

# Dynamical systems theory $\cap$ Knot theory

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# 1 Introduction and background

This report is heavily based on Robert Ghrist's and Robert Holmes' *An ODE whose solutions contain all knots and links* [1]. In that manuscript, Ghrist gives an excellent high-level exposition of a proof that an ODE containing all knots and links as periodic orbits exists, building heavily off of his existing work in this area. His manuscript is chock full of interesting side notes; the connection between knot theory and dynamical systems is so rich that it's difficult to not get distracted.

Rather than pointlessly recapitulate what Ghrist has already stated so elegantly, we seek to give a more compact and (mostly) self-contained exposition of the beautiful proof that the required ODE exists. The reader is encouraged to refer to Ghrist's review papers [1][2] to see many many interesting footnotes and connections that will not make it into this work.

We will inevitably be unable to resist the temptation to discuss at least a few interesting connections and questions. The last section of this manuscript is reserved for discussion of those distractions that most fascinate the authors during the writing.

As briefly stated above, our ultimate goal is to prove the existence of a universal ODE, an ODE whose periodic solutions contain all knots and links.

## 1.1 Templates

The story begins with *templates*, a beautiful and very powerful construction. It would be very difficult to prove the existence of a universal ODE working directly with ODEs, so following one of the main mantras of mathematics, we collapse out things that are the 'same' and obtain a new object. In this case, we collapse out along the stable direction of a given ODE, identifying all orbits which share the same asymptotic future. This transforms a flow on a three-dimensional manifold into a flow to a semiflow (a one directional flow) on a branched two-dimensional manifold.

We do not discuss the intricacies of this construction, for details, see [3].

**Definition 1** (Template). A *template* is a compact branched two-manifold fitted with a smooth expansive semiflow and built from a finite number of *joining* and *splitting* charts, as in Figure 1 [1].

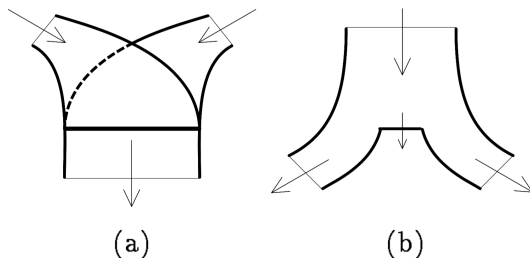


Figure 1: (a) joining and (b) splitting charts, reproduced from [1]<sup>1</sup>

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<sup>1</sup>Regrettably, without permission

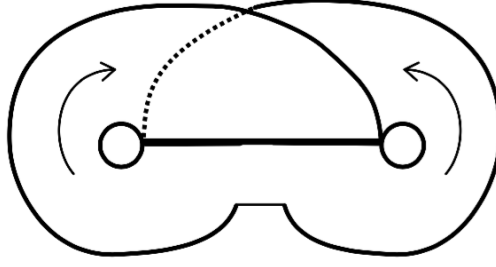


Figure 2: The Lorenz template,  $\mathcal{L}$ , reproduced from [1]. The dashed line indicates that the right subtemplate passes ‘under’ the left subtemplate. It’s worth noting that orbits may ‘fall off’ of templates, as those orbits that pass through the gap in the lower-middle of  $\mathcal{L}$  would. However, these orbits would not be periodic so they are immaterial to us.

Figure 2 shows the template for the familiar Lorenz system, the canonical chaotic system, described in equations 1-3. The simplicity of the Lorenz template is a testament to the elegance of templates - the Lorenz template exactly captures the repeated expanding and folding over of orbits that is the signature of chaos. For certain parameter values, periodic solutions to these equations form a link of infinitely many components, although we will see later that the Lorenz system is not universal.

$$\dot{x} = \sigma(y - x) \quad (1)$$

$$\dot{y} = \beta x - y - xz \quad (2)$$

$$\dot{z} = -\beta z + xy \quad (3)$$

The primary formal advantage of working with templates, is that they allow the use of *symbolic dynamics*. We can identify periodic orbits on a template  $\tau$  with the sequence of strips crossed.

