Random Numbers

Rudransh Mishra

Contents

I. Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

I.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Download the following files and execute the C program.

```
wget https://github.com/rudranshm/Random/
    blob/main/Code/exrand.c
wget https://github.com/rudranshm/Random/
    blob/main/Code/coeffs.h
```

Run the following commands:

```
cc exrand.c −lm
./a.out
```

I.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \le x) \tag{I.1}$$

Solution: The following code plots Fig I.2.

```
https://github.com/rudranshm/Random/blob/
    main/Code/cdf_plot.py
```

Run the following command in the terminal to run the code.

```
python uni_cdf.py
```

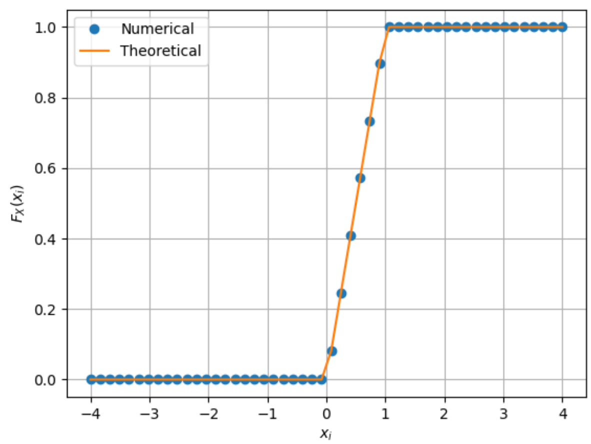

Fig. I.2: The CDF of U

I.3 Find a theoretical expression for $F_U(x)$.
Solution: Since U is an uniform random variable distribution, $P_U(x_i) = P_U(x_j) = k, \forall i, j$

$$\mathrm{F}_U(x) = \int P_U(x)dx \tag{I.2}$$

$$= \int k dx \tag{I.3}$$

$$\text{we know that } \int_0^1 k dx = 1 \tag{I.4}$$

$$\therefore k = 1 \tag{I.5}$$

$$\therefore F_U(x) = x \tag{I.6}$$

I.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{I.7}$$

and its variance as

$$\mathrm{var}[U] = E[U - E[U]]^2 \tag{I.8}$$

Write a C program to find the mean and variance of U.

Solution:

```
wget https://github.com/rudranshm/Random/
    blob/main/Code/meanvar1.4.c
```

Use below command to run file,

```
cc mean_var.c −lm
./a.out
```

running the code gives us the mean as 0.500031 an the variance as 0.083247

I.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{I.9}$$

$$dF_U(x) = dx \tag{I.10}$$

$$\therefore E[U^k] = \int_{-\infty}^{\infty} x^k dx \tag{I.11}$$

$$E[U] = \int_0^1 x dx = \frac{1}{2} \tag{I.12}$$

$$E[U^2] = \int_0^1 x^2 dx = \frac{1}{3} \tag{I.13}$$

$$\because P_X(x) = 0, \forall x \in (1, \infty) \cap (-\infty, 0) \tag{I.14}$$

$$Var(X) = E[U^2] - (E[U])^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \tag{I.15}$$

II. Central Limit Theorem

II.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{II.1}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

Solution:

```
wget https://github.com/rudranshm/Random/
    blob/main/Code/exrand.c
wget https://github.com/rudranshm/Random/
    blob/main/Code/coeffs.h
```

Running the above codes generates uni.dat and gau.dat file. Use the command

```
cc exrand.c −lm
.\a.out
```

II.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?

Solution: The CDF of X is plotted in II.2,Properties of the CDF:

- $\Phi(x) = P(Z \le x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left\{-\frac{u^2}{2}\right\} du$
- $\lim_{x\to\infty} \Phi(x) = 1, \quad \lim_{x\to-\infty} \Phi(x) = 0$
- $\Phi(0) = \frac{1}{2}$
- $\Phi(-x) = 1 - \Phi(x)$

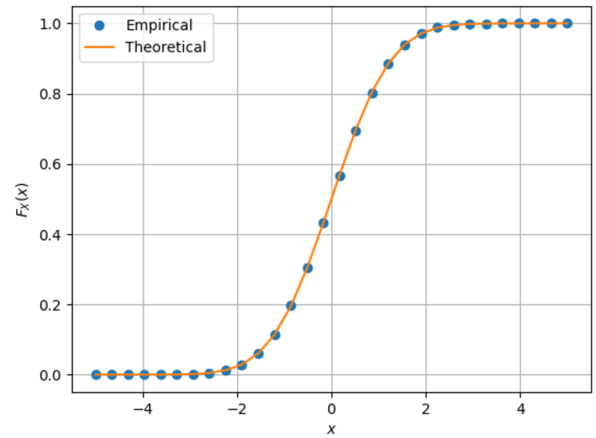

Fig. II.2: The CDF of X

II.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{II.2}$$

What properties does the PDF have?

