Quandle Colorings on (r, k)-torus knots

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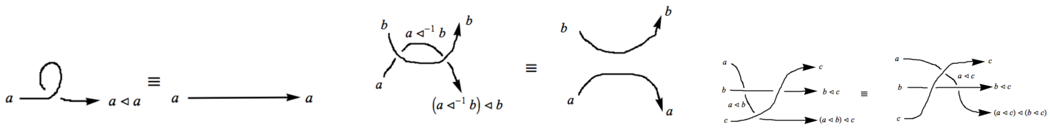
Abstract

Quandles are an algebraic structure equipped with axioms that mirror all the Reidemeister Moves. As such they play important role in knot theory. The dihedral quandle, for instance is used to compute the Fox n-coloring (or Z_p coloring) of a knot. The dihedral quandle is a special case of what is known as the Alexander Quandle. Furthermore, the Alexander Quandle is used to compute a more generalized version of For n-coloring known as Quandle Coloring (or $Z_{n,q}$ coloring). In this paper we devise methods of efficiently determining whether a (r, k)-torus knot is $Z_{n,q}$ colorable.

Contents

1 Introduction



(a) R1 (b) R2 (c) R3

Figure 1: The quandle axioms are chosen such that quandle colorings are naturally diagram invariant. Figures reproduced from [**?**]

1.1 Involutary Quandle

In this paper we concern ourselves with **Involutory Quandles**. An **involutory quandle** is any set K equiped with a binary operation $\triangleright$ that satisfies 3 axioms:

$$x \triangleright x = x \tag{1}$$

$$x \triangleright (x \triangleright y) = y \tag{2}$$

$$x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z) \tag{3}$$

These equations can be though of a symbolic representations of Reidemeister moves. (1) corresponds to the Type I Reidemeister move, (2) corresponds to the Type II Reidemeister move, and (3) corresponds to the Type III Reidemeister move. (See Fig)

1.2 Alexander Quandle

The **Alexander Quandle** is defined as:

$$x \triangleright y = ty + (1-t)a$$

t is usually a free variable in $\mathbb{C}$ however for this paper we consider a special case of the Alexander Quandle:

$$x \triangleright y = qy + (1-q)a$$

where q is a free variable in $\mathbb{Z}_n$. The reasons for this will be made clear when we consider the relationship between the Alexander Quandle and coloring.

1.3 Dihedral Quandle

The **Dihedral Quandle** is defined as:

$$x \triangleright y = 2x - y$$

it is a special case of the Alexander quandle when t is evaluated at -1.

1.4 Coloring

We say that a knot K is Z_p colorable if given an prime $p > 2$ every strand in the projection of K can be labeled using numbers 0 to $p-1$, with at least 2 of the labels distinct so that at each crossing we have:

$$2x - y - z = 0 \mod p$$

(TODO: Cite FinalPaper.pdf)

1.5 Quandle Coloring

(TODO: Complete the description) We say that a knot is $Z_{p,q}$ colorable if for some unit $q \in \mathbb{Z}_p$ there is a labeling of strands so that

2 Quandle Coloring of the (2,q)-torus knot

We first restrict ourselves to (r, n)-torus where $r = 2$. Furthermore we only concern Z_p when $p > 2$ is prime to ensure that every element of Z_p is a unit and every so every element has an inverse.

The $(2, n)$-torus knot satisfies the following recurrence relation:

$$a_0 = x$$
$$a_1 = y$$
$$a_{n+2} = a_n \cdot a_{n+1}$$

We can then substitute the Alexander quandle to conclude:

$$a_{n+2} = qa_n + (1-q)b_{n+1}$$

Solving for a_n we get:

$$a_n = \frac{(-1)^n qx + t^n x + (-1)^{n+1} y + q^n y}{1+q}$$

Because we restrict ourselves to p being prime division by $1 + q$ is allowed.

To determine whether a knot is $\mathbb{Z}_{p,q}$ colorable we must solve for the system of equations so that $\forall x, y \in \mathbb{Z}_n$ the following holds:

$$\begin{aligned} a_0 &\equiv a_n && \mod p \\ a_1 &\equiv a_{n+1} && \mod p \end{aligned}$$

We now know an explicit formula for a_n so we may substitute it in accordingly:

$$x \equiv \frac{(-1)^n qx + q^n x + (-1)^{n+1} y + q^n y}{1+q} \qquad \mod p$$
$$y \equiv \frac{(-1)^{n+1} qx + q^{n+1} x + (-1)^{n+2} y + q^{n+1} y}{1+q} \qquad \mod p$$

Because we are restricting ourselves to the quandle colorings of knots n must be odd. We can then simplify the congruences to the following:

$$x \equiv \frac{-qx + q^n x + y + q^n y}{1+q} \qquad \mod p$$
$$y \equiv \frac{qx + q^{n+1} x - y + q^{n+1} y}{1+q} \qquad \mod p$$

3 Further Questions

1. Consider what happens to $Z_{p,q}$ when p is not prime.

References

References