**Definition 2.** Following [1], given a template  $\tau$ , we define

- Branch lines  $\{l_j : j = 1, \dots, M\}$ , the one-dimensional lines strips are connected along
- Strips  $\{x_i : i = 1, \dots, N \geq 2M\}$ , the two-dimensional regions connecting branch lines
- Itinerary  $(x_{s_1} x_{s_2} x_{s_3} \dots)$ , a sequence of strips crossed by an orbit on  $\tau$
- Itinerary space  $\Sigma_\tau = \{a_0 a_1 a_2 \dots\} \subset \{x_1, x_2, \dots, x_N\}^{\mathbb{Z}^+}$ , the set of all possible itineraries on  $\tau$
- Transition matrix

$$A_\tau(i, j) = \begin{cases} 0 & \text{if } \nexists \text{ a strip from } x_i \text{ to } x_j \\ 1 & \text{if } \exists \text{ a strip from } x_i \text{ to } x_j \end{cases} \quad (4)$$

It’s interesting to note that powers of  $A_\tau$  capture allowable trajectories, or more precisely the  $(i, j)$ th element of the  $k$ th power of  $A_\tau$  will be 1 iff there is an orbit of  $\tau$  of length  $k$ . So

in principle we have succeeded in reducing questions about the periodic orbits of a flow to questions about powers of a single matrix. Of course, in practice it will be very difficult to compute  $A$  from an arbitrary dynamical system.

See the Figure 4 for an example of a template with labeled strips.

Ghrist confirms that the set of admissible sequences on  $\sigma_\tau$  corresponds exactly with the set of periodic orbits on  $\tau$ , as desired.

## 1.2 Ordering orbits on templates

It's possible, and will prove useful, to put an order on orbits on  $\tau$ . This is accomplished by choosing an ordering on strips emanating from the same branch line and using the induced lexicographic ordering to order orbits.

This is best illustrated with an example borrowed from [1]. Let  $\mathcal{V}$  be the template illustrated in Figure 4, outfitted with a Markov partition of strips  $\{x_1, x_2, x_3, x_4\}$ . Choose an ordering  $\triangleright$  on the branch lines:

$$l_1 : x_1 \triangleright x_2 \tag{5}$$

$$l_2 : x_3 \triangleright x_4 \tag{6}$$

Then a lexicographic ordering on  $\sigma_{\mathcal{V}}$  gives an ordering of orbits. For example,  $x_1^2 x_3 x_4 \cdots \triangleright x_1 x_2 x_3 x_4 \triangleright x_1 x_2 x_4 \cdots$ .

Although we have skirted over the details of this ordering here, rest assured that it is well-defined for all templates as confirmed in [4][5][1]

## 1.3 Template inflation

Our main tool in proving the existence of a universal template (and by extension a universal ODE) will be *template inflation*. The following definition are due to [1].

**Definition 3** (Subtemplate). A *subtemplate*  $S \subset \tau$  of a template  $\tau$  is a topological subset of  $\tau$  which, with the restriction of the semiflow of  $\tau$ , satisfies the definition of a template (Definition 1)

**Definition 4** (Template inflation). A *template inflation* of a template  $\tau$  is a map  $\mathbf{R} : \tau \rightarrow \tau$  taking orbits to orbits which is a diffeomorphism onto its image. The image of a template inflation is, by definition, a subtemplate.

See Figure 3 for a helpful illustration of how an inflation might be constructed.

Note that if a template  $\tau$  admits an inflation, then the image of that inflation (being diffeomorphic to the original template) can again be inflated, giving an increasingly complicated sequenced of embedded subtemplates each diffeomorphic to the original! This is the essential property that makes inflations such a powerful tool in proving the existence of a universal ODE, a bit later we will see exactly how its usefulness is manifest.

A *very* useful lemma proved in [1] follows

**Lemma 1** (Ghrist & Holmes 1993). A template inflation  $\mathbf{R} : \tau \rightarrow \rho$  induces a map  $\mathbf{R} : \sigma_\tau \rightarrow \sigma_\rho$  whose action is to inflate each element of the Markov partition of strips  $\{x_i : i = 1 \cdots N\}$  to a finite admissible word  $\{\mathbf{w}_i = w_1 w_2 \cdots w_{n(i)} : i = 1 \cdots N, w_i \in \{x_i\}\}$

