Dynamical systems theory $\bigcap$ Knot theory

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Contents

1 Introduction and background

This report is heavily based on Robert Ghrist's and Robert Holmes' *An ODE whose solutions contain all knots and links* [1]. In that manuscript, Ghrist gives an excellent high-level exposition of a proof that an ODE containing all knots and links as periodic orbits exists, building heavily off of his existing work in this area. His manuscript is chock full of interesting side notes; the connection between knot theory and dynamical systems is so rich that it's difficult to not get distracted.

Rather than pointlessly recapitulate what Ghrist has already stated so elegantly, we seek to give a more compact and (mostly) self-contained exposition of the beautiful proof that the required ODE exists. The reader is encouraged to refer to Ghrist's review papers [1][2] to see many many interesting footnotes and connections that will not make it into this work.

We will inevitably be unable to resist the temptation to discuss at least a few interesting connections and questions. The last section of this manuscript is reserved for discussion of those distractions that most fascinate the authors during the writing.

As briefly stated above, our ultimate goal is to prove the existence of a universal ODE, an ODE whose periodic solutions contain all knots and links.

1.1 Templates

The story begins with *templates*, a beautiful and very powerful construction. It would be very difficult to prove the existence of a universal ODE working directly with ODEs, so following one of the main mantras of mathematics, we collapse out things that are the 'same' and obtain a new object. In this case, we collapse out along the stable direction of a given ODE, identifying all orbits which share the same asymptotic future. This transforms a flow on a three-dimensional manifold into a flow to a semiflow (a one directional flow) on a branched two-dimensional manifold.

We do not discuss the intricacies of this construction, for details, see [3].

Definition 1 (Template)**.** A *template* is a compact branched two-manifold fitted with a smooth expansive semiflow and built from a finite number of *joining* and *splitting* charts, as in Figure 1 [1].

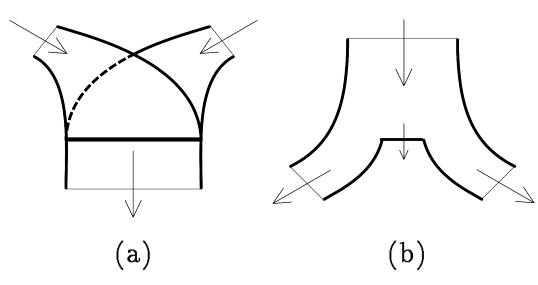

Figure 1: (a) joining and (b) splitting charts, reproduced from [1][1]

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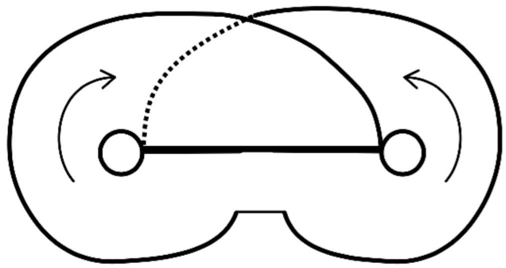

Figure 2: The Lorenz template, $\mathcal{L}$, reproduced from [1]. The dashed line indicates that the right subtemplate passes 'under' the left subtemplate. It's worth noting that orbits may 'fall off' of templates, as those orbits that pass through the gap in the lower-middle of $\mathcal{L}$ would. However, these orbits would not be periodic so they are immaterial to us.

Figure 2 shows the template for the familiar Lorenz system, the canonical chaotic system, described in equations 1-3. The simplicity of the Lorenz template is a testament to the elegance of templates - the Lorenz template exactly captures the repeated expanding and folding over of orbits that is the signature of chaos. For certain parameter values, periodic solutions to these equations form a link of infinitely many components [1], although we will see later that the Lorenz system is not universal.

$$\dot{x} = \sigma(y - x) \tag{1}$$

$$\dot{y} = \beta x - y - xz \tag{2}$$

$$\dot{z} = -\beta z + xy \tag{3}$$

The primary formal advantage of working with templates, is that they allow the use of *symbolic dynamics*. We can identify periodic orbits on a template τ with the sequence of strips crossed.

