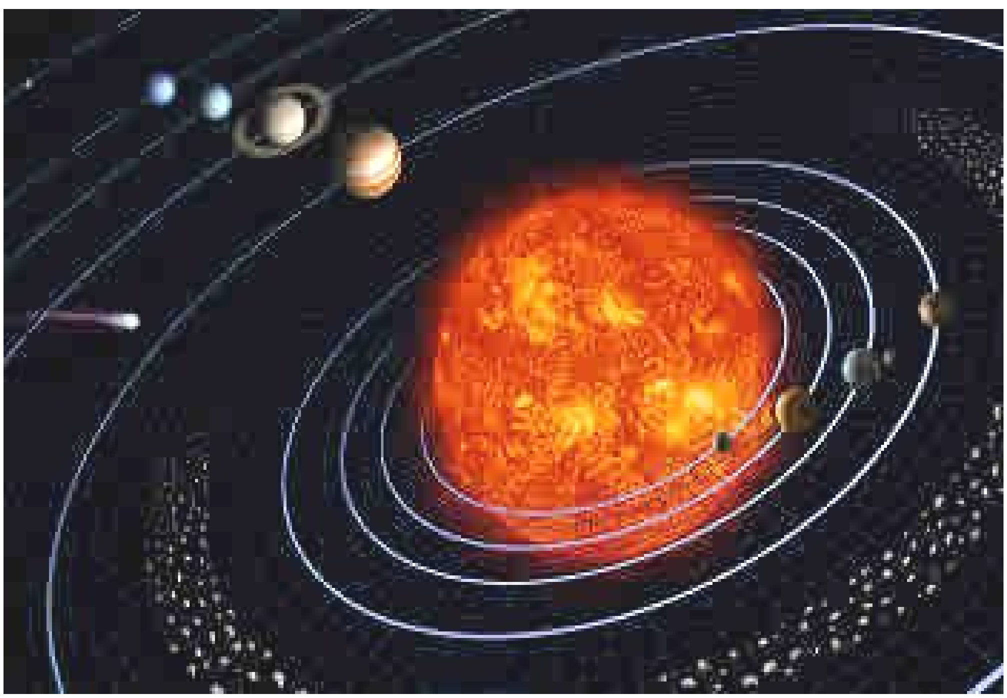
A Monograph

The n-body Problem – A Closed-Form Solution

4th Edition, June 2018

Abstract**:** This **Monograph** develops a general, closed-form algebraic *solution* for determining the magnitudes and directions for an unlimited number of vector reactions owing to an applied, static or quasi-static, concentrated load (force and moment)**.** In physics the problem is known as the n*-body problem***.** *The algorithm is really a generalization of* ***NEWTON****'s Law of Universal Gravitation***.** There is also a discussion on the mechanism of gravity, and owning to this mechanism it is inferred gravity is a manifestation of electromagnetic energy**.**



Anonymous, 2013

This **<u>Monograph</u>** is an ongoing, never-ending project**...**

Preface, 1st Ed. Oct. 2013

This **<u>Monograph</u>**'s main thrust is to provide a new, closed-form, algebraic algorithm for solving for n vector reactions **–** owning to a single, applied concentrated load**.** The load is a single vector force plus moments, applied at an arbitrary load point; and where the applied moments are reacted by reaction couples**.** More than one loading condition for the same configuration may be combined by superposition**.**

This is the general n*-body problem* when n is equal to, or greater than 3**.** Solutions can be obtained by employing one of the included, user-friendly, **Q*Basic*** or **FORTRAN** computer programs**.** Equilibrium is always checked via these programs for balance between the applied load and its reactions**.**

A fairly comprehensive *<u>outline</u>* of the algorithm's derivation is given; however, it is not necessary to follow this derivation to ***RUN*** the PC or main-frame computer programs and get quick results**.** Some modern applications are discussed; and a rivet loads analysis example is given of a 3D-field of twenty spaced rivets subject to an applied load, is first solved by hand; and then solved by the algorithm's computer program for comparisons (see **Appendix A**)**.** A suggested solution approach to the original, intractable Celestial Mechanics Solar System's n*-body problem* is also discussed**.**

NEWTON's *Empirical Force Laws* (Time-Geometry Laws) **–** his *Universal Law*, conjointly with his 2nd *Law of Motion* **–** has been shown to solve completely the forces for the classical *Two-body Problem* and the *<u>Restricted</u> Three-body Problem* (discussed below)**.** However, this algorithm can be employed to solve completely any n*-body problem* of unlimited size**.** *The algorithm is really a <u>generalization</u> of **NEWTON**'s Law of Universal Gravitation***.**

In general the mathematics employed is almost entirely algebraic**.** However, it is assumed the reader's mathematical skills include analytic and coordinate geometries, matrices; and especially *index notation***.** In addition, a *<u>small</u>,* amount of elementary calculus is included**.**[1]

The **<u>Monograph</u>** does assume the reader has some background knowledge of basic physical definitions, such as mass, position, velocity, etc.**;** plus *Kinematic*

1 See for example**: EISEL**, John A. and **MASON**, Robert M.**: <u>Applied Matrix and Tensor Analysis</u>**, *Wiley-Interscience*, 1970**.** This is similar in application to the methods use here; except tensors are not employed**.**

Principles; Sir Issac **NEWTON**'s (1643-1727) *Empirical Force Laws*; and Jean Le Rond **D'ALEMBERT** (1717-1783) two *Principles*. The only difference between this **Monograph**'s physics, and older, classical essays and textbooks, etc., are, in some cases, in how definitions and *Principles* are perceived and interpreted *physically* (see **Appendix E: NEWTON's *Laws of Motion***). That said, no definitions or *Principles* are changed mathematically. F is still equal to ma, etc.

It is further assumed the reader is familiar with the mathematical constructs of Euclidean three-dimensional space; and has some concept of clock *time*, mass[2], and *motion*[3]. Motion may owe to contact with other bodies; to gravitational attraction; or it may owe to electrical, thermal, chemical, or biological causes. Here it is restricted to *gravitational attraction*.

The **Monograph** is not a textbook, nor is the author trying to be pedantic; and it is nowhere near comprehensive in its discussion of *Newtonian physics*. In this effort though, it was finally realized: no n-body closed-form solution, nor variations thereof was forth coming now or in the future, so the algorithm needed to be documented and the perfect place was the Internet, so it could be shared with the whole world.

All errors of all types and kinds are the author.

Preface, 2nd Ed. 2015

The 2nd Edition includes some major rewriting as well as some grammar corrections to the narrative. One addition includes interpolating Kinematics represented graphical -to- algebraic representations of those results; and then linking the latter to Newtonian physics. This is followed by a discussion about gravity.

The revisions to the 1st Edition are described in the **Forward** following.

In that **Forward** for the 2nd Ed. it is stated "*the **Monograph** is not a textbook, nor is the author trying to be pedantic; and it is nowhere near comprehensive in its discussion of Newtonian physics. In this* (**Monograph**) *effort though, it was finally*

2 See **JAMMER**, Max: **Concepts of Space** and **Concepts of Mass**, *Harper Torchbooks*, 1954. Also see **EDDINGTON**, Sir Arthur: **Space, Time, and Gravitation**, *Harper Torchbooks,* 1959. "***Mass and Force in Newtonian Mechanics***," by L. N. G. **FILON**, ***Mathematical Gazette***, issue xxii, pp. 9-16, 1938.

3 See: "***Motion, Principles and Laws, Of***," by Robert B. **LINDSAY**, Volume 15, pp. 895-898, 1968 Edition, **Encyclopaedia Britannia**. Also see: "***Laws of Motion in Medieval Physics***," by Ernest A. **MOODY**, in ***The Scientific Monthly***, Vol, 72, No. 1, pp. 18-23, Jan. 1951. Two excellent articles written by experts.

realized prior to writing: *no solution or variations thereof*, of a closed-form *n*-body problem's solution *was forth coming now or in the* near *future* by mankind*, and the algorithm needed to be documented and the perfect place was the Internet*."

Preface, 3rd Ed. Sept. 2017

This **Monograph** is to a somewhat limited extent an obvious attempt to rewrite Newtonian physics. Feedback would be appreciated.

Now with 20-20 hindsight regarding the 2nd Ed., it is realized the inclusion of the Section, **Some Newtonian Physics Background** was effected because it was realized there was a need for the rather earthy approach to Newtonian physics – such as given in the Section **Synthesis**. In the 2nd Ed. it was stated: "*One addition included* interpolating *Kinematics represented graphical -to- algebraic representations of those results*; *and then linking the latter to Newtonian physics*." That Section is pedantic, but was added because it was felt almost all textbooks don't related Newtonian physics algebra -to- earthy kinematics representation of Newtonian physics, *at the very beginning of study*. How is the student to understand if not by an earthy example?

In this 3rd Ed. substantial rewrite has been effected in the Section "**Practical n-Body Problem Theoretical Solution**"; mainly in adding more mathematical steps in the derivation preceding Eq. **10**, the formula for the closed-form solution. And matrices and tensors are employed.

The example problem *has not changed* since the 1st Ed.; and none of the **Appendixes** *have changed* either. But the ***QBasic*** computer program ***Inertia2.bas*** print-sysout has changed: see **Revision Status** section below.

Other grammar corrections to the narrative have occurred throughout this 3rd Ed. of the **Monograph**, too numerous to list here. It is best just to download the latest version.

(Just for the record: when taking a graduate ***FEA*** course, it took only about 45 minutes or maybe a little less than that, about 50 years ago to derive the closed-form solution – because it is really simple; but it took about two-and-a-half days to write the **FORTRAN** program. The solution was used in 3D-rivet analyses, with the ***A***'s incorporating the shear and tension spring rates of the rivets continuously over the years but never shared until now. Don't use the A's = 1, which is a rigid body solution, which for rivet analyses is wrong.)

Preface, 4th Ed. June 2018

The 4th Edition has occurred principally owning to the switch between Internet servers (now ***ipage***); and a opportunity to add **Appendix F**: **MILANKOVITCH Cycles**. So changes to the 3rd Edition are slight. They include for the main, grammar corrections, which are too numerous to list here, and an updated **Index**; and a review and some rewrite of the **Planet Forces and Motions** section (suggest using the A's to represent the mass-ratios of the planetoids).

Also, download **QB64** from the Internet at https://thefreecountry.com/compilers/, which **Runs** on **Windows-10**, as it is an updated compiler for ***Microsoft***'s old ***Qbasic***.

1st Ed. Oct. 2013, 2nd Ed. 2015, 3rd Ed. Sept. 2017, 4th Ed. June 2018.

Table of Contents

Illustrations

Definitions And Symbols

Unless otherwise defined in the text, most definitions and symbols are explicitly given here.

Style

The narrative employs ***Open Office*** word-processor using *Arial* letters, size 13, with, generally, proportional line spacing set at 115%**.** Surnames, except Greek, are capitalized**.** Paragraph sections referred to in other paragraphs are **Arial Black**, size 13**.**

Cartesian tensors[4] employing index notation is used throughout the text**:** is a common, standard, algebraic notation for effecting ***3D*** analyses[5]**.** Normal line, mathematical scalar symbols use *Times New Roman* italic letters, scaled to size 16, to be somewhat equal in size to normal line Arial type, size 13.

Indexes at normal line use *Times New Roman,* italic text, size 16**;** normal line vector symbols use *Times New Roman,* italic-bold text, size 16; M*atrix notation* is used too, with symbols presented as italic-bold capitals.

Subscript letter indexes use *Times New Roman* italic letters, size 16**;** but the numerical indexes are in plain Arial text, size 16. For example**:** $\bar{\boldsymbol{r}}_\varsigma \Rightarrow \{\bar{r}_x, \bar{r}_y, \bar{r}_z\} \equiv \{\bar{r}_1, \bar{r}_2, \bar{r}_3\}$ (notice $\bar{\boldsymbol{r}}_\varsigma$ is bold and the bar always means the centroidal radius vector's position). ***MATHTYPE*** is employed for creating equations**.**

Symbols

- Brackets [ ] represent matrices (tensors), superscript –1 is the matrix inverse. And { } represent vectors.
- de*grees of freedom* [***DOF***]
- $\boldsymbol{\theta}_i$ represent angular displacements and technically are *axial vectors* since they depend on the order of rotation [radians]. $\boldsymbol{\theta}_i$ also denote the rotations**:** pitch (about

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4 See for example, **MYKLESTAD**, Nils O.**:** **Cartesian Tensors**, *D. Van Nostran Co.*, 1967**.**

5 See in **References:** in **KORENEU**'s book "***Supplement, Elements of Tensor Algebra and Indicial Notation in Mechanics***." See also in **References: EISELE** and **MASON**'s **Applied Matrix and Tensor Analysis**. **CREECH**, Dr. Merl D.**:** "***Tensor Analysis***," pp. 77-90, in **Your Turn…for mathematics and engineering**, Ed. by Douglas C. **GREENWOOD**, ***Product Engineering***, 1964**.** This is a once-over-lightly, but well-written, *practical* introduction to tensors, is targeted at the practical engineer**.**

the y-axis); roll (x-axis); and yaw (z-axis). $\boldsymbol{\theta}_i$ is the only axial vector defined in this **Monograph**.

- $\boldsymbol{a}$, acceleration [ft./sec.2 or meters/sec.^{2}].
- $A_i = A_{ix}\hat{\boldsymbol{i}} + A_{iy}\hat{\boldsymbol{j}} + A_{iz}\hat{\boldsymbol{k}}$; or equally, $\boldsymbol{A}_i \Rightarrow (A_{ix}, A_{iy}, A_{iz})$ as coordinate components; is a (*polar*) radius vector operator called a *Hookean*, wherein the components of $\boldsymbol{A}_i$ are bounded: $0 \le \boldsymbol{A}_i \le 1$, i = 1, 2, 3.
- $c.g.$ is *center of gravity* or barycenter.
- $\boldsymbol{F} = F_x\hat{\boldsymbol{i}} + F_y\hat{\boldsymbol{j}} + F_z\hat{\boldsymbol{k}}$ [Lb.]; or $\boldsymbol{F}_i \Rightarrow (F_{ix}, F_{iy}, F_{iz})$, is a (polar) vector force, a push or pull; equivalent symbols are $\vec{\boldsymbol{F}}_i$ or simply $\boldsymbol{F}_i$. $\boldsymbol{F}$ in lbf = (m in lbm)($\boldsymbol{a}$ in ft./sec.^{2}) / 32.174 lbm.-ft./lbf.-sec.^{2}
- $(\boldsymbol{F}_L)_i$ = applied load (Lb.), i = 1, 2, 3.
- $\boldsymbol{g} \equiv g_z\hat{\boldsymbol{k}}$, is the acceleration of gravity and is a vector, assumed to be 32.174 ft./sec.^{2}, or technically more correct, 32.174 lbm.-ft./lbf.-sec.^{2}; or 9.8067 m/sec.^{2}.
- G is the constant of proportionality of gravity, 3.44 x 10^{-8} ft^4/lbf-sec.^{4}, or 6.673 x 10^{-11} N m^2/kg^2.
- $(\boldsymbol{H}_o)_i$ is the *angular momentum* (in.-Lb.-sec.), i = 1, 2, 3.
- $\bar{\boldsymbol{I}}_{xx}, \bar{\boldsymbol{I}}_{yy}, \bar{\boldsymbol{I}}_{zz}, \bar{\boldsymbol{I}}_{xy}, \bar{\boldsymbol{I}}_{yz}, \bar{\boldsymbol{I}}_{yz}$ are centroidal, massless moments of inertia in terms of $\{x, y, z\}$ coordinates [in.^{2}].
- $\boldsymbol{I}_{pq} = a_{pl}a_{qj}\boldsymbol{I}_{ij}$ defines a congruent transformation of inertia.
- m = mass of a body [in kg; or slugs = lbm/32.174 lbm.-ft./lbf.-sec.^{2}]. The amount of units of m = units of $\boldsymbol{F}$/units of acceleration = lbf/(ft./sec.^{2})= lbf-sec.^{2}/ft.
- $m\boldsymbol{v}$ = momentum of a body (slugs-in./sec. or in.-Lb.-sec.), where $\boldsymbol{v}$ = translational velocity [ft./sec.].
- $\Delta m\boldsymbol{v} = +m\Delta\boldsymbol{v} + \boldsymbol{v}\Delta m = \boldsymbol{v}(m_1 - m_2) + m(\boldsymbol{v}_1 - \boldsymbol{v}_2)$, change in momentum [in.-Lb.-sec.].

- $\boldsymbol{M} = \boldsymbol{r}$ **x** $\boldsymbol{F}$ or $\boldsymbol{M}_i = \varepsilon_{ijk} \boldsymbol{r}_j \boldsymbol{F}_k$ moments, where **x** is the vector cross product [Lb.-in.].
- $(\boldsymbol{M}_\mathrm{o})_i$ applied moment [Lbs.-in.].
- N = number of reaction points or support points, where ζ = 1, …, N [Nondim.].
- $(\boldsymbol{P}_\zeta)_i$ = support point reactions (Lb.), where i = 1, 2, 3; ζ = 1, …, N supports.
- $\boldsymbol{R}$ is the resultant translational force [Lbf, or simply Lb.].
- Centroidal radius vector $\bar{\boldsymbol{r}}_\varsigma = \bar{x}_\varsigma \hat{\boldsymbol{i}} + \bar{y}_\varsigma \hat{\boldsymbol{j}} + \bar{z}_\varsigma \hat{\boldsymbol{k}}$, or $\bar{\boldsymbol{r}}_\varsigma \Rightarrow \left(\bar{x}_\varsigma, \bar{y}_\varsigma, \bar{z}_\varsigma\right)$, of a body's reaction pattern [in.] In symbols also .$\bar{\boldsymbol{r}}_\varsigma \Rightarrow \left\{\bar{r}_x, \bar{r}_y, \bar{r}_z\right\} \equiv \left\{\bar{r}_1, \bar{r}_2, \bar{r}_3\right\}$. The coordinates of each mass-point P_ς is $\left(\bar{x}_\varsigma,\ \bar{y}_\varsigma,\ \bar{z}_\varsigma\right)$, where the "*bar*" indicates a body's mass has been reduced to a point. A radius vector may be written as a column vector $\bar{\boldsymbol{r}}_\varsigma = \begin{Bmatrix} \bar{x}_\varsigma \\ \bar{y}_\varsigma \\ \bar{z}_\varsigma \end{Bmatrix}$; or as a coordinate $\bar{\boldsymbol{r}}_\varsigma \Rightarrow \left(\bar{x}_\varsigma, \bar{y}_\varsigma, \bar{z}_\varsigma\right)$.
- $\bar{\boldsymbol{r}}_\varsigma = \bar{x}_\varsigma (A_\varsigma)_x \hat{\boldsymbol{i}} + \bar{y}_\varsigma (A_\varsigma)_y \hat{\boldsymbol{j}} + \bar{z}_\varsigma (A_\varsigma)_z \hat{\boldsymbol{k}}$ is a *Hookean* radius vector for a ζ's point-mass [in.].
- $\Delta s_{\varsigma i} = \varepsilon_{ijk} \bar{\boldsymbol{r}}_{\varsigma j} \Delta\theta_{\varsigma k}$ incremental vector arc length of a 3-dimensional space curve, where $\bar{\boldsymbol{r}}_{\varsigma j}$ is *Hookean* [in.].
- $\boldsymbol{v} = d\boldsymbol{r}/dt$ translational velocity, equals the time rate of change of the position vector [in./sec., of m/sec.]
- $[\bar{x}_s]$ is the support transfer matrix, $[\bar{x}_s]$ = .$\begin{bmatrix} 0 & (z_\varsigma - \bar{z}_{xz}) & -(y_\varsigma - \bar{y}_{xy}) \\ -(z_\varsigma - \bar{z}_{yz}) & 0 & (x_\varsigma - \bar{x}_{xy}) \\ (y_\varsigma - \bar{y}_{yz}) & -(x_\varsigma - \bar{x}_{xz}) & 0 \end{bmatrix}$.

- $[\bar{x}_L]$ is the Loads transfer matrix $[\bar{x}_L] = \begin{bmatrix} 0 & (\bar{z}_L - \bar{z}_{yz}) & -(\bar{y}_L - \bar{y}_{yz}) \\ -(\bar{z}_L - \bar{z}_{xz}) & 0 & (\bar{x}_L - \bar{x}_{xz}) \\ (\bar{y}_L - \bar{y}_{xy}) & (\bar{x}_L - \bar{x}_{xy}) & 0 \end{bmatrix}$.

- P_ζ is the centroidal position of a reaction or support point , ζ = 1, 2, 3, …, N (in.)

- Weight $\boldsymbol{W}$, $\boldsymbol{W}$ in Lbf = (m in slugs) x (32.174 ft./sec^2). The poundal is 1/32.174 of the weight of the unit mass at the standard locality.

Subscripts

- 1, 2, 3 denotes a body's x-axis, y-axis, z-axis [non-dimensional].
- i, j, k = 1, 2, 3 range indices for ε_{ijk} .
- ⅍'s are the body's number for its reaction or support point [non-dim.].

Coordinate System

Basic coordinate system (x, y, z) is a single implicitly defined right-hand rectangular Cartesian coordinate system, everywhere *Euclidean*.

Tensors

Several Tensor Analysis references are given in the footnotes, as well as textbooks listed in the **References**.

Two special tensors are**:**

- δ_{ij} Kronecker delta symbol is either +1 if $i \neq j$; or 0 if $i = j$; where i, j = 1, 2.
- ϵ_{ijk} permutation symbol is either +1 for clockwise [***CW***] or –1 for counterclockwise [***CCW***]; or 0 if any two indices are equal.

The following page, not readily found in other textbooks, was taken from J. N. **REDDY**'s **Elastic Plates**, pp. 5-6, and combined into one page**:**

The symbol $\mathcal{E}_{ijk}$ is called the *alternating symbol* (permutation symbol) or *alternating tensor*, since it is a Cartesian component of a third-order tensor.

In an orthonormal basis, the scalar and vector products can be expressed in the index form using the Kronecker delta $\delta_{ij} \equiv \hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j$ and the alternating symbols

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &= (A_i \hat{\mathbf{e}}_i) \cdot (B_j \hat{\mathbf{e}}_j) = A_i B_j \delta_{ij} = A_i B_i \\ \mathbf{A} \times \mathbf{B} &= (A_i \hat{\mathbf{e}}_i) \times (B_j \hat{\mathbf{e}}_j) = A_i B_j \mathcal{E}_{ijk} \hat{\mathbf{e}}_k \end{aligned} \tag{1.2.13}$$

Note that

$$\mathcal{E}_{ijk} = \mathcal{E}_{kij} = \mathcal{E}_{jki}; \quad \mathcal{E}_{ijk} = -\mathcal{E}_{jik} = -\mathcal{E}_{ikj} \tag{1.2.14}$$

That is, a cyclic permutation of the indices does not change the sign, while the interchange of any two indices will change the sign. Further, the Kronecker delta and the permutation symbol are related by the identity, known as the $\mathcal{E} - \delta$ *identity*

$$\mathcal{E}_{ijk} \mathcal{E}_{imn} = \delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km} \tag{1.2.15}$$

The permutation symbol and the Kronecker delta prove to be very useful in proving vector identities and simplifying vector equations. The following example illustrates some of the uses of δ_{ij} and $\mathcal{E}_{ijk}$.

Example 1.2.1

We wish to simplify the vector operation $(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D})$ in an alternate vector form and thereby establish a vector identity. We begin with

$$\begin{aligned} (\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) &= (A_i \hat{\mathbf{e}}_i) \times (B_j \hat{\mathbf{e}}_j) \cdot (C_m \hat{\mathbf{e}}_m) \times (D_n \hat{\mathbf{e}}_n) \\ &= (A_i B_i \mathcal{E}_{ijk} \hat{\mathbf{e}}_k) \cdot (C_m D_n \mathcal{E}_{mnp} \hat{\mathbf{e}}_p) \\ &= A_i B_j C_m D_n \mathcal{E}_{ijk} \mathcal{E}_{mnp} \delta_{kp} \\ &= A_i B_j C_m D_n \mathcal{E}_{ijk} \mathcal{E}_{mnk} \\ &= A_i B_j C_m D_n (\delta_{im} \delta_{jn} - \delta_{in} \delta_{jm}) \\ &= A_i B_j C_m D_n \delta_{im} \delta_{jn} - A_i B_j C_m D_n \delta_{in} \delta_{jm} \end{aligned}$$

where we have used the $\mathcal{E} - \delta$ identity. Since $C_m \delta_{im} = C_i$ (or $A_i \delta_{im} = A_m$), and $A_i C_i = \mathbf{A} \cdot \mathbf{C}$, and so on, we can write

$$\begin{aligned} (\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) &= A_i B_j C_i D_j - A_i B_j C_j D_i \\ &= A_i C_i B_j D_j - A_i B_j D_j C_i \\ &= (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C}) \end{aligned}$$

Although the above vector identity is established using an orthonormal basis, it holds in a general coordinate system.

Revision Status

Jan. 16, 2014 minor changes**:**

- Update "*Preface*."
- Added "*dumbbell molecule*" model to **2-body Problem** section.
- Added E. T. W**HITTAKER** to **3-body Problem** section.
- Change footnote color.

Feb. 3-4, 2014**:** update **Symbols**; plus some minor corrections.

March 8,2014, added second paragraph at beginning.

4/1/14, added comments about **van der WAALS** forces.

Feb.-Mar. 2015**:** complete rewrite of front end. Mar. 29, 2015, Update **Planet Forces** section.

Sept. 2017**:** completely reviewed and rewrote some parts of the **Practical n-Body Problem Theoretical Solution** Section. The ***QBasic*** program ***Inertia2.bas*** (see section**: QBasic Code** Section below) was changed and updated Sept. 2017 to correct the **PRINT #3** sysout code. Further, corrected and clarified the narratives of other Sections, which are to numerous to mention.

Free compilers and interpreters for miscellaneous programming languages is at *https://www.thefreecountry.com/compilers/miscellaneous.shtml.* Download ***QBasic*** from there**:** this updated version of the old ***MS QBasic*** is much better than the old, with more syntax checks. Once up and running, which should take less than a half-hour, ***RUN Inertia2.bas*** using ***Display_Monitor.dat*** as ***INPUT*** (see section **A Numerical Example**. The load for this **Example** is in Lbs., is F_x = – cos15°, $\boldsymbol{F}_y$ = 0 and F_z = sin15°; applied at $(x_{c.g.}, y_{c.g.}, z_{c.g.})_{\text{ship}}$ = (– 7**.**79, 13**.**5625, 44**.**39), in inches.

June 2018**:** In general, total rewrite**;** added "4th Edition" to **Preface;** updated **TOC;** added clarification to **Style** paragraph in **Definitions and Symbols**; add reference, **BROOKE**, Wm. E. and **WILCOX**, Hugh B.**:** **Engineering Mechanics**, *Ginn and Co*., 1929, Chapter **XVI**, "***Dynamics***," paragraph **161**, "**D'ALEMBERT's *Principle***," pp. 249-250**.**

Rewrote section "*Gravitational Potential Function Development*" section**.** Added Appendix **F: MILANKOVITCH** Cycles**.** Updated Index**.**

The *n*-Body Problem

Although there were philosophical echoes from the ancient pass[6], and later during the Middle Ages – to the *n-body problem* – it only came into sharp focus at the end of the *Reformation* (17th century) and on into the early *Age of Reason* with the refined development of Newtonian physics.[7] The motive for solving the *problem* back then was in an effort to better understand how to predict correctly and reliably the motions of the planets. Initially the *n-body problem's* construct was based on some really bad astronomical problems with ancient astronomy:

- The Church's calendar was based upon **Ptolemy**'s error prone **Almagest**'[8], causing, by the 13th century, in spring planting being done a month-and-a-half late! Roger **BACON** (1214-1292) revamped the **Ptolemy** based calendar so dates coincided with observed astronomy[9]. Also, the Zodiac constellations were not in their predicted places at the specified times. These serious errors of course drew the **Almagest**'s correctness into question.
- By Nicholas **Copernicus**′ (1473-1543) time the complicated *Ptolemy Solar System model* with its epicycles came into question. **Copernicus** felt **Ptolemy's** model (based upon **Hipparchus**' work) was wrong and so constructed his own.[10,11] (See **Astronomy, Part I** on this webpage for

6 See **NEUGEBAUER**, Otto: (see **References** below), Chapter VI, "***Origin and Transmission of Hellenistic Science***," and *Appendix I*, "***The Ptolemaic System***." "*The center of 'ancient science' lies in the 'Hellenistic' period*."

7 **LEIMANIS** and **MINORSKY** (see **References):** Our interest is with **LEIMANIS**, who first discusses some history about the *n-body problem*, especially Ms. **KOVALEVSKAYA**'s ~1868 -to- 1888 (twenty-year!), *complex-variables* approach, which resulted in *failure*. **LEIMANIS** then follows with other known classical mathematical developments of the theory, which also have failed.

8 **SANTILLANA**, George **de:** "***Greek Astronomy***," ***Scientific American***, Vol. 180, No. 4, pp. 44-47, April 1949. This article is reproduced in its entirety in **Astronomy, Part I** on this web-page.

9 Jean **GIMPLE** in his **The Medieval Machine** discusses the role Roger **BACON** played in the development of the modern calendar; and **BACON**'s association at Oxford University with the origins of experimental science. Especially read Chapter **8**, "***Reason, Mathematics, and Experimental Science***," pp. 171-198.

10 **DANIELSON**, Dennis and **GRANEY**, Christopher M.: "***The Case Against Copernicus***," ***Scientific American***, Vol. 310, No. 1, Jan. 2014. "*The 17th century church objected to the idea Earth revolves around the Sun because it ran afoul of dogma. Scientists objected because of the facts against it*." Article discusses the *Geocentric Model* (fixed Earth with epicycles), the *Heliocentric Model* (**Copernicus**' model) and the *Geoheliocentric Model* (Tycho **Brahe**'s model). Also see **DANIELSON**, Dennis: **The First Copernican**, *Walker & Co*., 2006.

11, **ARMITAGE**, Angus: **World of Copernicus Sun Stand Thou Still**, subtitled **The Life and Work of Copernicus, the Astronomer**, *Henry Schuman Inc.*, 1947, *Mentor*, 1951.

many articles and books listed on **Copernicus** and the *Copernican Revolution*.)

- Sir Isaac **NEWTON**'s (1643-1727) conceptually formulated the *n-body problem* when trying to account and correct for **Copernicus**' model's observational perturbations in the motions of the planets. His main thrust however was to replace the **Almagest** and **Coperrnicus**' **The New Universe** (English title) with his own more accurate almanac (which he never did; his **Book III** is not an almanac).
- See **Astronomy, Part III** on this webpage for many articles and books listed on orbital mechanics. Also see Paul **HERGET**'s **The Computation of Orbits**, a 1949 textbook, now out-of-print, but reproduced on this webpage. **HERGET** covers form A -to- Z all the classical mathematical approaches for defining planetary orbits.

Defining the "*n*-body Problem"

The first concentrated efforts by **NEWTON** and others in trying to solve the *n*-body "*problem*" was made in the 17th century by employ his *Empirical Force Laws*. The *Empirical Force Laws* of *Newtonian physics* are **NEWTON**'s three *Laws of Motion*, and his *Universal Law* of *Gravitation* (see below). The first three are really *Time-Geometry* based relationships linked to force. The fourth, planet-to-planet and planet-to-Sun attracting forces, is the classical Newtonian interactive or "*forces-at-a-distance*" forces, which today are known as gravitational forces.

In physics the "*n*-body" term describes systems of independent bodies or particles – *not interacting with one another* – *except* through a *force-at-a-distance*, such as *gravitational forces*. For those conditions there are no rigid constraints between objects: like the "*free to move*" planets in space. Nor is there any concept of *elasticity* or elastic forces between the objects being studied. For instance, if the Moon were struck by a huge asteroid, the Moon's orbit around the Earth probably would change. If so, then there is no *restoring force* (elastic forces) to bring the Moon back into its original orbit.

Now to planetary *motion*: it is a function of *position*, *velocities*, *accelerations*, mass, *time*; and, in addition to the Sun's gravity field, *the planet-to-planet gravitational forces* (see **Appendix F: MILANKOVITCH Cycle**). Planetary positions, velocities, accelerations can be modeled as functions of discrete *time-steps*. The first problem in qualifying a planet's true orbit then is in ascertaining those parameters *and* critically, incorporating planet-to-planet gravitational forces associated with each planet. The latter effort is the classical *n-body problem*.

One way to solve the gravitational *n-body problem* is by assuming all *unsteady* or moving variables at an instant of *time* (in discrete time-steps) – are "*steady*," and thus approximate the *quasi-steady* magnitudes of those variable.[12] The mathematical technique is analogous to capturing a picture of a moving object using a super high shudder-speed camera, which concurrently records its instantaneous position relative to some reference point, and also records all nine instantaneous angular velocities and angular and translational accelerations. Today a commonly employed analytical method for obtaining loads or accelerations when an object (a planet) is in motion, is to assume the object is in a *quasi-steady* states.

Using the above *quasi-steady* concept, the *n-body problem* can be stated as: *given a series of time-step, quasi-steady orbital properties of a group of gravitationally connected celestial bodies, predict their continuously changing gravitational forces*; *and consequently, predict, when based in part upon those gravitational forces, their true orbital motions for all future times*.[13] Only three time-steps of their instantaneous positions, velocities and accelerations are needed as a function of time.

This **Monograph** provides the means to obtain those quasi-steady solutions for those gravitational forces.

That is to say, what the *n-body problem* closed-form, *algebraic* algorithm below does (see **Appendix B**) is allow the planet-to-planet gravitational forces to be determined, based solely on the mass-weighted, area-moment of inertia of the Sun, planetary bodies and other bodies, knowing their instantaneous positions, and weighted Sun-to-planet mass ratios. The algorithm must be run at least ten times (9 planets plus the Sun) to obtain the combined gravitational force on each planet from the other planetary bodies (i.e., add a loop). Each extra body included requires a ***RUN*** time. In this way the continuously changing planet -to- planet gravitational

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12 *Quasi-steady* state refers to the instantaneous (t: = 0) inertial loads generated by instantaneous translational accelerations (3 variables, but for each planet there are only two because it is essentially a planer problem); and the angular velocities and accelerations (α's and ω's: 2 each per axis time 3 axis = 6 variables, but for each planet in the Solar System there are really only two). In contrast, a steady-state condition refers to a system's state being invariant to *time*; or otherwise, the first derivatives and all higher derivatives are zero.

13 R. M. **ROSENBERG** states the *n-body problem* similarly (see **References**): Each particle in a system of a finite number of particles is subjected to a Newtonian gravitational attraction from all the other particles, and to no other forces. If the initial state of the system is given, how will the particles move? **ROSENBERG** failed to realize, like everyone else, that it is necessary to determine the forces first before the motions can be determined.

forces can be determined at discrete time-steps, which is the first step in the solution to the planetary *n-body problem*. The nine time-step instantaneous angular velocities and angular and translational accelerations about a fixed point (say the Sun) can be obtained by Earth based observations or via satellite observations. Knowing the kinematics plus the discrete forces will determine the true planetary orbits. The algorithm's derivation is outlined below; and more on the particular Solar System solution is discussed below too (see **Planet Forces and Moments**).

The **Monograph**'s main thrust is to develop a general, closed-form, static and/or quasi-steady, solution for determining a multitude of resultant reactions, wherein $n \geq 3$ reactions or supports having a total of 6 **DOF**) **–** owing to a concentrated applied load (an external force and moments). The geometry of the applied load has to be known with respect to the geometry of the reactions. More applied loads are effected by iteration and superposition. To this purpose the **Monograph** gives a simple, applied physics (algebraic) method for resolving these reactions.

As the reader will soon realize, the *algorithm* is a generalization of **NEWTON**'s *Law of Universal Gravitation*.

Current Application Efforts

Much of this **Monograph**, and especially the first parts, more or less deals with the development and understanding of **NEWTON**'s *Laws of Motion* and Newtonian gravitation (his three *Time-Geometry Laws*, plus gravity). In so doing it lays the foundation for the development of the closed-form solution to the *n-body problem*; or more succinctly, the *generalization* of **NEWTON**'s *Law of Universal Gravitation* in the form of an *algorithm*. Between the development -and- the algorithm are presented the classical applications of Newtonian gravitation: the **Two-body Problem** and the **Restricted Three-Body Problem**; and how the classical Newtonian approach for an unrestricted problem with $n \geq 3$ fails. On one hand is the analytical mechanics presentation; and on the other are several applications.

Currently the *n-body problem* application efforts now include trying to understand the dynamics of globular-cluster-like star systems[14] (the big); to forces between atoms in molecules (the small). To this purpose the idealized classical “*Two-body Problem”* has been completely solved analytically and is discussed below;

14 See **References** sited for **HEGGIE** and **HUT**. This **Monograph** makes their approach obsolete.

as is the famous "*Restricted 3-Body Problem*." As explained below, the effort to find any solution for 3 or more bodies without restrictions up to now using the classical Newtonian approach *has always* failed, as all those solutions, and what few there are, have been at best only approximations.[15, 16]

Other rigid-body *applications* (i.e., A's = 1, see below) include redistribution of useful loads in a *Finite Element Model* [***FEM***]; 3D rivet or bolt analyses using A's < 1 as springs (example problem included); determining molecular interactive (***van der WAALS*** like) forces; and classical celestial problems. Astrodynamics problems, and the like, can use this new, *general* n*-body problem*'s algorithm to determine forces applied to mass-points for n bodies. Once known, positions, velocities and accelerations; i.e. their motions, may be determined.

Determining equations and numerical values for astronomy problems using the n*-body problem*'s algorithm is beyond this **Monograph**'s present scope. Structural analyses or soil pile analyses elastic solutions using energy method approach and with the A's < 1 as springs, are possible too (analyses are within 2% of test values). This latter application is beyond the present scope too. Note especially: *rigid-body solutions for elastic structures, like rivet analyses, are not valid*, and should never be substitute for an elastic ***FEA***.

Historical Perspective: **NEWTON**'s **Principia**

Extracted From **COHEN**'s Article[17]: "*Development of **NEWTON**'s thinking on action and reaction* [phenomenon] *after he completed the first draft of **De Motu*** [a precursor to his masterpiece, the **Principia**], *is set out in the opening sections of the first book of the* **Principia**. *In the introduction to the 11th Section **NEWTON** explains he has confined himself so far to a situation that* [**NEWTON** words in italic-blue] '*hardly exist in the real world*,' *namely, the* '*motions of bodies attracted toward an unmoving center*.' *The situation is artificial because* '*attractions customarily are*

15 A general, classical solution in terms of first integrals is known to be impossible. An exact theoretical solution for arbitrary n can be approximated via *Taylor series*, but in practice such an *infinite series* must be truncated, giving at best only an approximate solution; and, an approach now obsolete. In addition, the *n-body problem* may be solved using *numerical integration*, but these, too, are approximate solutions; and again obsolete. For more background see Sverre J. **AARSETH**'s book **Gravitational N-body Simulations** listed in the **References**.

16 **HAMILTON**, Douglas A.: "***Fresh Solution to the Four-Body Problem***," ***Nature***, Vol. 533, pp. 187-188, 12 May 2016.

17 **COHEN**, I. Bernard: "***Newton's Discovery of Gravity***," in ***Scientific American***. A pdf copy is included on this Webpage. A key historical document and good reading too.

directed towards bodies and **–** *by the Third Law of motion* **–** *the actions of attracting and attracted bodies are always mutual and equal*.' *As a result*, '*if there are two bodies, neither the attracting nor the attracted body can be at rest*.' *Rather*, '*both bodies* (*by the Forth Corollary of the Laws*) *revolve about a common* center [i.e., barycenter], *as if by a mutual attraction*.'

"***NEWTON*** [realized] *if the Sun pulls on the Earth, the Earth must also pull on the Sun with a force of equal magnitude.*[18] *In this two-body system the Earth does not move in a simple orbit around the Sun. Instead the Sun and Earth each move about their mutual barycenter* (*center of gravity*)*. A further consequence of the* ***Third Law*** *is each planet is a center of attractive force as well as an attracted body; thus it follows* '*a planet not only attracts and is attracted by the Sun but also attracts and is attracted by each of the other planets*' [a key statement **–** the n-body problem]**.** *Here* ***NEWTON*** *has taken the momentous step from an* *interactive* *two-body system to an* *interactive* *many-body system* [the n-body problem]**.**

"In December 1684 ***NEWTON*** *completed a revised draft of* ***De Motu*** *describing planetary motion in the context of an* *interactive, many-body system. Unlike the earlier draft, the revised one concludes* '*the planets neither move exactly in ellipses nor revolve twice in the same orbit*.' *This conclusion led* ***NEWTON*** *to the following result*: '*There are as many orbits to a planet as revolutions, as in the Moon's motions; and the orbit of any one planet depends on the combined motion of all planets, not to mention the actions of all these on each other*.' He then wrote**:** '*To consider simultaneously all these causes of motion and to define these motions by exact laws* [*to*] *allow* [*for*] *convenient calculations, exceeds, unless I am mistaken, the force of the entire human intellect*.' " [No, he just needed a *PC***.** As for exceed the human intellect, a method for solving this n*-body problem* exists, called ***The FORCE***, is discussed later below.]

NEWTON's *Empirical Force Laws* have been employed in vain ever since his time, to determine the mutual gravitational forces of many (three or more) gravitationally connected celestial bodies; and if and whenever that is finally determined, then to incorporate their consequential force effects into a planet's orbital analyses.[19]

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18 That's so odd**:** why wouldn't the Sun pull with more force? The balancing phenomenon owes to the Sun's electromagnetic force field pushing the Earth away with the Earth is nested in the Sun's electromagnetic nodal ring , coupled with centrifugal force, while gravity pulls them together**.**

19 In the late 19th century King **Oscar II** of Sweden, advised by Gösta Mittag-**LEFFLER**, established a prize for anyone who could find the solution to the n*-body problem***.** The announcement was quite specific**:**

Some Newtonian Physics Background

Establishing the foundations of Newtonian physics was first effected by Galileo **GALILEI** (1564-1642)**.**[20] Later concurrently and somewhat jointly by René **DESCARTES** (1596-1650), **NEWTON**, Christiaan **HUYGENS** (1629-1695), and others, was even more challenging than the *n-body problem* is today**.** During **NEWTON**'s time, and even today, much difficulty existed in trying to understand the properties of matter (volume, density, weight, mass and chemistry, etc.)**;** and the individual physical phenomena associated with matter (here mainly position, velocity, acceleration, motion in general, and force)**;** and the conceptual difficulty of separating one phenomenon from another, as those verbs are all interactive**.** Particular difficulties were the concepts of force, mass, motion and inertia and how to *model* these variables**.** (C. A. **TRUESDALL** (1919-2000) in his writings[21] said around 1900 physicist who taught physics failed to understand fundamental Newtonian physics, and so generation after generation of students also failed to understand basic Principles**.** In addition, was understanding Jean Le Rond **D'ALEMBERT**'s (1717-1783) 1st and 2nd subtle *Principles*, a subject which belongs to *Mechanics* and will be discussed later[22]**.**)

NEWTON's 1st *Law*, discussed later, is all about *inertia***.** He never did define force or mass, and his 2nd *Law*, $\boldsymbol{F} = m\boldsymbol{a}$, is not a definition for force or mass; or even a "*Law*" for that matter, it is a *theorem* expressing a *relationship* between three variables**;** and not even its generalization**.** $m\boldsymbol{a}$ is referred to as an *inertial forces***.**

"*Given a system of arbitrarily many mass points that attract each according to* **NEWTON**'s *Law, under the assumption that no two points ever collide, try to find a representation of the coordinates of each point as a series in a variable that is some known function of time and for all of whose values the series converges uniformly***.**"

In case the problem could not be solved, any other important contribution to classical mechanics would then be considered to be prize-worthy**.**

The *Prize* was finally awarded to Henri **POINCARÉ** (1854-1912), even though he did not solve the original problem**.** (The first version of his contribution even contained a serious error; for details see the artic le by **DIACU**; i.e., see **References**)**.** The version finally printed contained many important ideas, which led to the development of chaos theory**.** The problem as stated originally was finally solved by Karl Fritiof **SUNDMAN** for n = 3 (see below)**.**

20 See**:** https://www.msn.com/en-us/video/wonder/newly-discovered-galileo-letter-shows-his-attempt-to-trick-the-inquisition/vi-AAAwFBA?ocid=spartanntp**.**

21 **TRUESDELL**, Clifford Ambrose, **III** (1919-2000)**:** **Essays in the History of Mechanics**, *Springer-Verlag*, 1968**.** A must-read**.**

22 **BROOKE**, Wm. E. and **WILCOX**, Hugh B.**:** **Engineering Mechanics**, *Ginn and Co*., 1929, Chapter **XVI**, "***Dynamics***," paragraph **161**, "**D'ALEMBERT's *Principle***," pp. 249-250**.** Actually, there is a Principle for Force and one for Moments**.** One of the best textbooks on Mechanics written**.**

As for mass: *mass* of a body in a gravity field (say Earth's), gives rise to *weight*, which is the mutual force of attraction of one body to another. *Mass*, a scalar quality is just a number without direction, is either <u>*not*</u> moving (at rest), here symbolized by m_0; or moving, symbolized by m. (Technically, there is no such thing as rest mass in this Universe, as everything in it is moving, so there really is no m_0; but it is a useful device.) In the Newtonian physics' working <u>*model*</u>, *mass* is *normalized* by removing the effects of *gravity,* i.e., by separating it from its weight $\boldsymbol{W}$ by the relationship $m_0 = \boldsymbol{W}/\boldsymbol{g}$. For a freely falling body in a vacuum subject to Earth's gravity field, $\boldsymbol{g}$ is its downward acceleration, or on Earth, $\boldsymbol{g} = 9.81$ m/s², (or $\boldsymbol{g} = 32.2$ ft./sec², see next section below). $m_0 = \boldsymbol{W}/\boldsymbol{g}$ is <u>*not*</u> a definition for *mass*, its a relationship.

Scottish mathematician Colin **MacLAURIN**'s (1698–1746) 1742 formulae for the variations of $\boldsymbol{g}$ at the equator and poles are:[23]

$$\boldsymbol{g}_{\text{equator}} = \gamma(\sin^{-1} e - \eta) \quad \text{and} \quad \boldsymbol{g}_{\text{pole}} = 2\gamma(e - \eta)$$

where $\gamma = 2\pi G\rho a \dfrac{(1-e^2)^{\frac{1}{2}}}{e^3}$, $\eta = e(1-e^2)^{\frac{1}{2}}$, and where G is the constant of gravity (see below), ρ is the density of the spheroid, a its semi-major axis, and e its eccentricity. Neither **NEWTON** and **MacLAURIN** actually knew the value of G, a value that had to wait until Lord **Henry CAVENDISH**'s experimental measuring of it occurred in 1798. More about G is discussed below.

