Quandle Colorings on (n, r)-torus knots

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Abstract

Quandles are an algebraic structure equipped with axioms that mirror the Reidemeister Moves. As such they play important role in knot theory. Dihedral quandles, for instance, are used to compute the Fox n-coloring (or Z_p coloring) of a knot. Dihedral quandles a special case of an important class of quandles known as Alexander Quandles. Furthermore, Alexander Quandles are used to compute a more generalized version of n-coloring known as Quandle Coloring. In this paper we restrict our focus to a particular subclass of Alexander Quandles known as $Z_{n,q}$ quandles and devise methods of efficiently determining whether a (r, k)-torus knot is $Z_{n,q}$ colorable.

Contents

1 Introduction

Quandles are algebraic structures that are deeply connected to knots. This connection is most strikingly presented by considering the arcs of a knot diagrams as objects acted on by the operation of undercrossing, $\triangleleft$. A set of axioms that $\triangleleft$ must satisfy in order to be a 'knot diagram invariant' (in a sense we will define shortly) can be easily derived from the Reidemeister moves, as shown in Section 1.3.

But this is not the only way that quandles can be seen to be deeply connected to knots - associated with every knot is a 'fundamental quandle'. This is obtained from the Wirtinger presentation of the fundamental knot group, as the relations involve only conjugation (Section **??**). The fundamental quandle is nearly a complete knot invariant[1]!

1.1 Notation

We set our notation for this manuscript in this section

- We refer to the set of possible knot diagrams for a given knot K as $\mathcal{D}(K)$.
- We refer to the set of arcs in a knot diagram $d \in \mathcal{D}(K)$ as $\mathcal{A}(d)$
- Let $\mathbb{C}(d)$ denote the set of crossings for a diagram d
- Let $\zeta_d : \mathbb{C}(d) \to \mathcal{A}(d) \times \mathcal{A}(d)$ denote the natural map that maps each crossing onto the pair of *undercrossing* arcs. We require that ζ_d preserve order in the undercrossing arcs in the sense that if a is an arc between crossings c_1 and c_2 then $\zeta_d(c_1) = (\alpha_1, a)$ and $\zeta_d(c_2) = (a, \alpha_2)$ for $\alpha_i \in \mathcal{A}(d)$. For instance, for a braid diagram (as in Figure 1), the first element of $\zeta_d(c)$ is always the arc on the left of the crossing, and the second element is the arc on the right of the crossing.
- θ_i are projection maps onto the ith cartesian product. For instance, $\theta_1 \circ \zeta_d$ gives the first undercrossing arc.
- We denote the remainder of a number n when divided by k as $n\%k$.
- We denote // to mean floor division.
- The variables p, q, n, k play different roles in different contexts.

1.2 Quandles generalize $\mathbb{Z}_p$ coloring

A $\mathbb{Z}_p$ *coloring* of a knot diagram $d \in \mathcal{D}(K)$ is a mapping $C : \mathcal{A}(d) \to \mathbb{Z}_p$ that does not map every arc onto a single element $p \in \mathbb{Z}_p$ and satisfies $z = 2x - y$ (in $\mathbb{Z}_p$) at each crossing where z is the label of the overcrossing and x, y are the labels of the undercrossings. For $\mathbb{Z}_3$ for instance, this relation demands that either all three colors or only one of them are present at every crossing.

Quandles (we will get to their definition in a minute) can also be used to color knots. In fact, quandle coloring includes $\mathbb{Z}_p$ coloring as a special case (Section 1.4.3). A *quandle*

[1] It doesn't care about orientation

Figure 1: Example of a braid diagram

coloring of a knot diagram $d \in \mathcal{D}(K)$ by a quandle Q is a mapping $C : \mathcal{A}(d) \to Q$ such that the labeling rules as described in Figure 2 are satisfied at all crossings.

