Eksamensopgaver 2012-2013

Mål- og integralteori

med vejledende besvarelser

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Exam 2012/2013

Measures and Integrals

University of Copenhagen
Department of Mathematical Sciences

Formalities

This is the final, individual, 27 hours take-home exam in the course *Measures and Integrals*, 2012/2013. It is made available electronically via Absalon at 9.00am, January 15, 2013 and the deadline for handing in the solution is 12.00am, January 16, 2013. Later handins are not accepted. The solution must be handed in at the secretarial office at the Department of Mathematical Sciences (office 04.1.03, E-building, HCØ). A printed copy of the exam and the official front page for the handin can be obtained from the secretarial office from 9.00am, January 15, 2013.

The solution can be written using pen, pencil or computer. The final handin must have numbered pages and it must be equipped with the official front page that you find electronically on Absalon or obtain from the secretarial office.

This is an individual exam. During the 27-hours exam you are not allowed to communicate in any way with anybody about anything related to the the exam.

The exam consists of 4 pages with 4 problems and a total of 15 questions all counted with equal weight. Throughout it can be assumed that all stochastic variables are defined on the background probability space $(\Omega, \mathbb{F}, P)$.

Problem 1

The first problem consists of 5 independent questions. A complete answer must be supported by a small argument, a counter example or a reference to the book.

Question 1.1. Let X be a real valued random variable. If $E|X|^2 < \infty$ does it then hold that X has finite first moment?

It is true that X has finite first moment, which follows from Lemma 16.13.

Question 1.2. If X, Y and Z are real valued stochastic variables and $X \perp\!\!\!\perp Y \perp\!\!\!\perp Z$ is it then true that $X \perp\!\!\!\perp Z$?

It is true that $X \perp\!\!\!\perp Z$ are independent, which follows by combining Theorem 18.8 and Theorem 18.9.

Question 1.3. Let X and Y be real valued stochastic variables with finite first moment and with $E(Y \mid X) = X$ a.e. Is it true that

$$E(X + Y \mid X) = 2X \quad \text{a.e.?}$$

It is true that $E(X + Y \mid X) = 2X$ a.e. as follows from Lemma 2.4 combined with Example 2.3, which gives

$$E(X + Y \mid X) = E(X \mid X) + E(Y \mid X) = X + X = 2X$$

with all equalities holding a.e.

Question 1.4. Is it true that the Lebesgue measure m on $\mathbb{R}$ is uniquely determined by its values on intervals of the form $(-\infty, x]$, that is, by

$$m((-\infty, x])$$

for all $x \in \mathbb{R}$?

It is **not** true that the Lebesgue measure is uniquely determined by its values on $(-\infty, x]$. It takes the value ∞ for all such sets and the measure given by $\nu(B) = \infty$ for $B \neq \emptyset$ and $\nu(\emptyset) = 0$ is thus equal to m for all $B = (-\infty, x]$, while ν is certainly not equal to the Lebesgue measure – take e.g. $B = [0, 1]$ or any other bounded, non-empty set.

Question 1.5. Let $B(x, r)$ denote the closed disc in $\mathbb{R}^2$ with center $x \in \mathbb{R}^2$ and radius $r > 0$. Is it true that

$$m_2(B((1, 1), 1)) = m_2(B((0, 0), 1))?$$

It is true that $m_2(B((1, 1), 1)) = m_2(B((0, 0), 1))$ by translation invariance of the Lebesgue measure (Theorem 10.11) as

$$B((0, 0), 1) = \tau_{(1,1)}^{-1}(B((1, 1), 1)).$$

That is, $B((1, 1), 1)$ is the translation of $B((0, 0), 1)$ by $w = (1, 1)$.

Problem 2

Let X_k and Y_k be two stochastic variables whose joint distribution is the regular normal distribution on $(\mathbb{R}^2, \mathbb{B}_2)$ with mean 0 and variance matrix

$$\Sigma_k = \begin{pmatrix} \frac{1}{k} & 0 \\ 0 & \frac{1}{k} \end{pmatrix}$$

for $k \in \mathbb{N}$. That is, $(X_k, Y_k)^T \sim \mathcal{N}(0, \Sigma_k)$.

Question 2.1. Show that $X_k + Y_k \sim \mathcal{N}(0, 2/k)$.

It follows from Corollary 18.29 with $B = (1, 1)$ that

$$X_k + Y_k = B(X_k, Y_k)^T \sim \mathcal{N}(0, B\Sigma_k B^T)$$

where

$$B\Sigma_k B^T = \frac{1}{k} + \frac{1}{k} = \frac{2}{k}.$$

Alternative, from Theorem 18.27, X_k and Y_k are independent each with a $\mathcal{N}(0, 1/k)$ distribution and Example 20.10 shows that $X_k + Y_k \sim \mathcal{N}(0, 1/k + 1/k) = \mathcal{N}(0, 2/k)$.

Question 2.2. Show that for all $\epsilon > 0$

$$P(|X_k + Y_k| > \epsilon) \leq \frac{2}{k\epsilon^2}$$

and conclude that

$$X_k + Y_k \xrightarrow{P} 0$$

for $k \to \infty$.

By the previous question $X_k + Y_k$ has a normal distribution, thus it has finite second moment and the variance is $2/k$. It follows from Chebyshev's inequality, Theorem 16.19, that for all $\epsilon > 0$

$$P(|X_k + Y_k| > \epsilon) \leq \frac{V(X_k + Y_k)}{\epsilon^2} = \frac{2}{k\epsilon^2}$$

where we have used that $E(X_k + Y_k) = 0$. As $2/(k\epsilon^2) \to 0$ for $k \to \infty$ for all $\epsilon > 0$ it follows by Definition 19.17 that

$$X_k + Y_k \xrightarrow{P} 0$$

for $k \to \infty$.

Question 2.3. Let $Z = X_2 Y_2$. Show that

$$EZ = 0 \quad \text{and} \quad VZ = \frac{1}{4}.$$

By Theorem 18.27 X_2 and Y_2 are independent each with a $\mathcal{N}(0, 1/2)$ distribution. As X_2 and Y_2 are independent it follows from slide 16, December 18 (Theorem 19.9) that $Z := X_2 Y_2$ has finite first moment and that

$$EZ = E(X_2 Y_2) = EX_2\, EY_2 = 0 \times 0 = 0.$$

By Theorem 18.12 also X_2^2 and Y_2^2 are independent with finite first moment and again by the same slide or Theorem 19.9 we have

$$E(X_2^2 Y_2^2) = EX_2^2\, EY_2^2 = VX_2\, VY_2 = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}.$$

Now, as $EZ = 0$ we have

$$VZ = EZ^2 = E(X_2^2 Y_2^2) = \frac{1}{4}.$$

Assume that $Z_1, Z_2, \ldots$ are independent and identically distributed all with the same distribution as Z.

Question 2.4. Show that

$$P\left(\frac{2}{\sqrt{n}}\sum_{i=1}^{n} Z_i \le 2\right)$$

is convergent for $n \to \infty$ and compute the limit.

As $EZ_i = 0$ and $\sigma^2 := VZ_i = 1/4$ and the Z's are assume independent and identically distributed it follows from Laplaces CLT, slide 7, January 7 (alternatively, Example 3.36) that

$$P\left(\frac{2}{\sqrt{n}}\sum_{i=1}^{n} Z_i \le 2\right) = P\left(\frac{1}{\sqrt{n}\sigma}\sum_{i=1}^{n} Z_i \le 2\right) \to \Phi(2)$$

for $n \to \infty$ where

$$\Phi(2) = \int_{-\infty}^{2} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,\mathrm{d}x \simeq 0.9772.$$

Problem 3

For the next question it can be assumed well known that

$$\int_{-\infty}^{\infty} \frac{1}{1+y^2}\,\mathrm{d}y = \pi.$$

Let $A = (-1, 1) \times \mathbb{R}$ and define the $\mathcal{M}^+$-function f by

$$f(x, y) = 1_A(x, y)\frac{|x|}{2\pi(1 + x^2y^2)}$$

for $(x, y) \in \mathbb{R}^2$.

Question 3.1. Show that

$$\int f\,\mathrm{d}m_2 = 1.$$

As the integrand is positive we use Tonelli's theorem to compute

$$\begin{aligned}\int_A \frac{|x|}{2\pi(1+x^2y^2)} \mathrm{d}m_2(x,y) &= \int_{(-1,1)} \int \frac{|x|}{2\pi(1+x^2y^2)} \,\mathrm{d}y\,\mathrm{d}x \\ &= \int_{(-1,1)\setminus\{0\}} \frac{1}{2}\,\mathrm{d}x = 1.\end{aligned}$$

Here we have used that $\{0\}$ is a m-nullset and that

$$\int \frac{1}{1+x^2y^2}|x|\,\mathrm{d}m(y) = \int \frac{1}{1+y^2}\,\mathrm{d}y = \pi$$

for $x \neq 0$. The first identity follows from Theorem 12.7 using, for fixed $x \neq 0$, that $h(y) = xy$ is C^1-function that maps $\mathbb{R}$ bijectively onto $\mathbb{R}$ with C^1-inverse and with $|h'(y)| = |x|$.