Definition 2. Following [1], given a template τ, we define

- Branch lines $\{l_j : j = 1, \cdots M\}$, the one-dimensional lines strips are connected along
- Strips $\{x_i : i = 1, \cdots N \geq 2M\}$, the two-dimensional regions connecting branch lines
- Itinerary $(x_{s_1}x_{s_2}x_{s_3}\dots)$, a sequence of strips crossed by an orbit on τ
- Itinerary space $\Sigma_\tau = \{a_0a_1a_2\cdots\} \subset \{x_1, x_2, \dots, x_N\}^{\mathbb{Z}^+}$, the set of all possible itineraries on τ
- The transition matrix A_τ

$$A_\tau(i,j) = \begin{cases} 0 \text{ if } \nexists \text{ a strip from } x_i \text{ to } x_j \\ 1 \text{ if } \exists \text{ a strip from } x_i \text{ to } x_j \end{cases} \tag{4}$$

It's interesting to note that powers of A_τ capture allowable trajectories, or more precisely the (i,j)th element of the kth power of A_τ will be nonzero iff there is an admissable itinerary of length k from x_i to x_j. In particular, $(A_\tau^k)_{i,j}$ is the number of different length k orbits linking x_i and x_j So in principle we have succeeded in reducing questions about the periodic orbits of a flow to questions about powers of a single matrix. Of course, in practice it will be very difficult to compute A from an arbitrary dynamical system.

Although crossing information is implicitly contained in A_τ, it does not directly give us much information about the knot type of periodic orbits. [andrew: come back to this] An interesting question that we briefly consider in Section 4 is what information should keep track of in addition to A_τ to (relatively) easily compute the link of knots contained in a given template τ.

See the Figure 4 for an example of a template with labeled strips.

Ghrist confirms that the set of admissible sequences on Σ_τ corresponds exactly with the set of periodic orbits on τ, as desired.

1.2 Ordering orbits on templates

It's possible, and will prove useful, to put an order on orbits on τ. This is accomplished by choosing an ordering on strips emanating from the same branch line and using the induced lexicographic ordering to order orbits.

This is best illustrated with an example borrowed from [1]. Let $\mathcal{V}$ be the template illustrated in Figure 4, outfitted with a Markov partition of strips $\{x_1, x_2, x_3, x_4\}$. Choose an ordering $\rhd$ on the branch lines:

$$l_1 : x_1 \rhd x_2 \tag{5}$$

$$l_2 : x_3 \rhd x_4 \tag{6}$$

Then a lexicographic ordering on $\Sigma_{\mathcal{V}}$ gives an ordering of orbits. For example, $x_1^2 x_3 x_4 \cdots \rhd x_1 x_2 x_3 x_4 \rhd x_1 x_2 x_4 \cdots$.

Although we have skirted over the details of this ordering here, rest assured that it is well-defined for all templates as confirmed in [4][5][1]

1.3 Template inflation

Our main tool in proving the existence of a universal template (and by extension a universal ODE) will be *template inflation.* The following definition are due to [1].

Definition 3 (Subtemplate)**.** A *subtemplate* $S \subset \tau$ of a template τ is a topological subset of τ which, with the restriction of the semiflow of τ, satisfies the definition of a template (Definition 1)

Definition 4 (Template inflation)**.** A *template inflation* of a template τ into a template ρ is a map $\boldsymbol{R} : \tau \to \rho$ taking orbits to orbits which is a diffeomorphism onto its image. The image of a template inflation is, by definition, a subtemplate.

Definition 5 (Template renormalization)**.** In the important case where an inflation $\boldsymbol{R}$ maps $\tau \to \tau$ we say that $\boldsymbol{R}$ is a *template renormalization* as $\boldsymbol{R}(\tau) \subset \tau$ is a subtemplate of τ which is diffeomorphic to τ.

See Figure 3 for a helpful illustration of how a template renormalization might be constructed.

Note that if a template τ admits an inflation, then the image of that inflation (being diffeormophic to the original template) can again be inflated, giving an increasingly complicated sequenced of embedded subtemplates each diffeormorphic to the original! This is the essential property that makes inflations such a powerful tool in proving the existence of a universal ODE, a bit later we will see exactly how its usefulness is manifest.