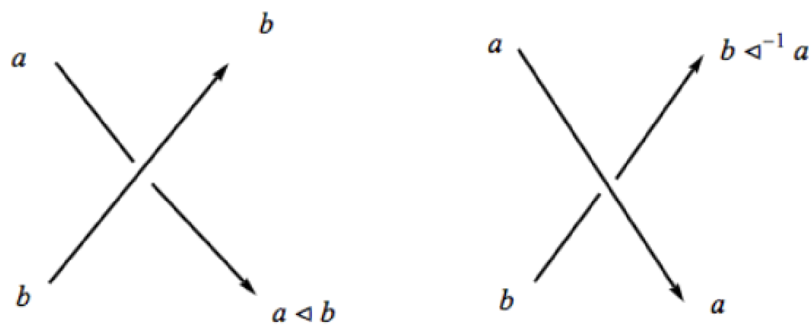

Figure 2: The convention we assume for quandle colorings. Figure reproduced from [1]

We say a knot is colorable by a quandle Q if such a map $C : \mathcal{A}(d) \to Q$ satisfying the labeling rules exists. It so happens that for braid diagrams of $T(n,r)$ knots only the type of crossing on the left in Figure 2 exists. This allows us to succintly express the colorability condition for $T(n,r)$ knots.

A braid diagram $b = (B_p)^q \in \mathcal{D}(T(p,q))$ (Figure 7) is colorable if $\forall c \in \mathbb{C}(b) \theta_2 \circ \zeta_b(c) = \theta_1 \circ \zeta_b(c) \lhd \theta_2 \circ \zeta_b(c)$.

Determining whether a given quandle Q colors a knot K is in general extremely difficult. In this work however, we restrict ourselves to a certain class of finite quandles (Section 1.4.1) as they have a very nice form. This has the side effect that checking whether a quandle in this class colors a knot boils down to a mechanical procedure that gives an answer in finite (but possibly very large) time - everything is finite! Our contribution in this work essentially just reduces the amount of computation that has to be done to check colorability through a careful treatment of the relations at hand.

One way to obtain a knot invariant from quandles is by counting the number of homomorphisms from the fundamental knot quandle into a fixed quandle Q. The Wirtinger has one generator for each arc in a knot diagram, so this reduces to counting the number of quandle colorings in the sense we have just defined! To reiterate, if Q admits n colorings for a diagram of K and $m \neq n$ colorings for K' then $K \ncong K'$.

1.3 Deriving the quandle axioms from the Reidemeister moves

In this section we 'derive' the quandle axioms from the Reidemeister moves, and in the process see how quandle colorings are naturally diagram invariants.

The first Reidemeister move (R1) allows to put in a 'twist' on any arc of a knot diagram without changing its equivalency class. But putting in a twist introduces a crossing - so to make quandles R1 invariant we demand that they satisfy $a \lhd a = a$ for all $a \in Q$. See Figure 3 for a very helpful illustration.

R2 allows us to pull non-interesting strands away from each other. Figure 4 illustrates how this gives rise to the second axiom, that $(a \lhd^{-1} b) \lhd b = b$.

I will not attempt to describe R3. Figure 5 illustrates how it gives rise to the third axiom, that $(a \lhd b) \lhd c = (a \lhd c) \lhd (b \lhd c)$.

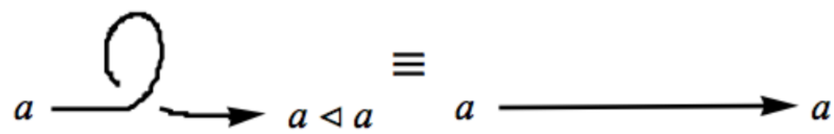

Figure 3: R1. Figure reproduced from [1].

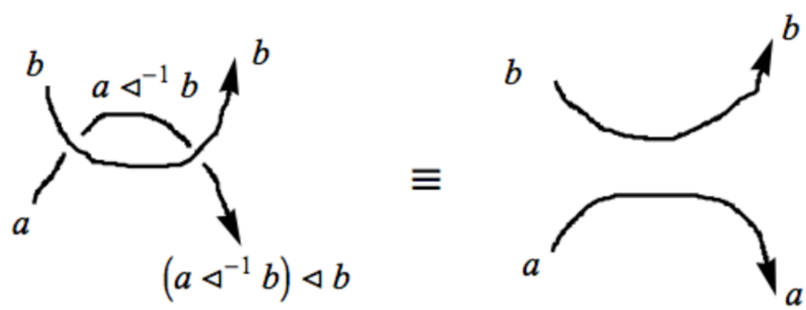

Figure 4: R2. Figure reproduced from [1].

