Lectures notes on knot theory

Andrew Berger
(instructor Chris Gerig)

April 14, 2016

Contents

1 Disclaimer

Caution - these lecture notes have not been proofread and may contain errors, due to either the lecturer or the scribe. Please send any corrections or suggestions to andbberger@berkeley.edu

2 1/19/16: Introduction + Motivation

Start with a joke: "Applied Math majors may have misread the course title"

- hand out syllabus & table of knots
- administration & logistics
- "idea" of class structure
- find out about class (their backgrounds & interests)

- introduce some material:

1) knots come from real life: tie ends of string together
- unknots
- simplest possible knot = trefoil

2) not everything is a circle
- want to distinguish them
- knot complements
- techniques?

3) links ? can be complicated while component knots simple

4) crossings / deformations
- swap crossings of trefoil to get its mirror (interesting fact: they're not equivalent!)

5) usefulness:
- want to know when objects are knotted
- get all 3-dimensional spaces
- applications to physics

3 1/21/16: 2nd class

3.1 Logistical things

Talk to Chris if you're uncomfortable with group theory. There are going to be two projects (long and short) - we'll start breaking up in to groups to work on those starting the second week of February.

3.2 Minimal introduction to point-set topology

Just to set terms and notation for future reference.

Definition 1. Some miscellaneous definitions:

- $\mathbb{R}^n := \mathbb{R} \oplus \ldots \oplus \mathbb{R}$
- $B^n = \{x \in \mathbb{R}^n | \, |x| \leq 1\}$ note that in general this includes the boundary (which we call $\partial B^n = S^{n-1}$, the $(n-1)$-sphere)
- There are many ways to get the inclusion $\mathbb{R}^m \subseteq \mathbb{R}^n$ for $m \leq n$, as a convention just take the first m elements from $\mathbb{R}^n$

Remark 1. The (closed) line and circle are the only 1-dimensional topological spaces (once we have defined appropriate notions of equivalence). More on this later

Definition 2 (Homeomorphism)**.** We say that f is a *homeomorphism* if it is a bicontinuous bijection (both f and its inverse are continuous). Two topological spaces X, Y are homeomorphic if there exists $f : X \to Y$ a homeomorphism, denoted $X \cong Y$

Remark 2. Given X, Y topological spaces and $f : X \to Y$ a homeomorphism we can 'pull' the notion of openness in X into a notion of openess in Y: $A \subseteq Y$ is open if $f^{-1}(A)$ is open in X

Example 1. Examples of homeomorphisms

- $(0, 1) \cong \mathbb{R}$. As an exercise find the homeomorphism that is witness to this fact (hint: use an arctangent)
- A square in the plane is homeomorphic to a circle in the plane
- $\mathbb{R}^n \ncong \mathbb{R}^m$ for $n \neq m$: dimension is invariant under homeomorphism. This is a deep result that is supposed to be hard to prove

Definition 3 (Knot)**.** A knot is a one-dimensional subset of $\mathbb{R}^3$ that is homeomorphic to S^1.

We can specify a knot K by specifying an embedding (smooth injective) $f : S^1 \to R^3$ so that $K = f(S^1)$. For f to be smooth, all of its derivatives must exist.

Example 2. Examples of embeddings specifying knots

- $f = \mathbb{1}$ (abuse of notation here) specifies a circle
- The infinite non-knot example we looked at yesterday fails to be a knot because the derivative does not exist at the limit point

Definition 4 (Link)**.** Given a collection of knots $\{K_i\}$, we define a link $L = \bigcup_i K_i$ (disjoint union)

The study of links is different from the study of knots, due to "linking behavior". Roughly speaking: knots can be very complicated as well their disjoint unions, but moreover, links can get very complicated while their connected components may all be unknots.

3.3 Equivalence of knots

Our word for equivalence of knots is ambient isotopy. This refers to the fact that the homeomorphism that is witness to the equivalence of the knots acts on the ambient space the knots lives and not only on the knot itself.

Definition 5 (Equivalence of knots)**.** For K_1, K_2 knots, we say that $K_1 \cong_{isotopic} K_2$ if there exists a (orientation-preserving) homeomorphism $f : \mathbb{R}^3 \to \mathbb{R}^3$ such that $f(K_1) = K_2$.

More precisely we require that there exists a 1-parameter family $\{f_t\}_{0 \leq t \leq 1}$ of smooth homeomorphisms such that $f_0 = \mathbb{1}$ and $f_1 = f$. In particular, we cannot have an isotopy that shrinks a knot to a point.

This aligns pleasingly for the intuitive notion that knots are equal when they can be deformed to each other: f here is just the global deformation.

Remark 3. Careful about the notation here: $K_1 \cong K_2 \cong S^1$ holds for any knots K_1, K_2

Definition 6 (Knotted)**.** We say that a knot K is *knotted* if $K \ncong_{isotopic} S^1$

Definition 7 (Planar diagram of K)**.** When we visualize knots we make some projection $\mathbb{R}^3 \to \mathbb{R}^2$ (with defined coordinate system). The naïve projection that discards information about the crossings is called the universe and is not necessarily unique. The planar diagram is this projection with crossing information captured.

Example 3. Examples of invariants (under equivaence)

- Dimension is an invariant of isotopy
- The crossing number is the minimal number of crossings in a given diagram

3.4 Reidemeister moves

What does an isotopy mean for planar diagrams? The famous Reidemeister moves

Definition 8 (Reidemeister moves)**.** Refer to figure 1

1. R0: You can straighten wiggly lines
2. R1: You can undo twists
3. R2: You can seperate underpasses (that don't intersect)
4. R3: You can move a line behind an intersection across the intersection

Theorem 1. *$K_1 \cong_{isotopic} K_2$ iff their diagrams can be obtained from each other using the Reidemeister moves*

Wherein Chris struggles to draw a trefoil.

Exercise 1.

1. Show that the figure eight knot is equivalent to its mirror.
2. Take your favorite knot diagram and show that it is equivalent to the diagram obtained as follows: Take an arc on the left-most side of your diagram, put a twist in it and then pull it all over to the right side of the diagram. (The point of this exercise is to give an abstract proof of equivalence using Reidemeister moves without worrying about the explicit diagram.)
3. Show that an unknot with a couple of loops is equivalent to the unknot *without* using R1

