Mål og Integrale Teori

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Abstract

These notes are intended as a resource for myself; past, present, or future students of this course, and anyone interested in the material. The goal is to provide an end-to-end resource that covers all material discussed in the course displayed in an organized manner. If you spot any errors or would like to contribute, please contact me directly.

1 Fundamentels

1.1 Paving

An arbitrary collection of subsets is a **paving**

1.2 Algebra

Definition 1.1. *A paving $\mathbb{A}$ on a set X is called an* ***algebra*** *if*

- $X \in \mathbb{A}$
- $A \in \mathbb{A} \Rightarrow A^c \in \mathbb{A}$
- $A, B \in \mathbb{A} \Rightarrow A \cup B \in \mathbb{A}$

Lemma 1.2. *If $\mathbb{A}$ is an* ***algebra*** *on X , then $\emptyset \in \mathbb{A}$*

Proof. We know that X itself is a member of $\mathbb{A}$, and we know that $\mathbb{A}$ is is stable under formation of complements. But the complements of X is indeed $\emptyset$. $\square$

Lemma 1.3. *If $\mathbb{A}$ is an* ***algebra*** *on X , is holds that*

$$A, B \in \mathbb{A} \Rightarrow A \cap B \in \mathbb{A}$$

Proof. Take A and B in $\mathbb{A}$. As $\mathbb{A}$ is stable under formation of complements, we see that A^c and B^c are two $\mathbb{A}$ -sets. As $\mathbb{A}$ is stable under formation of unions, we set that $A^c \cup B^c \in \mathbb{A}$. If we take the complement again, we see that

$$A \cap B = (A^c \cup B^c)^c \in \mathbb{A}$$

using de Morgan's law $\square$

Lemma 1.4. *If $\mathbb{A}$ is an algebra on X , it holds that*

$$A, B \in \mathbb{A} \Rightarrow A \backslash B \in \mathbb{A}$$

Proof. Take A and B in $\mathbb{A}$. As $\mathbb{A}$ is stable under the formation of complements, we see that B^c is in a $\mathbb{A}$ -set. As $\mathbb{A}$ is stable under the formation of intersections, we see that $A \cap B^c \in \mathbb{A}$. Per definition of the set difference, we have that

$$A \backslash B = A \cap B^c \in \mathbb{A}$$

$\square$

Lemma 1.5. *If $\mathbb{A}$ is an algebra on $\mathcal{X}$, and $A_1, \ldots, A_n$ are sets in $\mathbb{A}$, is holds that*

$$\bigcup_{i=1}^{n} A_i \in \mathbb{A}, \bigcap_{i=1}^{n} A_i \in \mathbb{A}$$

Proof. For $n = 2$ the claim is included in the definitation of an algebra. If the results is established for $n - 1$ sets, we have

$$\bigcup_{i=1}^{n} A_i = \left(\bigcup_{i=1}^{n-1} A_i \right) \cup A_n \in \mathbb{A}$$

□

1.3 sigma-algebras

The concept of algebras does not work under under approximate schemes. Therefore we introduce σ-algrebras.

Definition 1.6. *A paving $\mathbb{E}$ on a set $\mathcal{X}$ is called a σ-algebra if*

- $\mathcal{X} \in \mathbb{E}$
- $A \in \mathbb{E} \Rightarrow A^c \in \mathbb{E}$
- $A_1, A_2, \cdots \in \mathbb{E} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathbb{E}$

Definition 1.7. *Lad F være en abitrær familie a delmænger af $\mathcal{X}$. Der eksistere en unique mindste $\sigma-$algebra der indeholder alle mængder i F (F er ikke selv nødvendigvis en $\sigma -$ algebra). Foreningenmængden af alle $\sigma -$ algebra der indeholder F. Denne $\sigma -$ algebra $\sigma(F)$ er $\sigma -$ algebraen genereret af F.*