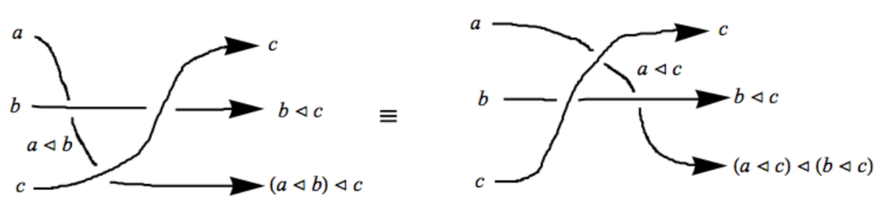

Figure 5: R3. Figure reproduced from [1]. This picture is worth at least several hundred words.

We can now give the definition of quandles.

Definition 1. An *involutory quandle* is a set K equiped with a binary operation $\lhd$ such that $\forall x, y, z \in K$,

$$x \lhd x = x \tag{1}$$

$$(x \lhd^{-1} y) \lhd y = y \tag{2}$$

$$(x \lhd y) \lhd z = (x \lhd y) \lhd (x \lhd z) \tag{3}$$

We usually refer to this structure just as a quandle. They are called 'involutory' because quandles are involutions (i.e. $\forall x \lhd (x \lhd x) = x$).

1.4 Classes of quandles

In this section we give a very brief and not exhaustive description of some classes of Quandles.

1.4.1 $\mathbb{Z}_{p,q}$ quandles

For any unit q (fixed) in a commutative ring K, we can define a quandle structure over K, denoted by K_q, by setting

$$a \lhd b := qa + (1-q)b \tag{4}$$

$$a \lhd^{-1} b := q^{-1}a + (1-q^{-1})b \tag{5}$$

In this work we concern ourselves with quandles of this type, taking $\mathbb{Z}_p$ as the base structure. We refer to this as a $\mathbb{Z}_{p,q}$ *quandle*.

In general, quandles defined this way are called *Alexander quandles*[2].

1.4.2 Conjugal quandles

We can define a quandle structure over any group G, using n-fold conjugation as the undercrossing operation.

$$a \lhd b := b^n a b^{-n} \tag{6}$$

1.4.3 Dihedral quandles

The order n dihedral group D_n gives rise to a quandle through conjugation, so dihedral quandles are a subset of conjugal quandles. However, they are notable for the following reason: $\mathbb{Z}_{n,n-1}$ gives rise to the same quandle! Furthermore, the dihedral undercrossing operation can be given as

$x \lhd y := 2x - y \mod n$

A knot is quandle colorable by the order n dihedral quandle if and only if it is $\mathbb{Z}_n$ colorable! In this sense, $\mathbb{Z}_p$ coloring $\subset$ quandle coloring.

[2]Alexander quandles are constructed from modules over $\mathbb{Z}_p[t, t^{-1}]$ with $a \lhd b := ta + (1-t)b$.