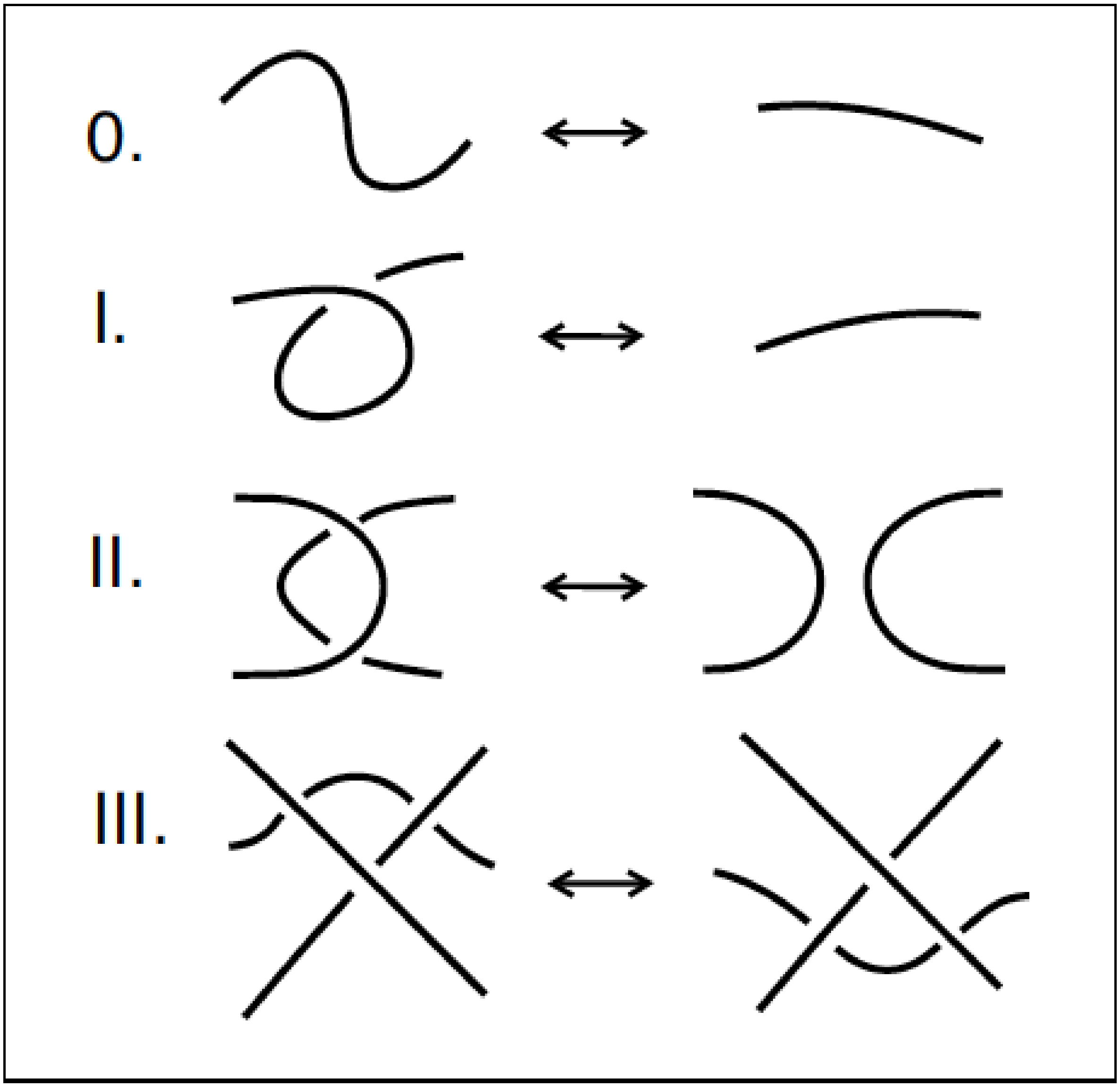


Figure 1: Sketch of allowed Reidemeister moves

4 1/26/16: recap of the last lecture

4.1 Recap of last lecture

What does it mean to be a knot? To be knotted?

Recall 1. Recall that a knot K is a subset of $\mathbb{R}^3$ that is homeomorphic to S^1.

We could equally as well have defined a knot to be the image $f(S^1)$ of a smooth embedding $f : S^1 \to \mathbb{R}^3$.

Notation 1. Let's fix our notation for ambient isotopy (the kind that captures a notion of knottedness) and homeomorphism (under which all knots are equivalent), being very careful to distinguish between them.

- For K_1, K_2 knots that are equivalently knotted (or isotopic) and write $K_1 \cong K_2$
- We write homeomorphism with $\approx$. For instance, all knots K are homeomorphic to S^1, $K \approx S^1$

4.2 Intro to knot complement

We expect that the knot complement $\mathbb{R}^3 \setminus K$ should somehow capture the notion of knottedness. Can we formalize that?

'It's not the space itself - it's what the knot is doing in 3-dimensional space that matters, and this theorem captures that precisely'

Theorem 2 (Gordon-Luecke)**.** *If* $\mathbb{R}^3 \setminus K_1 \approx \mathbb{R}^3 \setminus K_2$ *then* $K_1 \cong K_2$

Remark 4. The fundamental group sometimes allows us to answer the question '$\mathbb{R}^3 \setminus K_1 \approx \mathbb{R}^3 \setminus K_2$?' and therefore turn this into an algebraic problem.

Francesca pointed out that the converse of the preceding theorem is easily proved, directly from the definition of equivalence. Do it as an exercise!

Question 1. Marissa asked: when is a knot equal to its mirror? Chris isn't aware of any general classification results - maybe when the Jones polynomial has palyndromic coefficients. Another question: if $K_1 \cong K_2$ is it true that $K_1^* \cong K_2^*$ (here K^* indicates the knot mirror) Yet another question: if $K_1 \cong K_2$ and $K_1 \cong K_1^*$ is it true that $K_2 \cong K_2^*$? Someone pointed out that if you can prove the previous statement, then this statement is easily proved to be true.

4.3 Hard Unknots

Chris struggles again to draw a trefoil.

Let's fix a notion of complexity for knot diagrams - we say that the complexity of a diagram is the crossing number of that diagram.

Recall that a knot is equivalent to the unknot if it *has a diagram* with crossing number zero. A natural direction to head in towards proving that the trefoil is not the unknot might be to show that any Reidemiester move can not decrease the complexity of the trefoil diagram - but one needs to be very careful here as there may be a sequence that increases in complexity for a while before decreasing.

This motivates the concept of hard unknots: unknots that require a sequence of Reidemeister moves increasing the complexity of the diagram before the complexity finally falls to zero.

To begin with let's study a hard unknot called 'the culprit' from the paper *Hard Unknots and Collapsing Tangles* (Kauffman, Lambropoulou). Can you see how to untangle this knot?

5 1/28/16

Agenda: what sort of invariants; solved problem from last class; oriented knots, linking numbers, fundamental group of a knot

5.1 Logistical things

Start thinking about what you want to do for a project.

5.2 Question from last time

If $K \cong J$ does $K^* \cong J^*$?

One idea is that if D_K and D_J are diagrams for K, J, then we can find a sequence of R-moves $\{R_i\}$ transforming D_K to D_J. It's easily imagined that if we just 'flip' the moves in $\{R_i\}$ then that new flipped sequence transforms D_{K^*} to D_{J^*}. This idea can be used to prove that $K \cong J \implies K^* \cong J^*$.

Chris came up with a different proof:

First a false proof:

Fact: $\mathbb{R}^3 \setminus K \approx \mathbb{R}^3 \setminus K^*$. So if one naïvely applies the Gordon-Luecke theorem, they conclude $K \cong K^*$, which is obviously false.

What gives here is that the Gordon-Luecke theorem requires that the homeomorphism is orientation-preserving. The map that's witness to the fact that $\mathbb{R}^3 \setminus K \approx \mathbb{R}^3 \setminus K^*$ is orientation reversing. Thus the Gordon-Luecke theorem does not apply.

Let's do a real proof:

Lemma 1. If $K \cong J$ then $K^* \cong J^*$

Proof. Suppose that K, J are knots such that $K \cong J$. By the Gordon-Luecke theorem, $\mathbb{R}^3 \setminus K \approx \mathbb{R}^3 - J$ in an orientation preserving fashion.

$\mathbb{R}^3 - K^* \approx \mathbb{R}^3 - K$ with an orientation reversing homeomorphism; same for J. Now we can easily obtain the desired orientation preserving map that's witness to $\mathbb{R}^3 - K^* \approx \mathbb{R}^3 - J^*$ by the appropriate composition (note here, that the composition of an even number of orientation-reversing maps is orientation-preserving). □

5.3 Connect sum operation, knot cancelling, prime knots

Definition 9 (connect sum)**.** This is a general concept from algebraic topology, but for our purpose the connect sum of knots K and J (written $K\#J$) is given by cutting knots K and J at some point and pasting them together.

Remark 5. Is this operation well-defined? In particular, does it matter where we choose to cut and paste the knots together?

Yes it is, and no it does not. A pictoral proof: if we connected the knots in one place and we want to connect in another we can just slide the knots along each other until we arrive at the desired location.

One thing that does matter is orientation - if we don't assign an orientation there are two ways to join the arcs together at one location. We can resolve this by giving knots an orientation and making $\#$ preserve that.

Chris refined his technique for drawing the trefoil; it is now foolproof.

Question 2. Let $\mathcal{K}$ be the collection of knots (up to $\cong$). Is $(K, \#)$ a group?

- $K\#0 \cong K$, so there is an identity element. (0 indicates the unknot)
- $K_1\#K_2 \cong K_2\#K_1$, as can seen by a pictoral proof, so this operation is commutative.
- It's associative too (we can attach anywhere and at any time).
- Okay, do knot inverses exist? No. Knots cannot be cancelled.

Remark 6. In particular, it is not possible that $0\#0 \cong K$ with K knotted, despite how "complex" the knot diagrams for 0 are.

Theorem 3. *Knots cannot be cancelled under connect sum*

Proof. This is known as the Mazur-Swindle.
To prove this, we have to expand our notion of knots to include wild knots (those that shrink to a point).

Suppose that $K\#J \cong 0$. Now construct the infinite connect sum $K\#J\#K\#J\ldots$ letting the knots shrink in space as the sum goes along so that they occupy a closed topological space[1].

We have that $(K\#J)\#(K\#J)\ldots \cong 0$ but $K\#(J\#K)\#(J\#K) \cong K\#0 \cong K$, and thus our knots must be unknotted to begin with! $\square$

Question 3 (Unsolved)**.** Does $c(K\#J) = c(K)+c(J)$ where c is the minimal crossing number operation.

Definition 10 (Prime, composite knots)**.** K is called composite if it can be given as $K_1\#K_2 \cong K$ for $K_1 \ncong 0$ and $K_2 \ncong 0$.

Otherwise, K is called prime.

Exercise 2. Show that the trefoil and figure 8 knots are prime.

