Some Problems for Chapter 6 – Constraints Satisfaction

1. How many solutions there are for coloring the map of Australia with three colors?

   Solution
   There are 18 solutions for coloring Australia with three colors. To see this, start with SA, which can have any of three colors. Then moving clockwise, WA can have either of the other two colors. Then, everything else is strictly determined.
   Now, counting we have 6 possibilities for the mainland. Now, coloring Tasman is an independent problem, which means that we can color it with each of the three different colors. Therefore, we have a total of 18 possible colorings.

2. Consider the problem of placing k knights on an n x n chessboard such that no two knights are attacking each other, where k is given and $k \leq n$.
   (a) Choose a CSP formulation. In your formulation, what are the variables?
   (b) What are the possible values of each variable?
   (c) What sets of variables are constrained, and how?

   Solution A:
   (a) There is a variable corresponding to each of the n^2 positions on the board.
   (b) Each variable can take one of two values, {occupied, vacant}
   (c) Every pair of squares separated by a knight's move is constrained, such that both cannot be occupied. Furthermore, the entire set of squares is constrained, such that the total number of occupied squares should be k.

   Solution B:
   (a) There is a variable corresponding to each knight.
   (b) Each variable's domain is the set of squares.
   (c) Every pair of knights is constrained, such that no two knights can be on the same square or on squares separated by a knight's move.

3. Using the strategy of *backtracking* with *forward checking* and the *MRV* and *least-constraining-value heuristics*, solve by hand the cryptarithmetic problem

   TWO+
   <u>TWO</u>
   FOUR

   Solution: The exact steps depend on certain choices we make. For example:

   a. Choose the X3 variable. Its domain is {0, 1}: Choose the value 1 for X3. (We can't choose 0; it wouldn't survive forward checking, because it would force F to be 0, and the leading digit of the sum must be non-zero.)
   b. Choose F, because it has only one remaining value: Choose the value 1 for F (**F=1**).
   c. Now X2 and X1 are tied for minimum remaining values at 2: either could be 0 or 1.
   d. Either value survives forward checking: choose 0 for X2.
   e. Now X1 has the minimum remaining values: arbitrarily choose 0 for the value of X1.
   f. The variable O must be an even number (because it is the sum of T + T) less than 5; (because O + O = R + 10 x 0). That makes it most constrained.
   g. Arbitrarily choose 4 as the value of O (**O = 4**).
   h. Now, R has only 1 remaining value: choose the value 8 for R (**R = 8**).
   i. T now has only 1 remaining value: choose the value 7 for T (**T=7**).
   j. U must be an even number less than 9; choose U: The only value for U that survives forward checking is 6 (**U=6**).
   k. The only variable left is W: the only value left for W is 3 (**W=3**).
   l. This is a solution.

      734+
      734

      1468

COMMENT: Because this puzzle is under-constrained, and because we are allowed to use forward checking, we can arrive at a solution with no backtracking.

4. Any CSP can be transformed into a CSP with only binary constraints.
   To show this show how a single ternary constraint such as "A + B = C" can be turned into three binary constraints by using an auxiliary variable. You may assume finite domains. (Hint: Consider a new variable that takes on values that are pairs of other values, and consider constraints such as "X is the first element of the pair Y.")
   Next, show how constraints with more than three variables can be treated similarly.
   Finally, show how unary constraints can be eliminated by altering the domains of variables. This completes the demonstration that any CSP can be transformed into a CSP with only binary constraints.

   Solution: Eventually, the solution to this problem involves induction.
   1. Express the ternary constraint on A, B and C that A + B = C, by introducing a new variable, call it AB.
   2. If the domain of A and B is the set of numbers N, then the domain of AB is the set of pairs of numbers from N, i.e. N x N.
   3. Now form three binary constraints as follows:
      (a) between A and AB saying that the value of A must be equal to the first element of the pair-value of AB;

      (b) between B and AB saying that the value of B must equal the second element of the value of AB;

      (c) between the sum of the pair of numbers that is the value of AB and C.

      (d) Any other ternary constraints can be handled similarly.

   4. To reduce a 4-ary constraint on variables A, B, C, D, do the following:
      (a) first reduce A, B, C to binary constraints as shown above,
      (b) add back D to form a ternary constraint with AB and C
      (c) reduce this ternary constraint to binary by introducing CD.

Thus, by induction, we can reduce any n-ary constraint to an (n − 1)-ary constraint.

We can stop at binary: unary constraints can be dropped by redefining the domain of the variable involved.

5. Explain why it is a good heuristic to choose the **variable that is most constrained** but the **value that is least constraining** in a CSP search.

   Solution: Recall, that overall, the objective is to make assignments which avoid conflicts/failures.

   Thus, since the ***most constrained variable*** is that which is likely to cause a failure, once other variables have been assigned, it makes sense to select it for assignment.

   For the same reason, the least ***constraining value heuristic*** makes sense because it allows the most chances for future assignments to avoid conflict.