We define any rest mass by comparing it to one defined as the s*tandard-mass*, or $m_{\text{standard}} / m_{\text{unknown}} = |\boldsymbol{a}_2|/|\boldsymbol{a}_1|$, where $|\boldsymbol{a}_1|$ and $|\boldsymbol{a}_2|$ are the magnitudes of the respective accelerations of the two bodies as a result of their mutual interaction.[24] This permits the rational measurement of a mass for any body with respect to a

23 See S. **CHANDRASEKHAR**'s <u>**Ellipsoidal Figures of Equilibrium**</u>, *Yale Univ. Press*, 1969. A spinning, wholly liquid planet in the shape of a squashed ellipsoid, wherein its shape owes to gravitational equilibrium. **NEWTON** was the first to derive a dynamic equation of motion for a simple equilibrium model: an abstract, physical-toy planet (see <u>**Book III**</u>, page 288-294). Many physicists later added their efforts to this analyses. Attention here however is to **CHANDRASEKHAR**'s Chapter 1, which contains a "***Historical Introduction***," starting with **NEWTON**, followed by mathematician Colin **MacLAURIN** (b. Scotland, Feb. 1698–14 June 1746) in 1742, whose formulae for the variation of $\boldsymbol{g}$ are $\boldsymbol{g}_{\text{equator}}$ and $\boldsymbol{g}_{\text{pole}}$, and a generalized theory of **NEWTON**'s results. He showed an oblate spheroid was a possible equilibrium state for **NEWTON**'s gravity theory. **MacLAURIN**, who knew **NEWTON**, and is of the *Maclaurin series* fame, a special case of the *Taylor series*, made significant mathematical contributions.

24 See **LINDSAY**, <u>**Physical Mechanics**</u>, pg.13-16.

Standard body. Establishing this *Standard* has a long history and actually quite clever, but its history isn't germane here. The idea is the *Standard* is independent of $\boldsymbol{W}$ or $\boldsymbol{g}$; and via the Standard, masses may be compared with a platform balance or a spring balance.

Motion has meaning only in terms of *mass* m; that is to say, with any *velocity* $\boldsymbol{v}$ of *mass* m, comes *motion*, $m\boldsymbol{v}$. V*elocity* is a vector quality, and made artificially time dependent (for bookkeeping reasons) by qualifying $\boldsymbol{v}$ as a function of *time*. If *mass* m is moving (on an Earth assumed fixed),then there is a t_1 = start time, and a t_2 = finish time. By v*elocity*'s dependency on *time* makes it a function of *displacement* too, $\boldsymbol{v}$ = distance/time, and distance is function of positions.

The focus here however is *velocity magnifies rest mass* (on the assumption Earth is fixed), m_0 becomes $m\boldsymbol{v}$ with velocity, which is motion, but instead called by convention, *momentum*. (**NEWTON**'s "*motion"* is what is meant today as *momentum*. A change in *velocity* is called *acceleration*, even though it means a change in $m\boldsymbol{v}$. Any change in *momentum*, causes a *force*. Or, a change in *momentum* is effected by any *applied external force*. Position, velocity, acceleration elements may be separated and analyzed apart from mass (Kinematics); or included by default with mass coupled with *Force Laws* (Kinetics and Dynamics). Bodily contact forces are simply a push or pull; but gravitational, electrical and magnetic forces are field forces, i.e., a "*forces-at-a-distance*."

NEWTON's 2nd *Law* should be his 1st *Law*; and the 1st , which can be derived from his 2nd *Law*, should be the 2nd; and his 3rd *Law* which can also be derived from the 1st *Law* should remain the 3rd *Law*. For if $m_{\text{standard}} / m_{\text{unknown}} = |\boldsymbol{a}_2|/|\boldsymbol{a}_1|$, then equally:

$$m_1 / m_2 = |\boldsymbol{a}_2|/|\boldsymbol{a}_1|$$

or

$$m_1 |\boldsymbol{a}_1| = m_2 |\boldsymbol{a}_2|$$

But equilibrium (and common sense) requires the accelerations to be equal and opposite:

$$m_1 |\boldsymbol{a}_1| = - m_2 |\boldsymbol{a}_2|$$

hence:

$$\boldsymbol{F}_1 = m_1 |\boldsymbol{a}_1| = -\boldsymbol{F}_2 = -m_2 |\boldsymbol{a}_2|,$$

which is **NEWTON**'s 3rd *Law*: *If two bodies interact, the force exerted on body 1 by* *body 2 is equal to and opposite in direction to the force exerted on body 2 by body 1***.**

Synthesis

Galileo **GALILEI** (1564-1642) did the first gravitational experiment involving the acceleration of gravity, $\boldsymbol{g}$, and it is remarkable for its simplicity**.** Using, say, an 18-inch long, smooth hardwood 2X4 stick with a deep V-shaped groove in it, he would summarily set it at various inclination relative to a flat table**;** and roll from the top a hard marble ball down it**.** He did this incrementally for 10^0, 20^0, **…**, 80^0, and lastly, at 90^0**;** and recording each time the *time* it took for the ball to roll down the groove and onto the table**.** Obviously the last time he just dropped it**.** When he compared time recordings with inclinations he was able to show as the angle neared 90^0 it asymptotically neared the true value of $\boldsymbol{g}$**.**[25] In fact, he derived the *time-distance* relationship for falling bodies**:**

$$s = \tfrac{1}{2}\,\boldsymbol{g}t^2$$

where $\boldsymbol{s}$ is the *distance* traveled, t is the *time*, and $\boldsymbol{g}$ = 9**.**81 m/s**²** is the *acceleration of gravity***.**[26] His efforts also established the s*cientific method*[27] (and in so doing may have increased the Italian language too[28])**.** Significantly, this equation is ***independent*** of *weight* and *mass*, and that *odd* result was the real gist of his experiment**.**

NEWTON's Calculus

Then, thanks to René **DESCARTES**'s (1596-1650) analytical geometry, **NEWTON** may have graphed position, velocity, and acceleration with respect to time, say of **Galileo**'s freely falling ball**.** The sketch below illustrates this**:** with constant acceleration $= \boldsymbol{g}$, velocity varies linearly $\boldsymbol{v} = \boldsymbol{g}t$, and displacement nonlinearly, $\tfrac{1}{2}\,\boldsymbol{g}t^2$**.**

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25 See **MORN**'s 19th century test apparatus for demonstrating the *Laws* of falling bodies by means of uninterrupted inclinations, in Ernest **SPON**'s **Dictionary of Engineering**, Div.. I, "***Acceleration***," pp. 5-8, (1874)**.**

26 Let $x = \boldsymbol{v}t$ horizontal, and $s = \tfrac{1}{2}\boldsymbol{g}t^2$, the parametric forms of the equation $s = (\boldsymbol{g}/2\boldsymbol{v}^2)x^2$, which is an equation of a parabola.

27 The **Oxford American College Dictionary** (*Oxford Univ. Press*, 2002) defines the *scientific method* as "*a method or procedure that has characterized natural science since the 17th century, consisting in systematic observation, measurement, and experiment, and the formulation, testing, and modification of hypotheses*."

28 According to Clifford Ambrose **TRUESDELL** (1919-2000)**,** Leonardo da **VINCI** (1452-1519) was *not* a scientist, and he did only one experiment (with water)**.** During Leonardo time there was no word in Italian for "*experiment*," let alone for him to employ the scientific method**.** See **TRUESDELL**'s books **Essays in the History of Mechanics**, *Springer-Verlag*, 1968; and **Great Scientists of Old as Heretics in "The Scientific Method"**, *University Press of Virginia*, 1985.

After 4 second the body has fell 193.2 feet!

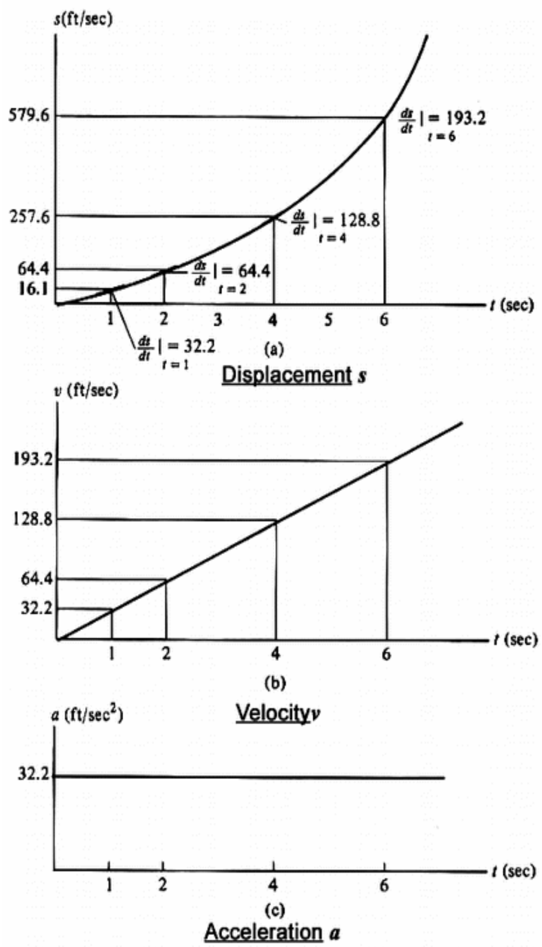

Figure **1:** Plots of a Freely Falling Body

The next figure below shows the *graphical slopes* (tangents to the curves), represented by straight lines on the displacement and velocity curves, are at common time-points, i.e., occurring at the same instant. **DESCARTES** would have written for those two common time-points equations of the form $(y_1 – y_2) = m(x_1 – x_2)$, where here m is the *tangent* value for those lines at those points.

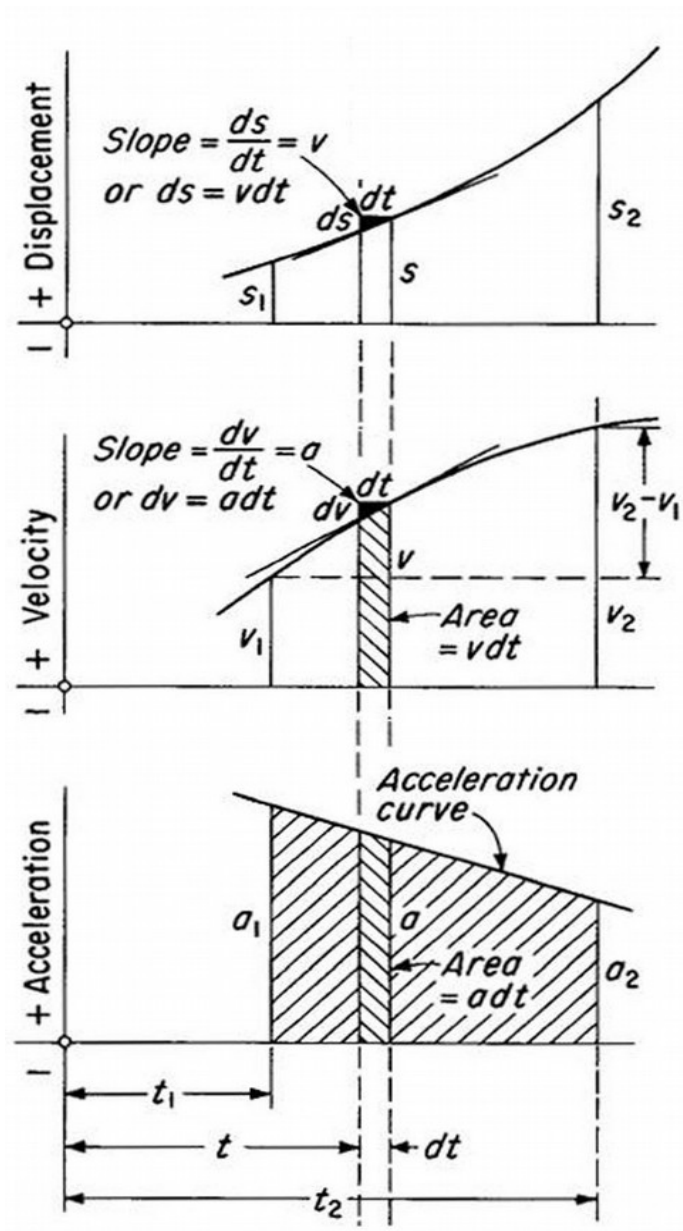

Figure **2:** Calculus Interpretations

These same tangent points are also shown in their equivalent mathematical calculus forms **–** it is a graphical interpretation of motion -to- a calculus interpretation of that graphics**.**

The average acceleration is $\bar{\boldsymbol{a}} = \Delta \boldsymbol{v} / \Delta t$, which is the *average* *rate-of-change* of the velocity with respect to time (see figure below).

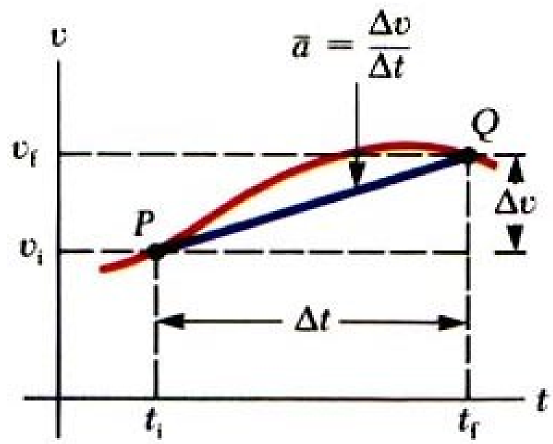

Figure **3:** Average Acceleration

Clearly, if $\Delta\boldsymbol{v}$ becomes smaller and smaller when Δt is made smaller and smaller, then, in the limit as Δt approaches zero, $\bar{\boldsymbol{a}}$ becomes the *instantaneous acceleration*:

$$\boldsymbol{a} \equiv \lim_{\Delta t \to 0} \Delta\boldsymbol{v} / \Delta t$$

Otherwise the instantaneous acceleration, and likewise the instantaneous velocity, are dependent upon the concept of a limit.[29]

NEWTON used, say, $\boldsymbol{s}$, to symbolize displacement, but in all cases the variable, whatever its symbol, is a *vector*, $\boldsymbol{s} \Rightarrow (x, y, z)$. He symbolized *instantaneous velocity* by ($\boldsymbol{v} =$) $\dot{\boldsymbol{s}}$ (reads s-dot), and ($\boldsymbol{a} =$) $\ddot{\boldsymbol{s}}$ (reads s-double-dot) for *instantaneous acceleration.*

29 See: **EDWARDS**, C. H., Jr.**:** **The Historical Development of the Calculus**, *Springer-Verlag*, 1979. The is without question one of the better accounts**.**

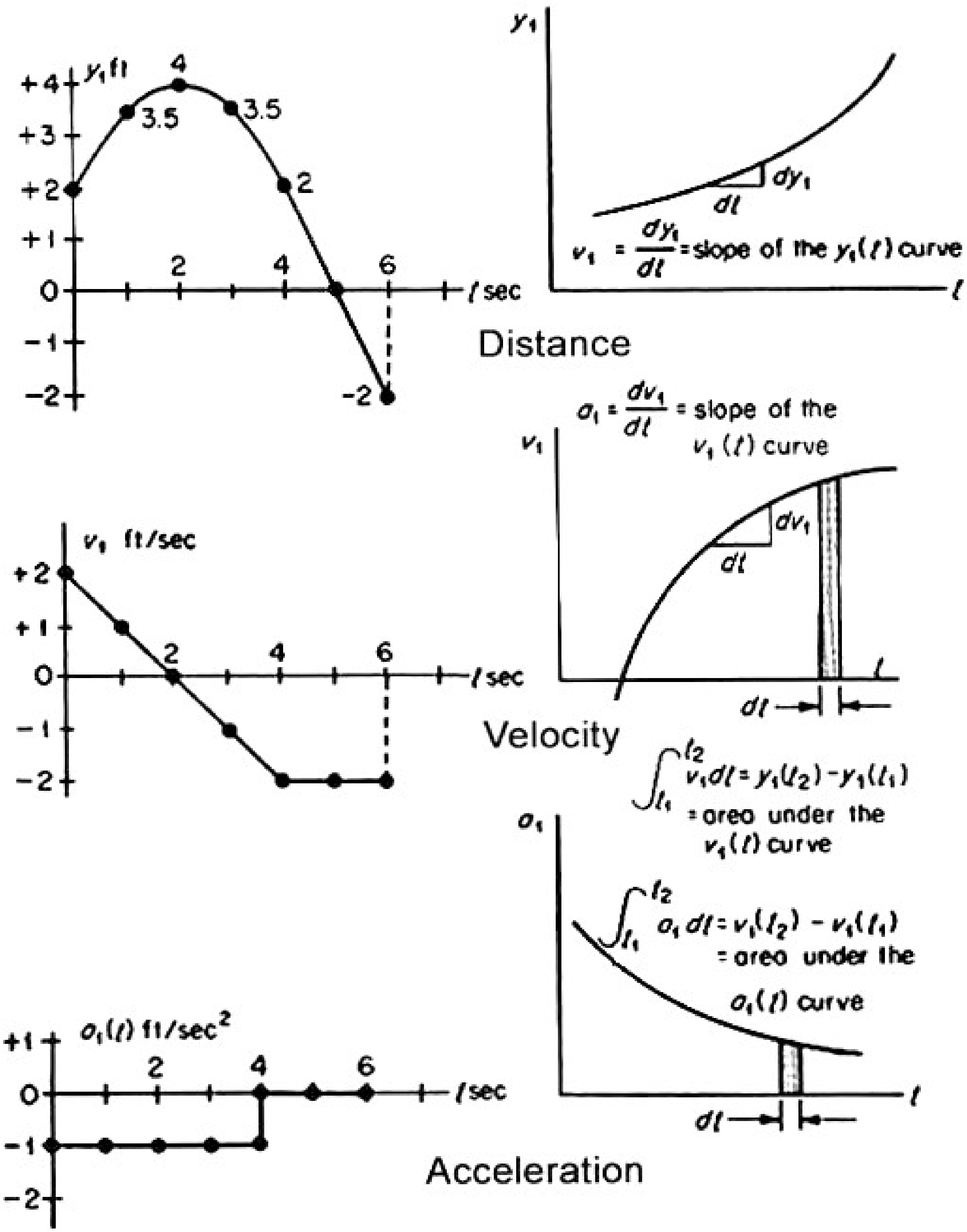

Figure **4:** Algebraic Interpretation of a real problem

NEWTON's Physics

German mathematician and philosopher Gottfried Wilhelm **LEIBNIZ** (1646-1716) used s, ds/dt and ds^2/dt^2 respectively, where d stands for *difference* and t stands for *time*. Leonhard **EULER** (1707-1783) adopted **LEIBNIZ**'s notation, which today is employed almost exclusively. But that said, **EULER**'s adoption was also a significant change in interpolating **NEWTON**'s *Laws of Motion*.

NEWTON thought mainly in physics terms, and a change in momenta was $m\Delta\boldsymbol{v}$. Whereas **EULER** thought in calculus terms – changing parameters were functions of other variables. **NEWTON**'s *time-rate-of-change* in momenta became **EULER**'s *time-rate-of-change* between variables. This was not just a change in notation – **EULER** used notation started by G. W. **LEIBNIZ** (1646-1716) – but a change in concepts too. Like $\boldsymbol{F} = m\Delta\boldsymbol{v}/\Delta t$ became $\boldsymbol{F} = md\boldsymbol{s}^2/dt^2$. From a physics perception -to- a mathematical perception. **NEWTON**'s calculus notation indicated *time-rate-of-change* in variables by dots: ($\boldsymbol{v}$ =) $\dot{\boldsymbol{s}}$ for velocity; ($\boldsymbol{a}$ =) $\ddot{\boldsymbol{s}}$ for acceleration. For example, **NEWTON**'s change in velocity $\ddot{s}$, became **EULER**'s *Eulerian* acceleration, with $\boldsymbol{a} = d^2\mathbf{s}/dt^2$.

Referring to Figure **4** above, and moving from the *top of the graph -to- bottom*, it is easy to see (now):

- The instantaneous *slope*, $\boldsymbol{v}_t = \dfrac{d\boldsymbol{y}_t}{dt}$, on the displacement curve at some *time*-point t, is the body's instantaneous velocity $\boldsymbol{v}_t$;
- And the instantaneous slope, $\boldsymbol{a}_t = {d^2\boldsymbol{y}_t}\big/{dt^2} = {d\boldsymbol{v}_t}\big/{dt}$, of the velocity curve at the same *time*-point as on the displacement curve, is the acceleration at that *time*-point too.

Similarly, it is also easily seen, moving from the *bottom -to- the top*,

- An area under the *acceleration* curve, say from t_1 -to- t_2, is equal to the difference between the start velocity and the end velocity, i.e., $\boldsymbol{v}_{t_2} - \boldsymbol{v}_{t_1} = \Delta\boldsymbol{v}_{\Delta t}$;
- The area under the *velocity* curve, from t_1 -to- t_2, is the difference between distant traveled $\boldsymbol{y}_{t_2} - \boldsymbol{y}_{t_1} = \Delta\boldsymbol{y}_{\Delta t}$ in $\Delta t = t_2 - t_1$ *time*.

Or, the mathematical version of the graph of the acceleration graph between $\Delta t = t_2 - t_1$ *times* is:

$$\Delta Area = \int_{t_1}^{t_2} \boldsymbol{a} dt = \int_{t_1}^{t_2} \left(\frac{d}{dt} \boldsymbol{v} \right) dt = \int_{\boldsymbol{v}_1}^{\boldsymbol{v}_2} d\boldsymbol{v} = \left(\boldsymbol{v}_2 - \boldsymbol{v}_1 \right) = \Delta \boldsymbol{v}_{\Delta t}$$

Multiplying the above by the mass m:

$$\boldsymbol{F} = m\int_{t_1}^{t_2} \boldsymbol{a} dt = m\int_{t_1}^{t_2} \left(\frac{d}{dt} \boldsymbol{v} \right) dt = m\int_{\boldsymbol{v}_1}^{\boldsymbol{v}_2} d\boldsymbol{v} = m\left(\boldsymbol{v}_2 - \boldsymbol{v}_1 \right) = m\Delta \boldsymbol{v} = m\boldsymbol{a}$$

Or:

$$\boldsymbol{F} = m\boldsymbol{a}$$

The *kinematics* of motion as presented graphically for a freely falling body, or any body in motion, has just been connected to **NEWTON**'s 2nd *Law***.**[30]

If $\boldsymbol{a} = ds^2/dt^2 = \boldsymbol{0}$ then the velocity is constant ($\boldsymbol{v}$ graphs horizontally)**;** and if the velocity $\boldsymbol{v} = ds/dt = \boldsymbol{0}$, then the body is at rest**.** These, top-down, bottom-up, and initial conditions *relationships* resulting from simple graphs clearly can be seen as a direct connection to calculus**.** Furthermore, the (*time*) *rate-of-change* concept is as intrinsical to the fundamentals of calculus, as it is to understanding Newtonian physics (instantaneous acceleration, etc.)**;** and vice-a-verse**.** To start the study of Newtonian physics with only kinematics considerations (position, velocity, acceleration), without the inclusions of the concept of *mass*, the *Laws of Motion*, and calculus in the analyses efforts, is a somewhat meaningless exercise because with those elements missing it misses the whole point of what Newtonian physics is about**.**[31] That is because there is no clear explanation of **NEWTON**'s Laws**.**

NEWTON didn't really describe or explain physical phenomena in a style found today in physics popularizations, or even in college texts**.** His **Principia** is more of a mathematics book, than a physics book**.** Nor did he explain his *Empirical Force Laws*, he just stated them**.** Clearly clearer is **CANNON**'s rendition (see **Appendix**

30 See**: WESTFALL**, Richard S.**:** "***Stages in the Development of Newton's Dynamics***," in **Perspectives in the History of Science and Technology**, edited by Duane H. **ROLLER**, *Univ. of Oklahoma Press*, 1971**.** Richard S. **WESTFALL** (April 22, 1924–Aug.21, 1996) was an American academic, biographer and historian of science**.** He is best known for his biography of Isaac **NEWTON** and his work on the scientific revolution of the 17th century**.**

31 Worst are the college physics textbooks without calculus**;** worst still, are first year encyclopedia-like texts (like **SEARS and ZEMANSKY**'s rewritten **University Physics**; or the **Purple Egg**)**.** Is it any wonder Newtonian physics is not understood by freshman (or undergraduates for that matter)**.** Colleges and universities have to teach mechanics three time**:** in Freshman *Physics*, sophomore *Statics-and-Dynamics*; and lastly, Junior *Engineering Mechanics***?** Most engineers don't know how to draw a free-body diagram! Isn't it better to use a simple text like Seibert **FAIRMAN** and Chester S. **CUTSHALL**'s **Engineering Mechanics**, 2nd Ed., *John Wiley & Sons*, 1938, 11th printing (!), 1958, in the Freshman year**?**

E.). **NEWTON** also failed to show the interconnectedness of these *Laws* (like the graphic example above).

Mass is *not* "*a measure of inertia*"; and *inertia* is *not* "*a quantitative measure of a body's resistance to movement*." These two phrases asserted in some textbooks as the meanings of mass and inertia are misleading: an "*inertia state*" is exactly as **NEWTON** stated it to be in his 1st *Law*, and no more (see below). In fact, and it is heresy of course, but the word "*inertia*" or "*inertia state*" should be dropped from the physicist's vocabulary just like "*violent forces*" or "*impressed forces*" have been dropped. And dropped too is **Galileo**'s discovery is his *Law of Gravitation*.

Inertia State

The concept of an *inertial state* is yet another relationship between motion and force. Of all the elementary physics concepts, *inertia* was probably the most problematic.[32] *To-resist-a-change-in-motion* is the concept of *inertia,* and only part of **NEWTON**'s 1st *Law* concept. **NEWTON**'s 1st *Law* simply put in the vernacular means: *a body can't put itself in motion nor bring itself to rest if in motion, to do so it must be acted upon by some external force*. The latter part also means a body resist a change to its motion (its inertia), but this is only half of the **NEWTON**'s 1st *Law*. Above all else these are simple statements of *fact*.

Most physics *Principles* are also just straightforward statements of facts, involving simple relationships between them.

Extracted from the **GOODSTEIN**'s book[33]:

32 See: **FRANKLIN**, Allan: **The Principle of Inertia in the Middle Ages**, *Colorado Associated University Press*, 1976. According to **FRANKLIN**, **Galileo** had the concept of inertial all wrong (which is true). Also see his essay "***The Roles of Experiment***," in ***Physics in Perspective*** magazine, Vol. 1, No. 1, pp. 35-53, March 1999, ISSN 1422-6944, published by *Birkhäuser*. **FRANKLIN** works on the history and philosophy of science, particularly on the roles of experiment in physics; unfortunately his book in some places to this reader is confusing.

33 **GOODSTEIN**, David I. And Judith R.: **Feynman's Lost Lecture, The Motion of Planets Around the Sun**, *W. W. NORTON, & Co*., "***Epilogue***," pp. 171-172, includes a CD, Indexed, 1996. See **References**.

Hamilton was part of a centuries-long tradition of refining Newton's mechanics into formulations of ever greater sophistication and elegance. For more than two hundred years after the publication of the *Principia*, the universe of Newton reigned supreme. Then, early in the twentieth century, a second scientific revolution took place in physics, almost as far-reaching as the first one. When it was over, Newton's laws could no longer be regarded as revealing the innermost nature of physical reality.

The second revolution took place on two separate fronts that have not yet, even today, been fully reconciled. One led to the theory of relativity. The other led to quantum mechanics.

The seeds of the theory of relativity can be traced as far back as Galileo's discovery that all bodies fall at the same rate regardless of mass. Newton's explanation was that the mass of a body plays two separate roles in physics: one role is to resist changes in the motion of the body; the other is to apply gravitational force to the body. Thus, the greater the mass of a body, the stronger the force of gravity on it, but also the more difficult to get it moving. Heavier bodies—falling toward the Earth, for example—have greater force on them but more strongly resist being accelerated. Lighter bodies have smaller forces but are more easily accelerated. The net effect is that all bodies fall at exactly the same rate. This peculiar coincidence was easy to accept as part of the price for the vast success of Newton's mechanics.

Niel **BOHR** said, "*It is wrong to think the task of physics is to find out how Nature is. Physics concerns* [itself with] *what we can say about Nature*."[34]

NEWTON's 1st *Law*

What **NEWTON** did essentially was mathematize known physical phenomena via three simple statements.

NEWTON's 1st *Law* addresses only linear motion. And his concept of *inertia* mathematized is distinct:

NEWTON's 1st *Law* (i.e., **DESCARTES**' *Law of Inertia*): a body at rest remains at rest (means $\boldsymbol{v} = dx/dt = \boldsymbol{0}$); or a body in motion will continue in motion with constant velocity (means $\boldsymbol{a} = dx^2/dt^2 = \boldsymbol{0}$), *unless acted upon by some external force* ($\boldsymbol{F} = m\boldsymbol{a} = m(dx^2/dt^2 \neq \boldsymbol{0}$).

The differential forms (calculus) of this *Law* owes to **EULER**, and these are *quantitative measurements.*

A better way of expressing **NEWTON**'s 1st *Law* is **CANNON**'s *conservation of*

34 See: Niels **BOHR:** **Atomic Physics and Human Knowledge**, *John Wiley & Sons, Inc*., 1958. A collection of his essays.

momentum approach: *the momentum of a particle is constant* ($m\boldsymbol{v}$ *and* $\boldsymbol{H}_o$ = *constant*) *in the absence of external forces* (period). (see **Appendix: Law of Motion**: **CANNON**'s modern way[35])

Corollary to (**CANNON**) 1st *Law*: *the linear motion and angular momentum of systems of particles* (*such as rigid bodies*) *is thus also constant in the absence of external forces*. Note here **CANNON** includes angular momentum.

Following **CANNON** then, a body may or may not have momentum: its (in the vernacular) *inertia* (its inertia state): thus it always resists change whatever its inertia state is.

NEWTON's *2nd Law*

NEWTON's 2nd *Law* (is a theorem): *A particle acted upon by a force moves such that the force vector is equal to the time rate of change of the linear momentum vector*:

$$\vec{\boldsymbol{F}}_{\text{inertia}} = \frac{d\vec{\boldsymbol{p}}}{dt} = m\frac{d\vec{\boldsymbol{v}}}{dt} = m\vec{\boldsymbol{a}},$$

where:

- $\vec{\boldsymbol{F}}_{\text{inertia}}$ is an *inertial force*, i.e., a *contact* (*vector*) *force* (rather than a field force, like gravity[36]); or, simply a push or pull force; and *inertial*, because it is a function of *mass*; and also a function of the *time rate of change* of velocity;

35 See **CANNON**'s **Dynamics of Physical Systems** (listed in **References**); or **Appendix: Law of Motion.**

36 In American physicist and teacher Peter Gabriel **BERGMANN** book, **Basic Theories of Physics, Mechanics and Electrodynamics**, (*Dover*, 1962), in Part **1**, "***Classical Mechanics***," Chapter **1**, "***Mechanics of Mass Points***," paragraphs **1.1** and **1.2** (pp. 3-17), declares all *inertial forces* are of the form $(m\ddot{x}_i)_n = -\left(\frac{\partial V}{\partial x_i}\right) + (F_i)_n$, where $\frac{\partial V}{\partial x_i}$ is the *potential force* for n = 1, 2, 3, … n bodies (i.e. a field force); and where for the external force experienced by the n^{th} particle by $(\boldsymbol{F}_i)_n$, the acceleration $(\ddot{x}_i)_n$ of the n^{th} particle is given by his above equation. In simpler notation he is saying $m\boldsymbol{a} = -\partial V/\partial \boldsymbol{r} + \boldsymbol{F}$. Otherwise, "*the total force acting on any one particle is the vector sum of all the* [field] *forces acting simultaneously on that particle – the internal forces owning to the other members of the system, and the external force*." Since he is using *index notation* – means the variables don't use *vector notation* (**bold**-italic letters). His third paragraph, **1.3**, for those interested, is titled "***Planetary motion***," and where he assumes the potential force $\partial V/\partial \boldsymbol{r}$ is equal to **NEWTON**'s *Law of Universal Gravitation* force $\boldsymbol{F}_{\text{gravity}}$. The mass *Equivalence Principle* relationship is discussed below. His assumption $\partial V/\partial \boldsymbol{r} = \boldsymbol{F}_{\text{gravity}}$ is proved in a separate **Monograph** (*pdf* file).

- $\vec{p} := m\vec{v}$, where the lower-case letter "$\boldsymbol{p}$" is the *linear momentum* (the modern words for **NEWTON**'s linear "*motion*")[37]**;** and, the right side of the relationship represents a spatial geometry mapping (this is discussed below)**.**

NEWTON's math model, $F = m\Delta\mathbf{v}/\mathbf{\Delta}t$, *only represents the forces associated with a* *__particle__*, not a rigid body**.** If a force acts on a particle, the change in motion (momentum) is directly proportional to that force and in the same direction**.** Stated somewhat differently, the vector change in momentum of a particle owning to an applied force is proportional to the impulse of that vector force, $\int \boldsymbol{F}dt$.

An aside**:** later Jean Le Rond **D'ALEMBERT**'s (1717-1783) generalized **NEWTON**'s 2nd *Law* – a subject discussed later**.** Read on**.**

NEWTON's 3rd *Law*

NEWTON's 3rd *Law* (a *Corollary* to Simon **STEVINUS**' (1548-1620) *PARALLELOGRAM Law*)**:** *Every collinear action and reaction between uncharged*[38]*, paired particles is equal and opposite*, $\boldsymbol{F}_{ij} = \boldsymbol{-F}_{ji}$

__Corollary__ to the 3rd *Law* (besides **STEVINUS**')**:** the (rigid) collision of two particles results in equal but opposite changes in their momenta (and thus in velocity changes inversely proportional to their masses)**.**

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37 German physicist and teacher Arnold **SOMERFIELD** textbook, now classic, **__MECHANICS, Lectures On Theoretical Physics__** (Vol. **1**, *Academic Press*, 1952), in Chapter 1, "***Mechanics of a Particle***," paragraph **I**, "***Newton's Axioms***," presents a clear, elegant presentation of **NEWTON**'s *Laws of Motion* from a *momentum* perspective**.** He was over 70 years of age when he wrote his now famous three volumes on physics**.**

38 Exceptions are electromagnetic forces between moving particles**.**

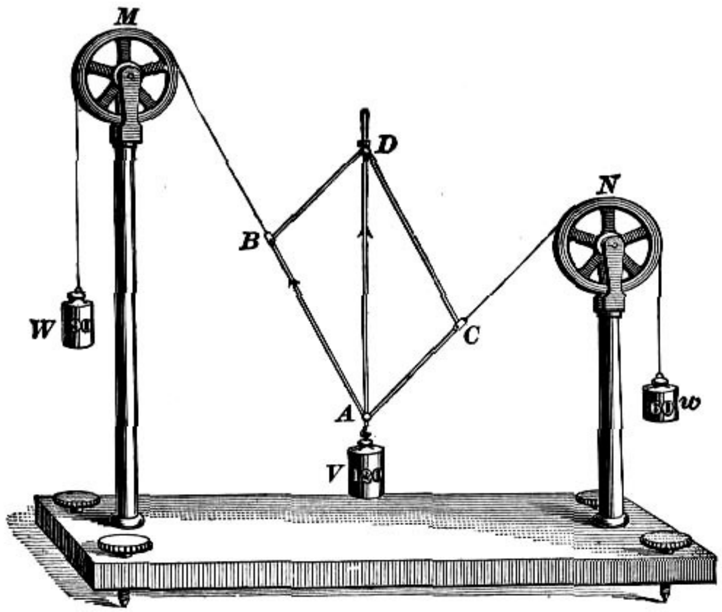


Figure **5:** Demonstration of the *Parallelogram Law*

Work and Energy

Consider now "*work done*," denoted by W**.** The measure of the net *work done* W by or against a constant, resultant vector force $\boldsymbol{F}$ is the arithmetic product (dot-product) of the force and the linear vector displacement $\boldsymbol{s}$, *in the direction of the force*, starting from its point of application**.** In symbols**:** $\boldsymbol{Fs} = W$**.** W is a scalar**.**

It seems a reasonable relationship but it is two separate phenomenon**:** an in-line push or pull *force* moves a body through a *geometric* distance $\boldsymbol{s}$**.** Work can also represent the *energy*, E, in the system during the application of $\boldsymbol{F}$**.** A particular energy E associated with motion[39] is kinetic energy, K**.**

The *Principle of Work* for a particle states that *the net work done by the forces acting upon it is equal to the increase in its kinetic energy***.** Kinetic energy when defined in terms of work is**:**

$$\boldsymbol{F} \cdot \boldsymbol{s} = W = E_k = K := \tfrac{1}{2} m v_2^2 - \tfrac{1}{2} m v_1^2,$$

where the body's kinetic energy has increases from $\boldsymbol{v}_1$ -to- $\boldsymbol{v}_2$ when the body is

39 Another closely related subject is impact energy, which is non-Newtonian owning to the *coefficient of restitution* properties of the impacting masses, is not cover here; but see the *excellent* article "***Equations and charts for analyzing Impact between masses***," by Dr. Karl W. **MAIER**, ***Product Engineering***, April 29, 1963**.** His 11-page paper cover comprehensively the basic physical phenomenon of impact**.** His article and more are in **Spring Design and Application**, *McGraw-Hill*, 344 pages, 1961**.**

subjected to a force acting in the direction of the initial displacement $\boldsymbol{s}$. Otherwise, the resultant external force's work applied to a body is equal to the change in the kinetic energy of the body. The *Principle* can be extended to any system of particles, meaning the net work done by both internal and external forces of the system is equal to the increment in the overall kinetic energy of the system.

Now $\boldsymbol{F} = m\boldsymbol{a}$ and from the above:

$$\boldsymbol{F} = m\frac{\boldsymbol{v}_2^2 - \boldsymbol{v}_1^2}{2\boldsymbol{s}}, \text{ where } \boldsymbol{a} = \frac{\boldsymbol{v}_2^2 - \boldsymbol{v}_1^2}{2\boldsymbol{s}}.$$

Or:

$$\boldsymbol{v}_2^2 = \boldsymbol{v}_1^2 + 2\boldsymbol{s}\boldsymbol{a}$$

Now distance, velocity and acceleration (kinematics) are connected to work and energy; and specifically, to kinetic energy, the energy of motion.

Conservation Laws

By ***NEWTON***'s 3rd *Law*[40]:

$$\boldsymbol{F}_{ij} = -\boldsymbol{F}_{ji}$$

Expanded:

$$\sum_i F_i = \sum_i \left(m_i \frac{d^2 x_i}{dt} \right) = \frac{d}{dt} \sum_i \left(m_i \frac{dx_i}{dt} \right) = 0$$

which is a statement of the conservation of linear momentum. This echoes **NEWTON**'s 1st *Law*: *the momentum of a particle is constant ($m\boldsymbol{v}$ = constant) in the absence of external forces.*

Angular Momentum

40 **TRUESDELL**, Clifford Ambrose, **III: Essays in the History of Mechanics**, *Springer-Verlag*, 1968.

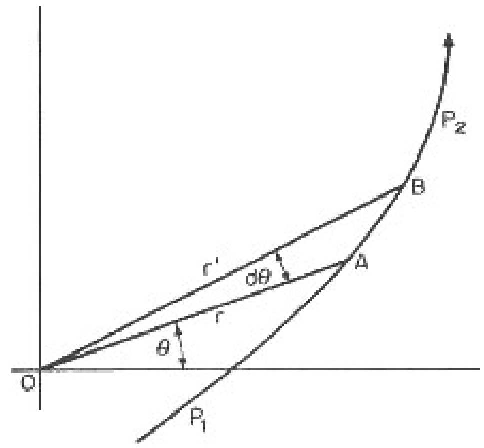


Figure **6:** Angular Momentum

Using the above figure, note**:**

$$r^2 = x^2 + y^2$$

And**:**

$$\theta = \arctan\left(\frac{y}{x}\right)$$

$$d\theta = \frac{xdy - ydx}{x^2 + y^2}$$

$$\frac{1}{2}r^2 d\theta = \frac{1}{2}(xdy - ydx)$$

Then**:**

$$M_i = m_i r_i^2 \frac{d\theta_i}{dt} = m_i \left(x_i \frac{dy_i}{dt} - y_i \frac{dx_i}{dt} \right).$$

where m_i are the masses**.**

Or, the total angular momentum M_i in expanded scalar form is**:**

$$\boldsymbol{M} = \sum_i m_i r_i^2 \frac{d\theta_i}{dt} = \sum_i m_i \left(x_i \frac{dy_i}{dt} - y_i \frac{dx_i}{dt} \right). \quad \text{..................Eq..1.1}$$

Eq.(1**.**1) would be differentiated incorrectly if ***r*** too changed with time!

Let**:** $\cos\theta = \frac{(x_i - x_j)}{r_{ij}}$ and $\sin\theta = \frac{(y_i - y_j)}{r_{ij}}$. Then**:**

$$R_{xj} = F_{xj}\cos\theta \quad , \quad R_{yj} = F_{yj}\sin\theta$$

$$x_i R_{ij} - y_i R_{ij} + x_j R_{ji} - y_j R_{ji} =$$

$$= F_{ij}\left[\left(x_i - x_j\right)\sin\theta - \left(y_i - y_j\right)\cos\theta\right]$$

$$= F_{ij} r_{ij}\left(\cos\theta\sin\theta - \sin\theta\cos\theta\right) = 0$$

But, by ***NEWTON***'s 3rd *Law*, $F_{ij} = -F_{ji}$, therefore**:**

$$\frac{dM}{dt} = 0$$

And M = constant**.** The above is a statement of the *conservation of angular momentum*, and echoes the *Corollary* to the 1st *Law* above**:** *the linear and angular momentum of systems of particles* (*such as rigid bodies*) *is thus also constant in the absence of external forces***.**

The value of *conservation* concepts lies in their generalities**.**

Some Mathematical Modeling Attributes of Gravity

Equivalence Principle

The Equivalence Principle is such an important aspect of basic physics (*Newtonian* as well as *Mechanics*), and odd the subject is not discussed in the standard college or university textbooks. The *Equivalence Principle* was demonstrate in a famous 1909 physics experiment by the Hungarian physicist Baron Von Roland (actually Loránd) **EÖTVÖS** (1848-1919), and it involves the concepts of inertia mass verses gravitational mass**:**

- *Inertial mass* is represented by **NEWTON**'s 2nd *Law*, $\boldsymbol{F} = (m_{\text{inertia}})\boldsymbol{a}$;

- *Gravitational mass* is represented by **NEWTON**'s *Law of Universal Gravitation,*

$$\vec{\boldsymbol{F}} = -G\frac{m_1 m_2}{r^3}\vec{\boldsymbol{r}},$$

Simply put the *Equivalence Principle* is $\boldsymbol{F}_{\text{inertial}} \equiv \boldsymbol{F}_{\text{gravitational}}$ – meaning the mass in **NEWTON**'s *Forces Laws* are, for all practical purposes[41], the same as the masses in his *Gravitation Law*.

This physically means, as an illustration, that in a windowless, freely falling laboratory embedded in a uniform gravity field, the laboratory experimenters would be unaware if the laboratory was in a state of nonuniform, ever increasing velocity when subject to an external, constant acceleration. All dynamical experiments yield the same results as obtained in an *inertial state* of rest, or of uniform motion when unaffected by gravity. Otherwise**:**

*Equivalence Principle***:** *There is no difference between inertial mass and gravitational mass*.

EÖTVÖS also demonstrated with great exactitude in 1909**:** *the attraction of bodies is independent of the quality of their substances, and therefore inertial mass and gravitational mass are the same for different materials*. This was something long suspected earlier, but never demonstrated with any accuracy.

The earliest experiments were done by **NEWTON** and improved upon later by

41 Robert H. **DICKE** with P. G. **ROLL** and R. **KROTKOV** re-ran the **EÖTVÖS** experiment using improved apparatus, consequently improved the accuracy of the measurements to 1 in 100 billion. See the Internet.

Friedrich Wilhelm **BESSEL** (1784-1846).[42] Already alluded too, **EÖTVÖS**' more accurate experiment using a torsion balance was carried out by starting around 1885, followed by further improvements in a lengthy run between 1906 and 1909. **EÖTVÖS**' team followed this with a series of similar but more accurate experiments, as well as experiments with different types of materials and in different locations around the Earth, with all demonstrating the same equivalence in mass.

His experiments led to the modern understanding of **EINSTEIN**'s *equivalence principle* embodied in *general relativity*, which essentially states there is no "*gravitational mass*," only *inertial mass*. His *Principle* asserts in free-fall the effect of gravity is totally abolished in all possible experiments ($\boldsymbol{g} = \mathbf{0}$); and general relativity reduces to special relativity, as in the *inertial state*.[43]

NEWTON's *Law of Universal Gravitation*

NEWTON's (or maybe **BULLIALDUS**') very powerful *Law* is expressed in some textbooks in *scalar form* as:

$$F = -G\frac{m_1 m_2}{r^2},$$

where F, a scalar, is, by **NEWTON**'s 3rd *Law*, the force of attraction (an the reason for the negative sign, see below) between the two masses. **NEWTON** proved in his **Principia** a spherically-symmetric body can be modeled as a mass-point. Hence the two mass bodies, m_1 and m_2, attracted to each other are each assumed reduced to mass-points; and r, a scalar, is the centroidal distance between these mass-points.

In most other textbooks it is expressed in *vector form* as:

$$\vec{\boldsymbol{F}} = -G\frac{m_1 m_2}{r^3}\vec{\boldsymbol{r}},$$

where both $\vec{\boldsymbol{F}}$ and $\vec{\boldsymbol{r}}$ are *vectors* having both *magnitude and direction* in some pre-defined Euclidean coordinate system. Arrow notation is employed to dramatize it is a *vector equation*. $\vec{\boldsymbol{r}}$ is the radius vector with origin at m_1 and points in the positive direction from m_1 to m_2. Since $\vec{\boldsymbol{F}}$ is a force of attraction it points in the direction

42 **CAPRIA**, Marco Mamone: **Physics Before and After Einstein**, *IOS Press*, pg. 167, 2005. ISBN 1586034626.