[1] If we let the knots go off to infinity without shrinking them, then the result does not form a loop and is not captured by our notion of knots. By shrinking them they become wild, but are still homeomorphic to S^1

6 2/2/16

6.1 Orientations

Put more structure on our knots:

1) Orient it by labelling the diagram with an arrow.

2) To each crossing we can associate a ± 1. To make this well-defined: rotate the over-strand CCW (counterclockwise) with respect to the under-strand, and assign $+1$ if the orientations (i.e. arrows) on those strands match up.

An *invariant* of a knot is one which doesn't depend on the diagram, which means it should be independent of the Reidemeister moves!

The crossing number (number of crossings) is not an invariant. Although R3 preserves it, R2 decreases (or increases) it by 2, and R1 decreases (or increases) it by 1.

OK, so what if we also consider the signs of the crossings? That is, maybe the *writhe* (number of crossings counted with their sign) is a better choice.... No, it is also not an invariant. Although R3 preserves it, and R2 preserves it (**exercise!**), but R1 changes it by ± 1.

Remark 7. Let K denote the unknot diagram obtained from the standard O by two R1-moves of opposite orientation, i.e. the unit circle but with two "opposing twists." *Remember our first exercise assigned at the beginning of the semester:* prove $K \cong O$ via R2+R3 moves only. Now, this is consistent with the fact that the writhe is invariant under R2+R3 moves, noting that the writhe of the standard diagram for the unknot is 0. But the writhe changes by ± 1 for the R1-move, so it is impossible to do the analogous exercise above where the construction of K involves two "twists" with the same orientation (or even with just a single twist!).

6.2 Linking number

So far, our different counts of crossings (with signs) doesn't get us an invariant. Well, we do get something for links!

The *linking number* of a link $L = K_1 \cup K_2$ is $lk(L) \equiv lk(K_1, K_2) = \frac{1}{2}\sum_c \text{sign}(c)$, where c sums over the crossings between K_1 and K_2 (not the self-crossings of K_i!). The factor $\frac{1}{2}$ is to be consistent with our intuition that K_1 "links" K_2 while K_2 "also links" K_1, so we shouldn't double-count.

Exercise 3. This is an invariant. R1 is obvious, it introduces self-intersections.

Example 4. $lk(\text{unlink}) = 0$ independent of orientation.

$lk(\text{Hopf link}) = \pm 1$ depending on orientation.

$lk(\text{Whitehead link}) = 0$... but it's "visually linked"! An invariant does not always provide a complete classification.

7 2/4/16

7.1 Logistical things

Introduce each other and discuss our interests for potential short projects (this will help us decide who to work with). Projects should begin by end of next week.

7.2 Seifert Surfaces

A *surface* (2-manifold) $\Sigma \subset \mathbb{R}^3$ can be roughly regarded as a collection of points (a "space"), such that for each point $x \in \Sigma$ there is a neighborhood of points which is homeomorphic either to $\mathbb{R}^2$ or $\mathbb{R}^2_{\geq 0}$ (the upper half-plane)... so the space is 2-dimensional. The points whose neighborhood is $\mathbb{R}^2_{\geq 0}$ form the *boundary* $\partial\Sigma$ of the surface, and is 1-dimensional.

Likewise, knots are 1-manifolds without boundary: they can be roughly regarded as a collection of points, such that for each point $x \in K$ there is a neighborhood of points which is homeomorphic to $\mathbb{R}$.

Remark 8. Manifolds can be either *compact* or *noncompact*. **Assume all of our spaces are *compact*.** For the non-topologists: think of a compact surface as a surface which "cannot be arbitrarily stretched to infinity" in the ambient space $\mathbb{R}^3$... the open interval $(0,1)$ can be arbitrarily stretched, as it is homeomorphic to $\mathbb{R}$, while the closed interval $[0,1]$ cannot.

Surfaces can be orientable or not. The Mobius strip is not orientable.

(Oriented) Surfaces without boundary are determined completely by the number of *holes* they have – this number is the *genus*. Sphere (0 holes), torus (1 hole), and n-holed tori.

Definition 11. *Seifert surface of a knot K* is an orientable surface Σ with boundary, such that $\partial\Sigma = K$.

Example 5. The unit circle (unknot) in the plane has the *disk* as a Seifert surface. The unlink (of two components) has the *cylinder* (or *annulus*) as a Seifert surface... and the same is true for the Hopf link (the annulus has 2 twists in this case).

Seifert surfaces always exist. Look up **Seifert's Algorithm.**

But Seifert surfaces are not unique – we can arbitrarily add holes to them! So the genus of a Seifert surface is not a well-defined number for a knot. Immediately you should question, what can we do to get something well-defined?

Definition 12. *Genus of a knot $g(K)$* is the smallest possible genus of all Seifert surfaces of the knot.

Theorem 4. *$g(K) = 0$ iff $K \cong O$.*

This theorem hinges on "Seifert surface" being orientable, to talk about "genus". That is, we have a reason to require orientability in the definition of Seifert surfaces: The theorem $g(K) = 0$ iff $K \cong O$, and the notion of genus makes sense for orientable surfaces.

Now, there is a notion of genus for nonorientable surfaces. Does the theorem extend if we allow Seifert surfaces to be nonorientable? Note that the standard picture of a Mobius strip (a strip with 1 twist) is a surface which bounds the unknot, but what would be its genus? And the trefoil is knotted but its standard diagram is bounded by a Mobius strip with 3 twists.

Exercise 4. The trefoil has the punctured torus as a Seifert surface. *Hint: First find a diagram for it which isn't the standard one (the "standard one" is the diagram which bounds the Mobius strip with 3 twists).*

We have a very limited amount of knowledge up to now in this course, so we can "easily" take all combinations of the things we learned to ask questions, such as what happens to the genus under the connect-sum operation. Well,

Theorem 5. $g(K\#J) = g(K) + g(J)$.

The inequality $g(K\#J) \leq g(K) + g(J)$ is obvious: the #-operation for 1-dimensional spaces also works for n-dimensional spaces, so wherever we "cut" the knots we can also "cut" the Seifert surfaces and then glue them. NOTE: This only gives us an inequality, because the resulting Seifert surface for $K\#J$ might not be the one having minimal possible genus (even though the surfaces for K and J had minimal genus)!

Corollary 1. Knots cannot be cancelled, i.e. there are no knot-inverses under #.

Our previous proof of this result involved passing to the realm of wild knots... here we don't have to!

7.3 Intro to research

Just as a tip: Question everything. See where proofs of theorems hinge on all of the hypotheses. Sometimes, the hypotheses are there to make life easier, rather than being the most general statement.

Typically, "orientability" is not necessary, and this is where a lot of statements you read in the literature can be generalized to handle non-orientable spaces.

When a proof of some theorem requires you to first move outside of a given framework and then come back into the given framework, then there is probably a way to prove the theorem without doing this. The "knots cannot be cancelled" is such an example (the given framework is "our study of tame (i.e. not wild) knots").

8 2/9/16 – The trefoil is knotted

8.1 The trefoil is not the unknot

Up until now has been the intro to knot theory. From now on we're going to pursue self studies and focus our attention on different topics to get depth. We'll get to polynomials next week as they're a good branching off point.

Believe it or not, we have yet to show that any given knot is knotted! We will now show (in a couple of ways) that

Theorem 6. *The trefoil $T \ncong O$.*

Some methods of proof: braiding, (tri)colorability, fundamental group, (Jones) polynomial.

8.2 Braids

Definition 13. A braid σ of n strings consists of n oriented arcs with fixed ordered endpoints. Take n unlinks passing in parallel through a region, cut all of the strings in some boxed region, and permute where they attach to on the other side (via σ). That gets you some link $\hat{\sigma}$.

Theorem 7 (Alexander)**.** *Any oriented link is given by $\hat{\sigma}$ for some σ.*
Every link can be obtained by a braid. Note that this includes knots, as they are 1 links.

Remark 9. What things can we do with this theorem (as researchers)? It says this map $\sigma \mapsto \hat{\sigma}$ is surjective - is it injective? No

Theorem 8. *Markov There exists moves $\sigma_1 \mapsto \sigma_2$ which don't affect $\hat{\sigma_1}$ (the produced link)*

Example 6. If $\sigma = \mathbb{1}_n$ (the identity permutation) then $\hat{\sigma}$ is the n unlink. But if $\hat{\sigma}$ is the n unlink does $\sigma = \mathbb{1}_n$? Yes (This is a theorem of Birman-Menasco) Note that this does not exclude permutations that produce less than n unlinks. There are certainly permutations that do this, for instance a simple twist of two strands.

8.2.1 The braid group

Are braids a group under concatenation? We discovered that connected sum on knots forms the structure of a monoid as knots cannot be canceled.

