Circuits and Transforms

EE3900: Linear Systems and Signal Processing

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1. Definitions

1.1 The unit step function is defined as

$$u(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases} \tag{1.1}$$

1.2 The Laplace transform of $g(t)$ is defined as

$$G(s) = \int_{-\infty}^{\infty} g(t)e^{-st}\,\mathrm{d}t \tag{1.2}$$

2. Laplace Transform

2.1. In the circuit, the switch S is connected to position P for a long time so that the charge on the capacitor becomes q_1 µC. Then S is switched to position Q. After a long time, the charge on the capacitor is q_2 µC

2.2. Draw the circuit using latex-tikz
Solution:

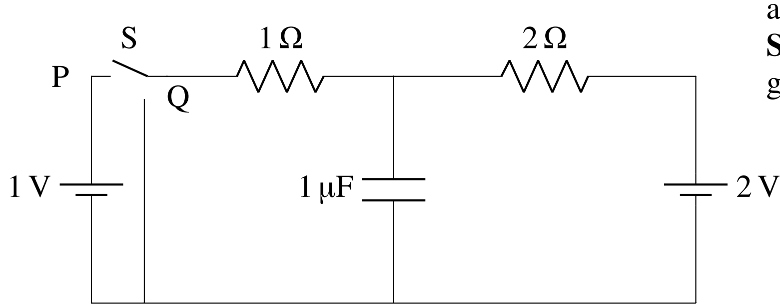

Fig. 2.2. Circuit diagram of the circuit in question

2.3. Find q_1
Solution: After a long time, when steady state is achieved, a capacitor behaves like an open circuit, i.e., current passing through it is zero

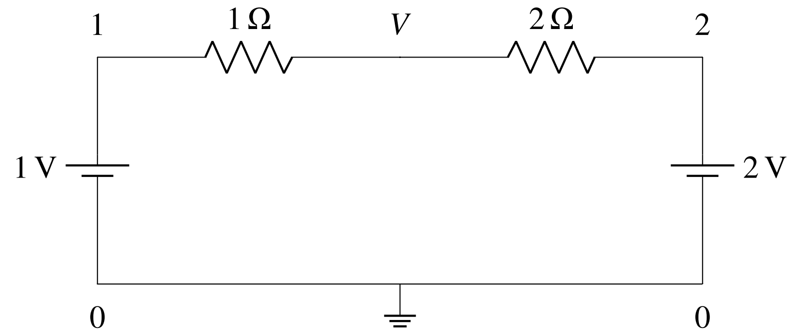

Fig. 2.3. Circuit diagram at steady state before flipping the switch

By Kirchoff's junction law, we get

$$\frac{V-1}{1} + \frac{V-2}{2} = 0 \tag{2.1}$$

$$\implies V = \frac{4}{3}\,\mathrm{V} \tag{2.2}$$

$$\implies q_1 = CV = \frac{4}{3}\,\mu\mathrm{C} \tag{2.3}$$

2.4. Show that the Laplace transform of $u(t)$ is $\frac{1}{s}$ and find the ROC
Solution: The Laplace transform of $u(t)$ is given by

$$\mathcal{L}\{u(t)\} = \int_{-\infty}^{\infty} u(t)e^{-st}\,\mathrm{d}t \tag{2.4}$$

$$= \int_{0}^{\infty} e^{-st}\,\mathrm{d}t \tag{2.5}$$

$$= \lim_{R\to\infty} \frac{1-e^{-sR}}{s} \tag{2.6}$$

This limit is finite only if $\Re(s) > 0$, which is going to be its ROC
Therefore

$$u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s} \qquad \Re(s) > 0 \tag{2.7}$$

2.5. Show that

$$e^{-at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a} \qquad a > 0 \tag{2.8}$$

and find the ROC
Solution: The Laplace transform of $e^{-at}u(t)$ for $a > 0$ is given by

$$\mathcal{L}\left\{u(t)\right\} = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}\, \mathrm{d}t \tag{2.9}$$

$$= \int_{0}^{\infty} e^{-(s+a)t}\, \mathrm{d}t \tag{2.10}$$

$$= \lim_{R\to\infty} \frac{1 - e^{-(s+a)R}}{s+a} \tag{2.11}$$

This limit is finite only if $\Re(s+a) > 0$, which is going to be its ROC
Therefore