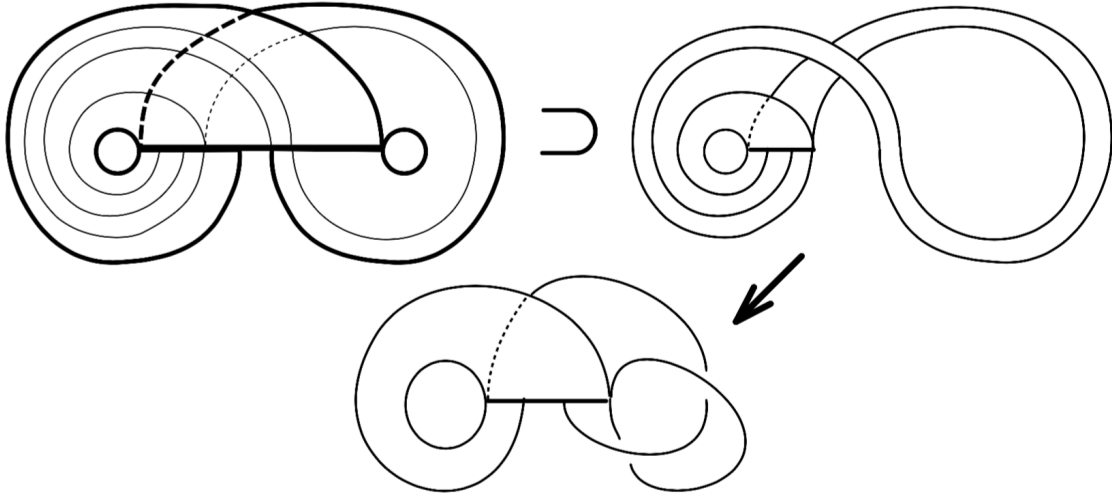


Figure 3: A subtemplate of the Lorenz subtemplate  $\mathcal{L}$ . When it is cut along the boundary and removed from  $\mathcal{L}$ , it forms a template diffeomorphic to the original one! Figure reproduced from [1]

For example the inflation  $L : \mathcal{L} \rightarrow \mathcal{L}$  shown in Figure 3 has the following action on the set of strips:

$$L : \mathcal{L} \rightarrow \mathcal{L} \begin{cases} x_1 \mapsto x_1 \\ x_2 \mapsto x_1 x_2 \end{cases} \quad (7)$$

Inflations take knots to knots but will typically change the knot type. We identify a special class of inflations that preserve knot type, as these are the inflations that we're really after.

**Definition 5.** Isotopic inflation An *isotopic inflation* of an embedded template  $\tau$  is a template inflation  $R$  such that  $R(\tau)$  is isotopic to  $\tau$

## 2 Proof that $\mathcal{V}$ is a universal template

In this section we outline the proof given in full detail in [6] that  $\mathcal{V}$  (shown in figure 4) is a universal template. Details will inevitably be omitted but our goal in this exposition is to straddle the line between the full proof and the very high-level outline given in [1], giving a concise and readable exposition that gets at the essence of the proof.

The essential idea is to find a special set of templates which have a special property that forces them to support all braids as orbits, and then to show that these templates can be found in  $\mathcal{V}$ . Since all knots and links can be realized as a braid, this immediately implies that  $\mathcal{V}$  is universal. The last step of showing that the special braid-supporting templates can

found in  $\mathcal{V}$  is the most difficult by far and where we will have to do the most hand-waving. Nonetheless, we will try to capture the essence.

Also important in this proof is the existence of the  $\mathcal{U}$  template, pictured below.  $U$  and  $\mathcal{V}$  are essentially equivalent, but there is one minor difference, and  $U$  is universal as well.

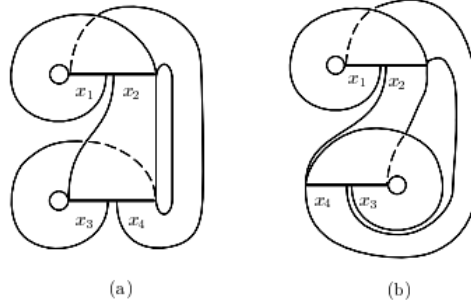


Figure 4: (a) The universal template  $\mathcal{V}$  and (b) The universal template  $U$ , reproduced from [?]

## 2.1 Braids and the theorem of Alexander

Recall that a closed braid is a collection of  $P$  disjoint simple closed curves in a standardly embedded torus  $D^2 \times S^1$  such that every  $D^2$  cross-section of the torus intersects the closed braid in exactly  $P$  points.

In a landmark paper [7], Alexander proved the following theorem which provides the crucial connection between braids and links.