Yes, $(B_n = \{\sigma_i\}), \cdot$ is in fact a group! It can be represented as $B_n = \langle \sigma_1, \ldots, \sigma_{n-1} \mid \text{relations below} \rangle$

The generators have the form $\sigma_i = (i[i+1])$ (they swap with their neighbor). If $|i - j| \geq 2$ then $\sigma_i\sigma_j = \sigma_j\sigma_i$ – when generators are far apart they commute. Else $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$

Example 7.

- Show that $(\sigma_1 \ldots \sigma_{n-1})^m$ produces the (n, m)-torus link.
- Show that the trefoil is not the unknot using braid theory

9 2/11: Coloring

9.1 Logistical happenings

- Does everyone have a partner? Subjects?
- Chris away March 15 - 17. Maybe we'll just meet anyway and do class presentations

We still don't know that the trefoil is knotted!

Example 8. Show with braid theory that the trefoil is knotted.

9.2 (Tri)Colorings

Assume for this section that our knots are connected (so ignore links).

Definition 14 (Strand)**.** A *strand* is a line in a diagram that goes from an undercross to an undercross (whereas an *arc* is a line in the diagram that goes from a crossing to crossing).

For instance the trefoil is made up of 3 strands.

Definition 15 (Color)**.** A "coloring" of a knot diagram is a choice of color for each strand.

Definition 16 (Tri-colorability)**.** A knot diagram has a "tri-coloring" if each strand can be colored one of 3 colors such that at each crossing either all three colors or only one color are present, and all 3 colors are present in the diagram.

Is (tri)colorability invariant under R-moves? Well-defined? Yes. It's an easy exercise to show.

Does every knot have a tri-coloring? No. The unknot is not tri-colorable. But the trefoil is! Thus $T \not\cong O$.

Notation 2. $\tau(K)$ denotes the number of colorings (not just tri-colorings!) of knot K

Question 4. Questions that come up about colorings

- Relation between colorability and coloring Seifert surfaces?
- Relation between prime knots and colorability?
- Relation to 4-Color Theorem of maps?

Example 9.

- $\tau(0) = 3 = 3^1$. The unknot has 3 colorings
- $\tau(T) = 3 + 3 \cdot 2 = 3^2$. The trefoil has 9 colorings (hence $T \not\cong O$ again)
- The trefoil is tri-colorable, but the figure 8 knot isn't. So now our problem is to see that the figure 8 knot is knotted... What is τ of it?
- In general, combinatorics says $\tau(K) \leq 3^{\text{number of strands of } K}$

Tuesday is going to be a crash course on homotopy and groups. We'll learn algebraically that knots are knotted because of their complements. That will segue into polynomials and all sorts of other fun stuff

10 2/16: π_1

10.1 Logistical things

March 15 + 17: the week before spring week, during which Chris will be absent and we should be emailing projects in. Know and/or begin final project

March 8 + 10: will be class presentations.

March 1 + 3: email Chris progress/thoughts 'report'

10.2 Crash course on the fundamental group

There are tons of differents kinds of mappings we could define from the set of knots to the integers, and to other sets such as integer polynomials. We have seen enough so far to believe that knots have a very rich structure - perhaps we'd also believe that the integers don't really have enough structure for mappings into them to be very useful, as far as revealing the structure of knots goes.

This motivates studying maps from knots into the polynomials, which obviously have much more structure than the integers.

Using the fundamental group for knots is motivated by: $K \cong J \iff \mathbb{R}^3 \setminus J \approx \mathbb{R}^3 \setminus K$.

1. π_1 takes n-dimensional 'spaces' and gives groups
2. π_1 is invariant under homeomorphism
3. For any knot K, $\pi_1(K) \cong \pi_1(S^1) \cong \mathbb{Z}$, which is why we look at the knot complement
4. $\pi_1(\mathbb{R}^3 - K)$ is called the **knot group**

Definition 17. For some 'space' Σ, $\pi(\Sigma) = \{$ loops (up to reparametrization)$\gamma(t) : S^1 \to \Sigma$ with respect to some basepoint$\}$. The group operation is composition of loops $\sigma_1 \cdot \sigma_2 : S^1 \to \Sigma$. The inverse $\bar{\gamma}$ is just the time reversed version of γ. The zero element is the constant map $S^1 \to * \subseteq \Sigma$ (where $*$ is the basepoint).

So far we're working up to reparametrization equivalence - but with that notion $\gamma \cdot \bar{\gamma}$ goes around γ once in the forwards direction and once in the reverse direction, it most certainly does not say this is a constant map as claimed. We want to work up to homotopic equivalence

Crash course in homotopy: a path γ is a map $[0,1] \to \Sigma$, a loop is a path with $\gamma(0) = \gamma(1)$. We say that γ_1 is homotopically equivalent to γ_2 if there exists a continuous family F_τ (with $\tau \in [0,1]$) of "deformations" of the paths such that $F_0 = \gamma_1$ and $F_1 = \gamma_2$. This is written $\gamma_1 \simeq \gamma_2$

Do you see how the fundamental group captures the holes in the 'space'?

Example 10.

- $\pi_1(\mathbb{R}^n) = 0$ (the trivial group) all loops contract to basepoint
- $\pi_1(\text{annulus}) = \mathbb{Z}$. $\gamma_1 \not\simeq *$ is the loop that circles the whole once, $\gamma^2 \not\simeq *$ winds twice (and is the same up to reparametrization because we require that reparametrizations are bijections $S^1 \to S^1$)
- $\pi_1(S^1) = \mathbb{Z}$. The annulus is nothing more than a thickened circle.

- What about $\pi_1(T^2)$? Is it $\mathbb{Z} \oplus \mathbb{Z}$? If α is the loop around one of the holes and β is the loops around the other hole, does $\alpha\beta = \beta\alpha$? It turns out that $\pi_1(T^2) = \mathbb{Z} \oplus \mathbb{Z}$. But π_1(double torus) is nonabelian!
- $\pi_1(\mathbb{R}^3 - O) = \mathbb{Z} = \langle g_1 | \emptyset \rangle$ the loops link around the unknot with any linking number

Remark 10. Computing π_1 is hard in general.

11 2/18: Wirtinger presentation

We showed that the knot group of the unknot is the integers, $\pi_1(\mathbb{R}^3 - O) \cong \mathbb{Z}$. Thus to show a knot is knotted, it suffices to show its knot group is not the integers. *Exercise: If the knot group surjects onto a group G such that $G \not\cong \mathbb{Z}_n$ nor $\mathbb{Z}$, then the knot is knotted.*

$\pi_1(\mathbb{R}^3 - O) = \mathbb{Z}$ in some sense tells us that nontrivial groups arise from knots due to crossings.

Knot group $\pi_1(\mathbb{R}^3 - K)$ can be computed using the *Wirtinger presentation* for any knot diagram of K. Given a diagram with $n \geq 1$ crossings, the presentation has n generators (plus specific relations). These generators correspond to loops that wrap around arcs of the diagram, and the relations correspond to the "obstruction" to homotoping the loops past the crossings.

Algorithm:

1. Orient the knot diagram.

2. Label the 3 arcs at each crossing (1 arc for the overcrossing and 2 arcs for the undercrossing), g_i and g_j and g_k. Here g_j represents the loop (with respect to same fixed basepoint in $\mathbb{R}^3 - K$) that encircles the j^{th} arc with linking number $+1$.

3. Stare hard enough to see the relations: Assuming g_k is the overcrossing-arc and the crossing is positive, $g_j = g_k g_i g_k^{-1}$. Basically, conjugation by the overcrossing-arc allows you to "pass a loop through the crossing."

Example 11. *The trefoil is knotted:* $\pi_1(\mathbb{R}^3 - T) \cong B_3$ the Braid group on 3 strands. Chris' proof in class doesn't establish this group-isomorphism, but he constructs a surjective homomorphism $\pi_1(\mathbb{R}^3 - T) \twoheadrightarrow S_3$ onto the symmetric group of 3 letters (and since we know there is no surjection $\mathbb{Z} \to S_3$, we're done). Chris' map is determined by the 3 generators of the Wirtinger presentation: $g_1 \mapsto (1\,2)$ and $g_2 \mapsto (2\,3)$ and $g_3 \mapsto (3\,1)$

Exercise: The figure 8 knot is not equivalent to the trefoil.
Exercise: The Hopf link is linked.

Remark 11. Note that the Wirtinger presentation is not usually optimal in terms of number of generators. The unknot can have n twists in it, but $\mathbb{Z}$ has 1 generator.