2 Quandle Coloring of the $T(n, 2)$ knot

2.1 Recurrence Relation

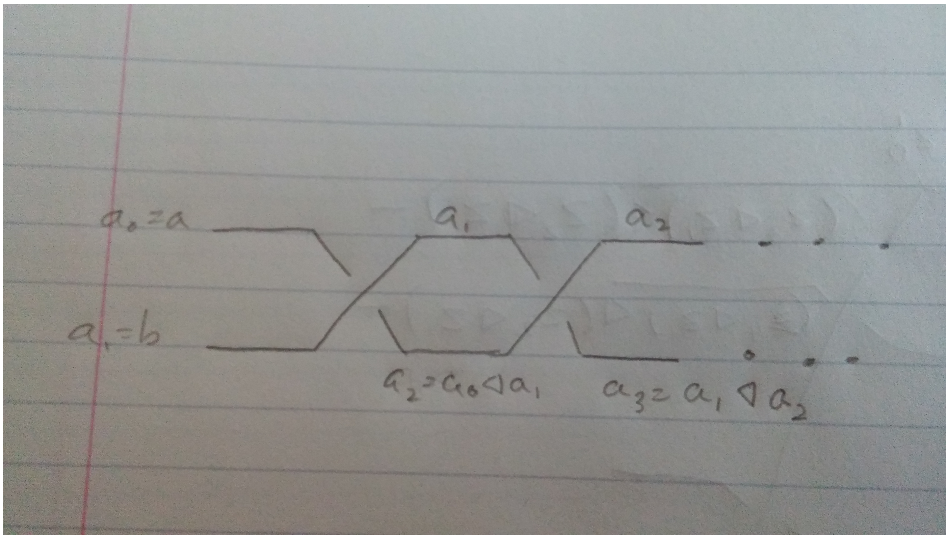

Figure 6: Visual representation of the recurrence relation $a_{n+2} = a_n \lhd a_{n+1}$ [1]

We first restrict ourselves to $T(r, n)$ where $r = 2$. Secondly, we only concern ourselves with $\mathbb{Z}_p$ when $p > 2$ is prime.

In Figure 6, we can see that the $T(n, 2)$ knot satisfies the following recurrence relation:

$$a_0 = x$$
$$a_1 = y$$
$$a_{n+2} = a_n \lhd a_{n+1}$$

We can then substitute the Alexander quandle to conclude:

$$a_{n+2} = qa_n + (1 - q)b_{n+1}$$

Solving for[3] a_n gives:

$$a_n = \frac{(-1)^n qx + q^n x + (-1)^{n+1} y + q^n y}{1 + q}$$

To determine whether a knot is $\mathbb{Z}_{p,q}$ colorable we must solve for the system of equations so that $\forall x, y \in \mathbb{Z}_n$ the following holds:

$$a_0 \equiv a_n \mod p$$
$$a_1 \equiv a_{n+1} \mod p$$

[3] We will provide a proof of this in the next subsection: Matrix Representation

We now know an explicit formula for a_n so we may substitute it in accordingly:

$$x \equiv \frac{(-1)^n qx + q^n x + (-1)^{n+1} y + q^n y}{1+q} \mod p$$
$$y \equiv \frac{(-1)^{n+1} qx + q^{n+1} x + (-1)^{n+2} y + q^{n+1} y}{1+q} \mod p$$

2.2 Matrix Representation

We consider an equivalent recurrence relation:

$$\begin{aligned} a_0 &= a_k \\ a_1 &= a_{k+1} \\ x &= a_{k-2} + (1-q)a_{k-1} \\ y &= a_{k-1} + (1-q)a_k \end{aligned}$$

To calculate a_k, we must find the matrix representation of

$$a_k = qa_{k-2} + (1-q)a_{k-1}$$

$$\begin{bmatrix} a_k \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} q & q-1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_{k-2} \\ a_{k-1} \end{bmatrix}$$
$$\begin{bmatrix} a_1 \\ a_0 \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$$
$$\begin{bmatrix} a_k \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} q-1 & q \\ 1 & 0 \end{bmatrix}^{k-1} \begin{bmatrix} a_{k-1} \\ a_{k-2} \end{bmatrix}$$

Thus:

$$\begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} a_{k+1} \\ a_k \end{bmatrix}$$
$$\begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} q & q-1 \\ 0 & 1 \end{bmatrix}^k \begin{bmatrix} y \\ x \end{bmatrix}$$
$$\begin{bmatrix} y \\ x \end{bmatrix} = A^k \begin{bmatrix} y \\ x \end{bmatrix}$$

$$A^k = \begin{bmatrix} q-1 & q \\ 1 & 0 \end{bmatrix}^k = PD^kP^{-1}$$

$$A = \begin{bmatrix} q-1 & q \\ 1 & 0 \end{bmatrix} = PDP^{-1}$$

$$P = \begin{bmatrix} -1 & q \\ 1 & 1 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} -(q+1)^{-1} & q(q+1)^{-1} \\ (q+1)^{-1} & (q+1)^{-1} \end{bmatrix}$$

$$D = \begin{bmatrix} -1 & 0 \\ 0 & q \end{bmatrix}$$

$$A^k = \begin{bmatrix} -1 & q \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & q \end{bmatrix}^k \begin{bmatrix} -(q+1)^{-1} & q(q+1)^{-1} \\ (q+1)^{-1} & (q+1)^{-1} \end{bmatrix}$$