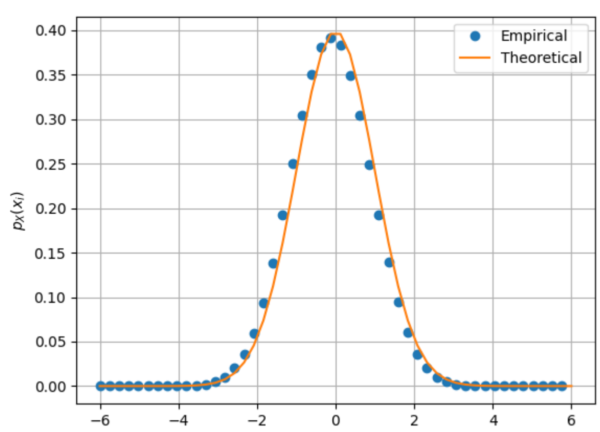

Fig. II.3: The PDF of X

Solution: The PDF of X is plotted in II.3 using the code below

```
https://github.com/rudranshm/Random/blob/
    main/Code/pdf_plot.py
```

Use the below command to run the code:

```
python pdf_plot.py
```

Properties of PDF:

- PDF is symmetric about $x = 0$
- graph is bell shaped
- mean of graph is situated at the apex point of the bell

II.4 Find the mean and variance of X by writing a C program.
Solution: Running the below code gives the mean as 0.000685 and the variance as 1.000024

```
wget https://github.com/rudranshm/Random/
    blob/main/Code/meanvargaussian.c
```

Command used:

```
cc meanvargaussian.c −lm
./a.out
```

II.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{II.3}$$

repeat the above exercise theoretically.

Given ,$p_X(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$

$$E[x] = \int_{-\infty}^{\infty} x p_X(x) dx \tag{II.4}$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x e^{-\frac{-x^2}{2}} \tag{II.5}$$

$$\because x e^{-\frac{-x^2}{2}} \text{is a odd function,} \tag{II.6}$$

$$E[x] = 0$$

$$E[x^2] = \int_{-\infty}^{\infty} x^2 p_X(x) dx \tag{II.7}$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x(x e^{-\frac{-x^2}{2}}) dx \tag{II.8}$$

Using integration by parts:

$$= x \int x e^{-\frac{-x^2}{2}} dx - \int \frac{d(x)}{dx} \int x e^{-\frac{-x^2}{2}} dx \tag{II.9}$$

$$I = \int x e^{-\frac{-x^2}{2}} \tag{II.10}$$

$$\text{Let} \frac{x^2}{2} = t \tag{II.11}$$

$$\implies x dx = dt \tag{II.12}$$

$$\implies = \int e^{-t} dt = -e^{-t} + c \tag{II.13}$$

$$\therefore \int x e^{-\frac{-x^2}{2}} = -e^{-\frac{-x^2}{2}} + c \tag{II.14}$$

Using (II.14) in (II.9)

$$= -x e^{-\frac{-x^2}{2}} + \int e^{-\frac{-x^2}{2}} dx \tag{II.15}$$

$$\text{Also,} \int_{-\infty}^{\infty} e^{-\frac{-x^2}{2}} dx = \sqrt{2\pi} \tag{II.16}$$

$$\therefore \text{substituting limits we get,} E[x^2] = 1 \tag{II.17}$$

$$Var(X) = E[x^2] - (E[x])^2 = 1 - 0 \tag{II.18}$$

III. From Uniform to Other

III.1 Generate samples of

$$V = -2 \ln (1 - U) \tag{III.1}$$

and plot its CDF.