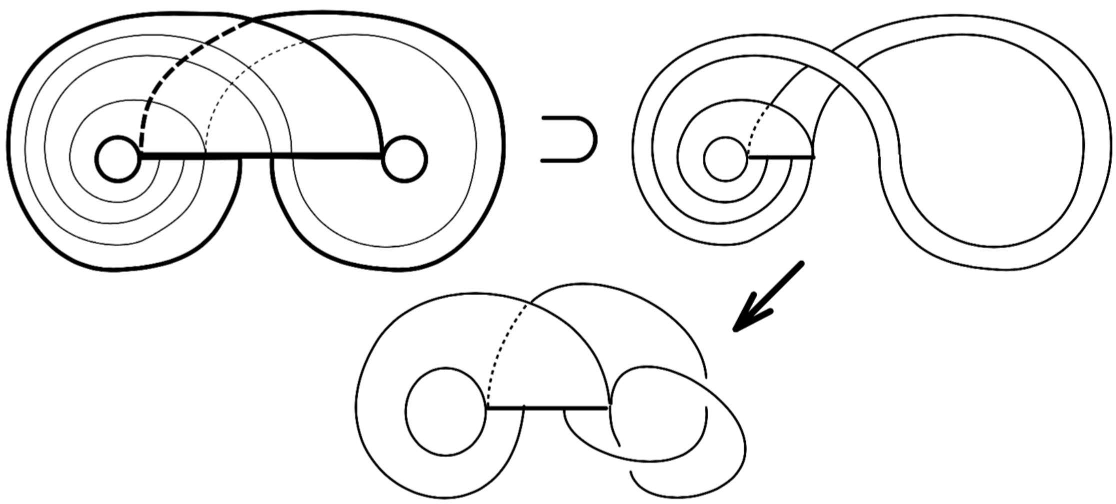

Figure 3: A subtemplate of the Lorenz subtemplate $\mathcal{L}$. When it is cut along the boundary and removed from $\mathcal{L}$, it forms a template diffeomorphic to the original one! Bold lines indicate the boundary of the $\mathcal{L}$ whereas the lighter lines indicate the boundary of the subtemplate, but can also be thought of as orbits in $\mathcal{L}$. Lines are dashed where one subtemplate passes behind another. The crossings in the template boundaries on the lower figure are intended to indicate twists in the strip. Figure reproduced from [1]

A *very* useful lemma proved in [1] follows

Lemma 1 (Ghrist & Holmes 1993)**.** A template inflation $\boldsymbol{R} : \tau \to \rho$ induces a map $\boldsymbol{R} : \Sigma_\tau \to \Sigma_\rho$ whose action is to inflate each element of the Markov partition of strips $\{x_i : i = 1 \cdots N\}$ to a finite admissible word $\{\boldsymbol{w_i} = w_1 w_2 \cdots w_{n(i)} : i = 1 \cdots N, w_i \in \{x_i\}\}$

For example the inflation $\boldsymbol{L} : \mathcal{L} \to \mathcal{L}$ shown in Figure 3 (with x_1 (resp. x_2) set as the left (right) strip) has the following action on itineraries in $\mathcal{L}$:

$$\boldsymbol{L} : \mathcal{L} \to \mathcal{L} \left\{ \begin{array}{c} x_1 \mapsto x_1 \\ x_2 \mapsto x_1 x_2 \end{array} \right. \tag{7}$$

Consider the Lorenz template $\mathcal{L}$ with the inflation $\tilde{\mathcal{L}} \subset \mathcal{L}$ as identified in Figure 3. Label the left and right strips of $\mathcal{L}$ x_1 and x_2 respectively, and the left and right strips of $\tilde{L}$ $\tilde{x_1}, \tilde{x_2}$.

