Discrete Mathematics & Graph Theory.

Set theory, Group Theory, Ring Theory, Graph Theory, Combinatorics, Number theory

Set : Unordered collection of well defined distinct object.

Eg : No. of good singer in class.

UNIT 1

Representation of Set

Roaster Method (listing Method) : V denotes the set of vowels in English alph

$N = \{a, e, i, o, u\}$

This is set builder form.

$A = \{n \ni N, 1 \leq n \leq 10\}$ → Set of natural no. less than or equal to 10.

$N = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ → when listing only for roaster.

Set builder form : $\{x \mid x \text{ has property } p\}$

$\mid$ → such that

$N = \{x \mid x \text{ is natural no. less than or equal to } 10\}$

Two Types of Set Finite Set & Infinite Set.

Example of Sets

Set of Natural Numb: $N = \{1, 2, 3 \ldots\}$
Set of whole No : $W = \{0, 1, 2, \ldots\}$
Set of Integer : $Z = \{\ldots -2, -1, 0, 1, 2 \ldots\}$
- Integer = $Z^{-} = \{\ldots -2, -1\}$
+ Integer = $Z^{+} = \{1, 2, 3 \ldots\}$
Nonnegative Integer = $Z = \{0, 1, 2 \ldots\}$

Set of Rational No : $Q = \{p/q \mid p \in Z, q \in Z \text{ and } q \neq 0\}$
Is 4 a Rational No? Yes because $4/1 = p/q$ form
$\sqrt{3}/2$ $\frac{1.4}{4}$? No because P is not Z
1.5 is ? $3/2$ therefore Rational no.
$2/3 = 0.\bar{6}$ → Repeating And non terminating is considered as Rational.

Real No. Set = $R = \{$ Unionn of Rational & Irrational No. $\}$
Set of Complex No : $C = \{$ any no in form of $x + yi \}$
Every Real No is a Complex No.

Size of Set : No. of distinct elements present in the set

Eg : S denotes the positive $Z \leq 10$ = $|S| = 10$

Empty Set = $|\phi| = 1$ Natural No : $|S| = \infty$

Size is called Caranity of set = $|S|$

Power Set : It is a set of All subset

Given Set S, then Powers set of S is the set of all subsets of S.

$S = \{0, 1, 2\}$. $\{0\}, \{1\}, \{2, 3\}, \{\}, \{0, 1\}, \{0, 2\}$
$\{1, 2\}, \{0, 1, 2\}$.

Denoted by $P(S)$ = Power of S.

$\therefore P(S) = \{\phi, \{1\}, \{2\}, \{3\}, \{0, 1\}, \{1, 2\}, \{0, 2\}, S\}$

Size of $P(S) = 2^n$ = n is no. of element in S

$\therefore |P(S)| = 8 = 2^3$

~~Contineed back Operation on Sets.~~

Continued back → Operation on Sets
→ types of Sets
→ Reprenstation of Set

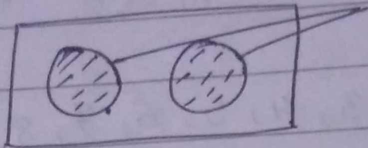
Operation

1) **Union** : It is denoted by $A \cup B$

Eg: $A = \{1, 3, 4, 5\}$ $B = \{2, 4, 6, 7\}$

= $A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$

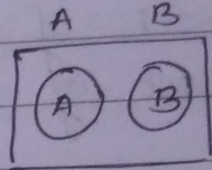
→ $A \cup B$ for disjoint



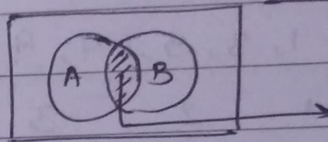
$A \cup B$

2) **Intersection** : It is denoted by $A \cap B$

For disjoint set $A \cap B = \phi$



For set (not disjoint) $A \cap B$ =

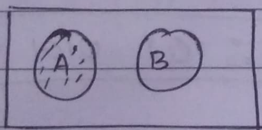


→ $A \cup B$ $A \cap B$

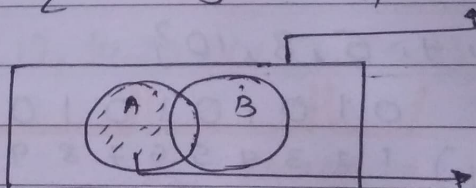
Eg $A = \{1, 2, 3\}$ = $B\{3, 4\}$ = $A \cap B = \{3\}$

③ **Difference of Sets $(A - B)$** (Elements of A but not B)

Eg = $A = \{1, 3, 5\}$ $B = \{1, 2, 3\}$ = Q) $A - B = \{1, 5\}$



for disjoint



A − B

(Not disjoint)

Q.) $B - A = \{2\}$

bit string is used to represent sets it contains only 1 & 0.