If $\mu = f \cdot m_2$, that is, μ is the measure with density f w.r.t. the Lebesgue measure, then it follows from the question above that μ is a probability measure.

Question 3.2. Show that

$$\int x\,\mathrm{d}\mu(x,y) = 0.$$

We first check that $(x,y) \mapsto x$ is integrable w.r.t. μ. By Theorem 11.7 and the fact that $|x| \leq 1$ for $(x,y) \in A$ we find that

$$\begin{aligned}\int |x|\,\mathrm{d}\mu &= \int_A |x| f \mathrm{d}m_2(x,y) \\ &\leq \int_A f \mathrm{d}m_2(x,y) = 1 < \infty.\end{aligned}$$

By Corollary 11.9 and Fubini's theorem

$$\begin{aligned}\int x\,\mathrm{d}\mu &= \int_A x f \mathrm{d}m_2(x,y) \\ &= \int_{(-1,1)} x \int \frac{1}{2\pi(1+x^2y^2)}|x|\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_{(-1,1)} \frac{x}{2}\,\mathrm{d}x\end{aligned}$$

where we used the same argument as above to compute the inner integral (we don't have to exclude the case $x = 0$ here). We compute the integral as

$$\int_{-1}^{1} \frac{x}{2}\,\mathrm{d}x = \frac{1}{4}x^2\Big|_{-1}^{1} = \frac{1}{4}(1^2 - (-1)^2) = 0.$$

Alternatively, we can use that

$$x \mapsto 1_{(-1,1)}(x)\frac{x}{2}$$

is an odd function and refer to Exercise 10.12(c) to conclude that the integral is 0.

Let X and Y denote stochastic variables whose joint distribution is μ.

Question 3.3. Compute the density w.r.t. the Lebesgue measure for the joint distribution of (X, XY).

Let $h(x,y) = (x, xy)$ which maps $U := A \backslash \{(x,y) \mid x = 0\}$ bijectively onto U itself with inverse $h^{-1}(z,w) = (z, w/z)$. We note that U is open, that h as well as h^{-1} are C^1 on U and that

$$Dh^{-1}(z,w) = \begin{pmatrix} 1 & 0 \\ -\frac{w}{z^2} & \frac{1}{z} \end{pmatrix}.$$

Thus, $|\det Dh^{-1}(z,w)| = |z|^{-1}$ for $(z,w) \in U$ (where $z \neq 0$). As $m_2(\{(x,y) \mid x = 0\}) = 0$ we have $\mu(U) = 1$ and by Theorem 12.14 (slide 7, December 18) the measure $h(\mu)$ has density

$$\tilde{f}(z,w) = 1_U(z,w)\frac{|z|}{2\pi(1+w^2)}|z|^{-1} = 1_U(z,w)\frac{1}{2\pi(1+w^2)}$$

w.r.t. m_2.

Problem 4

Let ν be the measure on $(0,\infty)$ with density

$$f(x) = \frac{1}{x^{3/2}}e^{-\frac{1}{x}}, \quad x > 0$$

w.r.t. the Lebesgue measure.

Question 4.1. Show that

$$\nu((0,\infty)) = \sqrt{\pi}.$$

Hint: Try substitution with $h(x) = 1/\sqrt{x}$.

We note that with h as in the hint, h is a C^1 bijective map of $(0,\infty)$ onto $(0,\infty)$ with inverse $h^{-1}(y) = 1/y^2$ also C^1. Moreover, $h'(x) = -\frac{1}{2x^{3/2}}$ hence

$$\nu((0,\infty)) = \int_0^\infty \frac{1}{x^{3/2}}e^{-\frac{1}{x}}\,\mathrm{d}x = 2\int_0^\infty \exp(-h(x)^2)|h'(x)|\,\mathrm{d}x = 2\int_0^\infty \exp(-y^2)\,\mathrm{d}y$$

by Theorem 12.7. The result follows since

$$2\int_0^\infty \exp(-y^2)\,\mathrm{d}y = \int_{-\infty}^\infty \exp(-y^2)\,\mathrm{d}y = \sqrt{\pi}$$

as we recognize the integral as the normalization constant for the density of the normal distribution with scale parameter $1/2$. We have, in principal, used Exercise 10.12(a) and the fact that $y \mapsto e^{-y^2}$ is an even function.

By the previous question we can introduce the probability measure

$$\mu = \frac{1}{\sqrt{\pi}} f \cdot m_{(0,\infty)},$$

that is, μ has density

$$g(x) = \frac{1}{\sqrt{\pi} x^{3/2}} e^{-\frac{1}{x}}, \quad x > 0$$

w.r.t. the Lebesgue measure. Let X be a stochastic variable with distribution μ.

Question 4.2. Find the density w.r.t. the Lebesgue measure for the distribution of $1/X$.

Let $h(x) = \frac{1}{x}$ for $x > 0$ then we have to find the density for the transformed measure $h(\mu)$, cf. Theorem 15.1. We have that h maps $(0,\infty)$ onto $(0,\infty)$ and that $h^{-1}(y) = \frac{1}{y}$ with $(h^{-1})'(y) = -\frac{1}{y^2}$ for $y > 0$. Since $\mu((0,\infty)) = P(X > 0) = 1$ it follows by Theorem 15.1 (alternatively, Theorem 12.6), that the density for $h(\nu)$ is

$$\tilde{f}(y) = f(y^{-1}) \frac{1}{y^2} = \frac{1}{\sqrt{\pi}} y^{1/2-1} e^{-y}.$$

We recognize this density as the density for the Γ-distribution with shape parameter $\lambda = 1/2$. (That the normalization constant, $\Gamma(1/2) = \sqrt{\pi}$, is correct follows from slide 5, November 23).

Question 4.3. Let Y be another stochastic variable with distribution μ such that X and Y are independent. Find the distribution of

$$\frac{1}{X} + \frac{1}{Y}.$$

It follows from the previous question that $1/X$ and $1/Y$ both are Γ-distributed with shape parameter $1/2$ (and scale parameter $\beta = 1$). It follows from Theorem 18.12 that $1/X$ and $1/Y$ are independent. Thus from Example 20.11 it follows that

$$\frac{1}{X} + \frac{1}{Y}.$$

has a Γ-distribution with shape parameter $\lambda = 1$ (and scale parameter $\beta = 1$). This is also the exponential distribution with density e^{-y} for $y > 0$ w.r.t. the Lebesgue measure. This density can also be found by direct usage of the convolution formula in Corollary 20.9 without reference to Example 20.11.

Reexam 2012/2013

Measures and Integrals

University of Copenhagen
Department of Mathematical Sciences

Formalities

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The exam consists of 5 problems with a total of 15 questions all counted with equal weight. Throughout it can be assumed that all stochastic variables are defined on the background probability space $(\Omega, \mathbb{F}, P)$.

Problem 1

The first problem consists of 5 independent questions. A complete answer must be supported by a small argument, a counter example or a reference to the book.

Question 1.1. If X and Y are real valued stochastic variables and $X \perp\!\!\!\perp Y$ is it then true that $X^2 \perp\!\!\!\perp Y^2$?

Question 1.2. Let X be a real valued stochastic variable with $EX = 0$ and such that $Ee^X < \infty$. Is it true that

$$1 \leq Ee^X?$$

Question 1.3. Let X and Y be independent real valued stochastic variables both with an exponential distribution with scale parameter $\beta = 1$. Is it true that $X + Y$ is exponentially distributed with scale parameter 2?

Question 1.4. Let $B = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ and let $f : [0,\infty) \to [0,\infty)$ be measurable. Is it true that

$$\int_B f(x^2+y^2)\, \mathrm{d}m_2(x,y) = 2\pi \int_0^1 f(r) r\, \mathrm{d}r$$

Question 1.5. Is it true that

$$F(x) = \arctan(x) + \pi/2$$

is the distribution function for a probability measure on $\mathbb{R}$?

Problem 2

Let X and Y be two real valued stochastic variables whose joint distribution is the regular normal distribution on $(\mathbb{R}^2, \mathbb{B}_2)$ with mean 0 and variance matrix

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \rho \\ \rho & \sigma_2^2 \end{pmatrix}$$

where we assume that $\sigma_1, \sigma_2 > 0$ and $-\sigma_1\sigma_2 < \rho < \sigma_1\sigma_2$. It can be shown that Σ is then positive definite, and Σ can thus be a variance matrix for a regular normal distribution. This can be used without further comments.