$$e^{-at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a} \qquad \Re(s) > -a \tag{2.12}$$

since a is real

2.6. Now consider the following resistive circuit transformed from Fig. 2.2

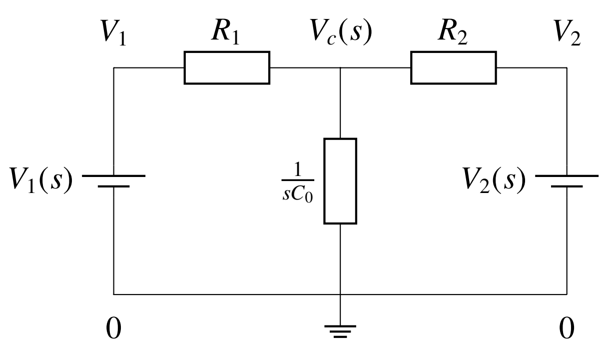

Fig. 2.6. Circuit diagram in s-domain before flipping the switch

where

$$u(t) \overset{\mathcal{L}}{\longleftrightarrow} V_1(s) \tag{2.13}$$

$$2u(t) \overset{\mathcal{L}}{\longleftrightarrow} V_2(s) \tag{2.14}$$

Find the voltage across the capacitor $V_c(s)$
Solution:

$$V_1(s) = \frac{1}{s} \qquad \Re(s) > 0 \tag{2.15}$$

$$V_2(s) = \frac{2}{s} \qquad \Re(s) > 0 \tag{2.16}$$

By Kirchoff's junction law, we get

$$\frac{V_c - V_1}{R_1} + \frac{V_c - V_2}{R_2} + \frac{V_c - 0}{\frac{1}{sC_0}} = 0 \tag{2.17}$$

$$\implies V_c\left(\frac{1}{R_1} + \frac{1}{R_2} + sC_0\right) = \frac{V_1}{R_1} + \frac{V_2}{R_2} \tag{2.18}$$

$$\implies V_c(s) = \frac{\frac{1}{sR_1} + \frac{2}{sR_2}}{\frac{1}{R_1} + \frac{1}{R_2} + sC_0} \tag{2.19}$$

$$= \frac{\frac{1}{R_1C_0} + \frac{2}{R_2C_0}}{s\left(s + \frac{1}{R_1C_0} + \frac{1}{R_2C_0}\right)} \tag{2.20}$$

2.7. Find $v_c(t)$. Plot using Python.
Solution: On performing partial fraction decomposition

$$V_c(s) = \frac{\frac{1}{R_1C_0} + \frac{2}{R_2C_0}}{\frac{1}{R_1C_0} + \frac{1}{R_2C_0}}\left(\frac{1}{s} - \frac{1}{s + \frac{1}{R_1C_0} + \frac{1}{R_2C_0}}\right), \Re(s) > 0 \tag{2.21}$$

On taking the inverse Laplace transform, we get

$$v_c(t) = \frac{2R_1 + R_2}{R_1 + R_2}\left(u(t) - e^{-\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\frac{t}{C_0}}u(t)\right) \tag{2.22}$$

$$= \frac{2R_1 + R_2}{R_1 + R_2}\left(1 - e^{-\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\frac{t}{C_0}}\right)u(t) \tag{2.23}$$

Substitute the values $R_1 = 1\,\Omega, R_2 = 2\,\Omega, C_0 = 1\,\mu\mathrm{F}$

$$v_c(t) = \frac{4}{3}\left(1 - \mathrm{e}^{-\frac{3}{2}\times 10^6 \mathrm{t}}\right)\mathrm{u(t)}\,\mathrm{V} \tag{2.24}$$

2.8. Verify your result using ngspice
Solution: Download the following codes for simulation and plotting Fig. 2.8 respectively

```
wget https://github.com/rudranshm/EE3900/
    raw/main/Circuit/codes/2.8.cir
wget https://github.com/rudranshm/EE3900/
    raw/main/Circuit/codes/2.7.py
```

Run the codes by executing

```
ngspice 2.8.cir
python 2.7.py
```

2.9. Obtain Fig. 2.6 using the equivalent differential equation
Solution: Using Kirchoff's junction law

$$\frac{v_c(t) - v_1(t)}{R_1} + \frac{v_c(t) - v_2(t)}{R_2} + \frac{\mathrm{d}q}{\mathrm{d}t} = 0 \tag{2.25}$$

