Mål og Integrale Teori

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Contents

Abstract

These notes are intended as a resource for myself; past, present, or future students of this course, and anyone interested in the material. The goal is to provide an end-to-end resource that covers all material discussed in the course displayed in an organized manner. If you spot any errors or would like to contribute, please contact me directly.

1 Fundamentels

1.1 Paving

An arbitrary collection of subsets is a **paving**

1.2 Algebra

Definition 1.1. *A paving $\mathbb{A}$ on a set X is called an* ***algebra*** *if*

- $X \in \mathbb{A}$
- $A \in \mathbb{A} \Rightarrow A^c \in \mathbb{A}$
- $A, B \in \mathbb{A} \Rightarrow A \cup B \in \mathbb{A}$

Lemma 1.2. *If $\mathbb{A}$ is an* ***algebra*** *on X , then $\emptyset \in \mathbb{A}$*

Proof. We know that X itself is a member of $\mathbb{A}$, and we know that $\mathbb{A}$ is is stable under formation of complements. But the complements of X is indeed $\emptyset$. □

Lemma 1.3. *If $\mathbb{A}$ is an* ***algebra*** *on X , is holds that*

$$A, B \in \mathbb{A} \Rightarrow A \cap B \in \mathbb{A}$$

Proof. Take A and B in $\mathbb{A}$. As $\mathbb{A}$ is stable under formation of complements, we see that A^c and B^c are two $\mathbb{A}$ -sets. As $\mathbb{A}$ is stable under formation of unions, we set that $A^c \cup B^c \in \mathbb{A}$. If we take the complement again, we see that

$$A \cap B = (A^c \cup B^c)^c \in \mathbb{A}$$

using de Morgan's law □

Lemma 1.4. *If $\mathbb{A}$ is an algebra on X , it holds that*

$$A, B \in \mathbb{A} \Rightarrow A \backslash B \in \mathbb{A}$$

Proof. Take A and B in $\mathbb{A}$. As $\mathbb{A}$ is stable under the formation of complements, we see that B^c is in a $\mathbb{A}$ -set. As $\mathbb{A}$ is stable under the formation of intersections, we see that $A \cap B^c \in \mathbb{A}$. Per definition of the set difference, we have that

$$A \backslash B = A \cap B^c \in \mathbb{A}$$

□

Lemma 1.5. *If $\mathbb{A}$ is an algebra on $\mathcal{X}$, and $A_1, \ldots, A_n$ are sets in $\mathbb{A}$, is holds that*

$$\bigcup_{i=1}^{n} A_i \in \mathbb{A}, \bigcap_{i=1}^{n} A_i \in \mathbb{A}$$

Proof. For $n = 2$ the claim is included in the definitation of an algebra. If the results is established for $n-1$ sets, we have

$$\bigcup_{i=1}^{n} A_i = \left(\bigcup_{i=1}^{n-1} A_i\right) \cup A_n \in \mathbb{A}$$

□

1.3 sigma-algebras

The concept of algebras does not work under under approximate schemes. Therefore we introduce σ-algrebras.

Definition 1.6. *A paving $\mathbb{E}$ on a set $\mathcal{X}$ is called a σ-algebra if*

- $\mathcal{X} \in \mathbb{E}$
- $A \in \mathbb{E} \Rightarrow A^c \in \mathbb{E}$
- $A_1, A_2, \dots \in \mathbb{E} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathbb{E}$

Definition 1.7. *Lad F være en abitrær familie a delmænger af $\mathcal{X}$. Der eksistere en unique mindste $\sigma-$algebra der indeholder alle mængder i F (F er ikke selv nødvendigvis en $\sigma-$algebra). Foreningenmængden af alle $\sigma-$algebra der indeholder F. Denne $\sigma-$algebra $\sigma(F)$ er $\sigma-$algebraen genereret af F.*

*A **measurable space** is a pair $(\mathcal{X}, \mathbb{E})$, consisting of the set $\mathcal{X}$ and a σ–algebra $\mathbb{E}$ on $\mathcal{X}$. We say that a subset $A \subset \mathcal{X}$ is $\mathbb{E}$–measurable if $A \in \mathbb{E}$*

Lemma 1.8. *If $\mathbb{E}$ is an σ-algebra on $\mathcal{X}$, the it is also an algebra.*

Proof. see book page 11. □

1.4 Borel Sigma algebra

Definition 1.9. *The **Borel Sigma algebra** $\mathbb{B}$ is the smallest $\sigma-$algebra generated by the open sets. symboliccally $\mathbb{B} = \sigma(\mathbb{O})$*

Remark 1.10. *As the borel algrebra $\mathbb{B}$ is a $\sigma-$algebra which is stabel under the formation of complements, $\mathbb{B}$ is also the* sigal *generated on the closed sets.*

1.5 Important distributions

2 Measures

Definition 2.1. *Borel σ-algebraen $\mathbb{B}_k$ på $\mathbb{R}_k$ er $\sigma-$algebra frembragt på de åbne mængder $\mathbb{O}_k$*

$\mathbb{B} = \sigma(\mathbb{O})$, dvs, den mindste sigal der indeholder $\mathbb{O}$. F.eks.

- De endelige mængder
- De åbne kasser $\mathbb{I}^k$
- De åbne kasser med rationelle hjørner
- For $k = 1$, intervaller af formen $(-\infty, b_1], \dots, \times(-\infty, b_i]$

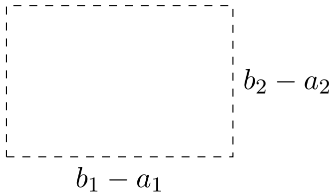
Lad $\mathcal{X}$ være en mængde og lad $\mathbb{E} = \sigma(\mathcal{X})$, så kaldes $(\mathcal{X}, \mathbb{E})$ et målbart rum.

Definition 2.2. *Et mål μ på $(\mathcal{X}, \mathbb{E})$ er en funktion på $\mathbb{E}$ som opfylder*

Distributions	Continues	Discrete	
Uniform	$\frac{1}{\beta}$, for $x \in (\alpha, \alpha+\beta)$	Bionomial	$\binom{n}{x} p^x(1-p)^{n-x}$
Exponential	$\frac{1}{\beta}e^{-x/\beta}$	H-gemometric	$\binom{N_1}{x}\binom{N-N_1}{n-x} \Big/ \binom{N}{N}$
Cauchy	$\frac{1}{\pi(1+x^2)}$	Poisson	$\frac{\lambda^x}{x!}e^{-\lambda}$
Normal	$\frac{1}{\sqrt{2\pi}}e^{-(x-\xi)^2/2\sigma^2}$		
Gamma Γ	$\frac{1}{\beta^\lambda\Gamma(\lambda)}x^{1-\lambda}e^{-x/\beta}$		
Beta	$\frac{1}{\lambda_1,\lambda_2}x^{\lambda_1-1}(1-)^{\lambda_2-1}$		
F	$\frac{\lambda_1^{\lambda_1}\lambda_2^{\lambda_2}}{B(\lambda_1,\lambda_2)}\frac{x^{\lambda_1-1}}{(\lambda_1 x+\lambda_2)^{\lambda_+\lambda_2}}$		
T	$\frac{1}{\sqrt{2\lambda}B(\lambda,\frac{1}{2})}\frac{1}{\left(1+\frac{x^2}{2\lambda}\right)^{\lambda+1/2}}$		

Table 1: distributions

- $\mu(A_n) \in [0, \infty]$
- $\mu(\emptyset) = 0$
- $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$



Example 2.3. *Arealet på* $(b_i - a_1)(b_2 - a_2)$

2.1 lebesgue målet

Definition 2.4.

$$m_k((a_1, b_1) \times (a_2, b_2) \times \ldots \times (a_k, b_k)) = \prod_{i=1}^{k} (a_i, b_i)$$

2.2 Intergation with respect to a measure

When we write something like dx in a integral $\int f(x)\,dx$ we mean the infinitesimal interval starting at x. So dx tell us 3 things. (i) which variable we are dealing with (ii) the value of that variable, and (iii) the length of a infinitesimal at that value. When we integrate with respect to a measure μ, we write $\mu(dx)$, which is short hand for $\mu(x, x + dx)$. That is the measure of a infinitesimal change in interval starting at x. So in a intergral

$$\int f(x)\mu(dx)$$

$f(x)$ is the height and $\mu(dx)$ is the lenght of the infinitisimal area $f(x)\mu(dx)$

3 The Uniquness Thoerem

The Uniqueness Theorem

The problem is that we would like to say something about then two measures on the same measurable space $\mathcal{X}, \mathbb{E}$ on "many" sets (e.g. the the power set). E.i, when $\mu(A) = \upsilon(A)$.
We cannot check all sets in $\mathbb{E}$. So we would like to say something about how many sets we need to check to astablish that $\mu(A) = \upsilon(A)$ for all $A \in \mathbb{E}$?
For some paving $\mathbb{D}$ and a $\mathbb{E} = \sigma(\mathbb{D})$ and two measures μ and υ we want to show that if $\mu(D) = \upsilon(D)\,\forall\, D \in \mathbb{D}$ then it also holds that $\mu(A) = \upsilon(A)\,\forall\, A \in \mathbb{E}$ trick is to:

(1) Show that $\mu(D) = \upsilon(D)\,\forall\, D \in \mathbb{D}$

(2) Then construct a set $\mathbb{H} = \{A \in \mathbb{E} \mid \mu(A) = \upsilon(A)\} \supset \mathbb{D}$

(3) Then show that $\mathbb{H} = \mathbb{E}$

The last part is however not easy unless we can show that $\mathbb{H}$ is a $\sigma-$algebra. So instead we make a loser requirement on $\mathbb{H}$ and then try to show that $\mathbb{H} = \mathbb{E}$ given some stronger requirements on $\mathbb{D}$ than just being a generator for the $\sigma-$algebra.

