UNIT 5 Counting

* Basic Counting techqnics –

1) Product rule

Suppose that procedure can be broken down into a sequence of two tasks, If there are 'n' ways, to do the first task and 'n' ways to do the second task so then product of 'n_1' x 'n_2' ways to so then do procedure & if and comes do multiplications.

Ex:1) The chairs of a auditorium are to be labbled with letters, and positive integers not exceeding 100. what is the largest no of chairs that can be labled differently

⇒ Here labels with
letters and integers $(1 \leq x \leq 100)$

{ Here and coms ∴ do multiplication }

$$\underbrace{26}_{\text{letters}} \times \underbrace{100}_{\text{integer}} = \underbrace{\underline{2600}}_{\text{result.}}$$

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Q:2 There are 32 micro computers in computer centre each micro computer has 24 parts. How many different parts to a micro-computers in the center are there?

=>

$$\underbrace{32}_{\text{micro computer}} \times \overset{\text{parts}}{24} = 768.$$

and

For Tm no of task we have nm no. of ways then $n_1 \times n_2 \times \ldots \ldots n_m$ ways are there to do procedure

Q.1 How many different bit strings of length seven are there?

=>

no of possibilities

| 2 | 2 | 2 | 2 | 2 | 2 | 2 |

Length of string 7

Ans =>

$2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$

$= 2^7$.

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Q.2 How many diff licence plates are available if each plate contains sequence of 3 letters followed by 3-digits.

=>

possibilites → licence plate having 6 spaces

$$\underbrace{|26|26|26|}_{\text{letters}}\underbrace{10|10|10|}_{\text{digits}}$$

$= 26 \times 26 \times 26 \times 10 \times 10 \times 10$

$= 26^3 \times 10^3$

* Sum Rule.

If task can be done either in ~~one~~ n_1 or n_2 ways then complete process can be done in $n_1 + n_2$ ways.

Ex: Suppose that either member of math faculity or a student who is a mathematic ~~measure~~ major is chosen & as a representative to univeros-ity commutity if thes 37 numbers & 83 mamthematic major & no one is & no one is both member & student.

=> $37 + 83 = 120$.

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Ex: Student can chose computer project from one of three list the three list contain 23, 15, 19 possible project no project is on more one list.

=) $23 + 15 + 19 = 57$

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

principal of inclusion & exclusion.

Ex: A computer company recieve 350 application from computer gradeate for job planning line for new web server suppose that 220 peple major in CS. 147 measure in bussiness & 51 measure in both.

=)

$= 220 + 147 - 51$

$= 367 - 51$

$= 316.$

$350 - 316$

$= 34$

Ex: How many positive integer not excedding 10000 are divisibly by 5 or 11.

⇒

Let A denotes set of Integer which is divisible by 5

$$|A| = \left\lfloor \frac{10000}{5} \right\rfloor = 2000$$

Let B denotes set of Integer which is divisible by 11

$$|B| = \left\lfloor \frac{10000}{11} \right\rfloor = \lfloor 909.09 \rfloor = 909$$

$$|A \cap B| = \left\lfloor \frac{10000}{55} \right\rfloor = 181$$

$$|A \cup B| = |A| + |B| - |A \cap B|$$
$$= 909 + 2000 - \cancel{2000}\ 181$$
$$= 2909$$
$$= 2728$$
$$= 2728$$

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Suppose A, B & C are three non empty finite sets then no of elements in A or B or C is

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

Ex: The total of 1232 students have taken course in spanish 879 have taken course in frech & 114 have taken course in russian further 103 have taken course both S & F

23 — 11 — S & R

14 — 11 — F & R

If 2092 student have taken at least on of spanish, french & russian how many student have taken course in all three languages

⇒

$|A \cup B \cup C| = 2092$

$|A| = 1232$

$|B| = 879$

$|C| = 114$

$||A| \cap |B|| = \cancel{23}\ 103$

$|A \cap C| = 23$

$|B \cap C| = 14.$

$2092 = 1232 + 879 + 114 - 140 + |A \cap B \cap C|$

$$2092 - 1232 - 879 - 114 + 140 = |A \cap B \cap C|$$

$$= 2232 - 1232 - 879 - 114$$
$$= 1000 - 879 - 114$$
$$= 1000 - 993$$
$$= 007$$
$$= 7.$$

The pigenhole Principle.