Question 2.1. Find the joint distribution of $X+Y$ and $X-Y$.

Question 2.2. Determine all values of σ_1, σ_2 and ρ for which $X+Y \perp\!\!\!\perp X-Y$.

Question 2.3. Assume that $\sigma_1 = \sigma_2 = 0.6$ and $\rho = 0.14$. Find the distribution of

$$(X+Y)^2 + \frac{1}{0.44}(X-Y)^2.$$

Problem 3

Let $n \in \mathbb{N}$ and consider, for each n, independent real valued stochastic variables $Z_{n1}, \ldots, Z_{nn}$ such that

$$P(Z_{nk} = n) = 1 - P(Z_{nk} = 0) = \frac{1}{n}$$

for $k = 1, \ldots, n$. Thus Z_{nk} takes almost surely only the two values n and 0. Define

$$Y_n = \frac{1}{n}\sum_{k=1}^{n} Z_{nk}.$$

Question 3.1. Show that $EY_n = 1$ and $VY_n = \frac{n-1}{n}$.

Question 3.2. Decide if $\sqrt{n}(Y_n - 1) \xrightarrow{\mathcal{D}} \mathcal{N}(0,1)$.

Hint: Consider the probability $P(\sqrt{n}(Y_n - 1) \leq -1)$.

Problem 4

Define the two $\mathcal{M}^+$-functions $f, g : \mathbb{R} \mapsto [0, \infty)$ by $f(x) = e^x$ and $g(y) = e^{-y}$. Define the two measures $\mu = f \cdot m$ and $\nu = g \cdot m$.

Question 4.1. Show that μ and ν are σ-finite measures.

By σ-finiteness there is a unique product measure $\mu \otimes \nu$ on $(\mathbb{R}^2, \mathbb{B}_2)$.

Question 4.2. Show that $\mu \otimes \nu$ has density

$$(x, y) \mapsto e^{x-y}$$

w.r.t. m_2 and compute

$$\mu \otimes \nu(\{(x, y) \mid |x + y| < 1, |x - y| < 1\}).$$

Question 4.3. Let $h(x, y) = 1_A(x, y)xy$ with $A = (-\infty, 0) \times (0, \infty)$. Compute

$$\int h \,\mathrm{d}\mu \otimes \nu.$$

Problem 5

Let X, Y and Z be three independent real valued stochastic variables all with finite second moment and all with mean 0 and variance 1. Define

$$W = \frac{X + YZ}{\sqrt{1 + Z^2}}.$$

Question 5.1. Show that

$$V(W \mid Z) = 1 \quad \text{a.e.}$$

Question 5.2. Find the distribution of W under the additional assumption that X, Y and Z all have a marginal $\mathcal{N}(0, 1)$-distribution.

Hint: Compute $P(W \leq w)$ by Tonelli, integrating out over the the joint distribution of (X, Y) first.

Trial exam 2012/2013

Measures and Integrals

University of Copenhagen
Department of Mathematical Sciences

Formalities

This is a trial exam. It corresponds in size, form and difficulty to the real exam, which is a 27 hours take-home exam.

This trial exam consists of 4 problems with a total of 15 questions all counted with equal weight. Throughout it can be assumed that all stochastic variables are defined on the background probability space $(\Omega, \mathbb{F}, P)$.

Problem 1

The first problem consists of 5 independent questions. A complete answer must be supported by a small argument, a counter example or a reference to the book.

Question 1.1. If X and Y are independent real valued random variables with finite second moment is it then true that

$$V(X-Y) = V(X) - V(Y)?$$

It is **not** true that $V(X-Y) = V(X) - V(Y)$. If X and Y are independent and identically distributed with finite second moment and non-zero variance then by Lemma 19.13

$$V(X-Y) = V(X) + V(-Y) = V(X) + V(Y) = 2V(X) > 0 = V(X) - V(Y).$$

Question 1.2. Let $\mathbb{D}$ be a σ-algebra and let Y be a real valued stochastic variable with finite first moment and with $E(Y \mid \mathbb{D}) = 1$. If X is $\mathbb{D}$-measurable and XY has finite first moment is it then true that

$$E(XY \mid \mathbb{D}) = X?$$

It is true that $E(XY \mid \mathbb{D}) = X$. By Corollary 2.8, if X is $\mathbb{D}$-measurable then

$$E(XY \mid \mathbb{D}) = XE(Y \mid \mathbb{D}) = X.$$

Question 1.3. Let the joint distribution of X_1 and X_2 be a regular normal distribution on $\mathbb{R}^2$. If $\text{cov}(X_1, X_2) = 0$ is it then true that $X_1 \perp\!\!\!\perp X_2$?

It is true that $X_1 \perp\!\!\!\perp X_2$. If $\text{cov}(X_1, X_2) = 0$ the variance matrix in the regular normal distribution is diagonal (has zero off-diagonal elements) by Example 19.15. It follows by Theorem 18.27 that X_1 and X_2 are independent.

Question 1.4. Let μ and ν be two probability measures on $(\mathbb{R}^2, \mathbb{B}_2)$. Assume that

$$\mu((-\infty, x] \times (-\infty, y]) = \nu((-\infty, x] \times (-\infty, y])$$

for all $x, y \in \mathbb{R}$. Is it then true that $\mu = \nu$?

It is true that the measures are identical. This is a direct consequence of Theorem 19.2 as the two measures then have the same (bivariate) distribution function.

Question 1.5. Is the function

$$(x, y) \mapsto 1_{[0,\infty)\times[0,\infty)}(x, y)(e^{-x} - e^{-y})$$

integrable w.r.t. the Lebesgue measure on $(\mathbb{R}^2, \mathbb{B}_2)$?

The function is **not** integrable. Note that for $y > x + 1$

$$|e^{-x} - e^{-y}| \geq e^{-x} - e^{-x}e^{-1} = e^{-x}(1 - e^{-1}) > 0,$$

thus

$$\begin{aligned}
\int_0^\infty \int_0^\infty |e^{-x} - e^{-y}| \mathrm{d}y \mathrm{d}x &\geq \int_0^\infty \int_{x+1}^\infty e^{-x}(1 - e^{-1}) \mathrm{d}y \mathrm{d}x \\
&\geq \int_0^\infty \underbrace{e^{-x}(1 - e^{-1})}_{>0} \underbrace{m([x + 1, \infty))}_{=\infty} \mathrm{d}x = \infty.
\end{aligned}$$

and it follows by Tonelli's theorem that the function is not integrable.

Problem 2

Let ν be the probability measure on $(0, \infty)$ with density

$$f(x) = \frac{1}{2\sqrt{x}} e^{-\sqrt{x}}, \quad x > 0$$

w.r.t. the Lebesgue measure. Define $h(x) = \sqrt{x}$ for $x > 0$.

Question 2.1. Show that $h(\nu)$ is the exponential distribution.

We note that $\nu((0,\infty)^c) = 0$ and that $I = (0,\infty)$ is an open interval. From Lemma 4.11 h has a measurable extension to $\mathbb{R}$ since h is continuous on the open interval I. On I h is a bijective function with image $J = (0,\infty)$ and inverse $h^{-1}(y) = y^2$. The derivative of h^{-1} is $(h^{-1})'(y) = 2y$, which is clearly continuous on J. Theorem 12.6 can then be applied and the transformed measure $h(\nu)$ has density

$$1_J(y) f(h^{-1}(y))|(h^{-1})'(y)| = 1_{(0,\infty)}(y) \frac{1}{2\sqrt{y^2}} e^{-\sqrt{y^2}} 2y = 1_{(0,\infty)}(y) e^{-y}$$

w.r.t. m where we have used that for $y \geq 0$ it holds that $\sqrt{y^2} = y$. We note that this is the density for the exponential distribution w.r.t. m.

Question 2.2. Compute

$$\int \sqrt{x} \, \mathrm{d}\nu(x).$$

We see that

$$\int \sqrt{x} \, \mathrm{d}\nu(x) = \int h(x) \, \mathrm{d}\nu(x) = \int y \, \mathrm{d}h(\nu)(y)$$

by the abstract-change-of-variable formula, Theorem 10.8. Hence, by Theorem 11.7 the integral equals

$$\int_0^\infty y e^{-y} \mathrm{d}y = 1$$

as we recognize this as the first moment for the exponential distribution with scale parameter $\beta = 1$, see Example 16.25.

Question 2.3. Assume that $X_1, \ldots, X_n$ are independent and identically distributed each with marginal distribution ν. Show that

$$\frac{1}{n} \sum_{i=1}^n \sqrt{X_i} \xrightarrow{P} 1$$

for $n \to \infty$.

