Constraint Satisfaction Problems

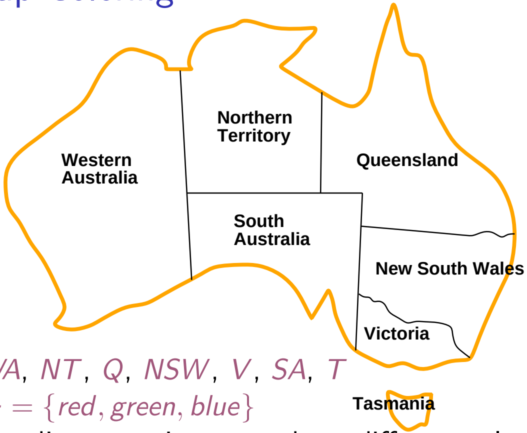
Outline

- $\diamondsuit$ CSP examples
- $\diamondsuit$ Backtracking search for CSPs
- $\diamondsuit$ Problem structure and problem decomposition
- $\diamondsuit$ Local search for CSPs

Constraint satisfaction problems (CSPs)

1. **Standard search problem:** state is a “black box”—any ”old” data structure that supports
   - goal test,
   - eval,
   - successor
2. **CSP:**
   - state is defined by variables X_i with values from domain D_i
   - goal test is a set of constraints specifying allowable combinations of values for subsets of variables
   - Simple example of a **formal representation language**
   - Allows useful **general-purpose** algorithms with more power than standard search algorithms

Example: Map-Coloring

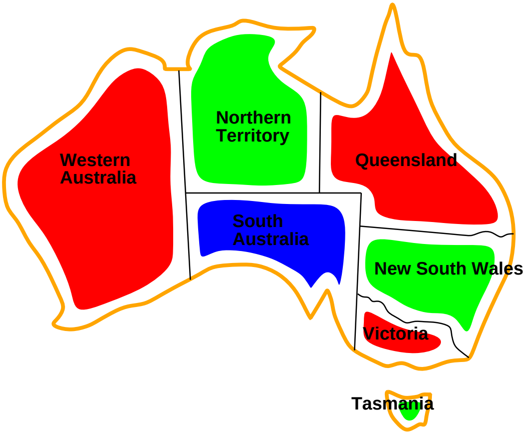


Variables WA, NT, Q, NSW, V, SA, T

Domains $D_i = \{red, green, blue\}$

Constraints: adjacent regions must have different colors: e.g., $WA \neq NT$ (if the language allows this), or $(WA, NT) \in \{(red, green), (red, blue), (green, red), (green, blue), \ldots\}$

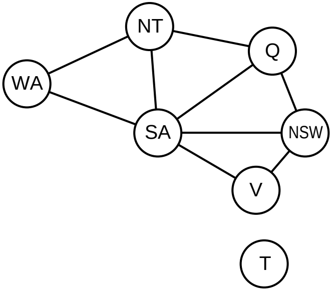
Example: Map-Coloring contd.



Solutions are assignments satisfying all constraints, e.g., $\{WA = red, NT = green, Q = red, NSW = green, V = red, SA = blue, T = green\}$

Constraint graph

Binary CSP: each constraint relates at most two variables
Constraint graph: nodes are variables, arcs show constraints



General-purpose CSP algorithms use the graph structure to speed up search: e.g., Tasmania is an independent subproblem!

Varieties of CSPs

- **Discrete variables**
  1. **Finite domains** size $d \implies O(d^n)$ complete assignments: e.g., Boolean CSPs, incl. Boolean satisfiability (NP-complete)
  2. **Infinite domains** (integers, strings, etc.): e.g., job scheduling, variables are start/end days for each job need a constraint language: $StartJob_1 + 5 \leq StartJob_3$ linear constraints solvable, nonlinear undecidable
- **Continuous variables**: e.g., start/end times for Hubble Telescope observations: linear constraints solvable in poly time by LP methods

Varieties of constraints

Unary constraints involve a single variable: $SA \neq green$

Binary constraints involve pairs of variables: $SA \neq WA$

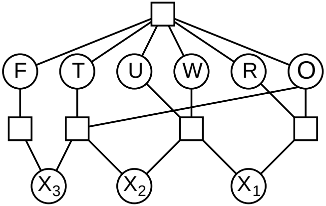
Higher-order constraints involve 3 or more variables: cryptarithmetic column constraints

Preferences (soft constraints): *red* is better than *green* often representable by a cost for each variable assignment

$\rightarrow$ constrained optimization problems

Example: Cryptarithmetic

		T	W	O
+		T	W	O
F		O	U	R



Variables: $F\ T\ U\ W\ R\ O\ X_1\ X_2\ X_3$
Domains: $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$
Constraints

1. $\mathit{alldiff}(F, T, U, W, R, O)$
2. $O + O = R + 10 \cdot X_1$,
3. etc.

