

We notice that for this example, the mean return times are given by the inverse of the limiting probabilities. In particular, we have

$$r_0 = \frac{1}{\pi_0}, \quad r_1 = \frac{1}{\pi_1}.$$

As we will see shortly, this is not a coincidence. In fact, we can explain this intuitively. The larger the  $\pi_i$  is, the smaller the  $r_i$  will be. For example, if  $\pi_i = \frac{1}{4}$ , we conclude that the chain is in state  $i$  one-fourth of the time. In this case,  $r_i = 4$ , which means that on average it takes the chain four time units to go back to state  $i$ .

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The two-state Markov chain discussed above is a "nice" one in the sense that it has a well-defined limiting behavior that does not depend on the initial probability distribution (PMF of  $X_0$ ). However, not all Markov chains are like that. For example, consider the same Markov chain; however, choose  $a = b = 1$ . In this case, the chain has a periodic behavior, i.e.,

$$X_{n+2} = X_n, \quad \text{for all } n.$$

In particular,

$$X_n = \begin{cases} X_0 & \text{if } n \text{ is even} \\ X_1 & \text{if } n \text{ is odd} \end{cases}$$

In this case, the distribution of  $X_n$  does not converge to a single PMF. Also, the distribution of  $X_n$  depends on the initial distribution. As another example, if we choose  $a = b = 0$ , the chain will consist of two disconnected nodes. In this case,

$$X_n = X_0, \quad \text{for all } n.$$

Here again, the PMF of  $X_n$  depends on the initial distribution. Now, the question that arises here is: when does a Markov chain have a limiting distribution (that does not depend on the initial PMF)? We will next discuss this question. We will first consider finite Markov chains and then discuss infinite Markov chains.

## Finite Markov Chains:

Here, we consider Markov chains with a finite number of states. In general, a finite Markov chain can consist of several transient as well as recurrent states. As  $n$  becomes large the chain will enter a recurrent class and it will stay there forever. Therefore, when studying long-run behaviors we focus only on the recurrent classes.

If a finite Markov chain has more than one recurrent class, then the chain will get absorbed in one of the recurrent classes. Thus, the first question is: in which recurrent class does the chain get absorbed? We have already seen how to address this when we discussed absorption probabilities (see [Section 11.2.5](#), and [Problem 2](#) of in [Section 11.2.7](#)).

Thus, we can limit our attention to the case where our Markov chain consists of one recurrent class. In other words, we have an irreducible Markov chain. Note that as we showed in [Example 11.7](#), in any finite Markov chain, there is at least one recurrent class. Therefore, in finite irreducible chains, all states are recurrent.

It turns out that in this case the Markov chain has a well-defined limiting behavior if it is aperiodic (states have period 1). How do we find the limiting distribution? The trick is to find a *stationary distribution*. Here is the idea: If  $\pi = [\pi_1, \pi_2, \dots]$  is a limiting distribution for a Markov chain, then we have

$$\begin{aligned}\pi &= \lim_{n \rightarrow \infty} \pi^{(n)} \\ &= \lim_{n \rightarrow \infty} [\pi^{(0)} P^n].\end{aligned}$$

Similarly, we can write

$$\begin{aligned}\pi &= \lim_{n \rightarrow \infty} \pi^{(n+1)} \\ &= \lim_{n \rightarrow \infty} [\pi^{(0)} P^{n+1}] \\ &= \lim_{n \rightarrow \infty} [\pi^{(0)} P^n P] \\ &= \left[ \lim_{n \rightarrow \infty} \pi^{(0)} P^n \right] P \\ &= \pi P.\end{aligned}$$

We can explain the equation  $\pi = \pi P$  intuitively: Suppose that  $X_n$  has distribution  $\pi$ . As we saw before,  $\pi P$  gives the probability distribution of  $X_{n+1}$ . If we have  $\pi = \pi P$ , we conclude that  $X_n$  and  $X_{n+1}$  have the same distribution. In other words, the chain has reached its *steady-state* (limiting) distribution. We can equivalently write  $\pi = \pi P$  as

$$\pi_j = \sum_{k \in S} \pi_k P_{kj}, \text{ for all } j \in S.$$

The righthand side gives the probability of going to state  $j$  in the next step. When we equate both sides, we are implying that the probability of being in state  $j$  in the next step is the same as the probability of being in state  $j$  now.