Theorem : 1 The pigenhole principle.

If k is positive integer and ~~key 1~~ k+1 or more objects are placed into k boxes then there is ~~of lowez~~ at least one box containing two or more object.

Ex: How many student must be in class to guratee that at least two student recieve same score on final exam. If the exam is graded on the scale 0 to 100 point

=> 102.

Generalize pigenhole principle

If suppose N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

Q. What is the minimum no of student required in DMGT class to be ~~sured~~ that at least 6 will recieve the same grade if there are 6 posible grade. (A, B, C, D, E, F).

⇒

$$\left\lceil \frac{N}{K} \right\rceil \geq 6 \qquad \left\lceil \frac{N}{6} \right\rceil = 6$$

$N = 31$

Q. Same 5 possible grade.

⇒

$$\left\lceil \frac{N}{K} \right\rceil \geq 6 \qquad \left\lceil \frac{N}{5} \right\rceil \geq 6$$

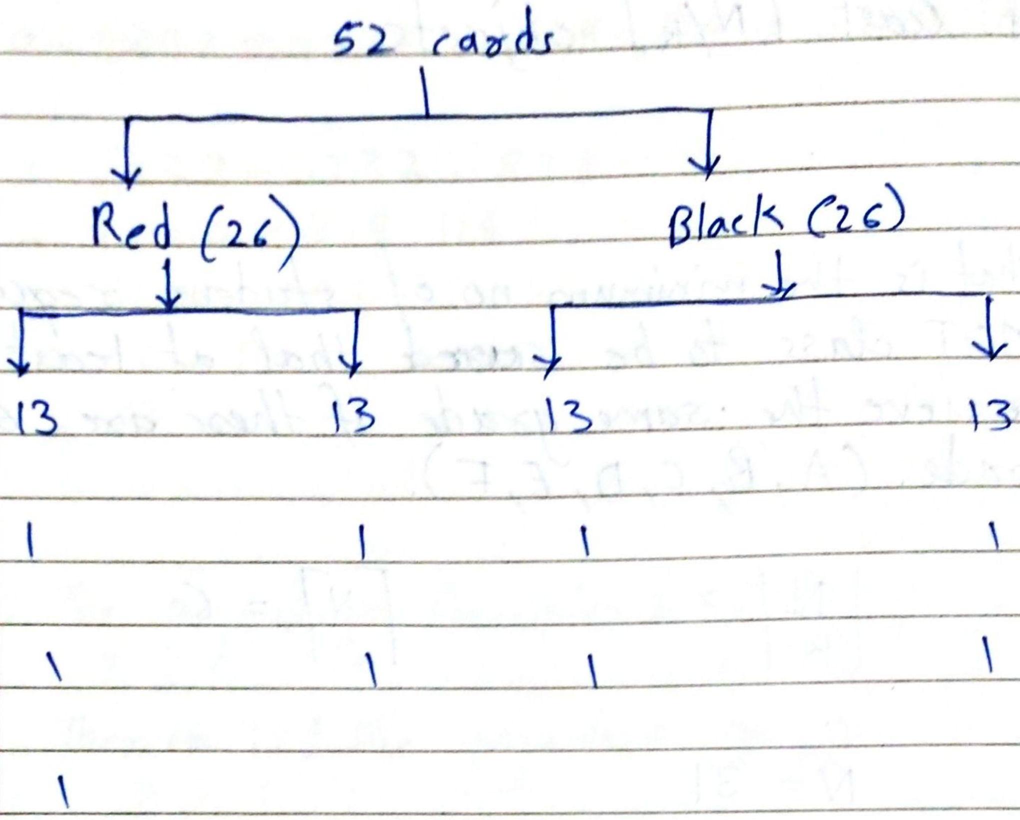
$N = 26.$

Q. How many cars must be selected from the standard ~~deck~~ of 52 cards to gurantee that at least 3 card same suit are chossen.