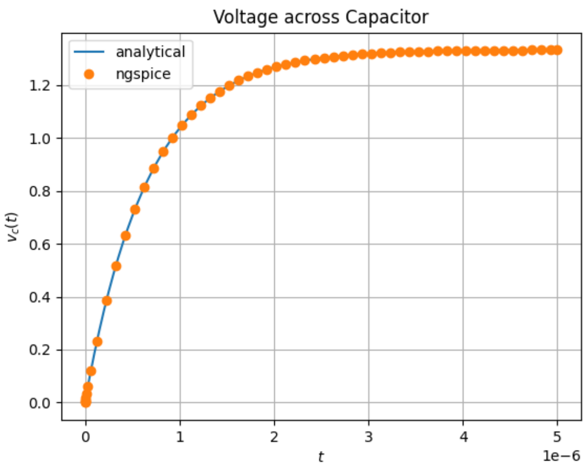

Fig. 2.8. Plot of $v_c(t)$ before flipping the switch

where $q(t)$ is the charge on the capacitor
On taking the Laplace transform on both sides of this equation

$$\frac{V_c(s) - V_1(s)}{R_1} + \frac{V_c(s) - V_2(s)}{R_2} + \left(sQ(s) - q(0^-)\right) = 0 \tag{2.26}$$

But $q(0^-) = 0$ and

$$q(t) = C_0 v_c(t) \tag{2.27}$$
$$\implies Q(s) = C_0 V_c(s) \tag{2.28}$$

Thus

$$\frac{V_c(s) - V_1(s)}{R_1} + \frac{V_c(s) - V_2(s)}{R_2} + sC_0 V_c(s) = 0 \tag{2.29}$$
$$\implies \frac{V_c(s) - V_1(s)}{R_1} + \frac{V_c(s) - V_2(s)}{R_2} + \frac{V_c(s) - 0}{\frac{1}{sC_0}} = 0 \tag{2.30}$$

which is the same equation as the one we obtained from Fig. 2.6

3. Initial Conditions

3.1. Find q_2 in Fig. 2.2
Solution: After a long time, when steady state is achieved, a capacitor behaves like an open circuit, i.e., current passing through it is zero

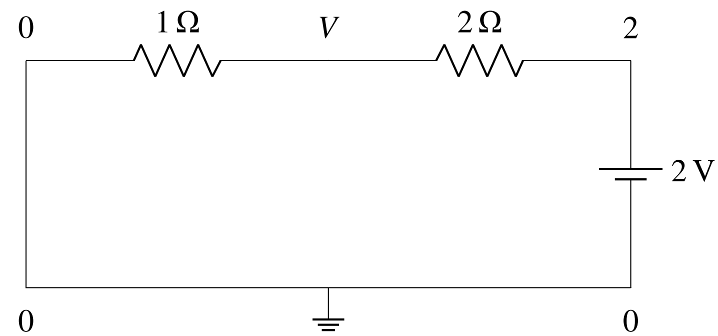

Fig. 3.1. Circuit diagram at steady state after flipping the switch

By Kirchoff's junction law, we get

$$\frac{V-0}{1} + \frac{V-2}{2} = 0 \tag{3.1}$$
$$\implies V = \frac{2}{3}\,\mathrm{V} \tag{3.2}$$
$$\implies q_2 = CV = \frac{2}{3}\,\mu\mathrm{C} \tag{3.3}$$

3.2. Draw the equivalent s-domain resistive circuit when S is switched to position Q. Use variables R_1, R_2, C_0 for the passive elements. Use latex-tikz
Solution:

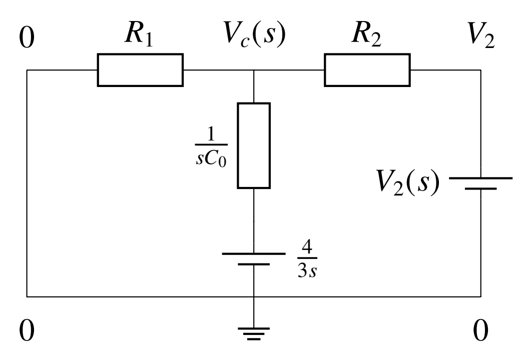

Fig. 3.2. Circuit diagram in s-domain after flipping the switch

The battery $\frac{4}{3s}$ corresponds to the intial potential difference of $\frac{4}{3}$ V across the capacitor just before switching it to Q