$U = \{a_1, a_2, a_3 ---- a_n\}$

$A \subseteq U$ if $a_1 \in A$ then 1

$a_1 \notin A$ then 0

Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

Subset of all odd integer

subset of all even integer

subset of integer not exceeding 5 in U.

$A = \{1, 3, 5, 7, 9\}$

[Joh A set meh hai use 1 in uni set

Rest 0

	1	2	3	4	5	6	7	8	9	10
bit string →	1	0	1	0	1	0	1	0	1	0

left side

make group of 4

$\boxed{\therefore \; 10\ 1010\ 1010}$

$B = \{2, 4, 6, 8, 10\}$

1-S = 01 0101 0101

2-S (Grouping) 1 2 3 4 5 6 7 8 9 10

$\boxed{\therefore \; 01\ 0101\ 0101}$

$C = \{1, 2, 3, 4, 5\}$

1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	0	0	0	0	0

$\therefore$ 11 1110 0000

TONS:

(OR) for union

$A \cup B = (10\ 1010\ 1010) \vee (01\ 0101\ 0101)$

$\boxed{A \cup B = (11\ 1111\ 1111)}$

$A \cup B = U$

I $A \cap B = (10\ 1010\ 1010) \wedge (01\ 0101\ 0101)$ $\boxed{\cap = \wedge \text{ (AND)}}$

$= (00\ 0000\ 0000)$

$\boxed{A \cap B = \phi}$

PRODUCT OF SET :/ CARTESIAN PRODUCT

Let A and B be the set (finite set)

$A = \{a_1, a_2 ---- a_n\}$ $|A| = n$

$B = \{b_1, b_2 ---- b_m\}$ $|B| = m$

then cartesian product of two set $(A \times B) = \{(a_i, b_j) / a_i \in A, b_j \in B, i = 1, 2 --- n, j = 1, 2, --- m\}$

$A = \{1, 2, 3\}$ $B = \{4, 5, 6\}$

1st

$A \times B = \{(1,4), (1,5), (1,6), (2,4), (2,5), (2,6), (3,4), (3,5), (3,6)\}$

first element should be of A / for B X A vv)

$|A \times B| = 9$

1st

$B \times A = \{(4,1), (5,1), (6,1), (4,2), (4,3), (5,2), (5,3), (6,2), (6,3)\}$

$|B \times A| = 9$

$(1,4) \neq (4,1)$

$\{1,4\} = \{4,1\}$

$\therefore$ $|A \times B| = |B \times A|$ then they have equal no. of element.

$\therefore$ $A \times B \neq B \times A$ not cummutative under cartesian product.

if $|A \times B|$; $A = B$ then $A \times B = B \times A$ $|A \times B| = |B \times A| = n$. $A = \{1, 2, 3\}$ $B = \{3, 2, 1\}$

if $|A| = n$ $|B| = m$ $|A \times B| = |B \times A| = m \times n$.

disjoint $\boxed{\text{i.e } A_i \cap A_j = \phi \text{ for (1) } i \neq j \text{ (2) } A = \bigcup_{i=I}^{n} A_i}$

Example: $A = \{1, 2, 3, 4\}$

Let $A_1 = \{1\}$, $A_2 = \{2, 3\}$ where $A_1 \subseteq A_2$ & $A_2 \subseteq A$

here A is partition set if and only if $A_1 \cup A_2 = A$

$A_1 \cup A_2 = A$ and $A_1 \cap A_2 = \phi$

i.e $A_1 \cup A_2 = \{1, 2, 3\}$ $A_1 \cap A_2 = \phi$

* **Conditions for partition sets**

→ if i) $\forall\ x \in S$, each element has to be non empty set

ii) $\forall\ x, y \in S$, either $x = y$ or $x \cap y = \phi$

set then it should be equal to A.