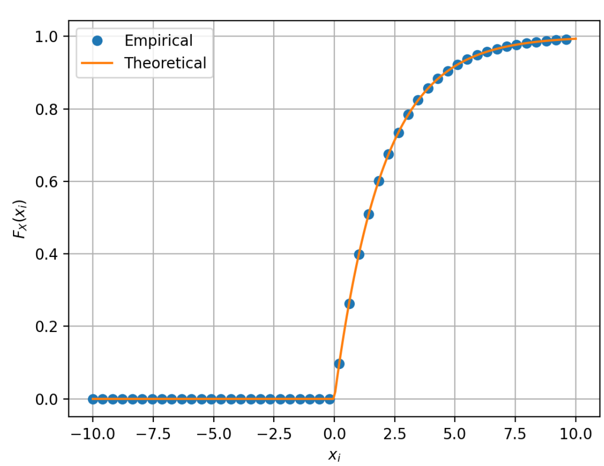

Fig. III.1: CDF for V

Solution: Running the below code generates samples of V from file uni.dat(U).

https://github.com/rudranshm/Random/blob/main/Code/var_v.py

Use the below command in the terminal to run the code:

```
python var_v.py
```

Now these samples are used to plot (III.1) by running the below code,

https://github.com/rudranshm/Random/blob/main/Code/cdfv.py

Use the below command to run the code:

```
python cdfv.py
```

III.2 Find a theoretical expression for $F_V(x)$.

$$F_V(x) = P(V \le x) \tag{III.2}$$
$$= P(-2ln(1-U) \le x) \tag{III.3}$$
$$= P(1 - e^{\frac{-x}{2}} \ge U) \tag{III.4}$$
$$P(U < x) = \int_0^x dx = x \tag{III.5}$$
$$\therefore P(1 - e^{\frac{-x}{2}} \ge U) = 1 - e^{\frac{-x}{2}}, \forall x \ge 0 \tag{III.6}$$

IV. Triangular Distribution

IV.1 Generate

$$T = U_1 + U_2 \tag{IV.1}$$

Solution: Run the below code to generate T.dat

https://github.com/rudranshm/Random/blob/main/Code/Tvar.c

Run the command below in the terminal

```
cc Tvar.c −lm
./a.out
```

IV.2 Find the CDF of T.

$$P_T(t) = (U1 + U2 = x) \quad = \int_{-\inf}^{\inf} P_{U1}(a) P_{U2}(t-a)\, da$$

As $P_{U1}(a) = 1$ for $0 < a < 1$ and 0 otherwise

$$\therefore P_T(t) = \int_0^1 P_{U2}(t-a)\, da$$

$P_{U2}(x-a) = 1$, for $0 \; x - a < 1 \therefore x - 1 < a < x$
If $x < 1, 0 < a < x$

$$P_T(t) = \int_0^x 1\, da$$

$$\therefore P_T(t) = \begin{cases} 0 & t < 0 \\ t & 0 \le t \le 1 \\ 2-t & 0 < t \le 2 \\ 0 & t > 2 \end{cases}$$

$$\therefore F_T(t) = P(U_1 + U_2 < t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 \le t \le 1 \\ 1 - \frac{(2-t)^2}{2} & 1 < t \le 2 \\ 1 & t > 2 \end{cases} \tag{IV.2}$$

Find the PDF of T.
Solution:

$$\therefore P_T(t) = \begin{cases} 0 & t < 0 \\ t & 0 \le t \le 1 \\ 2-t & 0 < t \le 2 \\ 0 & t > 2 \end{cases} \tag{IV.3}$$

Find the theoretical expressions for the PDF and CDF of T.
Solution:

$$P_T(t) = \begin{cases} 0 & t < 0 \\ t & 0 \le t \le 1 \\ 2-t & 0 < t \le 2 \\ 0 & t > 2 \end{cases} \tag{IV.4}$$

$$F_T(t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 \le t \le 1 \\ 1 - \frac{(2-t)^2}{2} & 1 < t \le 2 \\ 1 & t > 2 \end{cases} \tag{IV.5}$$