Real-world CSPs

- Assignment problems: e.g., who teaches what class
- Timetabling problems: e.g., which class is offered when and where?
- Hardware configuration
- Spreadsheets
- Transportation scheduling
- Factory scheduling
- Floorplanning

Notice that many real-world problems involve real-valued variables

Standard search formulation (incremental)

Start with the straightforward (dumb) approach, then fix it: States are defined by the values assigned so far

- Initial state: the empty assignment, $\emptyset$
- Successor function: assign a value to an unassigned variable that does not conflict with current assignment.
  $\implies$ fail if no legal assignments (cannot be fixed!)
- Goal test: the current assignment is complete
  1. This is the same for all CSPs!
  2. Every solution appears at depth n with n variables
     $\implies$ use depth-first search
  3. Path is irrelevant, so can also use complete-state formulation
  4. $b = (n - \ell)d$ at depth ℓ, hence $n!d^n$ leaves!!!!

Backtracking search

- Variable assignments are commutative:

  $$[WA = red \text{ then } NT = green] \equiv [NT = green \text{ then } WA = red]$$

- Only need to consider assignments to a single variable at each node

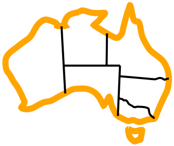
  $\implies$ $b = d$ and there are d^n leaves
- Depth-first search for CSPs with single-variable assignments is called backtracking search
- **Backtracking search is the basic uninformed algorithm for CSPs**
- Can solve n-queens for $n \approx 25$

Backtracking search

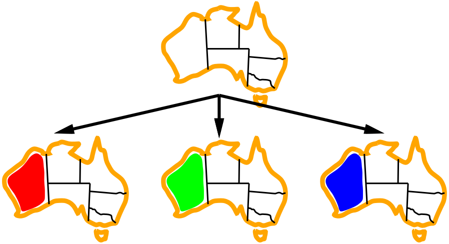
```
function BACKTRACKING-SEARCH(csp) returns solution/failure
   return RECURSIVE-BACKTRACKING({ }, csp)

function RECURSIVE-BACKTRACKING(assignment, csp) returns soln/failure
   if assignment is complete then return assignment
   var ← SELECT-UNASSIGNED-VARIABLE(VARIABLES[csp], assignment, csp)
   for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
      if value is consistent with assignment given CONSTRAINTS[csp] then
         add {var = value} to assignment
         result ← RECURSIVE-BACKTRACKING(assignment, csp)
         if result ≠ failure then return result
         remove {var = value} from assignment
   return failure
```

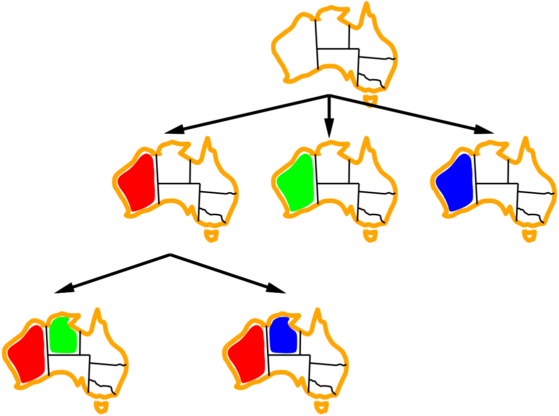
Backtracking example



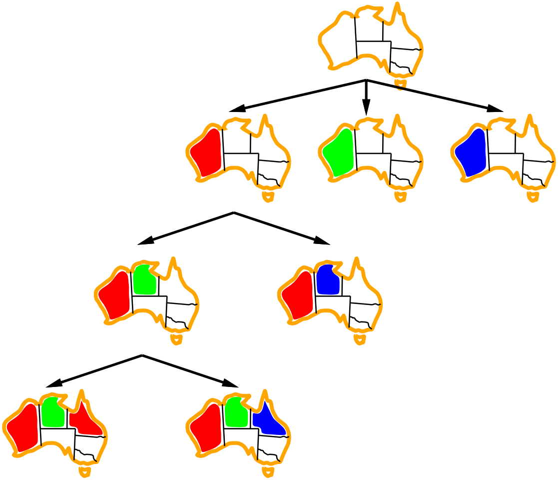
Backtracking example



Backtracking example



Backtracking example



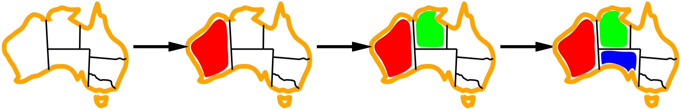
Improving backtracking efficiency

General-purpose methods can lead to important gains in speed:

1. Which variable should be assigned next?
2. In what order should its values be tried?
3. Can we detect inevitable failure early?
4. Can we take advantage of problem structure?