*A **measurable space** is a pair $(\mathcal{X}, \mathbb{E})$, consisting of the set $\mathcal{X}$ and a σ –algebra $\mathbb{E}$ on $\mathcal{X}$. We say that a subset $A \subset \mathcal{X}$ is $\mathbb{E}$ –measurable if $A \in \mathbb{E}$*

Lemma 1.8. *If $\mathbb{E}$ is an σ-algebra on $\mathcal{X}$, the it is also an algebra.*

Proof. see book page 11. □

1.4 Borel Sigma algebra

Definition 1.9. *The **Borel Sigma algebra** $\mathbb{B}$ is the smallest $\sigma -$ algebra generated by the open sets. symboliccally $\mathbb{B} = \sigma(\mathbb{O})$*

Remark 1.10. *As the borel algrebra $\mathbb{B}$ is a $\sigma -$ algebra which is stabel under the formation of complements, $\mathbb{B}$ is also the* $\mathtt{sigal}$ *generated on the closed sets.*

2 lecture 1

Definition 2.1. *Borel σ-algebraen $\mathbb{B}_k$ på $\mathbb{R}_k$ er $\sigma -$ algebra frembragt på de åbne mængder $\mathbb{O}_k$*

Exercise 2.1.1. Afgør om det er er borel mængder

(1) $(\mathbb{R}, \mathbb{B})$ er et målbart rum.

$$a, b \Rightarrow A \cup B^c \in \mathbb{B}$$

(2) Hvis $B \subseteq$ er en endelig mængde, så gælder at $B \in \mathbb{B}$

(3) $\{x\} \in \mathbb{B}$ for alle $x \in \mathbb{R}$

(4) Hvis $B \subseteq \mathbb{R}$ er en tællig mængde, så gælder det at $B \in \mathbb{B}$.

(5) Hvis $B \subseteq \mathbb{B}$ er overtællig mængde med tællig B^c, så $B \in \mathbb{B}$

Solution. Løsning på overstående

(1) Sandt: En $\sigma -$ algebra er **stabil** over for **endelige** og **tællige** mængdeoperation

(2) Falsk: $\mathbb{B} \neq \mathbb{P}(\mathbb{R})$ så findes $B \in \mathbb{P}(\mathbb{R})$ så $B \in \mathbb{B}$. Så $\mathbb{R} = B \cup B^c$. So if B er overtællig, så er B^c også overtællig. Derved er findes der et $B \notin \mathbb{B}$

(3) Sandt:$\{x\}$ er endelig og $\mathbb{B}$ indeholder alle **endelige mængder**

(4) $B \bigcup_{x \in B} \{x\}$, som er tællig, hvis B er tællig

(5) Hvis B er overtællig, B^c er tællig, så $B^c \in \mathbb{B}$, så $B = (B^c)^c \in B$ (B er stabil under komplimenter), så $B \in \mathbb{B}$

$\mathbb{B} = \sigma(\mathbb{O})$, dvs, den mindste $sigal$ der indeholder $\mathbb{O}$. F.eks.

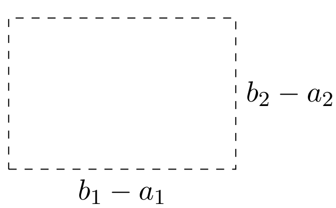
- De endelige mængder
- De åbne kasser $\mathbb{I}^k$
- De åbne kasser med rationelle hjørner
- For $k = 1$, intervaller af formen $(-\infty, b_1], \ldots, \times(-\infty, b_i]$

Lad $\mathcal{X}$ være en mængde og lad $\mathbb{E} = \sigma(\mathcal{X})$, så kaldes $(\mathcal{X}, \mathbb{E})$ et målbart rum.