Definition 3.1. *A paving* $\mathbb{H}$ *os a set* $\mathcal{X}$ *is a* ***Dynkin class*** *if*

(1) $\mathcal{X} \in \mathbb{H}$ *(the same as σ-algebra)*

(2) $A, B \in \mathbb{H}$, $A \subset B \Rightarrow B \setminus A \in \mathbb{H}$ *(stronger than the σ-algebra but as* $\mathcal{X} \setminus A = A^c$ *then it is also stable under complements)*

(3) $A_1, \ldots, A_n \in \mathbb{H}$, $A_1 \subset A_2 \subset \ldots \Rightarrow \bigcup_{n=1}^{\infty} A_n$ *(looser than σ-algebra)*

To show that a Dynkins class is af σ-algebra and vice-versa we need to lemmas

Lemma 3.2. *if* $\mathbb{A}$ *is an algebra it holds that*

$$A, B \in \mathbb{A} \Rightarrow A \setminus B \in \mathbb{A}$$

Lemma 3.3. *If* $\mathbb{E}$ *is a σ-algebra it is also an algebra*

So the property that $A, B \in \mathbb{A} \Rightarrow A \setminus B \in \mathbb{A}$ also holds for a σ-algebra.
We then introduce a new lemma that states $\mathbb{E} = \mathbb{H}$ and that $\mathbb{H} = \mathbb{E}$.

Lemma 3.4. *If* $\mathbb{E}$ *is a σ-algebra, then* $\mathbb{E}$ *is also a Dynkins Class. If* $\mathbb{H}$ *is a Dynkins class which is stable under intersections, then* $\mathbb{H}$ *is a σ-algebra.*

Proof. The first part follows easlialy. (1) is the same for both (2) follows from the two lemmas stating that if $A, B \in \mathbb{A} \Rightarrow A \setminus B \in \mathbb{A}$ holds for a σ-algebra. (3) holds, as a σ-algebra is stable under all types of unions, also unions of increasing sets. To show that $\mathbb{H}$ which is cap-stabe is a σ-algebra we first note that a DC which is stable under $\cap$ is also stable under finite $\cup$: if $A, B \in \mathbb{H}$ then

$$A \cup B = (A^c \cap B^c)^c \in \mathbb{H}$$

if $A_1, , A_2 \ldots \in \mathbb{H}$, we let

$$B_n = A_1 \cup \ldots \cup A_n$$

As a $\cap$-stable DC is stable under finite unions, then $B_n \in \mathbb{H}$. And as $B_1 \subset B_2 \subset \ldots$ we have that $\bigcup_{n=1}^{\infty} B_n \in \mathbb{H}$. But

$$\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$$

□

So if a DC $\mathbb{H}$ is $\cap$-stable it is in fact a σ-algebra.

Remark 3.5. *If* $(\mathbb{H}_i)_{i \in I}$ *is a family of DC, the intersection* $\bigcap_{i \in I} \mathbb{H}_i$ *is also a DC*

Remark 3.6. *A DC generated by* $\mathbb{D}$ *is the smallest DC containg all the elements of* $\mathbb{D}$

Lemma 3.7 (Dynkins lemma)**.** *Let* $\mathbb{D} \subset \mathbb{H} \subset \mathbb{E}$ *be pavings on the set* $\mathcal{X}$*, and assume that* $\mathbb{E} = \sigma(\mathbb{D})$*. If* $\mathbb{D}$ *is* $\cap$*-stable, and if* $\mathbb{H}$ *is a DC, then* $\mathbb{H} = \mathbb{E}$*.*

Proof. Let $\mathbb{K}$ be the smallest DC containing $\mathbb{D}$. Then

$$\mathbb{D} \subset \mathbb{K} \subset \mathbb{H} \subset \mathbb{E}$$

The proof is to show that $\mathbb{K}$ is $\cap$-stabel. If this is true, then it follows from above lemma that $\mathbb{K}$ is a σ-algebra. As $\mathbb{E}$ is the smallest σ-algebra containing $\mathbb{D}$ and $\mathbb{K}$ is a σ-algebra contaning $\mathbb{D}$ than it must be true that $\mathbb{K} = \mathbb{E}$. As $\mathbb{H}$ is squezzed between $\mathbb{K}$ and $\mathbb{E}$ then $\mathbb{H} = \mathbb{E}$. E.i. $\mathbb{H}$ is a σ-algebra.
The proof follows as

(1) Show that $\mathbb{K}$ is $\cap$-stabel

for each $A \in \mathbb{K}$, introduce a new paving

$$\mathbb{K}_A = \{B \in \mathbb{K} \mid A \cap B \in \mathbb{K}\}$$

$\mathbb{K}_A$ is a DC as

(a) $\mathcal{X} \in \mathbb{K}_A$ as $A \cap \mathcal{X} \Rightarrow A \subset \mathbb{D} \subset \mathbb{K}$. As $\mathbb{K}_A$ is the intersection of all elements of $\mathbb{K}$ and $\mathcal{X}$ is a element of $\mathbb{K}$

(b) $\tilde{A}, B, \tilde{A} \subset B \Rightarrow B \setminus \tilde{A} \in \mathbb{K}_A$. Then $\tilde{A} \cap A, B \cap A \in \mathbb{K}$ with $\tilde{A} \cap A \subset B \cap A$. So

$$(B \setminus \tilde{A}) \cap A = \underbrace{(B \cap A)}_{\in \mathbb{K}} \setminus \underbrace{(\tilde{A} \cap A)}_{\in \mathbb{K}}$$

So $B \setminus \tilde{A} \in \mathbb{K}$

(c) $B_1,, B_2 \ldots \in \mathbb{K}_A$, $B_1 \subset B_2 \subset \ldots \Rightarrow \bigcup_{n=1}^{\infty} B_n \in \mathbb{K}_A$

$$A \cap \bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} \underbrace{(A \cap B_n)}_{\in \mathbb{K}}$$

as $\mathbb{K}$ is a DC. Further $A \cap B_1 \subset A \cap B_2 \subset \ldots$, so $\bigcup_{n=1}^{\infty} A \cap B_n \in \mathbb{K}_A$

(2) Note that $\mathbb{D} \subset \mathbb{K}_A \subset \mathbb{K}$ and for $A, B \in \mathbb{D}$, then $A \cap B \in \mathbb{D} \subset \mathbb{K}$ which can given the definition of $\mathbb{K}_A$ this can be reformulated as: if $A \in \mathbb{D}$, then $\mathbb{D} \subset \mathbb{K}_A$ we must have that $\mathbb{K}_A = \mathbb{K}$. $\mathbb{K}$ is the smallest DC that contains $\mathbb{D}$ and $\mathbb{D} \subset \mathbb{K}_A$, then $\mathbb{K}_A = \mathbb{K}$. And a $\mathbb{K}_A$ is defined by the intersections of $\mathbb{K}$ then $\mathbb{K}$ is $\cap$-stabel.

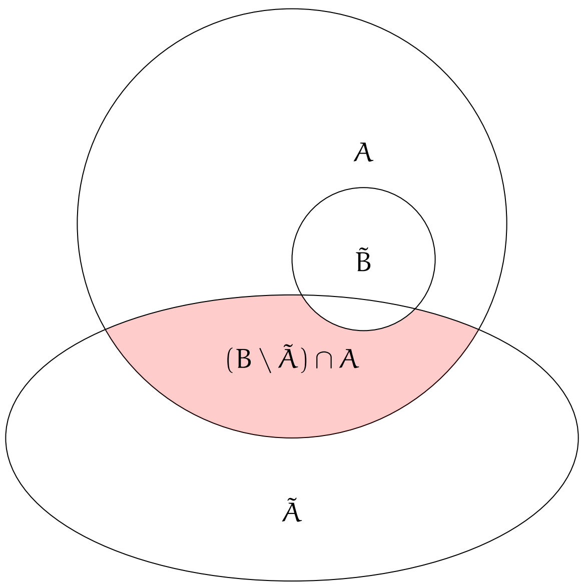


Figure 3.1: The $(B \setminus \tilde{A}) \cap A$

$\square$

Using the above we can then prove the **uniqueness theorem for probability measures** . The steps are as follows

(1) Let υ, μ be two measures of probability on a measurable space, and assume that $\mathbb{E} = \sigma(\mathbb{D})$ and that

$$\mu(D) = \upsilon(D)$$

if $\mathbb{D}$ is $\cap$-stabel, then $\mu = \upsilon$.