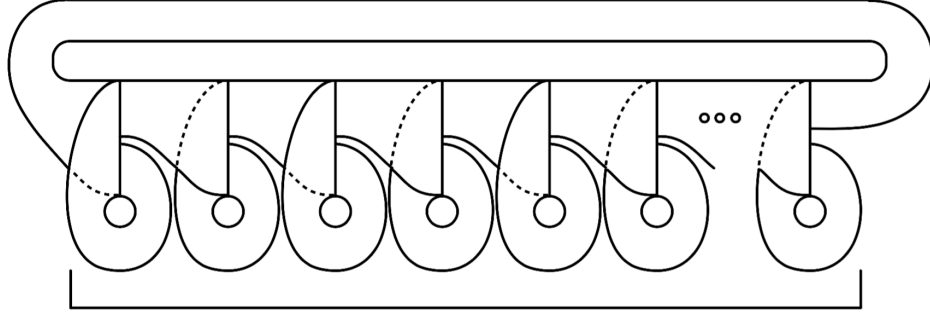
**Theorem 1** (Alexander 1923). *Each knot or link is isotopic to some closed braid on  $P$  strands for some  $P$*

## 2.2 The templates $\mathcal{W}_q$

The family of templates  $\{\mathcal{W}_q\}$  shown in figure 5 are those referred to earlier as the braid-supporting templates.  $\mathcal{W}_1$  is identically  $\mathcal{V}$  and increasing  $q$  by one has the effect of adding two *ears* to one side. The property that successive ears alternate in ‘sign’ is what makes this family of templates so useful, as this property makes proving that they support all braids delightfully simple.

**Lemma 2** (Ghrist 1996). An isotopic copy of any closed braid exists as a set of periodic orbits on some  $\mathcal{W}_q$  for sufficiently large  $q$ .

*Proof.* The concatenation of alternating positive and negative ears on  $\mathcal{W}_q$  mimics the braid group concatenation operation. The braid group  $B_{n+1}$  is naturally generated by  $\sigma_1, \sigma_2, \dots, \sigma_n$ , where  $\sigma_i$  is a single crossing of the  $i$ th strand over the  $(i+1)$ th strand and its inverse,  $\sigma_i^{-1}$ , is a single crossing of the  $(i+1)$ th strand over the  $i$ th strand. Naturally,  $\sigma_i \sigma_i^{-1} = \sigma_i^{-1} \sigma_i = I$ . We can consider the elements  $\pi_1, \pi_2, \dots, \pi_n, \pi'_1, \pi'_2, \dots, \pi'_n$  of  $B_{n+1}$ , where  $\pi_i = \sigma_1 \sigma_2 \cdots \sigma_i$  and  $\pi'_i = \sigma_1^{-1} \sigma_2^{-1} \cdots \sigma_i^{-1}$ . Note that  $\pi_1 = \sigma_1$ ,  $\pi'_1 = \sigma_1^{-1}$ , and that for  $i > 1$ ,



$2q$

Figure 5: The templates  $W_q$ , reproduced from [1]

$$\begin{aligned}
\pi_{i-1}^{-1} \pi_i &= (\sigma_1 \sigma_2 \dots \sigma_{i-1})^{-1} \sigma_1 \sigma_2 \dots \sigma_i \\
&= \sigma_{i-1}^{-1} \dots \sigma_2^{-1} \sigma_1^{-1} \sigma_1 \sigma_2 \dots \sigma_i \\
&= \sigma_{i-1}^{-1} \dots \sigma_2^{-1} \sigma_2 \dots \sigma_i \\
&= \sigma_{i-1}^{-1} \sigma_{i-1} \sigma_i \\
&= \sigma_i
\end{aligned}$$

$$\begin{aligned}
\pi'_{i-1} \pi'_i &= (\sigma_1^{-1} \sigma_2^{-1} \dots \sigma_{i-1}^{-1})^{-1} \sigma_1^{-1} \sigma_2^{-1} \dots \sigma_i^{-1} \\
&= \sigma_{i-1} \dots \sigma_2 \sigma_1 \sigma_1^{-1} \sigma_2^{-1} \dots \sigma_i^{-1} \\
&= \sigma_{i-1} \dots \sigma_2 \sigma_2^{-1} \dots \sigma_i^{-1} \\
&= \sigma_{i-1} \sigma_{i-1}^{-1} \sigma_i^{-1} \\
&= \sigma_i^{-1}
\end{aligned}$$

We can conclude that the elements  $\pi_1, \pi_2, \dots, \pi_n, \pi'_1, \pi'_2, \dots, \pi'_n$  of  $B_{n+1}$  are a generating set for  $B_{n+1}$ .