43 See the "***Equivalence Principle***" in Vol. **4** of the ***Micropedia***, pp. 535, of the 1988 Edition of the **Encyclopædia Britannia**.

from m_2 to m_1; and, by being oppositely directed to $\vec{\boldsymbol{r}}$, is negative. The *magnitude* of the radius vector is $r = |\vec{r}|$. The force on the right side is gravitational and proportional to the masses involved, and varies as the inverse square of the distance between these masses. The power in the denominator is three instead of two, to balance the vector difference in the numerator, which is used to specify the direction of the force.

The right-hand side of the formula is the net force $\boldsymbol{F}_{\text{gravitational}}$ on a particle relative to another particle; whereas the left-hand side can be set equal to the mass times acceleration for a particular j^{th} particle ($\boldsymbol{F}_{inertial} = m\boldsymbol{a}$ in vector formula), which is **NEWTON**'s 2nd *Law*. **EULER** actually transformed **NEWTON**'s 2nd *Law* equation – of changing momentum assumed equivalent to net force – into an ordinary, second order differential equation. These two forces can be set equal to one another by assuming the *Equivalence Principle*.

To see an application of these two vector formulae see the **Two-body Problem** below.

The gravitational *potential* corresponding to **NEWTON**'s *Universal Law* is**:**

$$V = -G\frac{m_1 m_2}{r},$$

This means it is also possible to fine the attraction between objects by first defining the potential and then using $F = -\nabla V$ (this is discussed in more detail below).

G, The Constant of Proportionality

G is special. G, the *constant of proportionality*, is the parameter making **NEWTON**'s *Law* empirialistic (i.e., capable of being verified or disproved by observation or experiment)**.** Remember, **NEWTON** actually didn't know the value of G and its value had to wait until Lord **CAVENDISH**'s experimental measuring of it in 1798[44]**.**

G allows mass (matter phenomenon) when combined with distance (spatial phenomenon) to be linked to "*force*" phenomenon**.** All three phenomena have

44 He measured the gravitational attraction between two large balls of lead and a pair of very small ones suspended on threads**.**

different units of measurement. It means **NEWTON**'s equation is *empirical* because G is empirical; but $(m_1 m_2 / r^3)\boldsymbol{r}$ is not a force and so $G(m_1 m_2 / r^3)\boldsymbol{r}$ is not equal to $\boldsymbol{F_g}$ (gravitational force) **–** but only *equivalent* to it (and that only by assuming we know what "*force*" is: in this case it is a *field force*[45,46]). His equation establishes only an equivalence relationship via G: the equation maps the mass and spatial phenomena -to- force. In the language of mathematics *force* and the *potential function* are functionals.

G allows **NEWTON**'s 2nd *Law*, and all other *Laws* of Mechanics, not to require incorporating their own constants of proportionality.

Remarkably the first accurate measurement of G, and therefore its validity, did not occur until Lord **CAVENDISH**'s experiment [47] in *1798*, 71 years after **NEWTON** had died! $\boldsymbol{F}_{\text{gravitational}} = \boldsymbol{F_g}$ in **NEWTON**'s time required a great leap in faith for physicist, including even **NEWTON**, in accepting his ***Theorem***. And **CAVENDISH**'s experiment also really dependents upon *comparing* *torsional force* resulting from torque, to the *force at a distance* (*attraction*). Further, it assumes these two *forces* are *equivalent*. It also proves G *is a physical entity*. His experiment also requires a dash of faith. Not that anyone every did, but neither *torsional force*, nor *gravitation attraction* can define "*force*" by this experiment.

Laws of Weight

A direct application of **NEWTON**'s *Law of Universal Gravitation* is the *Laws of Weights*. **NEWTON**'s *Universal Law* is generally considered as mainly applying to celestial objects, but is equally useful for Earth science (calculating Ocean tides, calculating the 13-mile high ellipsoidal budge at the equator, calculating the fuel requirements verses weight for rockets, etc.). For those applications there is the "*Laws of Weight*."[48] (see Appendix **D**), which is a modified form of his equation. Another odd phenomenon about gravity is expressed by the latter *Law*. It says bodies weigh most at the surface of the Earth. Below the surface the weight

45 See M. **JAMMER**'s **Concepts of Force**, *Harvard University Press*, 1957. Excellent discussion. Download book at https://ia600600.us.archive.org/18/items/ConceptsOfForce/Jammer-ConceptsOfForce.pdf

46 **WESTFALL**, Richard S.(1924-1996): **Force in Newton's Physics: The Science of Dynamics in the Seventeenth Century**, *Elsevier*, 1971. This is good background material.

47 **TAYLOR**, Barry N., **LANGENBERT**, Donald N. and **PARKER**, Wm. H., "***The Fundamental Physical Constants***," ***Scientific American***.

48 **The Elements of Mechanical And Electrical Engineering**, Volume **I**, Chapter "***Elementary Mechanics***," pp. 318-319, *The Colliery Engineer Co*., 1898.

decreases as to the *linear decrease* of the distance towards the Earth's center**;** above the surface the weight *decreases as to the square* of the distance increases (**NEWTON**'s *Law*)**.**

*Law of Weights***:** Let $\boldsymbol{R}$**:** = Earth's radius = 4000 miles; $\boldsymbol{W}$**:** = surface weight (Lb.); $\boldsymbol{w}$**:** = weight above or below surface (Lb.); $\boldsymbol{r}$**:** = distance between body centroid and Earth's center, in miles**.** Then**:**

Below the surface weigh decreases as the distance to the center decreases,

Below**:** $\boldsymbol{wR} = \boldsymbol{Wr}$,

or, $\boldsymbol{R} : \boldsymbol{r} :: \boldsymbol{W} : \boldsymbol{w}$**.**

Above the surface the weight decreases as to the square of the distance increases,

Above: $\boldsymbol{wd}^2 = \boldsymbol{WR}^2$,

or, $(\boldsymbol{r} : \boldsymbol{R})^2 :: \boldsymbol{W} : \boldsymbol{w}$**.**

As an example, the top of Mt. Hercules is 32 thousand feet, or about 6**.**1 miles above sea level**.** If a mountaineer weighs 165 pounds at sea level, what would she weigh on top of Mt. Hercules?

$$\boldsymbol{W} = (4000^2 \times 165) / 4006.1^2 = 164.5 \text{ Lb.}$$

Ismaël **BULLIALDUS** (1605-1694) must have known about the *Law of Weights* (*before* **NEWTON**)**.** To this point, in his principal work **Astronomia philolaica** published 1645, he proposed the *force of gravity* follows an *inverse-square law***:**

"As for the power by which the Sun seizes or holds the planets, and which, being corporeal, functions in the manner of hands, it is emitted in straight lines throughout the whole extent of the world, and like the species of the Sun, it turns with the body of the Sun. Now, seeing that it is corporeal, it becomes weaker and attenuated at a greater distance or interval, and the ratio of its decrease in strength is the same as in the case of light, namely, the duplicate proportion, but inversely, of the distances that is, $1/\boldsymbol{r}^2$**.**"

Sir Isaac **NEWTON** later claimed to have proved this idea mathematically in the **Principia**, published in 1687**.** In *Phenomenon* 4, Book 3, he only praises

BULLIALDUS' accurate tables, but does not give him any credit for his *inverse-square law* concept.

Conservative Forces

As shown, the *potential energy* is closely linked with forces.[49] If the work done moving along a path, which starts and ends in the same location, is zero, then the force is said to be *conservative* and it is possible to define a numerical value of potential associated with every point in space. A force field can be re-obtained by taking the vector gradient of the potential field.

Gravity is a *conservative force*[50]. The *work done*, W, by a unit mass going from point A with $W = a$ to point B with $W = b$ by gravity is $(b - a)$ and the work done going back the other way is $(a - b)$, so the total work done from A -to- B -to- A is:

$$W_{A \to B \to A} = (b - a) + (a - b) = 0$$

If we redefine the potential at A to be $a + c$ and the potential at B to be $b + c$, where c can be any number, positive or negative, but it must be the same number for all points, then the work done going from:

$$W_{A \to B} = (b + c) - (a + c) = b - a.$$

In practical terms, this means you can set the zero of W anywhere you like. You might set it to be zero at the surface of the Earth, or you might find it more convenient to set it zero at infinity. The work done going from A -to- B for conservative forces does not depend on the route taken. If it did, then it would be pointless to define a potential at each point in space.

Gravitational Potential Energy

A body of mass m and weight $\boldsymbol{w} = m\boldsymbol{g}$, moves up vertically from a point where its center of gravity ($c.g.$) is initially at a height z_1 above an arbitrarily chosen reference plane, to a point at a height z_2. ($z_2 > z_1$). The downward gravitational force on the body is its weight $\boldsymbol{w}$. Let $\boldsymbol{F}$ be the resultant of all other lifting forces acting on

49 See **BATTIN**'s book listed in the **References:** Chapter 2, pp. 97-99, "***Potential Functions***," for a discussion about the gravitational potential and the force function $\boldsymbol{F}^{\mathsf{T}} = m(\partial V/\partial \boldsymbol{r})$ as a row vector.

50 An example of a non-conservative force is friction. With friction, the route you take does affect the amount of work done, and it makes no sense at all to define a potential associated with friction.

the body, and W be the work of those forces. The direction of gravitational force $m\boldsymbol{g}$ is opposite the upward displacement, so the work of $\boldsymbol{F}$ is negative:

$$W_{\text{grav}} = -m\boldsymbol{g}(z_2 - z_1)$$

Since the total work W equals the change in kinetic energy, $W + W_{\text{grav}} = \Delta E_k$, and therefore:

$$W = \left(\tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2\right) + \left(-m\boldsymbol{g}(z_2 - z_1)\right)$$

Some Background About Gravity

Some background references are given below about gravitational phenomena before jumping directly into the other subjects following, such as the **Two-body Problem** and **Three-body Problem** applications; or the general *n-body problem*'s algorithm, and its example problem.

As an introduction into this subject read physicist George **GAMOW**'s (b. Russia, 2/20/1904–8/19/1968) **Gravity**, *Science Study Series,* **S22***,* pocketbook, *Doubleday Anchor Original*, 157 pages, 1962. A well written popularization written at a *YA* level, covers, in a non-mathematical way, all the basics, is now a classical and still enjoyable to read.

GAMOW's wrote other science popularizations[51], and his *Mr. Tompkins*, a main character in his popularizations, served as his device for bringing concepts in higher mathematics and physics to the general public (shades of **Galileo**).

Additional introductory background reading includes "***GRAVITATION***" in Vol. **10**, pp. 711-727 (~ 68 std. pages), of the 1967 or 1968 *Editions* of the **Encyclopædia Britannica**. This is a well-written overview on the subject, written by five (impressive) experts in the field: Thomas Frohock **GASKELL;** Wm. **BULLERWELL;** Alan Hugh **COOK;** Herman **SHAW;** and Ralph D. **WYCKOFF.** Read especially Section **I**, "***Introduction***"**;** and Section, **IV**, "***Theories of Gravitation***." Also take note of the article's references.

Equally well-written but slightly shorter (~ 36 std. pages) but more

51**GAMOW** also wrote **One, Two, Three...Infinity** (*Viking Press*, 1947, rev. 1961; *Dover*, 1974); and several ***Scientific American*** essays. He wrote several technical papers and one technical book, **Atomic Nucleus** (*Oxford Univ. Press*, 1931, revised 1949).

mathematical, is "***GRAVITATION***," in Vol. **20**, pp. 286-295, in the 1988 *Macropædia Edition* of the **Encyclopædia Britannica**. Their account of "***Newton's Law of Gravity***" is extremely interesting**.** Written by Kenneth I. **NORDTVEDT**, Jr.**;** James E. **FALLER;** and author Jesse W. **BEAMS.**

Another is **The Riddle of Gravitation**, Revised and Updated, by Peter G. **BERGMANN**, *Dover*, 1992. A classic**.**

Lastly is "***Gravitation Theory***," ***Scientific American***, by Clifford M. **WILL.**

In all of these not one word is said about the *mechanism of gravity***.**

A Mechanism For Gravity

One physical phenomenon **NEWTON** had to contend with (and everyone else) was how *odd* it is that *all bodies fall at the same rate*, *regardless* of size, mass, weight, volume**:** the distance x traveled by all falling bodies are *strictly* a function of ½ the *time* t squared times the acceleration of gravity $\boldsymbol{g}$, which owns to the force of gravity $\boldsymbol{F_g} = m\boldsymbol{g}$**.** This is **Galileo**'s *Law of Gravity***.**

$$x = \tfrac{1}{2}\boldsymbol{g}t^2$$

It is conjectured ***Galileo****'s Law's of Gravitation is the result of every atom's electromagnetic energy* **–** *energy emanating from the nucleus, which is just sufficient to barely keep the valence electrons in their orbits* **–** *energy passes beyond the valence electron's orbital boundary (shell), and thus weakly attracts other nearby electrons in other, sometimes non-contacting, atoms or molecules.* This is one of the *mechanisms of gravity* and this *by-pass or residue* attraction for other electrons is not a chemical bond per se**.** Also note **Galileo**'s *Law* is not a function of mass (and, as will be shown, neither is the generalization of **NEWTON**'s *Law of Universal Gravitation*)**.**

The attraction is or may be similar to the observable **van der WAALS** type forces (see the separate pdf file**:** "***van der WAALS Forces***")**.**

It is further surmised**:**

*Earth's mechanism of Gravity is the manifestation of the resulting sum of the total nuclei attractive energy of the Earth's by-pass electromagnetic flux***.**

This residue attraction results in one of the *mechanism of gravity***.** As a result, Earth's gravity field pulls on all the electrons attached to all the nuclei of every body equally, regardless of mass, weight, shape, etc.; and all bodies are pulled down equally at the same rate.

Experimental evidence via **ROTHEN**'s experiments, coupled with the observations of **van der WAALS**' forces, does provide more than a sufficient reason for the above conjectures; i.e., for the *mechanism of gravity*. The following related issues therefore, based on these ramifications of the mechanism of gravity implied above, is examined:

- The odd consequential equation form, $f(1/r)$, of **NEWTON**'s *Universal Law of Gravitation* for bodies below the Earth's surface. See appendix **Laws of Weights** below.
- The *n-body problem* algorithm allows complex molecular, **van der WAALS** type forces to be determined; and is an application of same to physical chemistry (see the last part of the **The Two Body Problem** section below).
- **NEWTON**'s *Law of Universal Gravitation* and **COULOMB**'s *Law*[52] can be linked together via the potential functions $V = f(m/\mathbf{r})$ and $V = f(q/\mathbf{r})$. A discussion of this subject follows below.
- The *n-body problem* algorithm's derivation below clear shows the distribution of forces via a *n-body problem* solution caused from an applied load, *is not a function of mass*, works; contrarily, the classical approach employing **NEWTON**'s 2nd *Law* combined with his *Universal Law of Gravitation* *is the wrong approach*: employing this latter approach makes the general *n-body problem* with $n \geq 3$ become *intractable*.

The Gravitational Potential Function

One direction of progress historically in physics following **NEWTON**'s efforts was the introduction of potential theory. Potential theory and its functions are based on earlier concepts of Newtonian physics as well as later concepts of kinetic energy, potential energy and work. In general, several different quantities used in physics are

52 **COULOMB**'s *Law*: the energy E obtained in bringing two ions with electric charges $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ from infinite separation to a distance r apart is $E = K\,\boldsymbol{q}_1\,\boldsymbol{q}_2/r$; and the vector force is $\boldsymbol{F} = (K\,\boldsymbol{q}_1\,\boldsymbol{q}_2/r^3)\mathbf{r}$.

all characterized by their connection with potential energy and the potential function.[53] One of the simplest is the *gravitational potential*, V**:** = $m/\boldsymbol{r}$**.** It allowed both practical as well as theoretical investigation of the gravitational variations in planetary motions, and in the anomalies owning to the irregularities and shape deformations of Earth, to be investigated**;** and more**.**[54]

Boardly speaking, potential theory for a particle**−** like $V(m, \boldsymbol{r}) = m/\boldsymbol{r}$ for gravity and $V(q,\boldsymbol{r}) = q/\boldsymbol{r}$ for electricity **−** serve as its potential functions, and is a means to describe different vector field effects when using **GAUSS**' *Law***.**[55] This Section suggests a way of connecting a *bridge* **−** via the *potential function* **−** between mass-gravitational phenomenon -and- charge-electromagnetic phenomenon**.**

The *gravitational potential* is also sometimes called the mass potential or the *potential function* for mass**.** Its usefulness and purpose is in *connecting* the gravitational potential -to- **NEWTON**'s *Law of Universal Gravitation;* and the rest of the world of potential theory too (see section ref. **PEIRCE**, B. O.)**.**

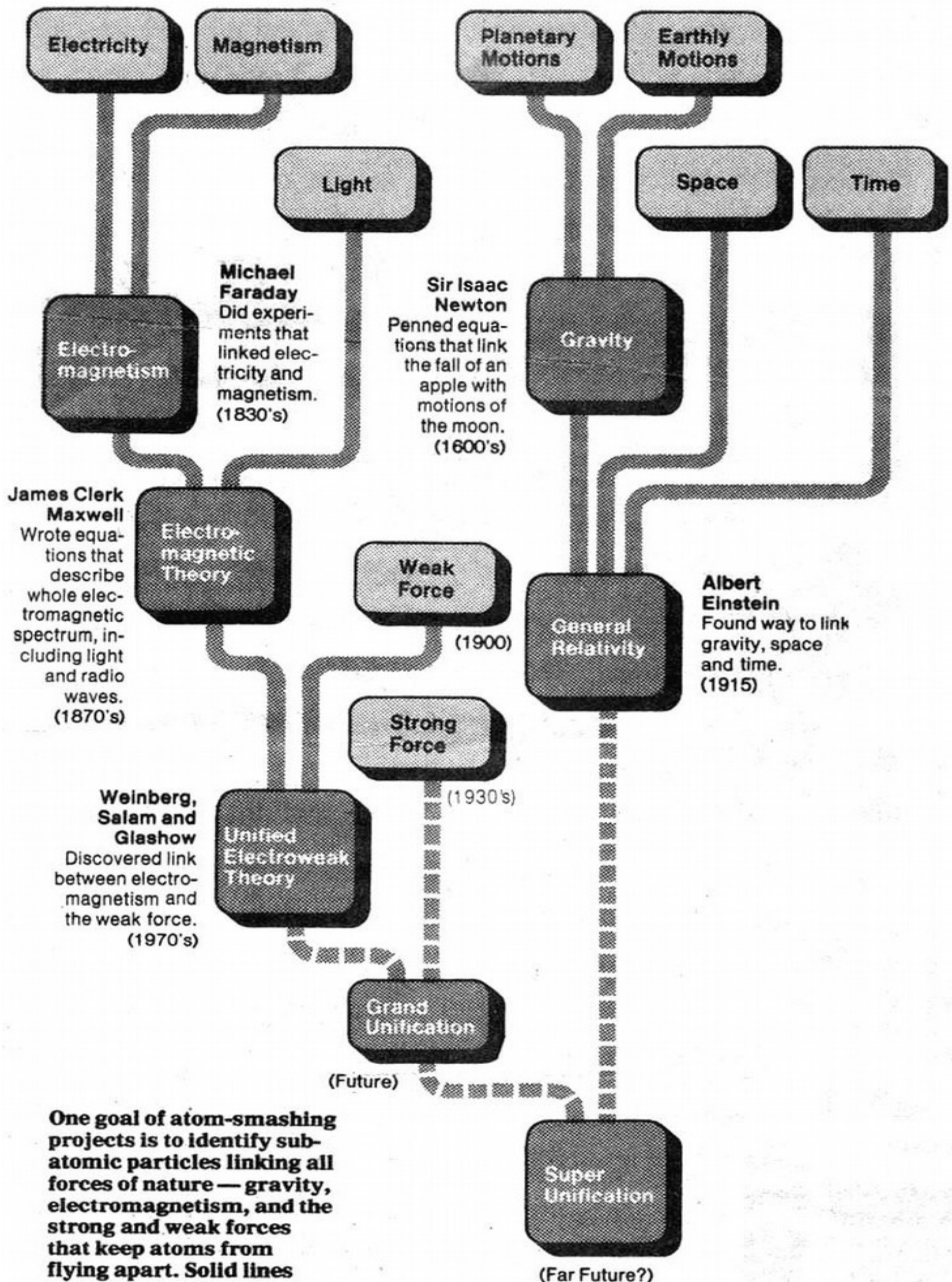
To glean an overview of this Section's ideas, the "***Explaining the Force of Nature***" diagram below is shown as an aid**.** The idea of this Section is to link electromagnetic phenomena -to- gravitational phenomena**.**

53 It should be noted however, much of the theoretical efforts with potential theory are more like exquisite, abstract mathematical toys of no useful purpose or practical usage**.** Potential theory today is utilized more in the fields of electronics and quantum mechanics, than with gravitational theory**.**

54 See W. A. **HEISKANEN** and F. A. Vening **MEINESZ**'s **The Earth and Its Gravity Field**, *McGraw-Hill*, 1958**.** Their methods are still used to find oil**.**

55 A good, simple discussion of **GAUSS**' *Law* is found in **HALLIDAY**, David**; RESNICK**, Robert**; EDWARDS**, W. Farrell and **MERRILL**, John**: Fundamentals of PHYSICS**, 1978, 3rd Ed. (hardcover, yellow), *John Wiley & Sons*, 827 pages**.** See the Chapter 24, "***GAUSS's Law***," pp. 449-464**.**

Explaining the Forces of Nature



One goal of atom-smashing projects is to identify subatomic particles linking all forces of nature — gravity, electromagnetism, and the strong and weak forces that keep atoms from flying apart. Solid lines trace history of past unifications. Dotted lines indicate hypothesized links that may lead to a totally unified theory.

Gravitational Potential Function Development

Starting with the gravitational potential function, $V(m,\boldsymbol{r})$, **NEWTON**'s *Law of Universal Gravitation* <u>can be derived</u>**.** It demonstrates **NEWTON**'s *Law of Universal Gravitation* is not really isolated, as it appears to be, from any of the other basic physical concepts, by this connectedness (Newtonian physics–potential theory–Universal Gravitation)**[56]**, and really forms part of the frame work of Mechanics**.[57]**

"*The significance of this approach is seen by observing that* [**LaPLACE** and Siméon-Denis] ***POISSON****'s equations can be solved under rather general conditions, which is not the case with* ***NEWTON****'s equation*" [***67Brit_***"***Gravitation***"]**.** A particularly remarkable phenomenon exposed in this potential function's math **[58]** is gravity's supposedly *natural harmonic (sinusoidal) character* and where calculated gravity wavelengths are equal to the planet's orbital distances**.[59]**

A separate **<u>The Gravity Waves, An Analysis</u>** monograph, which gives the complete connectedness analysis, develops **NEWTON**'s *Law of Universal Gravitation* starting with the *gravitational potential function,* $V(m, \boldsymbol{r})$**.** Some excerpts from that 17 paged **<u>Monograph</u>** are incorporated here to supplement the presentation of some new concepts following between **NEWTON**'s *Law* and **COULOMB**'s *Law***.**

The gravitational potential at any point $\boldsymbol{P}_1$ is measured by the energy necessary to carry a unit mass from that point to a region of space $\boldsymbol{P}_2$, *<u>infinitely removed</u>* from all matter**.** Its value is the volume integral $G\int_V d\rho/\boldsymbol{r}$, where G is the gravitation constant, or the *constant of proportionality*; $d\rho$ is any 3D-mass element and $\boldsymbol{r}$ is its spatial distance from the point in question**;** and the integration is extended

56 See **BERGMANN**, Peter Gabriel**:** **<u>Basic Theories of Physics</u>**, Vol. 1 or 2, *Dover*, 1949**.** Read his discussion about the potential function presented in Vol. 1, Chapter 1**;** except paragraph 1.3, "***Planetary Motion***," because it is wanting**.**

57 See Enzo **TONTI**'s diagram for "***Particle Dynamics***" is on the Internet**.** Also find many more of his diagrams in F. H. **BRANIN**, Jr. and K. **HUSEYIN**'s book (see **References** below)**.**

58 To develop the math start with **REDDICK**, H. W. and **MILLER**, F. H.**:** **<u>Advanced Mathematics For Engineers</u>**, 3rd Ed., *John Wiley & Sons*, 1960**:** Art. **78**, "***Potential due to charged sphere***," pp. 246-257, and pp. 325-328**;** "***Potential, Gravitational***," pp. 349, 365**.** This is a well-known textbook used in many colleges and universities**.**

59 Also see**: BROUWER** and **CLEMENCE**'s book, **<u>Methods of Celestial Mechanics</u>,** Chapter **III** (listed in **References** below). They do this same analyses as here, except just backwards to that given in **<u>Gravity Waves, An Analysis</u>**, by first assuming **NEWTON**'s *Universal Law*, which then yields the *potential function***.** Their solution is based in part upon François **TISSERAND**'s (1845-1896) Vol. 2, book, which has an exhaustive treatment on this subject**.**

throughout all existing matter. (The gravitational constant was discussed above.)[60]

The sketch below is based upon letting the *potential function* $V(x_2, y_2, z_2) = m/\boldsymbol{r}_{12}$, be a finite, continuous and single-valued function in a 3-D, Cartesian coordinate system, of a mass-point $\boldsymbol{P}_2 \Rightarrow (x_2, y_2, z_2)$ for the mass m, and letting $\boldsymbol{r}_{12}$ be the norm $\sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2 + \left(z_2 - z_1\right)^2}$ of $\boldsymbol{P}_2$ from another point $\boldsymbol{P}_1 \Rightarrow (x_1, y_1, z_1)$.[61]

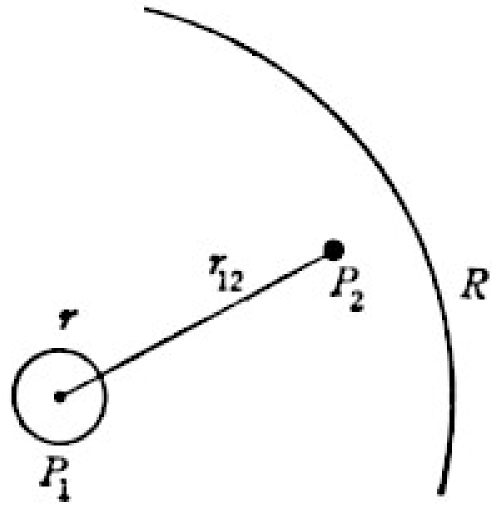


Figure **7:** Sketch for Potential Geometry

In the Figure two spheres centered about $\boldsymbol{P}_1$, with one very small, having radius r; and the other very large, with radius R – are defined as shown in the sketch. Note, since the potential function depends only on the distance to the other mass-points, it is consequently independent of the choice of coordinate axes.

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60 See **REDDICK**, W. H, and **MILLER**, F. H.: **Advanced Mathematics for Engineers**, 3rd Edition, 1960 (referenced above too). A similar problem statement for the value of $G\int dm/\boldsymbol{r}$ is presented in **REDDICK** and **MILLER**'s (priceless) book in Art. **78**, "***Potential due to charged sphere***," pp. 325-328; but also see Art. **65**, "***Legendre functions***," pp. 247-256. On page 251 they define the problem of determining the potential at a point ***P*** owning to two electric charges, numerically equal but opposite in sign. Their beginning analysis is similar to (taken from?) **PEIRCE**'s paragraph 18 problem (**PEIRCE** is referenced below). Their problem statement unfortunately doesn't have a sketch associated with Art **78**, but one is given below. Frederic H. **MILLER** also wrote **Calculus**, 1939; and **Partial Differential Equations**, 1941. All three books published by *John Wiley & Sons*. The latter two books greatly augments the first book. Similar to their books are Ivan. S. **SOKOLNIKOFF** and R. M. **REDHEFFER**'s **Mathematics of Physics and Modern Engineering**, 1958; and brother, Dr. Ivan S., and sister Dr. Elizabeth S. **SOKOLNIKOFF**'s famous, **Higher Mathematics for Engineers and Physicists**, 1941. The latter two published by *McGraw-Hill*. Their books are priceless too.

61 See **International Dictionary of Applied Mathematics**, "***Potential Operator***," page 711.

Also note in particular, $V(x,y,z)$, whether electrical or gravitational, is a *scalar field*[62]. Hence an element of mass of an *attracting body* is:

$$dm := \rho(\xi,\eta,\zeta)\,d\xi d\eta d\zeta = \rho d\rho,$$

and ρ is the body's density at the points (ξ,η,ζ), meaning $\rho(\xi,\eta,\zeta)$ is a function of position. Then the potential of V is defined at a spatial point $\boldsymbol{P}_1(x, y, z)$ owing to a distributed density $\rho(\xi,\eta,\zeta)$ throughout a closed volume B as:

$$V(x,y,z) = k\iiint \frac{\rho(\xi,\eta,\zeta)\,d\rho}{\sqrt{(\xi-x)^2+(\eta-y)^2+(\zeta-z)^2}}, \qquad \textbf{1a}$$

with integration extended over the entire body. The integration functions are:

- (ξ,η,ζ), is a point in the attracting body inside the volume B;
- $d\rho := d\xi d\eta d\zeta$ is an element of the body;
- (x,y,z) are coordinates of the attracted point and *independent* of (ξ,η,ζ).

This means the entire phenomenon above takes place in three-dimensional Euclidean space. Eq. (**1a**) in abbreviated form is:

$$V = \int_B \frac{\rho d\rho}{r}. \qquad \textbf{1b}$$

where it is understood $\boldsymbol{r}$ is the *norm*. Note the dot product or *norm* is:

$$r := \sqrt{\vec{\boldsymbol{r}}_i \bullet \vec{\boldsymbol{r}}_i} = \sqrt{d_i^2} = \|\vec{\boldsymbol{r}}\| = \sqrt{(\xi-x)^2+(\eta-y)^2+(\zeta-z)^2}. \qquad \textbf{2}$$

The integral is taken over the volume between the two spheres as $r \to 0$ and $R \to \infty$.

An analogous definitions hold for an electrical point-charge density $\boldsymbol{q}$.

Here $k = G$, is the *constant of proportionality* and depends on the type and intensity of the force at a unit distance, and on the system of units used. Hence k can be conveniently dropped from Eq. (**1a**) without any lost in content and restored

62 See **MORSE**, Philip M., and **FESHBACH** Herman classic: **Methods of Theoretical Physics**, Part 1 and 2, *McGraw-Hill*, 1978: Part 1, Chapter 1 has a good discussion on scalar and vector fields.

back whenever necessary later.

When $\rho = 0$, i.e., outside the sphere, **POISSON**'s equation $\nabla^2 V = 4\pi\varepsilon_0$ reduces to **LaPLACE**'s equation $\nabla^2 V = 0$, and has a general solution expressed as a series of powers of trigonometric cosine function.

The Gravity Waves, An Analysis monograph from here follows mainly the analyses used in **REDDICK** and **MILLER'**s textbook, coupled with **MacMILLAN**'s or **KELLOGG**'s (see **References**). It starts by doing partial differentiation under the integral sign with respect to independent x, y and z, which leads directly to **LaPLACE**'s Equation[63].

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0\,. \qquad \mathbf{3}$$

Next, rectangular coordinates are change to spherical coordinates; i.e., to $V(r, \theta)$ for the *potential function* $V(\boldsymbol{r}) = m/\boldsymbol{r}$, where m is mass and $\boldsymbol{r}$ is the spatial vector.

Solution of **LaPLACES**' partial differential equation, $\nabla^2 V = 0$ is not given here as it is given in **The Gravity Waves, An Analysis**. Suffice it to say **LaPLACE**'s equation after separation of variables via a product solution, say like $R(r)T(\theta)$ for Eq. (**3**), results in ***LeGENDRE'***s ordinary second-order linear differential equation, whose solution is via the *Method of **FROBENIUS***[64] [see **NAGLE**]; and whose outcome produces **NEWTON**'s *Law of Universal gravitation*.[65] (The rather long solution of **LeGRENDRE**'s equation is the next step but not developed here as it is given in many textbooks, for example see **REDDICK** and **MILLER**, Art. **65**, "***Legendre functions***," pp. 247-256, and note 60 above.)

If the electrical potential $V(q,\boldsymbol{r}) = q/\boldsymbol{r}$ and **POISSON**'s equation had been used in lieu of the gravitational potential, it would have resulted in **COULOMB**'s equation.

63 Pierre Simon Marquis de **LaPLACE**, (1749 – 1827).

64 Ferdinand Georg **FROBENIUS**, 1849–1917. Since **LeGENDRE**'s *Equation* is really a reduced hypergeometric, second-order linear differential equation, it can be solved by continued fraction expansions, which is a more modern treatment of the subject [**BATTIN**)]. Here we keep it simple.

65 See **BROUWER** and **CLEMENCE** book (listed in **References**): their Chapter **III** (the heart of the book), "***Gravitational Attraction Between Bodies of Finite Dimensions***," assumes **NEWTON**'s *Law of Universal Gravitation* first, to prove by energy methods it yields the *potential function* $V(\boldsymbol{r})$: = $m/\boldsymbol{r}$. Going the other way (derivation not in their book) and deriving **NEWTON**'s *Universal Law*, is given in a separate **Monograph**. The results raises big questions as to which is more fundamental, **NEWTON**'s Universal *Law*; or the *potential function* or **LaPlace**'s *equation*?

It has always been noted **NEWTON**'s *Law of Universal Gravitation* and **COULOMB**'s *Law* are of the same mathematical form**:**

NEWTON $\vec{F} = -G\dfrac{m_1 m_2}{r^3}\vec{r},$ and **COULOMB** $F = \dfrac{1}{4\pi\varepsilon_0}\dfrac{q_1 q_2}{r^2}$

Their common elements are force, distance, the gradient operator ∇ and ∇^2, and **NEWTON**'s and **COULOMB**'s equations**.** If the electromagnetic flux from the atom's nucleus (a natural unit[66]), is considered as the precursor (mechanism) of gravitation as inferred above, then the main difference between the two is obviously a matter of scale**.** Otherwise, the inferred concept here is electromagnetic phenomena and gravitation phenomena are really basically the same phenomena**:** electromagnetic radiation (shades of the **Cosmic Code**[67]); and the scale is a function of G -to- $4\pi\varepsilon_0$**.**

COULOMB's Constant**:**

The exact value of Coulomb's constant is:

$$k_e = \frac{1}{4\pi\varepsilon_0} = \frac{c_0^2\mu_0}{4\pi} = c_0^2 \times 10^{-7}\ \mathrm{H\cdot m^{-1}}$$
$$= 8.987\,551\,787\,368\,176\,4 \times 10^9\ \mathrm{N\cdot m^2\cdot C^{-2}}$$

Gravitational constant**:**

$$G = 6.673 \times 10^{-11}\ \mathrm{N\ m^2/Kg^2}$$

Listed below are some of the classical essays and textbook on *potential theory***:**

- **MARKINA**, Irina: "***Potential Theory: The Origin and Application***." On the Internet at .https://folk.uib.no/ima083/courses_files/potential.pdf**.**
- "***Gravitation***" by Kenneth I. **NORDTVEDT**, Jr.; James E. **FALLER**; and J. W.

66 **THOMSON**, Sir George**:** "***What You Should Know About Physics***," in **Adventures of the Mind**, *Second Series*, *Alfred A. Knopf*, 1961. Defines natural units as**:** electron, proton, neutron, nucleus, and the atom.

67 **PAGELS**, Heinz A.**:** **The Cosmic Code**, subtitled **Quantum Physics as the Language of Nature**, 3rd printing, *Bantam Books*, 1984**.** "*The visible world is neither matter nor spirit but the invisible organization of energy***.**" Otherwise, the Universe is not but electromagnetic energy in one form or another**.** Here a master explains the world of quantum physics, which is in reality, basic physics**.** He gives a non-mathematical presentation, but still slightly difficult to follow since quantum phenomena are difficult to understand**.** This is a "*must reading***.**"

BEAMS, in **Encyclopædia Britannica**, 15th Edition, Vol. **21**, pp. 286-295, 1988**.** Especially see page 288, paragraph "***Potential Theory.***"

- "***Gravitation***" in Vol. **10**, pp. 711-727 (~ 68 std. pages), of the 1967 or 1968 *Editions* of the **Encyclopædia Britannica.** This is a well-written overview on the subject, written by five (impressive) experts in the field**:** Thomas Frohock **GASKELL;** Wm. **BULLERWELL;** Alan Hugh **COOK;** Herman **SHAW;** and Ralph D. **WYCKOFF.** Read especially Section **I**, "***Introduction***"**;** and Section, **IV**, "***Theories of Gravitation.***" Also take note of the article's references**.**
- **PEIRCE**, B. O.**:** **Elements of the Theory of the Newtonian Potential Function**, 3rd revised and enlarged Edition, *Ginn & Co*., 1902**.** See Chapter 1, "***The Attraction of Gravitation***," and especially paragraph **18**, "***Attraction between Any Two Rigid Bodies.***" This is classic, applied calculus at its best, and enjoyable to read**.** His book is on the Internet at https://archive.org/details/elementsoftheory00peir**.**
- **MacMILLAN**, William Duncan**:** **The Theory of the Potential**, *McGraw-Hill*, 1930**.** Standard text**.** He also wrote **Statics and the Dynamics of a Particle**, *McGraw-Hill*, 1927**.**
- **KELLOGG**, Oliver Dimon**:** **Foundations of Potential Theory**, 1929, *Dover* reprint, 1953**.**
- **MISNER**, Charles W.; **THORNE**, Kip S.[68]**;** and **WHEELER**, John A.**:** **Gravitation**, *W. H. Freeman and Co*., 1973**.** A huge, 8½ by 11 inch book, 4½ inches thick**.** Half the books is devoted to the implication of "*differential forms.*" Supposedly at the time of its publication, and still ongoing, is its reputed as being "*the definitive*" book on gravity and research applications (and *cerebral*, and fits well into **BAGGETT**'s theses)**.**
- **ATKIN**, R. H.**:** **Mathematics and Wave Mechanics**, *John Wiley & Sons*, 1957**.** Contains a concise introduction to Wave Mechanics, eminently adopted to the needs of Physical Chemistry**.** Chapter **VI** is c*lassical Mechanics*, is a *must-reading***.** Also see paragraphs 7**.**11 (*Surface Integral*) -to- 7**.**21 (*Vector Potential*), pp 169-180**.**
- **LINDSAY**, Robert B.**:** "**Further Considerations on the Potential. GAUSS's Law and Laplace's Equation**," in paragraphs 4**.**5, pp. 111-117 **−** in his

68 Theoretical physicist **THORNE**, whose technical work partly inspired the film "***Interstellar***," a 2014 epic ***SF***, was one of the *Executive Producers* of the film, as well as functioning as a scientific consultant**.**

PHYSICAL MECHANICS, 3rd *Ed., D. Van Nostrand Co*.,1960**.** Read Chapter 4**:** "***Energy in Particle Dynamics***" (i.e., p*otential theory*)**.**

- **HALLIDAY**, David and **RESNICK**, Robert; 3rd Edition by **EDWARDS**, W. Farrell and **MERRILL**, John**:** "***GAUSS's Law***," Chapter 24, pp. 449-464, is in [*The Purple Egg*] **Fundamentals of PHYSICS**, 3rd Ed., *John Wiley & Sons*, 827 pages, 1978**.**
- **NAGLE**, R. Kent; and **SAFF**, Edward B.; and **SNIDER**, Arthur David**:** **Fundamental of Differential Equations and Boundary Value Problems**, ***Pearson International Edition***, 5th Edition, contains a CD, 2008**.** This *modern* textbook is not only current, but its notation and methods are current too**:** it covers material for a two-year, 3-hr./wk course in elementary differential equations, taken on the undergraduate level**.** "***Legendre's Equation***" is on pp. 516-518**.** "***Method of Frobenius***" is covered on pp. 487-498. Many solved example problems are physical in context**.**

Force and Potential

What follows was extracted from **The Gravity Waves, An Analysis** monograph**.**

The principal characteristic of the *gravitational potential* is**:** *the first derivative of the potential with respect to any direction at any point in space, is equal in magnitude to the component of the field intensity in that direction at the point in question***.** And, the potential involves inverse-square forces**.** That said, the gravitational potential importance derives from the gradient property of V_ζ, which gives the *force of attraction* of a mass-point at the point $\left(\bar{x}_\varsigma, \bar{y}_\varsigma, \bar{z}_\varsigma\right)$, thus $F = -\nabla V$.

Or**:** associated with every point $\boldsymbol{P}$ in the space around a mass M there is a scalar number for the gravitational potential, which is our V**.** This number represents the intensity of the scalar field at $\boldsymbol{P}$, since the potential is a scalar quantity**.** To determine the force $\boldsymbol{F}$ exerted by this field on a mass particle m placed in it, simply compute $dV/d\boldsymbol{r}$ at the point $\boldsymbol{P}$ in question and multiply it by m**.** The force has a magnitude $mdV/d\boldsymbol{r}$ and a direction radically inward toward the center of mass M**.**

$$\vec{\boldsymbol{F}} \equiv m\frac{dV}{dr} = m\frac{d}{dr}\left(\frac{M}{R} + \cdots\right) \approx \frac{Mm}{R^2}$$

And to this, if we add G (which, remember, was initially removed), the constant of proportionality for gravity, gives **NEWTON**'s *Law of University Gravitation***:**

$$\vec{\boldsymbol{F}} \equiv -G\frac{Mm}{R^3}\vec{\boldsymbol{R}}.$$

An aside, it is odd this result is not given in **MacMILLAN** or **KELLOGG**'s textbooks, the two principal textbooks dealing with *Potential Theory***.** There is another oversight missing from those books**.** Since the results of this analysis is **NEWTON**'s *Law of University Gravitation*, it is obvious the potential energy, and therefore the force of gravity, is a direct result the cumulative area under the potential curves; i.e., the areas under the integral signs**.** A plot, Figure **8**, of the **LeGENDRE** *Polynomials* will help dramatize this effect**.**

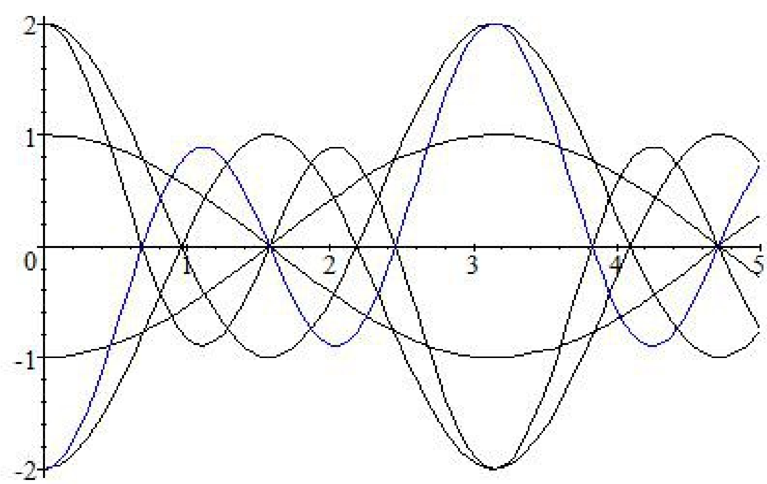

Figure **8:** Cross-sectional view of Gravity Waves Issuing from (say) the Sun

Gravity waves, 3D plots**:**

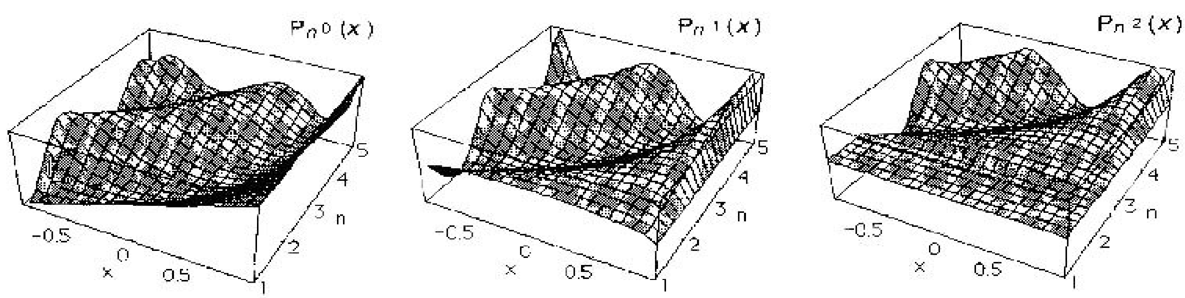

Figure **9:** Gravity Waves 3D Plots

The origin $\boldsymbol{O}$ on the left is, say, the Sun, with the vertical axis its poles, with a cross-sectional view showing its gravity waves extending outward from the poles. The Central horizontal axis or mid-plane is the equator and the plane of planetary rotations. The outer most wave is plotted in red, is $P_1 = \cos\theta$, which of course, besides P_0, dominates. Notice how there are nodal points or "*rings*" where $P_1 = \cos\theta$ crosses the equator. Perhaps a better visualization is by the following 3D-curves is shown in Figure **9** above.

The Sun's electromagnetic waves issuing from the North and South poles, encircle the Sun (Figure 9) and crisscross at the *invariable plane*, form nodal rings (Figure 8), is, on their far side, where the planets are nested. A planet's centripetal force, i.e., gravity, pulls it towards the Sun; while the Sun's electromagnetic field pushes it away; combined with the outward centrifugal force, are all perfectly balanced.

Gravity waves as such are equal to, are the same as electromagnetic waves and their wave lengths are equal to the distances of the planets to the Sun.

What is not graphed well in Figure **8** is, as R becomes larger and larger, the waves have less and less amplitude (fizzle-out). Near the orbits of the Asteroids the secondary potential field effects are rather ineffective and so bodies there are smeared into many satellites (like Saturn's rings). If these waves are "*real*" then the potential gravity field accounts for the Asteroid belt, Saturn's Rings and the Trojan planetoids (gravity wells or null points). Clearly however, above all else, the flux of gravity waves implies the existence of gravitational nodal rings the planets are trapped within.

The gravity wells the Trojan planetoids are trapped in is the result of the intersection of the gravity fields of Jupiter and the Sun – is additional proof of the existence of the Sun's electromagnetic field.