Verify your results through a plot.
Solution: Run the below code to get the cdf

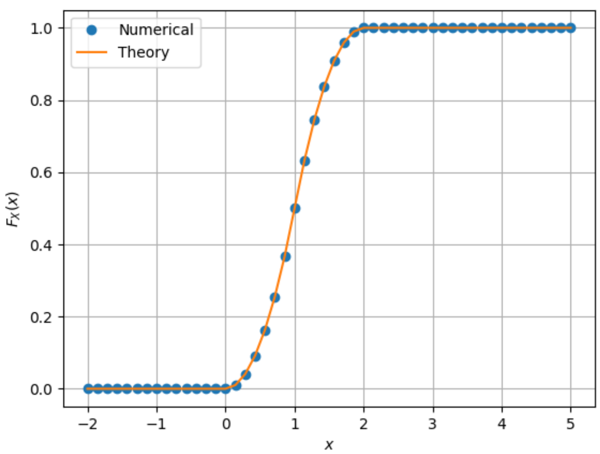

Fig. 2: CDF for (4)

```
https://github.com/rudranshm/Random/blob/main/
    Code/cdft.py
```

Use the following command in the terminal to run the code

```
python cdft.py
```

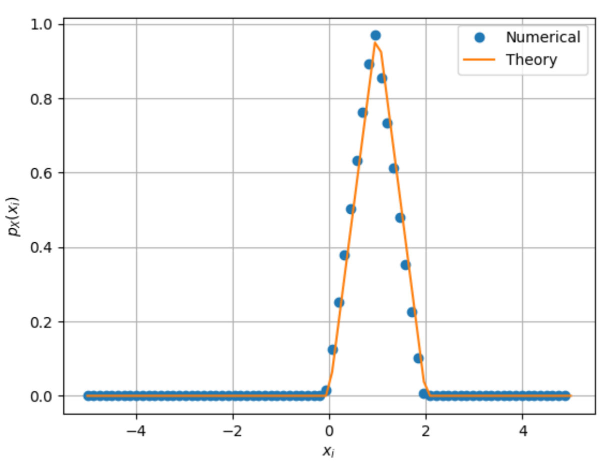

Fig. 2: PDF for (4)

Run the below code to get the pdf

```
https://github.com/rudranshm/Random/blob/main/
    Code/pdft.py
```

Use the following command in the terminal to run the code

```
python pdft.py
```

V. Maximul Likelihood

V.1 Generate equiprobable $X \in \{1, -1\}$.
Solution: Run the below code,

```
https://github.com/rudranshm/Random/blob/
    main/Code/bernoullivar.c
```

Use the below command in the terminal to run the code

```
cc bernoullivar.c −lm
./a.out
```

V.2 Generate

$$Y = AX + N, \tag{V.1}$$

where $A = 5$ dB, and $N \sim \mathcal{N}(0, 1)$.
Solution: Run the below code for generating samples of Y,

```
https://github.com/rudranshm/Random/blob/
    main/Code/Yvar.c
```

Use the below command in the terminal to run the code

```
cc Yvar.c −lm
./a.out
```

V.3 Plot Y using a scatter plot.
Solution:

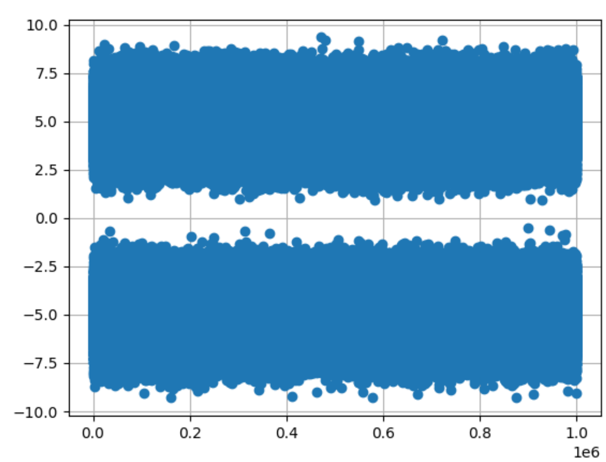

Fig. V.3: plot for (5.3)

Run the following code to generate the scatter plot

```
https://github.com/rudranshm/
    Random/blob/main/Code/y_plot.
    py
```

Use the below command to run the code,

```
python y_plot.py
```

V.4 Guess how to estimate X from Y.
Solution: if the received signal is greater than 0, we can assume that signal s_1, corresponding to $X = 1$ was transmitted.
if the received signal is less than or equal to 0, twe can assume that signal s_2, corresponding to $X = -1$ was transmitted. Here threshold 0 is taken to be the decision boundary.