(2) contruct a paving where the two measures are equal

$$\mathbb{H} = \{F \in \mathbb{E} \mid \mu(F) = \upsilon(F)\}$$

and show that $\mathbb{H}$ is a DC. This can be done with Dynkins lemma, which requires that

  (1) That $\mathbb{D}$ is $\cap$-stabel follows by the theorem
  (2) establish that $\mathbb{D} \subset \mathbb{H} \subset \mathbb{E}$. But this clearly follows from the contruction of $\mathbb{H}$
  (3) Then all we need to show is that $\mathbb{H}$ is a DC

To show that $\mathbb{H}$ is a DC we note that

- As μ, υ are probability measures, then clearly $\mathcal{X} \in \mathbb{H}$.
- If $A \subset B$ are two $\mathbb{H}$ sets, then

$$\mu(B \setminus A) = \mu(B) - \mu(A) = \upsilon(B - \upsilon(A)) = \upsilon(B \setminus A)$$

, so $B \setminus A \in \mathbb{H}$
- If $F_1 \subset F_2 \subset \ldots \in \mathbb{H}$, then

$$\mu\left(\bigcup_{n=1}^{\infty} F_n\right) = \lim_{n\to\infty} \mu(F_n) = \lim_{n\to\infty} \upsilon(F_n) = \upsilon\left(\bigcup_{n=1}^{\infty} F_n\right)$$

, then $\bigcup_{n=1}^{\infty} F_n \in \mathbb{H}$

3.1 Product Algebra

Looks a σ-algebra generated by several maps. That is, if you have many measurable maps and you take the cartisan product, will the product then still be measurable.

3.2 Measurability of intergrals

Examine under what conditions a integral is measurable

3.3 Measurable maps

Definition 3.8. *Lad $\mathcal{X}, \mathbb{E}$ and $\mathcal{Y}, \mathbb{K}$ be two measurable spaces, and let $f : \mathcal{X} \to \mathcal{Y}$ be a map. We say that f is **measurable** if*

$$f^{-1}(B) \in \mathbb{E} \text{ for all } B \in \mathbb{K}$$

Remark 3.9. *We say that a f is $\mathbb{E} - \mathbb{K}$ measurable if it satifies above*

Remark 3.10. *If there is no confusion about which algebra to use, we say that either f is $\mathbb{E}$-measurable if the σ-algebra $\mathcal{Y}$ is fixed and the only choice of confusion is the σ-algebra on $\mathcal{X}$. Similary we may say that that f is $\mathbb{K}$ measurable if $\mathcal{X}$ is fixed, and the only possible confusion is the choice of σ-algebra on $\mathcal{Y}$*

Often we cannot check all the sets in the σ-algebra as many of them are not acceseable for direct description. Luckly we only have to check the condition given in definition 3.8 on the generator for for the σ-algebra.

Lemma 3.11. *Let $(\mathcal{X}, \mathbb{E})$ and $\mathcal{Y}, \mathbb{K}$ be two measurable spaces, and let $f : \mathcal{X} \to \mathcal{Y}$ be a map. Let $\mathbb{D}$ be a paving on $\mathcal{Y}$ and assume that $\mathbb{K} = \sigma(\mathbb{D})$. If*

$$f^{-1}(D) \in \mathbb{E} \, \forall \, D \in \mathbb{D},$$

then f is $\mathbb{E} - \mathbb{K}$-measurable

3.4 Product measures

Thw product of two finite measures $\mu \otimes \upsilon$ is agian a measure. Especially: the product of two probability measures is agian a probability measure.

Theorem 3.12. *Let $(\mathcal{X}, \mathbb{E}, \mu)$ and $\mathcal{Y}, \mathbb{K}, \upsilon$ be σ-finite spaces. Then there is a unique measure $\mu \otimes \upsilon$ on the product space $\mathcal{X} \times \mathcal{Y}, \mathbb{E} \otimes \mathbb{K}$ satesfying that*

$$\mu \otimes \upsilon(A \times B) = \mu(A)\upsilon(B) \quad A \in \mathbb{E}, B \in \mathbb{K}$$

4 Intergration with respect to product measures

4.1 Integration of a product measure

Theorem 4.1 (Tonelli)**.** *Let $(\mathcal{X}, \mathbb{E}, \mu)$ and $(\mathcal{Y}, \mathbb{K}, \upsilon)$ be two σ-finite measurable spaces and let $d \in \mathcal{M}^+(\mathcal{X} \times \mathcal{Y}, \mathbb{E} \otimes \mathbb{K})$. It holds that*

$$\int f d\mu \otimes \upsilon = \int \left(\int f(y, x) d\mu(y) \right) d\upsilon(x)$$

5 Image measures

Definition 5.1 (Image measure)**.** *For two measurable spavces $(\mathcal{X}, \mathbb{E})$ with measure μ and $\mathcal{Y}, \mathbb{K}$. The **image measure** $t(\mu)$, as the measure on $\mathcal{Y}, \mathbb{K}$ is given by*

$$t(\mu) = \mu\left(t^{-1}(B)\right) \tag{5.1}$$

5.1 Intergration with respect ot image measures

Definition 5.2 (Abstract-change-of-variable formula)**.** *Let $\mathcal{X}, \mathbb{E}, \mu$ be a measure space and let $(\mathcal{Y}, \mathbb{K})$ be a measurable space, and let $t : \mathcal{X} \longrightarrow \mathcal{Y}$ be $\mathbb{E} - \mathbb{K}$ measurable. For every function $g \in \mathcal{M}^+(\mathcal{Y}, \mathbb{K})$ it holds that*

$$\int g \, dt(\mu) = \int g \circ t \, d\mu \tag{5.2}$$

Remark 5.3. *The above also holds for real-valued functions, but we have to make sure that the g function is $t(\mu)$-integrable. This holds iff $g \circ t$ is $t(\mu)$-integrable. See Collary 10.9 in EH.*

5.2 Translation invariance

Definition 5.4. *A **translation** in $\mathbb{R}^k$ is a map $\mathbb{R}^k \longrightarrow \mathbb{R}^k$ of the form*

$$\tau(x) = x + w \, \forall \, x \in \mathbb{R}^k \tag{5.3}$$

for fixed $w \in \mathbb{R}^k$

A map is **translation invariant** if

$$\tau_w(\mu) = \mu \text{ for every choice of } w \in \mathbb{R}^k \tag{5.4}$$

Remark 5.5. *Intuativly it does translation invariance means that the measure is determined by the shape of the set and not where it is located in* $\mathbb{R}^k$

Remark 5.6. *The lebesque measure is translation invariant. See EH theorem 10.11*

6 Measure with density and Transformation of densitys

6.1 Measures with density

Lemma 6.1. *Let* $(\mathcal{X}, \mathbb{E}, \mu)$ *be a measure space, and let* $f \in \mathcal{M}^+(\mathcal{X}\mathbb{E})$*. The set function* $\nu : \mathbb{E} \longrightarrow [0, \infty]$ *defined by*

$$\nu(A) = \int_A f \, d\mu \tag{6.1}$$

is a measure on $\mathcal{X}, \mathbb{E}$

Remark 6.2. *The measure in eq. (6.1) ussually written as* $\nu = f \cdot \mu$ *and is said to have* ***density*** f *wrt.* μ

6.2 Transformation of densitys

The problem of transforming variables with a measure. E.g. if a random variable has a density f and there is some transformation of X, such that $Y = h(X)$. How to we then find the density of Y?