Even though $\tilde{\mathcal{L}} \subset \mathcal{L}$, $\tilde{\mathcal{L}} \cong \mathcal{L}$! To understand why Lemma 1 holds, consider how the $\tilde{x_i}$ are contained in x_i. $\tilde{x_1} \subset x_1$ so one can easily see that orbits in $\mathcal{L}$ passing through x_1 will pass through $\tilde{x_1}$ under the inflation that takes $\mathcal{L}$ to $\tilde{\mathcal{L}}$. On the other hand, $\tilde{x_2} \subset x_1 \cup x_2$; orbits in $\tilde{\mathcal{L}}$ passing through $\tilde{x_2}$ will first pass through x_1 and then x_2 when pulled back along the inflation. This actually serves to prove the lemma if rewritten generally.

As a quick recap, we've established that renormalizing a template τ induces a map on the itineraries Σ_τ of τ that preserves the *dynamics* of τ. However we have *not* yet established that the renormalization (and by extension the induced map on itineraries) preserves the knot type of orbits. We now identify a special class of inflations that preserve knot type, as these are the inflations that we're really after.

Definition 6. Isotopic inflation An *isotopic inflation* of embedded templates τ and ρ is a template inflation $\boldsymbol{R} : \tau\ top$ such that $\boldsymbol{R}(\tau)$ is isotopic to ρ (in the ambient space)

2 Proof that $\mathcal{V}$ is a universal template

In this section we outline the proof given in full detail in [6] that $\mathcal{V}$ (shown in figure 4) is a universal template. Details will inevitably be omitted but our goal in this exposition is to straddle the line between the full proof and the very high-level outline given in [1], giving a concise and readable exposition that gets at the essence of the proof.

The essential idea is to find a special set of templates which have a special property that forces them to support all braids as orbits, and then to show that these templates can be found in $\mathcal{V}$. Since all knots and links can be realized as a braid, this immediately implies that $\mathcal{V}$ is universal. The last step of showing that the special braid-supporting templates can found in $\mathcal{V}$ is the most difficult by far and where we will have to do the most hand-waving. Nonetheless, we will try to capture the essence.

Also important in this proof is the existence of the $\mathcal{U}$ template, pictured below. U and $\mathcal{V}$ are essentially equivalent, but there is one minor difference, and U is universal as well.

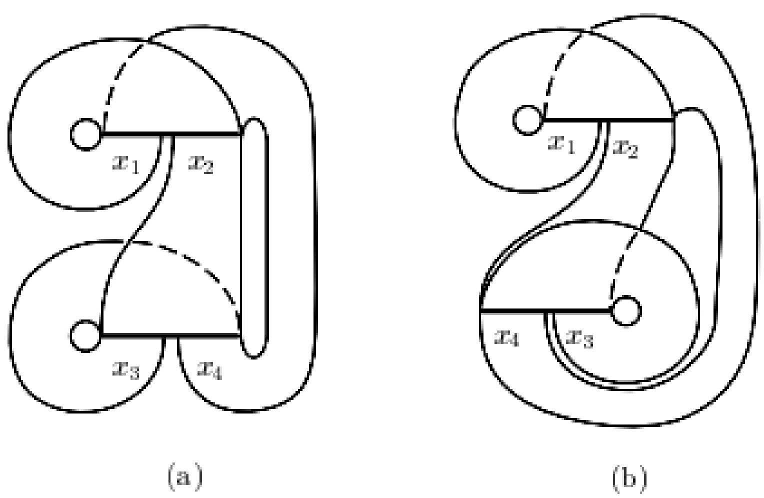

Figure 4: (a) The universal template $\mathcal{V}$ and (b) The universal template U, reproduced from [**?**]

2.1 Braids and the theorem of Alexander

Recall that a closed braid is a collection of P disjoint simple closed curves in a standardly embedded torus $D^2 \times S^1$ such that every D^2 cross-section of the torus intersects the closed braid in exactly P points.

In a landmark paper [7], Alexander proved the following theorem which provides the crucial connection between braids and links.

Theorem 1 (Alexander 1923)**.** *Each knot or link is isotopic to some closed braid on P strands for some P*

2.2 The templates $\mathcal{W}_q$

The family of templates $\{\mathcal{W}_q\}$ shown in figure 5 are those referred to earlier as the braid-supporting templates. $\mathcal{W}_1$ is identically $\mathcal{V}$ and increasing q by one has the effect of adding two *ears* to one side. The property that successive ears alternate in 'sign' is what makes this family of templates so useful, as this property makes proving that they support all braids delightfully simple.

