Reexam 2012/2013

Measures and Integrals

University of Copenhagen
Department of Mathematical Sciences

Formalities

This is the final, individual 27 hours take-home reexam in the course *Measures and Integrals*, 2012/2013. It is made available electronically via Absalon at 9.00am, April 16, 2013 and the deadline for handing in the solution is 12.00am, April 17, 2013. Later hand-ins are not accepted. The solution must be handed in at the secretarial office at the Department of Mathematical Sciences (office 04.1.03, E-building, HCØ).

The solution can be written using pen, pencil or computer. The final hand-in must have numbered pages and it must be equipped with the official front page that you find electronically on Absalon.

This is an individual exam. During the 27-hours exam you are not allowed to communicate in any way with anybody about anything related to the the exam.

The exam consists of 5 problems with a total of 15 questions all counted with equal weight. Throughout it can be assumed that all stochastic variables are defined on the background probability space $(\Omega, \mathbb{F}, P)$.

Problem 1

The first problem consists of 5 independent questions. A complete answer must be supported by a small argument, a counter example or a reference to the book.

Question 1.1. If X and Y are real valued stochastic variables and $X \perp\!\!\!\perp Y$ is it then true that $X^2 \perp\!\!\!\perp Y^2$?

Question 1.2. Let X be a real valued stochastic variable with $EX = 0$ and such that $Ee^X < \infty$. Is it true that

$$1 \leq Ee^X?$$

Question 1.3. Let X and Y be independent real valued stochastic variables both with an exponential distribution with scale parameter $\beta = 1$. Is it true that $X + Y$ is exponentially distributed with scale parameter 2?

Question 1.4. Let $B = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ and let $f : [0, \infty) \to [0, \infty)$ be measurable. Is it true that

$$\int_B f(x^2 + y^2)\,\mathrm{d}m_2(x, y) = 2\pi \int_0^1 f(r)r\,\mathrm{d}r$$

Question 1.5. Is it true that

$$F(x) = \arctan(x) + \pi/2$$

is the distribution function for a probability measure on $\mathbb{R}$?

Problem 2

Let X and Y be two real valued stochastic variables whose joint distribution is the regular normal distribution on $(\mathbb{R}^2, \mathbb{B}_2)$ with mean 0 and variance matrix

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \rho \\ \rho & \sigma_2^2 \end{pmatrix}$$

where we assume that $\sigma_1, \sigma_2 > 0$ and $-\sigma_1\sigma_2 < \rho < \sigma_1\sigma_2$. It can be shown that Σ is then positive definite, and Σ can thus be a variance matrix for a regular normal distribution. This can be used without further comments.

Question 2.1. Find the joint distribution of $X + Y$ and $X - Y$.

Question 2.2. Determine all values of σ_1, σ_2 and ρ for which $X + Y \perp\!\!\!\perp X - Y$.

Question 2.3. Assume that $\sigma_1 = \sigma_2 = 0.6$ and $\rho = 0.14$. Find the distribution of

$$(X + Y)^2 + \frac{1}{0.44}(X - Y)^2.$$

Problem 3

Let $n \in \mathbb{N}$ and consider, for each n, independent real valued stochastic variables $Z_{n1}, \ldots, Z_{nn}$ such that

$$P(Z_{nk} = n) = 1 - P(Z_{nk} = 0) = \frac{1}{n}$$

for $k = 1, \ldots, n$. Thus Z_{nk} takes almost surely only the two values n and 0. Define

$$Y_n = \frac{1}{n} \sum_{k=1}^{n} Z_{nk}.$$

Question 3.1. Show that $EY_n = 1$ and $VY_n = \frac{n-1}{n}$.

Question 3.2. Decide if $\sqrt{n}(Y_n - 1) \xrightarrow{\mathcal{D}} \mathcal{N}(0,1)$.

Hint: Consider the probability $P(\sqrt{n}(Y_n - 1) \leq -1)$.

Problem 4

Define the two $\mathcal{M}^+$-functions $f, g : \mathbb{R} \mapsto [0, \infty)$ by $f(x) = e^x$ and $g(y) = e^{-y}$. Define the two measures $\mu = f \cdot m$ and $\nu = g \cdot m$.

Question 4.1. Show that μ and ν are σ-finite measures.

By σ-finiteness there is a unique product measure $\mu \otimes \nu$ on $(\mathbb{R}^2, \mathbb{B}_2)$.

Question 4.2. Show that $\mu \otimes \nu$ has density

$$(x, y) \mapsto e^{x-y}$$

w.r.t. m_2 and compute

$$\mu \otimes \nu(\{(x,y) \mid |x+y| < 1, |x-y| < 1\}).$$

Question 4.3. Let $h(x,y) = 1_A(x,y)xy$ with $A = (-\infty, 0) \times (0, \infty)$. Compute

$$\int h \, \mathrm{d}\mu \otimes \nu.$$

Problem 5

Let X, Y and Z be three independent real valued stochastic variables all with finite second moment and all with mean 0 and variance 1. Define

$$W = \frac{X + YZ}{\sqrt{1 + Z^2}}.$$

Question 5.1. Show that

$$V\left(W \mid Z\right) = 1 \quad \text{a.e.}$$

Question 5.2. Find the distribution of W under the additional assumption that X, Y and Z all have a marginal $\mathcal{N}(0,1)$-distribution.

Hint: Compute $P(W \leq w)$ by Tonelli, integrating out over the the joint distribution of (X, Y) first.