Definition 2.2. *Et mål μ på $(\mathcal{X}, \mathbb{E})$ er en funktion på $\mathbb{E}$ som opfylder*

- $\mu(A_n) \in [0, \infty]$
- $\mu(\emptyset) = 0$
- $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$

Example 2.3. *Arealet på* $(b_i - a_1)(b_2 - a_2)$



2.1 lebesgue målet

Definition 2.4.

$$m_k((a_1, b_1) \times (a_2, b_2) \times \ldots \times (a_k, b_k)) = \prod_{i=1}^{k} (a_i, b_i)$$

Exercise 2.4.1. Afgør om følgende er sandt eller falsk

(1) hvis $A \subseteq B$ med $\mu(A) = 0$ og $B \subseteq A$ så $B \in \mathbb{B}$

(2) $\mu(\mathbb{R}) = \lim_{n\to\infty} \mu[-n, n]$

(3) $\mu(\{0\}) = \lim_{n\to\infty}(\left[-n^{-1}, n^{-1}\right])$

b

Solution. Løsninger på overstående

(1) Falsk: Der findes (for f.eks. lebesgue målet) ikke målige nul mængder. Dvs. $B \subseteq A$, hvor $\mu(A) = 0$, og $B \in \mathbb{R}$

(2) Sandt: $\mathbb{R} = \bigcup_{n\in\mathbb{N}}[-n, n]$ og $[-n, n]$ vokser opad (dvs. $[-1, 1] \subseteq [-2, 2] \subseteq \ldots \subseteq [-n, n]$. Da målet er opad kontinuert, så er $\mu \bigcup_n [-n, n] = \lim_{n\to\infty} \mu \bigcup_n [-n, n]$

(3) Falsk: $\{0\} = \bigcap_{n\in\mathbb{N}} \left[-n^{-1}, n^{-1}\right]$. Tag f.eks. tællemålet $\mu = \tau$. $\tau(\{0\}) = 1$ men $\tau\left[-n^{-1}, n^{-1}\right] = \infty$

3 Lecture 2

The Uniqueness Theorem

The problem is that we would like to say something about then two measures on the same measurable space $\mathcal{X}, \mathbb{E}$ on "many" sets (e.g. the the power set). E.i, when $\mu(A) = \upsilon(A)$.
We cannot check all sets in $\mathbb{E}$. So we would like to say something about how many sets we need to check to astablish that $\mu(A) = \upsilon(A)$ for all $A \in \mathbb{E}$?
For some paving $\mathbb{D}$ and a $\mathbb{E} = \sigma(\mathbb{D})$ and two measures μ and υ we want to show that if $\mu(D) = \upsilon(D)\,\forall\, D \in \mathbb{D}$ then it also holds that $\mu(A) = \upsilon(A)\,\forall\, A \in \mathbb{E}$ trick is to:

(1) Show that $\mu(D) = \upsilon(D)\,\forall\, D \in \mathbb{D}$

(2) Then construct a set $\mathbb{H} = \{A \in \mathbb{E} \mid \mu(A) = \upsilon(A)\} \supset \mathbb{D}$

(3) Then show that $\mathbb{H} = \mathbb{E}$

The last part is however not easy unless we can show that $\mathbb{H}$ is a $\sigma -$ algebra. So instead we make a loser requirement on $\mathbb{H}$ and then try to show that $\mathbb{H} = \mathbb{E}$ given some stronger requirements on $\mathbb{D}$ than just being a generator for the $\sigma -$ algebra.

Definition 3.1. *A paving $\mathbb{H}$ os a set $\mathcal{X}$ is a **Dynkin class** if*

(1) $\mathcal{X} \in \mathbb{H}$ (the same as σ-algebra)

(2) $A, B \in \mathbb{H}$, $A \subset B \Rightarrow B \setminus A \in \mathbb{H}$ (stronger than the σ-algebra but as $\mathcal{X} \setminus A = A^c$ then it is also stable under complements)

(3) $A_1, \ldots, A_n \in \mathbb{H}$, $A_1, A_2 \subset \ldots \Rightarrow \bigcup_{n=1}^{\infty} A_n$ (looser than σ-algebra)

To show that a Dynkins class is af σ-algebra and vice-versa we need to lemmas

Lemma 3.2. *if $\mathbb{A}$ is an algebra it holds that*

$$A, B \in \mathbb{A} \Rightarrow A \setminus B \in \mathbb{A}$$

Lemma 3.3. *If $\mathbb{E}$ is a σ-algebra it is also an algebra*

So the property that $A, B \in \mathbb{A} \Rightarrow A \setminus B \in \mathbb{A}$ also holds for a σ-algebra.
We then introduce a new lemma that states $\mathbb{E} = \mathbb{H}$ and that $\mathbb{H} = \mathbb{E}$.