If $\pi_1(\mathbb{R}^3 - K) \cong \pi_1(\mathbb{R}^3 - J)$ then does $K \cong J$? *No, look at the mirror of any knot!*

12 2/23: POLYNOMIALS

$\mathcal{K} := \{\text{knots}\}$. Our invariants so far took the form $\mathcal{K} \to \mathbb{Z}$ (such as the genus and minimal crossing number), or $\mathcal{K} \to \{\text{groups}\}$ via π_1 of the knot complement. We will now expand to polynomial rings, which should be more sensitive to the richness of $\mathcal{K}$ because $\mathbb{Z} \subset \mathbb{Z}[x]$.

12.1 Kauffman bracket polynomial

In analogy with *Seifert's algorithm*, we take a knot diagram for K and "break up" all of its crossings to end up with a collection of (Seifert) circles. We can "break up" each crossing via an A-split or B-split. Therefore, there are different results after all crossings are broken – each possible result is called a *state*, denoted σ.

σ has two pieces of information attached to it:

1) The number of Seifert circles $n(\sigma)$, and

2) The commutative product of labels for all of the crossings that were broken up (ex: A^2B for σ which resulted from a 3-crossing diagram that had two A-splits and one B-split), denoted $\langle K \,|\, \sigma \rangle$.

In analogy with the partition function of statistical thermodynamics (looking at weighted averages of the possible "states" of a given system), we define the **bracket polynomial**

$$\langle K \rangle = \textstyle\sum_\sigma \langle K \,|\, \sigma \rangle \cdot d^{n(\sigma)-1}$$

The point here is that if I give you all the states $\{\sigma\}$, you can recover K. Then $\langle K \rangle$ is a weighted average, so it should contain a lot of information about the knot... but it doesn't tell us everything, because it's only an average.

So far this has 3 indeterminate variables: A, B, d. But we must ask, is this an invariant of K, i.e. is it independent of the Reidemeister moves??? When analyzing the R2 and R3 moves, we will want $B = A^{-1}$ and $d = -A^2 - A^{-2}$. Thus we are left with 1 indeterminate variable A.

Example 12. $\langle O \rangle = 1$.

But what about the R1 move? It turns out that
$\langle \text{over-twist} \rangle = -A^3 \langle \text{no twist} \rangle$
$\langle \text{under-twist} \rangle = -A^{-3} \langle \text{no twist} \rangle$
Thus the Kauffman bracket polynomial isn't good enough! But we have learned that the writhe is sensitive to the (oriented) R1 move:
$w(\text{over-twist}) = 1 + w(\text{no twist})$
$w(\text{under-twist}) = -1 + w(\text{no twist})$
So we can alter the bracket polynomial to get a well-defined polynomial invariant $\mathcal{L}_K$ as a function of A:

$$\mathcal{L}_K = (-A^3)^{-w(K)} \langle K \rangle$$

Remark 12. This is not a map from $\mathcal{K}$ to $\mathbb{Z}[A]$, because there can be negative powers of A. We really have Laurent polynomials, and by a change of variables we will get the so-called **Jones polynomial**.

Now we can deduce a lot of stuff very quickly, through algorithmic computational algebra.

Theorem 9. *Achiral knots $K \cong K^*$ have palindromic coefficients in their polynomial.*

Indeed, reversing crossings will reverse the roles of A and A^{-1}, so that $\mathcal{L}_{K^*}(A) = \mathcal{L}_K(A^{-1})$.

Exercise 5. Compute $\mathcal{L}_T$ for the trefoil. You'll get the following results along the way:
1) The trefoil knot is knotted.
2) The trefoil knot is chiral (i.e. not equivalent to its mirror)!
3) The Hopf link is linked.
Answer: $\mathcal{L}_T(A) = A^{-4} + A^{-12} - A^{-16}$, $\mathcal{L}_{\text{Hopf}}(A) = -A^4 - A^{-4}$

12.2 Provoked questions

Based on the construction above, is there any relation between the bracket polynomial and (genus of) Seifert surfaces?

What information do we get if we plug in different values for A?

If the polynomial has palindromic coefficients, is the knot achiral?

13 2/25

Francesca's dinosaur is named Mavis.

Homework for next week: come up with an interesting question to ask in class

13.1 Axioms of the Jones polynomial

We denote the Jones polynomial of a knot K by $V(K)$, or $V_K(A)$ to emphasize the indeterminate

- $V_0 = 1$ normalization
- V_K is isotopy-invariant
- $t^{-1}V_{\text{cross over}} - tV_{\text{cross under}} = (\sqrt{t} - \frac{1}{\sqrt{t}})V_{\text{remove cross}}$ *Skein Relation*

Example 13. Put some twists in the unknot – what does that get you? You can easily compute the Jones polynomial of the unlink. $V_{00} = -t^{\frac{1}{2}} - t^{-\frac{1}{2}}$

For any link L, using R1 and the Skein relation you can compute $V_{L\cup 0}$

What about the trefoil? Its Skein relation involves the Hopf link and unknot!

13.2 Uniqueness of Jones polynomial

Theorem 10. *Suppose you have some $I : \mathcal{K} = \{\text{oriented links}\} \to \mathbb{Z}[\sqrt{t}, \frac{1}{\sqrt{t}}]$. If it satisfies the axioms of the Jones polynomials stated above, then it is the Jones polynomial.*

The point is that any desirable polynomial I needs to be an invariant (hence Axiom 2), and can always be normalized so that it evaluates the unknot as the unit 1 (hence Axiom 1). The Skein relation is the important one, and does all the work to compute stuff.

13.3 Just how sensitive is the Jones polynomial

Question 5. Does V 'detect' the unknot? $V_K = 1 \implies K \cong 0$? *This is still an open question!!!*

Question 6. Can you read off the unknotting number from the Jones polynomial? The Skein relation gives us reason to think that this might be the case.

There is an even more powerful object we could map the knots into, called the Khovonav homology. We won't be able to get into it in this class, but it essentially maps into some system of groups. And the reason we care? Khovonav homology is a "generalization" of the Jones polynomial, and detects the unknot.

Thistelthwaite found a 2-component link L such that $V_L = V_{00}$. Thus the Jones polynomial is not sensitive enough to detect the unlink. Does that make the "detecting unknot" conjecture more suspicious?

Mutation is an aptly named operation that given K outputs $M(K)$ produced by drawing a box around a region of the knot and rotating it by 180^o in 'different ways'. Specifically, the region of the knot is a tangle with four strands exiting the box, and we rotate to match up those endpoints on the box.

ANNOYING FACT: $V_{M(K)} = V_K$.

13.4 Other polynomials

K_-, K_+, K_0 denote the under-crossing, over-crossing, and no-crossing respectively. Again, to clarify, we take some knot and look at a single crossing, so that K_+ is the knot and K_- is swapping the crossing and K_0 is removing the crossing.

- Jones $V_K(t) \in \mathbb{Z}[\sqrt{t}, \frac{1}{\sqrt{t}}]$
- Alexander-Conway $\nabla_K(z) \in \mathbb{Z}[z]$. With Skein relation $\nabla_{K_+} - \nabla_{K_-} = z\nabla_{K_0}$
- HOMFLY $P_K(\alpha, z) \in \mathbb{Z}[\alpha, z]$. The previous two are special cases of this polynomial:
  - $V_K(t) = P_k(t^{-1}, \sqrt{t} - \frac{1}{\sqrt{t}})$
  - $\nabla_K(z) = P_K(1, z)$

  HOMFLY has Skein relation $\alpha P_{K_+} - \alpha^{-1}P_{K_-} = zP_{K_0}$

  Interestingly, $P_K(\alpha, z) = P_{K^*}(\alpha^{-1}, z)$ (where K^* is mirrored)

Exercise 6. Show that $P(K\#J) = P(K)P(J)$ in two ways: by definition of P and for V using the definition of the bracket polynomial.

Remark 13. $\nabla_K = 1 \iff G_K := \pi_1(\mathbb{R}^3 - K)$ is perfect. Perfect means that G is equal to its commutator subgroup $[G, G]$.

Thus, the Alexander polynomial cannot detect the unknot.

14 3/1: Quandles

Discuss random math questions about knots:

Is Alexander polynomial useful if we already knot about the Jones polynomial?

Can we define a polynomial for wild knots? Perhaps a limit of Jones polynomials for recursively defined wild knots.

Start thinking what topics you want to study for the final project, and who you want to work with.