$A = \{1, 2, 3, 4, 5, 6\}$

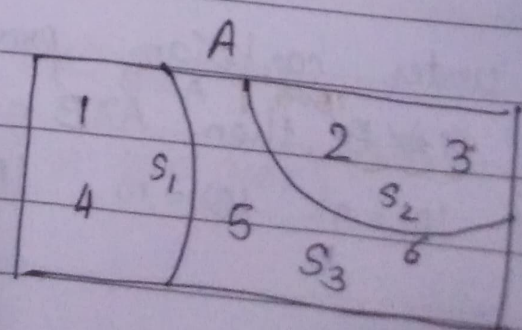
Here S is also a partitional set of A $S \subseteq P(A)$

here $S = \{S_1, S_2, S_3\}$

i.e $= S = \{\{1, 4\}, \{2, 3\}, \{5, 6\}\}$

S is a partition set of A as.

i) Each element of S i.e S_1, S_2, S_3 is non empty set



for $S_1, S_2 \in S$ either $S_1 = S_2$ or $S_1 \cap S_2 = \phi$ i.e.
$S_1 \cap S_2 = \{2,3\} \cap \{1,4\} = \phi$
$\therefore$ So 2^{nd} condition is true

Union of an element of partition set S is a universal set A
$S_1 \cup S_2 \cup S_3 = \{1,4\} \cup \{2,3\} \cup \{5,6\} = \{1,2,3,4,5,6\} = A$.
3^{rd} condition is true $\therefore$ S is partition set of A.

Covering of Set :-

Let A be the given set A and $A_1, A_2, \dots A_n$ is called covering of set A. subset of A $\bigcup_{i=1} A_i = P$
Then collection $\{A_1, A_2, \dots A_n\}$ is called covering of set A.

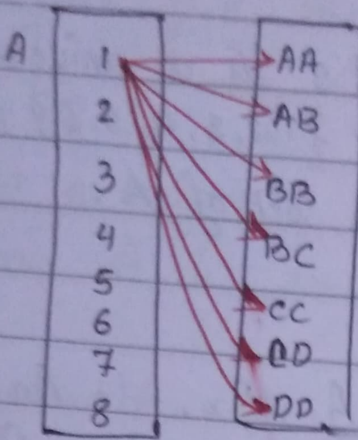
1) $A = \{1,2,3,4\}$ $A_1 = \{1,2\}$, $A_2 = \{2,3\}$, $A_3 = \{3,4\}$ $A_4 = \{2,3,4\}$
$A_1 \cup A_2 \cup A_3 \cup A_4 = \{1,2,3,4\} = A$
The collection of $\{A_1, A_2, A_3, A_4\}$ is called covering of A.

Every partition is covering. But every covering is not Partition

Relation of Set

Let $A = \{1, 2\}$. $B = \{3, 4, 5\}$

$A \times B = \{(1,3), (1,4), (1,5), (2,3), (2,4), (2,5)\}$



$\therefore$ Cartesian Product of $= 56$

$A \times B = \{(1, AA) - - - - (8, DD)\}$

Relation :- $\{(1, AA), (3, DD), (2, BC), (4, AB$
$(5, CD)\ (6, BB)\ (7, CC), (8, AA$

Relation is subset of Cartesian

Let A & B be the sets, A binary relation for A to B is a subset of $A \times B$.

$A = \{1, 2, 3, 4, 5\}$. $|A| = 5$
$B = \{A, B, C, D, E\}$ $|B| = 5$

$A \times B = \{(1, A)\ (1, B), (1, C), (1, D), (1, E), (2, A) - - - - (5, E)\}$

$|A \times B| = 25$ $|P(A \times B)| = 2^{2n}$ Subsets / relations

$R_1 = \phi$ ② $R_2 = A \times B$ ③ R_3

$A = \{1, 2, 3, 4\}$ $|P(A)| = 2^n = 2^4$ Subsets / relations

$A = \{1\}$, $B = \{2, 3\}$

$A \times B = \{(1, 2), (1, 3)\}$ $|A \times B| = 2$

$|A| = 1$ $|B| = 2$

1) $R_1 = \phi$ $R_2 = A \times B$ $R_3 = \{1,2\}$ $R_4 = \{1,3\}$

$|P| = 2^2 = 4$ $|P(A \times B)| = 2^n$ $, n = 2$ $= 2^2 = 4$

If $A = n$ $B = m$ $|A \times B| = nm$

then $|P(A \times B)| = 2^{nm}$

No. of Elements from A to B $= 2^{nm}$

→ If $A = B$, i.e $|A| = n$

$A = \{1, 2, 3\}$ $B = \{1, 2, 3\}$ [A = B in Relation is subset of $A \times A$

$R \subseteq A \times A$

No. of Relations from A to A $= 2^{n \times n} = 2^{n^2}$

→ Example $A = \{1, 2\}$ $B = A = \{1, 2\}$

$A \times A = \{(1,1), (1,2), (2,1), (2,2)\}$

possible relations → $R = \phi$ $R = A \times A$ $R =$ singleton sets or combination of those sets

$\boxed{\text{No. of Relations from A to A} = 2^{n^2}}$ $= 2^{(2)^2} = 2^4 = 16.$

* Reflexive Relation

A Relation R on set A is called Reflexive if (ordered pair) (a,a) ∈ R for every element a ∈ A.