Lemma 3.4. *If $\mathbb{E}$ is a σ-algebra, then $\mathbb{E}$ is also a Dynkins Class. If $\mathbb{H}$ is a Dynkins class which is stable under intersections, then $\mathbb{H}$ is a σ-algebra.*

Proof. The first part follows easlialy. (1) is the same for both (2) follows from the two lemmas stating that if $A, B \in \mathbb{A} \Rightarrow A \setminus B \in \mathbb{A}$ holds for a σ-algebra. (3) holds, as a σ-algebra is stable under all types of unions, also unions of increasing sets. To show that $\mathbb{H}$ which is cap-stabe is a σ-algebra we first note that a DC which is stable under $\cap$ is also stable under finite $\cup$: if $A, B \in \mathbb{H}$ then

$$A \cup B = (A^c \cap B^c)^c \in \mathbb{H}$$

if $A_1, , A_2 \ldots \in \mathbb{H}$, we let

$$B_n = A_1 \cup \ldots \cup A_n$$

As a $\cap$-stable DC is stable under finite unions, then $B_n \in \mathbb{H}$. And as $B_1, B_2 \subset \ldots$ we have that $\bigcup_{n=1}^{\infty} B_n \in \mathbb{H}$. But

$$\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$$

□

So if a DC $\mathbb{H}$ is $\cap$-stable it is in fact a σ-algebra.

Remark 3.5. *If $(\mathbb{H}_i)_{i \in I}$ is a family of DC, the intersection $\bigcap_{i \in I} \mathbb{H}_i$ is also a DC*

Remark 3.6. *A DC generated by $\mathbb{D}$ is the smallest DC containg all the elements of $\mathbb{D}$*

Lemma 3.7 (Dynkins lemma)**.** *Let $\mathbb{D} \subset \mathbb{H} \subset \mathbb{E}$ be pavings on the set $\mathcal{X}$, and assume that $\mathbb{E} = \sigma(\mathbb{D})$. If $\mathbb{D}$ is $\cap$-stable, and if $\mathbb{H}$ is a DC, then $\mathbb{H} = \mathbb{E}$.*

Proof. Let $\mathbb{K}$ be the smallest DC containing $\mathbb{D}$. Then

$$\mathbb{D} \subset \mathbb{K} \subset \mathbb{H} \subset \mathbb{E}$$

The proof is to show that $\mathbb{K}$ is $\cap$-stabel. If this is true, then it follows from above lemma that $\mathbb{K}$ is a σ-algebra. As $\mathbb{E}$ is the smallest σ-algebra containing $\mathbb{D}$ and $\mathbb{K}$ is a σ-algebra contaning $\mathbb{D}$ than it must be true that $\mathbb{K} = \mathbb{E}$. As $\mathbb{H}$ is squezzed between $\mathbb{K}$ and $\mathbb{E}$ then $\mathbb{H} = \mathbb{E}$. E.i. $\mathbb{H}$ is a σ-algebra.
The proof follows as

(1) Show that $\mathbb{K}$ is $\cap$-stabel

for each $A \in \mathbb{K}$, introduce a new paving

$$\mathbb{K}_A = \{B \in \mathbb{K} \mid A \cap B \in \mathbb{K}\}$$

$\mathbb{K}_A$ is a DC as

(a) $\mathcal{X} \in \mathbb{K}_A$ as $A \cap \mathcal{X} \Rightarrow A \subset \mathbb{D} \subset \mathbb{K}$. As $\mathbb{K}_A$ is the intersection of all elements of $\mathbb{K}$ and $\mathcal{X}$ is a element of $\mathbb{K}$

