Blatt 2

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5. Mai 2019

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Sei $f(x,y,z) = 4\cos z \arctan\left(ye^{-x^2}\right)$

Mit Definition:

$$\lim_{h\to 0} \frac{f(x_0+hr)-f(x_0)}{h} =$$

$$\lim_{h\to 0} \frac{4\cos\left(\frac{\pi}{4}+h\sqrt{14}\right)\arctan\left((1+h)e^{-(0+h)^2}\right) - 4\cos\left(\frac{\pi}{4}\right)\arctan\left(1e^{-0^2}\right)}{h} =$$

$$\lim_{h\to 0} \frac{4\cos\left(\frac{\pi}{4}+h\sqrt{14}\right)\arctan\left((1+h)e^{-h^2}\right) - 4\frac{1}{\sqrt{2}}\frac{\pi}{4}}{h} \overset{\text{Regel von de l'Hospital}}{=}$$

$$\lim_{h\to 0} \frac{-4\sin\left(\frac{\pi}{4}+h\sqrt{14}\right)\sqrt{14}\arctan\left((1+h)e^{-h^2}\right) + 4\cos\left(\frac{\pi}{4}+h\sqrt{14}\right)\frac{e^{h^2}(1-2h-2h^2)}{e^{2h^2}+(1+h)^2}}{1}$$

$$-4\frac{1}{\sqrt{2}}\sqrt{14}\frac{\pi}{4} + 4\frac{1}{\sqrt{2}}\frac{1}{2} = -\sqrt{7}\pi + \sqrt{2}$$

Mit Jacobi-Matrix:

$$\nabla f(x,y,z) = \begin{pmatrix} 4\cos z\frac{1}{1+\left(ye^{-x^2}\right)^2}\cdot ye^{-x^2}\cdot -2x = -8\cos z\cdot xy\frac{e^{-x^2}}{e^{-2x^2}\left(e^{2x^2}+y^2\right)} = -\frac{8\cos z\cdot xy\cdot e^{x^2}}{e^{2x^2}+y^2} \\ 4\cos z\frac{1}{1+\left(ye^{-x^2}\right)^2}\cdot e^{-x^2} = 4\cos z\frac{e^{-x^2}}{e^{-2x^2}\left(e^{2x^2}+y^2\right)} = \frac{4\cos z\cdot e^{x^2}}{e^{2x^2}+y^2} \\ -4\sin z\arctan\left(ye^{-x^2}\right) \end{pmatrix}$$

$$\nabla f(x_0) = \nabla f\left(0,1,\frac{\pi}{4}\right) = \begin{pmatrix} 0 \\ \frac{\frac{4}{\sqrt{2}}}{1+1} = \frac{2}{\sqrt{2}} \\ -4\frac{1}{\sqrt{2}}\frac{\pi}{4} = -\frac{\pi}{\sqrt{2}} \end{pmatrix}$$

$$\nabla f(x_0)^\top r = \begin{pmatrix} 0 \\ \frac{2}{\sqrt{2}} \\ -\frac{\pi}{\sqrt{2}} \end{pmatrix}^\top \begin{pmatrix} 1 \\ 1 \\ \sqrt{14} \end{pmatrix} = 0 + \frac{2}{\sqrt{2}} - \frac{\pi\sqrt{14}}{\sqrt{2}} = \sqrt{2} - \pi\sqrt{7}$$

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Sei $\varepsilon > 0$ beliebig und $\delta_\varepsilon = \varepsilon$, dann gilt für $f(x,y)$ stetig:

$$|f(x,y) - f(0,0)| = \left| \frac{|x|y}{|x| + y^2} - 0 \right| \overset{|x|\neq 0}{=} \left| \frac{y}{1 + \frac{y^2}{|x|}} \right| \overset{\frac{y^2}{|x|} \geqslant 0}{\leqslant} |y| < \delta_\varepsilon = \varepsilon$$

Sei $r = (r_x, r_y)^\top$ eine beliebige Richtung:

$$\begin{aligned}
\frac{\partial}{\partial r} f(x_0) &= \lim_{h \to 0} \frac{f(x_0 + hr) - f(x_0)}{h} = \lim_{h \to 0} \frac{f(0 + hr_x, 0 + hr_y) - 0}{h} \\
&= \lim_{h \to 0} \frac{\frac{|hr_x| hr_y}{|hr_x| + (hr_y)^2}}{h} = \lim_{h \to 0} \frac{|hr_x| r_y}{|hr_x| + h^2 r_y^2} = \lim_{h \to 0} \frac{|hr_x| r_y}{|hr_x|(1 + \underbrace{\frac{h^2 r_y^2}{|hr_x|}}_{\overset{h}{\to} 0})} = r_y
\end{aligned}$$

Die Funktion ist nicht total differenzierbar da die Jacobi-Matrix nicht existiert, denn:

$$\frac{\partial}{\partial x} \frac{|x|y}{|x| + y^2} \quad \Rightarrow \quad \text{nicht diff'bar da } |x| \text{ nicht diff'bar in } x = 0 \text{ ist}$$

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Sei $r = (r_x, r_y)^\top$ eine beliebige Richtung:

$$\begin{aligned}
\frac{\partial}{\partial r} f(x_0) &= \lim_{h \to 0} \frac{f(x_0 + hr) - f(x_0)}{h} = \lim_{h \to 0} \frac{f(0 + hr_x, 0 + hr_y) - 0}{h} \\
&= \lim_{h \to 0} \frac{\frac{hr_y \sin(h^2 r_x r_y)}{h^2 r_x^2 + h^4 r_y^4}}{h} = \lim_{h \to 0} \frac{r_y \sin(h^2 r_x r_y)}{h^2 r_x^2 + h^4 r_y^4} \overset{\text{l'H}}{=} \lim_{h \to 0} \frac{r_y \cos(h^2 r_x r_y) 2h r_x r_y}{2h r_x^2 + 4h^3 r_y^4} \\
&= \lim_{h \to 0} \frac{r_x r_y^2 \overbrace{\cos(h^2 r_x r_y)}^{\overset{h}{\to} 1}}{r_x^2 + \underbrace{2h^2 r_y^4}_{\overset{h}{\to} 0}} = \frac{r_y^2}{r_x}
\end{aligned}$$