Example: $A = \{1, 2, 3, 4\}$ Cartesian product $A \times A$

$A = \{1, 2, 3, 4\}$, $A = \{1, 2, 3, 4\}$

$$A \times A = \left\{ \begin{array}{l} (1,1), (1,2), (1,3)\ (1,4)\ (2,1)\ (2,3)\ (2,4)\ (2,2) \\ (3,1)\ (3,2)\ (3,3)\ (3,4)\ (4,1)\ (4,2)\ (4,3)\ (4,4) \end{array} \right\}$$

Eg. ① $R_1 = \{(1,1)\ (2,2)\ (3,3)\ (4,4)\}$ (Relation) ∈ R

but to check wether they are Reflexive

i.e $(a,a) \in R_1$ for every element $(a,a) \in A$.

Eg. ② $R_2 = \left\{ \begin{array}{l} (1,1), (2,2), (3,3), (4,4) \\ (2,3)\ (3,4) \end{array} \right\}$ Reflexive ?

∴ This is also a Reflexive Relation.

(If all ordered pair are present then it is Reflexive)

Eg. ③ $R_3 = \{(1,1)\ (2,2)\ (3,3)\}$ This is not Reflexive coz one pair is mi

Eg. ④ $R_4 = \{(1,1), (2,2), (3,3), (4,4), (5,5)\}$

This not Reflexive because $5,5 \notin A \times A$ ∴ It is not a Relation

∴ This not Reflexive because $5 \notin A$.

Eg. ⑤ $R_5 = A \times A$ It is Reflexive (Largest Reflexive Relation)

$R_2 = \{(1,1), (2,1), (1,2), (3,3) (1,3)\}$

$(1,3) \in R_2$ but $(3,1) \notin R_2$ $\therefore$ Not Symetric.

$R_3 = \{(1,1), (2,2), (3,3), (4,4)\}$ It is symmetric.

$R_4 =$ $A \times A$ It is symmetric

$R_5 = \{(2,2)\}$ "

$R_6 =$

$$\left\{ \begin{array}{l} (4,1), (1,4), (3,1) (1,3) \\ (5,1), (1,5) \end{array} \right\}$$

1. R6 is not a Relation because $(5,1) \notin A$

$\therefore$ It is not a ~~Re~~ Symmetric Relation.

Eg.2 $R_2 = \{(1,3), (3,1), (1,1), (3,2)\}$

R_2 is not transitive because $(1,2)$ is missing.

$(1,3), (3,1) \to (1,1)$

$(3,1), (1,1) \Rightarrow (3,1)$

$(1,3), (3,2) \Rightarrow (1,2) \notin$ ∴ This is missing in R_2.

Eg.3 $R_3 = \{(1,4), (4,1), (1,1), (4,3), (1,3)\}$

R_3 is ~~not~~ transitive ~~because~~ Relation.

Eg.4 $R_4 = A \times A = ((1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2)($

$(1,1) \to (1,2) = (1,2)$ $(1,2) \to (2,1) \to (1,1)$ $(1,2), (2,3) \to ($

$(1,1) \to (1,3) \to (1,3)$ $(1,2) \to (2,2) \to (1,2)$ $(1,3), (3,1) \to ($

$(1,3) \to (3,2) \Rightarrow (1,2)$ $(1,3), (3,3) \to (1,3)$ $(\cancel{2},3), (3,3) -$

R_4 is Transitive.

Eg.5) $R_5 = \{(1,1), (2,2), (3,3), (4,4)\}$ because $(1,1), (1,1) \to ($

If $(a,b) \in R$ & $(b,a) \in R$ then $(a=b)$ Similarly for other pairs too ∴ The Relation is Transitive.

Q $A = \{1,2,3,4\}$

$A = \{1,2,3,4\}$	Ref	Sym	Antisym	Trans.
① $R_1 = \{(1,1), (2,2), (3,3), (4,4)\}$	✓	✓	✓	✓
② $R_2 = A \times A$	✓	✓	✗	✓
③ $R_3 = \{(1,2), (2,3), (1,1), (3,3), (4,4)\}$	✗	✗	✓	✓
④ $R_4 = \{(1,1), (1,2), (2,1), (1,3), (2,4)\}$	✗	✗	✗	✗

Equivalence Relation

A Relation R on set A is called an equivalence Relation if it is reflexive, symmetric and transitive.