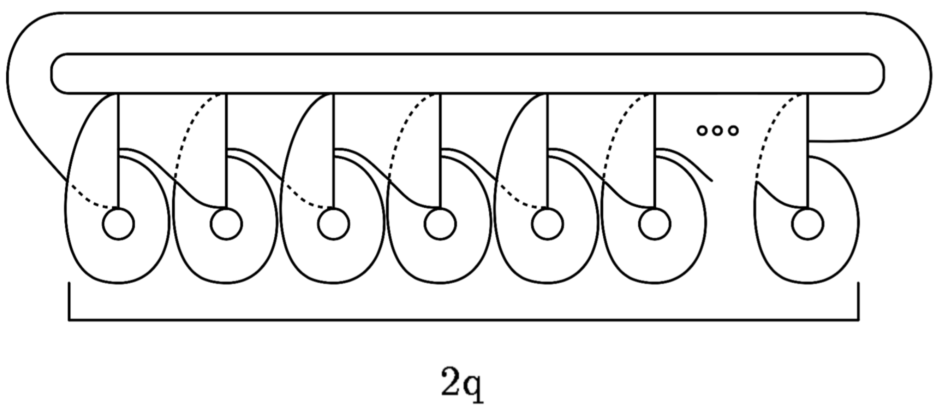

Figure 5: The templates W_q, reproduced from [1]

Lemma 2 (Ghrist 1996)**.** An isotopic copy of any closed braid exists as a set of periodic orbits on some W_q for sufficently large q.

Proof. The concatenation of alternating positive and negative ears on $\mathcal{W}_q$ mimics the braid group concatenation operation. The braid group B_{n+1} is naturally generated by $\sigma_1, \sigma_2, \ldots, \sigma_n$, where σ_i is a single crossing of the ith strand over the $(i+1)$th strand and its inverse, σ_i^{-1}, is a single crossing of the $(i+1)$th strand over the ith strand. Naturally, $\sigma_i \sigma_i^{-1} = \sigma_i^{-1} \sigma_i = I$. We can consider the elements $\pi_1, \pi_2, \ldots, \pi_n, \pi_1', \pi_2', \ldots, \pi_n'$ of B_{n+1}, where $\pi_i = \sigma_1 \sigma_2 \cdots \sigma_i$ and $\pi_i' = \sigma_1^{-1} \sigma_2^{-1} \cdots \sigma_i^{-1}$. Note that $\pi_1 = \sigma_1$, $\pi_1' = \sigma_1^{-1}$, and that for $i > 1$,

$$\begin{aligned}
\pi_{i-1}^{-1}\pi_i &= (\sigma_1\sigma_2\dots\sigma_{i-1})^{-1}\sigma_1\sigma_2\cdots\sigma_i \\
&= \sigma_{i-1}^{-1}\cdots\sigma_2^{-1}\sigma_1^{-1}\sigma_1\sigma_2\cdots\sigma_i \\
&= \sigma_{i-1}^{-1}\cdots\sigma_2^{-1}\sigma_2\cdots\sigma_i \\
&= \sigma_{i-1}^{-1}\sigma_{i-1}\sigma_i \\
&= \sigma_i
\end{aligned}$$

$$\begin{aligned}
\pi_{i-1}'^{-1}\pi_i' &= (\sigma_1^{-1}\sigma_2^{-1}\dots\sigma_{i-1}^{-1})^{-1}\sigma_1^{-1}\sigma_2^{-1}\cdots\sigma_i^{-1} \\
&= \sigma_{i-1}\cdots\sigma_2\sigma_1\sigma_1^{-1}\sigma_2^{-1}\cdots\sigma_i^{-1} \\
&= \sigma_{i-1}\cdots\sigma_2\sigma_2^{-1}\cdots\sigma_i^{-1} \\
&= \sigma_{i-1}\sigma_{i-1}^{-1}\sigma_i^{-1} \\
&= \sigma_i^{-1}
\end{aligned}$$

We can conclude that the elements $\pi_1, \pi_2, \dots, \pi_n, \pi_1', \pi_2', \dots, \pi_n'$ of B_{n+1} are a generating set for B_{n+1}.