Theorem 6.3 (Transformation of densities with random variables)**.** *Let* X *be a real vauled stochastic variable, defined on a background space* $(\Omega, \mathbb{F}, P)$*. Assume that the dsitributuion of* X *has a density* f *with respect to* m*. Let* $h : \mathbb{R} \longrightarrow \mathbb{R}$ *be a borel measurable map. If* I *and* J *are two open intervals, such that*

(1) h *maps* I *bijectivly to* J

(2) h^{-1} *is a* C^1*-map on* J

(3) $P(X \in I) = 1$

then the stochastic variable $Y = h(X)$ *has density* $\tilde{f}$ *with respect to* m*, where*

$$\tilde{f} = \begin{cases} f\left(h^{-1}(y)\right) \left| \left(h^{-1}\right)'(y) \right| & \text{for } y \in J \\ 0 & \text{for } y \notin J \end{cases}$$

7 Descriptive theory: Moments

7.1 Expectations

Definition 7.1. *The expectatin is given by*

$$EX = \int X \, dP$$

With $X \geqslant 0$ *or* $EX < \infty$

8 Descriptive theory: Quantiles

9 Multidimensional observations

9.1 Independence

Definition 9.1. *Let X and Y be stochastic variables, defined on a common background space $\Omega, \mathbb{F}, P$, with $\mathcal{X}, \mathbb{E}$ and $\mathcal{Y}, \mathbb{K}$ respectivley. The two variables X, Y are* ***independent*** *if*

$$(X, Y)(P) = X(P) \otimes Y(P)$$

Symbolically we write $X \perp\!\!\!\perp Y$ so signify independence

Example 9.2. *If X has density f wrt. μ and Y has density g wrt. λ, and if X, Y are independent, then the joint distribution of X, Y has density h wrt. $\mu \otimes \lambda$, where*

$$h(x, y) = f(x)g(y)$$

10 The Central Limit Theorem

Theorem 10.1 (Laplaces CLT). *If $X_1, X_2 \dots$ are IID real valued random variables, with mean ξ and variance σ^2, then*

$$P\left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{X_i - \xi}{\sigma} \leqslant x\right) \longrightarrow \Phi(x)$$

for $n \to \infty$

11 Exam 2012/2013

11.1 Problem 1

Problem 11.1. Let X be a real valued random variable. If $EX^2 < \infty$ does it then hold that X has finite first moment?

Solution: Yes. From Jensens inequality we know that $f(EX) \leqslant Ef(X)$. Let $f(x) = x^2$, then $(EX)^2 \leqslant EX^2$

Problem 11.2. Let X, Y and Z be real valued random variables and $X \perp\!\!\!\perp Y \perp\!\!\!\perp Z$. Is it then true that $X \perp\!\!\!\perp Z$?

Solution: Yes. From **thm 18.8** $X \perp\!\!\!\perp Y \perp\!\!\!\perp Z \Rightarrow X \perp\!\!\!\perp Z \perp\!\!\!\perp Y$ and from **thm 18.9** $X \perp\!\!\!\perp Z \perp\!\!\!\perp Y \Rightarrow X \perp \perp Z$

Problem 11.3. Let X and Y be real valued stochastic variables with finite first moment and $E(Y \mid X) = X$ a.e. Is it then true that

$$E(X + Y \mid X) = 2X$$

Solution: Yes.

$$E(X + Y \mid X) = E(X \mid X) + E(Y \mid X) = X + X = 2X$$

Problem 11.4. Is it true that the Lebesgue measure on $\mathbb{R}$ is uniquely determined by its value on intervals on the form $(-\infty, 0]$, that is by

$$m((-\infty, 0])$$

Solution: First we note that $m((-\infty, 0]) = \infty$ by **ex. 2.17** as

$$m((-\infty, 0]) = \lim_{n\to\infty} m((-n, 0]) = \lim_{n\to\infty} n = \infty$$

But we see that

$$2m((-\infty, 0]) = \lim_{n\to\infty} 2m((-n, 0]) = \lim_{n\to\infty} 2n = \infty$$

So the Lebesgue measure is not uniquely determined on $(-\infty, 0]$

Problem 11.5. Let $B(x, r)$ denote a closed disk in $\mathbb{R}^2$ with center in $x \in \mathbb{R}^2$ and radius $r > 0$. Is it true that

$$m_2(B((1,1),1)) = m_2(B((0,0),1))?$$

Solution: Yes. Let $\tau(1,1) : \mathbb{R}^2 \to \mathbb{R}^2$ være givet ved

$$\tau_{(1,1)}(x, y) = (x, y) + (1, 1), \qquad (x, y) \in \mathbb{R}^2$$

Det gælder da at

$$\tau^{-1}(B((1,1),1)) = B((0,0),1)$$

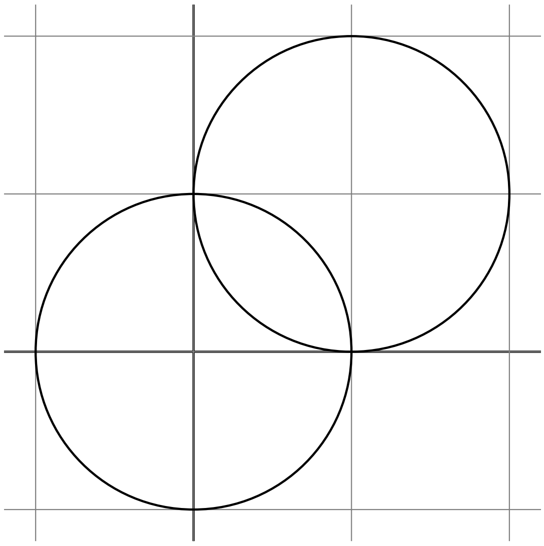


Figure 11.1: Translation invariance

11.2 Problem 2

Let X_k and Y_k be two random variables whose joint distribution is normal regular distribution on $(\mathbb{R}^2, \mathbb{B}_2)$ with mean 0 and variance matrix

$$\Sigma_k = \begin{pmatrix} \frac{1}{k} & 0 \\ 0 & \frac{1}{k} \end{pmatrix}$$

for $k \in \mathbb{N}$. That is $(X_k, Y_k)^T \sim \mathcal{N}(0, \Sigma_k)$.

Problem 11.6. Show that $X_k + Y_k \sim \mathcal{N}(0, 2/k)$

Solution: Let

$$M = \begin{pmatrix} X_k \\ Y_k \end{pmatrix}, \qquad \xi = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

Then using **Col. 18.29** we get that

$$BM \sim \mathcal{N}(B\xi, B\Sigma_k B^T) \tag{11.1}$$

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} X_k \\ Y_k \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{k} & 0 \\ 0 & \frac{1}{k} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \tag{11.2}$$

$$X_k + Y_k \sim \mathcal{N}(0, 2/k) \tag{11.3}$$

Problem 11.7. Show that for all $\epsilon > 0$

$$P \mid X_k + Y_k > \epsilon) \leqslant \frac{2}{k\epsilon^2}$$

Solution: From Chebychevs inequlity we have that

$$P(\mid X - EX \mid > \epsilon) \leqslant \frac{VX}{\epsilon^2}$$

Thereby it follows that

$$P(\mid X_k + Y_k - 0 \mid > \epsilon) \leqslant \frac{2/k}{\epsilon^2} \tag{11.4}$$

$$P(\mid X_k + Y_k \mid > \epsilon) \leqslant \frac{2}{k\epsilon^2} \tag{11.5}$$

Further. Clearly as $k \to \infty$, the rhs goes to zero, and thus the probability of $P(\mid X_k + Y_k - 0 \mid > \epsilon)$ must also go to zero, by Chebychevs inequality

Problem 11.8. Let $Z = X_2 Y_2$. Show that

$$EZ = 0 \qquad \text{and} \qquad VZ = \frac{1}{4}$$

Solution: We see from Σ that X_k, Y_k are independent and that $X_2 \sim \mathcal{N}(0, 1/2)$ and $X_2 \sim \mathcal{N}(0, 1/2)$. So

$$E(X_2 Y_2) = E(X_2)E(Y_2) = 0$$

Further we see that

$$V(X_2 Y_2) = E([X_2 Y_2]^2) - E(X_2 Y_2)^2 = E([X_2 Y_2]^2) = E(X_2^2)E(Y_2^2) = VX_2 VY_2 = \frac{1}{4}$$

Problem 11.9. Show that

$$P\left(\frac{2}{\sqrt{n}} \sum_{i=1}^{n} Z_i \leqslant 2 \right)$$

is convergent for $n \to \infty$ and compute the limit.