Example $A = \{1, 2, 3, 4\}$

Q.1 $R_1 = \{(1,1), (2,2), (3,3), (4,4)\}$ It is equivalence Reln

Q.2 $R_2 = \{(1,1), (2,2), (3,3), (4,4), (1,2), (2,1), (3,1)\}$ It is not equivalence

of $(3,1) \rightarrow (1,3)$ is not present $\therefore$ not symmetric

of $(3,1), (1,2) \rightarrow (3,2)$ not present $\therefore$ not transitive.

Q.3 $R_3 = A \times A$ It is equivalence Reln.

Q.4 $R_4 = \{(1,1), (2,2), (3,3), (4,4), (2,3), (3,2), (3,4), (4,3)\}$ It is not Equivalen

Because $(2,3)$ & $(3,4) \longrightarrow (2,4)$ is not present $\therefore$ not transitive

* Partial Order Relation

Relation R on set A is called a Partial order if it is reflexive anti symmetric, transitive.

Example $A = \{1, 2, 3, 4\}$.

$R_1 = \{(1,1), (2,2), (3,3), (4,4)\}$ It is Partial order Relation.

$(a, b) \in R$ & $(b, a) \notin R$ but if $(a, b) \in R$ & $(b, a) \in R$ then $a = b$

Q. $R_2 = A \times A$ Not antisymmetric $\therefore$ Not Partial order Relation.

Q. $R_3 = \{(1,1), (2,2), (3,3), (4,4), (1,3), (3,1), (2,4)\}$

This Relation is not antisymmetric

$\therefore$ Not a partial order Relation.

REPRESENTATION OF SET

Representing Relations Using Matrices :

Suppose that R is Relation from $A = \{a_1, a_2, a_3 - - - - a_n\}$
to $B = \{b_1, b_2 - - - - b_n\}$
The relation R can be represented by the matrix M_R

$\boxed{M_R = [m_{ij}]_{m \times n} \text{ where}}$

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$$

General form of Matrix

Order of matrix

$$M_R = \begin{bmatrix} m_{11} & m_{12} & - - - - - & m_{1n} \\ m_{21} & m_{22} & - - - - & m_{2n} \\ \vdots & \vdots & & \\ m_{m_1} & m_{m_2} & - - - - - - & m_{mn} \end{bmatrix}_{m \times n}$$

$$if = \begin{cases} 1 & \text{if } (\quad) \in R \\ 0 & \text{if } (\quad) \notin R \end{cases}$$

$A = \{1, 2, 3\} . B = \{4, 5\}$
Suppose $R = \{(1,4), (1,5), (2,5), (3,4)\}$
Order of Relation matrix ? 3×2

$$M_R = \begin{bmatrix} m_{14} & m_{15} \\ m_{24} & m_{25} \\ m_{34} & m_{35} \end{bmatrix}_{3 \times 2} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Representation of Set Using Matrix

$A = \{1, 2, 3, 4\}$

$R = \{(1,1), (2,2), (3,3), (4,4)\}$

order of matrix is 4×4

$$M_R = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix}_{4\times4} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R = A \times A$ then matrix is ? Identity Matrix

$A = \{1, 2, 3, 4\}$

$R = \phi$ then matrix is ? Null matrix / Zero matrix of 4×4

$A = \{1, 2, 3, 4\}$ $R = \{(1,2), (2,1), (3,1), (1,3)\}, (3,2), (2,3), (4,3)$
$(2,2), (4,4)$

Order of matrix is 4×4

$$M_R = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}_{4\times4}$$

Representation of Relation Using Daigraphs

→ Directed graph (have direction)

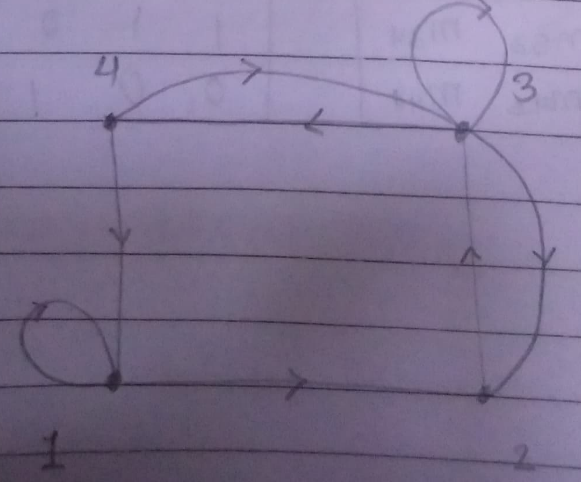
A directed graph or daigraphs consists of set V of ~~values~~ vertices together with set E of ordered pairs of elements of ~~E~~ V called ~~ages~~ edges.