3.3. Find $V_c(s)$

Solution: By Kirchoff's junction law, we get

$$\frac{V_c - 0}{R_1} + \frac{V_c - V_2}{R_2} + \frac{V_c - \frac{4}{3s}}{\frac{1}{sC_0}} = 0 \quad (3.4)$$

$$\implies V_c\left(\frac{1}{R_1} + \frac{1}{R_2} + sC_0\right) = \frac{V_2}{R_2} + \frac{4}{3}C_0 \quad (3.5)$$

$$\implies V_c(s) = \frac{\frac{2}{sR_2} + \frac{4}{3}C_0}{\frac{1}{R_1} + \frac{1}{R_2} + sC_0} \quad (3.6)$$

$$= \frac{\frac{2}{R_2C_0} + \frac{4}{3}s}{s\left(s + \frac{1}{R_1C_0} + \frac{1}{R_2C_0}\right)} \quad (3.7)$$

3.4. Find $v_c(t)$. Plot using Python
Solution: On performing partial fraction decomposition

$$V_c(s) = \frac{4}{3}\left(\frac{1}{s + \frac{1}{R_1C_0} + \frac{1}{R_2C_0}}\right) + \frac{\frac{2}{R_2C_0}}{\frac{1}{R_1C_0} + \frac{1}{R_2C_0}}\left(\frac{1}{s} - \frac{1}{s + \frac{1}{R_1C_0} + \frac{1}{R_2C_0}}\right) \quad (3.8)$$

for $\Re(s) > 0$
On taking the inverse Laplace transform, we get

$$v_c(t) = \frac{4}{3}e^{-\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\frac{t}{C_0}}u(t) + \frac{2R_1}{R_1 + R_2}\left(u(t) - e^{-\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\frac{t}{C_0}}u(t)\right) \quad (3.9)$$

Substitute the values $R_1 = 1\,\Omega, R_2 = 2\,\Omega, C_0 = 1\,\mu\text{F}$

$$v_c(t) = \frac{4}{3}e^{-\frac{3}{2}\times10^6 t}u(t) + \frac{2}{3}\left(1 - e^{-\frac{3}{2}\times10^6 t}\right)u(t) \quad (3.10)$$

$$= \frac{2}{3}\left(1 + \mathrm{e}^{-\frac{3}{2}\times10^6 \mathrm{t}}\right)\mathrm{u(t)\,V} \quad (3.11)$$

3.5. Verify your result using ngspice
Solution: Download the following codes for simulation and plotting Fig. 3.5 respectively

```
wget https://github.com/rudranshm/EE3900/
    raw/main/Circuit/codes/3.5.cir
wget https://github.com/rudranshm/EE3900/
    raw/main/Circuit/codes/3.4.py
```

Run the codes by executing

```
ngspice 3.5.cir
python 3.4.py
```

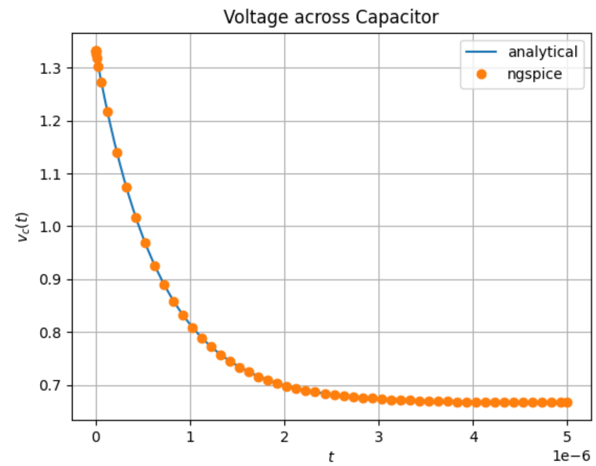

Fig. 3.5. Plot of $v_c(t)$ after flipping the switch

3.6. Find $v_c(0^-)$, $v_c(0^+)$ and $v_c(\infty)$
Solution: At $t = 0^-$, the switch still hasn't been switched to Q and the circuit is in steady state

