Module 9

DISTRIBUTION FUNCTION

AND

ITS PROPERTIES

- $\mathcal{E}$: a given random experiment;
- $(\Omega, \mathcal{P}(\Omega), P)$: probability space associated with $\mathcal{E}$;
- $X : \Omega \to \mathbb{R}$: a given random variable;
- $(\mathbb{R}, \mathcal{P}(\mathbb{R}), P_X)$; probability space induced by X;
- $$\begin{aligned} P_X(A) &= P(X^{-1}(A)) \\ &= P(\{\omega \in \Omega : X(\omega) \in A\}),\ A \in \mathcal{P}(\mathbb{R}) \\ &\to \text{induced probability function.} \end{aligned}$$

Definition 1: The distribution function (or cumulative distribution function) of a random variable X is a function $F_X \colon \mathbb{R} \to \mathbb{R}$, defined by

$$\begin{aligned} F_X(x) &= P_X((-\infty, x]) \\ &= P(\{\omega \in \Omega : X(\omega) \leq x\}), \;\; x \in \mathbb{R}. \end{aligned}$$

Remark 1:

(a) $0 \leq F_X(x) \leq 1, \ \forall \ x \in \mathbb{R}$;

(b) Clearly the distribution function (d.f.) is determined by induced probability function $P_X(\cdot)$. Using advanced probabilistic arguments it can be shown that the induced probability function $P_X(\cdot)$ is also uniquely determined by d.f. $F_X(\cdot)$. Thus studying induced probability space $(\mathbb{R}, \mathcal{P}(\mathbb{R}), P_X)$ (or induced probability function $P_X(\cdot)$) is equivalent to studying the d.f. $F_X(\cdot)$.

Example 1: Suppose that, for some real constant c, $P(\{X = c\}) = 1$. Then

$$F_X(x) = P(\{X \leq x\}) = \begin{cases} 0, & \text{if } x < c \\ \\ 1, & \text{if } x \geq c \end{cases}.$$

Example 2: Suppose

$$P(\{X=1\}) = P(\{X=-1\}) = \frac{1}{2}.$$

Then

$$F_X(x) = P(\{X \le x\}) = \begin{cases} 0, & \text{if } \ x < -1 \\ \\ \frac{1}{2}, & \text{if } \ -1 \le x < 1 \\ \\ 1, & \text{if } \ x \ge 1 \end{cases} .$$

Result 1: Let $F_X(\cdot)$ be the d.f. of a r.v. X. Then

(a) $F_X(\cdot)$ is non-decreasing;

(b) $F_X(\cdot)$ is right continuous, i.e., $\lim_{n\to\infty} F_X(x+\frac{1}{n})$ exists and

$$\lim_{n\to\infty} F_X(x+\frac{1}{n}) \doteqdot F_X(x+) = F_X(x), \quad \forall\, x \in \mathbb{R};$$

(c) $F_X(-\infty) \doteqdot \lim_{n\to\infty} F_X(-n) = 0$ and $F_X(+\infty) \doteqdot \lim_{n\to\infty} F_X(n) = 1$.

Proof:

(a) Let $-\infty < x < y < \infty$. Then $(-\infty, x] \subseteq (-\infty, y]$, and

$$\begin{aligned}
F_X(x) &= P(\{X \leq x\}) \\
&= P_X((-\infty, x]) \\
&\leq P_X((-\infty, y]) \\
&\qquad \text{(probability function is monotone)} \\
&= F_X(y).
\end{aligned}$$

(b) Let $x \in \mathbb{R}$ be fixed. Define, for $n = 1, 2, \ldots$,

$$A_n = \{X \leq x + \frac{1}{n}\} = X^{-1}((-\infty, x + \frac{1}{n}]).$$

Then $A_n \downarrow$ and

$$\begin{aligned} \bigcap_{n=1}^{n} A_n &= X^{-1}(\bigcap_{n=1}^{\infty}(-\infty, x + \frac{1}{n}]) \\ &= X^{-1}((-\infty, x]). \end{aligned}$$

Thus,

$$\begin{aligned} F_X(x) &= P(X^{-1}(-\infty, x]) \\ &= P(\bigcap_{n=1}^{\infty} A_n) \\ &= \lim_{n\to\infty} P(A_n) \quad (P \text{ is continuous}) \\ &= \lim_{n\to\infty} P(\{X \leq x + \frac{1}{n}\}) \\ &= \lim_{n\to\infty} F_x(x + \frac{1}{n}) = F_X(x+). \end{aligned}$$