For the braid group B_{n+1}, notice that for all $1 \leq i \leq n$, π_i can be drawn on any positive ear of $\mathcal{W}_q$, letting all but the bottom strand take the top strip and forcing the bottom strand to take the bottom strip, cross over exactly i strands, and take the top strip. Similarly, for all $i \leq i \leq n$, π_i' can be drawn on any negative ear of W_q, letting all but the bottom strand take the top strip, forcing the bottom strand to take the bottom strip, cross under exactly i strands, and take the top strip. Examples of π_2 and π_2' of the braid group B_5 are drawn in Figure 6.

Because this set of elements generates the braid group B_{n+1}, any braid $B \in B_{n+1}$ can be drawn as a sequence of π_i and π_i'. If B can be represented with m of these elements, then the closure of B lies in W_m, as W_m contains m pairs of positive/negative ears that can each support a π_i or π_i' structure. Thus, any closed braid on N strands lies in W_q for sufficiently large q.

□

2.3 Special Inflations: F and G

In his proof, Ghrist uses a few special inflations, F and G, and accompanies them with a proposition:

Proposition 1. The following inflations are isotopic.

$$F : U \hookrightarrow \mathcal{V} \begin{cases} x_1 \to x_1 \\ x_2 \to x_1x_2x_3 \\ x_3 \to x_4x_2 \\ x_4 \to x_4 \end{cases}$$

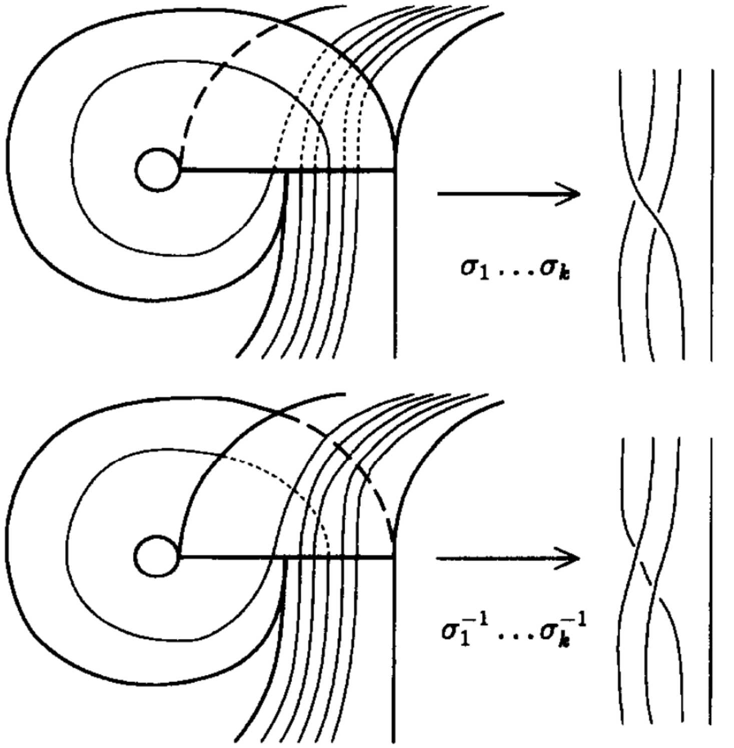

Figure 6: Top: π_2 in B_5 Bottom: π_2' in B_5. Figure reproduced from [6]

$$G : \mathcal{V} \hookrightarrow U \begin{cases} x_1 \to x_1 \\ x_2 \to x_1 \\ x_3 \to x_2 x_4 \\ x_4 \to x_2 x_3 x_4 \end{cases}$$

As proof of their isotopy, diagrams showing the inflation are given.
In addition, we can consider the following inflation:

$$\chi : \begin{matrix} U \to U \\ \mathcal{V} \to \mathcal{V} \end{matrix} \begin{cases} x_1 \to x_3 \\ x_2 \to x_4 \\ x_3 \to x_1 \\ x_4 \to x_2 \end{cases}$$

The purpose of this inflation is to switch the two branch lines.