$$v_c(0^-) = \frac{4}{3}\,\text{V} \quad (3.12)$$

For $t \geq 0$, we can use the above formula

$$v_c(0^+) = \lim_{t\to0^+} v_c(t) = \frac{4}{3}\,\text{V} \quad (3.13)$$

$$v_c(\infty) = \lim_{t\to\infty} v_c(t) = \frac{2}{3}\,\text{V} \quad (3.14)$$

3.7. Obtain Fig. 3.2 using the equivalent differential equation
Solution: Using Kirchoff's junction law

$$\frac{v_c(t) - 0}{R_1} + \frac{v_c(t) - v_2(t)}{R_2} + \frac{\mathrm{d}q}{\mathrm{d}t} = 0 \quad (3.15)$$

where $q(t)$ is the charge on the capacitor
On taking the Laplace transform on both sides of this equation

$$\frac{V_c(s) - 0}{R_1} + \frac{V_c(s) - V_2(s)}{R_2} + (sQ(s) - q(0^-)) = 0 \quad (3.16)$$

But $q(0^-) = \frac{4}{3}C_0$ and

$$q(t) = C_0 v_c(t) \quad (3.17)$$

$$\implies Q(s) = C_0 V_c(s) \quad (3.18)$$

Thus

$$\frac{V_c(s)-0}{R_1}+\frac{V_c(s)-V_2(s)}{R_2}+\left(sC_0V_c(s)-\frac{4}{3}C_0\right)=0 \tag{3.19}$$

$$\implies \frac{V_c(s)-0}{R_1}+\frac{V_c(s)-V_2(s)}{R_2}+\frac{V_c(s)-\frac{4}{3s}}{\frac{1}{sC_0}}=0 \tag{3.20}$$

which is the same equation as the one we obtained from Fig. 3.2

4. Bilinear Transform

4.1. In Fig. 2.2, consider the case when S is switched to Q right in the beginning. Formulate the differential equation

Solution: The differential equation is the same as before

$$\frac{v_c(t)-0}{R_1}+\frac{v_c(t)-v_2(t)}{R_2}+\frac{dq}{dt}=0 \tag{4.1}$$

$$\text{i.e., } \frac{v_c(t)}{R_1}+\frac{v_c(t)-v_2(t)}{R_2}+C_0\frac{dv_c}{dt}=0 \tag{4.2}$$

but with a different initial condition

$$q(0^-)=q(0)=0 \tag{4.3}$$

4.2. Find $H(s)$ considering the ouput voltage at the capacitor

Solution: On taking the Laplace transform on both sides of this equation

$$\frac{V_c(s)}{R_1}+\frac{V_c(s)-V_2(s)}{R_2}+sQ(s)-0=0 \tag{4.4}$$

$$\implies V_c(s)\left(\frac{1}{R_1}+\frac{1}{R_2}\right)+sC_0V_c(s)=\frac{V_2(s)}{R_2} \tag{4.5}$$

$$\implies \frac{V_c(s)}{V_2(s)}=\frac{\frac{1}{R_2}}{\frac{1}{R_1}+\frac{1}{R_2}+sC_0} \tag{4.6}$$

The transfer function is thus

$$H(s)=\frac{\frac{1}{R_2C_0}}{s+\frac{1}{R_1C_0}+\frac{1}{R_2C_0}} \tag{4.7}$$

On substituting the values, we get

$$H(s)=\frac{5\times10^5}{s+1.5\times10^6} \tag{4.8}$$

4.3. Plot $H(s)$. What kind of filter is it?

Solution: Download the following Python code that plots Fig. 4.3

```
wget https://github.com/rudranshm/EE3900/
    raw/main/Circuit/codes/4.3.py
```