The Two-Body Problem

This Section relates some of the real complexity in calculating planetary forces**.** It starts with the *Two-body Problem*, a problem which has historically always been the starting point for all *n-body problem* applications**.**

The *Two-body Problem*, n = 2, is outlined here, was completely solved by Johann **BERNOULLI** (1667-1748) by classical theory (and not by **NEWTON**) by assuming the main mass-point was *fixed* [69,70]**.**

Consider then the motion of two bodies, say Sun-Earth, with the Sun fixed, then**:**…..

$$\text{Sun -to- Earth……}\; m_1\boldsymbol{a}_1 = G\left(\frac{m_1 m_2}{r_{12}^3}\right)(\boldsymbol{r}_2 - \boldsymbol{r}_1),$$

$$\text{Earth -to- Sun......}\; m_2\boldsymbol{a}_2 = G\left(\frac{m_1 m_2}{r_{21}^3}\right)(\boldsymbol{r}_1 - \boldsymbol{r}_2).$$

On the left, the $m_i\boldsymbol{a}_i$ terms are the instantaneous (t = 0) expressions of **NEWTON**'s 2nd *Law* representing the inertial forces of the two point-masses, and only applies to a *fixed mass system* measured relative to an *inertial reference frame***.**[71] The two terms on the right are expressions of **NEWTON**'s *Law of Universal Gravitation*, representing the gravitational forces between the two masses**.** Notice *these two sets of equations are functions of geometry and mass only***.** $\boldsymbol{r}$ is the vector from the mass-point centroidal $c.g.$'s; and as such, according to classical theory, are valid only at a specific orbital time (*time-points*), as $\boldsymbol{r}$ keeps changing in this system**.**

In pure science the metric centimeter-gram-second (c.g.s.) system or the preferred meter-kilogram-second (m.k.s.) units are used for these equations**.** In the

69 See **BATE**, **MUELLER**, and **WHITE:** Chapter 1, "***Two-Body Orbital Mechanics***," pp 1-49**.** These authors were from the Dept. of Astronautics and Computer Science, *United States Air Force Academy***.** See Chapter 1 too**.** Their textbook is practical and not filled with advanced mathematics**.**

70 See **LAWDEN**, D. F.**:** **Analytical Mechanics**, ***Problem Solver*** series, *George Allen & Unwin LTD*, 1972**.** See Chapter 2, "***Plane Motions***," **Problem 2.2**, pp. 12-15**;** contains a detailed presentation of the *Two-body Problem***.** This and the other ***Problem Solver*** series books are somewhat like ***Schaum's Outline Series*****.** See **Theoretical Mechanics** by Murray R. **SPIEGEL**, *McGraw-Hill*, Chapter 5, "***Central Forces and Planetary Motion***," pp. 116-143**.** **SPIEGEL**'s approach is to assume the Sun is fixed and the planet's gravitational forces do not affect one another, reducing the Chapter to a mathematical toy**.** **SPIEGEL** has written several other excellent books too**.** See Paul **HERGET**'s **The Computation of Orbits** for a more realistic approach**.**

71 An *inertial reference frame* is a reference frame conforming to **NEWTON**'s *Force Laws***.**

latter the unit of force is the *newton* or one kilogram meter per second per second; and equal to 10^5 dynes in the c.g.s. System. G has the value 6.670×10^{-8} dyne cm^2/gm^2.

The equation describing the motion of mass m_2 relative to mass m_1 is readily obtained from the differences between these two equations and after canceling common terms gives $\boldsymbol{a} + (\eta/r^3)\,\boldsymbol{r} = 0$, where:

- $\boldsymbol{a}$ is the Eulerian acceleration $d^2\boldsymbol{r}/dt^2$;
- $\boldsymbol{r} = \boldsymbol{r}_2 - \boldsymbol{r}_1$ is the vector position of m_2 relative to m_1;
- and $\eta = G(m_1 + m_2)$.

$\boldsymbol{a} + (\eta/r^3)\boldsymbol{r} = 0$ is the classical fundamental differential equation for the *Two-body Problem* **BERNOULLI** initially solved in 1734.

Notice for this approach, forces have to be determined first, then the equation of motion resolved. This differential equation has elliptic, or parabolic or hyperbolic solutions too. In the classical Newtonian approach above the Sun is always assumed fixed; i.e., the model (differential equation) tacitly assumes one mass orbits about a *fixed* mass [72, 73, 74].

This "*one-mass fixed*" approach in almost all textbooks is misleading, because textbook authors do not state directly the unreal ramifications of one mass assumed fixed. At best it is a toy.

72 For example, in R. B. **LINDSAY**'s **Physical Mechanics** (a textbook classic) his central force differential equation approach may be (and is) mathematically beautiful, but it really has no physical reality. See Chapter 3, especially paragraph 3-6, "***Central Force***," pp. 74-81; paragraph 3-9, "***Planetary Motion***," pp. 83-88. **LINDSAY**'s presentation goes a long way in explaining those latter comments for the fixed *Two-body Problem*; i.e., when the Sun is assumed fixed. The conclusion is his physics model is a mathematical toy.

73 For the classical approach, if the common center of mass (i.e., the barycenter) of the two bodies is considered to be at rest, then each body travels along a conic section which has a focus at the barycenter of the system. In the case of a hyperbola it has the branch at the side of that focus. The two conics will be in the same plane. The type of conic (circle, ellipse, parabola or hyperbola) is determined by finding the sum of the combined kinetic energy of two bodies and the potential energy when the bodies are far apart. (This potential energy is always a negative value; energy of rotation of the bodies about their own axes is not counted here)

- If the sum of the energies is negative, then they both trace out ellipses.
- If the sum of both energies is zero, then they both trace out parabolas. As the distance between the bodies tends to infinity, their relative speed tends to zero.
- If the sum of both energies is positive, then they both trace out hyperbolas. As the distance between the bodies tends to infinity, their relative speed tends to some positive number. Ref. ***Wikipedia***.

74 Note: The fact a parabolic orbit has zero energy arises from the assumption the gravitational potential energy goes to zero as the bodies get infinitely far apart. One could assign *any* value to the potential energy in the state of infinite separation. That state is assumed to have zero potential energy *by convention*.

Another example of a two-body system is a rotating dumbbell molecule[75]. It has *exactly* the same math model equation *form* as the two-body planet model, (except $\boldsymbol{F} = + c\boldsymbol{x}$). Gordon M. **BARROW**'s book (see **References**) shows a simple way of calculating for a rotating dumbbell, some of its dynamic properties about its barycenter. For example, its total kinetic energy (see Figure below). He

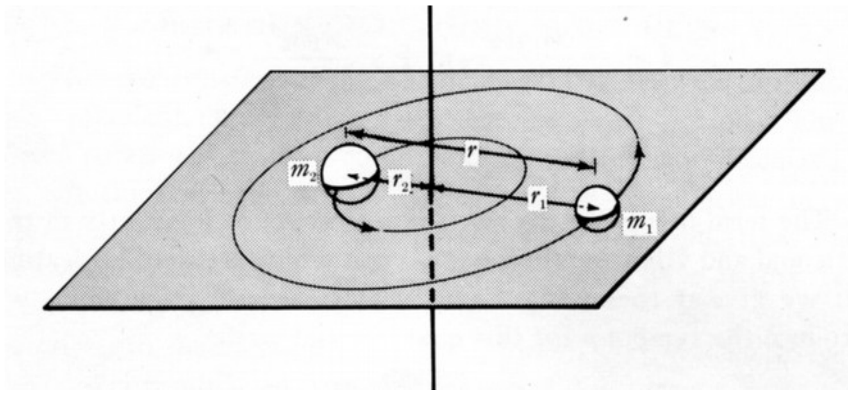

Figure **10:** Rotating Dumbbell

then goes on to calculate the total combined *mass moment of inertia*.

Suffice it to say his total angular kinetic energy for this system is $KE = \frac{1}{2}\omega^2\left(m_1\boldsymbol{r}_1^2 + m_2\boldsymbol{r}_2^2\right)$, where ω (rad./sec.) is the system's angular velocity. And the total centroidal mass moment of inertia of the system is $\overline{I} = m_1\boldsymbol{r}_1^2 + m_2\boldsymbol{r}_2^2$.

Now let $\boldsymbol{r}_1 = \boldsymbol{r}_1(t),\ \boldsymbol{r}_2 = \boldsymbol{r}_2(t),\ \boldsymbol{r} = \boldsymbol{r}(t)$ be the radius vectors of the mass-points m_1, m_2, respectively. Then **LYUBICH**'s "*Theorem* **6**" says: if the points move so that for all points of time, $\boldsymbol{r} = \boldsymbol{r}_1 + \boldsymbol{r}_2$, then their velocities are connected by a similar relation, $\boldsymbol{v} = \boldsymbol{v}_1 + \boldsymbol{v}_2$[76]. "*Theorem* **7**": If the points move so that $\boldsymbol{r}_2 = \lambda\boldsymbol{r}_1$, where λ is a constant, then their velocities are connected by a similar relation: $\boldsymbol{v}_2 = \lambda\boldsymbol{v}_1$. Simplifying then:

$$\boldsymbol{r}_1 = \frac{m_2}{m_1 + m_2}\boldsymbol{r} \text{ and } \boldsymbol{r}_2 = \frac{m_1}{m_1 + m_2}\boldsymbol{r}.$$

75 This same model is developed in Donald A. **McQUARRIE** and John D. **SIMON**'s **Physical Chemistry, A Molecular Approach**, *University Science books*, pp. 161-163, 1997. Their encyclopædia-like book stresses applied mathematics much more then physics; **BARROW** little book is just the opposite.

76 See **LYUBICH** in **Reference.** *Theorem* 6 and 7 (and more) are proved in his little book.

The center of gravity, $c.g.$, or barycenter, is located relative to $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, and therefore the relationship $\boldsymbol{r}_1 m_1 = \boldsymbol{r}_2 m_2$ must hold. If this latter expression is combined with $\boldsymbol{r}_1 + \boldsymbol{r}_2 = \boldsymbol{r}$, then, substituting back into $\overline{I}$, gives:

$$\overline{I} = \frac{m_1 m_2}{m_1 + m_2} \boldsymbol{r}^2 .$$

The term involving the two masses occurs frequently in physics and is named the *reduced mass*.

This completes the classical, mathematical modeling for the two *body problem*.

NEWTON's Effort in *n*-body Problem

With **NEWTON**'s 1687 publication of the **Principia Mathematica**, the *motion* problem of the heavenly bodies was thought finally resolved; but in truth his, and all following analyses did not include the effects of those gravitational *forces* between celestial bodies.

NEWTON, on his quest to replace the **Almagest**, went about the business of calculating the orbits of the 7 known planets. He did this straight away by knowing the orbital properties (position, mean diameter, velocity, and time) at three different positions (times) for each planet, and with analytical geometry was able to produce equations, which, on paper, predicted a planet's orbital motion[77]. **NEWTON** had obtained from the King's appointed astronomer John **FLAMSTEED** (1646-1719) the initial orbital properties, which were fairly accurate for those times.[78] However, **NEWTON** and others soon discovered over the course of a few years, those equations of motion did not predict positions very well; or in some cases, using them, some planets were lost[79]. And **NEWTON** soon realized it was because the gravitational forces amongst all the planets was effecting all their orbits.

However, even knowing the cause **NEWTON** did *not* provide a general

77 See **WOOLARD**'s "***The Calculation of Planetary Motions***," in **Reference.**

78 See David H. and Stephen P. H. **CLARK**'s **The Suppressed Scientific Discoveries of Stephen Gray and John Flamsteed, Newton's Tyranny**, *W. H. Freeman and Co*., 2001. A popularization of the historical events and bickering between those parties; but more importantly, about the results they produced.

79 An aside: these mathematically undefined planetary perturbations (wobbles) still exist undefined even today, so planetary orbits have to be constantly updated, usually yearly. See the current **Astronomical Ephemeris and the American Ephemeris and Nautical Almanac**, prepared jointly by the *Nautical Almanae Offices of the United Kingdom and the United States of America*.

deterministic method for determining those gravitational forces; which, if he had, would have potentially resolved analytically the observed perturbations[80] in **Copernicus**' model**. NEWTON** believed in his *Laws of motion* and the **Copernicus** model, but saying in the end the analytical problem was overwhelming and intractable, or words to that effect**.**

NEWTON does not say it directly but implies in his **Principia:** the *n-body problem* is unsolvable because of the multitude of those very gravitational forces[81]**. NEWTON** said in his **Principia**, paragraph 21**:** "*And hence it is that the attractive force is found in both bodies. The Sun attracts Jupiter and the other planets, Jupiter attracts its satellites and similarly the satellites act on one another. And although the actions of each of a pair of planets on the other can be distinguished from each other and can be considered as two actions by which each attracts the other, yet inasmuch as they are between the same, two bodies they are not two but a simple operation between two termini. Two bodies can be drawn to each other by the contraction of rope between them. The cause of the action is twofold, namely the disposition of each of the two bodies; the action is likewise twofold, insofar as it is upon two bodies; but insofar as it is between two bodies it is single and one*"...**NEWTON** concluded via his 3rd *Law* that "*according to this Law all bodies must attract each other*."[82] It is this last statement, which implies the existence of *gravitational interactive forces*, making the classical problem intractable**.** (See Sir David **BREWSTER** historical essay too[83]**.**)

Otherwise, it is not sufficient to just specify the initial position, velocity, the orbital diameter and period; or three orbital positions either, to determine accurately a planet's true orbit**:** *the interactive planetary gravitational forces must be known too***.**

In addition, a fundamental difference between **NEWTON**'s *cosmology model* and all those proceeding it, including **Copernicus**, was that *his* ***Copernican*** model

80 Rudolf **KURTH** has an extensive discussion in his book (see **References**) on planetary perturbations**.**

81 See **Principia**, Book Three, **System of the World**, **"***General Scholium*,**"** page 372, last paragraph**. NEWTON** was well aware his math model did not reflect physical reality**.** This edition referenced is from the **Great Books of the Western World**, Volume **34**, which was translated by Andrew **MOTTE** and revised by Florian **CAJORI.** This same paragraph is on page 1160 in Stephen **HAWKINS**' huge **On the Shoulders of Giants**, 2002 edition, which is a copy from Daniel **ADEE**'s 1848 addition**.** I. B. **COHEN** also has translated new editions**: Introduction to Newton's 'Principia'**, 1970**;** and **Isaac Newton's Principia, with Varian Readings**, 1972**. CAJORI** also wrote a **History of Science**, which is on the Internet**.**

82 See history of science historian, I. Bernard **COHEN**'s "***Newton's Discovery of Gravity***," ***Scientific American***, pp. 167–179, Vol. 244, No. 3, Mar. 1981**.**

83 For some more historical background about **NEWTON** see the essay "***Discovery of Gravitation (A.D. 1666)***" by Sir David **BREWSTER**, is in **The Great Events by famous Historians**, Volume **XII**, pp. 51-65, Rossiter **JOHNSON**, Editor-in-Chief, *The National Alumni*, 1905**.**

and *Principles* were based upon the fact that *all bodies on Earth obeying the* *__same__* *physical laws as all other celestial bodies*. This was a crucial philosophical advance in physical cosmology.[84]

Lastly, to make matters worst, after **NEWTON**'s time the classical *n-body problem* historically was not stated correctly (until now), *because it did not include a reference to those gravitational forces* causing planetary perturbations. Also Ironic, it has been this slavish attachment to his *Laws* which has resulted from the beginning the wrong approach.

Real Motion V.S. **KEPLER**'s *__Apparent__* Motion

This section discusses a historical overview with regard to descriptive astronomy, coupled with respect to Johannes **KEPLER**'s (1571-1630) curve-fitting *Laws*. For some more historical background also see **Appendix F**: **MILANKOVITCH** Cycles below. Milutin **MILANKOVIĆ** (1879-1959) theory in part supports the notion the Earth orbits the Sun circularly; not elliptically.

It is incorrect to think of m_1 (say the Sun) as fixed in space when applying **NEWTON**'s *Law of Universal Gravitation*, and to do so leads to erroneous results. Dr. Clarence **CLEMINSHAW** (1902-1985) calculated the approximate position of the Solar System's true barycenter (which is not the center of the Sun), a result achieved mainly by combining only the masses of Jupiter and the Sun (mathematically analogous to the rotating dumbbell molecule above). ***Science Program*** stated in reference to his work: "*The Sun contains 98 per cent of the mass in the solar system, with the superior planets beyond Mars accounting for most of the rest. On the average, the center of the mass of the Sun-Jupiter system, when the two most massive objects are considered alone, lies* *__462,000 miles from the Sun's center__*, *or some* *__30,000 miles above the solar surface__*! *Other large planets also influence the center of mass of the solar system, however. In 1951, for example, the systems' center of mass was not far from the Sun's center because Jupiter was on the opposite side from Saturn, Uranus and Neptune. In the late 1950s, when all four of these planets were on the same side of the Sun, the system's center of mass was more than 330,000 miles form the solar surface, Dr. C. H.* ***CLEMINSHAW*** *of Griffith*

84 See footnote **48**.

Observatory in Los Angeles has calculated."[85]

This means the Sun wobbles and Sun-spots are possibly caused via the movement of the barycenter, owing to Jupiter's 11-year cycles, which produce Sun-spots every 22 years. It further needs to be pointed out *the total mass orbiting the Sun is probably equal to the Sun's own mass*.

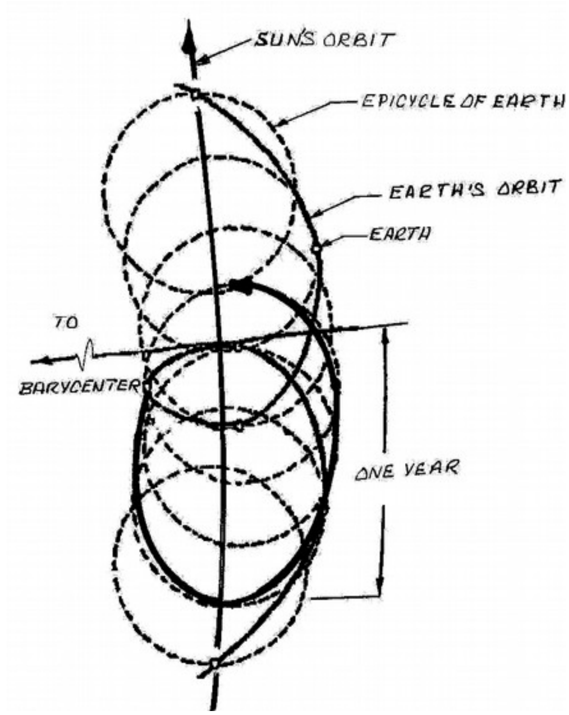

Figure **11:** Real Motion V.S. **KEPLER**'s *Apparent* Motion

The Sun's Wobble

The Sun wobbles as it rotates around the galactic center, dragging the Solar System and Earth along with it. What mathematician **KEPLER** did in arriving at his three famous equations was curve-fit the *apparent motions* of the planets using Tycho **BRAHE**'s data; and, excluding small planetary perturbations, *not* curve-fitting their true *circular* motions about the Sun (see Figure).

Both Robert **HOOKE** and **NEWTON** were well aware **NEWTON**'s *Law of Universal Gravitation* did not hold for the forces associated with elliptical orbits[86]. In fact, **NEWTON**'s *Universal Law* doesn't account for the orbit of Mercury, the Asteroid Belt's gravitational behavior, or Saturn's Rings[87].

NEWTON stated (in the 11th Section of the **Principia**) the main reason however for failing to predict the forces for elliptical orbits was his math model was for a body confined to a situation that "*hardly exist in the real world*," namely, the "*motions of bodies attracted toward an unmoving center*." Some present physics and astronomy textbooks don't emphasize the negative significance of **NEWTON**'s

85 ***Science Program*** magazine's "***The Nature of the Universe***" states Clarence **CLEMINSHAW** (1902-1985) served as Assistant Director of *Griffith Observatory* from 1938-1958 and as Director from 1958-1969. Some publications by **CLEMINSHAW**, C. H.: "**Celestial Speeds**," 4 1953, equation, **KEPLER**, orbit, comet, Saturn, Mars, velocity; "**The Coming Conjunction of Jupiter and Saturn**," 7 1960, Saturn, Jupiter, observe, conjunction; "**The Scale of The Solar System**," 7 1959, Solar system, scale, Jupiter, sun, size, light.

86 See. I. Bernard **COHEN**'s ***Scientific American*** article, "***Newton's Discovery of Gravity***," page 179.

87 **BRUSH**, Stephen G., Editor: **Maxwell on Saturn's Rings**, *MIT Press*, 1983.

assumption and end up teaching that his math model is in effect reality**.** Understand, in actuality the classical *Two-body Problem* solution above is naught but a mathematical toy too**.**

NEWTON conveniently fixed the Sun so he could do simple calculations (in effect he cheated), but all following after him have also made the same mistake by analytically fixing the Sun too**.** They perpetuated the mathematical toy**.**

An aside**:** Newtonian physics doesn't include (among other things) relative motion, which may be the root of the reason **NEWTON** "*fixed"* the Sun**.**

Another aside**:** I. Bernard **COHEN** states in a *footnote* [Ref. **16**, pg. 78] "***Galileo****'s observations of the phases and relative sizes of Venus, and of the occasional gibbous phase of Mars, proved Venus, and presumably the other planets, move in orbits around the Sun***.** *<u>There is no planetary observation by which we on Earth can prove the Earth is moving in an orbit around the Sun</u>***.** *Thus all* **Galileo***'s discoveries with the telescope can be accommodated to the system invented by Tycho* ***BRAHE*** *just before* **Galileo** *began his observations of the heavens***.** *In this Tychonic System, the planets Mercury, Venus, Mars, Jupiter and Saturn were in orbits around the Sun, while the Sun moves in an orbit around the Earth in a year* [for **Copernicus**' position on this, see below]***.*** *Furthermore, the daily rotation of the heavens is communicated to the Sun and planets, so the Earth itself neither rotates nor revolves in an orbit***.** *The Tychonic System appealed to those* [i.e., the Church] *who sought to save the immobility of the Earth while accepting some of the Copernican innovations***.**"[88]

COHEN's statement "*There is no ….*" above is wanting**:** the *retrograde motion* of Mars for example as seen from Earth would have implied even back than the Earth was in motion**.** In fact, the *Copernican Revolution* explained this retrograde motion**.**

A. **WOLF**'s famous two volume, **<u>A History of Science, Technology and Philosophy in the 16th and 17th Centuries</u>** (Peter **SMITH** publisher, 1968), clearly states in Chapter **VI** that **Copernicus** fixed the Sun, with all seven[89] then known planets orbiting it**.** Later Tycho revised the *Copernicus System* (his *Tychonic*

88 This same <u>*footnote*</u> is in **<u>Exploring the Universe</u>**, edited by Louise B. **YOUNG**, *McGraw-Hill Book Co*., 1963 (a book sponsored by the *American Foundation for Continuing Education*)**.** This thought provoking collections of scientific essays were written by experts **−** is a book well worth reading**.** Included is a short excerpt form I. B. **COHEN**'s book; and his essay about **Galileo:** "***Galileo's Discoveries of 1609*****.**" Also see **<u>The Achievements of Galileo</u>**, Edited with *Notes* by James **BROPHY** and Henry **PAOLUCCI**, with an *Introduction* by Henry **PAOLUCCI**, *The Bagehot Council*, 2001**.**
89 William **HERSCHEL** discovered Uranus , planet number 8, in 1781**.**

System) by back-stepping, by fixing the Earth, with the Sun rotating around it**.**[90] Still later, **KEPLER** returned to the *Copernicus System*; with **NEWTON** following**.**

(**KEPLER**'s math model is actually mathematically correct, except it doesn't relate to our system**.**)

90 "***Copernicus and Tycho***" by Owen **GINGERICH**, ***Scientific American***, V229. 6, pp. 86-101, June 1973**.**

The Three-body Problem

The case for n = 3, without any *a priori* assumptions or restrictions, has been the studied most. But still, not much is known analytically about the *n-body problem* for n equal to or greater than three. Many earlier attempts to understand and solve the *Three-body Problem* were quantitative, aiming at finding explicit solutions for special situations. This Section reviews historically a particular three-body problem solution, which has always involved simplifying assumptions. It is now referred to as the *Restricted Three-body* Problem; and it is the only valid solution when using Newtonian physics for $n = 3$.[91]

Historical Overview

The main players involved in the *Three-body Problem* were:

- In 1687 Isaac **NEWTON** published in the **Principia** the first steps in the study of the problem of the movements of three bodies subject to their mutual gravitational attractions, but his efforts only resulted in verbal descriptions and geometrical sketches. See especially Book 1, Proposition 66 and its corollaries.

- In 1767 **EULER** found collinear motions, in which three bodies of any size masses move proportionately along a fixed straight line. The circular *Restricted Three-body Problem* is the special case in which two of the bodies are in circular orbits (approximated by the Sun-Earth-Moon system; and many other systems).

- In 1772 Joseph-Louis **LAGRANGE** discovered two classes of periodic solution, each for three bodies of any masses. In one class, the bodies lie on a rotating straight line. In the other class, the bodies lie at the vertices of a rotating equilateral triangle. In either case, the paths of the bodies will be conic sections. Those solutions led to the study of central configurations, for which $\ddot{x} - kx$, for some constant $k > 0$.

- A major study of the Earth-Moon-Sun system was undertaken by Charles-Eugène **DELAUNAY**, who published two volumes on the topic, each of 900 pages in length, in 1860 and 1867! Among many other accomplishments, the work already hints at chaos theory, and clearly demonstrates the problem of so-called "*small denominators*" in *perturbation theory*.

- See French astronomer **TISSERAND**, François (Jan 13,1845-Oct. 20,

91 See **LEIMANIS** and **MINORSKY**'s historical comments in their book, **Dynamics and Nonlinear Mechanics** (see **References**).

1896) famous works, **Mecanique Celeste**, in 4-volumes, Gauthier-Villars, 1889-1896.[92] There were 13 French editions published between 1891 and 1990. His dissertation was in analyzing **DELAUNAY**'s lunar theory; and his famous 4- volume works are still being used as source books.

- **WHITTAKER**, E. T.: **Treatise on the Analytical Dynamics of Particles and Rigid Bodies**, 4th Ed., *Cambridge University Press* reprint, 1965. A classic, is really classical theoretical physics, originally published in 1905. Chapter **XIII** is on the mathematical aspects of the problem of three bodies. He has several other Chapters devoted to the analyses of the *Three-body Problem* too. It is now a *Dover* reprint.
- In 1917 Forest Ray **MOULTON** published his now classic, **An Introduction to Celestial Mechanics** (see **References**) with its plot of the *Restricted 3-Body Problem* (see figure below)[93]. An aside, see Leonard **MEIROVITCH**'s book **Methods of Analytical Dynamics**, *McGraw-Hill Book Co*., 1970., pages 414 and 413 for the *Three-body Problem*[94] solution via several simplified assumptions. **MEIROVITCH**'s book gives a modern, clean, detailed analysis.
- Victor G. **SZEBEHELY**[95] His Ph.D. thesis dealt with an analysis of the *Three-body Problem* and therefore foreshadowed his later work on this most important subject. He wrote or edited 18 books, among these was a definitive treatise on the **Three-body Problem, Theory of Orbits: The Restricted Problem of Three Bodies**, *Academic Press*, 1967[96].

Restricted Three-body Problem

MOULTON's solution may be easier to visualize (and definitely easier to solve)

92 His works is an update of **LaPLACES**' famous **Mécanique céleste**. See Britannica's **Micropædia**, Vol. **11**, page 798.
93 See **MOULTON**'s *Restricted Three-body problem* 's analytical and graphical solution.
94 See **MEIROVITCH**'s book: Chapters 11, ***Problems in Celestial Mechanics***; 12, ***Problem in Spacecraft Dynamics***; and Appendix **A:** ***Dyadics***.
95 Born in Budapest in 1921 and educated there, where he receiving a MS in Mechanical Engineering in 1943 from the Budapest Technical University and a Doctor of Science in Engineering from the same university in 1946. He was Professor of Aerospace Engineering at the University of Texas at Austin, and served as Department Chairman from 1977 to 1981. He was a prolific contributor to professional journals with about 200 articles to his credit. He died at his home in Austin on September 13, 1997. See https://www.ae.utexas.edu/faculty/faculty-memorials/99-faculty/631-victor-g-szebehely
96 He also published a textbook on celestial mechanics, **Adventures in Celestial Mechanics**, *University of Texas Press*, 1989. In 1994 a second edition, co-authored with Hans **MARK** was published. He was also Editor of ***Applications of Modern Dynamics to Celestial Mechanics and Astrodynamics***. He was responsible in the **Britannia** as coauthor of the articles "***Methods in Astrodynamics***" and "***Celestial Mechanics***"; and in part for "***Mechanics***." "***Mechanics, Celestial***," is in Vol. **11**, pp. 786d-762a, 1988 Edition of the **Encyclopædia Britannia**.

if one considers the more massive body (i.e., Sun) to be "*stationary*" in space, and the less massive body (i.e., Jupiter) to orbit around it, with the equilibrium points (*Lagrangian points*) in Jupiter's orbit, but maintaining the 60 degree-spacing ahead of, and behind it **–** containing the Trojan planetoids**.** The less massive body is almost in its true orbit; although in reality neither of the bodies are truly stationary, as they both orbit the center of mass or barycenter of the whole system **–** but mainly about the Sun-Jupiter barycenter**.** For sufficiently small mass ratio of the primaries, these triangular equilibrium points are stable, such that (nearly) massless particles will orbit about these points as they orbit around the larger primary (Sun)**.** The five equilibrium points of the circular problem are known as the *Lagrangian points***.** See figure below**:**

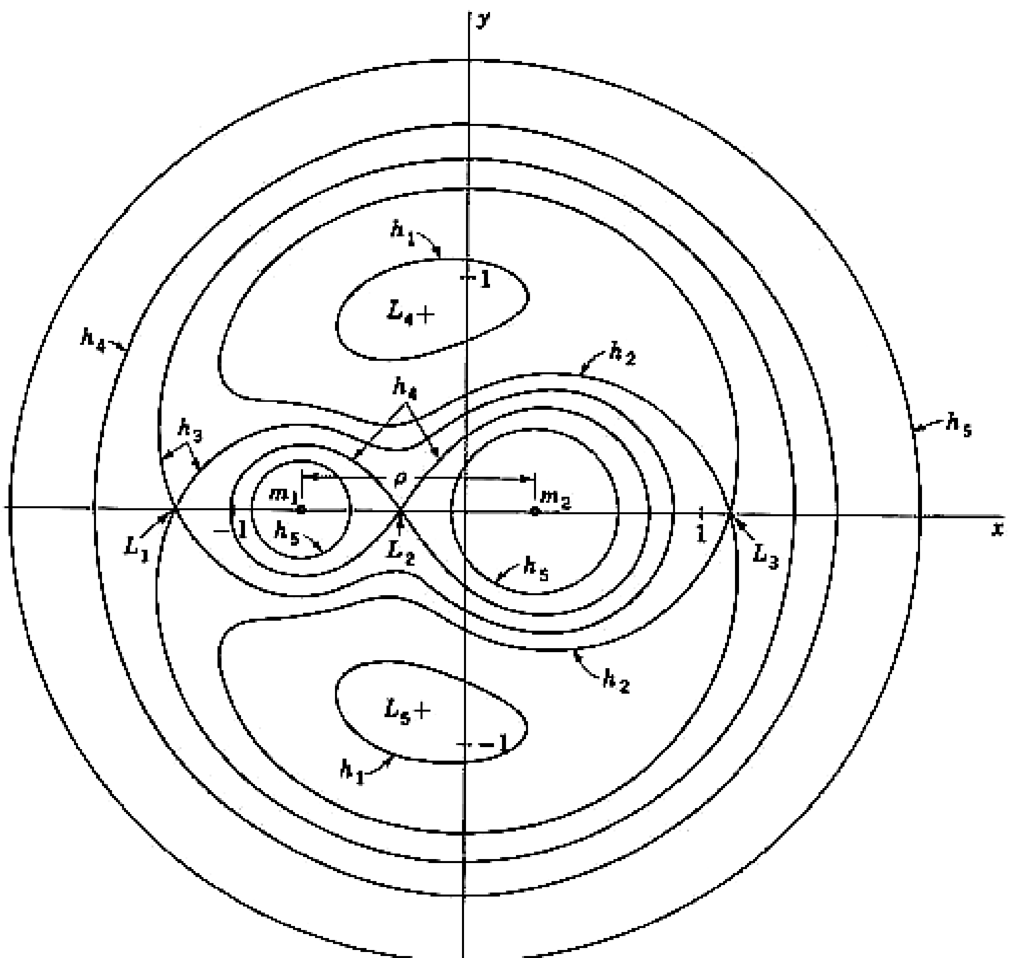

Figure **12:** *Restricted 3-Body Problem*.

In the *Restricted 3-Body Problem*'s math model figure above (ref. **MOULTON**), the Lagrangian points L_4 and L_5 are where the *Trojan planetoids* resided (Figure **13**); m_1 is the Sun and m_2 is Jupiter**.** L_2 is where the asteroid belt is**.** It must be realized for this model, this whole Sun-Jupiter diagram is rotating about its barycenter**.** The *Restricted 3-Body Problem* solution predicted the Trojan planetoids before they were first seen**.** The h-circles and closed loops echo the overlapping electromagnetic fluxes issued from the Sun and Jupiter**.** It is conjectured, contrary to Richard H. **BATTIN** conjecture (see **References**), the two h_1's are *gravity sinks*, in and where gravitational forces are zero, and the reason the Trojan planetoids are trap there**.** The total amount of mass of the planetoids is unknown**.**

For a discussion in the case where the negligible body is a satellite of the body of lesser mass, see *Hill sphere*; for binary systems, see *Roche lobe***.**

The Restricted 3-Body Problem (both circular and elliptical) was worked on extensively by many famous mathematicians and physicists, but most notably by Henri **POINCARÉ** at the end of the 19th century**.** Henri **POINCARÉ**'s work on the *Restricted 3-Body Problem* was the foundation of deterministic chaos theory**.** In the *Restricted Problem*, there exist five equilibrium points**.** Three are collinear with the masses (in the rotating frame) and are unstable**.** The remaining two are located on the third vertex of both equilateral triangles of which the two bodies are the first and second vertices**.**

The following article below (now in the public domain) by Dirk **BROUWER** was extracted from the "***Celestial Mechanics***" article in Vol. **15**, pp. 134d-137a, of the1968 Edition of the **Encyclopædia Britannia.** This particular article was rewritten later by Victor G. **SZEBEHELY** (see above) for the 1984 Edition, and maintained in Volume **11** in the *Britannia*'s 1988 **Macropædia** Edition**.** **SZEBEHELY**'s rewrite is excellent; but still, some is lost by deleting **BROUWER. BROUWER** also wrote **Methods of Celestial Mechanics** (see **References** below)**.**

The General Problem of Three Bodies.—This problem possesses ten known integrals, all of an algebraic character. Such an integral is a function of the co-ordinates and momenta of the three bodies that remains constant throughout the motion. The original equations of the problem form a system of the 18th order, nine differential equations of the 2nd order. With the aid of the ten integrals, the "elimination of the nodes" and the elimination of the time, the system may be reduced to one of the sixth order. This reduction was actually made by Lagrange and improved by later authors by the use of the canonical form of the equations. The reduction of the problem of three bodies with the aid of known integrals suggested that, if additional integrals were discovered, the problem might be further reduced and even completely solved. All such attempts failed; finally H. Bruns, in 1887, proved that no further algebraic integrals of the three-body problem exist. Soon afterward Poincaré proved that no further integrals uniform with respect to the elliptic elements exist. This result is of the greatest importance since it proves that the developments in trigonometric series used in the astronomical methods cannot converge for all values of the constants within a finite range. It does not exclude such a representation in the case of particular orbits; obvious examples are furnished by the periodic solutions and by the particular solutions which were first studied by Lagrange (*see* TROJAN PLANETS).

The simplified problem in which one of the three bodies has negligible mass and moves in the orbital plane of the two massive bodies, which are supposed to move in circular orbits, is the so-called restricted problem. The system of equations is one of the fourth order with one known integral, the Jacobian integral. Let $1 - m$ and m be the masses of the finite bodies; r, r_1, r_2, the distances of the infinitesimal mass from the centre of the mass and the two bodies respectively, and V the velocity of the infinitesimal mass in a co-ordinate system the origin of which is at the centre of mass, and which rotates uniformly with the period of revolution of the finite masses. The Jacobian integral, if the units of time and distance are conveniently chosen, is then

$$r^2 + \frac{2(1-m)}{r_1} + \frac{2m}{r_2} = V^2 + C,$$

C being an arbitrary constant. By putting $V^2 = 0$ one obtains a single family of curves with C as parameter. These "curves of zero velocity" may be looked upon as barriers in the sense that an orbit for which the constant of the Jacobian integral equals C' can never cross any of these curves of zero velocity for which C' exceeds C. The curves of zero velocity were first introduced by Hill with application to the moon's motion and have figured prominently in more recent studies of the restricted problem.

A totally different approach to the solution of the problem of three bodies was made by using developments in powers of a variable related to the time. If applied to the original equations the method fails owing to the singularities of the differential equations that correspond to collisions. These singularities may be removed by suitable changes of variables, a procedure known as regularization. The first significant step in this direction was made by P. Painlevé. In 1912 K. F. Sundmann obtained a solution for the general problem of three bodies that can be expanded as power series which are convergent, but not uniformly so, for all values of the time. This result is of great theoretical interest.

In the restricted problem, the use of the Jacobian integral permits the elimination of one of the velocity components. Hence the motion can be represented completely by a trajectory in a three-dimensional phase space comparable with a streamline in a noncompressible fluid. This approach permits an attack upon problems that were not accessible by other methods. The earlier developments are due to Poincaré; important advances were made by G. D. Birkhoff, especially on questions concerning the probability that a trajectory returns to the same small region in space.

BIBLIOGRAPHY.—An elementary exposition of the principles of celestial mechanics is found in Russell, Dugan and Stewart, *Astronomy*, vol. i, ch. x (1945). Introductory treatises are: F. R. Moulton, *Introduction to Celestial Mechanics*, 2nd ed. (1914); H. C. Plummer, *An Introductory Treatise on Dynamical Astronomy* (1918); W. M. Smart, *Celestial Mechanics* (1953). The standard treatise covering developments through the 19th century is F. Tisserand, *Traité de Mécanique Céleste*, 4 vol. (1889–96). Excellent chapters on the mathematical aspects of the problem of three bodies are contained in E. T. Whittaker, *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies*, 4th ed. (1937). Other important works for specialists are: H. Poincaré, *Les Méthodes Nouvelles de la Mécanique Céleste*, 3 vol. (1892–99); E. W. Brown, *An Introductory Treatise on the Lunar Theory* (1896); H. Poincaré, *Leçons de Mécanique Céleste*, 3 vol. (1905–10); G. D. Birkhoff, *Dynamical Systems*, Amer. Math. Soc. Colloquium Publ., vol. ix (1927); A. Wintner, *The Analytical Foundations of Celestial Mechanics* (1941); W. J. Eckert, Dirk Brouwer and G. M. Clemence, "Coordinates of the Five Outer Planets 1653–2060," *Astronomical Papers of the American Ephemeris*, vol. xii (1951).

(D. BR.)

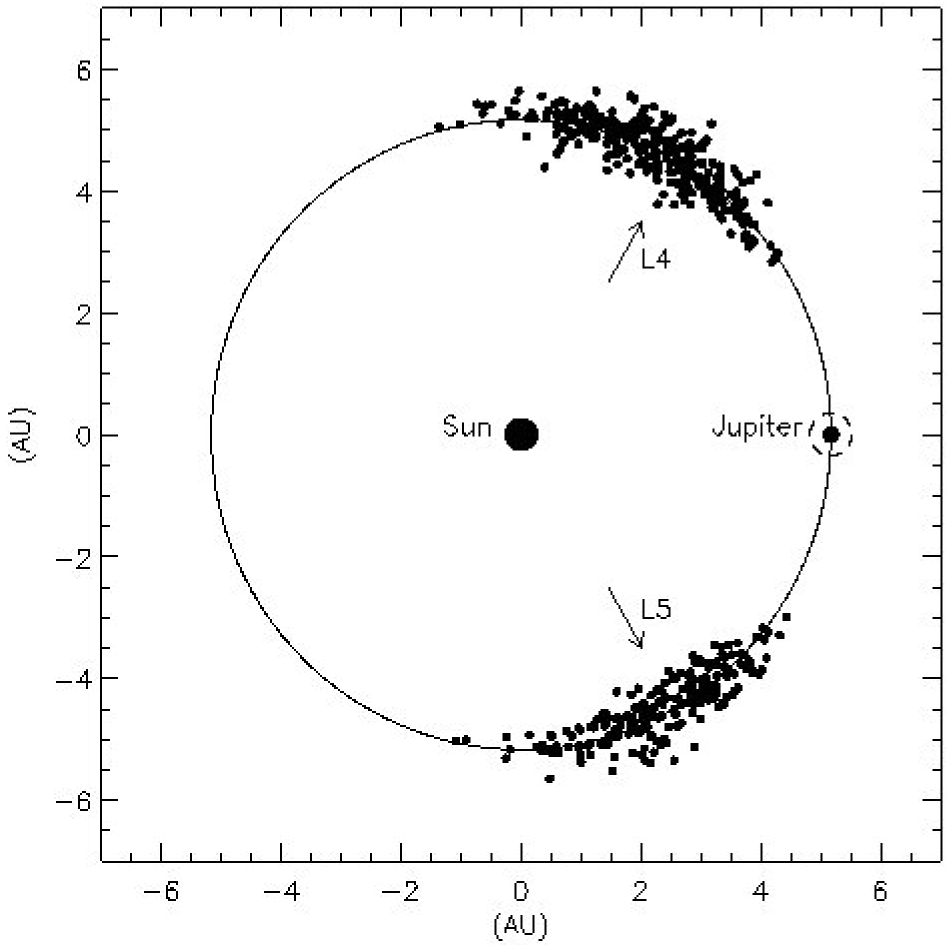

Figure **13:** The Trojan Planetoids

Any more discussion in regards to this classical approach is now rather pointless, considering the n-Body Problem solution following...

Generalization of **NEWTON**'s *Law of Universe Gravitation*

Many Bodies

Consider the classical case of one body being subjected to the gravitational forces of many other bodies. To obtain the total gravitational force on a body produced by many masses represented as m_1, m_2, m_3, ... m_ζ..., m_N, where ζ = 1, 2, 3, ... stands for the masses associated with the individual forces. These forces must be *vectorially* added together. Letting $\boldsymbol{r}$ be the spatial vector from the mass m to the m_ζ masses in association in the same way, and the force on m owning to several masses becomes the sum of the forces owning to each mass separately. Then, in simplified form, for the general case of the interactions and resulting motions of a system of N mass-points:

$$\vec{\boldsymbol{F}}_m = \vec{\boldsymbol{F}}_1 + \vec{\boldsymbol{F}}_2 + \vec{\boldsymbol{F}}_3 + \cdots = Gm\sum_{\zeta=1}^{N}\left(\frac{m_\zeta}{r_\zeta^3}\vec{\boldsymbol{r}}_\zeta\right)$$

Let the radius vector to these mass-points be:

$$\boldsymbol{r}_\varsigma = \bar{x}_\varsigma\hat{\boldsymbol{i}} + \bar{y}_\varsigma\hat{\boldsymbol{j}} + \bar{z}_\varsigma\hat{\boldsymbol{k}}$$

The radius vector may be written as a column vector, or as coordinate:

$$\boldsymbol{r}_\varsigma = \begin{Bmatrix}\bar{x}_\varsigma\\ \bar{y}_\varsigma\\ \bar{z}_\varsigma\end{Bmatrix}, \quad \text{or} \quad \boldsymbol{r}_\varsigma \Rightarrow \left(\bar{x}_\varsigma, \quad \bar{y}_\varsigma, \quad \bar{z}_\varsigma\right)$$

The coordinates of each mass-point P_ς is $\left(\bar{x}_\varsigma, \quad \bar{y}_\varsigma, \quad \bar{z}_\varsigma\right)$, where the "*bar*" indicates a body's mass has been reduced to a point. Then the distance[97] between any two mass-points, m_i and m_j is:

$$\boldsymbol{r}_{ij} = \left|r_j - r_i\right| := \sqrt{\left(r_j - r_i\right)^2}.$$

97 **MEIROVITCH**, Leonard: **Methods of Analytical Dynamics**, *McGraw-Hill*, "***Metric Spaces***," pp. 501-502.

The magnitude of the force of attraction between the i^{th} and j^{th} mass-points are $G \sum_{j=1,\, i \neq j}^{n} \frac{m_i m_j}{r_{ij}^2}$. The direction of the forces are conveniently expressed in terms of unit vectors**.** Thus, the force acting on m_i owning to m_j direction is $\left(\boldsymbol{r}_j - \boldsymbol{r}_i\right)/r_{ij}$; while the force on m_j owning to m_i is oppositely directed**.** Hence the total force $\boldsymbol{F}_\zeta$ affecting m_ζ, owning to the presence of the other N-1 mass-points, is**:**

$$\boldsymbol{F}_\varsigma = G \sum_{j=1,\, \varsigma \neq j}^{\mathrm{N}} \frac{m_\varsigma m_j}{r_{\varsigma j}^3} \left(\boldsymbol{r}_j - \boldsymbol{r}_\varsigma\right) .$$

Then, employing **NEWTON**'s 2nd *Law of Motion***:**

$$\boldsymbol{F}_\varsigma = m_\varsigma \frac{d^2 \boldsymbol{r}_\varsigma}{dt^2} = G \sum_{j=1,\, \varsigma \neq j}^{\mathrm{N}} \frac{m_\varsigma m_j}{r_{\varsigma j}^3} \left(\boldsymbol{r}_j - \boldsymbol{r}_\varsigma\right) .$$

which results in N vector differential equations, and represents the general case (which is in most Mechanics textbooks)**.** However, as will be explained below, the number of workable bodies has proven to be at best valid only when N ≤ 3**.** So in general the above *general* equation represents the *wrong* approach**.**

Failure of the Classical Approach

This Section relates why the classical approach to the *n-body problem won't work***.**

First, **NEWTON** correctly reasoned gravity is responsible for the motions of planets and stars, and he expressed their gravitational interactions in terms of his *Law of Universal Gravitation*; but soon found, like everyone else since, it was not a general solution to the *n-body problem* (see **LEIMANIS**' historical discussions). However, his conceptual thinking (i.e., his three *Laws***:** $\boldsymbol{F} = m\boldsymbol{a}$ and gravity) was the catalysis which started the quest of the modern approach to the solution to the *n-body problem***.**

$\boldsymbol{F}_{ij} = f(m_i, m_j \ddot{\boldsymbol{r}}_{ij})$, where i and j are integers, 1, 2, 3, **...** Otherwise, $\boldsymbol{F}$ is a function of *both* mass and geometry**.** Then, to employ **NEWTON**'s *Universal Law*, let $\boldsymbol{F}_{12} = G(m_1\, m_2 / r^3)\boldsymbol{r}$ be the force on m_1 owing to mass m_2; likewise, let $\boldsymbol{F}_{21} = G(m_1\, m_2 / r^3)\boldsymbol{r}$ be the force on m_2 owing to m_1. This way one arrives at two separate equations, and by **NEWTON**'s 3rd *Law*, $\boldsymbol{F}_{12} = -\boldsymbol{F}_{21}$ (see the **Two-body Problem** above). Imagine if there were three masses involved: there would be six separate equations, $\boldsymbol{F}_{12}$, $\boldsymbol{F}_{21}$, $\boldsymbol{F}_{13}$, $\boldsymbol{F}_{31}$, $\boldsymbol{F}_{23}$, $\boldsymbol{F}_{32}$. If four masses (or more) were involved there would be a mathematical mess. This is one reason why this approach fails.