For the braid group  $B_{n+1}$ , notice that for all  $1 \leq i \leq n$ ,  $\pi_i$  can be drawn on any positive ear of  $W_q$ , letting all but the bottom strand take the top strip and forcing the bottom strand to take the bottom strip, cross over exactly  $i$  strands, and take the top strip. Similarly, for all  $i \leq i \leq n$ ,  $\pi'_i$  can be drawn on any negative ear of  $W_q$ , letting all but the bottom strand take the top strip, forcing the bottom strand to take the bottom strip, cross under exactly  $i$  strands, and take the top strip. Examples of  $\pi_2$  and  $\pi'_2$  of the braid group  $B_5$  are drawn in Figure 6.

Because this set of elements generates the braid group  $B_{n+1}$ , any braid  $B \in B_{n+1}$  can be drawn as a sequence of  $\pi_i$  and  $\pi'_i$ . If  $B$  can be represented with  $m$  of these elements, then the closure of  $B$  lies in  $W_m$ , as  $W_m$  contains  $m$  pairs of positive/negative ears that can each



support a  $\pi_i$  or  $\pi'_i$  structure. Thus, any closed braid on  $N$  strands lies in  $W_q$  for sufficiently large  $q$ . □

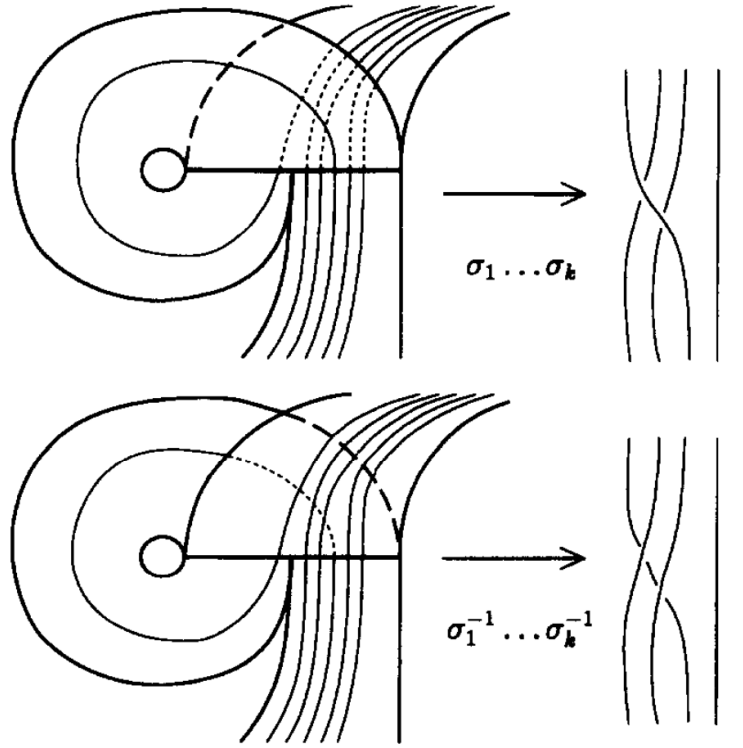


Figure 6: Top:  $\pi_2$  in  $B_5$  Bottom:  $\pi'_2$  in  $B_5$ . Figure reproduced from [6]

### 2.3 Special Inflations: $F$ and $G$

In his proof, Ghrist uses a few special inflations,  $F$  and  $G$ , and accompanies them with a proposition:

**Proposition 1.** The following inflations are isotopic.

$$F : U \hookrightarrow \mathcal{V} \begin{cases} x_1 \rightarrow x_1 \\ x_2 \rightarrow x_1 x_2 x_3 \\ x_3 \rightarrow x_4 x_2 \\ x_4 \rightarrow x_4 \end{cases}$$

$$G : \mathcal{V} \hookrightarrow U \begin{cases} x_1 \rightarrow x_1 \\ x_2 \rightarrow x_1 \\ x_3 \rightarrow x_2 x_4 \\ x_4 \rightarrow x_2 x_3 x_4 \end{cases}$$

As proof of their isotopy, diagrams showing the inflation are given.

