

Module 4

PROPERTIES OF PROBABILITY FUNCTION

- $(\Omega, \mathcal{P}(\Omega), P)$: a given probability space;
- For an event $E \in \mathcal{P}(\Omega)$, let $E^c = \Omega - E$ denote its complementary event (set of outcomes not favorable to E).

Result 1: For any event E

$$0 \leq P(E) \leq 1 \text{ and } P(E^c) = 1 - P(E).$$

Proof: Since $\Omega = E \cup E^c$ and, E and E^c are disjoint

$$\begin{aligned} P(\Omega) &= P(E \cup E^c) \\ 1 &= P(E) + P(E^c) \quad (\text{Axiom 2 and Axiom 3}) \\ &\geq P(E) \geq 0. \quad (\text{Axiom 1}) \end{aligned}$$

Thus $0 \leq P(E) \leq 1$. Since $0 \leq P(E) \leq 1 < \infty$, we also have $P(E^c) = 1 - P(E)$.

Result 2: Let E and F be events such that $E \subseteq F$. Then

$$P(E) \leq P(F) \quad (\text{monotonicity of probability function})$$

and

$$P(F - E) = P(F) - P(E).$$

Proof: Let $E, F \in \mathcal{P}(\Omega)$ and let $E \subseteq F$. Since $F = E \cup (F - E)$ and, E and $F - E$ are disjoint, we have

$$\begin{aligned} P(F) &= P(E \cup (F - E)) \\ &= P(E) + P(F - E) \quad (\text{Axiom 2}) \\ &\geq P(E). \quad (\text{Axiom 1}) \end{aligned}$$

Since $0 \leq P(E) \leq 1 < \infty$, we have

$$P(F - E) = P(F) - P(E).$$

Result 3: Let E_1 and E_2 be two events (not necessarily disjoint). Then

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2).$$

Proof: Since

$$E_1 \cup E_2 = E_1 \cup (E_2 - E_1),$$

and, E_1 and $E_2 - E_1$ are disjoint, we have

$$P(E_1 \cup E_2) = P(E_1) + P(E_2 - E_1).$$

Also, $E_2 - E_1 = E_2 - (E_1 \cap E_2)$ and $E_1 \cap E_2 \subseteq E_2$. Therefore,

$$\begin{aligned} P(E_2 - E_1) &= P(E_2) - P(E_1 \cap E_2) \quad (\text{Result 2}) \\ \implies P(E_1 \cup E_2) &= P(E_1) + P(E_2) - P(E_1 \cap E_2). \end{aligned}$$

Result 4 (Inclusion Exclusion Formula): For events E_1, E_2, \dots, E_n ($n \geq 2$), let

$$\begin{aligned}
 p_{1,n} &= \sum_{i=1}^n P(E_i) \\
 p_{2,n} &= \sum_{1 \leq i < j \leq n} P(E_i \cap E_j) \\
 &\vdots \\
 p_{r,n} &= \sum_{1 \leq i_1 < i_2 < \dots < i_r \leq n} P(E_{i_1} \cap E_{i_2} \cap \dots \cap E_{i_r}), \quad r = 2, \dots, n \\
 p_{n,n} &= P(E_1 \cap E_2 \cap \dots \cap E_n).
 \end{aligned}$$

Then

$$P\left(\bigcup_{i=1}^n E_i\right) = p_{1,n} - p_{2,n} + p_{3,n} - \dots + (-1)^{n-1} p_{n,n}. \quad (1)$$

Proof (By mathematical induction)

The result is clearly true for $n = 2$ (Result 3). Assume that equation (1) holds for any collection of m events. Then

$$\begin{aligned}
P\left(\bigcup_{i=1}^{m+1} E_i\right) &= P\left(\left(\bigcup_{i=1}^m E_i\right) \cup E_{m+1}\right) \\
&= P\left(\bigcup_{i=1}^m E_i\right) + P(E_{m+1}) - P\left(\left(\bigcup_{i=1}^m E_i\right) \cap E_{m+1}\right) \\
&= P\left(\bigcup_{i=1}^m E_i\right) + P(E_{m+1}) - P\left(\bigcup_{i=1}^m (E_i \cap E_{m+1})\right) \\
&= p_{1,m} - p_{2,m} + p_{3,m} \cdots + (-1)^{r-1} p_{r,m} + \cdots + (-1)^{m-1} p_{m,m} + P(E_{m+1}) \\
&\quad - \left(t_{1,m} - t_{2,m} + t_{3,m} \cdots + (-1)^{r-2} t_{r-1,m} + \cdots + (-1)^{m-1} t_m\right),
\end{aligned}$$

where, for $r = 1, \dots, m$,

$$\begin{aligned}
p_{r,m} &= \sum_{1 \leq i_1 < i_2 < \cdots < i_r \leq m} P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_r}) \\
t_{r,m} &= \sum_{1 \leq i_1 < i_2 < \cdots < i_r \leq m} P((E_{i_1} \cap E_{m+1}) \cap (E_{i_2} \cap E_{m+1}) \cap \cdots \cap (E_{i_r} \cap E_{m+1})) \\
&= \sum_{1 \leq i_1 < i_2 < \cdots < i_r \leq m} P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_r} \cap E_{m+1}).
\end{aligned}$$