Solution: insert the variance and mean into the CLT formula

$$P\left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{Z_i}{1/2} \leqslant 2 \right) = P\left(\frac{2}{\sqrt{n}} \sum_{i=1}^{n} Z_i \leqslant 2 \right) \to \Phi(2)$$

11.3 Problem 3

For the next question it can be assumed and well known that

$$\int_{-\infty}^{\infty} \frac{1}{1+y^2} = \pi$$

Let $A(-1,1) \times \mathbb{R}$ and definte the $\mathcal{M}^+$-function f by

$$f(x,y) = \mathbb{1}_A(x,y) \frac{|x|}{2\pi(1+x^2y^2)}$$

for $(x,y) \in \mathbb{R}^2$

Problem 11.10. Show that

$$\int f \, dm_2 = 1$$

Solution:

$$\int f \, dm_2 = \int_{-1}^{1} \int_{\mathbb{R}} \frac{|x|}{2\pi(1+x^2y^2)} \, dy \, dx$$

Using **thm 12.7** in "reverse" with $\phi = 1/(1+y^2)$ and $h(y) = xy$ for a fixed $x \neq 0$ to evaluate the inner integral. It follows that $h'(y) = x$, such that

$$\int_{\mathbb{R}} \phi(h(y)) \mid h'(y) \mid = \int_{h(\mathbb{R})} \phi(y) \, dy \tag{11.6}$$

$$\frac{1}{2\pi} \int_{\mathbb{R}} \frac{1}{1+x^2y^2} \mid x \mid = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{1}{1+y^2} = \frac{1}{2} \tag{11.7}$$

So

$$\int_{(-1,1)\setminus\{0\}} \frac{1}{2} \, dx = \frac{1}{2} x \Big|_{-1}^{1} = 1$$

where we have used that $\{0\}$ is a m null set.
It $\mu = f \cdot m_2$, that is, if μ is a measure with density f w.r.t. the Lebeques measure then then it follows from the question above that μ is a probability measure.

Problem 11.11. Show that

$$\int x \, d\mu(x,y) = 0$$

Solution: first we note from **thm. 11.7** that

$$\int x \, d\mu(x,y) = \int x \cdot f \, dm_2$$

To check that the function is integrable we note that for $|x| \leqslant 1$

$$\int_A |x| \, f(x,y) \, dm_2 \leqslant \int_A f(x,y) \, dm_2 = 1 < \infty$$

So we have that

$$\int_A x f(x,y) \, dm_2 = \int_{-1}^{1} x \int_{\mathbb{R}} \frac{|x|}{2\pi(1+x^2y^2)} \, dy \, dx = \int_{-1}^{1} \frac{x}{2} \, dx = \frac{1}{4} x^2 \Big|_{-1}^{1} = 0$$

Problem 11.12. compute the density w.r.t. the Lebesgue measure for the joint distribution of X, XY

Solution: let

$$h(x,xy) = \begin{pmatrix} x \\ xy \end{pmatrix} = \begin{pmatrix} z \\ w \end{pmatrix}$$

which maps $U := A \setminus \{(x,y) \mid x = 0\}$ bijectivly onto intselfs. We note that U is open, and that h and h^{-1} is C^1 on U. I then follows that

$$h^{-1}(z,w) = \begin{pmatrix} z \\ \frac{w}{z} \end{pmatrix}$$

Finding the Jacobian of $h^{-1}(x,xy)$

$$Dh^{-1}(z,w) = \begin{pmatrix} 1 & 0 \\ -\frac{w}{z^2} & \frac{1}{z} \end{pmatrix}$$

Thus $\mid \det Dh^{-1}(z,w) \mid = \mid z \mid^{-1}$. Then $h(\mu)$ has density

$$\tilde{f}(z,w) = \mathbb{1}_U(z,w)\frac{\mid z \mid}{2\pi(1+w^2)} \mid z \mid^{-1} = \mathbb{1}_U(z,w)\frac{1}{2\pi(1+w^2)}$$

12 Problem 4

Let υ be a measure on $(0,\infty)$ with density

$$f(x) = \frac{1}{x^{3/2}}e^{-1/x}, \qquad x > 0$$

w.r.t. the Lebesque measure.

Problem 12.1. Show that

$$\upsilon((0,\infty)) = \sqrt{\pi}$$

Hint: Try subsitution with $h(x) = 1/\sqrt{x}$

Solution: We use **thm. 12.7**. We note that h is an monotonicly dicreasing function on $(0,\infty)$ and $h^{-1}(y) = 1/y^2$. Thereby maps $(0,\infty)$ to $(0,\infty)$ bijectivly. Further h^{-1} is C^1 on $(0,\infty)$. Further, the density is 0 on $(0,\infty)^c$. We also note that

$$h(x) = \frac{1}{\sqrt{x}}, \qquad h'(x) = -\frac{1}{2x^{3/2}}$$

we now note that

$$\upsilon((0,\infty)) = \int_0^\infty \frac{1}{x^{3/2}}e^{-1/x}\,dx = 2\int_0^\infty \exp(-h(x)^2) \mid h'(x) \mid dx = \int_0^\infty \exp(-y^2)\,dy = 2 \cdot \frac{\sqrt{\pi}}{2}$$

by **thm 12.7** with $\phi(x) = \exp(x^2)$
By the previus question we can now introduce the probability measure

$$\mu = \frac{1}{\sqrt{\pi}} f \cdot m_{(0,\infty)}.$$

That is, μ has density

$$g(x) = \frac{1}{\sqrt{\pi}x^{3/2}}e^{-\frac{1}{x}}, \quad x > 0$$

w.r.t. Lebesgue measure. Let X be stochastic variable with distribution μ.

Problem 12.2. Find the density w.r.t. the Lebesgue measure for the distribution of $1/X$

Solution: We see that $h(x) = 1/x$ which maps $((0,\infty))$ onto $(0,\infty)$ bijectively. Further $h^{-1}(y) = 1/y$ is C^1 on $(0,\infty)$ with $\left(h^{(-1)}\right)' = -1/y^2$. We the see that

$$\tilde{g}(y) = f(h^{-1}(y)) \mid (h^{-1})' \mid = \frac{1}{\sqrt{\pi}x^{1/2-1}}e^{-y}$$

which is the Γ-distribution with shape parameter $\lambda = 1/2$

Problem 12.3. Let Y be another stochastic variable with distribution μ such that X and Y are independent. Find the distribution of

$$\frac{1}{X} + \frac{1}{Y}$$

Solution: From **thm. 18.12** we know that $1/X$ and $1/Y$ are independent. By **ex. 20.11** we have that

$$\frac{1}{X} + \frac{1}{Y}$$

have density $\Gamma(1,1)$

13 Reexam 2012/2013

13.1 Problem 1

Problem 13.1. If X and Y are real valued random variables and $X \perp\!\!\!\perp Y$, is it then true that $X^2 \perp\!\!\!\perp Y^2$?

i *Solution:* Yes. **Thm 18.12:** if $X \perp\!\!\!\perp Y \Rightarrow h_x(X) \perp\!\!\!\perp h_y(Y)$ with $h_x = h_y = X^2$

Problem 13.2. Let X be a real valued stochastic variable with $EX = 0$ and such that $Ee^X < \infty$. Is it true that

$$1 \leqslant Ee^X?$$

Solution: yes: **Thm 16.31:** From Jensen inequality we have that

$$\varphi(EX) \leqslant E\varphi(X)$$

Letting $\varphi(x) = e^x$, we get that

$$e^{EX} \leqslant Ee^X \Rightarrow e^0 \leqslant Ee^X \Rightarrow 1 \leqslant Ee^X$$

Problem 13.3. Let X and Y be independent real valued random variables. Both with exponential distribution with scale paramenter $\beta = 1$. Is it then true that $X + Y$ is exponentially distributed with scale paramter 2?

Solution: No. Let $Z = X + Y$. The distribution of Z is then

$$f_Z(Z) = \int_{-\infty}^{\infty} e^x e^y = \int_{-\infty}^{\infty} e^{x+y}$$

Problem 13.4. Let $B = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 <\}1\}$ and let $f : [0,\infty) \to [0,\infty)$ be measurable. Is it true that

$$\int_B f(x^2 + y^2)\, dm_2(x,y) = 2\pi \int_0^1 f(r) r\, dr$$

Solution: From **Example 12.17**, we see that

$$\begin{aligned}\int_B f(x^2+y^2)\,dm_2(x,y) &= \int_{(0,2\pi)\times(0,1)} f(r)r\,dm_2(r,\theta)\\ &= \int_0^{2\pi}\int_0^1 f(r)r\,dr\,d\theta\\ &= \int_0^{2\pi} d\theta \int_0^1 f(r)r\,dr\\ &= 2\pi\int_0^1 f(r)r\,dr\end{aligned}$$

Problem 13.5. Is it true that

$$F(x) = \arctan(x) + \pi/2$$

is a distribution function on $\mathbb{R}$?

Solution:

(1) F is clearly increasing as $d/dxF = 1/(x^2+1) > 0$

(2) F is continus and therby also right continus

(3) $\lim_{x\to\infty} F(x) = \pi$

(4) $\lim_{x\to-\infty} F(x) = 0$

As the function $F(x)$ does not converge to 1, but to $\pi > 1$ F is not a distribution function.