The vertex A is called initial vertex of the edge.
i.e (a, b) a → intial vertex
b → terminal vertex of the edge.

Example $A = \{1, 2, 3\}$ → this set of V (Vertices)
$E = R = \{(1,2), (3,3)\}$ → Here we have a loop.
Relation is set of Edges (E)
(1,2) → (a, b) ordered pair
i.e a → intial vertex
b → terminal vertex

edge of form (a, a)
intial vertex = terminal vertex
∴ It is said has loop. is formed.

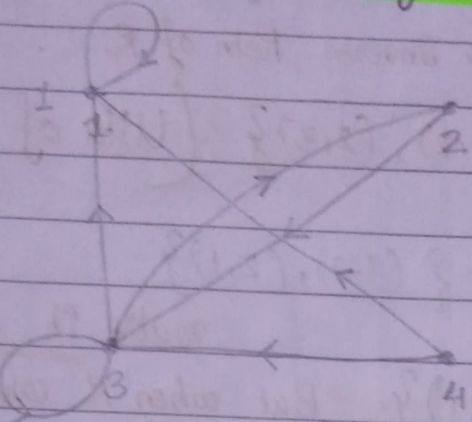
$A = \{1, 2, 3, 4\}$ $R = \{(1,1), (2,3), (3,3), (4,1), (3,2), (3,4), (1,2), \cancel{(3,2)}, (4,3)\}$



$A = \{1, 2, 3, 4\}$

$R = \{(1,1), (2,3), (3,1), (3,2), (4,1), (4,3), (3,3)\}$

Element in set : A = no. of vertex



Closure of Relation

$A = \{1, 2, 3, 4\}$

$R = \{(1,1), (2,2), (1,2), (3,2), (4,4)\}$

Is it Reflexive ? No, because (3,3) is to present.

$\overline{R_r} = \{(1,1), (2,2), (1,2), (3,2), (4,4), (3,3)\}$

$\longrightarrow$ reflexive closure of Relation R.

$\overline{R_r} = R \cup \Delta$, wher Δ is the daigonal reln on set A.

$\therefore R \subseteq \overline{R_r}$

[daigonal reln are (a,a) of all elements in set A]

$A = \{1, 2, 3, 4\}$ $R = \{(1,1), (2,3), (3,3)\}$ Make reflexive

$\overline{R_r} = \{(1,1), (2,3), (3,3), (4,4), (2,2)\}$

$\overline{R_r} = R \cup \Delta$

where $\Delta = \{(1,1), (2,2), (3,3), (4,4)\}$

Example 2) $A = \{1,2,3,4\}$ $R = \{(1,1), (1,2), (2,1), (3,2)\}$

make it symmetric Reln.

$\overline{R_S} = \{(1,1), (1,2), (2,1), (3,2), (2,3)\}$

$\longrightarrow$ Symmetric closure of R.

$\overline{R_S} = R \cup \underbrace{R^{-1}}_{\text{Inverse Reln of } R}$

$R^{-1} = \{(1,1), (2,1), (1,2), (2,3)\}$ (Ulta of order pair present in R)

Example 3) $A = \{1,2,3,4\}$ $R = \{(1,2), (2,1)\}$

make it transitive

$\overline{R_T} = \{(1,2), (2,1), (1,1)\}$ But when R contains more pair it get's difficult so we use warshell Algorithm.

* Warshell Algorithm

Find the transitive closure of Relation R where a is a set

$A = \{1, 2, 3, 4\}$ $R = \{(1,2), (2,1), (2,3), (3,4), (4,1)\}$

STEP 1

Write in matrix form

$$M_R = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

STEP 2: $A = M_R$

$$W_0 = A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

STEP 4: $W_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$

Replace $C_1 \times R_1$ ka elements ka position with 1, if 1 is alread present leave as it is.

~~$C_2 = \{2, 1) 1\}$ $R_2 = \{1, 3\}$ $C_2 \times R_2 = \{(1,1), (1$~~

STEP 5: Observe W_1 for C_2 & R_2

$C_2 = \{1, 2, 4\}$ $R_2 = \{1, 2, 3\} =$

$C_2 \times R_2 = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (4,1), (4,2), (4,3)\}$

STEP 6: $W_2 = \begin{bmatrix} 1 & 1 & \underline{1} & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & \underline{1} & 0 \end{bmatrix}$

marked (–) is replaced with 1 in every step.