In addition, employing **NEWTON**'s generalized *Universal Law* formula results in the *n-body problem* containing $6n$ variables, since each mass-point is represented by three spatial displacements and three momentum components. *First integrals* (for ordinary differential equations) are functions that remain constant along any given solution of the system, the constant depending on the solution. Otherwise, integrals provide relations between the variables of the system, so each scalar integral would normally allow the reduction of the system's dimension by one unit. Of course, this reduction can take place only if the integral is an uncomplicated *algebraic* function with respect to its variables.

If the integral is *transcendent* the reduction cannot be performed. Summing up, the *n-body problem*, it has 10 independent algebraic integrals:

- three for the center of mass;
- three for the linear momentum;
- three for the angular momentum;
- one for the energy.

This allows reduction of the variables to $6n - 10$. The question at the time this was first being addressed was whether there exist other integrals besides these 10; and the answer was given in the negative in 1887 by Heinrich **BRUNS**: *Theorem*: (First integrals of the *n-body problem*:) the only linearly independent integrals of the *n-body problem*, which are algebraic with respect to the variables are the 10 described above. This *Theorem* was later generalized by Henri **POINCARÉ** (1854-1912)[98].

⇒ Above all other considerations (as will be shown): *the n-body problem, when $n \geq 3$, is <u>not</u> a function of mass, just force and geometry*. Otherwise, as will be shown, **NEWTON**'s *Universal Law* has nothing to do with the <u>*general solution*</u> for the

98 These results however do not imply the perturbation series (*Lindstedt series*) diverges. Indeed, **SUNDMAN** provided such a solution by means of convergent series. (See **SUNDMAN**'s *Theorem* for the *3-body problem*). See **References** sited in Richard E. **CRANDALL**'s book **<u>Projects in Scientific Computation</u>**, pp. 94-96, and **ROSENBERG**'s book **<u>Analytical Dynamics</u>**, pp. 364-371, for more discussion about the ramifications of this approach.

n-body problem, period**.**

The *n-body problem* is just one aspect of the complexities of gravitation**.** The latter's solution via a Newtonian approach usually starts historically with the 2-*body problem* as above; but for three or more bodies, and *in general*, *that approach won't work***.** For the above reasons, a general, closed-form, *n-body problem* solution (i.,e., *n*-formulae) cannot be derived by employing **NEWTON**'s *Law of Universal Gravitation* equation in combination with his 2nd *Law*, *in any form***.** Period**.**

Practical n-body Problem Theoretical Solution

This Section presents a single *closed-form*, *algebraic* algorithm (see Equation **10** below) for calculating the reaction response of an arbitrary n-number ($n \geq 3$) of spatial reaction points or support points, owning to a single arbitrary concentrated applied load (a force and moment). The applied load is redistributed to the centroids of the reaction, mass-point *pattern* by employing in part, the massless geometric inertia properties of the support pattern.

Simply put, this solution to the *n-body problem* takes an external load and moves it over to the centroids of the reactions or supporting pattern; and by incorporating the massless geometric inertia properties of that pattern, beams (maps) the applied load to those reaction points, resulting in equilibrium between the applied load and the reactions.

Although somewhat messy algebraically, it's that simple. Proof of the *algorithm* (Eq. **10**) is by demonstrating (calculating) equilibrium for each particular problem, and the general equilibrium equations for the algorithm are given below.

Before preceding with the derivation of Equation **10**, consider the given cantilever beam shown below with its offset load F_1 at point A.

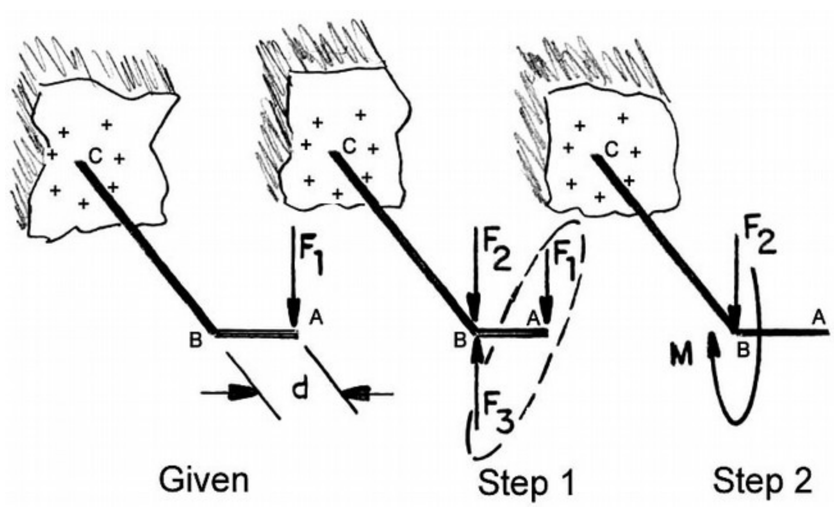

Figure **14:** Transferring a Force Produces a Moment

If the original force, F_1 at A is reduplicated and transferred to the tip at B as F_2, then F_2 has to be balanced by an up-load, F_3, for the two, F_2 and F_3, to remain in equilibrium. This up-load F_3, combined with the original down-load F_1, is a couple and equivalent to a moment $\boldsymbol{M}$ (torque). The final loading at the tip, point B, is $\boldsymbol{F}_2$ plus $\boldsymbol{M}$, or simply a force $\boldsymbol{F}$ plus a moment $\boldsymbol{M}$.

Second consideration is at the root of the cantilever: the reaction forces at the attachment points (seven shown) not only react with the vertical force $\boldsymbol{F}_2$ in pure upward vertical shear divided by the number of attachment points, but also by horizontal reaction _couples_ in- and out- of the wall. The moment $\boldsymbol{M}$ (torque) is reacted by only horizontal reaction couples at the root, parallel (in-plane) to the wall. In this way (symbolically) the external moment $\boldsymbol{M} = \boldsymbol{M}_{\text{external}} = \boldsymbol{M}_{\text{ internal}} = \boldsymbol{rF}_{\text{reaction}}$; i.e., the external moment(s) *equal* the in-plane and out-of-plane static reaction _couples_ at the attachments in the wall (via **D'ALEMBERT**'s 2nd *Principle* (*and not by **NEWTON**'s* 3rd *Law, which would be incorrect reasoning*).[99]

The n*-body problem solution* can be somewhat analogous to a ***3D***, rigid-body, rivet analysis; but it is more, because the "*load*" can be any physical quantity.[100] However, the algorithm, *_when used only as a function of load_*, is then only a function of geometry and _not_ mass. [As discussed below, a ***3D***, *_elastic_* rivet analysis employs the Hookean springs employing the $\boldsymbol{A}$'s vectors, which result in loads considerable different then those of a rigid body solution.]

The n*-body problem solution*, referred to as "***The Force,***" where ($n \geq 3$), is begun with the simple relationship everyone knows: $s = r\theta$. Multiplying the right side by $\boldsymbol{A}$, the *Hookean vector*, makes $\boldsymbol{s}$ a vector (discussed in the next subsection). Then Jean Le Rond **D'ALEMBERT**'s (1717-1783) 2nd *Principle* is applied. Turns out later, any solution to the n*-body problem* requires using both of **D'ALEMBERT**'s 1st and 2nd *Principles*, which are key to solving the general solution.

The next Section and remainder Sections develops an algorithm for determining the ***3D*** reaction loads owning to an single applied load. Superposition

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99 It is further conjectured, in some cases, some resultant internal moment sets, $\boldsymbol{M}$, may not be unique owning to the excessive number of reaction or support points associated with some n*-body problem*s **--** is a subject beyond the present paper's scope and development; nevertheless, if true, it is contrary to classical mechanics, which holds every moment set situation is unique.

100 This algorithm solution for the n*-body problem* occurred while trying to distribute useful applied loads (i.e., human body loads, luggage, attached extra hardware, etc.) to the grids points (spatial points) belonging to the surrounding basic structures of finite element models. How else would one distribute applied loads if not by a n*-body problem* type solution? The similarity for determining the reactive loads for a ***3D***-rivit analyses also became apparent. Originally it took 45 minutes to derive this solution but two and a half days to write the algorithm. This is to say the development of the algorithm is really simple.

can be employed for those case with more then one applied load.

Hookean Switches and Springs, Mass-Points and the Inertia Tensor

A vector operator $\boldsymbol{A}$, referred to as a *Hookean switch*, is introduced for controlling translational *degrees-of-freedom* [***DOF***]; and functions mathematically in a somewhat similar purpose as in grid point releases in a *Finite Element Analyses or Model* [***FEA*** or ***FEM***][101]. The $\boldsymbol{A}$'s three vector components are mathematical tools, which greatly simplify the static or quasi-steady equations of motion, which is $\boldsymbol{A}$'s purpose. *The components of the $\boldsymbol{A}$'s Hookean switches can remove when set to zero, translational **DOF**, thus allowing only feasible **DOF** reactions*. Hookean switches for rigid-*body problem*s in this way are logic, on-off switches.

In a ***3D*** rivet analysis for example, where some fasteners resist load only in one direction – the $\boldsymbol{A}$'s remove all other ***DOF*** (translational components) from the set of equations by setting the other $\boldsymbol{A}$ components to zero (see the example problem). The $\boldsymbol{A}$'s components can be be thought of as springs too, and used in combination with ***FEA GAP*** elements. Another example is if a problem is strictly planer in the, say x-y plane, then all the out-of-plane ***DOF*** are removed. That is, all the A_z's components are zero perpendicular to the x-y plane. This logic, on-off feature greatly simplifies area-moment of inertia calculations too.

The mathematical development of the $\boldsymbol{A}$'s is as follows:

- Let there be a vector function $\boldsymbol{A}_{\zeta i} \Rightarrow \{A_{\zeta x},\ A_{\zeta y},\ A_{\zeta z}\}$, where for each component range is $0 \le A_{\zeta i} \le 1.0$, and where $i := x, y, z$ or 1, 2, 3, represent the $\boldsymbol{A}$'s component magnitudes and directions; and where $\zeta := 1,...N$, are the number of mass-points' m_ζ.
- $A_{\zeta i}$ coefficients are functionally the feasible, translational reaction component's allowable ***DOF*** at each point $P_\zeta \Rightarrow (x_\zeta, y_\zeta, z_\zeta)$ specified for all mass-points m_ζ, and where the N mass-points m_ζ coordinates (x, y, z) are defined as being in the first quadrant in a right-hand, Euclidean coordinate system, denoted as E_1^3

101 In *Finite Element Analyses*, every grid (point in space) has six ***DOF*** and if some ***DOF***'s are not attached to the structure they spin, causing a singularity in the stiffness matrix. These loose grids are restricted from spinning by constraining them ("*pinning loose **DOF** to the wall*").

[One has to be careful when operating in different quadrants, see **GELMAN's** papers in **References**].

- Coupling forces are reacted at these mass-points too and are subject to the same feasible translational constraints (since couplings are naught but coupled forces).

Applied *Hookean switches* are vectors applied at the supports or reaction points but normally would not be shown on a free-body diagram per se. Of all three possible ***DOF*** translational vectors for each real $\boldsymbol{P}_{\zeta i}$ force at each $P_{\zeta i}$, triplet coordinate, only those feasible or allowable translational vectors would normally be shown as usual on a free-body diagram.

Physically speaking, a structural linear-analysis is based upon **HOOKE**'s *Law* $\boldsymbol{P} = k\boldsymbol{x}$, a concept employed in ***FEA***. In ***FEA***, the $\boldsymbol{x}$'s displacement vector's components $(x_i,\ y_i,\ z_i)$ for a particular spatial point $(x,\ y,\ z)$ are removed by specifying a grid point component's "*grid point constraint*," thus removing that variable from both the displacement vector *and* stiffness matrix k's row; thus resulting in no reaction for that direction. In ***FEA*** it is somewhat difficult to change configuration space for a particular ***FEM*** using constraints (its easy to make mistakes). With the $\boldsymbol{A}$'s, the size of configuration space can be changed with relative ease. A spatial model here using $\boldsymbol{A}$'s as springs could be translated into a ***FEM***, but that is a subject somewhat beyond the scope here.

Transformation aside, $\boldsymbol{A}$ values in between 0.0 and 1.0 can be envisioned as springs. A value of 1.0 represents a rigid body; but when one of the components of $\boldsymbol{A}_{\zeta I} \Rightarrow \{A_{\zeta x},\ A_{\zeta y},\ A_{\zeta z}\}$, for a particular spatial point $(x,\ y,\ z)$, is set to zero, the results is not only to eliminate that dimension from spatial geometry, but also its inertial properties for that dimension, and any reactive force in that particular direction associated with that component.

For any real, physical body there must be at least three $P_{\zeta j}$ points for initial stability having a minimum of <u>*six*</u> ***DOF***; meaning there must be at least six feasible reactions for these three points: three $P_{\zeta j}$ points with six non-zeroes $A_{\zeta i}$'s. (This latter requirement is the same as balancing an aircraft's ***FEM*** model on three points and then applying Jean Le Rond **D'ALEMBERT**'s two *Principles*, resulting in equilibrium when the externally applied loads just equals the internal inertial loads, resulting in no load on the three points.)[102]

Let the radius vector be the dot product:

$$\boldsymbol{r}_\zeta \boldsymbol{A}_\zeta = r_{\varsigma x} A_{\varsigma x} \hat{\boldsymbol{i}} + r_{\zeta y} A_{\zeta y} \hat{\boldsymbol{j}} + r_{\zeta z} A_{\zeta z} \hat{\boldsymbol{k}},$$

where $\hat{\boldsymbol{i}}, \hat{\boldsymbol{j}}, \hat{\boldsymbol{k}}$ are the unit directional vectors in E_1^3 and the $\boldsymbol{A}$'s are the Hookean switches or springs as defined above. The preferred way to write this dot product is:

$$\boldsymbol{r}_\zeta \boldsymbol{A}_\zeta \Rightarrow \left(r_{\varsigma x} A_{\varsigma x}, r_{\zeta y} A_{\zeta y}, r_{\zeta z} A_{\zeta z} \right),$$

Let $M := m_1 + m_2 + m_3 + ... + m_N$, for ζ = 1, 2, ..., N mass points P_ζ. For each m_ζ there corresponds a spatial triplet $P_\zeta \Rightarrow \left(\bar{x}_\varsigma, \bar{y}_\varsigma, \bar{z}_\varsigma \right)$. Each spatial variable of the triplet $(x,\ y,\ z)$ belonging to the set P_ζ with bars, $\left(\bar{x}_\varsigma, \bar{y}_\varsigma, \bar{z}_\varsigma \right)$ represents the centroid of its mass at that coordinate point. Meaning the volume of each mass is reduced to a point, but its magnitude need not be reduced; and in fact can be regulated via $\boldsymbol{A}$'s components being <1.0.

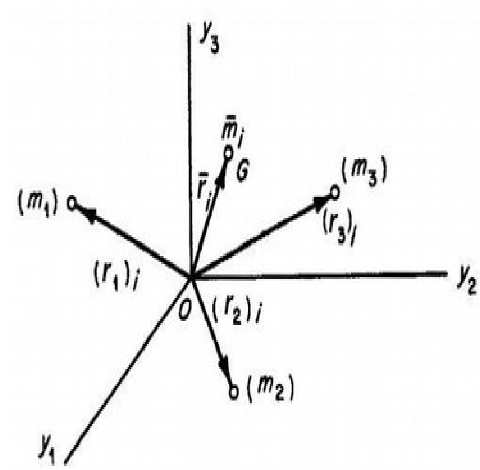

Figure **15:** Radius Vectors

Otherwise, the $\boldsymbol{A}$ components can represent mass. Let $\boldsymbol{r}_{\zeta\, i}$ at $P_{\zeta i}$ be a

102 For some rules about drawing free-body diagrams see **Engineering Mechanics**, 3rd Ed.. by W. G. **McLEAN** and E. W. **NELSON**, *Schaum's Outline Series*, *McGraw-Hill*, Chapter 5, "***Equilibrium of Coplanar Force Systems***," pp. 47-48, 1978. "*The force system being analyzed will be holding a body or system of bodies in equilibrium. A free-body diagram is a sketch of the body (bodies) showing (a) all active forces, such as applied loads, gravity forces and (b) all reactive forces. The latter forces are supplied by the ground, walls, pins, rollers, cables, or other means. A roller or knife-edge support means the reaction there* [at the support point] *is shown perpendicular to the member. A pin connection means the reaction can be at any angle – it is represented by a force at an unknown angle or by using components of the pin reaction, i.e., at point A_x and A_y in a plane*.

"*If the angle the reaction makes is known, the sense* [direction of the force] *is then assumed along the reaction line. A positive sign of a force from the results* [i.e., after summing ΣF and resolving all forces] *indicates the proper sense* [initially] *assumed* [was correct]; *a negative sign indicates the reaction force has the opposite sense to that assumed*." Don't worry if your initially assumed directions of the forces are correct or incorrect, because after the summation process and all forces are resolved, the correct signs (directions) will result. Redraw the free-body diagram, and in so doing correct all the incorrect force and reaction *directions*.

OMG: this is one of the *very few* physics textbooks specifying *free-body diagram* rules!

Hookean vector for each m_ζ in all $(x,\ y,\ z)$ If all $A_{\zeta i}$ = 0 then $\boldsymbol{r}_{\zeta i} = \mathbf{0}$; if all $A_{\zeta i}$ = 1.0 then there are three reaction components and the $\boldsymbol{A}$'s represent a rigid body point at the $P_{\zeta i}$. In between these values are $0 < A_{\zeta\ i} < 1.0$, the Hookean springs and together these three stipulations describe all possible Hookean switches.

It is assumed initially each point has associated with it both mass and force; later these points need not have any mass in determining the reaction forces when subject to a single applied force; in symbols, $\left(\boldsymbol{F}_{\text{Applied}}(x) \xrightarrow{\text{1-1}} \boldsymbol{P}_{\zeta i}\right)$. It will be shown the reacting forces are only a function of geometry. Later it will be realized these points can represent mass, or some other physical quality, whose magnitudes can be proportioned via the $\boldsymbol{A}$ components by ranking.

Excluding mass, the six *massless* barycenters (six naturally develops when using index notation) for $P_{\zeta i} \geq 3$, but including the $\boldsymbol{A}$'s, are, by simple statics[103]:

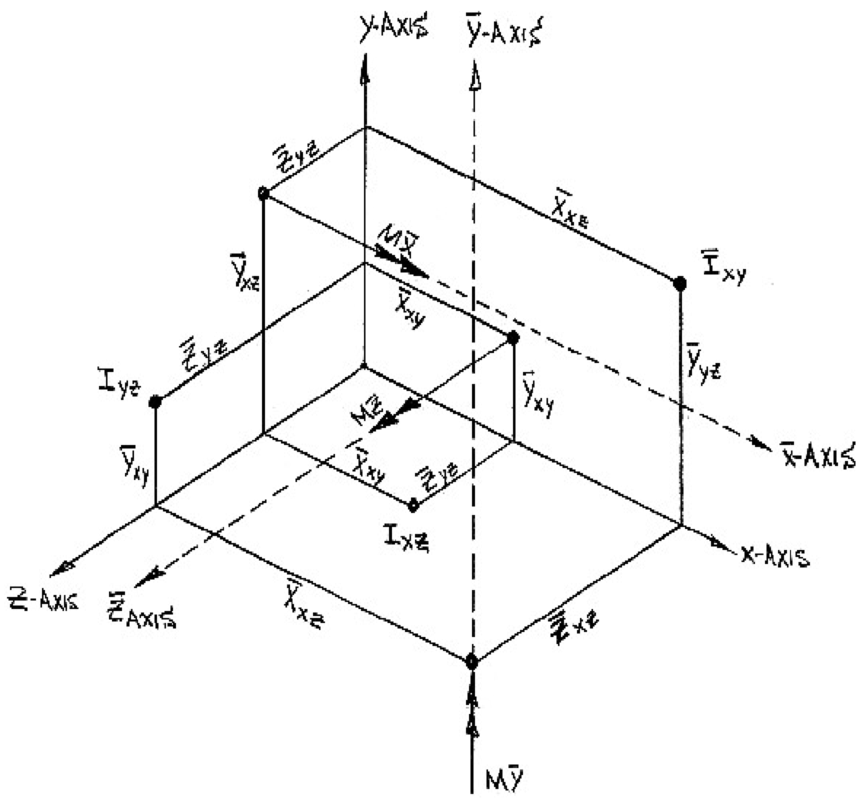

Figure **16:** Support Centroids

103 **BRUHN**, E. F.: **Analysis and Design of Flight Vehicle Structures**, Chapter **A3**, "***Properties of Sections – Centroids, Moments of Inertia, Etc.***," *Tri-State Offset Co*., pp. A3.1-3.16, 1965. Equations incorporate the *Parallel Axis Theorem* based on his Figure **6**. **BRUHN** is out-of-print.

Mathematically:

$$\bar{x}_{xz} = \frac{\sum A_{\varsigma z}\bar{x}_{\varsigma}}{\sum A_{\varsigma z}};\quad \bar{x}_{xy} = \frac{\sum A_{\varsigma y}\bar{x}_{\varsigma}}{\sum A_{\varsigma y}};\quad \bar{y}_{yz} = \frac{\sum A_{\varsigma z}\bar{y}_{\varsigma}}{\sum A_{\varsigma z}};$$
$$\bar{y}_{xy} = \frac{\sum A_{\varsigma x}\bar{y}_{\varsigma}}{\sum A_{\varsigma x}};\quad \bar{z}_{yz} = \frac{\sum A_{\varsigma y}\bar{z}_{\varsigma}}{\sum A_{\varsigma y}};\quad \bar{z}_{xz} = \frac{\sum A_{\varsigma x}\bar{z}_{\varsigma}}{\sum A_{\varsigma x}}. \qquad \text{.Eq. } \mathbf{4}$$

Where $\bar{\boldsymbol{r}} \Rightarrow \{\bar{x},\ \bar{\mathrm{y}},\ \bar{\mathrm{z}}\}$, but the values are given by Eq. 4**.**

If the reaction or support pattern is semi-symmetrical then the centroid set will be less than six; as, say for example, when x_{ij} = x_{ji}**.** If the pattern is entirely symmetrical then these six centroids will revert to the usual three**.** Continuing, the massless centroidal mass moments of inertia for N massless points, again by simple statics, are**:**

$$\begin{aligned}
\bar{I}_{xx} &= \sum A_{\varsigma y}\bar{z}_{\varsigma}^{2} + \sum A_{\varsigma z}\bar{y}_{\varsigma}^{2} - \bar{y}_{yz}^{2}\sum A_{\varsigma z} - \bar{z}_{yz}^{2}\sum A_{\varsigma y}\\
\bar{I}_{yy} &= \sum A_{\varsigma x}\bar{z}_{\varsigma}^{2} + \sum A_{\varsigma z}\bar{x}_{\varsigma}^{2} - \bar{x}_{xz}^{2}\sum A_{\varsigma z} - \bar{z}_{xz}^{2}\sum A_{\varsigma x}\\
\bar{I}_{zz} &= \sum A_{\varsigma x}\bar{y}_{\varsigma}^{2} + \sum A_{\varsigma y}\bar{x}_{\varsigma}^{2} - \bar{x}_{xz}^{2}\sum A_{\varsigma y} - \bar{y}_{xz}^{2}\sum A_{\varsigma x} \qquad \text{............Eq. } \mathbf{5}\\
\bar{I}_{xy} &= -\left(\sum A_{\varsigma z}\bar{x}_{\varsigma}\bar{y}_{\varsigma} - \bar{y}_{yz}\bar{x}_{xz}\sum A_{\varsigma z}\right)\\
\bar{I}_{yz} &= -\left(\sum A_{\varsigma x}\bar{y}_{\varsigma}\bar{z}_{\varsigma} - \bar{y}_{xy}\bar{z}_{xz}\sum A_{\varsigma x}\right)\\
\bar{I}_{xz} &= -\left(\sum A_{\varsigma y}\bar{x}_{\varsigma}\bar{z}_{\varsigma} - \bar{x}_{xy}\bar{z}_{yz}\sum A_{\varsigma y}\right)
\end{aligned}$$

where, say $\bar{x}_{xy}$, etc., are the barycenters calculated above[104, 105]**.** These equations are the geometric properties of the set of spatial points, and not associated with any

104 These equations, except for the addition of the A's, can be found in any standard dynamics textbooks; say one such as **MERIAM**'s **Engineering Mechanics**, Volume 2**.** See J. **HEADING**'s **Matrix Theory for Physicists**, *Longmans, Green and Co*., 1958**.** Required reading is James M. **GERE** and Wm. **WEAVER**, Jr., *priceless* **Matrix Algebra for Engineers**, *Van Nostrand Reinhold Co*., 1965, especially their last Chapter, "***Coordinate Transformations***."

105 The above equations are computerized (see below) so if the first, front, upper quadrant is used there is no need to worry about getting the signs wrong**.** If however, the left, front, upper quadrant is used too, then a matrix, and its inverse for an unsymmetrical matrix from the right side is not equal to its inverse and transpose on the left side**.** Each side must be independently developed**.** This problem occurs when an aircraft's main coordinate system splits it from the nose -to- the tail down the middle**.** Be careful using the n-body solution in other quadrants**.** Ref **GELMAN.**

mass.

Applied forces and moments may be Hookean too (generally by default, like planer forces).

A few words of caution: clarification and qualification for a right-hand coordinate system again arises because some U.S. Aerospace firms in some of their older documents without clarity used the left-hand coordinate system, making their signs for the cross products positive. If the inertia tensor is symmetric and positive-definite via the value of its determinant, then the coordinate system is right-handed – always check this because if not, then your cross-product signs are wrong.[106]

Don't use the following method presented below (***The Force***) to do a rigid body analysis for an *elastic* **FEM**, not even as an approximation, *as the latter **The Force** results will be wrong* (like apples and oranges). And use both series and parallel $\boldsymbol{A}$ springs between the various layers with all $A_{\zeta_i} < 1.0$, for ***3D*** rivet and bolt analyses.

The Force

In all that follows, is naught but simple physics: statics and no calculus. The *algorithm*'s development, called ***The Force***, following is somewhat similar in purpose to **GALLIAN** and **WILSON**'s whitepaper[107] for determining internal structual loads via ***FEA***. Some physical concepts here will be interpolated in more than one way when using their counterpart English (non-mathematical) word descriptions.

Interestingly, if *empirical force laws* and Euclidean geometry are true, and the *Principle of Equilibrium* is true of course, then any logical combination therein may reasonably be expected to also be true, if echoing correctly physical observations.

Consider for example a spring model. If the reaction points' material is structural; i.e., elastic $(\sigma = E\ \varepsilon)$, then it is further assumed in elementary analyses the structure will conform to a similar linear law, say **HOOKE**'s *Law*, $(\boldsymbol{F} = k\ \boldsymbol{x})$ of load-deformation proportionality, where k is the spring rate (stiffness). By inference the $\boldsymbol{A}$'s are considered analogous to k. Later it will be realized this assumption is not entirely necessary as $\boldsymbol{F} = f(k,\ \boldsymbol{x})$ can be equal to any functional without its exact

106 **GERE**, James M. and **WEAVER**, William, Jr.: **Matrix Algebra for Engineers**, *Van Nostrand Reinhold Co*., 1963. See especially Chapter 7, "***Coordinate Transformation***," pp. 137-160.

107 See **GALLIAN** and **WILSON** pdf file: this short, highly technical physics paper is a full-blown example of a real-world application of **D'ALEMBERT**'s 1st and 2nd *Principles* applied to a body floating in space just above the Earth's surface (and not in outer-space). The equations developed there are similar to some equations developed here; i.e., centroids and inertia equations.

definition given. But to start, the analogy is employed below in developing ***The Force***.

When an arc segment $\Delta s_{\varsigma i} = \varepsilon_{ijk}\bar{\boldsymbol{r}}_{\varsigma j}\Delta\boldsymbol{\theta}_{\varsigma k}$ (analogous to $s = r\theta$ from simple geometry) is combined with **D'ALEMBERT**'s two *Principles* $\boldsymbol{R} = 0$ and $\boldsymbol{M} = 0$,[108] then an algorithm (formula) called ***The Force***, can be developed giving the reactive forces for a pattern of points subject to a single external resultant force $\boldsymbol{F}_i$ and moment $\boldsymbol{M}_{oi}$, where $i = x, y, z$. The Δ's imply infinitesimal displacements.

$\bar{\boldsymbol{r}}_{\zeta j} \Rightarrow \{\bar{x}_\zeta, \bar{y}_\zeta, \bar{z}_\zeta\}$ may be a Hookean vector, and $\boldsymbol{s}_{\zeta i}$ covers all space. It will be shown ***The Force*** can be used to completely solve any *n-body problem* for any $n \geq 3$.

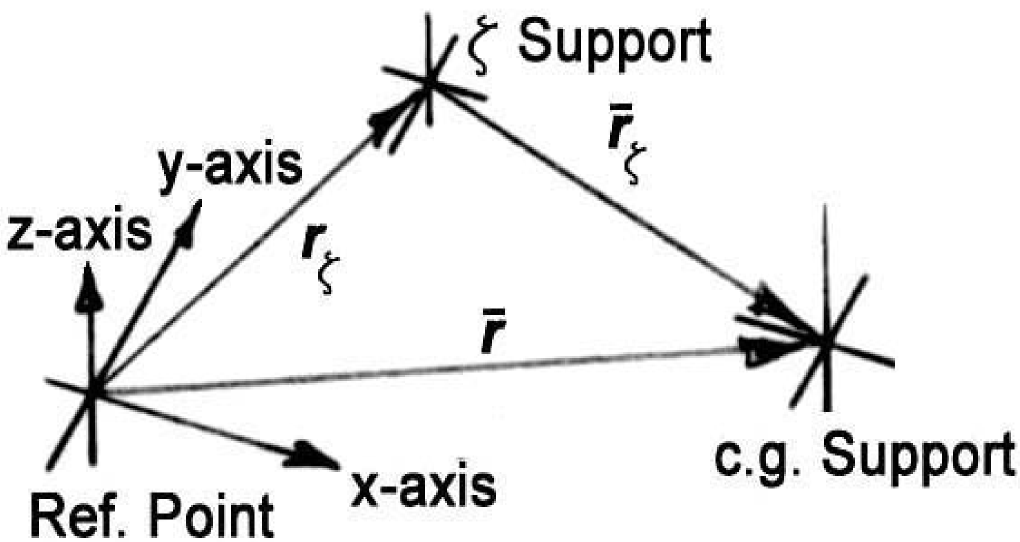

Figure **17:** Support Position Vectors: $\{\bar{r}_\varsigma\} = \{\bar{r}\} - \{r_\varsigma\}$

Referring to Figure **14** above, let the radius vector $\boldsymbol{r}_\zeta \Rightarrow \{\boldsymbol{r}_\zeta | \zeta = 1, 2, 3, ...N\}$ for $P_\zeta \geq 3$ massless-points be the radius vectors to the reaction or support points: are the set of radius vector, centroid coordinates of each reaction or supports. $\boldsymbol{r}_\zeta$ coordinates are basic input to the algorithm, and N is the total number of point in the

108 An allegory to **D'ALEMBERT**'s theory: if a body (an aircraft say) is supported via three external points, for a total of six-degrees of freedom — for complete initial stability — then, when the applied external loads are applied and equal to the reacting internal inertial loads (ma loads), there are no forces on the three points (the acid test). All forces, as well as couples, are balanced this way: external forces = internal forces.

set $\boldsymbol{r}_{\zeta}$. The reaction or support radius vector coordinate's c.g. Is at $\bar{\boldsymbol{r}}$. The c.g. of the reactions/supports is not $\bar{\boldsymbol{r}} \Rightarrow \{\bar{x},\ \bar{y},\ \bar{z}\}$, but the values given by Eq. 4.

Note also $\bar{\boldsymbol{r}}_{\varsigma i} \Rightarrow \{\bar{r}_{\varsigma x},\ \bar{r}_{\varsigma y},\ \bar{r}_{\varsigma z}\} \equiv \{\bar{r}_{\varsigma 1},\ \bar{r}_{\varsigma 2},\ \bar{r}_{\varsigma 3}\}$ are equivalent.

Assume an infinitesimal rigid-body displacement associated with these massless points owing to the infinitesimal angular rotations $\Delta\boldsymbol{\theta}_{\zeta i}$. The value and direction of the $\Delta\boldsymbol{\theta}_{\zeta i}$ rotations depends on the positions of the massless points with respect to the c.g. $\bar{\boldsymbol{r}}$. Then the reaction displacements are assumed to be**:**

$$\Delta s_{\varsigma i} = \varepsilon_{ijk}\bar{\boldsymbol{r}}_{\varsigma j}\Delta\boldsymbol{\theta}_{\varsigma k}\ , \qquad \text{Eq. } \mathbf{6}$$

where $\bar{\boldsymbol{r}}_{\varsigma j}$ are the position vectors of the reaction points or supports to the mass-point's $c.g.$ (Figure **14**). Real masses may be reduced to mass-points the same way **NEWTON** did it[109]. The underlying objective with $\bar{\boldsymbol{r}}_{\varsigma j}$ is to determine its relative motion. The $\boldsymbol{A}$'s will be added later to Eq. **6**..

Assume also the displacements occur in an instant of time; i.e., time is fixed, thus making the solution become one of *statics*, which may represent *quasi-steady* loads as well, reduced from a dynamic model.

The $\bar{\boldsymbol{r}}_{\varsigma} \Rightarrow \{\bar{r}_x, \bar{r}_y, \bar{r}_z\} \equiv \{\bar{r}_1, \bar{r}_2, \bar{r}_3\}$ coordinate set is use later to calculate the sum of the moments knowing the internal loads $\{\boldsymbol{P}_{\zeta i}\}$. The applied load center is defined to be at $(\bar{x}_{cg}, \bar{y}_{cg}, \bar{z}_{cg})$, who coordinates are basic input to the algorithm.

This completes the definition of $\Delta\boldsymbol{s}_{\zeta i}$.

Reaction or Support Side Equations

The Hookean $\gtrapprox\boldsymbol{s}_{\zeta i}$ equations in expanded matrix form are exactly analogous to $\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{F}$ or the bisor (second order tensor) $\boldsymbol{M}_i = \varepsilon_{ijk}\boldsymbol{r}_j\boldsymbol{F}_k$**:**

109 For a *proof* − the force of gravitational attraction of a homogeneous mass M and radius R is equivalent to a force exerted by a particle of mass m located at the center of the sphere − see D. T. **GREENWOOD: Classical Dynamics**, *Prentice-Hall*, 1988.

$$\begin{Bmatrix} \Delta s_{\varsigma 1} \\ \Delta s_{\varsigma 2} \\ \Delta s_{\varsigma 3} \end{Bmatrix} = \begin{Bmatrix} \bar{r}_{\varsigma 3}\Delta\theta_{\varsigma 2} - \bar{r}_{\varsigma 2}\Delta\theta_{\varsigma 3} \\ \bar{r}_{\varsigma 1}\Delta\theta_{\varsigma 3} - \bar{r}_{\varsigma 3}\Delta\theta_{\varsigma 1} \\ \bar{r}_{\varsigma 2}\Delta\theta_{\varsigma 1} - \bar{r}_{\varsigma 1}\Delta\theta_{\varsigma 2} \end{Bmatrix} = \left\{ \begin{bmatrix} 0 & \bar{r}_{\varsigma 3} & -\bar{r}_{\varsigma 2} \\ -\bar{r}_{\varsigma 3} & 0 & \bar{r}_{\varsigma 1} \\ \bar{r}_{\varsigma 2} & -\bar{r}_{\varsigma 1} & 0 \end{bmatrix} \begin{Bmatrix} \Delta\theta_{\varsigma 1} \\ \Delta\theta_{\varsigma 2} \\ \Delta\theta_{\varsigma 3} \end{Bmatrix} \right\}$$

The $\left[\bar{r}_{\zeta i}\right]$ matrix is known as a duel matrix. From Figure **14** above, $\left\{\bar{r}_{\varsigma}\right\} = \left\{\bar{r}\right\} - \left\{r_{\varsigma}\right\}$ is a vector equation. Then $\Delta s_{\zeta i} = \varepsilon_{ijk}\left\{\bar{r} - r_{\zeta j}\right\}\Delta\theta_{\zeta k}$, where

$$\begin{aligned} \bar{r}_{\varsigma 1} &= \bar{r}_1 - r_{\varsigma 1} \Rightarrow \bar{x}_{\varsigma} = \bar{x} - x_{\varsigma} \\ \bar{r}_{\varsigma 2} &= \bar{r}_2 - r_{\varsigma 2} \Rightarrow \bar{y}_{\varsigma} = \bar{y} - y_{\varsigma} \\ \bar{r}_{\varsigma 3} &= \bar{r}_3 - r_{\varsigma 3} \Rightarrow \bar{z}_{\varsigma} = \bar{z} - z_{\varsigma}. \end{aligned} \qquad \text{................Eq. } \mathbf{6a}$$

where the variables are shown in Eq. **6a** are comparable to those shown in Eq. **4** and **5:**

$$\left(\bar{x}_{\varsigma}, \bar{y}_{\varsigma}, \bar{z}_{\varsigma}\right) \Rightarrow \left(\bar{x}_{xy}, \quad \bar{x}_{xz}, \quad \bar{y}_{xy}, \quad \bar{y}_{yz}, \quad \bar{z}_{xz}, \quad \bar{z}_{yz},\right).$$

Substitution gives the reaction or support *duel matrix*; i.e., gives a *skewed Hermitian*:

$$\left[\bar{x}_s\right] = \begin{bmatrix} 0 & \left(\bar{z}_{\varsigma} - \bar{z}_{xz}\right) & -\left(\bar{y}_{\varsigma} - \bar{y}_{xy}\right) \\ -\left(\bar{z}_{\varsigma} - \bar{z}_{yz}\right) & 0 & \left(\bar{x}_{\varsigma} - \bar{x}_{xy}\right) \\ \left(\bar{y}_{\varsigma} - \bar{y}_{yz}\right) & -\left(\bar{x}_{\varsigma} - \bar{x}_{xz}\right) & 0 \end{bmatrix}. \qquad \text{............Eq. } \mathbf{6b}$$

The duel matrix, after some manipulation and rearranging to terms, represents the *distances* from the support's $c.g.$ -to- the individual reaction at each support point.

Assumed $\boldsymbol{P}_{\zeta i} = k\Delta \boldsymbol{s}_{\zeta i}$ **HOOKE**'s *Law* holds in an analogously way for all spatial centroidal point $P_{\zeta} \Rightarrow \left(\bar{x}_{\varsigma}, \bar{y}_{\varsigma}, \bar{z}_{\varsigma}\right)$, and wherein vector $\boldsymbol{P}_{\zeta i}$ is defined to represent the components of the reaction forces at the supports. k set to 1, i.e., to unity, since the mass points are massless. Then in index notation, and incorporating the $\boldsymbol{A}$'s, gives**:**

$$\boldsymbol{P}_{\zeta i} = \varepsilon_{ijk}\left(\overline{\boldsymbol{r}} - \boldsymbol{r}_{\zeta j}\right)\triangle\boldsymbol{\theta}_{\zeta k}\boldsymbol{A}_{\zeta i}, \qquad \text{..........Eq. } \mathbf{6c}$$

where $\left(\overline{\boldsymbol{r}} - \boldsymbol{r}_{\zeta j}\right)\triangle\boldsymbol{\theta}_{\zeta k}$ will represent the components of the stiffnesses and equivalent to k in **HOOKE**'s *Law* **–** associated with each component of force at the supports.

Taking moments at the support $c.g.$ owning to these reactions $\boldsymbol{P}_{\zeta i}$ now gives**:**

$$\left\{\boldsymbol{M}_{\zeta}\right\} = -\left[\overline{\boldsymbol{r}}_{\zeta}\right]\left\{\boldsymbol{P}_{\zeta}\right\} = -\left[\overline{\boldsymbol{r}}_{\zeta}\right]\left[\overline{\boldsymbol{r}}_{\zeta}\right]\left[A_{\zeta}\right]\left\{\triangle\boldsymbol{\theta}_{\zeta}\right\}. \qquad \text{Eq. } \mathbf{6d}$$

And when expanded gives**:**

$$\begin{Bmatrix} \boldsymbol{M}_{\zeta 1} \\ \boldsymbol{M}_{\zeta 2} \\ \boldsymbol{M}_{\zeta 3} \end{Bmatrix} = \begin{bmatrix} 0 & -\overline{\boldsymbol{r}}_{\zeta 3} & -\overline{\boldsymbol{r}}_{\zeta 2} \\ -\overline{\boldsymbol{r}}_{\zeta 3} & 0 & -\overline{\boldsymbol{r}}_{\zeta 1} \\ -\overline{\boldsymbol{r}}_{\zeta 2} & -\overline{\boldsymbol{r}}_{\zeta 1} & 0 \end{bmatrix} \begin{bmatrix} 0 & -\overline{\boldsymbol{r}}_{\zeta 3} & -\overline{\boldsymbol{r}}_{\zeta 2} \\ -\overline{\boldsymbol{r}}_{\zeta 3} & 0 & -\overline{\boldsymbol{r}}_{\zeta 1} \\ -\overline{\boldsymbol{r}}_{\zeta 2} & -\overline{\boldsymbol{r}}_{\zeta 1} & 0 \end{bmatrix} \begin{Bmatrix} A_{\zeta 1} & 0 & 0 \\ 0 & A_{\zeta 2} & 0 \\ 0 & 0 & A_{\zeta 3} \end{Bmatrix} \begin{Bmatrix} \triangle\boldsymbol{\theta}_{\zeta 1} \\ \triangle\boldsymbol{\theta}_{\zeta 2} \\ \triangle\boldsymbol{\theta}_{\zeta 3} \end{Bmatrix},$$

$$\begin{Bmatrix} \boldsymbol{M}_{\zeta 1} \\ \boldsymbol{M}_{\zeta 2} \\ \boldsymbol{M}_{\zeta 3} \end{Bmatrix} = -\begin{bmatrix} -\left(\overline{\boldsymbol{r}}_{\zeta 3}^2 + \overline{\boldsymbol{r}}_{\zeta 2}^2\right)A_{\zeta 1} & \left(\overline{\boldsymbol{r}}_{\zeta 2}\overline{\boldsymbol{r}}_{\zeta 1}\right)A_{\zeta 2} & \left(\overline{\boldsymbol{r}}_{\zeta 1}\overline{\boldsymbol{r}}_{\zeta 3}\right)A_{\zeta 3} \\ \left(\overline{\boldsymbol{r}}_{\zeta 1}\overline{\boldsymbol{r}}_{\zeta 2}\right)A_{\zeta 1} & -\left(\overline{\boldsymbol{r}}_{\zeta 3}^2 + \overline{\boldsymbol{r}}_{\zeta 1}^2\right)A_{\zeta 2} & \left(\overline{\boldsymbol{r}}_{\zeta 3}\overline{\boldsymbol{r}}_{\zeta 2}\right)A_{\zeta 3} \\ \left(\overline{\boldsymbol{r}}_{\zeta 3}\overline{\boldsymbol{r}}_{\zeta 1}\right)A_{\zeta 1} & \left(\overline{\boldsymbol{r}}_{\zeta 2}\overline{\boldsymbol{r}}_{\zeta 3}\right)A_{\zeta 2} & -\left(\overline{\boldsymbol{r}}_{\zeta 2}^2 + \overline{\boldsymbol{r}}_{\zeta 1}^2\right)A_{\zeta 3} \end{bmatrix} \begin{Bmatrix} \triangle\boldsymbol{\theta}_{\zeta 1} \\ \triangle\boldsymbol{\theta}_{\zeta 2} \\ \triangle\boldsymbol{\theta}_{\zeta 3} \end{Bmatrix},$$

There is the inertia! By substituting x, y, z indexes for 1, 2, 3 indexes, as was done above in Eq.s **4** and **5**, the $-\left[\overline{r}_{\zeta i}\right]\left[\overline{r}_{\zeta k}\right]\left\{A_{\zeta i}\right\}$ becomes, after some rearrangement of elements, the <u>*massless centroidal inertia tensor*</u> the of the support pattern, which is a constant. Consequently**:**

$$\begin{Bmatrix} \boldsymbol{M}_{\zeta x} \\ \boldsymbol{M}_{\zeta y} \\ \boldsymbol{M}_{\zeta z} \end{Bmatrix} = \begin{bmatrix} \overline{I}_{xx}A_{\zeta 1} & -\overline{I}_{xy}A_{\zeta 2} & -\overline{I}_{xz}A_{\zeta 3} \\ -\overline{I}_{yx}A_{\zeta 1} & \overline{I}_{yy}A_{\zeta 2} & -\overline{I}_{yz}A_{\zeta 3} \\ -\overline{I}_{zx}A_{\zeta 1} & -\overline{I}_{zy}A_{\zeta 2} & \overline{I}_{zz}A_{\zeta 3} \end{bmatrix} \begin{Bmatrix} \triangle\boldsymbol{\theta}_{\zeta 1} \\ \triangle\boldsymbol{\theta}_{\zeta 2} \\ \triangle\boldsymbol{\theta}_{\zeta 3} \end{Bmatrix} \ldots$$

Or $\{M_\varsigma\} = [\bar{I}_s]\{\Delta\theta_\varsigma\}$. The inertia tensor is exactly as given by Eq. **5** (and **4**), except for the embedded A's; and again notice the inertia tensor is only a function of geometry.

The trick now is to get rid of $\triangle\boldsymbol{\theta}_{\zeta k}$.