13.2 Problem 2

Let X and Y be two real valued stochastic variables who's joint distribution is the regular normal distribution on $(\mathbb{R}^2, \mathbb{B}_2)$ with mean 0 and variance matrix

$$\Sigma = \begin{pmatrix}\sigma_1^2 & \rho\\ \rho & \sigma_2^2\end{pmatrix}$$

where we assume that $\sigma_1^2, \sigma_2^2 > 0$ and $-\sigma_1\sigma_2 < \rho < \sigma_1\sigma_2$. Σ is positive definite, and can then be a variance matrix for a regular normal distribution.

Problem 13.6. Find the joint distribution of $X+Y$ and $X-Y$

Solution: Using **Col 18.29** with

$$B = \begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}, \qquad M = \begin{pmatrix}X\\ Y\end{pmatrix}, \qquad \xi = \begin{pmatrix}0\\ 0\end{pmatrix}$$

we see that

$$BM = \begin{pmatrix}X+Y\\ X-Y\end{pmatrix} \sim \mathcal{N}(B\xi, B\Sigma B^T) = \mathcal{N}(0_{2\times1}, \begin{pmatrix}2\sigma_1^2+2\rho & 0\\ 0 & 2\sigma_1^2-2\rho\end{pmatrix}))$$

Problem 13.7. Determine all values of σ_1^2, σ_2^2 and ρ for which $X+Y \perp\!\!\!\perp X-Y$

Solution: As the $Cov(X+Y, X-Y) = 0$ we see that $X+Y$ and $X-Y$ are always independent.

Problem 13.8. Assume that $\sigma_1^2 = \sigma_2^2 = 0.6$ and that $\rho = 0.14$. Find the distribution of

$$(X+Y)^2 + \frac{1}{0.44}(X-Y)^2$$

Solution: We know that if $X \sim \mathcal{N}(0,1)$ and all the X_i are independent, then

$$\chi^2 = \sum_{i=1}^{n} X_i^2$$

is χ^2 distributed with n-degrees of freedom. So let's standarize $X+Y$ and $X-Y$.

$$\frac{X+Y}{\sigma_1^2 + 2\rho} = \frac{X+Y}{2 \times 0.6^2 + 2 \times 0.14} = X+Y \tag{13.1}$$

$$\frac{1}{0.44}\frac{X-Y}{\sigma_1^2 - 2\rho} = \frac{X-Y}{2 \times 0.6^2 - 2 \times 0.14} = \frac{X-Y}{0.44}0.44 = X-Y \tag{13.2}$$

So we conclude that $(X+Y)^2 + \frac{1}{0.44}(X-Y)^2 \sim \chi^2(2)$

13.3 Problem 3

Let $n \in \mathbb{N}$ and consider, for each n, independent real valued stocastic variables $Z_{1n} \dots Z_n n$, such that

$$P(Z_{nk} = n) = 1 = 1 - P(Z_{nk} = 0) = \frac{1}{n}$$

for $k = 1, \dots, n$. Thus Z_{nk} takes on almost surly the two values n and 0. Define

$$Y_n = \sum_{i=1}^{n} Z_{nk}$$

Problem 13.9. Show that $EY_n = 1$ and that $VY_n = \frac{n-1}{n}$

Solution: First we use **Example 16.20**, and see that $EZ_{nk} = np = n\frac{1}{n}$=1. The we use that the expectations of sums is equal to the sums of expectations, so

$$EY_n = E\frac{1}{n}\sum_{k=1}^{n} Z_{nk} = \frac{1}{n}\sum_{k=1}^{n} EZ_{nk} = \frac{1}{n}n = 1$$

We also see that $VZ_{nk} = np(1-p) = 1 - \frac{1}{n}$. Further. Since all the $Z_n k$'s are uncorrelated, we have that the variance of the sum, is equal to the sum of the variances.

$$VY_n = V\frac{1}{n}\sum_{k=1}^{n} Z_{nk} = \frac{1}{n}\sum_{k=1}^{n} VZ_{nk} = \frac{1}{n}n(1-\frac{1}{n}) = \frac{n-1}{n}$$

Problem 13.10. Consider if $\sqrt{n}(Y_n - 1) \to \mathcal{N}(0,1)$
Hint: Consider $P(\sqrt{n}(Y_n - 1) \leqslant 1)$

Solution: ???

13.4 Problem 4

Define the two $\mathcal{M}^+$-functions $f, g : \mathbb{R} \to [0, -\infty)$ by $f(x) = e^x$ and $g(y) = e^{-y}$. Define two measures $\mu = f \cdot m$ and $\upsilon = g \cdot m$

Problem 13.11. Show that μ and υ are σ-finite

Solution: Let

$$A_n = [-n, n],\ n \in \mathbb{N}$$

We note that $A_1 \subset A_2 \subset \dots$
For σ-finiteness, we need to show that

(1) $\mu A_n < \infty$ for all $n \in \mathbb{N}$

(2) $\bigcup_{n=1}^{\infty} A_i = \mathbb{R}$

(2) follows disrectly from our definition of A_n. To show (1), we take the integral over the function and show that it is finite.

$$\mu(A_n) = \int_{-n}^{n} f \cdot dm = \int_{-n}^{n} e^x \, ddx = e^x \Big|_{-n}^{n} = e^n - e^{-n} < \infty$$
$$\upsilon(A_n) = \int_{-n}^{n} g \cdot dm = \int_{-n}^{n} e^{-y} \, dy = e^{-y} \Big|_{-n}^{n} = e^{-n} - e^n < \infty$$

By σ-finiteness there is a unique product measure $\mu \otimes \upsilon$ on $\mathbb{R}^2, \mathbb{B}_2$

Problem 13.12. Show that $\mu \otimes \upsilon$ has density

$$(x, y) \mapsto e^{x-y}$$

w.r.t. m_2 and compute

$$A = \mu \otimes \upsilon(\{(x, y) \big| \mid x + y \mid < 1, \mid x - y \mid < 1\})$$

Solution: Using **lem 11.5** we get that

$$\mu \otimes \upsilon = (f \cdot m) \otimes (g \cdot m) = (f \otimes g) \cdot m_2$$

where

$$(f \otimes g) \cdot m_2(x, y) = (f \otimes g)(x, y) = f(x)g(y) = e^x e^{-y} = e^{x-y}$$

To find the area on A define two auxillery sets

$$B = \{(x, y) \big| -1 < x < 0, -x - 1 < y < x + 1\}$$
$$C = \{(x, y) \big| 0 < x < 1, x - 1 < y < -x - 1\}$$ [1]

Note that

$$\mathbb{1}_{A(x,y)} = \mathbb{1}_{B \cup C(x,y)} = \mathbb{1}_{B(x,y)} + \mathbb{1}_{C(x,y)} = \mathbb{1}_{(-1,0)} x \mathbb{1}_{(-x-1, x+1)} y + \mathbb{1}_{(0,1)} x \mathbb{1}_{(x-1, -x-1)} y$$

Using tonelli

$$\int_{-1}^{0} \int_{-x-1}^{x-1} e^{x-y} \, dm_2 + \int_{0}^{1} \int_{x-1}^{-x+1} e^{x-y} \, dm_2 \qquad \text{replace the set A with B + C}$$
$$\int_{-1}^{0} e^x \int_{-x-1}^{x-1} e^{-y} \, dx \, dy + \int_{0}^{1} e^x \int_{x-1}^{-x+1} e^{-y} \, dx \, dy \qquad \text{Use Tonellie}$$
$$\int_{0}^{1} e^x \left(-e^{-x+1} + e^{x-1}\right) dx + \int_{0}^{1} e^x \left(-e^{x+1} + e^{-x+1}\right) dx \qquad \text{Evaluate the inner integral}$$
$$\int_{0}^{1} -e^{-1} + e^{2x+1} \, dx + \int_{0}^{1} e^x - e^{2x+1} + e^1 \, dx \qquad \text{Rearrange}$$