=

$$\cancel{\left\lceil \frac{N}{K} \right\rceil \geq 3} \qquad \cancel{\left\lceil \frac{N}{52} \right\rceil} \qquad \cancel{\left\lceil \frac{52}{K} \right\rceil \geq 3}$$

⇒



$$\left\lceil \frac{N}{4} \right\rceil = 3 \qquad N = 9.$$

* Recurrence ~~Reaction~~ Relation.

Recurrence relation for the sequence $\{a_n\}$ here is an equation that express a_n in terms of one or more ~~privious~~ term of the sequence.

$a_0, a_1 \dots a_{n-1}$

Ex: 1, 1, 2, 3, 5, 8, 13, 21

$a_0 = 1 \quad a_1 = 1 \quad a_2 = 2 \quad a_3 = 3 \quad \dots$

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$a_n = a_{n-1} + a_{n-2}$

Ex:

$a_n = 2a_{n-1} + 4a_{n-2} + 3a_{n-3}$ ✓

$a_n = 4a_{n-1}$ ✓

$\boxed{a_n = a_{n+1} + a_{n+2}} \rightarrow \times \boxed{(n+1)}$ next terms.

$a_n = a_{n-1} + (a_{n-2})^2$ ✓

Ex: Bacteria at time 0 is 5 after 1 hour it will be 10 calculate for 20 hours.

$a_0 = 5 \quad t = 0 \qquad a_1 = 10 \quad t = 1$

$a_2 = 20 \quad t = 2$

$a_n = 2(a_{n-1})$ — (1)

put $n=1$ $a(1)$ in (1)

$a_1 = 2a_0$ — (2)

put $n=2$ in (1)

$a_2 = 2a_1$

$a_2 = 2(2a_0) = 2^2 a_0$

put $n=3$ in (1)

$a_3 = 2^3 a_0$.

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$$a_n = 2^n a_0$$

$n = 20$

$$a_{20} = 2^{20} a_0$$

$$= 2^{20} \times 5$$

Ex: Suppose that a person deposite 10,000, in a saving account at a bank yielding 11% per year with interest compound annually How much will be amount after 30 year.

⇒

$$10000 + \left[\frac{11}{100} \times 10000\right]$$

$$= 11\cancel{1}00$$

$$a_0 = 10{,}000$$

$$a_1 = 10000 + \left(\frac{11}{100} \times 10000\right)$$

$$= a_0 + \frac{11}{100} \times a_0$$

$$a_2 = \cancel{10000} =$$

$$= a_1 + \frac{11}{100} \times a_1$$

$$a_n = a_{n-1} + \left(\frac{11}{100} \times a_{n-1}\right)$$

$a_n = a_{n-1} + (0.11)a_{n-1}$ —— ①

put $n=1$

$a_1 = a_0 + (0.11)a_0 = (1.11)a_0$

put $n=2$.

$a_2 = a_1 + (0.11)a_1 = (1.11)^2 a_0$.

put $n=3$

$a_3 = a_2 + (0.11)a_2 = (1.11)^3 a_0$

$a_n = (1.11)^n a_0$.

$a_{30} = (1.11)^{30} \times 10000$

$a_{30} = 22.89 \times 10000$

$= 2,28,900$.

✱ Linear homogenous recurrence relation.

A linear homogenous recurrence relation of degree k with constant coeffcient is a recurrence relation of the form

$a_n = C_1 a_{n-1} + C_2 a_{n-2} + \ldots\ldots C_k a_{n-k}$.

$a_n = \underline{n} a_{n-1} + 4a_{n-2}$ × (coefficient is variable)

$a_n = 4(a_{n-2})^2$ × (non linear)

$0 = a_n - a_{n-1} - a_{n-2}$ ✓

$a_n = a_{n-1} + a_{n-2}$. ✓ Degree 2

$a_n = a_{n-1} + 4$ × non homogeous

$a_n = 2a_{n-1} - 4a_{n-2} + 3a_{n-3}$ ✓

Degree 3

* Solving linear homogeneous recursive relation

$a_n = C_1 a_{n-1} + C_2 a_{n-2}$ —— (1)

Suppose $a_n = r^n$ is sol^n of (1)

$\Leftrightarrow r^n = C_1 r^{n-1} + C_2 r^{n-2}$

$\Leftrightarrow r^n - C_1 r^{n-1} - C_2 r^{n-2} = 0$.