Minimum remaining values

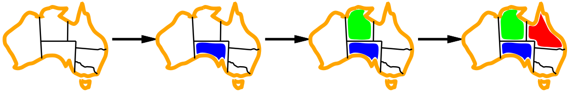
Minimum remaining values (MRV): choose the variable with the fewest legal values (most constrained variable)



Degree heuristic

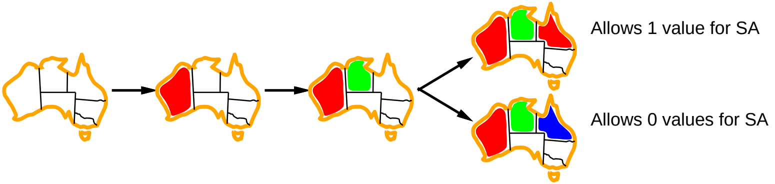
Tie-breaker among MRV variables

Degree heuristic: choose the variable with the most constraints on remaining variables



Least constraining value

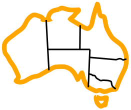
Given a variable, choose the least constraining value: the value that rules out the fewest values in the remaining variables



Combining these heuristics makes 1000 queens feasible

Forward checking

- Idea: Keep track of remaining legal values for unassigned variables
- Idea: Terminate search when any variable has no legal values



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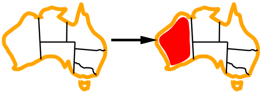

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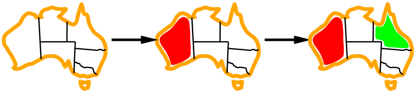

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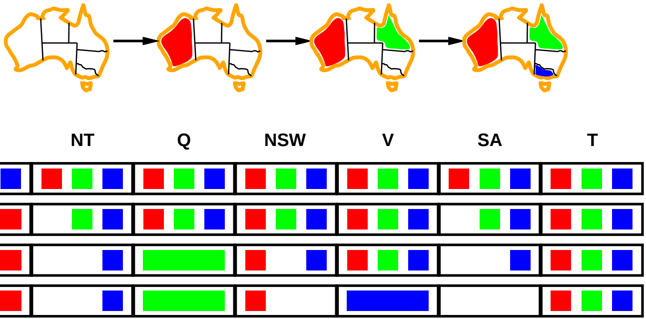
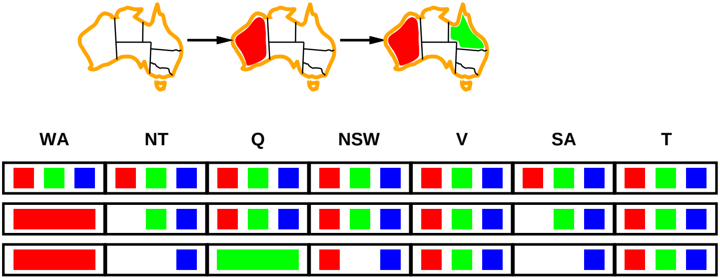


Constraint propagation

Forward checking propagates information from assigned to unassigned variables, but doesn't provide early detection for all failures:



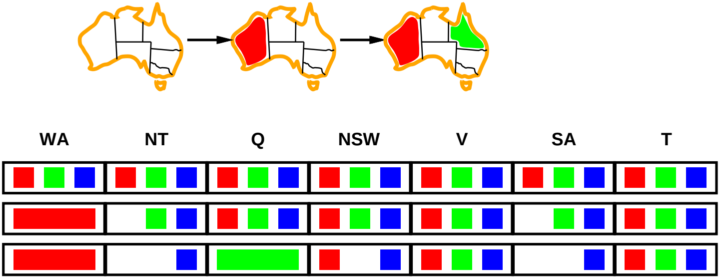
NT and *SA* cannot both be blue!