$$A^k = \begin{bmatrix} -1 & q \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & q^k \end{bmatrix} \begin{bmatrix} -(q+1)^{-1} & q(q+1)^{-1} \\ (q+1)^{-1} & (q+1)^{-1} \end{bmatrix}$$

$$A^k = \begin{bmatrix} \frac{q^{k+1}+(-1)^{k+1}}{q+1} & \frac{q^{k+1}+(-1)^k(q)}{q+1} \\ \frac{q^k+(-1)^{k+1}}{q+1} & \frac{q^k+(-1)^k(q)}{q+1} \end{bmatrix}$$

$$\begin{bmatrix} y \\ x \end{bmatrix} = A^k \begin{bmatrix} y \\ x \end{bmatrix}$$

$$\begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} y\frac{q^{k+1}+(-1)^{k+1}}{q+1} + x\frac{q^{k+1}+(-1)^k(q)}{q+1} \\ y\frac{q^k+(-1)^{k+1}}{q+1} + x\frac{q^k+(-1)^k(q)}{q+1} \end{bmatrix}$$

Which confirms

$$x = \frac{(-1)^n qx + q^n x + (-1)^{n+1} y + q^n y}{1+q} \mod p$$

A $T(2,n)$ knot must have $\gcd(2,n) = 1$, therefore, n must be odd. We can then simplify the congruences to the following:

$$x \equiv \frac{-qx + q^n x + y + q^n y}{1+q} \mod p$$
$$y \equiv \frac{qx + q^{n+1} x - y + q^{n+1} y}{1+q} \mod p$$

From this point, we can arrive upon the $\mathbb{Z}_{p,q}$ quandle colorings of any $T(n,2)$ torus knot by finding all q satisfying the equivalence relation above $\forall x, y \in \mathbb{Z}_p$

3 Quandle Coloring of the $T(n,3)$ knot

We first restricted ourselves to $T(n,2)$ knots. Conveniently, we were able to derive an easily diagonalizable recurrence relation matrix, A. Now, we are going to try and extend this $Z_{p,q}$ quandle coloring to all $T(n,3)$ knots.

Remember, the $T(n,2)$ knot satisfies the following recurrence relation:

$$\begin{aligned} a_0 &= x \\ a_1 &= y \\ a_{n+2} &= a_n \lhd a_{n+1} \end{aligned}$$

Using the braid word of a $T(n,3)$ knot, the quandle coloring satisfies two recurrence relations that depending on the parity of n:

$$\begin{aligned} a_0 &= x \\ a_1 &= y \\ a_2 &= z \end{aligned}$$

$$\begin{aligned} a_n &= a_{n-3} \lhd a_{n-2} \\ a_{n-1} &= a_{n-4} \lhd a_{n-2} \end{aligned}$$

Here, there recurrence relation for odd and even values of n are not equivalent. Thus, the recurrence relation matrix, A, for the $T(n,3)$ Alexander Quandle has two forms. A_0 for calculating a_n when $n \equiv 0 \mod 2$. A_1 is for calculating a_n when $n \equiv 1 \mod 2$:

$$A_0 = \begin{bmatrix} q-1 & q & 0 \\ q-1 & 0 & q \\ 1 & 0 & 0 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} q-1 & 0 & q \\ q-1 & q & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

If $n \equiv 0 \mod 2$:

$$\begin{bmatrix} a_n \\ a_{n-1} \\ a_{n-2} \end{bmatrix} = \begin{bmatrix} q-1 & q & 0 \\ q-1 & 0 & q \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_{n-2} \\ a_{n-3} \\ a_{n-4} \end{bmatrix}$$

If $n \equiv 1 \mod 2$:

$$\begin{bmatrix} a_n \\ a_{n-1} \\ a_{n-2} \end{bmatrix} = \begin{bmatrix} q-1 & 0 & q \\ q-1 & q & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_{n-2} \\ a_{n-3} \\ a_{n-4} \end{bmatrix}$$