Applied Load

The sum of the moments owning to all *reaction* components taken at the centroid of the support pattern are $\{M_\zeta\} = [\bar{I}_s]\{\Delta\theta_\zeta\}$. These reaction components (Eq. **6d**) are the resultant moments; or more properly, *couples*, which in turn owe to an applied external force and moment**:**

$$\boldsymbol{M}_{Li} = \varepsilon_{ijk}\bar{\boldsymbol{r}}_{Lj}\boldsymbol{F}_{Lk} + \boldsymbol{M}_{oi},$$

where $\bar{\boldsymbol{r}}_{Lj}$ are the *centroidal distances* from the applied force's point of application -to- the support $c.g.$; and where $\{\boldsymbol{F}_{Lk}\}$ and $\{\boldsymbol{M}_{oi}\}$ are the applied vector force and moment, respectively, k and i = 1, 2, 3, or x, y, z. The set of internal reactions $\{\boldsymbol{P}_\zeta\}$ reacts to these applied resultant force $\boldsymbol{F}_{Li}$ and moment $\boldsymbol{M}_{oi}$ components at the N reaction points $P_\zeta \Rightarrow \{\bar{x}_\zeta, \bar{y}_\zeta, \bar{z}_\zeta\}$, where the bar represents the centroid of the mass for that point.

$\{\boldsymbol{M}_\zeta\}$ will be equated to $\{\boldsymbol{M}_L\}$ directly, but in the meantime $\bar{\boldsymbol{r}}_{Li}$ is looked at more closely.

$$\bar{\boldsymbol{r}}_{Li} = \boldsymbol{r}_{Li} - \bar{\boldsymbol{r}}_i.$$

where $\boldsymbol{r}_{Li} \Rightarrow (\bar{x}_{cg}, \bar{y}_{cg}, \bar{z}_{cg})$ is the position of the applied force and moment. Rearranging the above equation somewhat and adding the vector $\{\boldsymbol{M}_0\}$ last, it is seen it takes the form when expanded in scalar form as**:**

$$\begin{aligned}
\boldsymbol{M}_{L1} &= \bar{\boldsymbol{r}}_{L2}\boldsymbol{F}_{L3} - \bar{\boldsymbol{r}}_{L3}\boldsymbol{F}_{L2} + \boldsymbol{M}_{o1},\\
\boldsymbol{M}_{L2} &= \bar{\boldsymbol{r}}_{L3}\boldsymbol{F}_{L1} - \bar{\boldsymbol{r}}_{L1}\boldsymbol{F}_{L3} + \boldsymbol{M}_{o2},\\
\boldsymbol{M}_{L3} &= \bar{\boldsymbol{r}}_{L1}\boldsymbol{F}_{L2} - \bar{\boldsymbol{r}}_{L2}\boldsymbol{F}_{L1} + \boldsymbol{M}_{o3}.
\end{aligned}$$

$$\begin{Bmatrix} \boldsymbol{M}_{L1} \\ \boldsymbol{M}_{L2} \\ \boldsymbol{M}_{L3} \end{Bmatrix} = \begin{bmatrix} 0 & -\bar{\boldsymbol{r}}_{L3} & \bar{\boldsymbol{r}}_{L2} \\ \bar{\boldsymbol{r}}_{L3} & 0 & -\bar{\boldsymbol{r}}_{L1} \\ -\bar{\boldsymbol{r}}_{L2} & \bar{\boldsymbol{r}}_{L1} & 0 \end{bmatrix} \begin{Bmatrix} \boldsymbol{F}_{L1} \\ \boldsymbol{F}_{L2} \\ \boldsymbol{F}_{L3} \end{Bmatrix} + \begin{Bmatrix} \boldsymbol{M}_{o1} \\ \boldsymbol{M}_{o2} \\ \boldsymbol{M}_{o3} \end{Bmatrix}.$$

But
$$\begin{bmatrix} 0 & -\bar{\boldsymbol{r}}_{L3} & \bar{\boldsymbol{r}}_{L2} \\ \bar{\boldsymbol{r}}_{L3} & 0 & -\bar{\boldsymbol{r}}_{L1} \\ -\bar{\boldsymbol{r}}_{L2} & \bar{\boldsymbol{r}}_{L1} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\left(\boldsymbol{r}_{L3} - \bar{\boldsymbol{r}}_3\right) & \left(\boldsymbol{r}_{L2} - \bar{\boldsymbol{r}}_2\right) \\ \left(\boldsymbol{r}_{L3} - \bar{\boldsymbol{r}}_3\right) & 0 & -\left(\boldsymbol{r}_{L1} - \bar{\boldsymbol{r}}_1\right) \\ -\left(\boldsymbol{r}_{L2} - \bar{\boldsymbol{r}}_2\right) & \left(\boldsymbol{r}_{L1} - \bar{\boldsymbol{r}}_1\right) & 0 \end{bmatrix}.$$

Switching now to (x, y, z) notation and substituting equations similar to Eq. **6a** above, gives the load transport matrix for the applied loads**:**

$$\left[\bar{x}_L\right] = \begin{bmatrix} 0 & \left(\bar{z}_L - \bar{z}_{yz}\right) & -\left(\bar{y}_L - \bar{y}_{yz}\right) \\ -\left(\bar{z}_L - \bar{z}_{xz}\right) & 0 & \left(\bar{x}_L - \bar{x}_{xz}\right) \\ \left(\bar{y}_L - \bar{y}_{xy}\right) & \left(\bar{x}_L - \bar{x}_{xy}\right) & 0 \end{bmatrix} \qquad \text{Eq. } \mathbf{7}$$

$\left[\bar{x}_L\right]_{3,3}$ is a *skewed Hermitian*. Then**:**

$$\begin{Bmatrix} \boldsymbol{M}_{Lx} \\ \boldsymbol{M}_{Ly} \\ \boldsymbol{M}_{Lz} \end{Bmatrix} = \begin{bmatrix} 0 & -\left(\bar{z}_L - \bar{z}_{yz}\right) & \left(\bar{y}_L - \bar{y}_{yz}\right) \\ \left(\bar{z}_L - \bar{z}_{xz}\right) & 0 & -\left(\bar{x}_L - \bar{x}_{xz}\right) \\ \left(\bar{y}_L - \bar{y}_{xy}\right) & \left(\bar{x}_L - \bar{x}_{xy}\right) & 0 \end{bmatrix} \begin{Bmatrix} \boldsymbol{F}_{Lx} \\ \boldsymbol{F}_{Ly} \\ \boldsymbol{F}_{Lz} \end{Bmatrix} + \begin{Bmatrix} \boldsymbol{M}_{ox} \\ \boldsymbol{M}_{oy} \\ \boldsymbol{M}_{oz} \end{Bmatrix}.$$

Closed-Form Solution

Returning to the question of how to get rid of $\triangle\boldsymbol{\theta}_{\zeta k}$ and looking at the big picture, *the total external loads are equal to the sum of all the internal coupling reactions (moments resisted by coupling forces), plus the direct shear reactions (translational reactions), with the latter divided by the number of massless points or attachments*.

In short (see Figure **11**): $\boldsymbol{M}_{\varsigma i} = \boldsymbol{M}_{Li}$

$$\begin{Bmatrix} \boldsymbol{M}_{\varsigma 1} \\ \boldsymbol{M}_{\varsigma 2} \\ \boldsymbol{M}_{\varsigma 3} \end{Bmatrix} = \begin{bmatrix} \bar{I}_{xx} A_{\varsigma 1} & -\bar{I}_{xy} A_{\varsigma 2} & -\bar{I}_{xz} A_{\varsigma 3} \\ -\bar{I}_{yx} A_{\varsigma 1} & \bar{I}_{yy} A_{\varsigma 2} & -\bar{I}_{yz} A_{\varsigma 3} \\ -\bar{I}_{zx} A_{\varsigma 1} & -\bar{I}_{zy} A_{\varsigma 2} & \bar{I}_{zz} A_{\varsigma 3} \end{bmatrix} \begin{Bmatrix} \Delta\boldsymbol{\theta}_{\varsigma 1} \\ \Delta\boldsymbol{\theta}_{\varsigma 2} \\ \Delta\boldsymbol{\theta}_{\varsigma 3} \end{Bmatrix}$$

$$= \begin{Bmatrix} \boldsymbol{M}_{Lx} \\ \boldsymbol{M}_{Ly} \\ \boldsymbol{M}_{Lz} \end{Bmatrix} = \begin{bmatrix} 0 & -\left(\bar{z}_L - \bar{z}_{yz}\right) & \left(\bar{y}_L - \bar{y}_{yz}\right) \\ \left(\bar{z}_L - \bar{z}_{xz}\right) & 0 & -\left(\bar{x}_L - \bar{x}_{xz}\right) \\ \left(\bar{y}_L - \bar{y}_{xy}\right) & \left(\bar{x}_L - \bar{x}_{xy}\right) & 0 \end{bmatrix} \begin{Bmatrix} \boldsymbol{F}_{Lx} \\ \boldsymbol{F}_{Ly} \\ \boldsymbol{F}_{Lz} \end{Bmatrix} + \begin{Bmatrix} \boldsymbol{M}_{\mathrm{o}x} \\ \boldsymbol{M}_{\mathrm{o}y} \\ \boldsymbol{M}_{\mathrm{o}z} \end{Bmatrix} \ldots \qquad \text{......Eq. } \mathbf{8}$$

Equate the right-hand-sides of $\boldsymbol{M}_{\varsigma i} = \boldsymbol{M}_{Li}$ and temporarily removing $\{\boldsymbol{M}_0\}$. Then multiply both sides by $\left[\bar{I}_s\right]^{-1}$. Lastly re-add $\{\boldsymbol{M}_0\}$, giving:

$$\{\Delta\boldsymbol{\theta}_\zeta\} = \left[\bar{I}_s\right]^{-1}\left[x_L\right]\{\boldsymbol{F}_L\} + \{\boldsymbol{M}_0\}. \qquad \text{.........Eq. } \mathbf{9a}$$

Every variable on the right-hand side is known: are applied loads. Remove $\{\boldsymbol{M}_0\}$ again and then back substitute $\{\Delta\boldsymbol{\theta}_\zeta\}$ back into the basic equation to eliminating it.

From Eq. **6c** above, $\{\boldsymbol{M}_\zeta\} = \left[\bar{r}_\zeta\right]\{\boldsymbol{P}_\zeta\}$, and:

$$\left[\bar{\boldsymbol{r}}_\zeta\right]\{\boldsymbol{P}_\zeta\} = \left[\bar{I}_s\right]^{-1}\left[x_L\right]\{F_L\}. \qquad . \qquad \text{........Eq. } \mathbf{9b}$$

Then letting $\left[\bar{\boldsymbol{r}}_\varsigma\right]^{-1} = \left[\bar{x}_s\right]$, that is:

$$\left[\bar{x}_s\right] = \begin{bmatrix} 0 & \left(\bar{z}_\varsigma - \bar{z}_{xz}\right) & -\left(\bar{y}_\varsigma - \bar{y}_{xy}\right) \\ -\left(\bar{z}_\varsigma - \bar{z}_{yz}\right) & 0 & \left(\bar{x}_\varsigma - \bar{x}_{xy}\right) \\ \left(\bar{y}_\varsigma - \bar{y}_{yz}\right) & -\left(\bar{x}_\varsigma - \bar{x}_{xz}\right) & 0 \end{bmatrix}. \qquad \text{.........Eq. } \mathbf{6b}$$

is the reaction or support, *duel matrix*; (a *skewed Hermitian*). Re-substitute $\{\boldsymbol{M}_0\}$, and then solve for the reaction couples $\{\boldsymbol{P}_\zeta\}$:

$$\left\{\boldsymbol{P}_{\varsigma}\right\}_{\text{coupling}} = \left\{\left[\bar{x}_s\right]\left[\bar{\boldsymbol{I}}_s\right]^{-1}\left\{\left[\bar{x}_L\right]\left\{\boldsymbol{F}_L\right\}+\left\{\boldsymbol{M}_{\text{o}}\right\}\right\}\right\}. \qquad \text{.........Eq. } \mathbf{9c}$$

These are the coupling forces; the translational forces will be added later below. It wasn't effected here but the external moments $\{\boldsymbol{M}_0\}$ are also functions of the $\boldsymbol{A}$'s and have to be resolved into applied couples.

The above equation accounts for all the coupling reactions owning to the applied loading. Usually these coupling reactions are neglected by analysts because of no known method for distributing them, and resort to only distributing the translational forces.

So, to the above coupling reactions must be added the translations reactions. Otherwise, ***D'ALEMBERT***'s two *Principles* are combined.

$$\begin{Bmatrix} \boldsymbol{P}_{\varsigma x} \\ \boldsymbol{P}_{\varsigma y} \\ \boldsymbol{P}_{\varsigma z} \end{Bmatrix}_{\text{Translational}} = \begin{Bmatrix} \dfrac{A_{\varsigma x}\boldsymbol{F}_{Lx}}{\sum A_x} \\ \dfrac{A_{\varsigma y}\boldsymbol{F}_{Ly}}{\sum A_y} \\ \dfrac{A_{\varsigma z}\boldsymbol{F}_{Lz}}{\sum A_z} \end{Bmatrix}.$$

Putting all the above together results in an equation call ***The Force***. ***The Force*** formula gives a set of reactions at the reaction points or support points (in Lb.).

$$\{\boldsymbol{P}_{\zeta}\}_i = \{\{\boldsymbol{F}_L\}[A_{\zeta i}]/\Sigma A_{\zeta}\} + \{[\bar{x}_s]_i\left[\bar{\boldsymbol{I}}_s\right]^{-1}\{[\bar{x}_L]\{\boldsymbol{F}_L\} + \{\boldsymbol{M}_{\text{o}}\}\}\}. \qquad \text{Eq. } \mathbf{9b}$$

The above in expanded matrix form, except for the inverse of the massless inertia tensor $\left[\bar{\boldsymbol{I}}_s\right]^{-1}$, for the three orthogonal reaction components at a ζ-reaction point, is (Equation **10**):

$$\begin{Bmatrix} \boldsymbol{P}_{\varsigma x} \\ \boldsymbol{P}_{\varsigma y} \\ \boldsymbol{P}_{\varsigma z} \end{Bmatrix} = \begin{Bmatrix} \dfrac{A_{\varsigma x}\boldsymbol{F}_{Lx}}{\sum A_x} \\ \dfrac{A_{\varsigma y}\boldsymbol{F}_{Ly}}{\sum A_y} \\ \dfrac{A_{\varsigma z}\boldsymbol{F}_{Lz}}{\sum A_z} \end{Bmatrix} + \left\{\begin{bmatrix} 0 & \left(\bar{z}_{\varsigma}-\bar{z}_{xz}\right) & -\left(\bar{y}_{\varsigma}-\bar{y}_{xy}\right) \\ -\left(\bar{z}_{\varsigma}-\bar{z}_{yz}\right) & 0 & \left(\bar{x}_{\varsigma}-\bar{x}_{xy}\right) \\ \left(\bar{y}_{\varsigma}-\bar{y}_{yz}\right) & \left(\bar{x}_{\varsigma}-\bar{x}_{xz}\right) & 0 \end{bmatrix}\left[\bar{\boldsymbol{I}}_s\right]^{-1}\left\{\begin{bmatrix} 0 & \left(\bar{z}_L-\bar{z}_{yz}\right) & -\left(\bar{y}_L-\bar{y}_{yz}\right) \\ -\left(\bar{z}_L-\bar{z}_{xz}\right) & 0 & \left(\bar{x}_L-\bar{x}_{xz}\right) \\ \left(\bar{y}_L-\bar{y}_{xy}\right) & \left(\bar{x}_L-\bar{x}_{xy}\right) & 0 \end{bmatrix}\begin{Bmatrix} \boldsymbol{F}_{Lx} \\ \boldsymbol{F}_{Ly} \\ \boldsymbol{F}_{Lz} \end{Bmatrix} + \begin{Bmatrix} \boldsymbol{M}_{\text{o}x} \\ \boldsymbol{M}_{\text{o}y} \\ \boldsymbol{M}_{\text{o}z} \end{Bmatrix}\right\}\right\},$$

where $i = x, y, z$; and ζ = 1, 2, ..., N reaction points.

In the expanded version, Eq. **10**, (except for the $\left[\bar{\boldsymbol{I}}_s\right]^{-1}$ term, see Eq. **6d**) one can clearly see the right hand side of the equation is composed of two parts: the translational forces (left) and the coupling forces (right).

The proof of ***The Force*** is by calculating equilibrium after the reactions After the internal forces have been determined, see **Equilibrium of Forces and Moments** section below for calculating equilibrium.

In the above Eq. **9b** and Eq. **10** equations:

- ζ = 1, 2, 3, ... N massless point coordinate number;
- Subscript L stands for applied load, s stands for supports;
- $\{F_L\}$ is the three column vector of the applied force (lb.); $\{M_o\}$ is the three column vector of the applied moment (lb.-in.), which are basic inputs to the algorithm;
- The external load center at $\left(\bar{x}_{cg}, \bar{y}_{cg}, \bar{z}_{cg}\right)$ is basic input (in.).
- $\{P_\varsigma\}$ are the vector reactions at each support $\left\{\bar{r}_{\zeta x}, \bar{r}_{\zeta y}, \bar{r}_{\zeta z}\right\} \equiv \left\{\bar{r}_{\zeta 1}, \bar{r}_{\zeta 2}, \bar{r}_{\zeta 3}\right\}$;
- $\left[\bar{I}_s\right]^{-1}$ is the inverse of the massless inertia tensor of the reaction pattern (via Crout's Method), is a symmetrical 3x3 matrix (in.$^{-2}$);
- $[A_{\zeta i}]_{3,3}$ are the Hookean springs or on-off switches [nondim.].
- $\left[\bar{x}_L\right]_{3,3}$ is the distance from the initial position of the applied force and moment to the centroid of the reaction pattern;
- $\left[\bar{x}_s\right]$ 3,3 are the centroids of the reaction pattern [in.].
- $\left\{\left[\bar{x}_s\right]\left[\bar{I}_s\right]^{-1}\left\{\left[\bar{x}_L\right]\{F_L\} + \{M_o\}\right\}\right\}$ is the applied force reacted as couples at the supports, is a 1x3 column matrix [lb.];
- $\{F_L\}[A_{\zeta i}]/(\Sigma A_i)$ are the symmetrical (translational) reactions, a a 1x3 column matrix [lb.].

The $\left[\bar{x}_s\right]\left[\bar{I}_s\right]^{-1}\left[\bar{x}_L\right]$ term is similar to a congruent transformation (see **MEIROVITCH**) and for all physical entities having like congruent transformations, like stress, strain, they are always positive definite if the right-hand rule is used.

If a particular motion is planar, then out-of-plane reaction forces will be zero by default. Determination of $\{\boldsymbol{P}_{\varsigma}\}$ has been computerized, see **Appendix B** below.

Equilibrium of Forces and Moments

Equilibrium calculations are completely independent of any n-body method of solution. The reaction points c.g.'s (center of gravities or barycenters), where, for the N masses m_ζ, have been assumed reduced to mass-points set, are at $\{\overline{x}_i, \overline{y}_i, \overline{z}_i\}$, for i =1, 2, ...,N coordinates. However, in most cases the mass-point coordinates are the same set as the non-reduced coordinates $\{x_i,\ y_i,\ z_i\} \equiv \{x_\varsigma,\ y_\varsigma,\ z_\varsigma\}$ of the ζ = 1, ... N reaction points.

Then, the sum of the forces for the entire support or reaction set $\boldsymbol{P}_{\varsigma i} \Rightarrow \{\overline{x}_{\varsigma i}, \overline{y}_{\varsigma i}, \overline{z}_{\varsigma i}\}$, for i = x, y, z vector components, and for the ζ = 1, ... N reaction points are:

$$\boldsymbol{F}_x - \sum_{\varsigma=1}^{\mathrm{N}} \boldsymbol{P}_{\varsigma x} = \boldsymbol{0},\ \boldsymbol{F}_y - \sum_{\varsigma=1}^{\mathrm{N}} \boldsymbol{P}_{\varsigma y} = \boldsymbol{0},\ \boldsymbol{F}_x - \sum_{\varsigma=1}^{\mathrm{N}} \boldsymbol{P}_{\varsigma z} = \boldsymbol{0}. \qquad \text{Eq. } \mathbf{11}$$

The applied load center is at $\left(\overline{x}_{cg}, \overline{y}_{cg}, \overline{z}_{cg}\right)$ and is a basic input. Then the sum of the moments for the entire supports (or reactions) plus the applied load are:

$$\begin{aligned}
\sum_{i=1}^{\varsigma} \boldsymbol{M}_{xx} &= 0 = \sum_{i=1}^{\varsigma}\left[\left(\overline{y}_i - \overline{y}_{cg}\right)\boldsymbol{P}_{zi} - \left(\overline{z}_i - \overline{z}_{cg}\right)\boldsymbol{P}_{yi}\right]; \\
\sum_{i=1}^{\varsigma} \boldsymbol{M}_{yy} &= 0 = \sum_{i=1}^{\varsigma}\left[\left(\overline{x}_i - \overline{x}_{cg}\right)\boldsymbol{P}_{zi} - \left(\overline{z}_i - \overline{z}_{cg}\right)\boldsymbol{P}_{xi}\right]; \qquad \text{Eq. } \mathbf{12}. \\
\sum_{i=1}^{\varsigma} \boldsymbol{M}_{zz} &= 0 = \sum_{i=1}^{\varsigma}\left[\left(\overline{y}_i - \overline{y}_{cg}\right)\boldsymbol{P}_{xi} - \left(\overline{x}_i - \overline{x}_{cg}\right)\boldsymbol{P}_{yi}\right].
\end{aligned}$$

The algorithm automatically calculates equilibrium, but regardless of the method of how the support reactions were obtain – if both force and moment equilibria, which are independent calculations for a particular problem, can be demonstrated, then its reaction set is the correct set, Q.E.D.; but it may not necessarily be unique. With n somewhat large, it is conceivable the coupling reactions could be reversed, a mirror image or partly a mirror image, and still be in equilibrium, and this would be contrary to classical mechanics, which says every solution is unique.

Planet Forces and Motions

There has always been much interest in classical, Celestial Mechanics problems (see **GOODSTEIN**, ref. **21**). And as we know, it came into sharp focus during Sir Isaac **NEWTON** (1643-1727) time in the 17th century. Simply stated, the problem was to determine analytically all the gravitational forces of attraction between the seven known (back then) planets for the purpose of predicting all their orbits, correctly. They realized seven planets in an ever changing gravitationally dynamic environment connects 21 force-to-force combinations. These additional planetary forces, besides the Sun's common gravitational force on all the planets, obviously changes each planet's orbit. Thus predicting their orbital paths for longer than, say, one orbital period at best without their gravitational inclusion, is not viable when using only analytical geometry, **NEWTON**'s simplified "*fixed*" Sun assumption, and his Force Laws. Even **NEWTON** said the real analytical problem, with a moving Sun (and he knew it was moving), coupled with those perturbational forces, was overwhelming, or words to that effect.

Today, definitive solutions employing even some improved analytical methods of Mechanics, combined with numerical analyses, computers, but still employing the old methods of Newtonian physics, are still to no avail either; and planetary orbits to this day are recalculated at least yearly; and some, using computers, calculated daily; or as needed. For Celestial Mechanics then, the first problem is to determine the planetary, and their moons, interactive[110] gravitational forces[111]. The second problem is determining correct planetary orbits**:** time dependent positions, velocities, and accelerations of the planets. Here only the first problem, and the main problem, determining the interactive forces, is discussed further**.**

In short, **NEWTON**'s *Law of Universal Gravitation*, or variations thereof (approximate methods, etc.), cannot be used for determining planetary motions with any degree of accuracy over any extended time for three or more moving bodies. This paper is not needed to prove that point. Succinctly then, any solution requires incorporating the all encompassing spatial inertial properties of the orbiting bodies, properties lacking in **NEWTON**'s *Laws*, but available using the algebraic algorithm (Equation **10** above)**.**

If we define the Solar System's domain as having expanded as far as the

110 "*Interactive*" meaning forces conform to the 3rd *Law*)

111 Including the *Trojan planetoids*. The *Trojan planetoids* are trapped in two gravitational wells created by the Sun, and mainly Jupiter's, gravity fields; and together with Jupiter, effects all the other planet orbits. See the **Restricted Three-body Problem** above.

gravitational fields of the Sun, Jupiter, the other outer-planets, including Pluto; then the Solar System's complete domain beyond Pluto probably extends perhaps a thousand times further out than we have yet explored.

Out on the edge we know there are billions of small objects in a zone known as Oort Cloud. If all the known mass (the planets, asteroids, moons, Trojan planetoids, etc.) – except the Sun and Oort Cloud mass – is subtracted from the Sun's known mass, then it is conjectured the remaining would be equal to all of the zone consisting of Oort Cloud's mass.

This is not unreasonable gravitationally, as the Sun's gravitational field is equal to the task of absorbing this additional mass.

Planetary motion is a function of position and inertial properties, velocities, accelerations, mass, time; and, in addition to the Sun's gravity field, a function of the planet-to-planet gravitational forces. Position and inertial properties, velocities, accelerations must all be functions of discrete time-step times. The first problem in qualifying a planet's true orbit then is in ascertaining those above parameters and the collectively, determining the planet-to-planet gravitational forces on each planet (don't use **NEWTON**'s *Law of Universal Gravitation*). The latter task is the classical n-body problem.

One way to obtain time-step parameter inputs is by assuming all unsteady or moving variables at an instant of time (at a time-step) are "*steady*" (*quasi-steady*). The technique for doing this is analogous to capturing by photograph a picture of a moving object using a super high shudder speed camera, which concurrently records a planet's instantaneous position relative to some reference point (the fixed stars); and also records all nine instantaneous angular velocities, and three angular and three translational accelerations. These quasi-steady values[112] would only be approximations to the real values of those variables. Several time-points spaced a month or months apart would be need to approximate those variables, but that information is currently already available.

When the quasi-steady time-step coordinate sets of the planetoid positions of the main masses, and additional bodies, are known, then the Solar System's general n-body solution may be effected by:

- Calculate in pairs, all known planet -to- planet forces and other body forces via

112 *Quasi-steady* state refers to the instantaneous (t: = 0) inertial loads generated by instantaneous angular velocities and accelerations, as well as the instantaneous translational accelerations (9 variables). In contrast, a steady-state condition refers to a system's state being invariant to *time*; or otherwise, the first derivatives and all higher derivatives are zero.

NEWTON's *Law of Universal Gravitation*, whose values are to be used in the algorithm as initial force inputs, one at a time discretely as the applied external force $\boldsymbol{F}$. The applied force $\boldsymbol{F}$ is the sum of the other bodies collectively (use a spreadsheet). The algorithm has to be run for each body $\boldsymbol{F}$; and the forces so accumulated have to be adjusted until equilibrium is obtained for the system (write a loop). After balance do a comparisons of the n-body solution forces -to- initial Newtonian forces.

● At the inner diameter of Oort's Cloud, establish, say, a hundred (or more) mass-points uniformly distributed around the inside of its sphere, whose total mass sum is equal to the Sun's mass minus the the known masses. The bulk of Oort's Cloud mass is actually probably near the Solar plain.

● Run the n-body computer program first using the Sun as the known force; then the planets, etc., as unknown reactions; and record the position of the Solar System's barycenter (shades of **CLEMINSHAW**). It will (should) be independent of Oort's Cloud.

● Use the quasi-steady positions of the known masses combined with the mass-points of Oort's Cloud mass-points, as input to the n-body program. Recalculate using Oort's Cloud input. If these forces are known then the eccentricities of the planet's orbits can be determined.

● Repeat computer runs incrementally time-wise until the orbit of the Solar System's barycenter is known (this will time-wise encompass more the several years to include all known bodies), then calculate the orbital paths of the planets.

This task is way to large to be presented here.

See "***Dwarf Galaxies and the Dark Web***," by astrophysicist Noam I. **LIBESKIND**, ***Scientific American***, Vol. 310, No. 3, pp. 46-51, Mar. 2014. He says "*Beginning in the 1970's researchers in the field of computational cosmology have attempted to simulate the history of the Universe using computer codes* [algorithms]. *The technique is straightforward: define an imaginary box in a computer. Place imaginary point particles (representing clumps of dark matter) in a near-perfect lattice inside the box. Calculate the gravitational pull on each particle from every other particle in the box and move each particle according to the net gravity in feels. Iterate this process for 13 billion years*." The Solar System model is similar.

n-body particle simulations can also be effected via computer too using Henley

QUADLING program.[113] This program demonstrates body-to-body interactions; i.e., very close head-on collisions, which almost always result in bodies orbiting bodies, rather than collisions.

Use weighted mass ratios**:** mi = planeti/Sun; then applied Fi = mi ratios; Hookean switches A can be weighted too.

113 See**:** "***Gravitational N-Body Simulation***" written by Henley **QUADLING**, is a simple 16 bit ***DOS*** application, allowing simulation of N particles subject to their gravitational interactions. Initial velocities and different masses can be entered. There is no limit on the number of particles; the only limiting factor is the speed of your ***CPU***. The calculation is a true ***3D*** calculation and there is an explicit accuracy control.

Appendix **A:** Numerical Example

This example problem concerns a large video (game) monitor, whose design weight is 130 Lbs. (120 Lbs. actual weight), fastener and structural support loads. A console bolted to the floor and as big as a small refrigerator, is cantilevered off the floor like one, has the monitor installed into it near the top: eye-level when seated in a chair. The monitor is supported internally by locked slides and by 20 attachment (support) points. It is sloped downward at a 15^o FWD slope with respect to the floor. Both Local support and Global (ships) coordinates in inches, with local coordinates given in Tables 1 -to- 3 below. The externally applied force (i.e., the 130 Lbs. monitor) has a local load c.g. at (3.96442, 13.5625, 44.89365) inches.

The problem is to determine three rigid-body, global unit-load sets $\hat{\boldsymbol{F}}_x, \hat{\boldsymbol{F}}_y, \hat{\boldsymbol{F}}_z$, applied at the support pattern needed for the analyses of the attachment loads and for determining console internal stresses. Three sets are needed to obtain a complete loading profile for, say, a ***FEM***.

The whole console apparatus is airborne with an ultimate design load of 9.0 G's forward, 6.5 G's side and down, and 3.0 G's up and aft load. Critical condition for the analyses of attachments for crash safety is 30G's at half-sine shock (ultimate), T = 11+/- 1 ms. Attachment reaction assumptions include**:**

- No shear pins in front face. Front face captive screws carry tension only.
- No shear pins in rear of monitor.
- Slides supports monitor for vertical and side load shears (no tension).

All attachment allowable D.O.F.'s are established via the above reaction assumptions (see Tables below for A's values). The six support c.g.'s (i.e., $\overline{x}_{xy}$, etc.) as well as the inertia components ($\boldsymbol{I}$'s) are calculated (only once) below.

ζ	x_ς	y_ς	z_ς	$A_{\varsigma x}$	$A_{\varsigma y}$	$A_{\varsigma z}$	$A_{\varsigma z}\overline{x}_\varsigma$	$A_{\varsigma z}\overline{y}_\varsigma$	$A_{\varsigma y}\overline{x}_\varsigma$	$A_{\varsigma x}\overline{y}_\varsigma$	$A_{\varsigma x}\overline{z}_\varsigma$
1	13.21278	1.94	38.87491	1	0	0	0	0	0	1.94	38.87491
2	13.21246	1.94	42.37423	1	0	0	0	0	0	1.94	42.37423
3	13.21177	1.94	46.37058	1	0	0	0	0	0	1.94	46.37058
4	13.21145	1.94	49.86990	1	0	0	0	0	0	1.94	49.86990
5	13.21278	25.185	38.87491	1	0	0	0	0	0	25.185	38.87491
6	13,21246	25.185	42.37423	1	0	0	0	0	0	25.185	42.37423
7	13.21177	25.185	46.37058	1	0	0	0	0	0	25.185	46.37058
8	13.21145	25.185	49.86990	1	0	0	0	0	0	25.185	49.86990
9	10.33788	3.0	38.58149	0	1	1	10.33788	3.0	10.33788	0	0
10	4.42158	3.0	38.58150	0	1	1	4.42158	3.0	4.42158	0	0
11	0.67865	3.0	38.58149	0	1	1	0.67865	3.0	0.67865	0	0
12	−3.66806	3.0	38.58150	0	1	1	−3.66806	3.0	−3.66806	0	0
13	−7.53171	3.0	38.58148	0	1	1	−7.53171	3.0	−7.5317	0	0
14	10.33788	24.125	38.58149	0	1	1	10.33788	24.125	10.33788	0	0
15	4.42138	24.125	38.58150	0	1	1	4.42138	24.125	4.42158	0	0
16	0.67865	24.125	38.58149	0	1	1	0.67865	24.125	0.67865	0	0
17	−3.66806	24.125	38.58150	0	1	1	−3.66806	24.125	−3.66806	0	0
18	−7.53171	24.125	38.58148	0	1	1	−7.53171	24.125	−7.53171	0	0
19	−5.95725	5.5625	36.93022	0	1	1	−5.95725	5.5625	−5.95725	0	0
20	−5.95725	21.5625	36.93022	0	1	1	−5.95725	21.5625	−5.95725	0	0
			Sum	**8**	**12**	**12**	**-3.43822**	**162.75**	**-3.43822**	**108.50**	**354.97924**

ζ	x_ς	y_ς	z_ς	$A_{\varsigma x}$	$A_{\varsigma y}$	$A_{\varsigma z}$	$A_{\varsigma y}\bar{z}_\varsigma$	$A_{\varsigma x}\bar{y}_\varsigma\bar{z}_\varsigma$	$A_{\varsigma y}\bar{x}_\varsigma\bar{z}_\varsigma$	$A_{\varsigma x}\bar{x}_\varsigma\bar{y}_\varsigma$	$A_{\varsigma x}\bar{y}_\varsigma^2$
1	13.21278	1.94	38.87491	1	0	0	0	0	0	0	0
2	13.21246	1.94	42.37423	1	0	0	0	0	0	0	0
3	13.21177	1.94	46.37058	1	0	0	0	0	0	0	0
4	13.21145	1.94	49.86990	1	0	0	0	0	0	0	0
5	13.21278	25.185	38.87491	1	0	0	0	0	0	0	0
6	13,21246	25.185	42.37423	1	0	0	0	0	0	0	0
7	13.21177	25.185	46.37058	1	0	0	0	↓	0	0	↓
8	13.21145	25.185	49.86990	1	0	0	0	= 4814.40595	0	0	= 2552.19130
9	10.33788	3.0	38.58149	0	1	1	38.58149	0	398.85081	0	0
10	4.42138	3.0	38.58150	0	1	1	38.58150	0	170.58347	0	0
11	0.67865	3.0	38.58149	0	1	1	38.58149	0	26.18333	0	0
12	−3.66806	3.0	38.58150	0	1	1	38.58150	0	−141.51926	0	0
13	−7.53171	3.0	38.58148	0	1	1	38.58148	0	−290.58452	0	0
14	10.33788	24.125	38.58149	0	1	1	38.58149	0	398.85081	0	0
15	4.42138	24.125	38.58150	0	1	1	38.58150	0	170.58347	0	0
16	0.67865	24.125	38.58149	0	1	1	38.58149	0	26.18333	0	0
17	−3.66806	24.125	38.58150	0	1	1	38.58150	0	−141.51926	↓	0
18	−7.53171	24.125	38.58148	0	1	1	38.58148	0	−290.58452	=114.95954	0
19	−5.95725	5.5625	36.93022	0	1	1	36.93022	0	−220.00255	↓	0
20	−5.95725	21.5625	36.93022	0	1	1	36.93022	0	−220.00255	= −161.59041	0
			Sum	**8**	**12**	**12**	**459.67536**	**4814.40595**	**-112.97744**	**-46.63086**	**2552.19130**

$\varsigma = i$	$\bar{x}$	$\bar{y}$	$\bar{z}$	$A_{\bar{x}}$	$A_{\bar{y}}$	$A_{\bar{z}}$	$A_{\bar{y}}\bar{x}^2$	$A_{\bar{z}}\bar{x}^2$	$A_{\bar{z}}\bar{y}^2$	$A_{\bar{x}}\bar{z}^2$	$A_{\bar{y}}\bar{z}^2$
1	13.21278	1.94	38.87491	1	0	0	0.0	0.0	0.0	1511.25824	0.0
2	13.21246	1.94	42.37423	1	0	0	0.0	0.0	0.0	1795.57537	0.0
3	13.21177	1.94	46.37058	1	0	0	0.0	0.0	0.0	46.37058	0.0
4	13.21145	1.94	49.86990	1	0	0	0.0	0.0	0.0	49.86990	0.0
5	13.21278	25.185	38.87491	1	0	0	0.0	0.0	0.0	38.87491	0.0
6	13.21246	25.185	42.37423	1	0	0	0.0	0.0	0.0	42.37423	0.0
7	13.21177	25.185	46.37058	1	0	0	0.0	0.0	0.0	46.37058	0.0
8	13.21145	25.185	49.86990	1	0	0	0.0	0.0	0.0	49.86990	0.0
9	10.33788	3.0	38.58149	0	1	1	106.87176	106.87176	9.0	0.0	1488.53137
10	4.42138	3.0	38.58150	0	1	1	19.54860	19.54860	9.0	0.0	1488.53214
11	0.67865	3.0	38.58149	0	1	1	0.46057	0.46057	9.0	0.0	1488.53137
12	-3.66806	3.0	38.58150	0	1	1	13.45466	13.45466	9.0	0.0	1488.53214
13	-7.53171	3.0	38.58148	0	1	1	56.72666	56.72666	9.0	0.0	1488.53060
14	10.33788	24.125	38.58149	0	1	1	106.87176	106.87176	582.01563	0.0	1488.53137
15	4.42138	24.125	38.58150	0	1	1	19.54860	19.54860	582.01563	0.0	1488.53214
16	0.67865	24.125	38.58149	0	1	1	0.46057	0.46057	582.01563	0.0	1488.53137
17	-3.66806	24.125	38.58150	0	1	1	13.45466	13.45466	582.01563	0.0	1488.53214
18	-7.53171	24.125	38.58148	0	1	1	56.72666	56.72666	582.01563	0.0	1488.53060
19	-5.95725	5.5625	36.93022	0	1	1	35.48883	35.48883	30.94141	0.0	1363.84115
20	-5.95725	21.5625	36.93022	0	1	1	35.48883	35.48883	464.94141	0.0	1363.84115
			Sum =	8	12	12	465.10215	465.10215	3450.96094	1396.47669	17612.99755

Centroids and then inertias must be calculated first. Extracting values from the above tables gives**:**

$$\sum A_{\varsigma x}=8;\ \ \sum A_{\varsigma y}=\sum A_{\varsigma z}=12\ (\text{non-dim.})$$
$$\sum A_{\varsigma z}\bar{x}_\varsigma=\sum A_{\varsigma y}\bar{x}_\varsigma=-3.43822\ \text{in.};$$
$$\sum A_{\varsigma z}\bar{y}_\varsigma=162.75\ \text{in.};\ \ \sum A_{\varsigma x}\bar{y}_\varsigma=108.50\ \text{in.};$$
$$\sum A_{\varsigma y}\bar{z}_\varsigma=459.67536\ \text{in.};\ \ \sum A_{\varsigma x}\bar{z}_\varsigma=354.97924\ \text{in.};$$
$$\bar{x}_{xz}=\frac{\sum A_{\varsigma z}\bar{x}_\varsigma}{\sum A_{\varsigma z}}=\frac{-3.43822}{12}=-0.28652\ \text{in.};\ \ \bar{x}_{xy}=\frac{\sum A_{\varsigma y}\bar{x}_\varsigma}{\sum A_{\varsigma y}}=\frac{-3.43822}{12}=-0.28652\ \text{in.};$$
$$\bar{y}_{yz}=\frac{\sum A_{\varsigma z}\bar{y}_\varsigma}{\sum A_{\varsigma z}}=\frac{162.75}{12}=13.56250\ \text{in.};\ \ \bar{y}_{xy}=\frac{\sum A_{\varsigma x}\bar{y}_\varsigma}{\sum A_{\varsigma x}}=\frac{108.50}{8}=13.56250\ \text{in.};$$
$$\bar{z}_{yz}=\frac{\sum A_{\varsigma y}\bar{z}_\varsigma}{\sum A_{\varsigma y}}=\frac{459.67536}{12}=38.30628\ \text{in.};\ \ \bar{z}_{xz}=\frac{\sum A_{\varsigma x}\bar{z}_\varsigma}{\sum A_{\varsigma x}}=\frac{354.97924}{8}=44.372405\ \text{in.}$$

And these values are used to calculate the the massless inertias**:**

$$\begin{aligned}
\bar{I}_{xx}&=\sum A_{\varsigma y}\bar{z}_\varsigma^2+\sum A_{\varsigma z}\bar{y}_\varsigma^2-\bar{y}_{yz}^2\sum A_{\varsigma z}-\bar{z}_{yz}^2\sum A_{\varsigma y}\\
&=17612.99755+3450.96094-13.56250^2x12-38.30628^2x12=1248.20857\ \text{in.}^2\\
\bar{I}_{yy}&=\sum A_{\varsigma x}\bar{z}_\varsigma^2+\sum A_{\varsigma z}\bar{x}_\varsigma^2-\bar{x}_{xz}^2\sum A_{\varsigma z}-\bar{z}_{xz}^2\sum A_{\varsigma x}\\
&=15888.14322+465.10215-.28652^2x12-44.372405^2x8=600.97765\ \text{in.}^2\\
\bar{I}_{zz}&=\sum A_{\varsigma x}\bar{y}_\varsigma^2+\sum A_{\varsigma y}\bar{x}_\varsigma^2-\bar{x}_{xz}^2\sum A_{\varsigma y}-\bar{y}_{xz}^2\sum A_{\varsigma x}\\
&=2552.19130+465.10215-.28652^2x12-13.56250^2x8=1544.7708\ \text{in.}^2\\
\bar{I}_{xy}&=-\left(\sum A_{\varsigma z}\bar{x}_\varsigma\bar{y}_\varsigma-\bar{y}_{yz}\bar{x}_{xz}\sum A_{\varsigma z}\right)=-(-46.63086+13.56250x.28652x12)=0\\
\bar{I}_{yz}&=-\left(\sum A_{\varsigma x}\bar{y}_\varsigma\bar{z}_\varsigma-\bar{y}_{xy}\bar{z}_{xz}\sum A_{\varsigma x}\right)=-(4814.40595-13.56250x44.372405x8)=0\\
\bar{I}_{xz}&=-\left(\sum A_{\varsigma y}\bar{x}_\varsigma\bar{z}_\varsigma-\bar{x}_{xy}\bar{z}_{yz}\sum A_{\varsigma y}\right)=-(-112.97744+.28652x38.30628x12)=-18.72874\ \text{in.}^2
\end{aligned}$$

And the inertia tensor with the Hookean vectors already incorporated into it is**:**

$$\left[\bar{I}_\varsigma\right]=\begin{bmatrix}\bar{I}_{xx}&\bar{I}_{xy}&\bar{I}_{xz}\\ \bar{I}_{yx}&\bar{I}_{yy}&\bar{I}_{yz}\\ \bar{I}_{zx}&\bar{I}_{zy}&\bar{I}_{zz}\end{bmatrix}=\begin{bmatrix}1248.20857&0&-18.72874\\0&600.97765&0\\-18.72874&0&1544.77708\end{bmatrix}$$

And the inverse inertia tensor is via ***CROUT****'s Method***:**

$$\left[\bar{I}_\varsigma\right]^{-1} = \begin{bmatrix} 8.01294 & 0 & .09715 \\ 0 & 16.93955 & 0 \\ .09715 & 0 & 6.47460 \end{bmatrix} \text{x}10^{-4}$$

The load transfer matrix (from the load $c.g.$ -to- support $c.g.$) is:

$$[x_L] = \begin{bmatrix} 0 & -\left(\bar{z}_L - \bar{z}_{yz}\right) & \left(\bar{y}_L - \bar{y}_{yz}\right) \\ \left(\bar{z}_L - \bar{z}_{xz}\right) & 0 & -\left(\bar{x}_L - \bar{x}_{xz}\right) \\ -\left(\bar{y}_L - \bar{y}_{xy}\right) & \left(\bar{x}_L - \bar{x}_{xy}\right) & 0 \end{bmatrix} \qquad \text{Eq. 8}$$

Or,

$$[x_L] = \begin{bmatrix} 0 & -6.58787 & 0 \\ .52124 & 0 & -4.25094 \\ 0 & 4.25094 & 0 \end{bmatrix}$$

And the support transfer matrix, which transfers the applied moments as couples to the support points, is:

$$[x_\varsigma] = \begin{bmatrix} 0 & \left(\bar{z}_\varsigma - 44.37241\right) & -\left(\bar{y}_\varsigma - 13.56250\right) \\ -\left(\bar{z}_\varsigma - 38.30628\right) & 0 & \left(\bar{x}_\varsigma + .28652\right) \\ \left(\bar{y}_\varsigma - 13.56250\right) & -\left(\bar{x}_\varsigma + .28652\right) & 0 \end{bmatrix}$$

where, for the masses, m_ζ , reduced to a point-set of mass-points, are at ζ =1, 2, ...,N. In most cases these mass-point coordinates are the same set as the coordinates (x_ς, y_ς, z_ς) set for ζ = 1, ... N reaction points.

Following two tables calculates, via spreadsheets, give the values of the $\boldsymbol{P}_\varsigma$'s and moments $\boldsymbol{M}$'s for two different loading cases. In these two cases ($x_{c.g.}$, $y_{c.g.}$, $z_{c.g.}$) = (3.96442, 13.5625, 44.89365) in. In this first example, a unit, local (horizontal) load is F_x = – cos15º , $\boldsymbol{F}_y$ = 0 and F_z = sin15º .(globally equivalent to F_x = 1.0 Lb.), is applied to determine its unit reaction set.