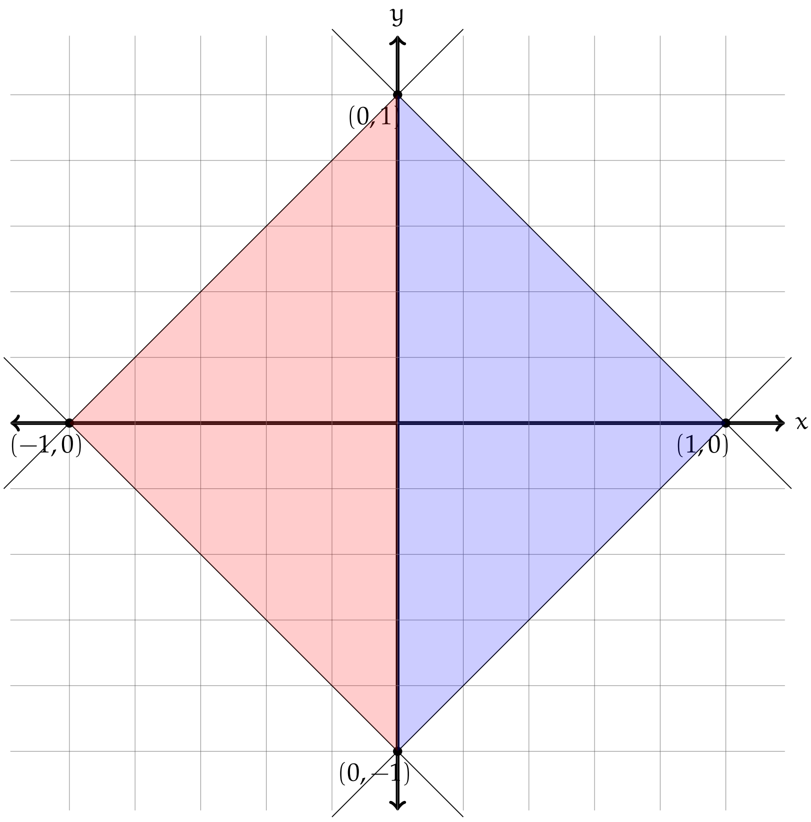


Figure 13.1: B set in red and C set in blue. Note that $C \cup B = A$

$$e \cdot \int_0^1 e^{2x}\,dx - e^{-1} + e - e^{-1} \cdot \int_0^1 -e^{2x}\,dx \qquad \text{Rearrange}$$
$$e\frac{1-e^{-2}}{2} - e^{-}1 + e - e^{-}1\frac{e^2-1}{2}$$
$$\frac{e}{2} - \frac{e^{-1}}{2} - e^{-1} + e\frac{e}{2} + \frac{e^{-1}}{2} = e + e^{-1}$$

Problem 13.13. Let $h(x,y) = \mathbb{1}_A(x,y)xy$ with $A = (-\infty, 0) \times (0, \infty)$. Compute

$$\int h \, d\mu \otimes \upsilon$$

Solution: First note that h is a function defined on a negative set A. So h is a negative function. Define a new function $g := -h \in \mathcal{M}^+$.

$$\begin{aligned}
\int g \, d(\mu \otimes \upsilon) &= \int g(x,y)e^{x-y}\,dm_2(x,y) \\
&= \int\int -\mathbb{1}_A(x,y)xye^{x-y}\,dx\,dy \\
&= \int_0^\infty \int_{-\infty}^0 -xye^{x-y}\,dx\,dy \\
&= \int_0^\infty -xe^x\,dx \int_{-\infty}^0 ye^{-y}\,dy \\
&= \int_{-\infty}^0 xe^x\,dx \int_0^\infty -ye^{-y}\,dy
\end{aligned}$$

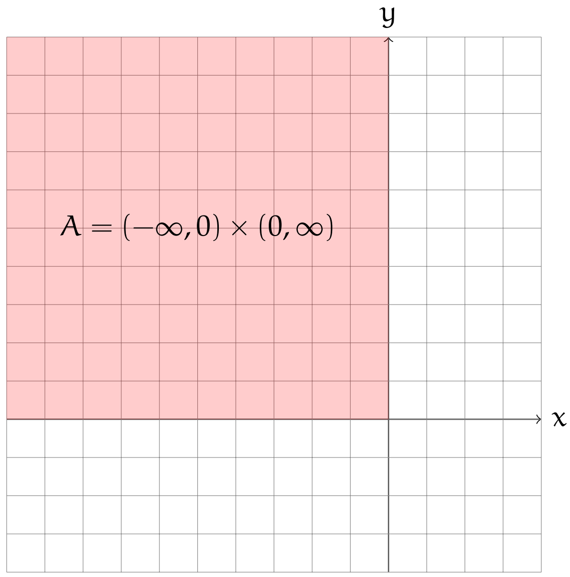


Figure 13.2: The set $A = (-\infty, 0) \times (0, \infty)$

$$
\begin{aligned}
&= \left(\int_0^\infty -ye^{-y}\, dy \right)^2 \\
&= \left([-y(-e^{-y})]_0^\infty + \int_0^\infty e^{-y} \right)^2 \\
&= \left([-e^{-y}]_0^\infty \right)^2 \\
&= 1
\end{aligned}
$$

So $g =| h |$ is h-integrable w.r.t $\mu \otimes \nu$ and

$$\int h\, d(\mu \otimes \nu) = -\int g\, d\mu \otimes \nu = -1$$

13.5 Problem 5

Let X, Y and Z be three indenependent real valued random variables. Al with finite second momennt and all with mean 0 and variance 1. Define

$$W = \frac{X + YZ}{\sqrt{1 + Z^2}}$$

Problem 13.14. Show that $V(W \mid Z) = 1$ a.e.

Solution: We have that

$$V(W \mid Z) = \frac{V(X + YZ \mid Z)}{1 + Z^2} \tag{13.3}$$

$$V(X + YZ \mid Z) = V(X \mid Z) + V(YZ \mid Z) = V(X) + Z^2 V(Y) = 1 + Z^2 \tag{13.4}$$

Problem 13.15. Find the distribution of W under the additional assumption that X, Y and Z all have a marginal $\mathcal{N}(0, 1)$-distribution
Hint: Compute $P(W \leqslant w)$ by Tonellie, intergrating out of the joint distribution of (X, Y) first.

Solution:

For independent $T_1 \sim \mathcal{N}(\mu_{T_1}, \sigma^2_{T_1})$ and $T_2 \sim \mathcal{N}(\mu_{T_2}, \sigma^2_{T_2})$ we have

$$\Phi\left(\frac{\mu_{T_2} - \mu_{T_1}}{\sqrt{\sigma^2_{T_1} + \sigma^2_{T_2}}}\right) = P\left(T_1 < T_2\right) = \frac{1}{\sigma_{T_2}\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \Phi\left(\frac{t - \mu_{T_1}}{\sigma_{T_1}}\right) e^{-\frac{1}{2}\left(t-\mu_{T_2}\right)^2/\sigma^2_{T_2}}\, dt\,, \quad (13.5)$$

where Φ is the CDF for the standard normal distribution

So, after using the definition of conditional probability, the independence of x and $\{Y, Z\}$ and Tonelli's theorem (to justify the iterated integrals and "associativity" thereof), we can use the observation above to solve the integral as follows:

$$\begin{aligned}
P\left(W < w\right) &= P\left(\frac{X + YZ}{\sqrt{1+Z^2}} < w\right) = P\left(X + YZ < w\sqrt{1+Z^2}\right) \\
&= P\left(X < w\sqrt{1+Z^2} - YZ\right) \\
&= \frac{1}{\left(\sqrt{2\pi}\right)^2} \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} P\left(X < w\sqrt{1+z^2} - yz \,\middle|\, Y = y, Z = z\right) e^{-\frac{1}{2}\left(y^2+z^2\right)} dy\, dz \\
&= \frac{1}{\left(\sqrt{2\pi}\right)^2} \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} P\left(X < w\sqrt{1+z^2} - yz\right) e^{-\frac{1}{2}\left(y^2+z^2\right)} dy\, dz \\
&= \frac{1}{\left(\sqrt{2\pi}\right)^2} \int\limits_{-\infty}^{\infty} \left(\int\limits_{-\infty}^{\infty} P\left(X < w\sqrt{1+z^2} - yz\right) e^{-\frac{1}{2}y^2} dy\right) e^{-\frac{1}{2}z^2}\, dz \\
&= \frac{1}{\left(\sqrt{2\pi}\right)^2} \int\limits_{-\infty}^{\infty} \left(\int\limits_{-\infty}^{\infty} \Phi\left(w\sqrt{1+z^2} - yz\right) e^{-\frac{1}{2}y^2} dy\right) e^{-\frac{1}{2}z^2}\, dz \\
&= \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \frac{1}{z\sqrt{2\pi}} \left(\int\limits_{-\infty}^{\infty} \Phi\left(u\right) e^{-\frac{1}{2}\left(u - w\sqrt{1+z^2}\right)^2/z^2} dy\right) e^{-\frac{1}{2}z^2}\, dz \\
&= \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \Phi\left(w\right) e^{-\frac{1}{2}z^2}\, dz \\
&= \Phi\left(w\right)
\end{aligned}$$

Therefore, $W \sim \mathcal{N}\left(0, 1\right)$.

14 Exam 2013/2014

14.1 Problem 1

Problem 14.1. Let $X \sim \mathcal{N}(0, 1)$. Argue that the distribution for $\exp(X)$ has density w.r.t. the Lebesgue measure. You do not need to find the density.

Solution: From **thm. 15.1** we know that the distribution has density as

(1) h maps $\mathbb{R}$ onto $\mathbb{R}^+$

(2) $h^{-1} = \log(x)$ is C^1 on $\mathbb{R}^+$

(3) $P(X \in \mathbb{R}) = 1$

Problem 14.2. Assume that

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\right)$$

Determine if $X_1 \perp\!\!\!\perp X_2$.