By Theorem 18.12 (technically the comment after) it holds that also $\sqrt{X_1}, \ldots, \sqrt{X_n}$ are independent and identically distributed, and the distribution of $\sqrt{X_i}$ is $h(\nu)$. We have shown above that the expectation of $\sqrt{X_i}$ equals 1, and we have, in fact, shown that the distribution of $\sqrt{X_i}$ is exponential and thus the stochastic variables have finite second moment. It then follows from the Law of Large numbers, Theorem 19.19, that

$$\frac{1}{n} \sum_{i=1}^n \sqrt{X_i} \xrightarrow{P} 1.$$

Question 2.4. Show that

$$P\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(\sqrt{X_i}-1)\leq 1\right)$$

is convergent for $n \to \infty$ and compute the limit.

As $\sqrt{X_i}$ was exponentially distributed (with scale parameter $\beta = 1$) it follows from Example 16.25, again, that the variance equals 1. It then follows from Laplace's CLT that

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(\sqrt{X_i}-1) \xrightarrow{\mathcal{D}} \mathcal{N}(0,1),$$

cf. Example 3.36 or slide 8, January 7. From slide 7, January 7, it follows specifically that

$$P\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(\sqrt{X_i}-1)\leq 1\right) \to \Phi(1) \simeq 0.8413.$$

The value of $\Phi(1)$ is computed using the function call `pnorm(1)` in R.

Problem 3

Let μ be the measure on $(\mathbb{R}^2, \mathbb{B}_2)$ with density

$$f(x_1,x_2) = 1_{[1,\infty)\times[1,\infty)}(x_1,x_2)c\,e^{-x_1x_2}$$

w.r.t. m_2 where $c > 0$ is an arbitrary constant. In other words, $\mu = f \cdot m_2$.

Question 3.1. Show that for $A \in \mathbb{B}$

$$\mu(A\times\mathbb{R}) = \int_{A\cap[1,\infty)} \frac{c}{x_1}e^{-x_1}\mathrm{d}x_1$$

and show that $\mu(\mathbb{R}^2) < \infty$.

By definition $\mu(A\times\mathbb{R}) = \int_{A\times\mathbb{R}} f\mathrm{d}m_2$ and since f is positive we can use Tonelli's theorem to find that

$$\mu(A\times\mathbb{R}) = \int\int 1_{A\cap[1,\infty)}(x_1)1_{[1,\infty)}(x_2)ce^{-x_1x_2}\mathrm{d}x_2\mathrm{d}x_1 = \int_{A\cap[1,\infty)}\int_1^{\infty} ce^{-x_1x_2}\mathrm{d}x_2\mathrm{d}x_1.$$

The inner integral for given $x_1 \geq 1$ is found as

$$\int_1^{\infty} ce^{-x_1x_2}\mathrm{d}x_2 = \left[-\frac{c}{x_1}e^{-x_1x_2}\right]_1^{\infty} = \frac{c}{x_1}e^{-x_1}$$

using standard integration techniques. Hence

$$\mu(A \times \mathbb{R}) = \int_{A \cap [1,\infty)} \frac{c}{x_1} e^{-x_1} \mathrm{d}x_1.$$

Taking $A = \mathbb{R}$ and noting that for $x_1 \geq 1$ $\frac{c}{x_1} e^{-x_1} \leq c e^{-x_1}$ we find that

$$\mu(\mathbb{R}^2) = \mu(\mathbb{R} \times \mathbb{R}) = \int_1^\infty \frac{c}{x_1} e^{-x_1} \mathrm{d}x_1 \leq \int_1^\infty c e^{-x_1} \mathrm{d}x_1 = c e^{-1} < \infty.$$

In the remaining questions it is assumed that $c > 0$ is chosen such that μ becomes a probability measure, which is possible because we have shown in general that $\mu(\mathbb{R}^2) < \infty$. The actual value, $c \simeq 4.56$, can be found by numerical integration but is not needed. Let, furthermore, $X = (X_1, X_2)$ denote a stochastic variable with values in $\mathbb{R}^2$ and with distribution μ. That is, $X(P) = \mu$.

Question 3.2. Show that the distribution of X_1 has density

$$g(z) = \frac{c}{z} e^{-z}, \quad z > 1$$

w.r.t. the Lebesgue measure. Argue that X_2 has the same distribution as X_1.

Observe that for $A \in \mathbb{B}$ we have $X_1(P)(A) = \mu(A \times \mathbb{R})$. Hence from the previous question

$$X_1(P)(A) = \int_{A \cap [1,\infty)} \frac{c}{x_1} e^{-x_1} \mathrm{d}x_1 = \int_A 1_{[1,\infty)}(x_1) \frac{c}{x_1} e^{-x_1} \mathrm{d}x_1 = \int_A g(z) \mathrm{d}z,$$

By definition, g is thus a density for $X_1(P)$ w.r.t. m. Interchanging the roles of x_1 and x_2 above will not change anything as $f(x_1, x_2) = f(x_2, x_1)$ and in particular $X_1(P) = X_2(P)$.

Question 3.3. Argue that X_1 has finite second moment and show that

$$EX_1 = ce^{-1} \quad \text{and} \quad EX_1^2 = 2ce^{-1}.$$

To verify that X_1 has finite second moment we use Example 16.8 and compute

$$E|X_1|^2 = \int_1^\infty |z|^2 \frac{c}{z} e^{-z} \mathrm{d}z = c \int_1^\infty z e^{-z} \mathrm{d}z$$

where we have used that for $z \in [1, \infty)$ we have $|z| = z$. By standard integration by parts we find that

$$\int_1^\infty z e^{-z} \mathrm{d}z = \left[-z e^{-z} - e^{-z}\right]_1^\infty = 2e^{-1} < \infty.$$

This shows that X_1 has finite second moment, and since X_1 is positive P-almost surely we have actually computed the second moment as

$$EX_1^2 = 2ce^{-1}.$$

Since X_1 has finite second moment it also has finite first moment, and using again Example 16.8 we find that

$$EX_1 = \int_1^\infty z\frac{c}{z}e^{-z}\mathrm{d}z = c\int_1^\infty e^{-z}\mathrm{d}z = ce^{-1}.$$

Question 3.4. Show that X_1X_2 has finite first moment and that

$$E(X_1X_2) = ce^{-1} + 1.$$

Using Example 16.6 with $t(x_1, x_2) = x_1x_2$ we find that

$$E|X_1X_2| = \int |x_1x_2|1_{[1,\infty)\times[1,\infty)}(x_1, x_2)ce^{-x_1x_2}\mathrm{d}m_2(x_1, x_2).$$

Since the integrand is positive, and since $|x_1x_2| = x_1x_2$ for $x_1 \in [1, \infty)$ and $x_2 \in [1, \infty)$ we can use Tonelli's theorem to find that

$$\int |x_1x_2|1_{[1,\infty)\times[1,\infty)}(x_1, x_2)ce^{-x_1x_2}\mathrm{d}m_2(x_1, x_2) = c\int_1^\infty\int_1^\infty x_1x_2e^{-x_1x_2}\mathrm{d}x_1\mathrm{d}x_2.$$

The inner integral is computed for $x_2 \geq 1$ using integration by parts as

$$\int_1^\infty x_1x_2e^{-x_1x_2}\mathrm{d}x_1 = \left[-x_1e^{-x_1x_2} - \frac{1}{x_2}e^{-x_1x_2}\right]_1^\infty = e^{-x_2} + \frac{1}{x_2}e^{-x_2}.$$

From this we find that

$$E|X_1X_2| = c\int_1^\infty e^{-x_2} + \frac{1}{x_2}e^{-x_2}\mathrm{d}x_2 = c\left[-e^{-x_2}\right]_1^\infty + 1 = ce^{-1} + 1 < \infty.$$

Hence X_1X_2 has finite first moment. Again, since X_1 and X_2 and hence X_1X_2 is positive P-almost surely, we have in fact computed the first moment of X_1X_2 as

$$E(X_1X_2) = ce^{-1} + 1.$$

For the final question you can without further arguments use that since X_2 has the same distribution as X_1 it holds that also $EX_2 = ce^{-1}$ and $EX_2^2 = 2ce^{-1}$.

Question 3.5. Show that $X_1 + X_2$ has finite second moment and compute $V(X_1 + X_2)$.

We have $(X_1+X_2)^2 = X_1^2+X_2^2+2X_1X_2$ and all three terms have finite first moment (finite expectation), hence by Theorem 16.3 the sum has finite first moment and

$$E(X_1+X_2)^2 = EX_1^2 + EX_2^2 + 2E(X_1X_2) = 6ce^{-1} + 2.$$

Then $V(X_1+X_2) = E(X_1+X_2)^2 - (EX_1+EX_2)^2$, hence

$$V(X_1+X_2) = 6ce^{-1} + 2 - 4c^2e^{-2} = 2ce^{-1}(3-2ce^{-1}) + 2 \ (\simeq 0.8136).$$

Problem 4

Let X and Y be two independent stochastic variables both with the uniform distribution on $(-1/2,,1/2)$. That is, their common distribution has density $f = 1_{(-1/2,1/2)}$ w.r.t. the Lebesgue measure on $(\mathbb{R},\mathbb{B})$.