$$y > 0 \implies s_1 \tag{V.2}$$
$$y \leq 0 \implies s_0 \tag{V.3}$$

V.5 Find

$$P_{e|0} = \Pr\left(\hat{X} = -1|X = 1\right) \tag{V.4}$$

and

$$P_{e|1} = \Pr\left(\hat{X} = 1|X = -1\right) \tag{V.5}$$

Solution: Here s_1 and s_2 are equally probable ie, $p(s_1) = p(s_0) = \frac{1}{2}$

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{\frac{-x^2}{2}} dx \tag{V.6}$$
$$p(e|s_1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 e^{-\frac{(y-A)^2}{2}} dy$$
$$= Q(A) \tag{V.7}$$
$$p(e|s_0) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{(y+A)^2}{2}} dy$$
$$= Q(A) \tag{V.8}$$

V.6 Find P_e assuming that X has equiprobable symbols.
Solution: Total probability of bit error:

$$P_e = p(s_1)p(e|s_1) + p(s_0)p(e|s_0) \tag{V.9}$$
$$= \frac{1}{2}[Q(A) + Q(A)] \tag{V.10}$$
$$\because p(s_1) = p(s_0) = \frac{1}{2}, \text{X has equiprobable symbols} \tag{V.11}$$
$$= Q(A)$$
$$= Q(5) \tag{V.12}$$

Since A=5

V.7 Verify by plotting the theoretical P_e with respect to A from 0 to 10 dB.
Solution:

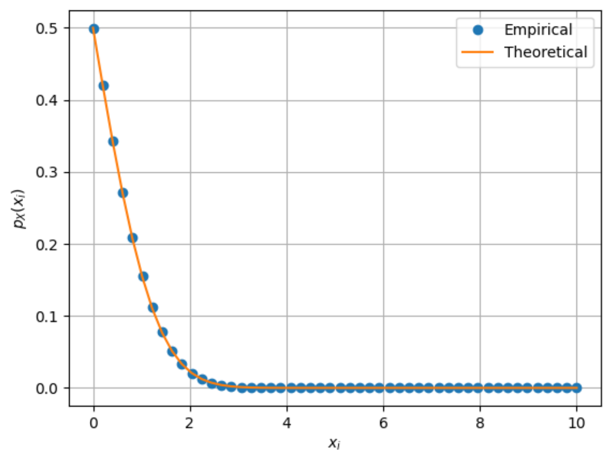

Fig. V.7: plot for (5.7)

Run the following code to generate the scatter plot

```
https://github.com/rudranshm/
    Random/blob/main/Code/pe_plot
    .py
```

Use the below command to run the code,

```
python pe_plot.py
```

V.8 Now, consider a threshold δ while estimating X from Y. Find the value of δ that minimizes the theoretical P_e.
Solution: Threshold=δ,

$$y > \delta \implies s_1 \quad \text{(V.13)}$$
$$y \leq \delta \implies s_0 \quad \text{(V.14)}$$
$$p(e|s_1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\delta} e^{-\frac{(y-A)^2}{2}} dy \quad \text{(V.15)}$$
$$p(e|s_0) = \frac{1}{\sqrt{2\pi}} \int_{\delta}^{\infty} e^{-\frac{(y+A)^2}{2}} dy$$
$$P_e = \frac{1}{2\sqrt{2\pi}} (\int_{-\infty}^{\delta} e^{-\frac{(y-A)^2}{2}} dy + \int_{\delta}^{\infty} e^{-\frac{(y+A)^2}{2}} dy)$$
$$P_e = \frac{Q(\delta + A) + Q(A - \delta)}{2} \quad \text{(V.16)}$$
$$P_e = f(\delta) \quad \text{(V.17)}$$
$$\text{to minimize} P_e, \frac{d(f(\delta))}{d\delta} = 0 \text{ and} f''(\delta) > 0 \quad \text{(V.18)}$$
$$e^{\frac{-(A-\delta)^2}{2}} - e^{\frac{-(A+\delta)^2}{2}} = 0 \quad \text{(V.19)}$$
$$\therefore A - \delta = A + \delta, \implies \delta = 0 \quad \text{(V.20)}$$
$$f''(\delta) = k((A-\delta)e^{\frac{-(A-\delta)^2}{2}} + (A+\delta)e^{\frac{-(A+\delta)^2}{2}}) > 0 \quad \text{(V.21)}$$