$R_3 = \{4\}$ $C_3 = \{1, 2, 4\}$

$C_3 \times R_3 = \{(1,4), (2,4) (4,4)\}$

STEP 7: $W_3 = \begin{bmatrix} 1 & 1 & 1 & \underline{1} \\ 1 & 1 & 1 & \underline{1} \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & \underline{1} \end{bmatrix}$

$C_4 = \{1, 2, \cancel{4}\} 4\}$

$R_4 = \{1, 2, 3, 4\}$

$C_4 \times R_4 = \{ A \times A$

STEP 8: $W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ \underline{1} & \underline{1} & \underline{1} & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

STEP 9: $\therefore R_T = \begin{cases} (1,1), (1,2), (1,3), (1,4) \\ (2,1), (2,2), (2,3), (2,4) \\ (3,1), (3,2), (3,3), (3,4) \\ (4,1), (4,2), (4,3), (4,4) \end{cases}$

Write Transitive closure by observing the last matrix.

R_T contains order pair ~~where~~ from matrix where 1 is present.

$A = \{1,2,3,4\}$ $R_1 = \{(2,1),(2,3),(3,1),(3,4),(4,1),(4,3)\}$

Find the Transitive Closure.

$$M_R = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}_{4\times4}$$

$$W_0 = A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

(1) $C_1 = \{2,3,4\}$

$R_1 = \{\phi\}$

$\therefore C_1 \times R_1 = \{\phi\}$

(2) $\therefore W_0 = W_1$

(3) $C_2 \times R_2 = \phi$

$C_2 = \{\phi\}$ $R_2 = \{1,3\}$

$\therefore W_0 = W_1 = W_2$

(4) $C_3 = \{2,4\}$ ~~$R_3\{$~~ $R_3 = \{1,4\}$

ε $C_3 \times R_3 = \{(2,1),(2,4),(4,1),(4,4)\}$

$$W_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & \underline{1} \\ 1 & 0 & 0 & \underline{1} \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

marked with (−) got changed.

$C_4 = \{2,3,4\}$ $R_4 = \{1,3,4\}$

$C_4 \times R_4 = \{\cancel{(1,1)}, \cancel{(1,3)}, \cancel{(1,4)}, (3,1),(3,3),(3,4),(4,1),(4,3),(4,4)$
$(2,1),(2,3),(2,4)\}$

$$W_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & \underline{1} \\ 1 & 0 & \underline{1} & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

$\therefore R_T = \{$ $\cancel{(1,1)}, \cancel{(1,3)}, \cancel{(1,4)}$
$(2,1),(2,3),(3,1)$
$(3,3),(3,4),(4,1)$
$(4,3),(4,4),(2,4)$ $\}$

Examples $A = \{1,2,3,4\}$ $R = \{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}$ find Transitive Closure

Solution

$$M_R = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix}_{4\times4} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{4\times4}$$

$\rightarrow$ $W_0 = A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

step 1 $C_1 = \{\phi\}$ $\therefore C_1 \times R_1 = \{\phi\}$

$R_1 = \{2,3,4\}$

$\therefore W_0 = W_1$

step 2 $C_2 \times R_2 = \{(1,3),(1,4)\}$ $C_2 = \{1\}$, $R_2 = \{3,4\}$

$\therefore W_2 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

step 3 $C_3 = \{1,2\}$ $R_3 = \{4\}$ $C_3 \times R_3 = \{(1,4),(2,4)\}$

$W_3 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$C_4 = \{1,2,3\}$ $R_4 = \{\phi\}$

$\therefore W_3 = W_4$

step

$A = \{2, 8, 7, 6\}$ $R = \{(2,2), (8,8), (7,7), (6,6), (7,2), (2,7), (8,7)\} \subseteq A\times$

$A = \{2, 8, 7, 6\}$

$$M_R = \begin{bmatrix} m_{22} & m_{28} & m_{27} & m_{26} \\ m_{82} & m_{88} & m_{87} & m_{86} \\ m_{72} & m_{78} & m_{77} & m_{76} \\ m_{62} & m_{68} & m_{67} & m_{66} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{4\times 4}$$

$C_1 = \{2, 7\}$ $R_1 = \{2, 7\}$ $C_1 \times R_1 = \{(2,7), (2,2), (7,2), (7,7)\}$

$$W_1 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_2 = \{8\}$ $R_2 = \{8, 7\}$

$C_2 \times R_2 = \{(8,8), (8,7)\}$

$$W_2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_3 = \{2, 8, 7\}$ $R_3 = \{2, 7\}$

$C_3 \times R_3 = \{(2,2), (2,7), (8,2), (8,7), (7,7)\}$

$$W_3 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_4 = \{6\}$ $R_4 = \{6\}$

$C_4 \times R_4 = \{6, 6\}$

$W_3 = W_4$

$\therefore$ $T_R = \{(2,2), (8,2), (7,2), (2,7), (8,8), (8,7), (8,6),$

A relation R on set A is called partial order relation if R is reflexive, antisymmetric and transitive

$A = \{1,2,3,4\}$ $R = \{(1,1),(2,2),(3,3),(4,4),(1,2),(1,3),(4,2)\}$ → This relation is partially ordered.