Question 4.1. Compute the density for the joint distribution of $(X+Y, X-Y)$. Are $X+Y$ and $X-Y$ independent?

Note that since X and Y are independent their joint distribution has density $h(x,y) = f(x)f(y) = 1_{(-1/2,1/2)}(x)1_{(-1/2,1/2)}(y)$ w.r.t. $m_2 = m\otimes m$ by Lemma 11.5. Let

$$A = \left\{ \begin{matrix} 1 & 1 \\ 1 & -1 \end{matrix} \right\}$$

and consider the linear transformation $s(x) = Ax$, $s:\mathbb{R}^2\to\mathbb{R}^2$. Then $(X+Y, X-Y)^T = s(X,Y)$ and, noting that A is invertible with $A^{-1} = \frac{1}{2}A$, then $|\det A^{-1}| = \frac{1}{2}$ and either by referring to Theorem 12.13 it follows that

$$s(h\cdot m_2) = \frac{1}{2}h\circ s^{-1}\cdot m_2.$$

Thus the joint distribution of $(X+Y, X-Y)$ has density

$$\frac{1}{2}h\circ s^{-1}(z,w) = \frac{1}{2}1_{(-1,1)}(z+w)1_{(-1,1)}(z-w) = \frac{1}{2}1_{\{(z,w)||z|+|w|<1\}}(z,w).$$

The joint distribution of $(X+Y, X-Y)$ is precisely the probability measure considered in Exercise 12.6, and from that exercise it follows that this measure is not a product measure, hence $X+Y$ and $X-Y$ are **not** independent. One can either use the exercise as stated or repeat the argument (the solution of this exercise will come as a model solution). Note that $\mathrm{cov}(X+Y, X-Y) = V(X) - V(Y) = 0$, hence covariance computations cannot be used to show that the two variables are dependent, nor does it follow from the fact that the covariance is 0 that the variables are independent.

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Formalia

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Eksamen er en 4 timers skriftlig eksamen med hjælpemidler. Dvs. bøger, kompendier og andet undervisningsmateriale kan benyttes. Det er ligeledes tilladt at benytte lommeregner eller computer. Elektroniske hjælpemidler må **ikke** på nogen måde bruges til kommunikation med andre, og det er ligeledes **ikke** tilladt at etablere forbindelse til internettet eller andre netværk under eksamen. Det er tilladt at skrive med blyant.

Opgave 1

Denne opgave består af 4 uafhængige spørgsmål. Hvert spørgsmål besvares med et kort argument, et modeksempel eller en reference til undervisningsmaterialet. Alle stokastiske variable er defineret på målrummet $(\Omega, \mathbb{F}, P)$.

Spørgsmål 1.1. Lad $X \sim \mathcal{N}(0,1)$. Gør rede for at fordelingen af $\exp(X)$ har en tæthed m.h.t. lebesguemålet på $(\mathbb{R}, \mathbb{B})$. Det er ikke nødvendigt at finde tætheden.

Funktionen $h(x) = \exp(x)$ afbilder det åbne interval $\mathbb{R}$ bijektivt på det åbne interval $(0, \infty)$ med C^1-invers $h^{-1}(y) = \log(y)$. Per definition er $P(X \in \mathbb{R}) = 1$, og fordelingen af X har tæthed m.h.t. m, så $\exp(X)$ har tæthed m.h.t. lebesguemålet iflg. sætning 15.1

Spørgsmål 1.2. Antag at

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right).$$

Afgør hvorvidt $X_1 \perp\!\!\!\perp X_2$.

Da variansmatricen i den 2-dimensionale normalfordeling er diagonal følger det af sætning 18.27 at $X_1 \perp\!\!\!\perp X_2$.

Spørgsmål 1.3. Lad μ og ν være to σ-endelige mål på $(\mathbb{R}, \mathbb{B})$. Hvis $\mu \otimes \nu$ er et sandsynlighedsmål på $(\mathbb{R}^2, \mathbb{B}_2)$, er μ og ν så også sandsynlighedsmål?

Nej, ikke nødvendigvis. Tag f.eks. $\mu = 2\delta_0$ og $\nu = \frac{1}{2}\delta_0$, så er $\mu \otimes \nu = \delta_0 \otimes \delta_0 = \delta_{(0,0)}$ et sandsynlighedsmål, men $\mu(\mathbb{R}) = 2$ og $\nu(\mathbb{R}) = \frac{1}{2}$.

Spørgsmål 1.4. Antag at $X_1, \ldots, X_n$ er uafhængige og identisk fordelte med

$$P(X_i = 2) = P(X_i = -2) = \frac{1}{2}.$$

Afgør hvorvidt

$$P\left(\sum_{i=1}^{n} X_i \leq 2x\sqrt{n}\right) \to \Phi(x)$$

for $n \to \infty$.

Da X_i er begrænset (n.o.), har den alle momenter, $\xi = EX_i = -2/2 + 2/2 = 0$ og

$$\sigma^2 = VX_i = EX_i^2 = 2^2\frac{1}{2} + (-2)^2\frac{1}{2} = 4.$$

Dvs. $\sigma = \sqrt{4} = 2$. Det følger nu af Laplace's CLT, da X_i'erne er antaget uafhængige og identisk fordelte, at

$$P\left(\sum_{i=1}^{n} X_i \leq 2x\sqrt{n}\right) = P\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} \frac{X_i}{2} \leq x\right) \to \Phi(x).$$

Opgave 2

Antag at X og Y er uafhængige reelle stokastiske variable, at X er ligefordelt på $(-1, 1)$, og at Y er Γ-fordelt med formparameter $\lambda = 3$.

Spørgsmål 2.1. Vis at

$$E\left(\frac{1}{Y}\right) = \frac{1}{2}.$$

Vi konstaterer først at Y^{-1} er positiv, at tætheden for Γ-fordelingen med formparameter $\lambda = 3$ er

$$\frac{1}{\Gamma(3)} y^{3-1} e^{-y} = \frac{1}{2} y^2 e^{-y},$$

og at

$$E\left(\frac{1}{Y}\right) = \frac{1}{2}\int_0^\infty \frac{1}{y} y^2 e^{-y} \mathrm{d}y = \frac{1}{2}\int_0^\infty y e^{-y} \mathrm{d}y = \frac{1}{2}\Gamma(1) = \frac{1}{2}.$$

Det viser dels at Y^{-1} har første moment, da Y^{-1} er positiv, og dels at $EY^{-1} = \frac{1}{2}$.

Spørgsmål 2.2. Udregn

$$E\left(\frac{X}{Y}\,\middle|\,X\right) \quad \text{og vis at} \quad E\left(\frac{X}{Y}\right) = 0.$$

Observer først at

$$\left|\frac{X}{Y}\right| \leq \frac{1}{Y},$$

hvilket viser, at $\frac{X}{Y}$ har middelværdi. Da X er målelig m.h.t. $\sigma(X)$ følger det af korollar 2.8 samt opgave 2.6. i notatet om betingede middelværdier at

$$E\left(\frac{X}{Y}\,\middle|\,X\right) = XE\left(\frac{1}{Y}\,\middle|\,X\right) = XE\left(\frac{1}{Y}\right) = \frac{X}{2}.$$

Det følger nu at

$$E\left(\frac{X}{Y}\right) = E\left(E\left(\frac{X}{Y}\,\middle|\,X\right)\right) = \frac{EX}{2} = 0$$

idet

$$EX = \frac{1}{2}\int_{-1}^{1} x \mathrm{d}x = 0.$$

Alternativt kan vi bruge sætning 18.12 til at slutte at X og Y^{-1} er uafhængige, så

$$E\left(\frac{X}{Y}\right) = EXE\left(\frac{1}{Y}\right) = \frac{EX}{2} = 0.$$

Opgave 3

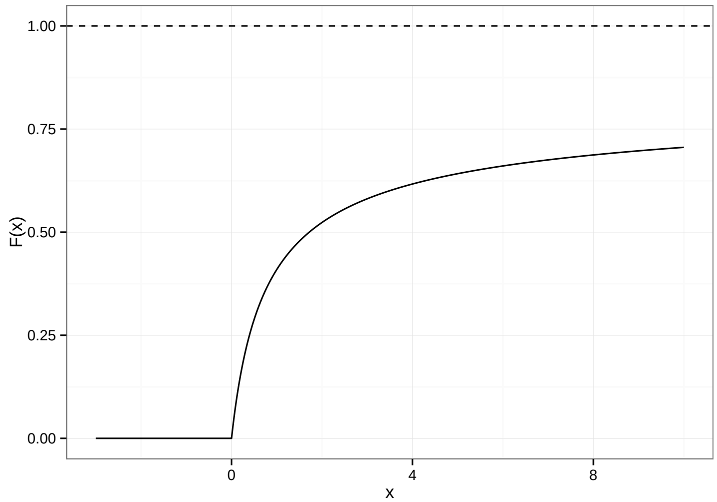
Definer funktionen $F : \mathbb{R} \to [0, \infty)$ ved

$$F(x) = 1 - \frac{1}{1 + \log(x + 1)}$$

for $x \geq 0$ og $F(x) = 0$ for $x < 0$. Det kan uden bevis benyttes at F er en fordelingsfunktion for et sandsynlighedsmål på $(\mathbb{R}, \mathbb{B})$. Lad ν være sandsynlighedsmålet med fordelingsfunktion F. Dvs.

