Exam 2012/2013

Measures and Integrals

University of Copenhagen
Department of Mathematical Sciences

Formalities

This is the final, individual, 27 hours take-home exam in the course *Measures and Integrals*, 2012/2013. It is made available electronically via Absalon at 9.00am, January 15, 2013 and the deadline for handing in the solution is 12.00am, January 16, 2013. Later handins are not accepted. The solution must be handed in at the secretarial office at the Department of Mathematical Sciences (office 04.1.03, E-building, HCØ). A printed copy of the exam and the official front page for the handin can be obtained from the secretarial office from 9.00am, January 15, 2013.

The solution can be written using pen, pencil or computer. The final handin must have numbered pages and it must be equipped with the official front page that you find electronically on Absalon or obtain from the secretarial office.

This is an individual exam. During the 27-hours exam you are not allowed to communicate in any way with anybody about anything related to the the exam.

The exam consists of 4 pages with 4 problems and a total of 15 questions all counted with equal weight. Throughout it can be assumed that all stochastic variables are defined on the background probability space $(\Omega, \mathbb{F}, P)$.

Problem 1

The first problem consists of 5 independent questions. A complete answer must be supported by a small argument, a counter example or a reference to the book.

Question 1.1. Let X be a real valued random variable. If $E|X|^2 < \infty$ does it then hold that X has finite first moment?

Question 1.2. If X, Y and Z are real valued stochastic variables and $X \perp\!\!\!\perp Y \perp\!\!\!\perp Z$ is it then true that $X \perp\!\!\!\perp Z$?

Question 1.3. Let X and Y be real valued stochastic variables with finite first moment and with $E(Y \mid X) = X$ a.e. Is it true that

$$E(X + Y \mid X) = 2X \quad \text{a.e.?}$$

Question 1.4. Is it true that the Lebesgue measure m on $\mathbb{R}$ is uniquely determined by its values on intervals of the form $(-\infty, x]$, that is, by

$$m((-\infty, x])$$

for all $x \in \mathbb{R}$?

Question 1.5. Let $B(x, r)$ denote the closed disc in $\mathbb{R}^2$ with center $x \in \mathbb{R}^2$ and radius $r > 0$. Is it true that

$$m_2(B((1,1),1)) = m_2(B((0,0),1))?$$

Problem 2

Let X_k and Y_k be two stochastic variables whose joint distribution is the regular normal distribution on $(\mathbb{R}^2, \mathbb{B}_2)$ with mean 0 and variance matrix

$$\Sigma_k = \begin{pmatrix} \frac{1}{k} & 0 \\ 0 & \frac{1}{k} \end{pmatrix}$$

for $k \in \mathbb{N}$. That is, $(X_k, Y_k)^T \sim \mathcal{N}(0, \Sigma_k)$.

Question 2.1. Show that $X_k + Y_k \sim \mathcal{N}(0, 2/k)$.

Question 2.2. Show that for all $\varepsilon > 0$

$$P(|X_k + Y_k| > \varepsilon) \leq \frac{2}{k\varepsilon^2}$$

and conclude that

$$X_k + Y_k \xrightarrow{P} 0$$

for $k \to \infty$.

Question 2.3. Let $Z = X_2 Y_2$. Show that

$$EZ = 0 \quad \text{and} \quad VZ = \frac{1}{4}.$$

Assume that $Z_1, Z_2, \ldots$ are independent and identically distributed all with the same distribution as Z.

Question 2.4. Show that

$$P\left(\frac{2}{\sqrt{n}} \sum_{i=1}^{n} Z_i \leq 2\right)$$

is convergent for $n \to \infty$ and compute the limit.

Problem 3

For the next question it can be assumed well known that

$$\int_{-\infty}^{\infty} \frac{1}{1+y^2} \, \mathrm{d}y = \pi.$$

Let $A = (-1, 1) \times \mathbb{R}$ and define the $\mathcal{M}^+$-function f by

$$f(x, y) = 1_A(x, y) \frac{|x|}{2\pi(1 + x^2 y^2)}$$

for $(x, y) \in \mathbb{R}^2$.

Question 3.1. Show that

$$\int f \, \mathrm{d}m_2 = 1.$$

If $\mu = f \cdot m_2$, that is, μ is the measure with density f w.r.t. the Lebesgue measure, then it follows from the question above that μ is a probability measure.

Question 3.2. Show that

$$\int x \, \mathrm{d}\mu(x, y) = 0.$$

Let X and Y denote stochastic variables whose joint distribution is μ.

Question 3.3. Compute the density w.r.t. the Lebesgue measure for the joint distribution of (X, XY).

Problem 4

Let ν be the measure on $(0, \infty)$ with density

$$f(x) = \frac{1}{x^{3/2}} e^{-\frac{1}{x}}, \quad x > 0$$

w.r.t. the Lebesgue measure.

Question 4.1. Show that

$$\nu((0, \infty)) = \sqrt{\pi}.$$

Hint: Try substitution with $h(x) = 1/\sqrt{x}$.

By the previous question we can introduce the probability measure

$$\mu = \frac{1}{\sqrt{\pi}} f \cdot m_{(0,\infty)},$$

that is, μ has density

$$g(x) = \frac{1}{\sqrt{\pi} x^{3/2}} e^{-\frac{1}{x}}, \quad x > 0$$

w.r.t. the Lebesgue measure. Let X be a stochastic variable with distribution μ.

Question 4.2. Find the density w.r.t. the Lebesgue measure for the distribution of $1/X$.

Question 4.3. Let Y be another stochastic variable with distribution μ such that X and Y are independent. Find the distribution of

$$\frac{1}{X} + \frac{1}{Y}.$$