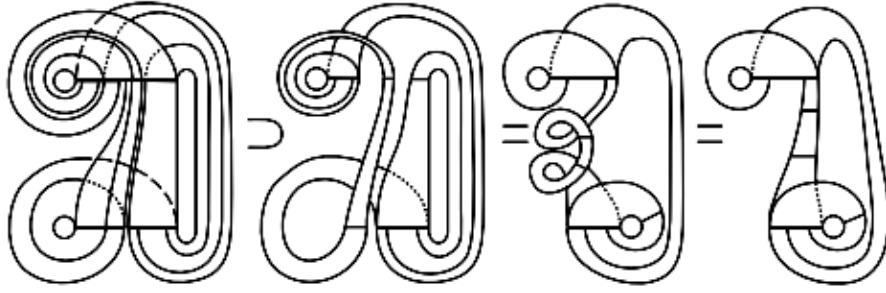


Figure 7: Left to Right: A subtemplate of  $\mathcal{V}$  as given by  $F$ , then cut along the boundary and removed from  $\mathcal{V}$ , simplified to show that it equals  $U$ . Figure reproduced from [?]

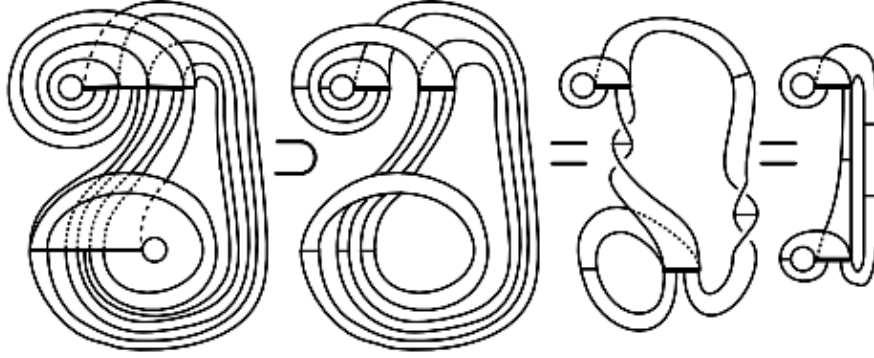


Figure 8: Left to Right: A subtemplate of  $U$  as given by  $G$ , then cut along the boundary and removed from  $U$ , simplified to show that it equals  $\mathcal{V}$ . Figure reproduced from [?]

In addition, we can consider the following inflation:

$$\chi : \begin{matrix} U \rightarrow U \\ \mathcal{V} \rightarrow \mathcal{V} \end{matrix} \begin{cases} x_1 \rightarrow x_3 \\ x_2 \rightarrow x_4 \\ x_3 \rightarrow x_1 \\ x_4 \rightarrow x_2 \end{cases}$$

The purpose of this inflation is to switch the two branch lines.

**Lemma 3** (Ghrist 1997). Given an isotopic inflation  $A$  having either  $U$  or  $\mathcal{V}$  as a domain and range (independently of each other), the inflation  $A^* = \chi A \chi$  is also isotopic.

For example, given  $F$  as defined above,  $F^*$  is defined as

$$F^* : U \hookrightarrow \mathcal{V} \begin{cases} x_1 \rightarrow x_2x_4 \\ x_2 \rightarrow x_2 \\ x_3 \rightarrow x_3 \\ x_4 \rightarrow x_3x_4x_1 \end{cases}$$

Finally, in the next section, we can include the inflation  $\mathfrak{H} = F^*GFG^*$  in the next section to continue our proof.

## 2.4 Find $\mathcal{W}_q \subset \mathcal{V}$ for all $q$

We must start with a lemma, provided by Ghrist.

**Lemma 4** (Ghrist 1997). Given that a subtemplate of  $\mathcal{V}$  doesn't contain  $x_1^\infty$ , it is possible to add an ear near the top branch line, like that in Figure 9, which is a positive ear. Similarly, given that a subtemplate of  $\mathcal{V}$  doesn't contain  $X_3^\infty$ , it is possible to add an ear near the bottom branch line, which is a negative ear.