Relation R is reflexive, antisymmetric and transitive → as this is true

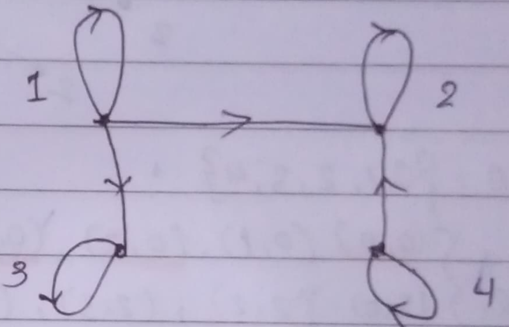
then order pair (A, R) → Partial order Pair on set A

R → partial order Relation

$A = \{1,2,3,4,5,6,7,8,9,10\}$

$R = \{(a,b) \mid a \le b\}$ This relation is also partially ordered set.

① Digraph for Example 1



for partial ordered Relation.

Hasse Diagram - Simplest version of daigraph.

1) Remove all the loops from vertices/daigram
2) Remove transitive pair Eg: (1,2), (2,3) → then remove (1,3)
3) Always arrange no. as intial and terminal.
4) Terminal should always be above initial.
5) Remove direction.

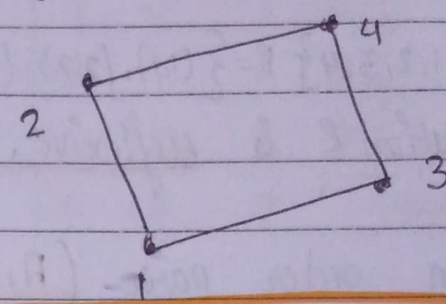
Re follow this step 1 by 1

Not to consider transitive pair to remove For Hasse daigram
Eg: (1,1) (1,2) → (1,2)
(This cannot be removed)

Let $A = \{1,2,3,4\}$ $R = \{(1,1), (1,2), (2,2), (2,4), (1,3), (3,3),$
$(3,4), (1,4), (4,4)\}$

Then show that R is partial order and draw it's Hasse Daigram.

[Except Reflexive pair]

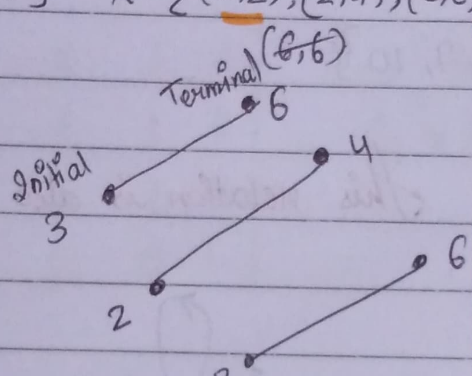


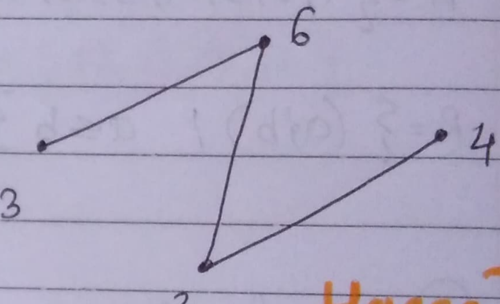
(1) Remove Transitive pair

$(1,2)(2,4) \rightarrow (1,4)$

$(1,3)(3,4) \rightarrow (1,4)$

$A = \{2,3,4,6\}$ $R = \{(2,2), (2,4), (2,6), (3,3), (3,6), (4,4), \cancel{(6,4)}\}$ (6,6)





Hasse Dai

$A = \{0,1,2,3,4\}$

$R = \{(0,0), (0,1), (0,2), (0,3), (0,4), (1,1), (1,2), (1,3), (1,4)$
$\cancel{(2,0)}$ $(2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$

draw it's Hesse daigram.

$(0,1)(1,2) \rightarrow (0,2)$

$(0,2)(2,3) \rightarrow (0,3)$

$(0,2)(2,4) \rightarrow (0,4)$

$(1,2)(2,3) \rightarrow (1,3)$

$(1,3)(3,4) \rightarrow (1,4)$

$(1,2)(2,4) \rightarrow (1,4)$

$\therefore R = \{(0,1), (1,2), (2,3)$
$(2,4), (3,4)\}$