Constraint propagation repeatedly enforces constraints locally

Arc consistency – 1

Simplest form of propagation makes each arc consistent
$X \rightarrow Y$ is consistent iff

for **every** value x of X there is **some** allowed y



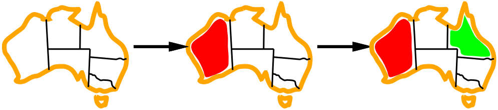
Arc consistency – 2

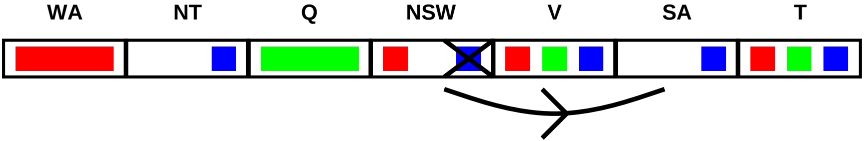
Simplest form of propagation:

arc consistent $X \to Y$ is consistent

if and only if

$\forall$ value x of X, $\exists$ **some** allowed value y of Y





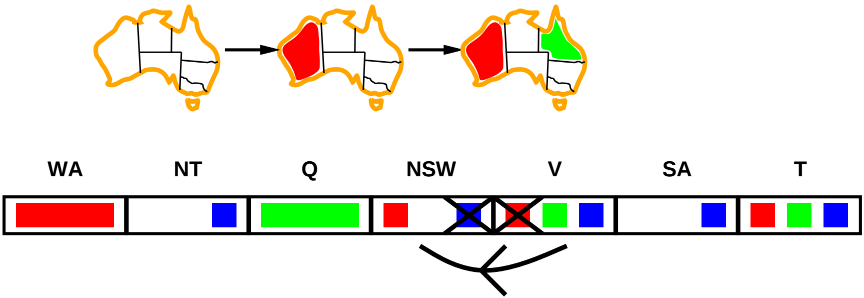
Arc consistency – 3

Simplest form of propagation makes each arc consistent

arc consistent $X \to Y$ is consistent

if and only if

$\forall$ value x of X, $\exists$ **some** allowed value y of Y



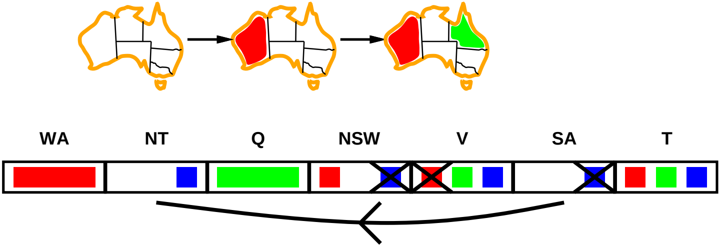
If X loses a value, neighbors of X need to be rechecked

Arc consistency – 4

arc consistent $X \rightarrow Y$ is consistent

if and only if

$\forall$ value x of X, $\exists$ **some** allowed value y of Y



- If X loses a value, neighbors of X need to be rechecked
- Arc consistency detects failure earlier than forward checking
- Can be run as a preprocessor or after each assignment

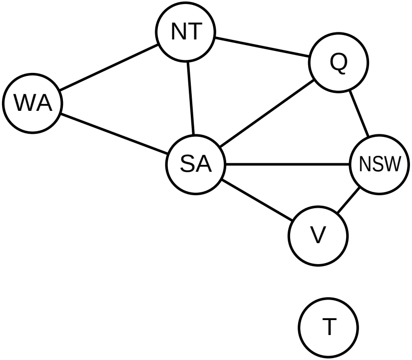
Arc consistency algorithm

```
function AC-3( csp) returns the CSP, possibly with reduced domains
   inputs: csp, a binary CSP with variables {X_1, X_2, ..., X_n}
   local variables: queue, a queue of arcs, initially all the arcs in csp

   while queue is not empty do
      (X_i, X_j) ← REMOVE-FIRST(queue)
      if REMOVE-INCONSISTENT-VALUES(X_i, X_j) then
         for each X_k in NEIGHBORS[X_i] do
            add (X_k, X_i) to queue
function REMOVE-INCONSISTENT-VALUES( X_i, X_j) returns true iff succeeds
   removed ← false
   for each x in DOMAIN[X_i] do
      if no value y in DOMAIN[X_j] allows (x,y) to satisfy the constraint X_i ↔ X_j
         then delete x from DOMAIN[X_i]; removed ← true
   return removed
```

$O(n^2 d^3)$, can be reduced to $O(n^2 d^2)$ (but detecting **all** is NP-hard)

Problem structure – 1



Tasmania and mainland are independent subproblems
Identifiable as connected components of constraint graph