Run the codes by executing

```
python 4.3.py
```

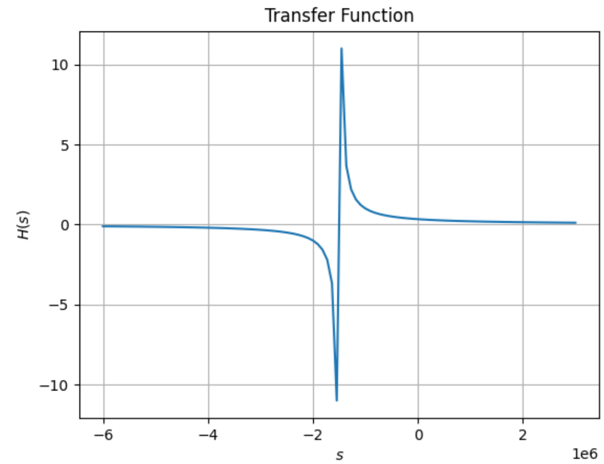

Fig. 4.3. Plot of $H(s)$

Consider the frequency-domain transfer function by putting $s=\jmath\omega$

$$H(\jmath\omega)=\frac{5\times10^5}{\jmath\omega+1.5\times10^6} \tag{4.9}$$

$$\implies \left|H(\jmath\omega)\right|=\frac{5\times10^5}{\sqrt{\omega^2+2.25\times10^{12}}} \tag{4.10}$$

As ω increases, $\left|H(\jmath\omega)\right|$ decreases

In other words, the amplitude of high-frequency signals gets diminished and they get filtered out

Therefore, this is a low-pass filter

4.4. Using trapezoidal rule for integration, formulate the difference equation by considering

$$y(n)=y(t)|_{t=n} \tag{4.11}$$

Solution:

$$\frac{v_c(t)}{R_1}+\frac{v_c(t)-v_2(t)}{R_2}+C_0\frac{dv_c}{dt}=0 \tag{4.12}$$

$$\implies C_0\frac{dv_c}{dt}=\frac{2u(t)-v_c(t)}{R_2}-\frac{v_c(t)}{R_1} \tag{4.13}$$

$$\implies v_c(t)|_{t=n}^{n+1}=\int_n^{n+1}\left(\frac{2u(t)-v_c(t)}{R_2C_0}-\frac{v_c(t)}{R_1C_0}\right)dt \tag{4.14}$$

By the trapezoidal rule of integration

$$\int_a^b f(t)\mathrm{d}t \approx \frac{b-a}{2}(f(a)+f(b)) \tag{4.15}$$

Consider $y(t) = v_c(t)$

$$y(n+1) - y(n) = \frac{1}{R_2C_0}(u(n)+u(n+1)) - \frac{1}{2}(y(n+1)+y(n))\left(\frac{1}{R_1C_0}+\frac{1}{R_2C_0}\right) \tag{4.16}$$

Thus, the difference equation is

$$y(n+1)\left(1+\frac{1}{2R_1C_0}+\frac{1}{2R_2C_0}\right) = y(n)\left(1-\frac{1}{2R_1C_0}-\frac{1}{2R_2C_0}\right) + \frac{1}{R_2C_0}(u(n)+u(n+1)) \tag{4.17}$$

4.5. Find $H(z)$
Solution: Let $\mathcal{Z}\{y(n)\} = Y(z)$
On taking the Z-transform on both sides of the difference equation

$$zY(z)\left(1+\frac{1}{2R_1C_0}+\frac{1}{2R_2C_0}\right) = Y(z)\left(1-\frac{1}{2R_1C_0}-\frac{1}{2R_2C_0}\right) + \frac{1}{R_2C_0}\left(\frac{1}{1-z^{-1}}+\frac{z}{1-z^{-1}}\right) \tag{4.18}$$

$$Y(z)\left(z+\frac{z}{2R_1C_0}+\frac{z}{2R_2C_0}-1+\frac{1}{2R_1C_0}+\frac{1}{2R_2C_0}\right) = \frac{1}{R_2C_0}\frac{1+z}{1-z^{-1}} \tag{4.19}$$

Also

$$v_2(t) = 2 \qquad \forall t \geq 0 \tag{4.20}$$

$$\implies x(n) = 2u(n) \tag{4.21}$$

$$\implies X(z) = \frac{2}{1-z^{-1}} \qquad |z| > 1 \tag{4.22}$$

Thus, the transfer function in z-domain is

$$H(z) = \frac{Y(z)}{X(z)} \tag{4.23}$$

$$= \frac{\frac{1+z}{2R_2C_0}}{z+\frac{z}{2R_1C_0}+\frac{z}{2R_2C_0}-1+\frac{1}{2R_1C_0}+\frac{1}{2R_2C_0}} \tag{4.24}$$

$$= \frac{\frac{1+z^{-1}}{2R_2C_0}}{1+\frac{1}{2R_1C_0}+\frac{1}{2R_2C_0}-z^{-1}+\frac{z^{-1}}{2R_1C_0}+\frac{z^{-1}}{2R_2C_0}} \tag{4.25}$$