Divide by lowest power i.e r^{n-2}

$$\frac{r^n}{r^{n-2}} - \frac{C_1 r^{n-1}}{r^{n-2}} - \frac{C_2 r^{n-2}}{r^{n-2}} = 0$$

$$r^2 - c_1 r - c_2 = 0 \quad -(2)$$

Suppose r_1 & r_2 be root of equation (2)

Case (1) r_1, r_2 distict.

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$$

Case (2) r_1, r_2 repeated ($r_1 = r_2$)

$$a_n = \alpha_1 r_0^n + \alpha_2 r_0^n . n.$$

Q What is the solution of the recurssion relation
$a_n = a_{n-1} + 2a_{n-2}$
with $a_0 = 2$ and $a_1 = 7$?

→

$$a_n = a_{n-1} + 2a_{n-2}$$

$$a_n = r^n$$

$$r^2 - c_1 r - c_2 = 0$$

$$r^2 - r - 2 = 0$$
$$r^2 - 2r + r - 2 = 0$$
$$r(r-2) + 1(r-2) = 0$$
$$(r+1)(r-2) = 0$$
$$r = -1 \quad r = 2.$$

$$r_1 = -1, \quad r_2 = 2$$

$$a_n = \alpha_1 (x_1)^n + \alpha_2 (x_2)^n$$

$$a_n = \alpha_1 (-1)^n + \alpha_2 (2)^n$$

intial cond^n $a_0 = 2$ $a_1 = 7$

put $n = 0$ in ②

$$a_0 = \alpha_1 (-1)^0 + \alpha_2 (2)^0$$

$$\alpha_1 + \alpha_2 = 2 \quad \text{--- ③}$$

put $n=1$ in ②

$$a_1 = \alpha_1 (-1)^1 + \alpha_2 (2)^1$$

$$7 = -\alpha_1 + 2\alpha_2 \quad \text{--- ④}$$

Add 3 ~~&~~ 4

$$7 = 2\alpha_2 - \alpha_1$$

$$2 = \alpha_2 + \alpha_1$$

$$3\alpha_2 = 9$$

$$\alpha_2 = 3 \qquad \alpha_1 = -1$$

$$a_n = (-1)(-1)^n + 3(2)^n$$

$$a_n = (-1)^{n+1} + 3 \cdot 2^n.$$

Ex: what is the solution of the recurrence relation

$a_n = 6a_{n-1} - 9a_{n-2}$ with $a_0 = 1$ & $a_1 = 6$.

=>

$a_n = 6a_{n-1} - 9a_{n-2}$ — (1)

$r^2 - c_1 r - c_2 = 0$

$r^2 - 6r + 9 = 0$

$(r-3)^2 = 0$

$r = 3, 3$

$r_1 = r_2 = 3$

repeated root

$a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$

$a_n = \alpha_1 3^n + \alpha_2 n 3^n$ — (2)

put $n = 0$ in (2)

$1 = \alpha_1 + 0$

$\alpha_1 = 1$

put $n=1$ in (2)

$6 = \alpha_1 3 + \alpha_2 3$

$2 = \alpha_1 + \alpha_2$

$\alpha_2 = 1$.

$a_n = 3^n(1+n)$.

Generating function

generating function for the sequence of $a_0, a_1, a_2 \ldots a_n$ of sequence of real number is the infinte series.

$$G(x) = \sum_{k=0}^{\infty} a_k x^k$$

$$= a_0 + a_1 x + a_2 x^2 \ldots + a_n x^n + \ldots$$

Ex: Find the generating function for the sequence 1, 2, 3, 4, 5, 6.

=>

$a_0 = 1$, $a_1 = 2$, $a_2 = 3$, $a_3 = 4$, $a_4 = 5$, $a_5 = 6$

$$G(x) = \sum_{k=0}^{\infty} a_k x^k$$

$$G(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5$$

$= 1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5$