very useful fact. We will see examples of how we use it shortly. Right now let's state this fact more precisely as a theorem. We omit the proof here.

Theorem 6.1 Consider two random variables X and Y. Suppose that there exists a positive constant c such that MGFs of X and Y are finite and identical for all values of s in $[-c, c]$. Then,

$$F_X(t) = F_Y(t), \text{ for all } t \in \mathbb{R}.$$

Example 6.7

For a random variable X, we know that

$$M_X(s) = \frac{2}{2-s}, \text{ for } s \in (-2, 2).$$

Find the distribution of X.

Solution

We note that the above MGF is the MGF of an exponential random variable with $\lambda = 2$ (Example 6.5). Thus, we conclude that $X \sim Exponential(2)$.

Sum of Independent Random Variables:

Suppose X_1, X_2, ..., X_n are n independent random variables, and the random variable Y is defined as

$$Y = X_1 + X_2 + \cdots + X_n.$$

Then,

$$\begin{aligned} M_Y(s) &= E[e^{sY}] \\ &= E[e^{s(X_1+X_2+\cdots+X_n)}] \\ &= E[e^{sX_1}e^{sX_2}\cdots e^{sX_n}] \\ &= E[e^{sX_1}]E[e^{sX_2}]\cdots E[e^{sX_n}] \quad \text{(since the } X_i\text{'s are independent)} \\ &= M_{X_1}(s)M_{X_2}(s)\cdots M_{X_n}(s). \end{aligned}$$

If X_1, X_2, ..., X_n are n independent random variables, then

$$M_{X_1+X_2+\cdots+X_n}(s) = M_{X_1}(s)M_{X_2}(s)\cdots M_{X_n}(s).$$

Example 6.8

If $X \sim Binomial(n, p)$ find the MGF of X.

Solution

We can solve this question directly using the definition of MGF, but an easier way to solve it is to use the fact that a binomial random variable can be considered as the sum of n independent and identically distributed (i.i.d.) Bernoulli random variables. Thus, we can write

$$X = X_1 + X_2 + \cdots + X_n,$$

where $X_i \sim Bernoulli(p)$. Thus,

$$\begin{aligned} M_X(s) &= M_{X_1}(s)M_{X_2}(s)\cdots M_{X_n}(s) \\ &= \big(M_{X_1}(s)\big)^n \quad \text{(since the } X_i\text{'s are i.i.d.)} \end{aligned}$$

Also,

$$M_{X_1}(s) = E[e^{sX_1}] = pe^s + 1 - p.$$

Thus, we conclude

$$M_X(s) = \big(pe^s + 1 - p\big)^n.$$

Example 6.9

Using MGFs prove that if $X \sim Binomial(m, p)$ and $Y \sim Binomial(n, p)$ are independent, then $X + Y \sim Binomial(m + n, p)$.

Solution

We have

$$
\begin{aligned}
M_X(s) &= \big(pe^s + 1 - p\big)^m, \\
M_Y(s) &= \big(pe^s + 1 - p\big)^n.
\end{aligned}
$$

Since X and Y are independent, we conclude that

$$
\begin{aligned}
M_{X+Y}(s) &= M_X(s)M_Y(s) \\
&= \big(pe^s + 1 - p\big)^{m+n},
\end{aligned}
$$

which is the MGF of a $Binomial(m+n,p)$ random variable. Thus, $X+Y \sim Binomial(m+n,p)$.