Lemma 3 (Ghrist 1997)**.** Given an isotopic inflation A having either U or $\mathcal{V}$ as a domain and range (independently of each other), the inflation $A^* = \chi A \chi$ is also isotopic.

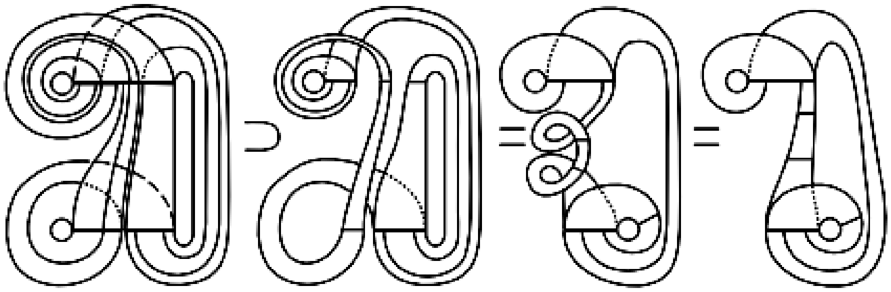

Figure 7: Left to Right: A subtemplate of $\mathcal{V}$ as given by F, then cut along the boundary and removed from $\mathcal{V}$, simplified to show that it equals U. Figure reproduced from [**?**]

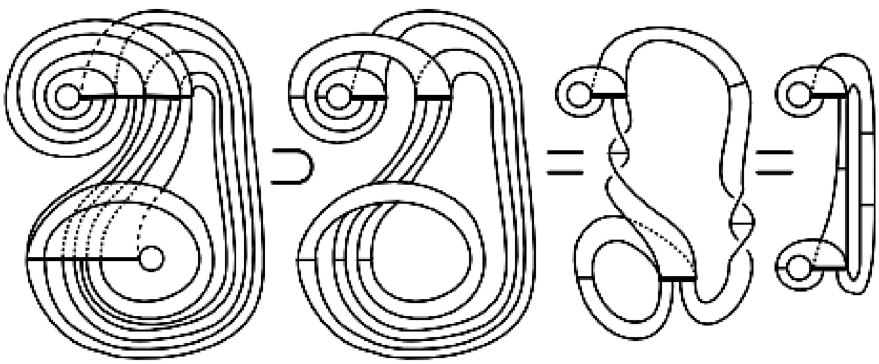

Figure 8: Left to Right: A subtemplate of U as given by G, then cut along the boundary and removed from U, simplified to show that it equals $\mathcal{V}$. Figure reproduced from [**?**]

For example, given F as defined above, F^* is defined as

$$F^* : U \hookrightarrow \mathcal{V} \begin{cases} x_1 \to x_2 x_4 \\ x_2 \to x_2 \\ x_3 \to x_3 \\ x_4 \to x_3 x_4 x_1 \end{cases}$$

Finally, in the next section, we can include the inflation $\mathfrak{H} = F^* G F G^*$ in the next section to continue our proof.

2.4 Find $\mathcal{W}_q \subset \mathcal{V}$ for all q

We must start with a lemma, provided by Ghrist.

Lemma 4 (Ghrist 1997)**.** Given that a subtemplate of $\mathcal{V}$ doesn't contain x_1^∞, it is possible to add an ear near the top branch line, like that in Figure 9, which is a positive ear. Similarly, given that a subtemplate of $\mathcal{V}$ doesn't contain X_3^∞, it is possible to add an ear near the bottom branch line, which is a negative ear.