Clearly,

$$\begin{aligned}p_{1,m} + P(E_{m+1}) &= p_{1,m+1} \\p_{2,m} + t_{1,m} &= p_{2,m+1} \\p_{r,m} + t_{r-1,m} &= p_{r,m+1}, \quad r = 2, 3, \dots, m \\t_{m,m} &= p_{m+1,m+1}.\end{aligned}$$

Thus

$$P\left(\bigcup_{i=1}^{m+1} E_i\right) = p_{1,m+1} - p_{2,m+1} + p_{3,m+1} + \cdots + (-1)^m p_{m+1,m+1}.$$

Result 5: For events E_1, E_2, \dots, E_n

(i)

$$P\left(\bigcup_{i=1}^n E_i\right) \leq \sum_{i=1}^n P(E_i); \quad (\text{Boole's inequality})$$

(ii)

$$P\left(\bigcap_{i=1}^n E_i\right) \geq \sum_{i=1}^n P(E_i) - (n-1). \quad (\text{Bonferroni's inequality})$$

Proof: (i) We have

$$\bigcup_{i=1}^n E_i = E_1 \cup (E_2 \cap E_1^c) \cup (E_3 \cap E_1^c \cap E_2^c) \cup \dots \cup (E_n \cap E_1^c \cap E_2^c \cap \dots \cap E_{n-1}^c),$$

where $E_1, (E_2 \cap E_1^c), (E_3 \cap E_1^c \cap E_2^c), \dots, (E_n \cap E_1^c \cap E_2^c \cap \dots \cap E_{n-1}^c)$ are disjoint events. Also $E_r \cap E_1^c \cap E_2^c \cap \dots \cap E_{r-1}^c \subseteq E_r$, $r = 2, \dots, n$.

Thus

$$\begin{aligned} P\left(\bigcup_{i=1}^n E_i\right) &= P(E_1) + P(E_2 \cap E_1^c) + P(E_3 \cap E_1^c \cap E_2^c) + \dots + P(E_n \cap E_1^c \cap E_2^c \cap \dots \cap E_{n-1}^c) \\ &\leq P(E_1) + P(E_2) + P(E_3) + \dots + P(E_n). \end{aligned}$$

(ii)

$$\begin{aligned} P\left(\bigcap_{i=1}^n E_i\right) &= P\left(\left(\bigcup_{i=1}^n E_i^c\right)^c\right) \\ &= 1 - P\left(\bigcup_{i=1}^n E_i^c\right) \\ &\geq 1 - \sum_{i=1}^n P(E_i^c) \quad (\text{using Boole's inequality}) \\ &= 1 - \sum_{i=1}^n (1 - P(E_i)) \\ &= 1 - n + \sum_{i=1}^n P(E_i) \\ &= \sum_{i=1}^n P(E_i) - (n - 1). \end{aligned}$$

Remark 1: : Under the notation of Result 4, it can be shown that

$$\begin{aligned}
p_{1,n} - p_{2,n} &\leq P \left(\bigcup_{i=1}^n E_i \right) \leq p_{1,n}; \\
p_{1,n} - p_{2,n} + p_{3,n} - p_{4,n} &\leq P \left(\bigcup_{i=1}^n E_i \right) \leq p_{1,n} - p_{2,n} + p_{3,n}; \\
&\vdots \\
\sum_{i=1}^{2r} (-1)^{i-1} p_{i,n} &\leq P \left(\bigcup_{i=1}^n E_i \right) \leq \sum_{i=1}^{2r-1} (-1)^{i-1} p_{i,n}, \quad r = 1, 2, \dots
\end{aligned}$$

~~Take Home Problems:~~

(1) Let E be an event such that $P(E) = 1$. Show that

$$P(F) = P(E \cap F), \quad \forall F \in \mathcal{P}(\Omega).$$

(2) Let E be an event such that $P(E) = 0$. Show that

$$P(F) = P(E \cup F) \quad \forall F \in \mathcal{P}(\Omega).$$

(3) If $P(E_i) = 1$, $i = 1, \dots, n$, show that

$$P\left(\bigcap_{i=1}^n E_i\right) = 1.$$

(4) If $P(E_i) = 0$, $i = 1, \dots, n$, show that

$$P\left(\bigcup_{i=1}^n E_i\right) = 0.$$

Abstract of Module 5

We will discuss:

- Continuity of probability function.
- Equally likely probability models.

Thank you for your patience

