# Module 7

## INDEPENDENT EVENTS

**Definition 1:** Events $E_1$, $E_2$, ..., $E_n$ are said to be

**(a)** pairwise indenpendent if

$$P\left(E_i \bigcap E_j\right) = P(E_i)P(E_j), \forall i \neq j;$$

**(b)** mutually independent if $\forall\ k \in \{2, 3, \ldots, n\}$ and distinct $d_1, d_2, \ldots,\ d_k \in \{1, 2, \ldots, n\}$

$$P\left(E_{d_1} \bigcap E_{d_2} \bigcap \cdots \bigcap E_{d_k}\right) = P\left(E_{d_1}\right)\left(E_{d_2}\right)\cdots\left(E_{d_k}\right)$$

($\sum_{j=2}^{n} \binom{n}{j} = 2^n - n - 1$ conditions).

- Mutual independence of events $E_1, \ldots, E_n \Rightarrow$ pairwise independence of events $E_1, \ldots, E_n$. Converse may not be true, i.e., in general

  pairwise independence of events $E_1, \ldots, E_n \not\Rightarrow$ mutual independence of events $E_1, \ldots, E_n$; as the following example illustrates.

## Example 1:

- Let $\Omega = \{1, 2, 3, 4\}$ and let $P(\cdot)$ be such that

$$P(\{i\}) = \frac{1}{4}, \quad i = 1, 2, 3, 4.$$

- Let $E_1 = \{1, 4\}$, $E_2 = \{2, 4\}$ and $E_3 = \{3, 4\}$. Then

$$P(E_1) = P(E_2) = P(E_3) = \frac{1}{2}$$

$$P\left(E_1 \bigcap E_2\right) = P\left(E_1 \bigcap E_3\right) = P\left(E_2 \bigcap E_3\right) = P(\{4\}) = \frac{1}{4}$$

and

$$P\left(E_1 \bigcap E_2 \bigcap E_3\right) = P(\{4\}) = \frac{1}{4}.$$

- Clearly

$$P\left(E_i \bigcap E_j\right) = P(E_i)\, P(E_j) = \frac{1}{4} \quad \forall\, i \neq j,$$

implying that $E_1$, $E_2$ and $E_3$ are pairwise independent.

- However

$$P\left(E_1 \bigcap E_2 \bigcap E_3\right) = \frac{1}{4} \neq \frac{1}{8} = P(E_1)\, P(E_2)\, P(E_3),$$

implying that $E_1$, $E_2$ and $E_3$ are not mutually independent.

**Remark 1:**

**(a)** Events in any subcollection of independent events are independent.

**(b)** Suppose that $E_1$, $E_2$, ..., $E_n$ are independent, $k \in \{1, 2, \ldots, n-1\}$ and $1 \leq i_1 < i_2 < \cdots < i_k \leq n-1$. Then

$$\begin{aligned}
P\left(E_{i_1} \bigcap E_{i_2} \bigcap \ldots \bigcap E_{i_k} \bigcap E_n^c\right) &= P\left(E_{i_1} \bigcap E_{i_2} \bigcap \ldots \bigcap E_{i_k}\right) \\
&\quad -P\left(E_{i_1} \bigcap E_{i_2} \bigcap \ldots \bigcap E_{i_k} \bigcap E_n\right) \\
&= \prod_{j=1}^{k} P\left(E_{i_j}\right) - \left(\prod_{j=1}^{k} P\left(E_{i_j}\right)\right) P\left(E_n\right) \\
&= \prod_{j=1}^{k} P\left(E_{i_j}\right)\left[1 - P\left(E_n\right)\right] \\
&= \left(\prod_{j=1}^{k} P\left(E_{i_j}\right)\right) P\left(E_n^c\right).
\end{aligned}$$

Thus

$E_1$, $E_2$, …, $E_n$ are independent $\implies$ $E_{i_1}, \ldots, E_{i_k}$, $E^c_{i_{k+1}}, \ldots, E^c_{i_n}$ are independent, where $1 \leq k \leq n-1$ and $\{i_1, i_2, \ldots, i_n\} = \{1, 2, \ldots, n\}$.

Also $E_1$, $E_2$, …, $E_n$ are independent $\implies$ $E^c_1, E^c_2, \ldots, E^c_n$ are independent.

**(c)** When we say that two random experiments are performed independently what it means is that associated events are independent

**(d)** Suppose that $P(E_1) > 0$. Then $E_1$ and $E_2$ are independent if, and only if,

$$
\begin{aligned}
P\left(E_1 \bigcap E_2\right) &= P(E_1)\,P(E_2) \\
\Leftrightarrow \qquad \frac{P(E_1 \bigcap E_2)}{P(E_1)} &= P(E_2) \\
\Leftrightarrow \qquad P(E_2|E_1) &= P(E_2)
\end{aligned}
$$

$\Leftrightarrow$ Conditional probability of $E_2$ given $E_1$ is the same as unconditional probability of $E_2$.

**(e)** If $E_1$, $E_2$, …, $E_n$ are independent events then

- $E_1^c$ and $E_2 \bigcup E_3^c \bigcup E_4$ are independent;
- $E_1 \bigcup E_2^c$ and $E_3^c$ and $E_4 \bigcap E_5^c$ are independent.

Let $E_1$, $E_2$ and $E_3$ be independent events with $P(E_i) = \frac{1}{i+1}$, $i = 1, 2, 3$. Find the value of $P(E_1 \bigcup E_2^c \bigcup E_3)$.

**Take Home Problems:**

Let $\{E_n\}_{n\geq 1}$ be a sequence of independent events (i.e., event in any finite sub-collection of $\{E_n\}_{n\geq 1}$ are independent).

**(a)** Show that

$$P\left(\bigcup_{i=1}^{n} E_i\right) \geq 1 - e^{-\sum_{i=1}^{n} P(E_i)}, \quad n = 1, 2, \ldots$$

**(b)** If $\sum_{i=1}^{\infty} P(E_i) = \infty$, show that

$$P\left(\bigcap_{i=1}^{\infty} E_i^c\right) = 0.$$

## Abstract of Next Module

- In many situations we may not be directly interested in the sample space $\Omega$. Rather we may be interested in some numerical aspect of $\Omega$, i.e., we may be interested in a function $X : \Omega \to \mathbb{R}$. Such functions are called random variables (r.v.s)

- We will formally define r.v. and study the properties of probability functions induced by them.

# Thank you for your patience