Solution: As the joint distribution of X_1 and X_2 has $Cov(X_1, X_2) = 0$, it follows from **thm. 18.27** that $X_1 \perp\!\!\!\perp X_2$.

Problem 14.3. Assume that $X_1 \dots X_n$ are iid. with

$$P(X_i = 2) = P(X_i = -2) = \frac{1}{2}$$

Determine if

$$P\left(\sum_{i=1}^{n} X_i \leqslant 2x\sqrt{n}\right) \to \Phi(x)$$

for $n \to \infty$

Solution: As X_i's are bounded by $\{-2, 2\}$ all moments exists. $EX = -2\frac{1}{2} + 2\frac{1}{2} = 0$ and $VX = (-2)^2\frac{1}{2} + 2^2\frac{1}{2} = 4$, so $\sigma = \sqrt{4} = 2$. It the follows from the CLT that

$$P\left(\sum_{i=1}^{n} X_i \leqslant 2x\sqrt{n}\right) = P\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} \frac{X_i}{2} \leqslant x\right) \to \Phi(x)$$

14.2 Problem 2

Assume that X and Y are independent real valued random variables and that X is uniformly distributed on $(-1, 1)$ and that Y is Γ-distributed with shape parameter $\lambda = 3$

Problem 14.4. Show that

$$E\left(\frac{1}{Y}\right) = \frac{1}{2}$$

Solution: Use **Thm. 15.1** with $h = 1/Y$. We see that h and $h^{-1} = 1/X$ is a monotonically decreasing function on $(0, \infty)$ and thereby also bijective. Further we follows that $P(Y \in (0, \infty)) = 1$. Thereby we can apply the theorem. So

$$f\left(h^{-1}\left(\frac{1}{y}\right)\right)\left|\left(h^{-1}\right)'\left(\frac{1}{y}\right)\right| = \frac{1}{2}\left(\frac{1}{x}\right)^2 e^{-1/x}\left(\frac{1}{x^2}\right) \tag{14.1}$$

$$= \frac{1}{2}\frac{1}{x^4}e^{1/-x} \tag{14.2}$$

so the distribution of $1/Y$ is

$$\tilde{f} = \begin{cases} \frac{1}{2}\frac{1}{x^4}e^{1/-x} & \text{for } x \in (0, \infty) \\ 0 & \text{for } x \notin (0, \infty) \end{cases}$$

Calculating the expectation

$$\begin{aligned}
\int_0^{-\infty} x \cdot \frac{1}{2}\frac{1}{x^4} e^{1/-x}\,dx &= \int_0^{-\infty} \frac{1}{2}\frac{1}{x^3} e^{1/-x}\,dx \\
&= \frac{1}{2}\int_0^{-\infty} \frac{1}{x^3} e^{1/-x}\,dx && \text{Factor out the constant} \\
&= \frac{1}{2}\int_0^{-\infty} ue^{-u}\,du && \text{change } u = 1/x \\
&= \frac{1}{2}\underbrace{\left[ue^{-u}\right]_0^\infty}_{=0} - \frac{1}{2}\int_0^\infty e^{-u}\,du && \text{integration by parts} \\
&= -\frac{1}{2}\left[-e^{-u}\right]_0^\infty && \text{evaluate the integral} \\
&= -\frac{1}{2}\left[-e^{-1/x}\right]_0^\infty && \text{revert} \\
&= \frac{1}{2}
\end{aligned}$$

A faster way is to use **Example 16.6**. Let $t(x)$ be a transformation, then

$$Et(x) = \int t(h\,d\mu)$$

So with $t(x) = 1/y$ and $f\mu = \int \frac{1}{2}x^2e^{-x}$ we get that

$$\int_{-\infty}^{\infty} \mathbb{1}_{(0,\infty)} \frac{1}{y}\frac{1}{2}y^2e^{-y}\,dy = \int_0^\infty \frac{1}{2}ye^{-y}\,dy \tag{14.3}$$

$$= \frac{1}{2}\underbrace{\int_0^\infty ye^{-y}\,dy}_{\text{Exponential density}} \tag{14.4}$$

$$= \frac{1}{2}\cdot 1 = \frac{1}{2} \tag{14.5}$$

Problem 14.5. Compute

$$E\left(\frac{X}{Y}\,\middle|\,X\right) \text{ ans show that } E\left(\frac{X}{Y}\right) = 0$$

Solution: As $X \perp\!\!\!\perp Y$ we have that

$$E\left(\frac{X}{Y}\,\middle|\,X\right) = E(X \mid X)\,E\left(\frac{1}{Y}\,\middle|\,X\right) = X\,E\left(\frac{1}{Y}\right) = 0\frac{1}{2} = 0$$

14.3 Problem 3

Define the function $F : \mathbb{R} \to [0, \infty)$ by

$$F(x) = 1 - \frac{1}{1 + \log(x+1)}$$

for $x \geqslant 0$ and $F(x) = 0$ for $x < 0$. It can be used, without proof that F is a distribution function for a measure on $(\mathbb{R}, \mathbb{B})$. Let υ be a probability measure with distribution function F. That is

$$F(x) = \upsilon((-\infty, x])$$

Problem 14.6. Draw a graph of the distribution function F. Explain that υ has density w.r.t. the Lebesgue measure and find the density.

Solution: We further note that F is C^1 on $[0, \infty)$ with

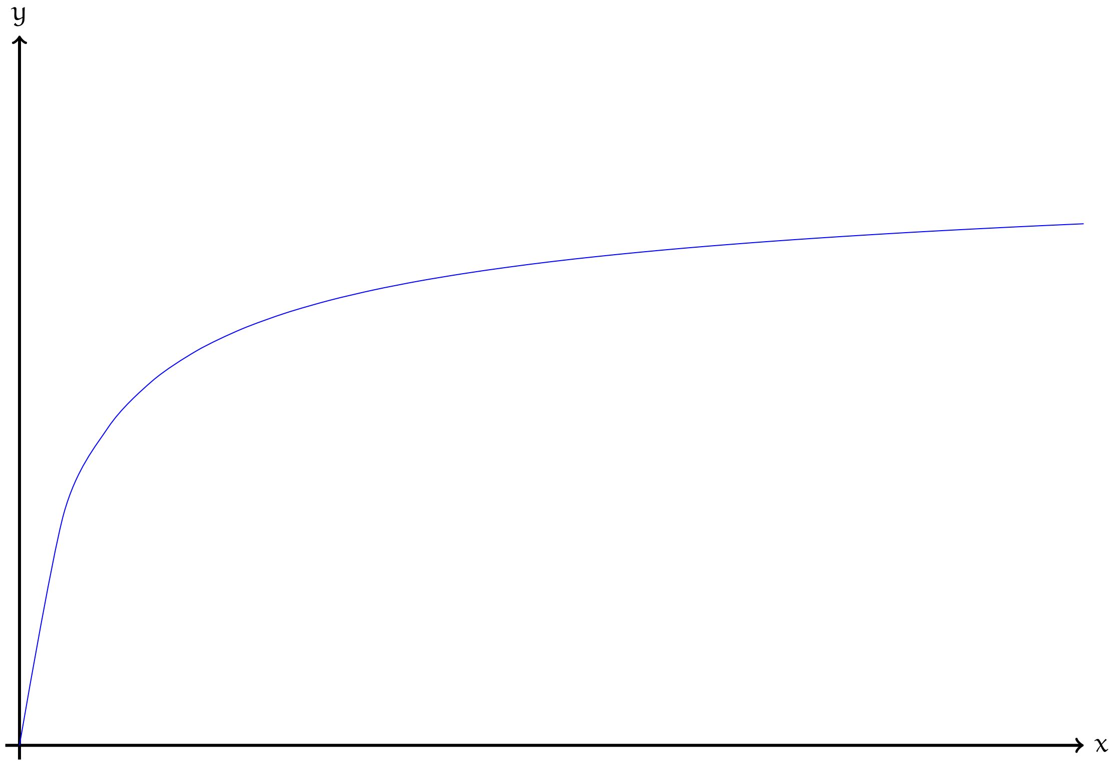


Figure 14.1: Plot of $F(x) = 1 - \dfrac{1}{1 + \log(x+1)}$

$$f(x) = F'(x) = \frac{1}{(1 + \log(x+1))^2(x+1)}$$

We see that

$$\int_0^\infty f(x) = F(x)\Big|_0^\infty = 1$$

We thereby conclude that υ has density

$$f(x) = \begin{cases} \frac{1}{(1+\log(x+1))^2(x+1)} & \text{for } x > 0 \\ 0 & \text{for } x \leqslant 0 \end{cases}$$

w.r.t. the Lebesgue measure.

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