$\zeta = i$	$\bar{x}$	$\bar{y}$	$\bar{z}$	$A_{\bar{x}}$	$A_{\bar{y}}$	$A_{\bar{z}}$	$\hat{P}_x$	$\hat{P}_y$	$\hat{P}_z$	$\hat{M}_{xx}$	$\hat{M}_{yy}$	$\hat{M}_{zz}$
1	13.21278	1.94	38.87491	1	0	0	-0.10607	0	0	0.00000	0.63841	-1.23281
2	13.21246	1.94	42.37423	1	0	0	-0.11541	0	0	0.00000	0.29076	-1.34134
3	13.21177	1.94	46.37058	1	0	0	-0.12607	0	0	0.00000	-0.18620	-1.46528
4	13.21145	1.94	49.86990	1	0	0	-0.13541	0	0	0.00000	-0.67384	-1.57381
5	13.21278	25.185	38.87491	1	0	0	-0.10607	0	0	0.00000	0.63841	1.23281
6	13.21246	25.185	42.37423	1	0	0	-0.11541	0	0	0.00000	0.29076	1.34134
7	13.21177	25.185	46.37058	1	0	0	-0.12607	0	0	0.00000	-0.18620	1.46528
8	13.21145	25.185	49.86990	1	0	0	-0.13541	0	0	0.00000	-0.67384	1.57381
9	10.33788	3.0	38.58149	0	1	1	0.00000	0	0.04992	-0.52727	-0.31816	0.00000
10	4.42138	3.0	38.58150	0	1	1	0.00000	0	0.03413	-0.36051	-0.01560	0.00000
11	0.67865	3.0	38.58149	0	1	1	0.00000	0	0.02414	-0.25502	0.07933	0.00000
12	-3.66806	3.0	38.58150	0	1	1	0.00000	0	0.01254	-0.13250	0.09575	0.00000
13	-7.53171	3.0	38.58148	0	1	1	0.00000	0	0.00223	-0.02360	0.02569	0.00000
14	10.33788	24.125	38.58149	0	1	1	0.00000	0	0.04992	0.52727	-0.31816	0.00000
15	4.42138	24.125	38.58150	0	1	1	0.00000	0	0.03413	0.36051	-0.01560	0.00000
16	0.67865	24.125	38.58149	0	1	1	0.00000	0	0.02414	0.25502	0.07933	0.00000
17	-3.66806	24.125	38.58150	0	1	1	0.00000	0	0.01254	0.13250	0.09575	0.00000
18	-7.53171	24.125	38.58148	0	1	1	0.00000	0	0.00223	0.02360	0.02569	0.00000
19	-5.95725	5.5625	36.93022	0	1	1	0.00000	0	0.00644	-0.05149	0.06386	0.00000
20	-5.95725	21.5625	36.93022	0	1	1	0.00000	0	0.00644	0.05149	0.06386	0.00000
			Sum =	8	12	12	-0.96592	0	0.25880	0.00000	0.00001	0.00000

where:

- – 0.96592 +→ 0.25880 = 1.0;
- P_x = (– COS(15*PI()/180)/8 – 0.00266849*(AF7 – 44.37241))*AG7

  cell AF7 = 38.87491; cell AG7 = A_x = 1.0.
- P_z = (SIN(15*PI()/180)/12 + 0.00266849*(AD15 + 0.28652))*AI15)

  cell AD15 = 10.33788; cell AI15 = A_z = 1.0

All internal reactions at the N support points are given by the following three equations, with the 15º angle of rotation (local plane -to- global plane) already incorporated:

$$\hat{P}_{\zeta x} = \{\frac{\hat{F}_x}{8} + .00086732(\bar{z}_\zeta - 44.37241)\hat{F}_x - .00268832(\bar{y}_\zeta - 13.5625)\hat{F}_y - .007073373(\bar{z}_\zeta - 44.37241)\hat{F}_z\}A_x$$

$$\hat{P}_{\zeta y} = \{\frac{\hat{F}_y}{12} + 0 + \left[.00523712(\bar{z}_\zeta - 38.30628) + .00268832(\bar{x}_\zeta + .28652)\right]\hat{F}_y + 0\}A_y$$

$$\hat{P}_{\zeta z} = \{\frac{\hat{F}_z}{12} - .00086732(\bar{x}_\zeta + 0.28652)\hat{F}_x - .00523712(\bar{y}_\zeta - 13.5625)\hat{F}_y + .007073373(\bar{x}_\zeta + 0.28652)\hat{F}_z\}A_z$$

The table below employs $F_x = F_y = 0$ and $F_z = 1.0$.

$\zeta = i$	$\bar{x}$	$\bar{y}$	$\bar{z}$	A_x	A_y	A_z	$\hat{P}_x$	$\hat{P}_y$	$\hat{P}_z$	$\hat{M}_{xx}$	$\hat{M}_{yy}$	$\hat{M}_{zz}$
1	13.21278	1.94	38.87491	1	0	0	0.03889	0	0.00000	0.00000	-0.23407	0.45200
2	13.21246	1.94	42.37423	1	0	0	0.01413	0	0.00000	0.00000	-0.03560	0.16423
3	13.21177	1.94	46.37058	1	0	0	-0.01411	0	0.00000	0.00000	-0.02084	-0.16403
4	13.21145	1.94	49.86990	1	0	0	-0.03889	0	0.00000	0.00000	-0.19353	-0.45200
5	13.21278	25.185	38.87491	1	0	0	0.03889	0	0.00000	0.00000	-0.23407	-0.45200
6	13.21246	25.185	42.37423	1	0	0	0.01411	0	0.00000	0.00000	-0.03556	-0.16403
7	13.21177	25.185	46.37058	1	0	0	-0.01413	0	0.00000	0.00000	-0.02087	0.16423
8	13.21145	25.185	49.86990	1	0	0	-0.03889	0	0.00000	0.00000	-0.19353	0.45200
9	10.33788	3.0	38.58149	0	1	1	0.00000	0	0.15848	-1.67395	-1.01007	0.00000
10	4.42138	3.0	38.58150	0	1	1	0.00000	0	0.11663	-1.23190	-0.05330	0.00000
11	0.67865	3.0	38.58149	0	1	1	0.00000	0	0.09016	-0.95232	0.29625	0.00000
12	-3.66806	3.0	38.58150	0	1	1	0.00000	0	0.05941	-0.62752	0.45345	0.00000
13	-7.53171	3.0	38.58148	0	1	1	0.00000	0	0.03209	-0.33895	0.36891	0.00000
14	10.33788	24.125	38.58149	0	1	1	0.00000	0	0.15848	1.67395	-1.01007	0.00000
15	4.42138	24.125	38.58150	0	1	1	0.00000	0	0.11663	1.23190	-0.05330	0.00000
16	0.67865	24.125	38.58149	0	1	1	0.00000	0	0.09016	0.95232	0.29625	0.00000
17	-3.66806	24.125	38.58150	0	1	1	0.00000	0	0.05941	0.62752	0.45345	0.00000
18	-7.53171	24.125	38.58148	0	1	1	0.00000	0	0.03209	0.33895	0.36891	0.00000
19	-5.95725	5.5625	36.93022	0	1	1	0.00000	0	0.04322	-0.34576	0.42881	0.00000
20	-5.95725	21.5625	36.93022	0	1	1	0.00000	0	0.04322	0.34576	0.42881	0.00000
			Sum =	8	12	12	0.00000	0	0.99998	0.00000	5E-005	0.0

It is important to note the internal moments (are actually couples) don't disappear in structures, but product internal bending stresses in same, even though the gross external moment effects are zero. These additional stresses are usually missed in analyses because of the lack of not knowing how to determine them.

The equilibrium moments in ship's coordinates are:

$$\sum_{i=1}^{\zeta} M_{xx} = 0 = \sum_{i=1}^{\zeta}\left[\left(\bar{y}_i - \bar{y}_{c.g.}\right)P_{zi} - \left(\bar{z}_i - \bar{z}_{c.g.}\right)P_{yi}\right] = \sum_{i=1}^{\zeta}\left[\left(\bar{y}_i - 13.5625\right)P_{zi} - \left(\bar{z}_i - 44.39\right)P_{yi}\right];$$

$$\sum_{i=1}^{\zeta} M_{yy} = 0 = \sum_{i=1}^{\zeta}\left[-\left(\bar{x}_i - \bar{x}_{c.g.}\right)P_{zi} + \left(\bar{z}_i - \bar{z}_{c.g.}\right)P_{xi}\right] = \sum_{i=1}^{\zeta}\left[-\left(\bar{x}_i + 7.7900\right)P_{zi} + \left(\bar{z}_i - 44.39\right)P_{xi}\right];$$

$$\sum_{i=1}^{\zeta} M_{zz} = 0 = \sum_{i=1}^{\zeta}\left[-\left(\bar{y}_i - \bar{y}_{c.g.}\right)P_{xi} + \left(\bar{x}_i - \bar{x}_{c.g.}\right)P_{yi}\right] = \sum_{i=1}^{\zeta}\left[-\left(\bar{y}_i - 13.5625\right)P_{xi} + \left(\bar{x}_i + 7.7900\right)P_{yi}\right].$$

where $(x_{c.g.}, y_{c.g.}, z_{c.g.})_{\text{ship}}$ = (– 7.79, 13.5625, 44.39) in.

See **Appendix B**, "**Numerical Example *Output File*.**"

Appendix **B:** Computer Programs, Input/Output Files

Numerical Example ***Input File***

QBasic ***INPUT*** file for computer ***RUN***s for the *Numerical Example* above is *Monitor.dat*. It list only the title and the twenty of the monitor's fastener locations, plus their ***DOF***, in local display monitor coordinates (not ship coordinates)**:**

Display Monitor [Line one: title]

[input**:** Grid, x, y, z, A_1, A_2, A_3]

```
1,13.21278,1.94,38.87491,1.0,0.0,0.0
2,13.21246,1.94,42.37423,1.0,0.0,0.0
3,13.21177,1.94,46.37058,1.0,0.0,0.0
4,13.21145,1.94,49.86990,1.0,0.0,0.0
5,13.21278,25.185,38.87491,1.0,0.0,0.0
6,13.21246,25.185,42.37423,1.0,0.0,0.0
7,13.21177,25.185,46.37058,1.0,0.0,0.0
8,13.21145,25.185,49.86990,1.0,0.0,0.0
9,10.33788,3.0,38.58149,0.0,1.0,1.0
10,4.42138,3.0,38.58150,0.0,1.0,1.0
11,0.67865,3.0,38.58149,0.0,1.0,1.0
12,-3.66806,3.0,38.58150,0.0,1.0,1.0
13,-7.53171,3.0,38.58148,0.0,1.0,1.0
14,10.33788,24.125,38.58149,0.0,1.0,1.0
15,4.42138,24.125,38.58150,0.0,1.0,1.0
16,0.67865,24.125,38.58149,0.0,1.0,1.0
17,-3.66806,24.125,38.58150,0.0,1.0,1.0
18,-7.53171,24.125,38.58148,0.0,1.0,1.0
19,-5.95725,5.56250,36.93022,0.0,1.0,1.0
20,-5.95725,21.5625,36.93022,0.0,1.0,1.0
```

[End of data]

Copy the above date to a ***Notepad*** file called Display_Monitor.dat or whatever.dat as an ***INPUT*** file.

The load in Lbs. in the "**A Numerical Example**" above is F_x = – cos15º , $\boldsymbol{F}_y$ = 0 and F_z = sin15º, applied at $(x_{c.g.}, y_{c.g.}, z_{c.g.})_{\text{ship}}$ = (– 7.79, 13.5625, 44.39) in.; better agree is the results shown below.

Numerical Example *OUTPUT File*

The ***OUTPUT*** file or sysout (called *Output_Fz.dat*) below was created by a ***MS***' ***QBasic*** program ***Inertia2.bas***, which has, besides screen, and the ***LPRINT*** code for a printer, an additional "***PRINT #3***" code for writing to a file (here *Output_Fz.dat*). This latter file is easily inserted into a work-document (as was done here). [There are some very small numerical differences in the 5th decimal places between the ***OOFC***'s calculated spreadsheets cell values and the new ***QBasic*** calculated values.] ***QBasic***'s ***OUTPUT*** follows**:**

INERTIA LOADS — UNIT SOLUTION

All units in inches and pounds.

Echo support coordinates for...Input file**:** *Display_*M*onitor.dat*

		xi	yi	zi	Axi	Ayi	Azi
		in.	in.	in.	------ D.O.F.-----		
GRID	1	13.21278	1.94000	38.87491	1.00000	0.00000	0.00000
GRID	2	13.21246	1.94000	42.37423	1.00000	0.00000	0.00000
GRID	3	13.21177	1.94000	46.37058	1.00000	0.00000	0.00000
GRID	4	13.21145	1.94000	49.86990	1.00000	0.00000	0.00000
GRID	5	13.21278	25.18500	38.87491	1.00000	0.00000	0.00000
GRID	6	13.21246	25.18500	42.37423	1.00000	0.00000	0.00000
GRID	7	13.21177	25.18500	46.37058	1.00000	0.00000	0.00000
GRID	8	13.21145	25.18500	49.86990	1.00000	0.00000	0.00000
GRID	9	10.33788	3.00000	38.58149	0.00000	1.00000	1.00000
GRID	10	4.42138	3.00000	38.58150	0.00000	1.00000	1.00000
GRID	11	0.67865	3.00000	38.58149	0.00000	1.00000	1.00000
GRID	12	-3.66806	3.00000	38.58150	0.00000	1.00000	1.00000
GRID	13	-7.53171	3.00000	38.58148	0.00000	1.00000	1.00000
GRID	14	10.33788	24.12500	38.58149	0.00000	1.00000	1.00000

GRID	15	4.42138	24.12500	38.58150	0.00000	1.00000	1.00000
GRID	16	0.67865	24.12500	38.58149	0.00000	1.00000	1.00000
GRID	17	-3.66806	24.12500	38.58150	0.00000	1.00000	1.00000
GRID	18	-7.53171	24.12500	38.58148	0.00000	1.00000	1.00000
GRID	19	-5.95725	5.56250	36.93022	0.00000	1.00000	1.00000
GRID	20	-5.95725	21.56250	36.93022	0.00000	1.00000	1.00000

Loading C.G.'s**:** Xcg, Ycg, Zcg =
3.96 13.56 44.89

Applied Loads**:** Fx, Fy, Fz =
0.00 0.00 1.00

Applied moments: Mx, My, Mz
0.00 0.00 0.00

XBARxy in.	XBARxz in.	YBARyx in.	YBARyz in.	ZBARxz in.	ZBARyz in.
-0.28652	-0.28652	13.56250	13.56250	44.37240	38.30628

INERTIA TENSOR of the support system

Ixx in^2	Iyy in^2	Izz in^2	Ixy in^2	Ixz in^2	Iyz in^2
1248.20855	600.97770	1544.77705	0.00000	-18.72799	0.00000

INVERSE TENSOR of the support system

Ixx 1/in^2	Iyy 1/in^2	Izz 1/in^2	Ixy 1/in^2	Ixz 1/in^2	Iyz 1/in^2
0.00080	0.00166	0.00065	-0.00000	0.00001	-0.00000

[Notice in the “Px” column following (FWD force components), how the forces are couples, which product no side-ways or vertical translational forces**:** only couples cancel. ***OUTPUT*** is in the local coordinates system.]

Unit Vertical Load

This is another example ***RUN***.

APPLIED LOADS...[$F_x = F_y = 0.0$, $F_z = 1.0$.]

GRIDS	Px	Py	Pz
	Lb.	Lb.	Lb.
1	0.03889	0.00000	0.00000
2	0.01413	0.00000	0.00000
3	-0.01413	0.00000	0.00000
4	-0.03889	0.00000	0.00000
5	0.03889	0.00000	0.00000
6	0.01413	0.00000	0.00000
7	-0.01413	0.00000	0.00000
8	-0.03889	0.00000	0.00000
9	0.00000	-0.00000	0.15848
10	0.00000	-0.00000	0.11663
11	0.00000	-0.00000	0.09016
12	0.00000	-0.00000	0.05941
13	0.00000	-0.00000	0.03209
14	0.00000	-0.00000	0.15848
15	0.00000	-0.00000	0.11663
16	0.00000	-0.00000	0.09016
17	0.00000	-0.00000	0.05941
18	0.00000	-0.00000	0.03209
19	0.00000	0.00000	0.04322
20	0.00000	0.00000	0.04322

	Mxx	Myy	Mzz
	Lb.-in.	Lb.-in.	Lb.-in.
1	0.00000	-0.23404	0.45195
2	0.00000	-0.03561	0.16427
3	0.00000	-0.02087	-0.16427

4	0.00000	-0.19351	-0.45195
5	0.00000	-0.23404	-0.45195
6	0.00000	-0.03561	-0.16427
7	0.00000	-0.02087	0.16427
8	0.00000	-0.19351	0.45195
9	-1.67398	-1.01009	-0.00000
10	-1.23195	-0.05330	-0.00000
11	-0.95232	0.29625	0.00000
12	-0.62756	0.45348	0.00000
13	-0.33890	0.36886	0.00000
14	1.67398	-1.01009	-0.00000
15	1.23195	-0.05330	-0.00000
16	0.95232	0.29625	0.00000
17	0.62756	0.45348	0.00000
18	0.33890	0.36886	0.00000
19	-0.34578	0.42884	-0.00000
20	0.34578	0.42884	-0.00000

SUMMATION OF FORCES AND MOMENTS ABOUT THE LOAD POINT

FX	FY	FZ
Lb.	Lb.	Lb.
0.00000	0.00000	1.00000

Mxx	Myy	Mzz
Lb.-in.	Lb.-in.	Lb.-in.
-0.00000	0.00000	-0.00000

Unit *FWD* Load example

[In the next case (second ***RUN***) the unit set ($\boldsymbol{F}_x$ = 1.0) has been rotated 15^o upward to match the floor, i.e. to match the global coordinate system.]

APPLIED LOADS...[$\boldsymbol{F}_x = -\cos 15^o$, $\boldsymbol{F}_y = 0$ and $\boldsymbol{F}_z = \sin 15^o$.]

GRIDS	Px	Py	Pz
	Lb.	Lb.	Lb.
1	-0.10607	0.00000	0.00000
2	-0.11541	0.00000	0.00000
3	-0.12607	0.00000	0.00000
4	-0.13541	0.00000	0.00000
5	-0.10607	0.00000	0.00000
6	-0.11541	0.00000	0.00000
7	-0.12607	0.00000	0.00000
8	-0.13541	0.00000	0.00000
9	0.00000	0.00000	0.04992
10	0.00000	0.00000	0.03413
11	0.00000	0.00000	0.02414
12	0.00000	-0.00000	0.01254
13	0.00000	-0.00000	0.00223
14	0.00000	0.00000	0.04992
15	0.00000	0.00000	0.03413
16	0.00000	0.00000	0.02414
17	0.00000	-0.00000	0.01254
18	0.00000	-0.00000	0.00223
19	0.00000	-0.00000	0.00644
20	0.00000	-0.00000	0.00644

	Mxx	Myy	Mzz
	Lb.-in.	Lb.-in.	Lb.-in.
1	0.00000	0.63841	-1.23281
2	0.00000	0.29076	-1.34134
3	0.00000	-0.18620	-1.46528
4	0.00000	-0.67384	-1.57381
5	0.00000	0.63841	1.23281
6	0.00000	0.29076	1.34134
7	0.00000	-0.18620	1.46528
8	0.00000	-0.67384	1.57381

9	-0.52727	-0.31816	0.00000
10	-0.36051	-0.01560	0.00000
11	-0.25502	0.07933	-0.00000
12	-0.13250	0.09575	0.00000
13	-0.02360	0.02569	0.00000
14	0.52727	-0.31816	0.00000
15	0.36051	-0.01560	0.00000
16	0.25502	0.07933	-0.00000
17	0.13250	0.09575	0.00000
18	0.02360	0.02569	0.00000
19	-0.05149	0.06385	0.00000
20	0.05149	0.06385	0.00000

SUMMATION OF FORCES AND MOMENTS ABOUT THE LOAD POINT

FX	FY	FZ
Lb.	Lb.	Lb.
– 0.96593	0.00000	0.25882

Mxx	Myy	Mzz
Lb.-in.	Lb.-in.	Lb.-in.
0.00000	– 0.00000	– 0.00000

FORTRAN Code

The code following is identical to the ***QBasic*** code in scope and intent, except for the ***Sysout*** printing, which has been limited to just a printer. This short code initially was written for the mainframe years ago, and still can, but can also be directly embedded into **C++** on a ***PC***.

```
C
C             INERTIA LOADS UNIT SOLUTION. FORTRAN VERSION . . . .
C
C
       DIMENSION GRID(250). X(250). Y(250). Z(250). TITLE(13).
      1 AX(250), AY(250), AZ(250),
      2 PX(250), PY(250), PZ(250), TEMPX(250),
      3 MXX(250), MYY(250), MZZ(250)
C
       REAL*8 X,Y,Z,XCG/O./,YCG/O./,ZCG/O./,FX/O./,FY/O./,FZ/O./,
      1 PX,PY,PZ, MXX,MYY,MZZ, TEMPX,
      2 IIXX/O./,IIYY/O./,IIZZ/O./,IIXY/O./,IIXZ/O./,IIYZ/O./,
      3 I3/O./, K1/O./, K2/O./, K3/O./
C
       DOUBLE PRECISION IXX, IYY,IZZ,IXY,IXZ,IYZ,
      1 IIXX, IIYY, IIZZ, IIXY, IIXZ, IIYZ, THETA
C
        REAL*8 SAX, SAY, SAZ, TAX, TAY, TAZ, TPX, TPY, TPZ,
      5 AXYY/O./,AXZZ/O./,AXYZ/O./,SAXY/O./,SAXZ/O./,SAXYY,SAXZZ,SAXYZ,
      6 AYXX/O./,AYZZ/O./,AYXZ/O./,SAYX/O./,SAYZ/O./,SAYXX,SAYZZ,SAYXZ,
      7 AZXX/O./,AZYY/O./,AZXY/O./,SAZX/O./,SAZY/O./,SAZXX,SAZYY,SAZXY
        REAL*8 SAXYY/O./,SAXZZ/O./,SAXYZ/O./,
      8 SAYXX/O./,SAYZZ/O./,SAYXZ/O./,SAZXX/O./,SAZYY/O./,SAZXY/O./,
      9 YBYX/O./, ZBXZ/O./, XBXY/O./, ZBYZ/O./, XBXZ/O./, YBYZ/O./,
      * AXY/O./,AXZ/O./,AYX/O./,AYZ/O./,AZX/O./,AZY/O./
       REAL*8 SMXX,SMYY,SMZZ,TMXX,TMYY,TMZZ,
      * SPX, SPY, SPZ
C
       INTEGER I/O/, SUPRT/O/, GRID, AX, AY, AZ
C
       IIXX  =  0.0
       IIYY  =  0.0
       IIZZ  =  0.0
       IIXY  =  0.0
       IIXZ  =  0.0
       IIYZ  =  0.0
C
C
       IXX  =  0.0
       IYY  =  0.0
       IZZ  =  0.0
       IXY  =  0.0
       IXZ  =  0.0
       IYZ  =  0.0
C
C
C
C
C   ........................ READ UNIT LOADING  .....................
       READ(2,90) FX, FY, FZ, TITLE
       WRITE(6,99) TITLE, FX, FY, FZ
       WRITE(7,99) TITLE, FX, FY, FZ
 90    FORMAT(3F8.3, 13A4)
 99    FORMAT(/12X,13A4,//5X,'APPLIED LOAD: ,3F9.5)
C
C
C    ........................ LOADING  CENTER   .....................
       READ(2,91) SUPRT, XCG, YCG, ZCG
       WRITE(6,98) SUPRT, XCG, YCG, ZCG
       WRITE(7,98) SUPRT, XCG, YCG, ZCG
 91    FORMAT(I3,5X,3F8.3)
 98    FORMAT(5X,'NO. OF SUPPORTS: ',I3,',',5X,'LOAD C.G.',3F9.5)
C
C
       DO 10 I = 1, SUPRT
             GRID(I)   =  O
             AX(I) = O
             AY(I) = O
             AZ(I) = O
             X(I) = 0.0
             Y(I) = 0.0
             Z(I) = 0.0
             PX(I) = 0.0
```

```
          PY(I) = 0.0
          PZ(I) = 0.0
          MXX(I) = 0.0
          MYY(I) = 0.0
          MZZ(I) = 0.0
 10   CONTINUE
C
C   ....................... SUPPORT GEOMETRY .......................
C
         WRITE(6,399)
         WRITE(7,399)
 399     FORMAT(/15X,'SUPPORT COORDIATES ....')
C
      DO 102 I = 1, SUPRT
         READ(2,100) GRID(I), X(I), Y(I), Z(I), AX(I), AY(I), AZ(I)
         WRITE(6,199) GRID(I), X(I), Y(I), Z(I), AX(I), AY(I), AZ(I)
         WRITE(7,199) GRID(I), X(I), Y(I), Z(I), AX(I), AY(I), AZ(I)
 100     FORMAT(8X,I3,13X,3F8.3,I1,7X,I1,7X,I1)
 199     FORMAT(1X,'GRID',4X,I3,13X,3F8.3,3X,I1,3X,I1,3X,I1)
 102  CONTINUE
C
      SAX  = 0.0
      SAY  = 0.0
      SAZ  = 0.0
      TAX  = 0.0
      TAY  = 0.0
      TAZ  = 0.0
C
      DO 103 I = 1, SUPRT
         TAX      =  AX(I)
         SAX      =  SAX     +  TAX
         TAY      =  AY(I)
         SAY      =  SAY     +  TAY
         TAZ      =  AZ(I)
         SAZ      =  SAZ     +  TAZ
 103  CONTINUE


      DO 104 I = 1, SUPRT
C
        AXY           =  AX(I)*Y(I)
        SAXY          =  SAXY     +  AXY
C
        AXZ           =  AX(I)*Z(I)
        SAXZ          =  SAXZ     +  AXZ
C
        AXYY          =  AX(I)*Y(I)*Y(I)
        SAXYY         =  SAXYY +     AXYY
        AXZZ          =  AX(I)*Z(I)*Z(I)
        SAXZZ         =  SAXZZ +     AXZZ
        AXYZ          =  AX(I)*Y(I)*Z(I)
        SAXYZ         =  SAXYZ +     AXYZ
C
C
        AYX           =  AY(I)*X(I)
        AYZ           =  AY(I)*Z(I)
        AYXX          =  AY(I)*X(I)*X(I)
        AYZZ          =  AY(I)*Z(I)*Z(I)
        AYXZ          =  AY(I)*X(I)*Z(I)
        SAYX          =  SAYX     +  AYX
        SAYZ          =  SAYZ     +  AYZ
        SAYXX         =  SAYXX +     AYXX
        SAYZZ         =  SAYZZ +     AYZZ
        SAYXZ         =  SAYXZ +     AYXZ
C
        AZX           =  AZ(I)*X(I)
        AZY           =  AZ(I)*Y(I)
        AZXX          =  AZ(I)*X(I)*X(I)
        AZYY          =  AZ(I)*Y(I)*Y(I)
        AZXY          =  AZ(I)*X(I)*Y(I)
        SAZX          =  SAZX      +  AZX
        SAZY          =  SAZY      +  AZY
        SAZXX         =  SAZXX     +  AZXX
```

```
         SAZYY       =  SAZYY    +  AZYY
         SAZXY       =  SAZXY    +  AZXY
C
 104  CONTINUE
C
C
C ........................ CALCULATE C.G. .........................
C
C
          XBXY     =    SAYX/SAY
          XBXZ     =    SAZX/SAZ
C
          YBYX     =    SAXY/SAX
          YBYZ     =    SAZY/SAZ
C
          ZBXZ     =    SAXZ/SAX
          ZBYZ     =    SAYZ/SAY
C
      WRITE(6,398) XBXY, XBXZ, YBYX, YBYZ, ZBXZ, ZBYZ
      WRITE(7,398) XBXY, XBXZ, YBYX, YBYZ, ZBXZ, ZBYZ
 398  FORMAT(/10X,' CENTROIDS OF SUPPORT SYSTEM .....'./
     1    9X,'XBXY',5X,F8.4,5X,'XBXZ',5X,F8.4,5X,'YBYX',5X,F8.4,/
     2    9X,'YBYZ',5X,F8.4,5X,'ZBXZ',5X,F8.4,5X,'ZBYZ',5X,F8.4/)
C
          IIXX =  SAYZZ + SAZYY - YBYZ**2*SAZ - ZBYZ**2*SAY
          IIYY =  SAXZZ + SAZXX - XBXZ**2*SAZ - ZBXZ**2*SAX
          IIZZ =  SAXYY + SAYXX - XBXY**2*SAY - YBYX**2*SAX
C
          IIXY  =  -(SAZXY - YBYZ*XBXZ*SAZ)
          IIYZ  =  -(SAXYZ - YBYX*ZBXZ*SAX)
          IIXZ  =  -(SAYXZ - XBXY*ZBYZ*SAY)
C
          WRITE(6,395) IIXX, IIYY, IIZZ , IIXY, IIXZ, IIYZ
          WRITE(7,395) IIXX, IIYY, IIZZ , IIXY, IIXZ, IIYZ
 395  FORMAT(10X' INERTIA TENSOR .....'/.
     1  9X,'IIXX',4X,F11.5,5X,'IIYY',5X,F11.5,5X,'IIZZ',5X,F11.5/,
     2  9X,'IIXY',4X,F11.5,5X,'IIXZ',5X,F11.5,5X,'IIYZ',5X,F11.5)
C
C ........................  INVERSE  ...............................
C
      I3  =  IIXX*IIYY*IIZZ - IIXX*IIYZ**2
     1         -IIYY*IIXZ**2 - IIZZ*IIXY**2 + 2*IIXY*IIYZ*IIXZ
      IF (I3 .EQ. 0) GOTO 999
C
          IXX =  (IIYY*IIZZ - IIYZ**2)/I3
          IYY =  (IIXX*IIZZ - IIXZ**2)/I3
          IZZ =  (IIXX*IIYY - IIXY**2)/I3
          IXY =  (IIXY*IIZZ - IIXZ*IIYZ)/I3
          IXZ =  (IIXY*IIYZ - IIXZ*IIYY)/I3
          IYZ =  (IIYZ*IIXX - IIXY*IIXZ)/I3
C
          WRITE(6,394) IXX, IYY, IZZ ,IXY, IXZ, IYZ
          WRITE(7,394) IXX, IYY, IZZ ,IXY, IXZ, IYZ
 394  FORMAT(/10X,' INVERSE ....',/
     1  9X,'IXX',5X,E12.5,5X,'IYY',5X,E12.5,5X,'IZZ',5X,E12.5/,
     2  9X,'IXY',5X,E12.5,5X,'IXZ',5X,E12.5,5X,'IYZ',5X,E12.5/)
C
C
C ...........................  K FACTORS  ..........................
C
      K1  = (IXY*(ZCG-ZBXZ)-IXZ*(YCG-YBYX))*FX
     1                 +(-IXX*(ZCG-ZBYZ)+IXZ*(XCG-XBXY))*FY
     2                 +(IXX*(YCG-YBYZ)-IXY*(XCG-XBXZ))*FZ
C
      K2  = (IYY*(ZCG-ZBXZ)-IYZ*(YCG-YBYX))*FX
     1                 +(-IXY*(ZCG-ZBYZ)+IYZ*(XCG-XBXY))*FY
     2                 +(IXY*(YCG-YBYZ)-IYY*(XCG-XBXZ))*FZ
C
      K3  = (IYZ*(ZCG-ZBXZ)-IZZ*(YCG-YBYX))*FX
     1                 +(-IXZ*(ZCG-ZBYZ)+IZZ*(XCG-XBXY))*FY
     2                 +(IXZ*(YCG-YBYZ)-IYZ*(XCG-XBXZ))*FZ
C
C
```

```
      SPX  =  0.0
      SPY  =  0.0
      SPZ  =  0.0
      TPX  =  0.0
      TPY  =  0.0
      TPZ  =  0.0
C
      WRITE(6,301) TITLE
      WRITE(7,301) TITLE
 301  FORMAT(1X,13A4,/' LOCAL COORDINATES')
      TPZ  =  0.0
C
         WRITE(8,211) TITLE
 211     FORMAT('$',/'$',10X,13A4/,'$')
C
      DO 202 I = 1, SUPRT
C
         PX(I)  = (FX/SAX -(Y(I)-YBYX)*K3 + (Z(I)-ZBXZ)*K2)*AX(I)
         TPX      =  PX(I)
         SPX      =  TPX     +  SPX
         PY(I)  = (FY/SAY +(X(I)-XBXY)*K3 - (Z(I)-ZBYZ)*K1)*AY(I)
         TPY      =  PY(I)
         SPY      =  TPY     +  SPY
         PZ(I)  = (FZ/SAZ -(X(I)-XBXZ)*K2 + (Y(I)-YBYZ)*K1)*AZ(I)
         TPZ      =  PZ(I)
         SPZ      =  TPZ     +  SPZ
C
         WRITE(6,203) GRID(I), PX(I), PY(I), PZ(I)
         WRITE(7,203) GRID(I), PX(I), PY(I), PZ(I)
 203     FORMAT(' FORCE',3X,I4,20X,3F10.5)
C
         WRITE(8,214) GRID(I), PX(I), PY(I), PZ(I)
 214     FORMAT('FORCE',3X,8X,I4,4X,8X,'1.0',5X,3F8.5)
C
 202  CONTINUE
C
C
      WRITE(6,204) SPX, SPY, SPZ
      WRITE(7,204) SPX, SPY, SPZ
 204  FORMAT(/'    SUM OF THE FORCES: ',6X,3F10.5)
C
C
C ....................... CALCULATE  MOMENTS ......................
C
      SMXX    =  0.0
      SMYY    =  0.0
      SMZZ    =  0.0
      TMXX    =  0.0
      TMYY    =  0.0
      TMZZ    =  0.0
C
      DO 205 I = 1, SUPRT
          MXX(I)   =   (Y(I) - YCG)*PZ(I) - (Z(I) - ZCG)*PY(I)
          TMXX     =   MXX(I)
          SMXX     =   TMXX + SMXX
          MYY(I)   =   (Z(I) - ZCG)*PX(I) - (X(I) - XCG)*PZ(I)
          TMYY     =   MYY(I)
          SMYY     =   TMYY + SMYY
          MZZ(I)   =   (X(I) - XCG)*PY(I) - (Y(I) - YCG)*PX(I)
          TMZZ     =   MZZ(I)
          SMZZ     =   TMZZ + SMZZ
 205  CONTINUE
C
      WRITE(6,206) SMXX, SMYY, SMZZ
      WRITE(7,206) SMXX, SMYY, SMZZ
 206  FORMAT('    SUM OF THE MOMENTS: ',5X,3F10.5//)
C
 999  STOP
      END
```

QBasic Code

This program was changed and updated Sept. 2017 to correct the **PRINT #3** sysout code.[114] Copy this code directly into ***QBasic***'s ***menu*** and ***RUN*** with *Display_Monitor.dat*.

```
'
'        INERTIA LOADS UNIT SOLUTION, QBASIC VERSION, 12/2/89
'
    TYPE parameters
        D AS STRING * 3
        T AS STRING * 20
        F AS STRING * 12
        i AS INTEGER
        j AS INTEGER
        H AS STRING * 3
        HH AS STRING * 3
        HARD AS INTEGER
        HARDTO AS INTEGER
        G AS STRING * 3
        P AS STRING * 3
        Kount AS INTEGER
        SKIP AS STRING * 3
        SKIPIT AS INTEGER
        PAGE2 AS INTEGER
        SUPRT AS INTEGER
'
        X AS SINGLE
        Y AS SINGLE
        Z AS SINGLE
        GRID AS INTEGER
        Ax AS SINGLE
        Ay AS SINGLE
        Az AS SINGLE
        Px AS DOUBLE
```