We can start calculating successive entries in the sequence a_n:

$$\begin{bmatrix} a_3 \\ a_2 \\ a_1 \end{bmatrix} = \begin{bmatrix} q-1 & 0 & q \\ q-1 & q & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} z \\ y \\ x \end{bmatrix}$$

$$\begin{bmatrix} a_4 \\ a_3 \\ a_2 \end{bmatrix} = \begin{bmatrix} q-1 & q & 0 \\ q-1 & 0 & q \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} q-1 & 0 & q \\ q-1 & q & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} z \\ y \\ x \end{bmatrix}$$

$$\begin{bmatrix} a_n \\ a_{n-1} \\ a_{n-2} \end{bmatrix} = \begin{bmatrix} q-1 & q & 0 \\ q-1 & 0 & q \\ 1 & 0 & 0 \end{bmatrix}^{n//2} \begin{bmatrix} q-1 & 0 & q \\ q-1 & q & 0 \\ 1 & 0 & 0 \end{bmatrix}^{(n//2)+n\%2)} \begin{bmatrix} z \\ y \\ x \end{bmatrix}$$

$$\begin{bmatrix} a_n \\ a_{n-1} \\ a_{n-2} \end{bmatrix} = \begin{bmatrix} q-1 & 0 & q \\ q-1 & q & 0 \\ 1 & 0 & 0 \end{bmatrix}^{n\%p} \begin{bmatrix} (q-1)^2+q(q-1) & q^2 & (q-1)q \\ (q-1)^2+q & 0 & (q-1)q \\ q-1 & 0 & q \end{bmatrix}^{n//2} \begin{bmatrix} z \\ y \\ x \end{bmatrix}$$

Replicating the calculations of the $T(n,2)$ Alexander quandle equivalence relations, you can find the following equivalence relations:

$$\begin{aligned} x &\equiv P(x,y,z,q) \mod p \\ y &\equiv P(x,y,z,q) \mod p \\ z &\equiv P(x,y,z,q) \mod p \end{aligned}$$

Given the complexity of the diagonal of A^p, we were unable to find a simplified form for

$$A^n = \begin{bmatrix} q-1 & 0 & q \\ q-1 & q & 0 \\ 1 & 0 & 0 \end{bmatrix}^{n\%2} \begin{bmatrix} (q-1)^2+q(q-1) & q^2 & (q-1)q \\ (q-1)^2+q & 0 & (q-1)q \\ q-1 & 0 & q \end{bmatrix}^{n//2}$$

We suspect there are alternative or brute force methods available to solve the $T(n,3)$ Alexander quandle equivalence relations. We have not calculated the explicit solution in terms of q for this equivalence set due to computational capacity. Additional endeavors in efficient computing or decomposition of A^n are needed to find a timely solution to the equivalence set for the $T(n,3)$ Alexander quandle representations.

4 A quandle relation for $T(p,q)$ knots

In this section we derive a recurrence relation that a quandle coloring $T(p,q)$ must satisfy. We take advantage that every $T(p,q)$ knot has the very simple braid diagram representation $(B_p)^q$ (see Figure 7). First, we set some notation, adopting that of Figure 8. For a $(B_p)^q$ braid diagram, we call the strands, starting from the far left of the first B_p $z_1^0, z_2^0, \ldots, z_p^0$ from top to bottom. The right strands of the first copy of B_p, or equivalently, the left strands of the second copy ofB_p we notate analogously, changing the superscript to 1. For instance, the right most strands in the diagram are called z_i^{q-1}.

Additionally, we adopt O_i as notation for the overcrossing strand in the $i-1$th copy of B_p. The first overcrossing is called O_0.