V.9 Repeat the above exercise when

$$p_X(0) = p \quad \text{(V.22)}$$

Solution: $p_X(0) = p$
$\implies p_X(1) = 1 - p$

$$P_e = pP(e|s_0) + (1-p)P(e|s_1) \quad \text{(V.23)}$$
$$= pQ(A+\delta) + (1-p)Q(A-\delta) \quad \text{(V.24)}$$
$$\frac{d(P_e)}{d(\delta)} = 0 \quad \text{(V.25)}$$
$$\implies e^{\frac{(A+\delta)^2-(A-\delta)^2}{2}} = \frac{p}{1-p} \quad \text{(V.26)}$$
$$\therefore \delta = \frac{1}{2A} log(\frac{p}{1-p}) \quad \text{(V.27)}$$
$$\frac{d(P_e)}{d(\delta)} \quad at \quad \delta + \epsilon > 0 \quad \text{(V.28)}$$
$$\frac{d(P_e)}{d(\delta)} \quad at \quad \delta - \epsilon < 0$$
$$\therefore \delta = \frac{1}{2A} log\left(\frac{p}{1-p}\right) \longrightarrow minima$$
$$A = 5 \implies \delta = \frac{1}{10} log\left(\frac{p}{1-p}\right) \quad \text{(V.29)}$$

V.10 Repeat the above exercise using the MAP criterion.

Solution:

$$P_{X|Y}(x|y)\Big|_{X=1} = \frac{P(Y=y|X=1)P(X=1)}{P(Y=y)} \quad \text{(V.30)}$$
$$\begin{aligned} P(Y=y) = P(Y=y|X=1)P(X=1) \\ +P(Y=y|X=-1)P(X=-1) \end{aligned} \quad \text{(V.31)}$$
$$P(Y=y|X=1)P(X=1) = pP(Y=A+N) \quad \text{(V.32)}$$
$$= p\left(\frac{1}{\sqrt{2\pi}} e^{\frac{-(y-A)^2}{2}}\right) \quad \text{(V.33)}$$
$$\therefore P_{X|Y}(x|y)\Big|_{X=1} = \frac{p\left(\frac{1}{\sqrt{2\pi}} e^{\frac{-(y-A)^2}{2}}\right)}{P(Y=y)} \quad \text{(V.34)}$$
$$P_{X|Y}(x|y)\Big|_{X=-1} = \frac{P(Y=y|X=-1)P(X=-1)}{P(Y=y)} \quad \text{(V.35)}$$
$$P(Y=y|X=-1)P(X=-1) = (1-p)P(Y=-A+N)$$
$$= (1-p)\left(\frac{1}{\sqrt{2\pi}} e^{\frac{-(y+A)^2}{2}}\right) \quad \text{(V.36)}$$
$$\therefore P_{X|Y}(x|y)\Big|_{X=-1} = \frac{(1-p)\left(\frac{1}{\sqrt{2\pi}} e^{\frac{-(y+A)^2}{2}}\right)}{P(Y=y)} \quad \text{(V.37)}$$

Now comparing $a = P_{X|Y}(x|y)\Big|_{X=-1}$ and $b = P_{X|Y}(x|y)\Big|_{X=1}$, if $a > b, X = -1$ is more likely ,$a < b, X = 1$ is more likely.
$pe^{\frac{-(y-A)^2}{2}} \gtrless (1-p)e^{\frac{-(y+A)^2}{2}}$
$\implies e^{2Ay} \gtrless \frac{1-p}{p}$
$\implies y \gtrless \frac{1}{2A} log\left(\frac{1-p}{p}\right)$
$\delta = \frac{1}{2A} log\left(\frac{1-p}{p}\right)$
$y > \delta \implies$ X=1 is more likely
$y < \delta \implies$ X=-1 is more likely

VI. Gaussian to Other

VI.1 Let $X_1 \sim \mathcal{N}(0,1)$ and $X_2 \sim \mathcal{N}(0,1)$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \quad \text{(VI.1)}$$

Solution:
Method1:The sum of squares of k independent standard random normal variables is nothing but a χ^2 distribution with k degrees of freedom.
$\chi^2(k) = \frac{x^{\frac{n}{2}-1}}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})} e^{\frac{-x}{2}}, \forall x \geq 0$