$$F(x) = \nu((-\infty, x]).$$

Spørgsmål 3.1. Tegn en skitse af grafen for fordelingsfunktionen F. Gør rede for at ν har tæthed m.h.t. lebesguemålet m og find tætheden.



Bemærk at $F(0) = 0$, så $\nu((0,\infty)) = 1$, og F er kontinuert differentiabel på $(0,\infty)$ med

$$f(x) = F'(x) = \frac{1}{(x+1)(1+\log(x+1))^2}$$

for $x > 0$. Dvs.

$$F(x) - F(y) = \int_y^x f(z)\mathrm{d}z$$

for alle $x, y > 0$. Definer $f(x) = 0$ for $x \leq 0$. Lad nu $y \searrow 0$, så følger det af monoton konvergens (da f er positiv), samt højrekontinuitet af F, at

$$F(x) = \lim_{y \searrow 0} F(x) - F(y) = \lim_{y \searrow 0} \int_y^x f(z)\mathrm{d}z = \int_0^x f(z)\mathrm{d}z = \int_{-\infty}^x f(z)\mathrm{d}z$$

for $x > 0$. Eftersom $F(x) = 0 = \int_{-\infty}^x f(z)\mathrm{d}z$ for $x \leq 0$ konkluderer vi, at ν har tæthed

$$f(x) = \begin{cases} \frac{1}{(x+1)(1+\log(x+1))^2} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$$

m.h.t. lebesguemålet.

Spørgsmål 3.2. Lad $h : \mathbb{R} \to \mathbb{R}$ være givet ved

$$h(x) = \begin{cases} \log(x+1) & \text{for } x > 0 \\ 0 & \text{for } x \leq 0. \end{cases}$$

Vis at $h(\nu)$ har tæthed m.h.t. lebesguemålet m og find tætheden.

I spørgsmål 3.1 konstaterede vi at $\nu((0,\infty)^c) = 0$. Endvidere er h bijektiv fra $I = (0,\infty)$ på $J = (0,\infty)$ med C^1-invers

$$h^{-1}(y) = \exp(y) - 1.$$

Observer at $(h^{-1})'(y) = \exp(y)$. Da $\nu = f \cdot m$ følger det af sætning 12.6 at $h(\nu)$ har tæthed m.h.t. lebesguemålet givet ved

$$\tilde{f}(y) = \frac{1}{(1+e^y-1)(1+\log(e^y-1+1))^2}|e^y| = \frac{1}{(1+y)^2}$$

for $y > 0$ (og 0 for $y \leq 0$).

Alternativt kan vi først finde fordelingsfunktionen for $h(\nu)$.

$$h(\nu)((-\infty, y]) = \nu(h^{-1}((-\infty, y])) = F(e^y - 1) = 1 - \frac{1}{1+y}.$$

Ved differentiation som i spørgsmål 3.1 finder vi tætheden

$$\tilde{f}(y) = \frac{1}{(1+y)^2}$$

for $y > 0$.

Opgave 4

I denne opgave betegner

$$B = \{(x,y) \in \mathbb{R}^2 \mid \sqrt{x^2+y^2} \leq 1,\ y \geq 0\}$$

den øvre halvdel af den afsluttede enhedsdisk i planen, og $g : [0,\infty) \to [0,\infty]$ antages at være en målelig funktion. Funktionen $f : \mathbb{R}^2 \to [0,\infty]$ defineret ved

$$f(x,y) = g(\sqrt{x^2+y^2})$$

er ligeledes målelig. Bemærk at

$$1_B(x,y) = 1_{[0,\infty)}(y)1_{[0,1]}(\sqrt{x^2+y^2}).$$

Spørgsmål 4.1. Vis at

$$\int_B f \mathrm{d}m_2 = \pi \int_0^1 g(r) r \mathrm{d}r.$$

Eftersom f er en $\mathcal{M}^+$-funktion giver eksempel 12.17 (polær integration) at

$$\begin{aligned}
\int_B f \mathrm{d}m_2 &= \int 1_{[0,\infty)}(y)1_{[0,1]}(\sqrt{x^2+y^2})g(\sqrt{x^2+y^2})\mathrm{d}m_2(x,y) \\
&= \int_0^{2\pi}\int_0^{\infty} 1_{[0,\infty)}(r\sin\theta)1_{[0,1]}(r)g(r)r\mathrm{d}r\mathrm{d}\theta \\
&= \int_0^{2\pi} 1_{(0,\pi)}(\theta)\mathrm{d}\theta \int_0^1 g(r)r\mathrm{d}r \\
&= \pi \int_0^1 g(r)r\mathrm{d}r.
\end{aligned}$$

Vi har undervejs udnyttet at $(r\cos\theta)^2 + (r\sin\theta)^2 = r^2$ samt at for $\theta \in (0, 2\pi)$ er $r\sin\theta \geq 0$ hvis og kun hvis $\theta \in (0, \pi)$.

Spørgsmål 4.2. Udregn

$$\int_B \frac{1}{\sqrt{x^2+y^2}} \mathrm{d}m_2(x,y).$$

Med $g(r) = r^{-1}$, som er en målelig funktion $g : [0,\infty) \to [0,\infty]$, ser vi, at

$$\frac{1}{\sqrt{x^2+y^2}} = g(\sqrt{x^2+y^2}).$$

Det følger af spørgsmål 4.1 at

$$\int_B \frac{1}{\sqrt{x^2+y^2}} \mathrm{d}m_2(x,y) = \pi \int_0^1 \frac{1}{r} r \mathrm{d}r = \pi.$$

Spørgsmål 4.3. Vis at

$$\int_B f \mathrm{d}m_2 = \int_{-1}^1 \int_0^{\sqrt{1-x^2}} g(\sqrt{x^2+y^2}) \mathrm{d}y \mathrm{d}x.$$

Observer at $y \geq 0$ og $\sqrt{x^2+y^2} \leq 1$ hvis og kun hvis $-1 \leq x \leq 1$ og $0 \leq y \leq \sqrt{1-x^2}$. Dvs.

$$1_B(x,y) = 1_{[-1,1]}(x) 1_{[0,\sqrt{1-x^2}]}(y).$$

Da f er en $\mathcal{M}^+$-funktion og $m_2 = m \otimes m$ giver Tonellis sætning at

$$\begin{aligned} \int_B f \mathrm{d}m_2 &= \int \int 1_{[-1,1]}(x) 1_{[0,\sqrt{1-x^2}]}(y) g(\sqrt{x^2+y^2}) \mathrm{d}y \mathrm{d}x \\ &= \int_{-1}^1 \int_0^{\sqrt{1-x^2}} g(\sqrt{x^2+y^2}) \mathrm{d}y \mathrm{d}x. \end{aligned}$$

Det oplyses nu, at funktionen $H(y) = \log(y + \sqrt{a+y^2})$ for $a > 0$ opfylder

$$H'(y) = \frac{1}{\sqrt{a+y^2}}.$$

Spørgsmål 4.4. Vis at

$$\int_{-1}^1 \log \frac{\sqrt{1-x^2}+1}{|x|} \mathrm{d}x = \pi.$$

Her er konventionen for $x = 0$ at $2/0 = \infty$ og $\log(\infty) = \infty$.