14.1 Generalized colorings

Recall: **3-coloring** of knot diagram

1) each strand assigned color R G B
2) use all 3 colors, i.e. use at least 2 colors
3) each crossing has all 3 colors or only 1 color

Generalize to **$\mathbb{Z}_p$-labeling** (p prime)

1) assign element in $\mathbb{Z}_p = \{0, \ldots, p-1\}$
2) use at least 2 elements
3) If y crosses under x to become z, then $z \equiv 2x - y \mod p$

Generalize to **quandle**

Motivate: group has 2 operations via conjugation, $(x, y) \mapsto y^{-1}xy =: x \triangleright y$ and $yxy^{-1} =: x \triangleright^{-1} y$ which satisfy

1) $x \triangleright x = x$
2) $(x \triangleright y) \triangleright^{-1} y = x = (x \triangleright^{-1} y) \triangleright y$
3) $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$

A **quandle** is a set with 2 binary operations satisfying the above. "Isomorphism" for quandles is the analog of that of groups.

Example 14. $Conj(G)$ for group G above (if G is abelian then $\triangleright = \triangleright^{-1}$)

Example 15. $\mathbb{Z}_p$-labeling of knot can be viewed as the quandle associated to the ring $\mathbb{Z}_p$ (unit $q = -1$) with $x \triangleright y = -x + 2y = x \triangleright^{-1} y$

15 3/3: Knot Quandle

The *knot quandle* $Q(K)$ is a particular quandle associated to a knot K. It consists of a set of labels of the strands for an oriented diagram of K, such that if a crosses under b to become c then you get $a \triangleright b$, and if a crosses over b to become c then you get $b \triangleright^{-1} a$.

If we ignore the orientation, then we get the *involuntary* knot quandle $IQ(K) := Q(K)/\{\triangleright = \triangleright^{-1}\}$.

$Q(K)$ is an isotopy-invariant: R1 and R2 and R3 are the Axioms 1 and 2 and 3 above! Beautiful!

More importantly, $Q(K)$ is a complete invariant: $Q(K) \cong Q(J)$ iff $K \cong J$.

Example 16. $IQ(\text{trefoil}) \cong Conj(\mathbb{Z}_3)$

15.1 Geometric Motivation

A *meridian* $\gamma \in \pi_1(\mathbb{R}^3 - K)$ is a loop (with basepoint $*$) that has linking number $+1$ with a given strand of K. View K and γ as the independent loops of a torus.

The *Wirtinger presentation* had generators $a, b, c, \ldots$ being meridians up to homotopy, such that if a crosses under b to become c then

$b^{-1}ab = c$ if crossing is positive

$bab^{-1} = c$ is crossing is negative

Well, the knot group cannot distinguish certain knots, such as the Square Knot from the Granny Knot. Analyzing this example further, we see that the isomorphism of their knot groups won't send the meridians to the meridians.

So $Q(K)$ is the analog of $\pi_1(\mathbb{R}^3 - K)$ except instead of using meridians, you use "disks with paths". Meridians can be homotopic while the associated "disks with paths" are non-homotopic, and this extra obstruction makes $Q(K)$ more sensitive to the richness of knots than the knot group.

That is, the knot quandle is the "fundamental quandle" of the knot complement, while the *knot group* is the fundamental group of the knot complement. This quandle is a "larger" algebraic object than the group, and captures all information that the group doesn't recognize. For example, the knot group doesn't tell apart K from K^*.

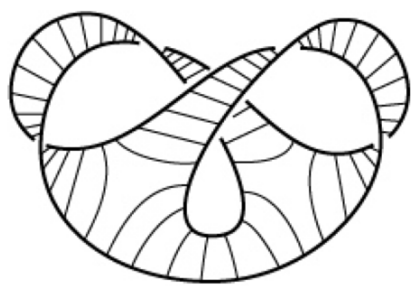
Remark 14. We have seen now seen these weird algebraic objects, quandles, appear as a generalization of *colorings* as well as π_1 of the knot / knot complement. **Any other way for quandles to appear in knot theory?**

16 3/29+31: Invariants via Seifert Matrix (notes by Chris)

Assume K is connected, and oriented.

16.1 Signature

Take (for example) the Figure 8 knot K. Here is a particular Seifert surface S for it:



Our S has genus 1 (**Question:** Is it also the genus of the Figure 8 knot?) It is a disk attached with 2 handles, each handle having 2 twists in it.

Fixing a basepoint somewhere, there are two independent loops $\gamma_1, \gamma_2 \subset S$ wrapping around each handle. We can compute a *linking number* between them as follows. Take γ_1, push it off of S (sitting inside of ambient $\mathbb{R}^3$) in the positive direction normal to S, and call that loop γ_1^+. Then we're done: $lk(\gamma_1^+, \gamma_2)$.

In general, a Seifert surface of genus g will have $2g$ "independent" loops (wrapping around handles). Group-theoretically, take the abelianization of $\pi_1(S)$, denoted $H_1(S)$. This turns out to be isomorphic to the group $\mathbb{Z}^{2g}$, and a basis $\gamma_1, \ldots, \gamma_{2g}$ are our desired loops.

The *Seifert matrix* M_S has ij-entry $lk(\gamma_i^+, \gamma_j)$. Note that this makes sense when $i = j$, and that M_S depends on S.

How can we get something that depends only on K? Take that matrix M_S, and compute its signature $\sigma(M_S)$ – that is called the *signature* of the knot. It is an invariant, but we will not prove it (we would need to relate the matrices M_S and $M_{S'}$ for different Seifert surfaces).

16.2 Polynomial

Conway-Alexander polynomial $\triangle_K(t) \in \mathbb{Z}[t^{1/2}, t^{-1/2}]$ of (oriented) knot satisfies
Skein relation $\triangle_{L_+} - \triangle_{L_-} = (t^{1/2} - t^{-1/2})\triangle_{L_0}$
Normalization $\triangle_O = 1$
Remember: L_+ denotes the knot and looks at a positive crossing, L_- swaps that crossing (making it negative), and L_0 removes that crossing (coherently with knot's orientation).

This isn't as "good" as the Jones polynomial, but we're talking about it as an excuse to talk about the Seifert matrix:
$\triangle_K(t) = \det(t^{1/2}M_S - t^{-1/2}M_S^T)$

Exercise 7. If K is connected and S has genus g, then $\triangle_K(t) = t^{-g}\det(tM_S - M_S^T)$.

17 3/29: in class notes

17.1 Invariants from Matrices

For now let, K the figure 8 knot, but everything that we say will hold for oriented connected knots.

Exercise 8. Draw a Seifert surface for K. For the figure eight this will be diffeomorphic to a disk with two 'handles' attached that have a twist in them. Show that this surface has genus 1. Call this surface S_8.

Recall that the genus of a knot is the minimal genus of all seifert surfaces for that knot. Since we know that the figure eight knot is not the unknot, and there's a theorem that the unknot is the only knot with genus 0, we now know that the figure eight knot has genus 1.

There are two loops that will contribute to the fundamental group for S_8, they go around each handle. Call them γ_1, γ_2. These loops are 'independent' in the sense that they are the generators for the $\pi_1(S_8)$, this is important for what follows to be well-defined. Formally, these loops are smooth maps $S^1 \to S_8$.

In general, a Seifert surface S will have $2g(S)$ 'independent' loops, $\gamma_1, \ldots, \gamma_{2g}$

We want to compute the linking number $\mathrm{lk}(\gamma_1^+, \gamma_2)$ viewed inside $\mathbb{R}^3$, where the + superscript on γ_1 indicates that we're going to 'push' it off of S_8 along the positive (we have an orientation) normal vector of S_8. Now it no longer intersects γ_2 and we can compute the linking number. The linking number is after all a 1×1 matrix, and linking numbers are knot invariants so we've obtained the first matrix invariant of the day.

We can now obtain larger matrix invariants by looking at all possible combinations of γ_1 and γ_2

$$\begin{bmatrix} \mathrm{lk}(\gamma_1^+, \gamma_1) & \mathrm{lk}(\gamma_1^+, \gamma_2) \\ \mathrm{lk}(\gamma_2^+, \gamma_1) & \mathrm{lk}(\gamma_2^+, \gamma_2) \end{bmatrix} v \tag{1}$$

This is our first example of a Seifert matrix.

Recall $G_{ab} = G/[G, G]$ (quotient out the commutative subgroup). In general, we work with $\pi_1(S)_{ab} = H_1(S) \cong \mathbb{Z}^{2g}$. The 'independent' loops that we would like to work are precisely those that generate $\pi_1(S)$ and are nonzero in $\pi_1(S)_{ab}$. Let $\{\gamma_i\}$ be the set of such $2g(S)$ 'independent' generators that exist.

We define the matrix elements of the Seifert surface as $(M_S)_{i,j} = \mathrm{lk}(\gamma_i^+, \gamma_j)$. This matrix depends on $g(S)$, and probably on S. At the end of the day, for two Seifert surfaces S, S' of the same knot, M_S and $M_{S'}$ *must* be related in some way.

