left(X \leq \frac{z}{Y}\right) &= \int_0^1 g(y) \;\; dy \\ &= \int_0^z 1 \;\; dy + \int_z^1 \frac{z}{y} \;\; dy \\ &= z - z \ln z. \end{aligned}$$

The second way to find $P(X \leq \frac{z}{Y})$ is to use the law of total probability. We have

$$\begin{aligned} P(X \leq \frac{z}{Y}) &= \int_0^1 P(X \leq \frac{z}{Y} | Y = y) f_Y(y) \;\; dy \\ &= \int_0^1 P\left(X \leq \frac{z}{y}\right) f_Y(y) \;\; dy \qquad \text{(since } X \text{ and } Y \text{ are independent)} \end{aligned} \tag{5.20}$$

Note that

$$P\left(X \leq \frac{z}{y}\right) = \begin{cases} 1 & \text{for } 0 < y < z \\ \\ \frac{z}{y} & \text{for } z \leq y \leq 1 \end{cases}$$

Therefore,

$$\begin{aligned} P\left(X \leq \frac{z}{Y}\right) &= \int_0^1 P\left(X \leq \frac{z}{y}\right) f_Y(y) \;\; dy \\ &= \int_0^z 1 \;\; dy + \int_z^1 \frac{z}{y} \;\; dy \\ &= z - z \ln z. \end{aligned}$$

Thus, in the end we obtain

$$F_Z(z) = \begin{cases} 0 & z \leq 0 \\ z - z \ln z & 0 < z < 1 \\ 1 & z \geq 1 \end{cases}$$

You can check that $F_Z(z)$ is a continuous function. To find the PDF, we differentiate the CDF. We have

$$f_Z(z) = \begin{cases} -\ln z & 0 < z < 1 \\ 0 & \text{otherwise} \end{cases}$$