(c) Define, for $n = 1, 2, \ldots$,

$$B_n = \{\omega \in \Omega : X(\omega) \leq -n\} = X^{-1}((-\infty, -n])$$

and

$$C_n = \{\omega \in \Omega : X(\omega) \leq n\} = X^{-1}((-\infty, n]).$$

Then $B_n \downarrow$, $C_n \uparrow$,

$$\bigcap_{n=1}^{\infty} B_n = X^{-1}(\bigcap_{n=1}^{\infty}(-\infty, -n]) = X^{-1}(\phi) = \phi,$$

and

$$\bigcup_{n=1}^{\infty} C_n = X^{-1}(\bigcup_{n=1}^{\infty}(-\infty, n]) = X^{-1}(\mathbb{R}) = \Omega.$$

Thus

$$\begin{aligned}
0 &= P(\phi) \\
&= P(\bigcap_{n=1}^{\infty} B_n) \\
&= \lim_{n\to\infty} P(B_n) \quad (P \text{ is continuous}) \\
&= \lim_{n\to\infty} P(\{X \le -n\})
\end{aligned}$$

Remark 2:

(a) Using advanced mathematical arguments, it can be shown that any function $G(\cdot)$ satisfying properties (a)-(c) of the above theorem (non-decreasing, right continuous,
$G(-\infty) = 0$, $G(+\infty) = 1$)) is a d.f of some r.v.

(b) From the knowledge of calculus we know that any monotone function is either continuous on $\mathbb{R}$ or it has atmost countable number of discontinuities. Thus any d.f $F_X(\cdot)$ is either continuous on $\mathbb{R}$ or has only countable number of discontinuities.

(c) We have, for every $x \in \mathbb{R}$,

$$
\begin{aligned}
F_X\left(x - \frac{1}{n}\right) \quad &\leq \quad F_X(x) = F_X(x+), \;\; n \in \mathbb{N} \\
\lim_{n\to\infty} F_X\left(x - \frac{1}{n}\right) \quad &\leq \quad F_X(x) = F_X(x+) \\
\text{i.e.,}\;\; F_X\left(x-\right) \quad &\leq \quad F_X\left(x\right) = F_X\left(x+\right).
\end{aligned}
$$

(d) For $x \in \mathbb{R}$

$$
\begin{aligned}
P(\{X < x\}) &= P_X((-\infty, x)) \\
&= P_X\left(\bigcup_{n=1}^{\infty}\left(-\infty, x - \frac{1}{n}\right]\right) \\
&= \lim_{n\to\infty} P_X\left(\left(-\infty, x - \frac{1}{n}\right]\right) \\
&\qquad \left(\left(-\infty, x - \frac{1}{n}\right] \uparrow\right) \\
&= \lim_{n\to\infty} F_X\left(x - \frac{1}{n}\right) \\
&= F_X(x-).
\end{aligned}
$$

Thus

$$P(\{X < x\}) = F_X(x-), \quad \forall\, x \in \mathbb{R}.$$

(e) For $-\infty < a < b < \infty$

$$\begin{aligned} P\left(\{X \leq b\}\right) &= P\left(\{X \leq a\}\right) \\ &\quad + P\left(\{a < X \leq b\}\right) \end{aligned}$$

$$\begin{aligned} P\left(\{a < X \leq b\}\right) &= P\left(\{X \leq b\}\right) \\ &\quad - P\left(\{X \leq a\}\right) \\ &= F_X\left(b\right) - F_X\left(a\right). \end{aligned}$$

Thus

$$P\left(\{a < X \leq b\}\right) = F_X\left(b\right) - F_X\left(a\right), \ \forall \ -\infty < a < b < \infty.$$

Similarly, for $-\infty < a < b < \infty$,

$$\begin{aligned} P(\{a < X < b\}) &= P(\{X < b\}) - P(\{X \le a\}) \\ &= F_X(b-) - F_X(a), \end{aligned}$$

$$\begin{aligned} P(\{a \le X \le b\}) &= P(\{X \le b\}) - P(\{X < a\}) \\ &= F_X(b) - F_X(a-), \end{aligned}$$

$$\begin{aligned} P(\{a \le X < b\}) &= P(\{X < b\}) - P(\{X < a\}) \\ &= F_X(b-) - F_X(a-), \end{aligned}$$

$$\begin{aligned} P(\{X > b\}) &= 1 - P(\{X \le b\}) \\ &= 1 - F_X(b), \end{aligned}$$

$$\begin{aligned} P(\{X \ge b\}) &= 1 - P(\{X < b\}) \\ &= 1 - F_X(b-). \end{aligned}$$