(b) $\tilde{A}, B, \tilde{A} \subset B \Rightarrow B \setminus \tilde{A} \in \mathbb{K}_A$. Then $\tilde{A} \cap A, B \cap A \in \mathbb{K}$ with $\tilde{A} \cap A \subset B \cap A$. So

$$(B \setminus \tilde{A}) \cap A = \underbrace{(B \cap A)}_{\in \mathbb{K}} \setminus \underbrace{(\tilde{A} \cap A)}_{\in \mathbb{K}}$$

So $B \setminus \tilde{A} \in \mathbb{K}$

(c) $B_1,, B_2 \ldots \in \mathbb{K}_A$, $B_1, B_2 \subset \ldots \Rightarrow \bigcup_{n=1}^{\infty} B_n \in \mathbb{K}_A$

$$A \cap \bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} \underbrace{(A \cap B_n)}_{\in \mathbb{K}}$$

as $\mathbb{K}$ is a DC. Further $A \cap B_1, A \cap B_2 \subset \ldots$, so $\bigcup_{n=1}^{\infty} A \cap B_n \in \mathbb{K}_A$

(2) Note that $\mathbb{D} \subset \mathbb{K}_A \subset \mathbb{K}$ and for $A, B \in \mathbb{D}$, then $A \cap B \in \mathbb{D} \subset \mathbb{K}$ which can given the definition of $\mathbb{K}_A$ this can be reformulated as: if $A \in \mathbb{D}$, then $\mathbb{D} \subset \mathbb{K}_A$ we must have that $\mathbb{K}_A = \mathbb{K}$. $\mathbb{K}$ is the smallest DC that contains $\mathbb{D}$ and $\mathbb{D} \subset \mathbb{K}_A$, then $\mathbb{K}_A = \mathbb{K}$. And a $\mathbb{K}_A$ is defined by the intersections of $\mathbb{K}$ then $\mathbb{K}$ is $\cap$-stabel.

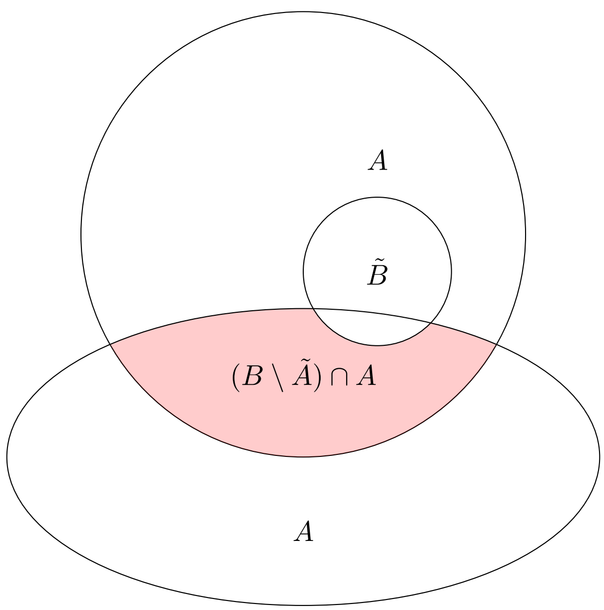


Figure 3.1: The $(B \setminus \tilde{A}) \cap A$

□

Using the above we can then prove the **uniqueness theorem for probability measures** . The steps are as follows

(1) Let υ, μ be two measures of probability on a measurable space, and assume that $\mathbb{E} = \sigma(\mathbb{D})$ and that

$$\mu(D) = \upsilon(D)$$

if $\mathbb{D}$ is $\cap$-stabel, then $\mu = \upsilon$.