On substituting the values

$$H(z) = \frac{2.5\times10^5(1+z^{-1})}{7.5\times10^5+1+(7.5\times10^5-1)z^{-1}} \tag{4.26}$$

with the ROC being

$$|z| > \max\left(1, \left|\frac{7.5\times10^5-1}{7.5\times10^5+1}\right|\right) \tag{4.27}$$

$$\implies |z| > 1 \tag{4.28}$$

4.6. How can you obtain $H(z)$ from $H(s)$?
Solution: The Z-transform can be obtained from the Laplace transform by the substitution

$$s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}} \tag{4.29}$$

where T is the step size of the trapezoidal rule (1 in our case)
This is known as the bilinear transform
Thus

$$H(z) = \frac{\frac{1}{R_2C_0}}{2\frac{1-z^{-1}}{1+z^{-1}}+\frac{1}{R_1C_0}+\frac{1}{R_2C_0}} \tag{4.30}$$

$$= \frac{\frac{1+z^{-1}}{2R_2C_0}}{1-z^{-1}+\left(\frac{1}{2R_1C_0}+\frac{1}{2R_2C_0}\right)(1+z^{-1})} \tag{4.31}$$

$$= \frac{\frac{1+z^{-1}}{2R_2C_0}}{1+\frac{1}{2R_1C_0}+\frac{1}{2R_2C_0}-z^{-1}+\frac{z^{-1}}{2R_1C_0}+\frac{z^{-1}}{2R_2C_0}} \tag{4.32}$$

$$= \frac{2.5\times10^5(1+z^{-1})}{7.5\times10^5+1+(7.5\times10^5-1)z^{-1}} \tag{4.33}$$

which is the same as what we obtained earlier

4.7. Find $y(n)$ from $H(z)$ and verify whether $y(n) = y(t)|_{t=n}$

Solution:

$$Y(z) = H(z)X(z) \tag{4.34}$$
$$= \frac{2.5\times10^5(1+z^{-1})}{7.5\times10^5+1+(7.5\times10^5-1)z^{-1}}\frac{2}{1-z^{-1}} \tag{4.35}$$
$$= \frac{\frac{2}{3}}{1-z^{-1}} - \frac{\frac{2}{3}}{7.5\times10^5+1+(7.5\times10^5-1)z^{-1}} \tag{4.36}$$

On taking the inverse Z-transform by considering the ROC to be $|z| > 1$, we get

$$y(n) = \frac{2}{3}u(n) - \frac{2}{3}\left(-\frac{7.5\times10^5-1}{7.5\times10^5+1}\right)^n u(n) \tag{4.37}$$
$$= \frac{2}{3}\left(1-\left(\frac{1-7.5\times10^5}{1+7.5\times10^5}\right)^n\right)u(n) \tag{4.38}$$

If we are sampling the signal at intervals of $T \ll 10^{-5}$, say 10^{-7} s, i.e., $n = 10^{-7}, 2\times10^{-7}, \dots$

$$y(n) \approx \frac{2}{3}\left(1-\frac{1-7.5\times10^5 n}{1+7.5\times10^5 n}\right)u(n) \tag{4.39}$$

Now

$$Y(s) = H(s)X(s) \tag{4.40}$$
$$= \frac{5\times10^5}{s+1.5\times10^6}\frac{2}{s} \tag{4.41}$$
$$= \frac{10^6}{1.5\times10^6}\left(\frac{1}{s}-\frac{1}{s+1.5\times10^6}\right) \tag{4.42}$$

On taking the inverse Laplace transform by considering the ROC to be $\Re(s) > 0$, we get

$$y(t) = \frac{2}{3}\left(1-e^{-1.5\times10^6 t}\right)u(t) \tag{4.43}$$

But

$$e^{-1.5\times10^6 t} = \frac{e^{-0.75\times10^6 t}}{e^{0.75\times10^6 t}} \tag{4.44}$$
$$\approx \frac{1-7.5\times10^5 t}{1+7.5\times10^5 t} \quad \text{when } t \ll 10^{-6} \tag{4.45}$$

Therefore

$$y(t) \approx \frac{2}{3}\left(1-\frac{1-7.5\times10^5 t}{1+7.5\times10^5 t}\right)u(t) \tag{4.46}$$
$$\therefore y(n) = y(t)|_{t=n} \tag{4.47}$$