*Proof.* Let us begin with  $\mathcal{W}_1 = \mathcal{V}$ . We perform the inflation  $\mathfrak{H}$ :

$$\mathfrak{H} : \mathcal{V} \hookrightarrow \mathcal{V} \begin{cases} x_1 \rightarrow x_2x_3^2x_4x_1(x_2x_4)^2x_2x_3x_4x_1 \\ x_2 \rightarrow x_2x_3^2x_4x_1(x_2x_4)^3x_2x_3x_4x_1 \\ x_3 \rightarrow x_2x_3^2x_4x_1x_2x_4 \\ x_4 \rightarrow x_2x_3^2x_4x_1x_2x_4 \end{cases}$$

This inflation satisfies the condition that this subtemplate does not contain  $x_1^\infty$ . Then, after performing this inflation, we will be able to add a positive ear at the top branch line; after doing so, we now have  $\mathcal{W}_1^+$ .

There exists a very similar inflation  $\mathfrak{H}^*$  from  $\mathcal{V}$  onto  $\mathcal{V}$  that will allow us to add a negative ear at the bottom branch line. Now, we have added a positive ear and a negative ear, so we are at  $\mathcal{W}_2$ , yet this subtemplate is still contained within  $\mathcal{V}$ ; this implies that  $\mathcal{W}_2 \subset \mathcal{V}$ . Now, we can repeat this procedure, adding another positive and negative ear after the respective inflations to get  $\mathcal{W}_3$ .

This procedure can be repeated  $q$  times to show that  $\mathcal{W}_q \subset \mathcal{V}$  for any positive  $q$ ; thus, by induction,  $\mathcal{W}_q \subset \mathcal{V}$  for all  $q$ .  $\square$

## 2.5 Universal Templates and V

**Theorem 2** (Ghrist 1996). *The template  $\mathcal{V}$  contains an isotopic copy of every tame knot and link as a periodic orbit of the semiflow.*

*Proof.* This proof comes immediately following the fact that  $\mathcal{W}_q \subset \mathcal{V}$  for all  $q$ , and that all closed braids (and thus all tame links and knots) can be contained within  $\mathcal{W}_q$  for large enough  $q$ . Thus,  $\mathcal{V}$  contains every tame knot and link.  $\square$

Ghrist then extends the theorem above to prove the following:

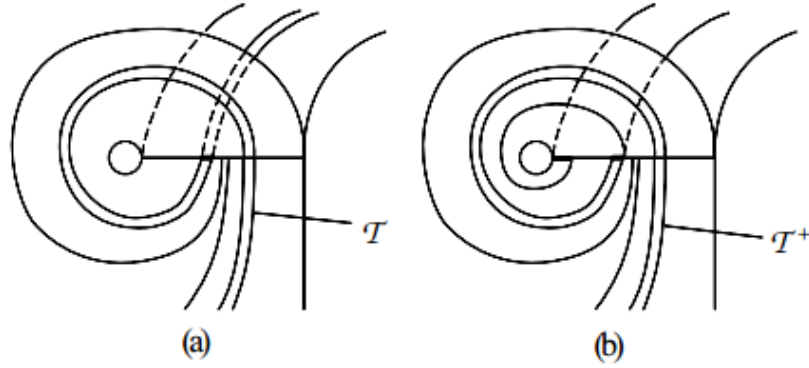


Figure 9: Appending a positive ear to  $T$  (left) yields  $T^+$  (right). Figure reproduced from [6]

**Theorem 3** (Ghrist 1996). *The template  $\mathcal{V}$  contains all orientable templates as subtemplates of  $\mathcal{V}$ .*

It must follow that if there are any other universal templates  $T$ , then  $\mathcal{V}$  must contain  $T$  as a subtemplate. As  $T$  contains all tame links and knots as well, as it is universal, the same theorems can be applied to  $T$  as were applied to  $\mathcal{V}$ , so that  $T$  also contains all templates (including  $\mathcal{V}$ ).

### 3 Implications for ODEs in general

Very brief exposition on section 5 in a knotty ode discussing what the universality of  $\mathcal{V}$  implies in general. Proving universality is reduced to showing that a template can be renormalized to  $\mathcal{V}$

### 4 Miscellaneous discussions and interesting open questions

The last theorem of Section 2.4 hints at a requirement for two templates to be considered equal: given templates  $A$  and  $B$ ,  $A$  and  $B$  are equal if  $A \subset B$  and  $B \subset A$ . If each template may be contained within the other, then all knots in  $A$  must be contained in  $B$  as well, and all knots in  $B$  must be contained in  $A$  as well.

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