Med $g(r) = r^{-1}$ som i spørgsmål 4.2 konstaterer vi, at for $x \neq 0$ er

$$\begin{aligned}
\int_0^{\sqrt{1-x^2}} \frac{1}{\sqrt{x^2+y^2}} \mathrm{d}y &= \log(\sqrt{1-x^2} + \sqrt{x^2+1-x^2}) - \log(0 + \sqrt{x^2+0^2}) \\
&= \log \frac{\sqrt{1-x^2}+1}{|x|}.
\end{aligned}$$

For $x = 0$ er

$$\int_0^1 \frac{1}{\sqrt{0^2+y^2}} \mathrm{d}y = \int_0^1 \frac{1}{y} \mathrm{d}y = \infty = \log \frac{\sqrt{1-0^2}+1}{|0|},$$

i henhold til konventionen i opgaven. Ved at kombinere spørgsmål 4.2 og 4.3 følger det, at

$$\pi = \int_B \frac{1}{\sqrt{x^2+y^2}} \mathrm{d}m_2(x,y) = \int_{-1}^1 \int_0^{\sqrt{1-x^2}} \frac{1}{\sqrt{x^2+y^2}} \mathrm{d}y \mathrm{d}x = \int_{-1}^1 \log \frac{\sqrt{1-x^2}+1}{|x|} \mathrm{d}x.$$

Bemærk at da etpunktsmængden $\{0\}$ er en m-nulmængde, er det ikke strengt nødvendigt at beregne det indre integral for $x = 0$ endsige at tillægge $\log \frac{\sqrt{1-x^2}+1}{|x|}$ en værdi i 0. Opgaven kan således besvares fuldt ud ved at se bort fra tilfældet $x = 0$ med argumentet at $\{0\}$ er en m-mulmængde.

Bemærk også, at det er muligt at beregne integralet ved at finde stamfunktioner. Man skal så være omhyggelig med at håndtere singulariteten i 0 korrekt, og for de fleste er et program som Maple nok nødvendigt til at "gætte" en stamfunktion. Det er relativt imponerende, at Maple såvel som WolframAlpha uden videre beregner integralet korrekt til at være π – altså symbolsk π, og ikke en numerisk approksimation!

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Denne opgave består af 4 uafhængige spørgsmål. Hvert spørgsmål besvares med et kort argument, et modeksempel eller en reference til undervisningsmaterialet. Alle stokastiske variable er defineret på målrummet $(\Omega, \mathbb{F}, P)$.

Spørgsmål 1.1. Lad μ være et mål på $(\mathbb{R}, \mathbb{B})$ som opfylder

$$\mu((-\infty, x]) = \begin{cases} x^2 & \text{for } x \geq 0 \\ 0 & \text{for } x < 0. \end{cases} \tag{1.1}$$

Afgør hvorvidt (1.1) entydigt bestemmer μ.

Mængderne $(-\infty, x]$ for $x \in \mathbb{R}$ udgør et $\cap$-stabilt frembringersystem for $\mathbb{B}$. Endvidere er $\mathbb{R} = \cup_{n \in \mathbb{N}}(-\infty, n]$ og $\mu((-\infty, n]) = n^2 < \infty$. Det følger derfor af sætning 3.9 at (1.1) entydigt bestemmer målet μ.

Spørgsmål 1.2. Lad X og Y være uafhængige reelle stokastiske variable. Antag $E|Y| < \infty$ og $E|X| < \infty$ samt at $\mathbb{D} \subseteq \sigma(Y) \subseteq \mathbb{F}$ for en σ-algebra $\mathbb{D}$. Vis at

$$E(XY \mid \mathbb{D}) = EXE(Y \mid \mathbb{D}).$$

Det følger af uafhængigheden at $E|XY| < \infty$. Fra sætning 2.6 (tårnegenskaben) fås at

$$E(XY \mid \mathbb{D}) = E(E(XY \mid \sigma(Y)) \mid \mathbb{D}) = E(\underbrace{E(X \mid \sigma(Y))}_{=EX} Y \mid \mathbb{D}) = EXE(Y \mid \mathbb{D}) \quad \text{n.s.}$$

hvor vi udnytter at Y er $\sigma(Y)$-målelig (tredje lighed) og at $X \perp\!\!\!\perp Y$ (fjerde lighed).

Spørgsmål 1.3. Lad X og Y være reelle stokastiske variable således at fordelingen af (X, Y) har tæthed

$$f(x,y) = \begin{cases} 2 & \text{for } 0 \le y \le x \le 1 \\ 0 & \text{ellers} \end{cases}$$

m.h.t. lebesguemålet m_2. Afgør hvorvidt $X \perp\!\!\!\perp Y$.

Fordelingen af (X, Y) er ikke et produkt af de marginale fordelinger, derfor er X og Y ikke uafhængige. Konkret kan vi se på $B = (0, 0.5) \times (0.5, 1)$. Tætheden f er 0 på denne kasse, så $P((X,Y) \in B) = 0$. Derimod ses det ved direkte udregning at $P(X \in (0, 0.5)) = P(Y \in (0.5, 1)) = 0.25 > 0$, så hvis X og Y havde været uafhængige, så havde $P((X,Y) \in B) = 0.25^2 > 0$.

Spørgsmål 1.4. Afgør hvorvidt funktionen $f : \mathbb{R}^2 \to \mathbb{R}^2$ givet ved

$$f(x,y) = e^{-x^4 - y^4 - 2x^2y^2}$$

er integrabel m.h.t. lebesguemålet m_2 på $(\mathbb{R}^2, \mathbb{B}_2)$.
Vink: Sammenlign med tætheden for en normalfordeling.

Observer at

$$x^4 + y^4 + 2x^2y^2 = (x^2+y^2)^2 \ge x^2 + y^2 - 1.$$

Heref følger at

$$\int f \,\mathrm{d}m_2 \le e \int e^{-x^2-y^2} \,\mathrm{d}m_2(x,y) = e \left(\int e^{-x^2} \,\mathrm{d}x \right)^2 = e\pi < \infty$$

idet vi genkender integralet som normeringskonstanten $\sqrt{\pi}$ i normalfordelingen med middelværdi 0 og varians 0.5. Vi har undervejs brugt at f er positiv, samt Tonellis sætning i den første lighed, jf. også eksempel 12.18.

Opgave 2

Antag at $X_1, X_2, \ldots$ er uafhængige identisk fordelte stokastiske variable med

$$P(X_i = 1) = P(X_i = -1) = \frac{1}{2}.$$

Spørgsmål 2.1. Vis at

$$P\left(\left|\sum_{i=1}^{n} X_i\right| \leq 2\sqrt{n}\right) \to 1 - 2\Phi(-2) \simeq 0.9545$$

for $n \to \infty$.

Vi skal bruge CLT. Vi konstaterer først at $EX_i = 1/2 - 1/2 = 0$, og at

$$VX_i = \frac{1^2}{2} + \frac{(-1)^2}{2} = 1.$$

Vi har derfor at

$$P\left(\left|\sum_{i=1}^{n} X_i\right| \leq x\sqrt{n}\right) = P\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i - 0\right| \leq x\frac{1}{\sqrt{n}}\right) \to 1 - 2\Phi(-x)$$

ved CLT – mere præcist, bemærkningerne på side 2 og 3 i notatet. Resultatet følger med $x = 2$ idet $\Phi(-2) \simeq 0.022750$.

Spørgsmål 2.2. Vis at

$$P\left(\left|\sum_{i=1}^{n} X_i\right| \leq 2\sqrt{n}\right) \geq \frac{3}{4}.$$

Den stokastiske variabel

$$Y = \sum_{i=1}^{n} X_i$$

har middelværdi 0, jf. udregningerne i opgave 2.1. Den har envidere endeligt 2. moment og $VY = nVX_1 = n$, da X_i'erne er uafhængige med varians 1. Det følger derfor at Chebychevs ulighed, sætning 16.19, med $\varepsilon = 2\sqrt{n}$ at

$$P(|Y - 0| > 2\sqrt{n}) \leq \frac{n}{(2\sqrt{n})^2} = \frac{1}{4}.$$

Heraf følger, at

$$P\left(\left|\sum_{i=1}^{n} X_i\right| \leq 2\sqrt{n}\right) = 1 - P(|Y| > 2\sqrt{n}) \geq 1 - \frac{1}{4} = \frac{3}{4}.$$

Opgave 3

Definer funktionen $f : \mathbb{R} \to \mathbb{R}$ ved

$$f(x) = \begin{cases} \frac{1}{\pi\sqrt{1-x^2}} & \text{for } x \in (-1,1) \\ 0 & \text{ellers.} \end{cases}$$

Lad $\nu = f \cdot m$ være målet med tæthed f m.h.t. lebesguemålet m på $(\mathbb{R}, \mathbb{B})$.

Spørgsmål 3.1. Find tætheden for det transformerede mål $\arccos(\nu)$. Gør rede for at ν er et sandsynlighedsmål.
Bemærk: Funktionen arccos, der betegner den inverse til cos, er som udgangspunkt kun defineret på $[-1, 1]$.