Here k=2,

$$\therefore \chi^2(2) = P_V(v) = \frac{e^{\frac{-x}{2}}}{2} \tag{VI.2}$$

$$\implies F_V(v) = \int_0^v \frac{e^{\frac{-x}{2}}}{2} dx \tag{VI.3}$$

$$= 1 - e^{\frac{-x}{2}} \tag{VI.4}$$

Method2:

$$X_1 = R\cos\theta \tag{VI.5}$$

$$X_2 = R\sin\Theta \tag{VI.6}$$

$R \in [0,\infty), \Theta \in [0, 2\pi)$. Jacobian Matrix is given by

$$J = \begin{pmatrix} \frac{\partial X_1}{\partial R} & \frac{\partial X_2}{\partial R} \\ \frac{\partial X_1}{\partial \Theta} & \frac{\partial X_2}{\partial \Theta} \end{pmatrix} \tag{VI.7}$$

$$= \begin{pmatrix} \cos\Theta & \sin\Theta \\ -R\sin\Theta & R\cos\Theta \end{pmatrix} \tag{VI.8}$$

$$\implies |J| = R \tag{VI.9}$$

We also know that

$$|J| p_{X_1,X_2}(x_1, x_2) = p_{R,\Theta}(r, \theta) \tag{VI.10}$$

$$\implies p_{R,\Theta}(r,\theta) = R p_{X_1}(x_1) p_{X_2}(x_2) \tag{VI.11}$$

$$= \frac{R}{2\pi} e^{\left(-\frac{X_1^2+X_2^2}{2}\right)} \tag{VI.12}$$

$$= \frac{R}{2\pi} e^{\left(-\frac{R^2}{2}\right)} \tag{VI.13}$$

X_1, X_2 are iid random variables. Thus,

$$p_R(r) = \int_0^{2\pi} p_{R,\Theta}(r,\theta) d\theta \tag{VI.14}$$

$$= Re^{\left(-\frac{R^2}{2}\right)} \tag{VI.15}$$

However, $V = X_1^2 + X_2^2 = R^2 \geq 0$, thus $F_V(x) = 0$ for $x \geq 0$.

$$F_V(x) = F_R(\sqrt{x}) \tag{VI.16}$$

$$= \int_0^{\sqrt{x}} re^{\left(-\frac{r^2}{2}\right)} dr \tag{VI.17}$$

$$= \int_0^{\frac{x}{2}} e^{-t} dt = 1 - e^{-\frac{x}{2}} \tag{VI.18}$$

To generate data for V , run the following code,

```
https://github.com/rudranshm/Random/blob/
    main/Code/var_v.py
```

Run the below command in terminal,

```
cc var_v.c -lm
./a.out
```

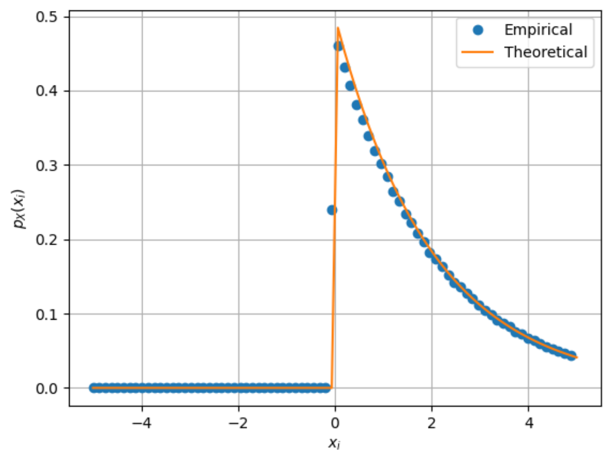

Fig. VI.1: PDF for (6.1)

The PDF plot of the $\chi^2(2)$ can be obtained by running the code below,

```
https://github.com/rudranshm/Random/blob/
    main/Code/chi_pdf.py
```

Use the following command in the terminal to run the code

```
python chi_pdf.py
```

The CDF plot of the $\chi^2(2)$ can be obtained by running the code below,

```
https://github.com/rudranshm/Random/blob/
    main/Code/chi_cdf.py
```

Use the following command in the terminal to run the code

```
python chi_cdf.py
```

VI.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases} \tag{VI.19}$$

find α.