114 This program is ***Inertia2.bas*** (the only difference between ***Inertia1.bas*** and ***Inertia2.bas*** is the inclusion of the **PRINT #3** lines, which enables the code to print a file to a folder). The original version was coded in **BASICA**, 12/2/84.

```
Py AS DOUBLE
Pz AS DOUBLE
Mxx AS DOUBLE
Myy AS DOUBLE
Mzz AS DOUBLE
'
Xcg AS SINGLE
Ycg AS SINGLE
Zcg AS SINGLE
Fx AS SINGLE
Fy AS SINGLE
Fz AS SINGLE
Mx AS SINGLE
My AS SINGLE
Mz AS SINGLE
SAx AS DOUBLE
SAy AS DOUBLE
SAz AS DOUBLE
TAx AS DOUBLE
TAy AS DOUBLE
TAz AS DOUBLE
IIxx AS DOUBLE
SAyzz AS DOUBLE
SAzyy AS DOUBLE
YByz AS DOUBLE
ZByz AS DOUBLE
IIyy AS DOUBLE
SAxzz AS DOUBLE
SAzxx AS DOUBLE
XBxz AS DOUBLE
ZBxz AS DOUBLE
IIzz AS DOUBLE
SAxyy AS DOUBLE
SAyxx AS DOUBLE
XBxy AS DOUBLE
YByx AS DOUBLE
```

```
        IIxy AS DOUBLE
        SAzxy AS DOUBLE
        IIyz AS DOUBLE
        SAxyz AS DOUBLE
        IIxz AS DOUBLE
        SAyxz AS DOUBLE
        I3 AS DOUBLE
        K1 AS DOUBLE
        K2 AS DOUBLE
        K3 AS DOUBLE
        SPx AS DOUBLE
        SPy AS DOUBLE
        SPz AS DOUBLE
        SMxx AS DOUBLE
        SMyy AS DOUBLE
        SMzz AS DOUBLE
        TMxx AS DOUBLE
        TMyy AS DOUBLE
        TMzz AS DOUBLE
'
      END TYPE
'
      OPTION BASE 1
'
      DECLARE SUB Outputfile1 (T$, SUPRT%, GRID(), X(), Y(), Z(), Ax(), Ay(), Az(), Xcg, Ycg, Zcg, Fx, Fy, Fz, Mx, My, Mz)
      DECLARE SUB supportgeometry (T$, SUPRT%, GRID(), X(), Y(), Z(), Ax(), Ay(), Az())
      DECLARE SUB inputfile (T$, SUPRT%, GRID(), X(), Y(), Z(), Ax(), Ay(), Az())
      DECLARE SUB printtitlehead (T$, F$) STATIC

      DIM GRID(40), X(40), Y(40), Z(40)
      DIM Ax(40), Ay(40), Az(40)
      DIM Px#(40), Py#(40), Pz#(40)
      DIM Mxx#(40), Myy#(40), Mzz#(40)

      CLS
```

```
PRINT: PRINT: PRINT:
PRINT TAB(22); "INERTIA LOADS UNIT SOLUTION ..."
PRINT: PRINT:
PRINT TAB(12); "All UNITS in inches and pounds."

PRINT: PRINT: PRINT:
PRINT TAB(12); "INPUT: is there an INPUT FILE FOR THE SUPPORT GEOMETRY?"
PRINT:
PRINT TAB(12); "INPUT: /YES/NO/ ";: INPUT ; G$

IF G$ = "YES" OR G$ = "yes" OR G$ = "Y" OR G$ = "y" THEN
  CALL inputfile(T$, SUPRT%, GRID(), X(), Y(), Z(), Ax(), Ay(), Az())
ELSE
  CALL supportgeometry(T$, SUPRT%, GRID(), X(), Y(), Z(), Ax(), Ay(), Az())
END IF

CLS
PRINT: PRINT: PRINT:
PRINT TAB(12); "DO YOU WANT A HARD COPY? /YES/NO/ ";: INPUT ; H$

IF H$ = "YES" OR H$ = "yes" OR H$ = "Y" OR H$ = "y" THEN

  HARD% = 1
  PRINT: PRINT:
  PRINT TAB(12); "WANT PRINTING ON TWO PAGES? /YES/NO/ ";: INPUT ; P$
  IF P$ = "YES" OR P$ = "yes" OR P$ = "Y" OR P$ = "y" THEN PAGE2% = 1
  PRINT:
  PRINT TAB(12); "Do you want to skip page 1 ";: INPUT ; SKIP$
  IF SKIP$ = "YES" OR SKIP$ = "yes" OR SKIP$ = "Y" OR SKIP$ = "y" THEN
    SKIPIT% = 1
  ELSE END IF
END IF

PRINT: PRINT:
PRINT TAB(12); "Do you want an OUTPUT file written to a folder?  /YES/NO/ ";: INPUT ; HH$
HARDTO% = 1
```

```
IF HH$ = "YES" OR HH$ = "yes" OR HH$ = "Y" OR HH$ = "y" THEN
   PRINT TAB(12); "What mame shall your SYSOUT file be called?";: INPUT ; INERTIAOUT$
   OPEN "OUTPUT", #3, INERTIAOUT$
ELSE END IF
'
'      ..................  INPUT LOADING  .....................
CLS
PRINT: PRINT: PRINT:
PRINT TAB(12); "INPUT: Loading C.G.'s: Xcg, Ycg, Zcg = ";
INPUT ; Xcg, Ycg, Zcg
PRINT: PRINT:
PRINT TAB(12); "INPUT: Applied Loads: Fx, Fy, Fz = ";
INPUT ; Fx, Fy, Fz
PRINT: PRINT:
PRINT TAB(12); "INPUT: Applied moments: Mx, My, Mz = ";
INPUT ; Mx, My, Mz
PRINT: PRINT

'       ..................  LOADING  SUPPORTS  POINTS  ..................
'
CLS
PRINT: PRINT: PRINT:
PRINT TAB(15); " ECHO SUPPORT COORDINATES ....."
PRINT TAB(12); T$
PRINT TAB(12); "           Xi      Yi     Zi      Axi   Ayi   Azi"
PRINT TAB(12); "           in.     in.    in.    ------ D.O.F.----"
FOR i% = 1 TO SUPRT%
   PRINT:
   PRINT TAB(12); "GRID "; USING "###"; GRID(i%);
   PRINT USING "####.#####"; X(i%); Y(i%); Z(i%);
   PRINT USING "##.#####"; Ax(i%); Ay(i%); Az(i%)
NEXT i%

101 PAUS$ = INKEY$: IF PAUS$ = "" THEN GOTO 101
CLS
```

```
IF HARD% <> 1 THEN GOTO 760
IF SKIPIT% = 1 THEN GOTO 760
CALL printtitlehead(T$, F$)
'       LPRINT CHR$(27); CHR$(15);      ' small print
LPRINT TAB(15); " SUPPORT COORDINATES ....."
LPRINT TAB(12); "           Xi       Yi      Zi        Axi   Ayi   Azi"
LPRINT TAB(12); "           in.      in.     in.       ------ D.O.F.----"
FOR i% = 1 TO SUPRT%
   LPRINT TAB(12); "GRID "; USING "###"; i%;
   LPRINT USING "####.#####"; X(i%); Y(i%); Z(i%);
   LPRINT USING "#.#####"; Ax(i%); Ay(i%); Az(i%)
NEXT i%


760 IF HARDTO% <> 1 THEN GOTO 761
PRINT #3, TAB(15); " SUPPORT COORDINATES ....."
PRINT #3, TAB(12); "           Xi       Yi      Zi        Axi   Ayi   Azi"
PRINT #3, TAB(12); "           in.      in.     in.       ------ D.O.F.----"
FOR i% = 1 TO SUPRT%
   PRINT #3, TAB(12); "GRID "; USING "###"; i%;
   PRINT #3, USING "####.#####"; X(i%); Y(i%); Z(i%);
   PRINT #3, USING "#.#####"; Ax(i%); Ay(i%); Az(i%)
NEXT i%

761 SAx# = 0!: SAy# = 0!: SAz# = 0!: TAx# = 0!: TAy# = 0!: TAz# = 0!

FOR i% = 1 TO SUPRT%
   TAx# = Ax(i%)
   SAx# = SAx# + TAx#
   TAy# = Ay(i%)
   SAy# = SAy# + TAy#
   TAz# = Az(i%)
   SAz# = SAz# + TAz#
NEXT i%
```

```
FOR i% = 1 TO SUPRT%

  Axy# = Ax(i%) * Y(i%)
  SAxy# = SAxy# + Axy#

  Axz# = Ax(i%) * Z(i%)
  SAxz# = SAxz# + Axz#

  Axyy# = Ax(i%) * Y(i%) * Y(i%)
  SAxyy# = SAxyy# + Axyy#
  Axzz# = Ax(i%) * Z(i%) * Z(i%)
  SAxzz# = SAxzz# + Axzz#
  Axyz# = Ax(i%) * Y(i%) * Z(i%)
  SAxyz# = SAxyz# + Axyz#



  Ayx# = Ay(i%) * X(i%)
  Ayz# = Ay(i%) * Z(i%)
  Ayxx# = Ay(i%) * X(i%) * X(i%)
  Ayzz# = Ay(i%) * Z(i%) * Z(i%)
  Ayxz# = Ay(i%) * X(i%) * Z(i%)
  SAyx# = SAyx# + Ayx#
  SAyz# = SAyz# + Ayz#
  SAyxx# = SAyxx# + Ayxx#
  SAyzz# = SAyzz# + Ayzz#
  SAyxz# = SAyxz# + Ayxz#

  Azx# = Az(i%) * X(i%)
  Azy# = Az(i%) * Y(i%)
  Azxx# = Az(i%) * X(i%) * X(i%)
  Azyy# = Az(i%) * Y(i%) * Y(i%)
  Azxy# = Az(i%) * X(i%) * Y(i%)
  SAzx# = SAzx# + Azx#
  SAzy# = SAzy# + Azy#
  SAzxx# = SAzxx# + Azxx#
  SAzyy# = SAzyy# + Azyy#
```

```
   SAzxy# = SAzxy# + Azxy#

NEXT i%

'    ......................... CALCULATE C.G. .........................

YByx# = SAxy# / SAx#: ZBxz# = SAxz# / SAx#: XBxy# = SAyx# / SAy#
ZByz# = SAyz# / SAy#: XBxz# = SAzx# / SAz#: YByz# = SAzy# / SAz#

PRINT
PRINT TAB(12); "    XBARxy     XBARxz      YBARyx "
PRINT TAB(12); "      in.        in.         in. "
PRINT TAB(12); USING "#####.#####"; XBxy#; XBxz#; YByx#: PRINT
PRINT
PRINT TAB(12); "    YBARyz     ZBARxz      ZBARyz "
PRINT TAB(12); "      in.        in.         in. "
PRINT TAB(12); USING "#####.#####"; YByz#; ZBxz#; ZByz#
PRINT: PRINT

IF HARD% <> 1 THEN GOTO 1540
IF SKIPIT% = 1 THEN GOTO 1540
LPRINT: LPRINT:
LPRINT TAB(12); "    XBARxy     XBARxz      YBARyx "
LPRINT TAB(12); "      in.        in.         in. "
LPRINT TAB(12); USING "#####.#####"; XBxy#; XBxz#; YByx#
LPRINT: LPRINT:
LPRINT TAB(12); "    YBARyz     ZBARxz      ZBARyz "
LPRINT TAB(12); "      in.        in.         in. "
LPRINT TAB(12); USING "#####.#####"; YByz#; ZBxz#; ZByz#
LPRINT: LPRINT:

1540 IF HARDTO% <> 1 THEN GOTO 102
PRINT #3,:
PRINT #3, TAB(12); "    XBARxy     XBARxz      YBARyx "
PRINT #3, TAB(12); "      in.        in.         in. "
PRINT #3, TAB(12); USING "#####.#####"; XBxy#; XBxz#; YByx#
```

```
PRINT #3,: PRINT #3,
PRINT #3, TAB(12); "    YBARyz     ZBARxz      ZBARyz "
PRINT #3, TAB(12); "      in.        in.         in. "
PRINT #3, TAB(12); USING "#####.#####"; YByz#; ZBxz#; ZByz#
PRINT #3,: PRINT #3,

102 PAUS$ = INKEY$: IF PAUS$ = "" THEN GOTO 102
CLS

IIxx# = SAyzz# + SAzyy# - YByz# ^ 2 * SAz# - ZByz# ^ 2 * SAy#
IIyy# = SAxzz# + SAzxx# - XBxz# ^ 2 * SAz# - ZBxz# ^ 2 * SAx#
IIzz# = SAxyy# + SAyxx# - XBxy# ^ 2 * SAy# - YByx# ^ 2 * SAx#

IIxy# = -(SAzxy# - YByz# * XBxz# * SAz#)
IIyz# = -(SAxyz# - YByx# * ZBxz# * SAx#)
IIxz# = -(SAyxz# - XBxy# * ZByz# * SAy#)

PRINT:
PRINT TAB(12); "  THE INERTIA TENSOR OF THE SUPPORT SYSTEM"
PRINT TAB(12); "      Ixx        Iyy          Izz   "
PRINT TAB(12); "      in^2       in^2         in^2"
PRINT TAB(12); USING "#####.#####"; IIxx#; SPC(2); IIyy#; SPC(2); IIzz#
PRINT
PRINT TAB(12); "      Ixy        Ixz          Iyz   "
PRINT TAB(12); "      in^2       in^2         in^2"
PRINT TAB(12); USING "#####.#####"; IIxy#; SPC(2); IIxz#; SPC(2); IIyz#
PRINT: PRINT:

IF HARD% <> 1 THEN GOTO 1790
IF SKIPIT% = 1 THEN GOTO 1790
LPRINT:
LPRINT TAB(12); "  THE INERTIA TENSOR OF THE SUPPORT SYSTEM"
LPRINT TAB(12); "      Ixx          Iyy         Izz   "
LPRINT TAB(12); "      in^2        in^2        in^2"
LPRINT TAB(12); USING "#####.#####"; IIxx#; SPC(2); IIyy#; SPC(2); IIzz#
LPRINT
```

```
LPRINT TAB(12); "      Ixy         Ixz         Iyz   "
LPRINT TAB(12); "      in^2        in^2        in^2"
LPRINT TAB(12); USING "#####.#####"; IIxy#; SPC(2); IIxz#; SPC(2); IIyz#
LPRINT: LPRINT:

1790 IF HARDTO% <> 1 THEN GOTO 106
PRINT #3,:
PRINT #3, TAB(12); "  THE INERTIA TENSOR OF THE SUPPORT SYSTEM"
PRINT #3, TAB(12); "      Ixx        Iyy          Izz   "
PRINT #3, TAB(12); "      in^2       in^2         in^2"
PRINT #3, TAB(12); USING "###.########"; IIxx#; SPC(2); IIyy#; SPC(2); IIzz#
PRINT #3,:
PRINT #3, TAB(12); "      Ixy        Ixz          Iyz   "
PRINT #3, TAB(12); "      in^2       in^2         in^2"
PRINT #3, TAB(12); USING "###.########"; IIxy#; SPC(2); IIxz#; SPC(2); IIyz#
PRINT #3,: PRINT #3,:

106 PAUS$ = INKEY$: IF PAUS$ = "" THEN GOTO 106

'       ..........................  INVERSE  .............................

I3# = IIxx# * IIyy# * IIzz# - IIxx# * IIyz# ^ 2
I3# = I3# - IIyy# * IIxz# ^ 2 - IIzz# * IIxy# ^ 2 + 2! * IIxy# * IIyz# * IIxz#
IF I3# = 0! THEN END

Ixx# = 0!
Ixx# = (IIyy# * IIzz# - IIyz# ^ 2) / I3#
Iyy# = 0!
Iyy# = (IIxx# * IIzz# - IIxz# ^ 2) / I3#
Izz# = 0!
Izz# = (IIxx# * IIyy# - IIxy# ^ 2) / I3#
Ixy# = (IIxz# * IIyz# - IIxy# * IIzz#) / I3#
Ixz# = (IIxy# * IIyz# - IIxz# * IIyy#) / I3#
Iyz# = (IIxy# * IIxz# - IIyz# * IIxx#) / I3#

PRINT:
```

```
PRINT TAB(12); "  THE INVERSE TENSOR OF THE SUPPORT SYSTEM"
PRINT TAB(12); "      Ixx        Iyy         Izz   "
PRINT TAB(12); "     1/in^2     1/in^2      1/in^2"
PRINT TAB(12); USING "###.########"; Ixx#; SPC(2); Iyy#; SPC(2); Izz#
PRINT:
PRINT TAB(12); "      Ixy        Ixz         Iyz   "
PRINT TAB(12); "     1/in^2     1/in^2      1/in^2"
PRINT TAB(12); USING "###.########"; Ixy#; SPC(2); Ixz#; SPC(2); Iyz#
PRINT: PRINT:

70 PAUS$ = INKEY$: IF PAUS$ = "" THEN GOTO 70

IF HARD% <> 1 THEN GOTO 2090
IF SKIPIT% = 1 THEN GOTO 2090
LPRINT:
LPRINT TAB(12); "  THE INVERSE TENSOR OF THE SUPPORT SYSTEM"
LPRINT TAB(12); "      Ixx        Iyy         Izz   "
LPRINT TAB(12); "     1/in^2     1/in^2      1/in^2"
LPRINT TAB(12); USING "#####.#####"; Ixx#; SPC(2); Iyy#; SPC(2); Izz#
LPRINT:
LPRINT TAB(12); "      Ixy        Ixz         Iyz   "
LPRINT TAB(12); "     1/in^2     1/in^2      1/in^2"
LPRINT TAB(12); USING "#####.#####"; Ixy#; SPC(2); Ixz#; SPC(2); Iyz#
LPRINT: LPRINT:

2090 IF HARDTO% <> 1 THEN GOTO 2091
PRINT #3,:
PRINT #3, TAB(12); "  THE INVERSE TENSOR OF THE SUPPORT SYSTEM"
PRINT #3, TAB(12); "      Ixx        Iyy         Izz   "
PRINT #3, TAB(12); "     1/in^2     1/in^2      1/in^2"
PRINT #3, TAB(12); USING "#####.#####"; Ixx#; SPC(2); Iyy#; SPC(2); Izz#
PRINT #3,:
PRINT #3, TAB(12); "      Ixy        Ixz         Iyz   "
PRINT #3, TAB(12); "     1/in^2     1/in^2      1/in^2"
PRINT #3, TAB(12); USING "#####.#####"; Ixy#; SPC(2); Ixz#; SPC(2); Iyz#
PRINT #3,: PRINT #3,:
```

```
'        ....................  K FACTORS  .............................

2091 K1# = (Ixy# * (Zcg - ZBxz#) - Ixz# * (Ycg - YByx#)) * Fx
K1# = K1# + (-Ixx# * (Zcg - ZByz#) + Ixz# * (Xcg - XBxy#)) * Fy
K1# = K1# + (Ixx# * (Ycg - YByz#) - Ixy# * (Xcg - XBxz#)) * Fz

K2# = (Iyy# * (Zcg - ZBxz#) - Iyz# * (Ycg - YByx#)) * Fx
K2# = K2# + (-Ixy# * (Zcg - ZByz#) + Iyz# * (Xcg - XBxy#)) * Fy
K2# = K2# + (Ixy# * (Ycg - YByz#) - Iyy# * (Xcg - XBxz#)) * Fz

K3# = (Iyz# * (Zcg - ZBxz#) - Izz# * (Ycg - YByx#)) * Fx
K3# = K3# + (-Ixz# * (Zcg - ZByz#) + Izz# * (Xcg - XBxy#)) * Fy
K3# = K3# + (Ixz# * (Ycg - YByz#) - Iyz# * (Xcg - XBxz#)) * Fz

SPx# = 0!: SPy# = 0!: SPz# = 0!: TPx# = 0!: TPy# = 0!: TPz# = 0!
FOR i% = 1 TO SUPRT%
   Px#(i%) = 0!: Py#(i%) = 0!: Pz#(i%) = 0!
NEXT i%

105 PAUS$ = INKEY$: IF PAUS$ = "" THEN GOTO 105
CLS

FOR i% = 1 TO SUPRT%

   Px#(i%) = (Fx / SAx# - (Y(i%) - YByx#) * K3# + (Z(i%) - ZBxz#) * K2#) * Ax(i%)
   Px#(i%) = Px#(i%) + (((Z(i%) - ZBxz#) * Iyz# - (Y(i%) - YByx#) * Ixz#) * Mx + ((Z(i%) - ZBxz#) * Iyy# - (Y(i%) - YByx#) * Iyz#) * My + ((Z(i%) - ZBxz#) * Iyz# - (Y(i%) - YByx#) * Izz#) * Mz) * Ax(i%)
   TPx# = Px#(i%)
   SPx# = TPx# + SPx#

   Py#(i%) = (Fy / SAy# + (X(i%) - XBxy#) * K3# - (Z(i%) - ZByz#) * K1#) * Ay(i%)
   Py#(i%) = Py#(i%) + ((-(Z(i%) - ZByz#) * Ixx# + (X(i%) - XBxy#) * Ixz#) * Mx + (-(Z(i%) - ZByz#) * Ixy# + (X(i%) - XBxy#) * Iyz#) * My + (-(Z(i%) - ZByz#) * Ixz# + (X(i%) - XBxy#) * Izz#) * Mz) * Ay(i%)
   TPy# = Py#(i%)
   SPy# = TPy# + SPy#
```

```
    Pz#(i%) = (Fz / SAz# - (X(i%) - XBxz#) * K2# + (Y(i%) - YByz#) * K1#) * Az(i%)
    Pz#(i%) = Pz#(i%) + (((Y(i%) - YByz#) * Ixx# - (X(i%) - XBxz#) * Iyz#) * Mx + ((Y(i%) - YByz#) * Ixy#
- (X(i%) - XBxz#) * Iyy#) * My + ((Y(i%) - YByz#) * Ixz# - (X(i%) - XBxz#) * Iyz#) * Mz) * Az(i%)
    TPz# = Pz#(i%)
    SPz# = TPz# + SPz#
NEXT i%

PRINT TAB(12); "APPLIED REACTIONS..."
PRINT:
PRINT TAB(12); "GRIDS     Px          Py          Pz  "
PRINT TAB(12); "          Lb.         Lb.         Lb. "
FOR i% = 1 TO SUPRT%
    PRINT TAB(12); USING "###"; i%;
    PRINT USING "#######.#####"; Px#(i%); Py#(i%); Pz#(i%)
NEXT i%

IF HARD% <> 1 THEN GOTO 2590
IF SKIPIT% = 1 THEN GOTO 2471
IF PAGE2% <> 1 THEN GOTO 2591

LPRINT CHR$(12): LPRINT: LPRINT
2471 LPRINT: LPRINT: LPRINT:
CALL printtitlehead(T$, F$)

2591 LPRINT: LPRINT:
LPRINT TAB(12); "EXTERNAL LOADS....";
LPRINT TAB(12); "FORCES  ";
LPRINT USING "#######.##"; Fx; Fy; Fz;
LPRINT ; " Lb."
LPRINT TAB(12); "MOMENTS ";
LPRINT USING "########.##"; Mx; My; Mz;
LPRINT ; " Lb.-in."
LPRINT TAB(12); "LOCATION";
LPRINT USING "####.#####"; Xcg; Ycg; Zcg;
LPRINT ; " in."
```

```
LPRINT:
LPRINT TAB(12); "APPLIED LOADS...": LPRINT
LPRINT TAB(12); "GRIDS     Px          Py          Pz  "
LPRINT TAB(12); "          Lb.         Lb.         Lb. "

FOR i% = 1 TO SUPRT%
   LPRINT TAB(12); USING "###"; i%;
   LPRINT USING "#######.#####"; Px#(i%); Py#(i%); Pz#(i%)
NEXT i%

2590 IF HARDTO% <> 1 THEN GOTO 2092
PRINT #3, TAB(12); "APPLIED LOADS..."
PRINT #3,:
PRINT #3, TAB(12); "GRIDS     Px          Py          Pz  "
PRINT #3, TAB(12); "          Lb.         Lb.         Lb. "
FOR i% = 1 TO SUPRT%
   PRINT #3, TAB(12); USING "###"; i%;
   PRINT #3, USING "#######.#####"; Px#(i%); Py#(i%); Pz#(i%)
NEXT i%

'       ................. CALCULATE  MOMENTS .........................

2092 SMxx# = 0!: SMyy# = 0!: SMzz# = 0!
TMxx# = 0!: TMyy# = 0!: TMzz# = 0!

FOR i% = 1 TO SUPRT%
   Mxx#(i%) = -(Y(i%) - Ycg) * Pz#(i%) + (Z(i%) - Zcg) * Py#(i%)
   TMxx# = Mxx#(i%)
   SMxx# = TMxx# + SMxx#

   Myy#(i%) = -(Z(i%) - Zcg) * Px#(i%) + (X(i%) - Xcg) * Pz#(i%)
   TMyy# = Myy#(i%)
   SMyy# = TMyy# + SMyy#

   Mzz#(i%) = -(X(i%) - Xcg) * Py#(i%) + (Y(i%) - Ycg) * Px#(i%)
   TMzz# = Mzz#(i%)
```

```
   SMzz# = TMzz# + SMzz#
NEXT i%

PRINT:
PRINT TAB(12); "     SUMMATION OF FORCES AND MOMENTS ABOUT THE LOAD POINT"
PRINT TAB(12); "       FX           FY          FZ  "
PRINT TAB(12); "       Lb.          Lb.         Lb."
PRINT TAB(12); USING "######.#####"; SPx#; SPy#; SPz#
PRINT
PRINT TAB(12); "       Mxx          Myy         Mzz"
PRINT TAB(12); "      Lb.-in.      Lb.-in.     Lb.-in."
PRINT TAB(12); USING "#######.#####"; -SMxx#; SPC(2); -SMyy#; SPC(2); -SMzz#

IF HARD% = 1 THEN
   LPRINT:
   LPRINT TAB(12); "     SUMMATION OF FORCES AND MOMENTS ABOUT THE LOAD POINT"
   LPRINT
   LPRINT TAB(12); "       Fx           Fy          Fz   "
   LPRINT TAB(12); "       Lb.          Lb.         Lb."
   LPRINT TAB(12); USING "######.#####"; SPx#; SPC(2); SPy#; SPC(2); SPz#
   LPRINT
   LPRINT TAB(12); "       Mxx          Myy           Mzz"
   LPRINT TAB(12); "      Lb.-in.      Lb.-in.       Lb.-in."
   LPRINT TAB(12); USING "#######.#####"; -SMxx#; SPC(2); -SMyy#; SPC(2); -SMzz#
END IF

IF HARDTO% <> 1 THEN GOTO 103
PRINT #3,:
PRINT #3, TAB(12); "     SUMMATION OF FORCES AND MOMENTS ABOUT THE LOAD POINT"
PRINT #3, TAB(12); "       FX           FY          FZ  "
PRINT #3, TAB(12); "       Lb.          Lb.         Lb."
PRINT #3, TAB(12); USING "######.#####"; SPx#; SPy#; SPz#
PRINT #3,:
PRINT #3, TAB(12); "       Mxx          Myy         Mzz"
PRINT #3, TAB(12); "      Lb.-in.      Lb.-in.     Lb.-in."
PRINT #3, TAB(12); USING "#######.#####"; -SMxx#; SPC(2); -SMyy#; SPC(2); -SMzz#
```

```
103 PAUS$ = INKEY$: IF PAUS$ = "" THEN GOTO 103
END
'  Program's end

SUB inputfile (T$, SUPRT%, GRID(), X(), Y(), Z(), Ax(), Ay(), Az()) STATIC

FOR i% = 1 TO 40
   X(i%) = 0!: Y(i%) = 0!: Z(i%) = 0!
   Ax(i%) = 0!: Ay(i%) = 0!: Az(i%) = 0!
NEXT i%

CLS
PRINT: PRINT:
COLOR 14
PRINT TAB(12);: FILES "C:\MATHEM~1\QB64\qb64-1.1-20170120.51-win\qb64\*.DAT"
COLOR 7

PRINT
PRINT TAB(12); "INPUT name of file: ";: INPUT ; F$
OPEN "I", #1, F$
INPUT #1, T$

j% = 0
35 IF EOF(1) THEN SUPRT% = j%: GOTO 40
j% = j% + 1

INPUT #1, GRID(j%), X(j%), Y(j%), Z(j%), Ax(j%), Ay(j%), Az(j%)
GOTO 35
40 CLOSE #1

CLS
PRINT: PRINT
PRINT TAB(12); T$
FOR i% = 1 TO SUPRT%
   PRINT TAB(2); GRID(i%); SPC(2);
```

```
   PRINT USING "####.###"; X(i%); SPC(2); Y(i%); SPC(2); Z(i%); SPC(3);
   PRINT USING "##.#####"; Ax(i%); Ay(i%); Az(i%)
NEXT i%

PRINT: PRINT:
PRINT TAB(12); "THERE ARE "; SUPRT%; " GRIDS."

78 PAUS$ = INKEY$: IF PAUS$ = "" THEN GOTO 78

END SUB
'
SUB Outputfile1 (T$, SUPRT%, GRID(), X(), Y(), Z(), Ax(), Ay(), Az(), Xcg, Ycg, Zcg, Fx, Fy, Fz, Mx, My, Mz)

PRINT #3,: PRINT #3,: PRINT #3,:
PRINT #3, TAB(22); "INERTIA LOADS UNIT SOLUTION ..."
PRINT #3,: PRINT #3,:
PRINT #3, TAB(12); "All UNITS in inches and pounds."

PRINT #3,: PRINT #3,: PRINT #3,:
PRINT #3, TAB(15); " ECHO SUPPORT COORDINATES FOR ....."
PRINT #3,: PRINT #3,:
PRINT #3, TAB(22); T$
PRINT #3, TAB(12); "         Xi      Yi      Zi      Axi   Ayi   Azi"
PRINT #3, TAB(12); "         in.     in.     in.     ------ D.O.F.----"
FOR i% = 1 TO SUPRT%
   PRINT #3,:
   PRINT #3, TAB(12); "GRID "; USING "###"; GRID(i%);
   PRINT #3, USING "####.#####"; X(i%); Y(i%); Z(i%);
   PRINT #3, USING "##.#####"; Ax(i%); Ay(i%); Az(i%)
NEXT i%

PRINT #3,: PRINT #3,: PRINT #3,:
PRINT #3, TAB(12); " Loading C.G.'s: Xcg, Ycg, Zcg = ";
PRINT #3, TAB(22); USING "####.#####"; Xcg; Ycg; Zcg
PRINT #3,: PRINT #3,:
```

```
PRINT #3, TAB(12); " Applied Loads: Fx, Fy, Fz = ";
PRINT #3, TAB(22); USING "####.#####"; Fx; Fy; Fz
PRINT #3,: PRINT #3,:
PRINT #3, TAB(12); " Applied moments: Mx, My, Mz = ";
PRINT #3, TAB(22); USING "####.#####"; Mx; My; Mz
PRINT #3,: PRINT #3,:

END SUB

SUB printtitlehead (T$, F$) STATIC

LPRINT: LPRINT:
LPRINT TAB(12); "INERTIA LOADS UNIT SOLUTION..."
LPRINT TAB(12); DATE$, , TIME$
LPRINT: LPRINT:
LPRINT TAB(12); T$
LPRINT TAB(12); F$
LPRINT:

END SUB

SUB supportgeometry (T$, SUPRT%, GRID(), X(), Y(), Z(), Ax(), Ay(), Az()) STATIC

PRINT: PRINT:
PRINT TAB(12); "INPUT: Title: ";: INPUT ; T$

PRINT TAB(12); "INPUT: Number of Supports (RIVETS; ETC.) = ";
INPUT ; SUPRT%
PRINT:

FOR i% = 1 TO SUPRT%
   X(i%) = 0!: Y(i%) = 0!: Z(i%) = 0!
   Ax(i%) = 0!: Ay(i%) = 0!: Az(i%) = 0!
NEXT i%

'  ...................  SUPPORT GEOMETRY  ........................
```

```
PRINT
PRINT TAB(12); "DOES EACH SUPPORT HAVE ALL THREE DEGREES OF FREEDOM?"
PRINT TAB(12); "INPUT: 'YES' IF TRUE ";: INPUT ; D$
PRINT
IF D$ = "YES" OR D$ = "yes" OR D$ = "Y" OR D$ = "y" THEN
   PRINT TAB(12); "INPUT ONLY THE x,y,z COORDINATES:": PRINT
   FOR i% = 1 TO SUPRT%
      PRINT USING "###"; i%;
      GRID(i%) = i%
      INPUT ; X(i%), Y(i%), Z(i%)
      PRINT
      Ax(i%) = 1!: Ay(i%) = 1!: Az(i%) = 1!
   NEXT i%

ELSE
   PRINT TAB(12); "INPUT: Xi, Yi, Zi, Axi, Ayi, Azi ="
   FOR i% = 1 TO SUPRT%
      PRINT USING "###"; i%;
      GRID(i%) = i%
      INPUT ; X(i%), Y(i%), Z(i%), Ax(i%), Ay(i%), Az(i%)
      PRINT
   NEXT i%
END IF
END SUB
```

Appendix **C:** Mechanics

In physics, Mechanics[115] is the study of a material body's motions, and today that aspect of Mechanics, i.e., motion, is referred to as "*Classical*" Mechanics **–** to distinguish it from theories concerned with the structure of matter, such as "*Quantum*" Mechanics and "*Wave*" Mechanics. Only Classical Mechanics will be discussed here; and as such, only in reference to the "*n-body problem*."

Classical Mechanics involves concepts of force, mass, distance and time, and how these physical verbs interrelate. One purpose of Mechanics is to resolve forces and be able to predict future behavior caused by these forces. A force has both magnitude and direction, but physically is simply defined as a "*push*" or "*pull*," A force is represented mathematically by a vector applied to a specific point. A vector however is a mathematical construct more complex than a force with many other properties. Vector concepts when where needed are fully developed in the **Monograph**.

Physical principles and methods employed in mechanics are developed primarily via an application of **NEWTON**'s *Laws Of Motion* (*Time-Geometry Laws*), particular his 2nd *Law* (which is actually a *theorem*); along with **D'ALEMBERT**'s *Principles of Equilibrium of Forces and Moments*. **NEWTON**'s *Time-Geometry Laws* (Newtonian physics) are nowhere near complete, a fact later well recognized by later generations of physicist and mathematicians, like **D'ALEMBERT** and **EULER**.

These physical constructs for mechanical systems also incorporate free-body diagrams to aide in deducing forces. It employs sometimes the Principle of a Cut, a finite, discrete portion from a much larger mechanism broken away to form an idealized, free-body diagram (sketch). As such, this sketch is a graphical representation (a cartoon) showing all the external and internally imposed forces on the broken away system. All internal and external forces acting on this restricted domain are specified using **D'ALEMBERT**'s *Principles*, by writing equilibrium equations and then resolving these for the individual forces. Key to this approach *is knowing how to draw a free-body diagram* (a must) and to show the system is in equilibrium.

A system is crouched in terms of its space and velocity coordinates. The system's State is specified via its space, velocity and behavior; i.e., the system's

115 See "***Mechanics, Classical***," Vol. 11, pp. 762-779, 1984 Edition, **Encyclopædia Britannia**. Also "***Mechanics***," in Vol. 23, pp. 802-827, 1988 Edition of the **Macropaedia**, *Encyclopædia Britannia*. Written by experts these are really quite good discussions and should not be overlooked.

change from one state -to- another owing to its interaction with its external surroundings. A system will not change its position or velocity in space unless it is acted upon by some net external force (**NEWTON**'s 1st *Law*). The 1st *Law* can actually be derived via calculus from the 2nd Law.

Important to note is the overall organizational thinking and approach for solving mechanical problems**:** it is all focused around the above statements.

Lastly, the assumptions usually true in a multi-body problem or system on Earth **–** semi-rigid constraints, elastic interactions among the multiple bodies **–** do not hold in these n-body systems that physicist and mathematicians study. So the n-body problem represents a special class of problems in the domain of physics; i.e., in Mechanics.

Appendix **D:** *Law of Weights*

The following was taken from: **The Elements of Mechanical And Electrical Engineering**, Volume I, Chapter "***Elementary Mechanics***," pp. 318-319, *The Colliery Engineer Co*., 1898.

889. Law of Gravitation :—

The force of attraction by which one body tends to draw another body towards it, is directly proportional to its mass, and inversely proportional to the square of the distance between their centers.

890. Laws of Weight :—

Bodies weigh most at the surface of the earth. Below the surface, the weight decreases as the distance to the center decreases.

Above the surface the weight decreases as the square of the distance increases.

ILLUSTRATION.—If the earth's radius is 4,000 miles, a body that weighs 100 pounds at the surface will weigh nothing at the center, since it is attracted in every direction with equal force. At 1,000 miles from the center, it will weigh 25 pounds, since

$$4{,}000 : 1{,}000 = 100 : 25.$$

At 2,000 miles from the center, it will weigh 50 pounds, since

$$4{,}000 : 2{,}000 = 100 : 50.$$

EXAMPLES FOR PRACTICE.

1. How much would 1,000 tons of coal weigh one mile below the surface? Ans. 1,999.50 lb.

2. How much would the coal in example 1 weigh one mile above the surface? Ans. 1,999.00 lb., nearly.

3. How far above the earth's surface would it be necessary to carry a body in order that it may weigh only half as much?

Ans. 1,656.854 miles, nearly.

4. A man weighs 160 pounds at the surface; how much will he weigh 50 miles below the surface? Ans. 158 lb.

5. If a body weighs 100 pounds 400 miles above the earth's surface. how much will it weigh at the surface? Ans. 121 lb.

NOTE.—*Use 4,000 miles as the radius of the earth.*

$$w\,R = d\,W. \qquad \textbf{(11.)}$$

Formula for weight when the body is above the surface:

$$w\,d^2 = W\,R^2. \qquad \textbf{(12.)}$$

EXAMPLE.—How far below the surface of the earth will a 25-pound ball weigh 9 pounds?

SOLUTION.—Use formula **11**, $w\,R = d\,W$.

Substituting the values of R, W, and w, we have

$$9 \times 4{,}000 = d \times 25, \text{ or}$$

$$d = \frac{9 \times 4{,}000}{25} = 1{,}440 \text{ miles from the center. Ans.}$$

EXAMPLE.—If a body weighs 700 pounds at the surface of the earth, at what distance above the earth's surface will it weigh 112 pounds?

SOLUTION.—Use formula **12**, $w\,d^2 = W\,R^2$.

Substituting the values of R, W, and w, we have

$$112 \times d^2 = 700 \times 4{,}000^2, \text{ or}$$

$$d = \sqrt{\frac{700 \times 4{,}000^2}{112}} = 10{,}000 \text{ miles.}$$

Therefore, $10{,}000 - 4{,}000 = 6{,}000$ miles above the earth's surface. Ans.

EXAMPLE.—The top of Mt. Hercules was said to be 32,000 feet, say 6 miles above the level of the sea. If a body weighs 1,000 pounds at sea-level, what would it weigh if carried to the top of the mountain?

SOLUTION.— $w\,d^2 = W\,R^2$, or $w \times 4{,}006^2 = 1{,}000 \times 4{,}000^2$; whence,

$$w = \frac{4{,}000^2 \times 1{,}000}{4{,}006^2} = 997 \text{ pounds. Ans.}$$

Appendix **E: NEWTON**'s *Laws of Motion*

The following was extracted from **Dynamics of Physical Systems**, by Robert H. **CANNON**, Jr., *McGraw-Hill Book Co.*, 1967.

Appendix D

NEWTON'S LAWS OF MOTION

Newton's three laws of motion are given in his *Principia*† as follows:

Lex I (in edition of 1726). Corpus omne perseverare in statu suo quiescendi vel movendi uniformiter in directum, nisi quatenus illud a viribus impressis cogitur statum suum mutare.

Lex II. Mutationem motis proportionalem esse vi motrici impressae, et fieri secondum lineam rectam qua vis illa imprimitur.

Lex III. Actioni contrariam semper et aequalem esse reactionem: sive corporum duorum actiones in se mutuo semper esse aequales et in partes contrarias dirigi.

These are translated by Cajorian,‡ together with Newton's elaborations on them (which add to our understanding of his meaning), as follows:

Law I. Every body continues in its state of rest, or of uniform motion in a straight line, unless it is compelled to change that state by forces impressed upon it.

Projectiles continue in their motions, so far as they are not retarded by the resistance of the air, or impelled downwards by the force of gravity. A top, whose parts by their cohesion are continually drawn aside from rectilinear motions, does not cease its rotation, otherwise than as it is retarded by the air. The greater bodies of the planets and comets, meeting with less resistance in freer spaces, preserve their motions both progressive and circular for a much longer time.

Law II. The change of motion is proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed.

If any force generates a motion, a double force will generate double the motion, a triple force triple the motion, whether that force be impressed altogether and at once, or gradually and successively. And this motion (being always directed the same way with the generating force), if the body moved before, is added to or subtracted from the former motion, according as they directly conspire with or are directly contrary to each other; or obliquely joined, when they are oblique, so as to produce a new motion compounded from the determination of both.

† I. Newton, *Principia Mathematica Philosophiae Naturalis*, 1686.
‡ F. Cajorian, "Sir Isaac Newton's Mathematical Principles of Natural Philosophy and His System of the World," University of California Press, Berkeley, 1960.

Law III. To every action there is always opposed an equal reaction: or, the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.

> Whatever draws or presses another is as much drawn or pressed by that other. If you press a stone with your finger, the finger is also pressed by the stone. If a horse draws a stone tied to a rope, the horse (if I may so say) will be equally drawn back towards the stone; for the distended rope, by the same endeavor to relax or unbend itself, will draw the horse as much towards the stone as it does the stone towards the horse, and will obstruct the progress of the one as much as it advances that of the other. If a body impinge upon another, and by its force change the motion of the other, that body also (because of the equality of the mutual pressure) will undergo an equal change, in its own motion, towards the contrary part. The changes made by these actions are equal, not in the velocities but in the motions of bodies; that is to say, if the bodies are not hindered by any other impediments. For, because the motions are equally changed, the changes of the velocities made towards contrary parts are inversely proportional to the bodies. This law takes place also in attractions, as will be proved in the next Scholium.†

It is clear from the above that Newton used the word *motion* to mean precisely what we call *momentum* ($m\mathbf{v}$, for a point mass). By "inversely proportional to the bodies," he meant inversely proportional to their masses. The term *vector* makes a concise substitute for "in a right line." Finally, we find it helpful to specify explicitly a point mass or particle. The extension to rigid bodies and other mass systems is straightforward but not trivial. [For example, the laws may be shown to apply verbatim to the mass centers of rigid bodies.]

Thus, in modern parlance, the laws may be stated as follows:

Law I. (*Conservation of momentum*). The momentum of a particle is constant in the absence of external forces.

> *Corollary:* The linear and angular momentum of systems of particles (such as rigid bodies) is thus also constant in the absence of external forces.

Law II. ("$\mathbf{f} = m\mathbf{a}$"). The vector change in the momentum of a particle due to an applied force is proportional to the impulse of that force [where impulse $\triangleq \int \mathbf{f}\, dt$].

Law III. ("*Action equals reaction*"). To every action there is always opposed an equal reaction; or, if body A is exerting a force on body B, then body B is exerting an equal and (vectorially) opposite force on body A.

> *Corollary:* The collision of two particles results in equal but opposite changes in their momenta (and thus in velocity changes inversely proportional to their masses).

Actually, we commonly use a derivative of Law II. That is, differentiating $m(\mathbf{v} - \mathbf{v}_{\text{initial}}) = \int \mathbf{f}\, dt$, we obtain $(d/dt)(m\mathbf{v}) = \mathbf{f}$, or for a particle (constant m),

$$\textbf{Law II} \qquad m\dot{\mathbf{v}} = \mathbf{f} \tag{D.1}$$

or $\mathbf{f} = m\mathbf{a}$, as it is commonly phrased. Equations (2.1) and (2.2) are obtained for a rigid body by space integration of (D.1) over all its mass particles, invoking Law III to cancel internal forces properly.‡

† The presentation to this point is from a compilation by J. M. Slater.

‡ See, for example, G. W. Housner and D. E. Hudson, "Applied Mechanics–Dynamics," Chaps. 6 and 7, Second Edition, D. Van Nostrand Company, Inc., Princeton, N.J., 1959.

It is clear from the above that Newton used the word *motion* to mean precisely what we call *momentum* ($m\mathbf{v}$, for a point mass). By "inversely proportional to the bodies," he meant inversely proportional to their masses. The term *vector* makes a concise substitute for "in a right line." Finally, we find it helpful to specify explicitly a point mass or particle. The extension to rigid bodies and other mass systems is straightforward but not trivial. [For example, the laws may be shown to apply verbatim to the mass centers of rigid bodies.]

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Appendix F: **MILANKOVITCH** Cycles

***MILANKOVITCH** cycles* describe the collective effects of changes in the Earth's movements on its climate over thousands of years.[116] The term is named for Serbian geophysicist and astronomer Milutin **MILANKOVIĆ** (1879-1959). In the 1920s, he hypothesized variations in eccentricity, axial tilt, and precession of the Earth's orbit resulted in cyclical variation in the solar radiation reaching the Earth, and that this orbital forcing strongly influenced climatic patterns on Earth.

Similar astronomical hypotheses had been advanced in the 19th century by Joseph **ADHEMAR**, James **CROLL** and others; but verification was difficult because there was no reliably dated evidence, and because it was unclear which periods were important.

Now, materials on Earth unchanged for millennia are being studied to indicate the history of Earth's climate. Though they are consistent with the **MILANKOVITCH** hypothesis, there are still several observations the hypothesis does not explain.

A major contradiction to **MILANKOVITCH**'s theory, besides **BolŠshakov**'s (see **References**), is for not including the work of Dr. Clarence **CLEMINSHAW** (1902-1985)[117], who calculated the approximate position of the Solar System's true barycenter, a result achieved mainly by combining only the masses of Jupiter and the Sun (See above).

Earth's movements

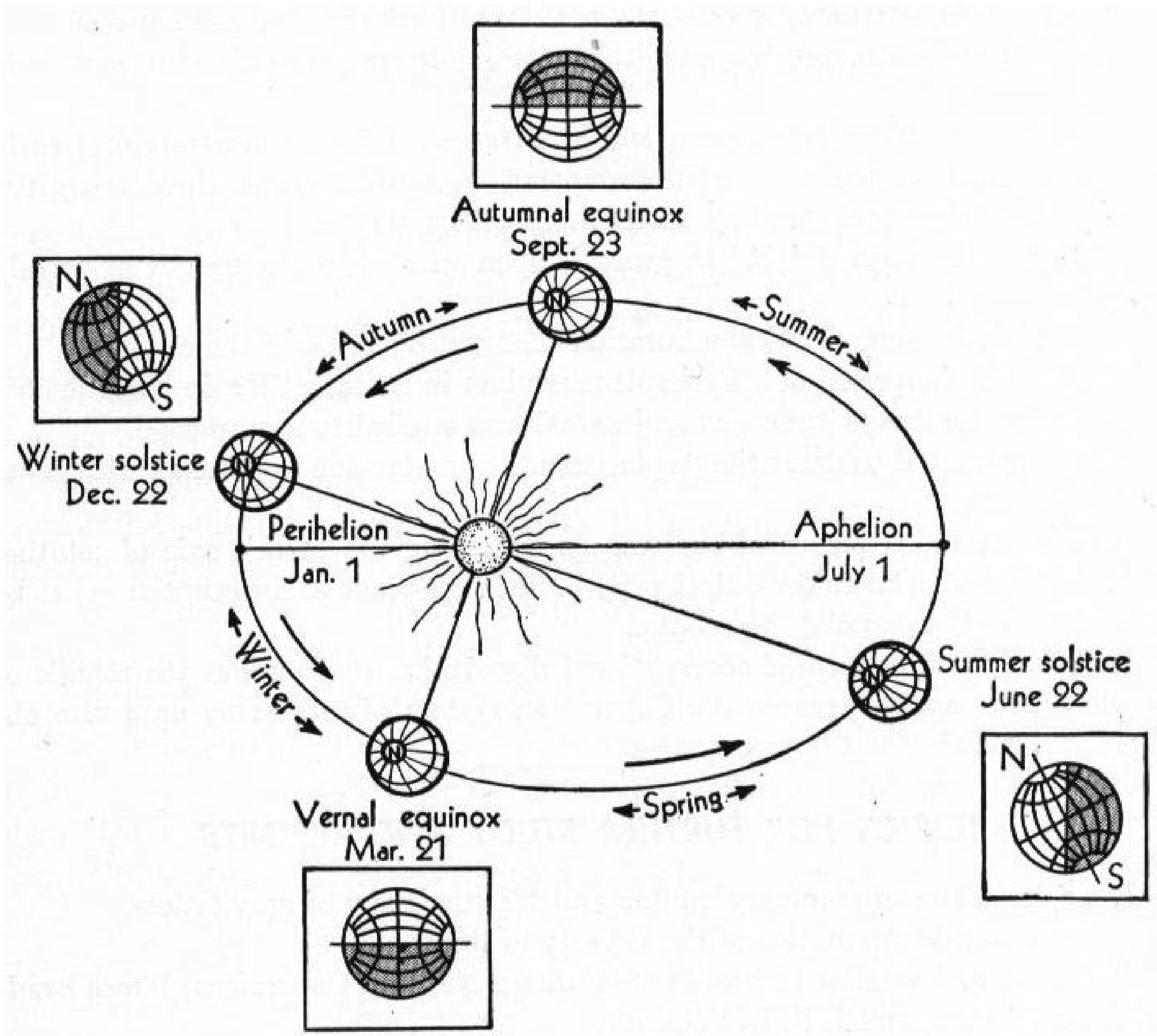
The Earth's rotation around its axis, and revolution around the Sun, evolve over time owning to gravitational interactions with other bodies in the solar system. The variations are complex, but a few cycles are dominant.[**GIRKIN**, Amy Negich: "***A Computational Study on the Evolution of the Dynamics of the Obliquity of the Earth***," (**PDF**) *Master of Science thesis*, Miami University, 2005.]

The Earth's orbit varies between nearly circular and mildly elliptical (its eccentricity varies). When the orbit is more elongated, there is more variation in the distance between the Earth and the Sun, and in the amount of solar radiation at

116 Some narrative for this Section comes from: https://en.wikipedia.org/wiki/Milankovitch_cycles#cite_note-3

117 ***Science Program*** magazine's "***The Nature of the Universe***" states Clarence **CLEMINSHAW** (1902-1985) served as Assistant Director of *Griffith Observatory* from 1938-1958 and as Director from 1958-1969. Some publications by **CLEMINSHAW**, C. H.: "**Celestial Speeds**," 4, 1953, equation, **KEPLER**, orbit, comet, Saturn, Mars, velocity; "**The Coming Conjunction of Jupiter and Saturn**," 7, 1960, Saturn, Jupiter, observe, conjunction; "**The Scale of The Solar System**," 7, 1959, Solar system, scale, Jupiter, sun, size, light.

different times in the year. In addition, the rotational tilt of the Earth (its obliquity) changes slightly. A greater tilt makes the seasons more extreme. Finally, the direction in the fixed stars pointed to by the Earth's axis changes (axial precession), while the Earth's elliptical orbit around the Sun rotates (apsidal precession). The combined effect is that proximity to the Sun occurs during different astronomical seasons.



MILANKOVIĆ studied changes in these movements of the Earth, which alter the amount and location of solar radiation reaching the Earth. This is known as solar forcing (an example of radioactive forcing). **MILANKOVIĆ** emphasized the changes experienced at 65° north owning to the great amount of land at that latitude. Land masses change temperature more quickly than oceans, because of the mixing of surface and deep water and the fact that soil has a lower volumetric heat capacity than water.

Orbital shape (eccentricity)

The Earth's orbit approximates an ellipse. Eccentricity measures the departure of this ellipse from circularity. The shape of the Earth's orbit varies between nearly circular (with the lowest eccentricity of 0.000055) and mildly elliptical (highest eccentricity of 0.0679). Its geometric or logarithmic mean is 0.0019. The major component of these variations occurs with a period of 413,000 years (eccentricity variation of ±0.012). Other components have 95,000-year and 125,000-year cycles (with a beat period of 400,000 years). They loosely combine into a 100,000-year cycle (variation of −0.03 to +0.02). The present eccentricity is 0.017 and decreasing.

<u>*Eccentricity varies primarily owning to the gravitational pull of Jupiter and*</u> <u>*Saturn.*</u> However, the semi-major axis of the orbital ellipse remains unchanged; according to perturbation theory, which computes the evolution of the orbit, the semi-major axis is invariant. The orbital period (the length of a sidereal year) is also invariant, because according to Johannes **KEPLER**'s (1571-1630) 3rd *Law*, it is determined by the semi-major axis.

KEPLER's theory of Earth's elliptical orbits (eccentricities) is not really correct because the Earth's basic orbit is circular...except when pulled out or its circular orbit by the gravitation attractions of Jupiter and Saturn; and to some extent by the other planetoids. All the forces are always in equilibrium.

Effect on temperature

The semi-major axis is away constant. Therefore, when Earth's orbit becomes more eccentric, the semi-minor axis shortens. This increases the magnitude of seasonal changes.

Incoming solar radiation varies by about 6.8% for Earth's current orbital eccentricity, while the distance from the Sun currently varies by only 3.4% (5.1 million km). Meaning the relative increase in solar irradiation at closest approach to the Sun (at perihelion) compared to the irradiation at the furthest distance (aphelion) is slightly larger than four times the eccentricity. Perihelion presently occurs around January 3, while aphelion is around July 4. When the orbit is at its most eccentric, the amount of solar radiation at perihelion will be about 23% more than at aphelion. However, the Earth's eccentricity is always so small, the variation in solar irradiation is a minor factor in seasonal climate variation, compared to axial tilt and even compared to the relative ease of heating the larger land masses of the Northern hemisphere.

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118 Theoretical physicist **THORNE**, whose technical work partly inspired the film "***Interstellar***," a 2014 epic ***SF***, was one of the *Executive Producers* of the film, as well as functioning as a scientific consultant**.**

119 The massive first edition, titled **PHYSICS**, **Pt.'s I** and **II**, 1966, was referred to as "*The Purple Egg*" because the original edition's book-cover was purple, thus the reference**.** The original authors called their massive textbook "*The Great Eggplant*" or "*The Great Pumpkin***.**" The 1st and 2nd Editions were really not well written (lemons)**;** and has been complete re-written with this third edition**.** Some 600 Colleges and Universities used and still use the textbook, which is a record-best seller for *John Wiley and Sons* publishing co**.**

Into Helicopter Airframe Design/Analysis," *American Helicopter Society* Pub., Reprint No. 780, May 1973**.** This is a paper, not a book**.**

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Summary**:** An investigation on the reason of the analogies between physical theories is carried out**.** The reason is found in the fact that in every physical theory there are basic physical quantities referring to geometrical and chronometrical elements**.** The analyses is pursued further using the rudiments of algebraic topology, i.e. the notions of "*chain"* and that of "*coboundary*." A rational classification scheme for the physical variables and equations of whatever physical theory is then constructed**.** Some of the schemes close the paper**.** An example of a "***TONTI*** *Diagram"* is shown below from his book**.**

"***TONTI*** *Diagram,"* created by the Italian physicist and mathematician Enzo **TONTI**, is a diagram classifying variables and equations of physical theories of classical and relativistic physics**.** The theories involved are**:** particle dynamics, analytical mechanics, mechanics of deformable solids, fluid mechanics, electromagnetism, *gravitation*, heat conduction, and irreversible thermodynamics. The classification stems from the observation that each physical variable has a well-defined association with a space and a time element, which can be grasped from the corresponding global variable and from its measuring process**.**

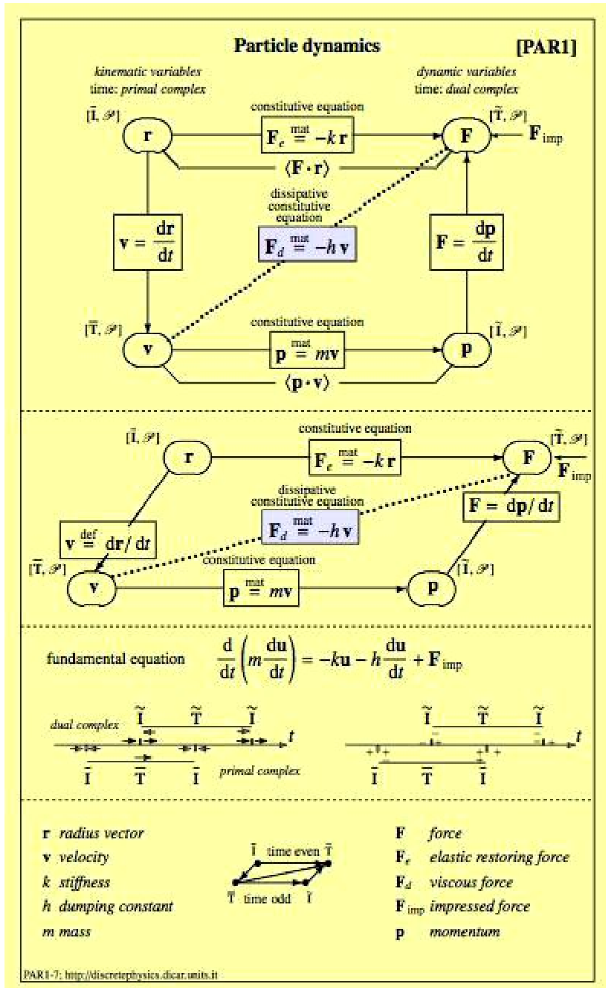

Figure**18:** ***TONTI*** *Diagram***:** "***Particle Dynamics***"

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