Vi bemærker først at $\nu((-1,1)^c) = 0$, og at $h = \arccos$ afbilder $I = (-1,1)$ bijektivt på $J = (0, \pi)$ med C^1-invers $h^{-1} = \cos : (0,\pi) \to (-1,1)$. Endvidere er $|(h^{-1})'(y)| = |\sin(y)|$. Det følger nu af sætning 12.6 at $\mu := \arccos(\nu) = \tilde{f} \cdot m$ hvor tætheden $\tilde{f}$ er givet ved

$$\tilde{f}(y) = f(\cos(y))|\sin(y)| = \frac{1}{\pi}\frac{|\sin(y)|}{\underbrace{\sqrt{1-\cos(y)^2}}_{=|\sin(y)|}} = \frac{1}{\pi}$$

for $y \in (0, \pi)$. Det er tætheden for ligefordelingen på $(0, \pi)$, og altså et sandsynlighedsmål. Derfor er $\nu = \cos(\mu)$ også et sandsynlighedsmål.

Spørgsmål 3.2. Vis at ν har k'te moment for alle $k \in \mathbb{N}$.

Observer at $|x|^k f(x) \leq f(x)$ for alle $x \in \mathbb{R}$ idet $|x|^k \leq 1$ for $x \in (-1,1)$ og $f(x) = 0$ for $|x| > 1$. Det følger nu af sætning 11.7 at

$$\int |x|^k \,\mathrm{d}\nu(x) = \int |x|^k f(x)\,\mathrm{d}x \leq \int f(x)\,\mathrm{d}x = 1 < \infty$$

for alle $k \in \mathbb{N}$. Det viser, jf. definition 16.7, at ν har k'te moment for alle $k \in \mathbb{N}$.

Spørgsmål 3.3. Lad X være en stokastisk variabel med fordeling ν. Vis at

$$V(X) = \frac{1}{2}.$$

Iflg. opgave 3.2 har ν alle momenter, specielt har X endelig middelværdi og varians. Funktionen $x \mapsto xf(x)$ ses at være ulige, så det følger af opgave 10.12(c) at

$$EX = \int xf(x)\,\mathrm{d}x = 0.$$

Vi har så

$$VX = \int x^2 f(x)\,\mathrm{d}x = \frac{1}{\pi}\int_{-1}^{1} x^2(1-x^2)^{-1/2}\,\mathrm{d}x.$$

Integranden er lige, så ved opgave 10.12(a) samt substitutionen $z = x^2$ fås

$$VX = \frac{1}{\pi}\int_0^1 z^{1/2}(1-z)^{-1/2}\,\mathrm{d}z = \frac{B(3/2,1/2)}{\pi} = \frac{1}{2}$$

hvor

$$B(3/2,1/2) = \Gamma(3/2)\Gamma(1/2) = \pi/2$$

er B-funktionen i $(3/2, 1/2)$, jf. eksempel 12.16.

Alternativt (i såvel opgave 3.2 som 3.3) kan man bruge at $X = \cos(Y)$ hvor Y er ligefordelt på $(0,\pi)$ iflg. opgave 2.1. Heraf følger f.eks. ved stamfunktionsbestemmelse at

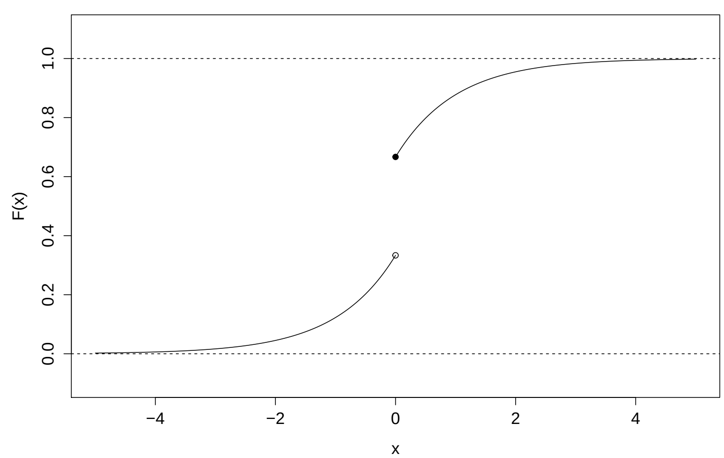
$$VX = \frac{1}{\pi}\int_0^\pi \cos(y)^2\,\mathrm{d}y = \frac{1}{\pi}\left[\frac{1}{2}(y+\sin(y)\cos(y))\right]_0^\pi = \frac{1}{2}.$$

Opgave 4

Lad funktionen $F : \mathbb{R} \to \mathbb{R}$ være givet ved

$$F(x) = \begin{cases} \frac{1}{3}e^x & \text{for } x < 0 \\ 1 - \frac{1}{3}e^{-x} & \text{for } x \geq 0. \end{cases}$$

Spørgsmål 4.1. Tegn en skitse af grafen for F. Vis at F er en fordelingsfunktion for et sandsynlighedsmål.



Vi skal eftervise de fire betingelser i sætning 17.3. Vi starter med at konstatere at $e^x \to 0$ for $x \to -\infty$, så betingelse 3) er opfyldt. Ligeledes vil $e^{-x} \to 0$ for $x \to \infty$, så $F(x) = 1 - e^{-x}/3 \to 1$, og 4) er også opfyldt. På hvert af de åbne intervaller $(-\infty, 0)$ og $(0, \infty)$ er F kontinuert, og F er også højrekontinuert i 0, så 3) følger. Endvidere er $F(0) = 2/3 > F(0-0) = 1/3$ og F er kontinuert differentiabel på $(-\infty, 0)$ såvel som $(0, \infty)$ med $F'(x) = e^{-|x|}/3 > 0$. Heraf følger det at F er voksende, så 1) gælder. Sætning 17.4 sikrer at F er fordelingsfunktion for et sandsynlighedsmål.

Lad ν betegne sandsynlighedsmålet på $(\mathbb{R}, \mathbb{B})$ med fordelingsfunktion F.

Spørgsmål 4.2. Beregn $\nu((-\infty, 0))$ og $\nu(\{0\})$. Afgør hvorvidt ν har tæthed m.h.t. lebesguemålet på $\mathbb{R}$.

Det følger af formlerne side 397 at

$$\nu((-\infty, 0)) = F(0-0) = \frac{1}{3}.$$

Heraf følger nu at

$$\nu(\{0\}) = \nu((-\infty, 0] \setminus (-\infty, 0)) = F(0) - F(0-0) = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}.$$

Hvis ν havde tæthed m.h.t. lebesguemålet ville alle etpunktsmængder havde ν-mål 0. Ovenstående viser, at $\{0\}$ har positivt ν-mål. Vi konkluderer, at ν ikke har tæthed m.h.t. lebesguemålet.

Spørgsmål 4.3. Gør rede for at ν har første moment og beregn

$$\int x \, \mathrm{d}\nu(x).$$

Observer først at for $n \in \mathbb{N}$ er

$$\nu([-n, -n+1)) = \nu((n-1, n]) = \frac{1}{3} e^{-n}(e-1)$$

Eftersom $|x| \leq n$ på $[-n, -n+1) \cup (n-1, n]$ har vi følgende vurdering

$$\int |x| \, \mathrm{d}\nu(x) \leq \frac{2}{3}(e-1) \sum_{n=1}^{\infty} n e^{-n} < \infty.$$

Det følger direkte af definitionen af F at

$$F(-x) + F(x-0) = 1$$

for alle $x \in \mathbb{R}$, dvs. fordelingen er symmetrisk omkring 0, jf. eksempel 17.24, formel (17.18). Dvs. med $h(x) = -x$ opfylder ν at $h(\nu) = \nu$. Integraltransformationssætningen giver nu at

$$\int x \, \mathrm{d}\nu(x) = \int x \, \mathrm{d}h(\nu)(x) = \int h(x) \, \mathrm{d}\nu(x) = -\int x \, \mathrm{d}\nu(x),$$

og vi slutter at integralet er 0.

Alternativt giver analysen i spørgsmål 4.1 at

$$\nu = \frac{1}{3}e^{-|x|} \cdot m + \frac{1}{3}\delta_0.$$

Heraf fås at

$$\int |x| \, \mathrm{d}\nu(x) = \frac{1}{3}\int xe^{-|x|} \, \mathrm{d}x + \frac{1}{3}\underbrace{\int x\mathrm{d}\delta_0(x)}_{=0} = \frac{2}{3}\underbrace{\int_0^\infty xe^{-x} \, \mathrm{d}x}_{=1} = \frac{2}{3} < \infty.$$

Eftersom $x \mapsto xe^{-|x|}$ er ulige er integralet

$$\int x \, \mathrm{d}\nu(x) = \frac{1}{3}\int xe^{-|x|} \, \mathrm{d}\nu(x) = 0$$

iflg. opgave 10.12(c).