Download the following codes for simulation and plotting Fig. 4.7 respectively

```
wget https://github.com/rudranshm/EE3900/
    raw/main/Circuit/codes/4.7.cir
wget https://github.com/rudranshm/EE3900/
    raw/main/Circuit/codes/4.7.py
```

Run the codes by executing

```
ngspice 4.7.cir
python 4.7.py
```

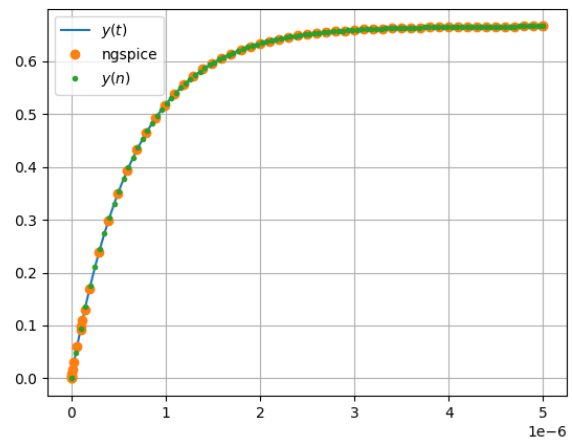

Fig. 4.7. Plots of $y(t)$ and $y(n)$

4.8. Find $y(n)$ by solving the difference equation
Solution:

$$y(n+1)\left(1+\frac{1}{2R_1C_0}+\frac{1}{2R_2C_0}\right) = y(n)\left(1-\frac{1}{2R_1C_0}-\frac{1}{2R_2C_0}\right) + \frac{1}{R_2C_0}\left(u(n)+u(n+1)\right) \tag{4.48}$$

For $n \geq 0$, $u(n)+u(n+1) = 2$

$$y(n+1) = y(n)\left(\frac{1-\frac{1}{2R_1C_0}-\frac{1}{2R_2C_0}}{1+\frac{1}{2R_1C_0}+\frac{1}{2R_2C_0}}\right) + \frac{\frac{2}{R_2C_0}}{1+\frac{1}{2R_1C_0}+\frac{1}{2R_2C_0}} \tag{4.49}$$

Let

$$a = \frac{1-\frac{1}{2R_1C_0}-\frac{1}{2R_2C_0}}{1+\frac{1}{2R_1C_0}+\frac{1}{2R_2C_0}} = -\frac{7.5\times10^5-1}{7.5\times10^5+1} \tag{4.50}$$

$$b = \frac{\frac{2}{R_2C_0}}{1+\frac{1}{2R_1C_0}+\frac{1}{2R_2C_0}} = \frac{10^6}{7.5\times10^5+1} \tag{4.51}$$

Therefore, the difference equation is

$$
\begin{aligned}
y(n+1) &= ay(n) + b && (4.52)\\
\implies y(n) &= ay(n-1) + b && (4.53)\\
&= a(ay(n-2) + b) + b && (4.54)\\
&= a^2(ay(n-3) + b) + b(1+a) && (4.55)
\end{aligned}
$$

On repeating this decomposition, we finally get

$$
\begin{aligned}
y(n) &= a^n y(0) + b(1 + a + \cdots + a^{n-1}) && (4.56)\\
&= 0 + b\left(\frac{1-a^n}{1-a}\right) && (4.57)
\end{aligned}
$$

$$
\begin{aligned}
y(n) = {} & \frac{10^6}{\cancel{7.5\times 10^5 + 1}} \times \frac{\cancel{7.5\times 10^5 + 1}}{2\times 7.5\times 10^5}\\
& \times\left(1 - \left(-\frac{7.5\times 10^5 - 1}{7.5\times 10^5 + 1}\right)^n\right) \quad (4.58)
\end{aligned}
$$

$$
y(n) = \frac{10}{15}\left(1 - \left(-\frac{7.5\times 10^5 - 1}{7.5\times 10^5 + 1}\right)^n\right) \qquad n \geq 0 \tag{4.59}
$$

$$
\therefore\ y(n) = \frac{2}{3}\left(1 - \left(\frac{1 - 7.5\times 10^5}{1 + 7.5\times 10^5}\right)^n\right)u(n) \tag{4.60}
$$

which is the same as what we obtained earlier