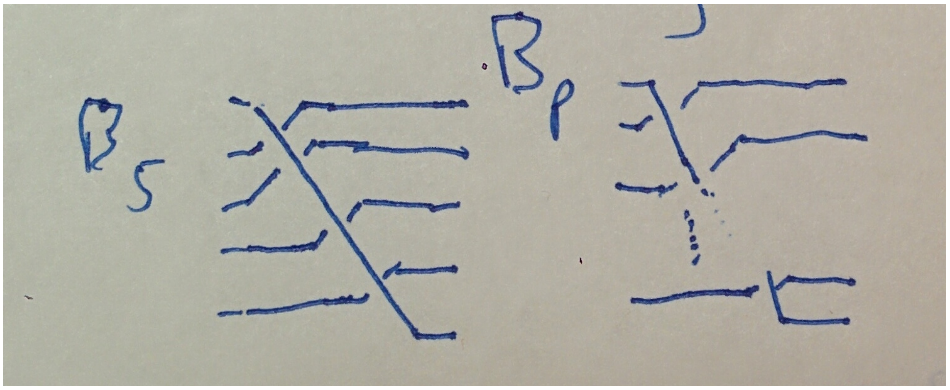

Figure 7: On the left, the braid B_5. On the right, B_p

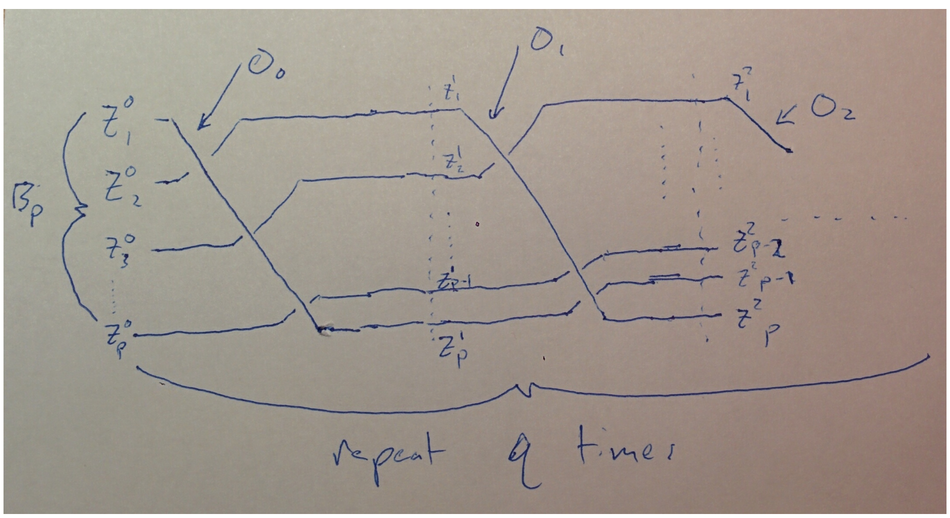

Figure 8: The notation we adopt for braid diagrams.

With this setup, we easily derive the following recurrence relation

$$z_l^{n+1} = \begin{cases} z_{l+1}^n \lhd O_n & \text{if } l < p \\ z_1^n & \text{if } l = p \end{cases} \tag{7}$$

Furthermore it is easily seen that

$$O_n = (\dots((z_n \lhd O_1) \lhd O_2)\dots) \lhd O_{n-1} \tag{8}$$

Through successive application of the third quandle axiom (Equation 3), or alternatively, by playing with the knot diagram[4] this can be seen to simplify into the possibly more transparent form

$$O_n = (z_n \lhd O_1) \lhd (O_2 \lhd O_{n-1}) \lhd \ldots \lhd (O_{n-2} \lhd O_{n-1}) \tag{9}$$

Despite our best efforts, these relations resisted further simplification. Were we more skilled with Mathematica we would have substituted the $Z_{n,q}$ undercrossing operation into these relations and attempted to fully solve (by setting $z_i^q = z_i^0$) and simplify the generalized equivalence relation given in Equation 7. As it stands, this relation gives a prescription for determining if $T(p,q)$ is $\mathbb{Z}_{n,q}$ colorable, and for counting colorings.

However, in order to exercise our general quandle relation, we used it to compute another relation (over $\mathbb{Z}_n$) for $T(2,k)$ knots. We omit the calculation, as it is fairly straightforward. The relation is, for fixed p and q a unit $\mathbb{Z}_n$

$$z_l = q^2 z_{l+2} + q(1-q)z_2 + (1-q)z_1 \text{ for } l < p-1 \tag{10}$$

$$z_{p-1} = (1-q+q^2)z_1 + (1+q-q^2)z_2 \tag{11}$$

$$z_p = qz_2 + (1-q)z_2 \tag{12}$$

References

[1] Larry Cusick, *Knot Quandles & Quandle knots*, `http://zimmer.csufresno.edu/{~}larryc/pubs/quandletallkpost.pdf`, 2012

[4]Since quandles by construction satisfy the Redeimeister moves, you can do algebra on quandle colorings visually by playing with the knot diagram! Cool!