Module 11

CONTINUOUS

AND

ABSOLUTELY CONTINUOUS R.V.S

- X : a given r.v. defined on some probability space $(\Omega, \mathcal{P}(\Omega), P)$;
- F_X: d.f. of X.

Definition 1 :

(a) The r.v. X is said to be continuous if d.f. $F_X(\cdot)$ is continuous on $\mathbb{R}$;

(b) The r.v. X is said to be absolutely continuous (A.C.) if there exists a non-negative integrable function $f_X : \mathbb{R} \rightarrow [0, \infty)$ such that

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\, dt,\ \ x \in \mathbb{R}.$$

The function $f_X(\cdot)$ is called a probability density function (p.d.f.) of X and the set

$$\begin{aligned} S_X &= \{x \in \mathbb{R} : F_X(x+\epsilon) - F_X(x-\epsilon) > 0, \\ &\qquad \forall\, \epsilon > 0\} \\ &= \{x \in \mathbb{R} : P(\{x-\epsilon < X \leq x+\epsilon\}) > 0, \\ &\qquad \forall\, \epsilon > 0\} \end{aligned}$$

is called the support of X.

Remark 1:

(a) Since the integral of any function remains unaltered if we change the values of function at a finite (in fact even countable) number of points, the p.d.f. of A.C. r.v. is not unique. For example, for the d.f.

$$F_X(x) = \begin{cases} 0, & \text{if } x < 0 \\ x, & \text{if } 0 \le x < 1 \\ 1, & \text{if } x \ge 1 \end{cases},$$

the functions

$$g_1(x) = \begin{cases} 1, & \text{if } 0 < x < 1 \\ 0, & \text{otherwise} \end{cases},$$

and

$$g_2(x) = \begin{cases} 1, & \text{if } 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

are both p.d.f.s of X.

In the above example, the support of X is $S_X = [0,1]$.

(b) The p.d.f. $f_X(\cdot)$ of an A.C. r.v. X satisfies

$$\begin{aligned} f_X(x) &\geq 0, \ \forall \ x \in \mathbb{R} \\ \text{and} \quad \int_{-\infty}^{\infty} f_X(x)\, dx &= F_X(\infty) \\ &= 1. \end{aligned}$$

Conversely, it can be shown that, any function $g : \mathbb{R} \to \mathbb{R}$ that satisfies the above two conditions (i.e., $g(x) \geq 0, \ \forall \ x \in \mathbb{R}$ and $\int_{-\infty}^{\infty} g(x)dx = 1$) is a p.d.f of some r.v. Y.

(c) If X is A.C. then

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\, dt, \;\; x \in \mathbb{R}$$

is continuous on $\mathbb{R}$. Thus every A.C. r.v. is continuous. Converse may not be true. There are r.v.s which are continuous but not A.C. Such r.v.s are difficult to handle and will not be studied in this course.

(d) Since the d.f. of a r.v. X is determined by its p.d.f. $f_X(\cdot)$, to study the probability function $P_X(\cdot)$ induced by r.v. X, it suffices to study p.d.f. $f_X(\cdot)$ of X.

(e) Wherever $F_X(x)$ is differentiable

$$f_X(x) = F_X'(x), \quad x \in \mathbb{R}.$$

In particular if $f_X(x)$ is continuous at $x = x_0$ then

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt, \quad x \in \mathbb{R},$$

is differentiable at $x = x_0$ and $f_X(x_0) = F_X'(x_0)$.

(f) In this course we will consider only those A.C. r.v.s whose p.d.f.s are continuous everywhere except (possibly) at finite number of points where it has jump discontinuities ($f_X(x_0+) = \lim_{x \downarrow x_0} f_X(x)$ and $f_X(x_0-) = \lim_{x \uparrow x_0} f_X(x)$ exist but may be different). Such functions are called piecewise continuous functions. Then the d.f. $F_X(\cdot)$ is differentiable everywhere except (possibly) at finite number of points, say $x_1, \ldots, x_n$. In such situations we take

$$f_X(x) = \begin{cases} F'_X(x), & \text{if } x \notin \{x_1, \ldots, x_n\} \\ 0, & \text{otherwise} \end{cases}.$$

(g) Let X be a continuous (or A.C.) r.v. with d.f. $F_X(\cdot)$. Then $F_X(\cdot)$ is continuous everywhere and therefore

$$P(\{X = x\}) = F_X(x) - F_X(x-) = 0, \ \forall \ x \in \mathbb{R}.$$

In general for any countable set C

$$\begin{aligned} P(\{X \in C\}) &= P\left(\bigcup_{x \in C} \{X = x\}\right) \\ &= \sum_{x \in C} P(\{X = x\}) = 0. \end{aligned}$$

