Trial Exam 2012/2013

Measures and Integrals

Formalities

This is a trial exam. It corresponds in size, form and difficulty to the real exam, which is a 27 hours take-home exam.

This trial exam consists of 4 problems with a total of 15 questions all counted with equal weight. Throughout it can be assumed that all stochastic variables are defined on the background probability space $(\Omega, \mathbb{F}, P)$.

Problem 1

The first problem consists of 5 independent questions. A complete answer must be supported by a small argument, a counter example or a reference to the book.

Question 1.1. If X and Y are independent real valued random variables with finite second moment is it then true that

$$V(X - Y) = V(X) - V(Y)?$$

Question 1.2. Let $\mathbb{D}$ be a σ-algebra and let Y be a real valued stochastic variable with finite first moment and with $E(Y \mid \mathbb{D}) = 1$. If X is $\mathbb{D}$-measurable and XY has finite first moment is it then true that

$$E(XY \mid \mathbb{D}) = X?$$

Question 1.3. Let the joint distribution of X_1 and X_2 be a regular normal distribution on $\mathbb{R}^2$. If $\operatorname{cov}(X_1, X_2) = 0$ is it then true that $X_1 \perp\!\!\!\perp X_2$?

Question 1.4. Let μ and ν be two probability measures on $(\mathbb{R}^2, \mathbb{B}_2)$. Assume that

$$\mu((-\infty, x] \times (-\infty, y]) = \nu((-\infty, x] \times (-\infty, y])$$

for all $x, y \in \mathbb{R}$. Is it then true that $\mu = \nu$?

Question 1.5. Is the function

$$(x, y) \mapsto 1_{[0,\infty)\times[0,\infty)}(x, y)(e^{-x} - e^{-y})$$

integrable w.r.t. the Lebesgue measure on $(\mathbb{R}^2, \mathbb{B}_2)$?

Problem 2

Let ν be the probability measure on $(0, \infty)$ with density

$$f(x) = \frac{1}{2\sqrt{x}} e^{-\sqrt{x}}, \quad x > 0$$

w.r.t. the Lebesgue measure. Define $h(x) = \sqrt{x}$ for $x > 0$.

Question 2.1. Show that $h(\nu)$ is the exponential distribution.

Question 2.2. Compute

$$\int \sqrt{x} \, \mathrm{d}\nu(x).$$

Question 2.3. Assume that $X_1, \ldots, X_n$ are independent and identically distributed each with marginal distribution ν. Show that

$$\frac{1}{n} \sum_{i=1}^{n} \sqrt{X_i} \xrightarrow{P} 1$$

for $n \to \infty$.

Question 2.4. Show that

$$P\left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (\sqrt{X_i} - 1) \leq 1 \right)$$

is convergent for $n \to \infty$ and compute the limit.

Problem 3

Let μ be the measure on $(\mathbb{R}^2, \mathbb{B}_2)$ with density

$$f(x_1, x_2) = 1_{[1,\infty)\times[1,\infty)}(x_1, x_2) c\, e^{-x_1 x_2}$$

w.r.t. m_2 where $c > 0$ is an arbitrary constant. In other words, $\mu = f \cdot m_2$.

Question 3.1. Show that for $A \in \mathbb{B}$

$$\mu(A \times \mathbb{R}) = \int_{A\cap[1,\infty)} \frac{c}{x_1} e^{-x_1} \mathrm{d}x_1$$

and show that $\mu(\mathbb{R}^2) < \infty$.

In the remaining questions it is assumed that $c > 0$ is chosen such that μ becomes a probability measure, which is possible because we have shown in general that $\mu(\mathbb{R}^2) < \infty$. The actual value, $c \simeq 4.56$, can be found by numerical integration but is not needed. Let, furthermore, $X = (X_1, X_2)$ denote a stochastic variable with values in $\mathbb{R}^2$ and with distribution μ. That is, $X(P) = \mu$.

Question 3.2. Show that the distribution of X_1 has density

$$g(z) = \frac{c}{z} e^{-z}, \quad z > 1$$

w.r.t. the Lebesgue measure. Argue that X_2 has the same distribution as X_1.

Question 3.3. Argue that X_1 has finite second moment and show that

$$EX_1 = ce^{-1} \quad \text{and} \quad EX_1^2 = 2ce^{-1}.$$

Question 3.4. Show that $X_1 X_2$ has finite first moment and that

$$E(X_1 X_2) = ce^{-1} + 1.$$

For the final question you can without further arguments use that since X_2 has the same distribution as X_1 it holds that also $EX_2 = ce^{-1}$ and $EX_2^2 = 2ce^{-1}$.

Question 3.5. Show that $X_1 + X_2$ has finite second moment and compute $V(X_1 + X_2)$.

Problem 4

Let X and Y be two independent stochastic variables both with the uniform distribution on $(-1/2, , 1/2)$. That is, their common distribution has density $f = 1_{(-1/2,1/2)}$ w.r.t. the Lebesgue measure on $(\mathbb{R}, \mathbb{B})$.

Question 4.1. Compute the density for the joint distribution of $(X + Y, X - Y)$. Are $X + Y$ and $X - Y$ independent?