Proof. Let us begin with $\mathcal{W}_1 = \mathcal{V}$. We perform the inflation $\mathfrak{H}$:

$$\mathfrak{H} : \mathcal{V} \hookrightarrow \mathcal{V} \begin{cases} x_1 \to x_2 x_3^2 x_4 x_1 (x_2 x_4)^2 x_2 x_3 x_4 x_1 \\ x_2 \to x_2 x_3^2 x_4 x_1 (x_2 x_4)^3 x_2 x_3 x_4 x_1 \\ x_3 \to x_2 x_3^2 x_4 x_1 x_2 x_4 \\ x_4 \to x_2 x_3^2 x_4 x_1 x_2 x_4 \end{cases}$$

This inflation satisfies the condition that this subtemplate does not contain x_1^∞. Then, after performing this inflation, we will be able to add a positive ear at the top branch line; after doing so, we now have $\mathcal{W}_1^+$.

There exists a very similar inflation $\mathfrak{H}^*$ from $\mathcal{V}$ onto $\mathcal{V}$ that will allow us to add a negative ear at the bottom branch line. Now, we have added a positive ear and a negative ear, so we are at $\mathcal{W}_2$, yet this subtemplate is still contained within $\mathcal{V}$; this implies that $\mathcal{W}_2 \subset \mathcal{V}$. Now, we can repeat this procedure, adding another positive and negative ear after the respective inflations to get $\mathcal{W}_3$.

This procedure can be repeated q times to show that $\mathcal{W}_q \subset \mathcal{V}$ for any positive q; thus, by induction, $\mathcal{W}_q \subset \mathcal{V}$ for all q. □

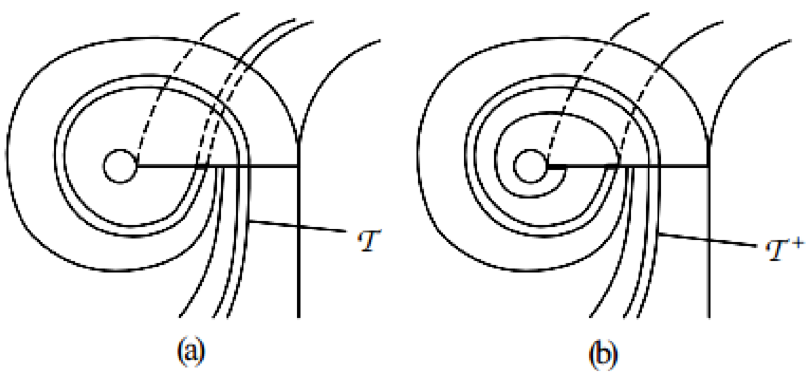

Figure 9: Appending a positive ear to T (left) yields T^+ (right). Figure reproduced from [6]

2.5 Universal Templates and V

Theorem 2 (Ghrist 1996)**.** *The template $\mathcal{V}$ contains an isotopic copy of every tame knot and link as a periodic orbit of the semiflow.*

Proof. This proof comes immediately following the fact that $\mathcal{W}_q \subset \mathcal{V}$ for all q, and that all closed braids (and thus all tame links and knots) can be contained within $\mathcal{W}_q$ for large enough q. Thus, $\mathcal{V}$ contains every tame knot and link. □

Ghrist then extends the theorem above to prove the following:

Theorem 3 (Ghrist 1996)**.** *The template $\mathcal{V}$ contains all orientable templates as subtemplates of $\mathcal{V}$.*

It must follow that if there are any other universal templates T, then $\mathcal{V}$ must contain T as a subtemplate. As T contains all tame links and knots as well, as it is universal, the same theorems can be applied to T as were applied to $\mathcal{V}$, so that T also contains all templates (including $\mathcal{V}$).

3 Implications for ODEs in general

Very brief exposition on section 5 in a knotty ode discussing what the universality of $\mathcal{V}$ implies in general. Proving universality is reduced to showing that a template can be renormalized to $\mathcal{V}$

4 Miscellaneous discussions and interesting open questions

The last theorem of Section 2.4 hints at a requirement for two templates to be considered equal: given templates A and B, A and B are equal if $A \subset B$ and $B \subset A$. If each template may be contained within the other, then all knots in A must be contained in B as well, and all knots in B must be contained in A as well.

Discussion of A_τ. How much information does it tell us? How hard is to compute from a given ODE (very hard)? What additional information should we keep track off (ie crossings) to make it relatively easy to compute the link of knots contained in a given template?

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