For $-\infty < a < b < \infty$

$$P\left(\{a < X \leq b\}\right) = P\left(\{a < X < b\}\right)$$

$$= P\left(\{a \leq X < b\}\right) = P\left(\{a \leq X \leq b\}\right)$$

$$= F_X\left(b\right) - F_X\left(a\right)$$

$$= \int_{-\infty}^{b} f_X(t)dt - \int_{-\infty}^{a} f_X(t)dt$$

$$= \int_{a}^{b} f_X(t)dt.$$

Also

$$
\begin{aligned}
P(X > a) &= 1 - P(X \leq a) \\
&= F_X(\infty) - F_X(a) \\
&= \int_a^{\infty} f_X(t)dt \\
\text{and } P(\{X < a\}) &= P(X \leq a) \\
&= \int_{-\infty}^{a} f_X(t)dt.
\end{aligned}
$$

Result 1: Let X be a r.v. with d.f. $F_X(\cdot)$ which is differentiable everywhere except possibly at finite number of points (say, $x_1, \ldots, x_n$). Define

$$g(x) = \begin{cases} F_X'(x), & \text{if } x \notin \{x_1, \ldots, x_n\} \\ 0, & \text{otherwise} \end{cases}.$$

Then X is A.C. iff

$$\int_{-\infty}^{\infty} g(x)\, dx = 1.$$

In that case $g(\cdot)$ is a p.d.f. of X.

Example 1: Let X be a r.v. with d.f.

$$F_X(x) = \begin{cases} 0, & \text{if } x < 0 \\ x, & \text{if } 0 \leq x < 1 \\ 1, & \text{if } x \geq 1 \end{cases}.$$

- Clearly $F_X(\cdot)$ is continuous, non-decreasing, $F_X(-\infty) = 0$ and $F_X(\infty) = 1$;
-

$$\begin{aligned} S_X &= \{x \in \mathbb{R} : F_X(x+\epsilon) - F_X(x-\epsilon) > 0, \\ &\qquad \forall \varepsilon > 0\} \\ &= [0,1]. \end{aligned}$$

- $F_X(\cdot)$ is differentiable everywhere except at $x = 0, 1$. Define

$$g(x) = \begin{cases} F'_X(x), & \text{if } x \notin \{0, 1\} \\ 0, & \text{otherwise} \end{cases}$$
$$= \begin{cases} 1, & \text{if } 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}.$$

- Clearly

$$\int_{-\infty}^{\infty} g(x)\, dx = 1.$$

Therefore X is A.C. with p.d.f. $f_X(x) = g(x), x \in \mathbb{R}$.

Example 2: Let X be a r.v. with d.f.

$$F_X(x) = \begin{cases} 0, & \text{if } x < 0 \\ \frac{x+2}{4}, & \text{if } 0 \leq x < 1 \\ \frac{4x}{5}, & \text{if } 1 \leq x < \frac{9}{8} \\ 1, & \text{if } x \geq \frac{9}{8} \end{cases}.$$

- Clearly $F_X(\cdot)$ is right continuous, non-decreasing, $F_X(-\infty) = 0$ and $F_X(\infty) = 1$;
- $$\begin{aligned} S_X &= \{x \in \mathbb{R} : F_X(x+\epsilon) - F_X(x-\epsilon) > 0, \\ &\qquad \forall\ \epsilon > 0\} \\ &= \left[0, \frac{9}{8}\right]. \end{aligned}$$
- $D_X = \left\{0, 1, \frac{9}{8}\right\} \neq \phi$. So X is not continuous (and hence also not A.C.).

- $$\begin{aligned} P(\{X \in D_X\}) &= P(\{X = 0\}) + P(\{X = 1\}) \\ &\quad + P\left(\{X = \frac{9}{8}\}\right) \\ &= \left[\frac{2}{4} - 0\right] + \left[\frac{4}{5} - \frac{3}{4}\right] + \left[1 - \frac{36}{40}\right] \\ &= \frac{13}{20} < 1. \end{aligned}$$

  Therefore X is neither a discrete r.v..

Remark 2: The above example demonstrates that there are r.v.s that are neither discrete nor continuous (or A.C).

Let X be a r.v. with d.f.

$$F_X(x) = \begin{cases} 0, & \text{if } x < 0 \\ \frac{x}{4}, & \text{if } 0 \leq x < 1 \\ \frac{x}{2}, & \text{if } 1 \leq x < \frac{4}{3} \\ \frac{3x}{5}, & \text{if } \frac{4}{3} \leq x < \frac{5}{3} \\ 1, & \text{if } x \geq 1 \end{cases}.$$

(i) Then $S_X = \ldots\ldots\ldots\ldots$

(ii) Then X is

(a) discrete (b) Continuous
(c) A.C. (d) neither discrete nor A.C.

(iii) $P\left(\{1 \leq X < \frac{4}{3}\}\right) = \ldots\ldots\ldots\ldots;$

(iv) $P\left(\{\frac{1}{2} < X < \frac{3}{4}\}\right) = \ldots\ldots\ldots\ldots$

Abstract of Next Module

- X: a discrete r.v. with d.f. $F_X(\cdot)$ and p.m.f. $f_X(\cdot)$;
- $Y = g(X)$, where $g : \mathbb{R} \to \mathbb{R}$ is a given function.
- How to find p.d.f/p.m.f of Y?

Thank you for your patience