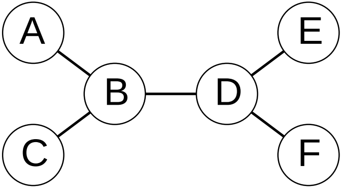
Problem structure – 2

Suppose each subproblem has c variables out of n total
Worst-case solution cost is $n/c \cdot d^c$, **linear** in n
E.g., $n = 80$, $d = 2$, $c = 20$

2^{80} = 4 billion years at 10 million nodes/sec

$4 \cdot 2^{20}$ = 0.4 seconds at 10 million nodes/sec

Tree-structured CSPs



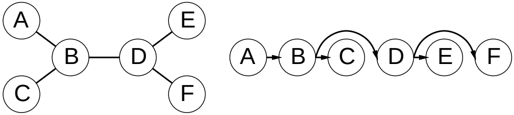
Theorem: If the constraint graph has no loops, the CSP can be solved in $O(n\,d^2)$ time

Compare to general CSPs, where worst-case time is $O(d^n)$

This property also applies to logical and probabilistic reasoning: an important example of the relation between syntactic restrictions and the complexity of reasoning.

Algorithm for tree-structured CSPs

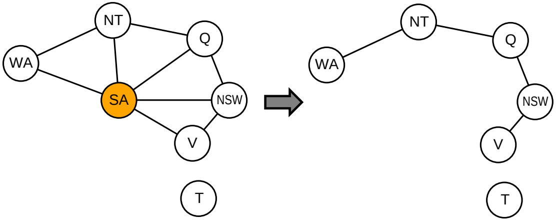
1. Choose a variable as root, order variables from root to leaves such that every node's parent precedes it in the ordering



2. For j from n down to 2, apply $\textsc{RemoveInconsistent}(Parent(X_j), X_j)$
3. For j from 1 to n, assign X_j consistently with $Parent(X_j)$

Nearly tree-structured CSPs

Conditioning: instantiate a variable, prune its neighbors' domains



Cutset conditioning: instantiate (in all ways) a set of variables such that the remaining constraint graph is a tree

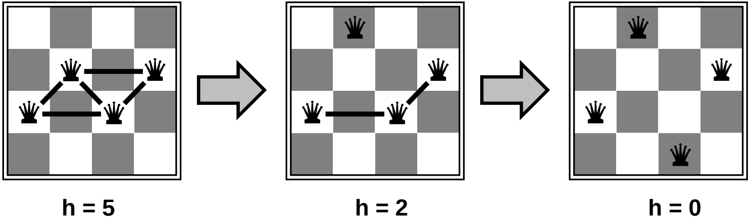
Cutset size $c \implies$ runtime $O(d^c \cdot (n-c)d^2)$, very fast for small c

Iterative algorithms for CSPs

- Hill-climbing, simulated annealing typically work with "complete" states, i.e., all variables assigned.
- To apply to CSPs:
  - allow states with unsatisfied constraints
  - operators **reassign** variable values
- Variable selection: randomly select any conflicted variable
- Value selection by min-conflicts heuristic:
  - choose value that violates the fewest constraints, i.e., hillclimb with $h(n)$ = total number of violated constraints

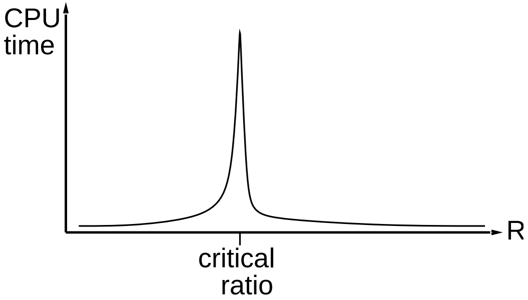
Example: 4-Queens

- States: 4 queens in 4 columns ($4^4 = 256$ states)
- Operators: move queen in column
- Goal test: no attacks
- Evaluation: $h(n)$ = number of attacks



Performance of min-conflicts

- Given random initial state, can solve n-queens in almost constant time for arbitrary n (e.g., n=10^7) with high probability.
- The same appears to be true for any randomly-generated CSP, **except** in a narrow range of the ratio
  $R = \frac{\text{number of constraints}}{\text{number of variables}}$



Summary

- CSPs are a special kind of problem:
  - states defined by values of a fixed set of variables
  - goal test defined by constraints on variable values
- Backtracking = depth-first search with one variable assigned per node
- Variable ordering and value selection heuristics help significantly
- Forward checking prevents assignments that guarantee later failure
- Constraint propagation (e.g., arc consistency) does additional work to constrain values and detect inconsistencies
- The CSP representation allows analysis of problem structure
- Tree-structured CSPs can be solved in linear time
- Iterative min-conflicts is usually effective in practice