(f) For $x \in \mathbb{R}$

$$\begin{aligned} P(\{X = x\}) &= P(\{X \leq x\}) - P(\{X < x\}) \\ &= F_X(x) - F_X(x-). \end{aligned}$$

(g) Let D_X be the set of discontinuity points of F_X. Since F_X is always right continuous,

$$\begin{aligned} x \in D_X &\Leftrightarrow F_X(x) - F_X(x-) > 0 \\ &\Leftrightarrow P(\{X = x\}) > 0. \end{aligned}$$

Thus a d.f. has only jump discontinuities and

$$D_X = \{x \in \mathbb{R} : P(\{X = x\}) > 0\}.$$

Example 3: Let X be a r.v. with d.f

$$F_X(x) = \begin{cases} 0, & \text{if } x < 0 \\ \frac{x}{3}, & \text{if } 0 \leq x < 1 \\ \frac{1}{2}, & \text{if } 1 \leq x < 2 \\ \frac{2}{3}, & \text{if } 2 \leq x < 3 \\ 1, & \text{if } x \geq 3 \end{cases}.$$

Clearly $F_X(\cdot)$ is $\uparrow$, right continuous, $F_X(-\infty) = 0$, and $F_X(+\infty) = 1$. Also $D_X = \{1, 2, 3\}$.

$$P(\{X = 1\}) = F_X(1) - F_X(1-) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6},$$

$$P(\{X = 2\}) = F_X(2) - F_X(2-) = \frac{2}{3} - \frac{1}{2} = \frac{1}{6},$$

$$P(\{X = 3\}) = F_X(3) - F_X(3-) = 1 - \frac{2}{3} = \frac{1}{3},$$

$$P(\{X < 3\}) = F_X(3-) = \frac{2}{3},$$

$$P(\{X \geq 1\}) = 1 - F_X(1-) = 1 - \frac{1}{3} = \frac{2}{3}.$$

For any $c \notin \{1, 2, 3\}$

$$P(\{X = c\}) = 0,$$

$$P(\{2 < X \leq 3\}) = F_X(3) - F_X(2) = 1 - \frac{2}{3} = \frac{1}{3},$$

$$P(\{1 < X < 2\}) = F_X(2-) - F_X(1) = \frac{1}{2} - \frac{1}{2} = 0$$

$$P(\{1 \leq X < 3\}) = F_X(3-) - F_X(1-) = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

$$P(\{2 \leq X \leq 4\}) = F_X(4) - F_X(2-) = 1 - \frac{1}{2} = \frac{1}{2}.$$

Let X be a r.v. with d.f.

$$F_X(x) = \begin{cases} 0, & \text{if } x < 1 \\ 1 - \frac{1}{x}, & \text{if } x \geq 1 \end{cases}.$$

Let $S = \{1, 2, 3, ...\}$. Find the value of $P(\{X \in S\})$.

Take Home Problem

1. Let $G(\cdot)$ be d.f. of some r.v. Show that

   (a)
$$G(x-) \leq G(x) = G(x+)\,,\ \forall\ x \in \mathbb{R};$$

   (b)
$$G(x+) \leq G(y-)\,, \forall -\infty < x < y < \infty.$$

2. Let $F_X(\cdot)$ be continuous on $\mathbb{R}$ and let S be any countable set. Show that $P(\{X \in S\}) = 0$.

Abstract of Next Module

1. We have seen that

$$\begin{aligned} D_X &= \text{set of discontinuity points of } F_X(\cdot) \\ &= \{x \in \mathbb{R} : P(\{X = x\}) > 0\} \\ &\quad \text{(a countable set)} \end{aligned}$$

   There may be situations where

$$P(\{X \in D_X\}) = \sum_{x \in D_X} P(\{X = x\}) = 1.$$

   We call such r.v.s as discrete r.v.

2. In next module, we will introduce discrete r.v.s and study their properties.

Thank you for your patience