Solution: From (VI.4) $\alpha = 0.5$

VI.3 Plot the CDF and PDf of

$$A = \sqrt{V} \tag{VI.20}$$

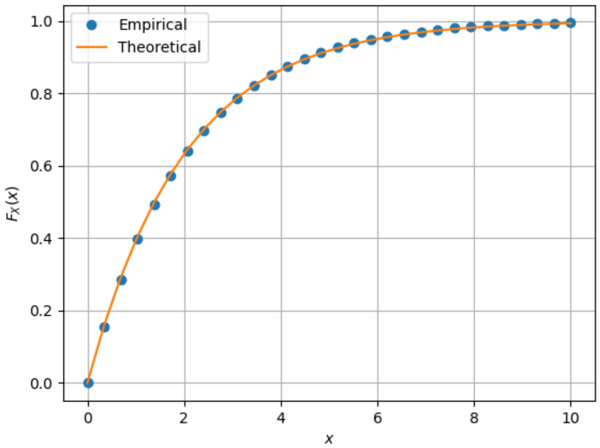

Fig. VI.1: CDF for (6.1)

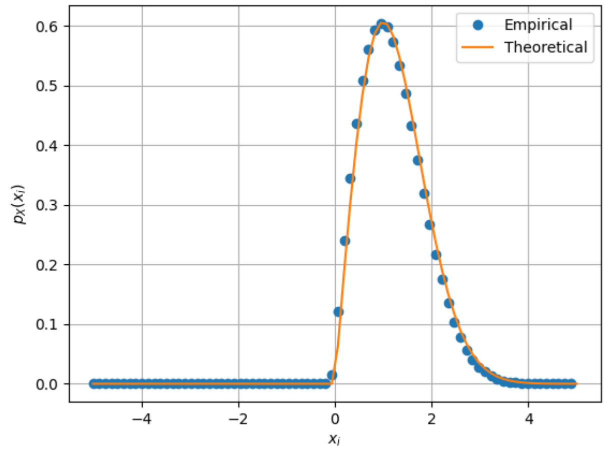

Fig. VI.3: PDF for (6.3)

Solution:

$$F_A(a) = P(A < a) = P(V < a^2) \quad \text{(VI.21)}$$

$$\text{from(VI.4)}, = \begin{cases} 1 - e^{\frac{-a^2}{2}} & a > 0 \\ 0 & a <= 0 \end{cases} \quad \text{(VI.22)}$$

$$\implies P_A(a) = \frac{d(F_A(a))}{da} \quad \text{(VI.23)}$$

$$= \begin{cases} ae^{\frac{-a^2}{2}} & a > 0 \\ 0 & a <= 0 \end{cases} \quad \text{(VI.24)}$$

To generate data for A , run the following code,

```
https://github.com/rudranshm/Random/blob/
    main/Code/varA.c
```

Run the below command in terminal,

```
cc varA.c −lm
./a.out
```

The PDF plot of A can be obtained by running the code below,

```
https://github.com/rudranshm/Random/blob/
    main/Code/A_pdf.py
```

Use the following command in the terminal to run the code

```
python A_pdf.py
```

The CDF plot of the A can be obtained by running the code below,

```
https://github.com/rudranshm/Random/blob/
    main/Code/A_cdf.py
```

Use the following command in the terminal to run the code

```
python A_cdf.py
```

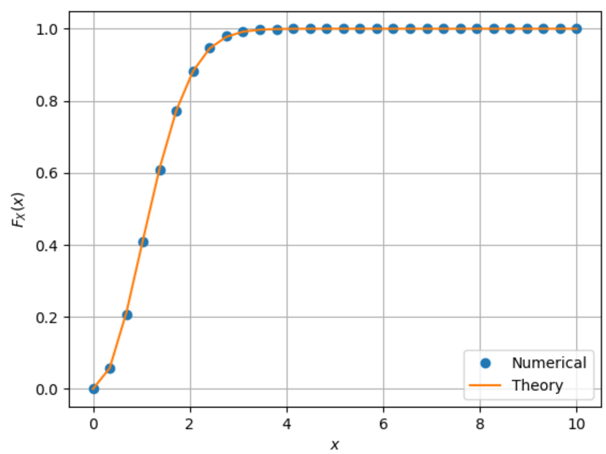

Fig. VI.3: CDF for (6.3)