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### Example 11.14

Consider a Markov chain in [Example 11.12](#): a Markov chain with two possible states,  $S = \{0, 1\}$ , and the transition matrix

$$P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix},$$

where  $a$  and  $b$  are two real numbers in the interval  $[0, 1]$  such that  $0 < a + b < 2$ . Using

$$\pi = \pi P,$$

find the limiting distribution of this Markov chain.

**Solution**

Let  $\pi = [\pi_0, \pi_1]$ . Then, we can write

$$\begin{aligned} [\pi_0, \pi_1] &= \pi P = [\pi_0, \pi_1] \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix} \\ &= [\pi_0(1-a) + \pi_1 b \quad \pi_0 a + \pi_1(1-b)]. \end{aligned}$$

We obtain two equations; however, they both simplify to

$$\pi_0 a = \pi_1 b.$$

We remember that  $\pi$  must be a valid probability distribution, i.e.,  $\pi_0 + \pi_1 = 1$ . Thus, we can obtain a unique solution, i.e.,

$$\pi = [\pi_0 \quad \pi_1] = \left[ \frac{b}{a+b} \quad \frac{a}{a+b} \right]$$

which is the same answer that we obtained previously.

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We now summarize the above discussion in the following theorem.

Consider a finite Markov chain  $\{X_n, n = 0, 1, 2, \dots\}$  where  $X_n \in S = \{0, 1, 2, \dots, r\}$ . Assume that the chain is irreducible and aperiodic. Then,

1. The set of equations

$$\begin{aligned}\pi &= \pi P, \\ \sum_{j \in S} \pi_j &= 1\end{aligned}$$

has a unique solution.

2. The unique solution to the above equations is the limiting distribution of the Markov chain, i.e.,

$$\pi_j = \lim_{n \rightarrow \infty} P(X_n = j | X_0 = i),$$

for all  $i, j \in S$ .

3. We have

$$r_j = \frac{1}{\pi_j}, \quad \text{for all } j \in S,$$

where  $r_j$  is the mean return time to state  $j$ .

In practice, if we are given a finite irreducible Markov chain with states  $0, 1, 2, \dots, r$ , we first find a stationary distribution. That is, we find a probability distribution  $\pi$  that satisfies

$$\begin{aligned}\pi_j &= \sum_{k \in S} \pi_k P_{kj}, \quad \text{for all } j \in S, \\ \sum_{j \in S} \pi_j &= 1.\end{aligned}$$

In this case, if the chain is also aperiodic, we conclude that the stationary distribution is a limiting distribution.

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### Example 11.15

Consider the Markov chain shown in Figure 11.14.

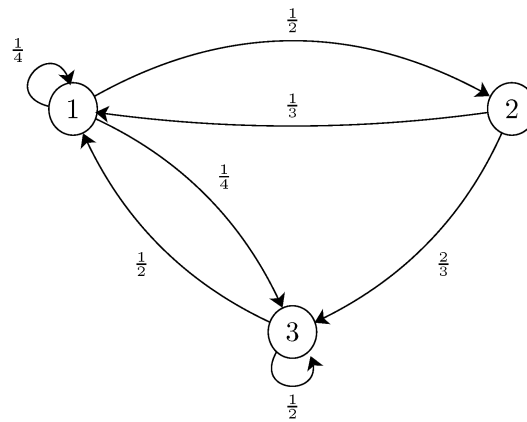


Figure 11.14- A state transition diagram.

- Is this chain irreducible?
- Is this chain aperiodic?
- Find the stationary distribution for this chain.
- Is the stationary distribution a limiting distribution for the chain?

### Solution

- The chain is irreducible since we can go from any state to any other states in a finite number of steps.
- Since there is a self-transition, i.e.,  $p_{11} > 0$ , we conclude that the chain is aperiodic.
- To find the stationary distribution, we need to solve

$$\pi_1 = \frac{1}{4}\pi_1 + \frac{1}{3}\pi_2 + \frac{1}{2}\pi_3,$$

$$\pi_2 = \frac{1}{2}\pi_1,$$

$$\pi_3 = \frac{1}{4}\pi_1 + \frac{2}{3}\pi_2 + \frac{1}{2}\pi_3,$$

$$\pi_1 + \pi_2 + \pi_3 = 1.$$

We find

$$\pi_1 = \frac{3}{8}, \pi_2 = \frac{3}{16}, \pi_3 = \frac{7}{16}.$$

- Since the chain is irreducible and aperiodic, we conclude that the above stationary distribution is a limiting distribution.