Call $\sigma(M_{S_K})$ the signature of the Seifert matrix. Chris believes that its the number of negative eigenvalues $-$ the number of positive eigenvaluoes. This is an invariant so we can speak meaningfully of $\sigma(K)$.

On Thursday we'll get more invariants out of M_S, and discuss how useful they are.

On a final note, it's much less cumbersome to work with $\pi_1(S)_{ab}$ than directly with $\pi_1(S)$. The reason for this is that for *any* group, we can find a space X such that $\pi_1(X)$ is that group. However, by the Fundamental theorem of finitely generated abelian groups, $\pi_1(X)_{ab}$ is direct sums of either $\mathbb{Z}^n$ or $\mathbb{Z}/n\mathbb{Z}$

18 4/5: Dehn surgery (notes by Chris)

View a knot K in S^3 instead of $\mathbb{R}^3$, by adding a point at infinity.

The boundary of a (small) tubular neighborhood N_K of K in S^3 is a (knotted) torus $\partial N_K \approx \mathcal{T}^2$. There are two generators of $\pi_1(\mathcal{T}^2) \cong \mathbb{Z} \oplus \mathbb{Z}$, namely a copy γ_K of K (that runs along the torus) and a meridian γ_m (wrapping around an arm of the torus) that links K positively once.

Take $S^3 - N_K$ and "glue back in" N_K by identifying their boundary torus. This gluing is given by a diffeomorphism (i.e. homeomorphism with all derivatives defined) $f : \mathcal{T}^2 \to \mathcal{T}^2$ of the torus! Assume that f maps γ_K to γ_K, and maps γ_m to some (p, q)-torus knot on $\mathcal{T}^2$, meaning $f(\gamma_m) = p\gamma_K + q\gamma_m$.

We just operated in the sense of doing **Dehn surgery**.

If the surgery/gluing is trivial, i.e. the diffeomorphism is the identity, i.e. $p = 0$ and $q = 1$, then you get back S^3. Otherwise, you will get something else!

Theorem 11. *If f is not the identity, you cannot get S^3 back.*

Proof. Note that $S^3 - N_K$ is the same as the knot complement of K (up to deformation). If there was a nontrivial Dehn surgery that also gives S^3, then $S^3 - (S^3 - N_K)$ is a solid torus, but in two different ways (different solid tori, one of which is N_K (for the trivial surgery)). Since the "core" of a solid torus is a knot, this means we have two different knots with the same knot complement. Thanks to the Gordon-Luecke theorem, this can't happen. □

Theorem 12 (Lickorish-Wallace)**.** *Any compact 3-manifold can be obtained by performing Dehn surgery on a link $L \subset S^3$.*

Get our research-minds thinking (they should ALWAYS be thinking). How is this all useful? We can study 3-dimensional spaces via 1-dimensional spaces!

Is the surgery unique? That is, are there different Dehn surgeries on different links that give the same 3-manifold? YES. The process of identifying which links/surgeries give the same 3-manifold is **Kirby Calculus**, invented by our fellow UC Berkeley professor Rob Kirby.

Exercise 9. Why did we use S^3 and not $\mathbb{R}^3$ for the ambient manifold?

19 4/5: Surgery in class notes

Is everyone okay with sharing their projects (for your peers' curiosity)?

Coming up: Surgery on 3-dim spaces, Jones' paper, knots in other dimensions

19.1 Surgery on 3-dim spaces

Take $K \subset \mathbb{R}^3 \subset S^3$ (if you're confused how it can be the case that $\mathbb{R}^3 \subset S^3$, think of $\mathbb{R}^2$ and S^2 where S^2 is obtained from $\mathbb{R}^2$ by 'wrapping up' the point at infinity)

Wherein Chris struggles to draw a trefoil.

To the prime the discussion of how higher-dimensional manifolds can be knotted, consider $\mathcal{D}^2 \times S^1$ ($\mathcal{D}^2$ is the unit disk). For any knot K this construction gives a tubular neighborhood of K, let's denote it $\mathcal{N}_K$. The boundary of this manifold is a torus (denoted $\mathcal{T}^2$). In the past we've computed that $\pi_1(\mathcal{T}^2) = \mathbb{Z} \times \mathbb{Z}$; intuitively $\pi_1(\mathcal{T}^2)$ is generated by two loops γ_k, γ_m which respectively go around the 'long way' of $\mathcal{T}^2$ and the 'short way'. γ_k is nothing more than a copy of K that lives on the boundary of $\mathcal{D}^2 \times K$. γ_m is the meridian

Fact: $S^3 \setminus \mathcal{N}_k$ is a solid torus. More precisely, it's closure is ($\mathcal{N}_k$ is closed, its complement is open). $S^3 =$ (solid torus) $\cup$ (solid torus). Try visualizing it this way: you have a donut, and going through the hole of the donut is a mega-donut going off to infinity and filling all of the space in the complement of the normal donut. But remember $\mathbb{R}^3 \subset S^3$ not the other way around, so you're cheating a bit when you visualize it this way.

Now let's do some voodoo nonsense: $(S^3 - \mathcal{N}_k) \cup \mathcal{N}_k = S^3$ where the 'union will be along a torus' $\partial \mathcal{N}_k$ (I'm having a hard time notating this but you should imagine cutting a disk out of a plane and putting it back; the interesection is the boundary of the disk you cut out, so the 'gluing' operation amounts to specifying how the boundary is mapped when you put the disk back). For our purposes the gluing operation is a homeomorphism $g : \mathcal{T}^2 \to \mathcal{T}^2$ (because the boundary we glue along is a torus). In the case where g is the identity, the gluing operation gives us back S^3, but in general g will wangjangle $\mathcal{N}_k$ in some complicated fashion to fit the boundary and do all sorts of crazy things to the topology of $(S^3 \setminus \mathcal{N}_k) \cup_g \mathcal{N}_k$ (here $\cup_g$ means glue as prescribed by g).

We just don't want any g though, we want $g(\gamma_K) = \gamma_K$. Meanwhile $g(\gamma_m) = p\gamma_m + q\gamma_k$ (it's a (p,q) torus knot!). Congratulations, we jsut performed Dehn surgery!

Example 17. If g is the identity then $X := (S^3 \setminus \mathcal{N}_K) \cup_g \mathcal{N}_K = S^3$. In this case, $g(\gamma_m)$ is a $(1, 0)$ torus knots.

Is there any other g that also gives $X = S^3$ after gluing?? The answer turns out to be no!

Proof. Note that $S^3 \setminus \mathcal{N}_k \cong S^3 \setminus K$ (I'm being sloppy with $\cong$, but you know what I mean). Suppose there exists g not equal to the identity so that $X \approx S^3$. $X \setminus (S^3 \setminus \mathcal{N}_K) = S^3 \setminus (S^3 \setminus \mathcal{N}_k) = \mathcal{N}_k \cong K$, but $X \setminus (S^3 \setminus \mathcal{N}_k)$ is a solid torus which can be deformed into a knot J. J and K have the same complements, which by the Gordon-Luecke theorem means that $J \cong K$ which then implies g has to be the identity. Boom. $\square$

Okay, cool. What other sorts of spaces can we get with different g?

Theorem 13 (Lickorish-Wallace)**.** *Any compact 3-manifold X can be obtained by Dehn surgery on a link L*

Question 7. Is this unique? Turns out no. Kirby calculus analyzes this uniqueness

20 4/7: Other types of "knotting"

Why do we look at $S^1 \hookrightarrow \mathbb{R}^3$ versus any other types of embeddings?

20.1 Range $\mathbb{R}^n$

$S^1 \hookrightarrow \mathbb{R}^{n>3}$ has no "knotting" behavior. The extra dimensions (above 3) can be used to push over-crossings to under-crossings and allow us to isotope all embeddings to the standard unknot. [The same reason applies to $S^1 \hookrightarrow X$ with $\dim X > 3$.]

Analogy: We cannot move $\{-1\}$ to $\{1\}$ on the punctured line $\mathbb{R} - \{0\}$. But look at $\mathbb{R}^2 - \{(0,0)\}$ and view $\mathbb{R} - \{0\}$ as the punctured x-axis, where $\{-1\}$ becomes $\{(-1,0)\}$ and $\{1\}$ becomes $\{(1,0)\}$. Now we can move the points around the origin.

$S^1 \hookrightarrow \mathbb{R}^1$ doesn't exist (it would have to collapse the circle, which is not injective).