(2) contruct a paving where the two measures are equal

$$\mathbb{H} = \{F \in \mathbb{E} \mid \mu(F) = \upsilon(F)\}$$

and show that $\mathbb{H}$ is a DC. This can be done with Dynkins lemma, which requires that

(1) That $\mathbb{D}$ is $\cap$-stabel follows by the theorem
(2) establish that $\mathbb{D} \subset \mathbb{H} \subset \mathbb{E}$. But this clearly follows from the contruction of $\mathbb{H}$
(3) Then all we need to show is that $\mathbb{H}$ is a DC

To show that $\mathbb{H}$ is a DC we note that

- As μ, υ are probability measures, then clearly $\mathcal{X} \in \mathbb{H}$.
- If $A \subset B$ are two $\mathbb{H}$ sets, then

$$\mu(B \setminus A) = \mu(B) - \mu(A) = \upsilon(B - \upsilon(A)) = \upsilon(B \setminus A)$$

, so $B \setminus A \in \mathbb{H}$
- If $F_1, F_2 \subset \ldots \in \mathbb{H}$, then

$$\mu\left(\bigcup_{n=1}^{\infty} F_n\right) = \lim_{n\to\infty} \mu(F_n) = \lim_{n\to\infty} \upsilon(F_n) = \upsilon\left(\bigcup_{n=1}^{\infty} F_n\right)$$

, then $\bigcup_{n=1}^{\infty} F_n \in \mathbb{H}$

3.1 Product Algebra

Looks a σ-algebra generated by several maps. That is, if you have many measurable maps and you take the cartisan product, will the product then still be measurable.

3.2 Measurability of intergrals

Examine under what conditions a integral is measurable

3.3 Measurable maps

Definition 3.8. *Lad $\mathcal{X}, \mathbb{E}$ and $\mathcal{Y}, \mathbb{K}$ be two measurable spaces, and let $f : \mathcal{X} \to \mathcal{Y}$ be a map. We say that f is* ***measurable*** *if*

$$f^{-1}(D) \in \mathbb{E} \textit{ for all } B \in \mathbb{K}$$

Remark 3.9. *We say that a f is $\mathbb{E} - \mathbb{K}$ measurable if it satifies above*

Remark 3.10. *If there is no confusion about which algebra to use, we say that either f is $\mathbb{E}$-measurable if the σ-algebra $\mathcal{Y}$ is fixed and the only choice of confusion is the σ-algebra on $\mathcal{X}$. Similary we may say that that f is $\mathbb{K}$ measurable if $\mathcal{X}$ is fixed, and the only possible confusion is the choice of σ-algebra on $\mathcal{Y}$*

Often we cannot check all the sets in the σ-algebra as many of them are not acceseable for direct description. Luckly we only have to check the condition given in definition 3.8 on the generator for for the σ-algebra.

Lemma 3.11. *Let* $(\mathcal{X}, \mathbb{E})$ *and* $\mathcal{Y}, \mathbb{K}$ *be two measurable spaces, and let* $f : \mathcal{X} \to \mathcal{Y}$ *be a map. Let* $\mathbb{D}$ *be a paving on* $\mathcal{Y}$ *and assume that* $\mathbb{K} = \sigma(\mathbb{D})$ *. If*

$$f^{-1}(D) \in \mathbb{E} \, \forall \, D \in \mathbb{D},$$

then f *is* $\mathbb{E} - \mathbb{K}$*-measurable*

3.4 Product measures

Thw product of two finite measures $\mu \otimes \upsilon$ is agian a measure. Especially: the product of two probability measures is agian a probability measure.

4 Lecture 3

4.1 Integration of a product measure

Theorem 4.1 (Tonelli). *Let* $(\mathcal{X}, \mathbb{E}, \mu)$ *and* $(\mathcal{Y}, \mathbb{K}, \upsilon)$ *be two* σ*-finite measurable spaces and let* $d \in \mathcal{M}^{+}(\mathcal{X} \times \mathcal{Y}, \mathbb{E} \otimes \mathbb{K})$.*It holds that*

$$\int f d\mu \otimes \upsilon = \int \left(\int f(y, x) d\mu(y) \right) d\upsilon(x)$$

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