$S^1 \hookrightarrow \mathbb{R}^2$ has no crossings. The theory reduces to two key results:
1) *Jordan Curve Theorem* – the knot separates the plane into two regions
2) *Schonflies Theorem* – one of those regions is homeomorphic to a disk

20.2 Range X

S^3 is "nothing more" than $\mathbb{R}^3$ with an additional point, so that all "infinities" are identified with each other. The region becomes bounded, allowing more tools to be built to study knots. The image of S^1 in S^3 must miss some point, so knot theory in S^3 is really knot theory in $\mathbb{R}^3$. ***Exercise:*** *Can you unknot $K \subset S^3$ if K is knotted in $\mathbb{R}^3$?*

$S^1 \hookrightarrow S^1 \times S^2$ is interesting. View $S^1 \times S^2$ as the solid region between two concentric spheres, where those two spheres are identified with each other, i.e. take $[0,1] \times S^2$ and identify $\{(0,x)\} \sim \{(1,x)\}$ for all $x \in S^2$. Consider any "tangle" as a single (knotted) strand $[0,1] \times S^2$ whose endpoints on $\{0,1\} \times S^2$ are identified... it's isotopic to the unknot!

20.3 Domain S^i

Analyze the linking behavior of knots of the form $S^0 \hookrightarrow S^1$ or $S^0 \hookrightarrow S^2$, based on the number of (pairs of) points and their locations.

$S^2 \hookrightarrow \mathbb{R}^3$ has no "knotting" behavior, because there are two few dimensions to move around in, just like $S^1 \hookrightarrow \mathbb{R}^2$. But $S^2 \hookrightarrow \mathbb{R}^4$ works, and this is a special case of **higher-dimensional knot theory**: $S^n \hookrightarrow \mathbb{R}^{n+2}$. Think of $\mathbb{R}^4$ as 3-dimensional space moving through time, and think of S^2 as a collection of circles. Now consider embeddings $S^2 \subset \mathbb{R}^4$ as a "movie" of embeddings of $S^1 \subset \mathbb{R}^3$ where unknots turn into knots and back to unknots.

20.4 Domain X

More generally, there is knotting of objects $X \hookrightarrow \mathbb{R}^n$ for appropriate n based on the dimension of X. For example, tori can be knotted in $\mathbb{R}^3$, since tori are nothing more than the boundaries of thickened knots!

Exercise: *Find two 2-holed tori that are not isotopic to each other in $\mathbb{R}^3$.*

21 4/12: Jones' original paper

Backtrack to braid theory and the Jones polynomial. Let's connect them, and in particular, see how Jones originally defined his polynomial (before Kauffman did)!

Definition 18. The *Jones algebra* A_n for $n > 1$ is the free additive algebra on the (multiplicative) generators $e_1, \ldots, e_{n-1}$ viewed as a $\mathbb{C}[\tau, \tau^{-1}]$-module. Here τ is a variable that commutes with all generators, and the generators satisfy the relations
1) $e_i^2 = e_i$
2) $e_i e_{i\pm 1} e_i = \tau e_i$
3) $e_i e_j = e_j e_i$ when $|i - j| > 2$

Jones noticed that A_n looks like the Braid group B_n. He constructed a particular map $\rho : B_n \to A_n$. He also constructed a particular "trace" map $tr : A_n \to \mathbb{C}[\tau, \tau^{-1}]$, which means $tr(ab) = tr(ba)$. So we get a trace map

$$tr \circ \rho : B_n \to \mathbb{C}[\tau, \tau^{-1}]$$

and this is the *Jones polynomial* after a particular normalization!

Let's analyze this more closely. By Alexander's theorem, any oriented link is given by the closure $\hat{\sigma}$ of some braid σ. The non-uniqueness is handled by Markov's theorem, so we have to make sure that $tr \circ \rho$ is independent of the "Markov moves", in order to get an invariant of the link (closure of the braid). Let's look at the Markov moves:
1) Vacuously, two braids are identified up to isotopy-equivalence in the braid group.
2) Conjugate a braid $\sigma \in B_n$ by another braid $b \in B_n$. **Exercise:** $\widehat{b\sigma b^{-1}} = \hat{\sigma}$
3) An actual *Markov move*: Given $\sigma \in B_n$, view it inside B_{n+1} by attaching an isolated strand at the end, take the generator $\sigma_n \in B_{n+1}$, and form $\sigma\sigma_n^{\pm 1}$. **Exercise:** $\widehat{\sigma\sigma_n^{\pm 1}} = \hat{\sigma}$
4) The "inverse" *Markov move*, undoing (3), where a braid of the form $\sigma\sigma_n^{\pm 1}$ becomes σ.

The definition of "trace" already handles conjugation of braids. To handle the actual Markov moves, Jones' trace map tr for A_n is chosen to satisfy the additional property $tr(we_i) = \tau \cdot tr(w)$ whenever $w \in A_{i-1}$. To get our desired link invariant, we need to normalize the trace map to account for these τ's that appear under Jones' additional property when a Markov move occurs.

Define the writhe $w(\sigma)$ of a braid σ in the same way as the writhe of a knot. Check $w(\sigma) = w(\hat{\sigma})$.

Theorem 14. $V_{\widehat{\sigma}}(\tau) = \tau^{-w(\sigma)} tr \circ \rho(\sigma)$

Remark 15. We can do something similar for the Kauffman bracket polynomial, using other algebras.

22 4/14: Trichotomy of knots

Remark 16. Chris claims that the study of (1-dimensional) knots in the "Whitehead manifold" W is nothing more than the study of knots in $\mathbb{R}^3$, even though W is not homeomorphic to $\mathbb{R}^3$. Reasoning: due to W being contractible, there is no "topology" to obstruct deforming the knot into a "Euclidean patch" sitting inside of W.

22.1 (p, q)-torus knots

Pros:

- Easy description: single circle "wraps" in only two possible directions.
- Crossings are easily studied.

Here, p refers to wrappings around the meridian and q refers to the number of times the center hole is circled, and $gcd(p, q) = 1$. In particular, a trefoil is a $(3, 2)$-torus knot. This construction is not an injective mapping into the space of knots: $(1, 1), (0, 1), (1, 0)$ are all the unknot.

Also, $(p, q) \cong (q, p)$.

Also, torus knots are prime.

22.2 Satellite knots

Pros:

- Gives simple procedure for producing knots from other knots.

Cons:

- Difficult to know when a given knot is a Satellite, because you need two other knots!

To construct a satellite knot, take any nontrivial knot K and view it as sitting inside of some solid torus. Now take the solid torus and knot it into the tubular neighborhood of some other nontrivial knot J. This warps K into some possibly different knot, let's call it $\mathrm{Sat}_J(K)$.

There is ambiguity in the notation $\mathrm{Sat}_J(K)$, because K can sit inside of a solid torus in multiple different ways!

Question 8. Is it true that $c(K) \leq c(\mathrm{Sat}_J(K))$? Does $c(\mathrm{Sat}_J(K))$ depend on J?

Proposition 1. Every composite knot is satellite knot. In particular $K_1 \# K_2$ is of the form $\mathrm{Sat}_{K_2}(K_1)$. To see this, stretch out an arc of K_1, so that the majority of K_1 is in one small region of the solid torus while the single arc wraps around the hole of the torus.

Whitehead doubling: $Wh(K)$ is of the form $\mathrm{Sat}_K(O)$, where the description of the unknot O (sitting inside of the solid torus) is really two strands that wrap around the hole of the solid torus and "hook" together.

Equivalently, take K and "double" the string and then "hook" those two strings together somewhere. In other words, I took two strands that followed along K and then hooked the two strands (in order to obtain a single knot).

Moral: Satellite knots are a pretty broad class, whereas (p, q) torus knots are much more restricted.

22.2.1 Hyperbolic knots

Theorem 15 (William Thurston)**.** *If a knot K is neither a (p, q)-torus knot or a satellite knot, then it is a hyperbolic knot.*

This class dwarfs the other classes in size. There is reason to believe something along the lines of: as the number of crossings grows, the probability of being a hyperbolic knot asymptotically approaches 1.

Definition 19. A knot K is hyperbolic if $\mathbb{R}^3 \setminus K$ has a hyperbolic metric.

Example 18. The figure-8 knot is hyperbolic. One potential (but possibly impossible) way to do this is: Check that the figure-8 could not arise from a satellite procedure (this is doable if the satellite procedure does *not* decrease crossings), and easily check that the figure-8 cannot be (p, q)-torus.

Corollary 2. Hyperbolic knots are prime. (*proof:* composite knots are satellites)

This is counterintuitive: It suggests that there are many more prime knots than composite knots! (As opposed to prime numbers.)

As a result, expect many instances of the form $K_1 \# K_2 \cong K_3 \# K_4$ (for K_i prime).

22.3 Thoughts and Interpretations

So every (connected) knot falls into one of these 3 categories. That's simple. But each category can be very complex (except maybe for torus knots). That's hard.