Chapter 3

Spillover Effects in Spatial Models*

In this chapter we present and discuss various types of spatial spillovers. The usual interpretation of β as the effect of a one unit change in x on y, *ceteris paribus*, is no longer valid in spatial models when ρ_1 in (3.1.1) below is different from zero. In the next section we define the spatial effects emanating from a given unit. Specifically, we specify three types of effects: a direct effect, an emanating effect, and an own spillover effect. In Section 3.2 we look at these effects from a different angle, and we define a vulnerability index. This index relates to the sensitivity of a unit to changes in its environment.

3.1 EFFECTS EMANATING FROM A GIVEN UNIT

With evident notation, consider the model

$$y = X\beta + \rho_1 W y + u \tag{3.1.1}$$

where X is matrix of only exogenous variables, and $E(u) = 0$. Since in this chapter we do not consider issues of parameter estimation, we do not focus on particular properties of the error term such as spatial correlation.[1]

The solution of the model in (3.1.1) for the dependent variable y is

$$y = (I_N - \rho_1 W)^{-1}[X\beta + u] \tag{3.1.2}$$

and so

$$E(y) = (I_N - \rho_1 W)^{-1} X\beta. \tag{3.1.3}$$

$*$ As one might expect, the specification, estimation, and indexing of spatial spillovers are of prime importance in spatial modeling. Some references which relate to the material in this chapter are Easterly and Levine (1998), Fujita et al. (1999), Persson and Tabellini (2009), LeSage and Fischer (2008), LeSage and Pace (2009), Autant-Bernard and LeSage (2011), Kim et al. (2003), Abreu et al. (2005), and Halleck Vega and Elhorst (2015).

1. As an aside, the particular pattern of spatial correlation of the error term does not affect the interpretation of β. Therefore, the fact that we are not considering any particular structure for the error term should not be viewed as a limitation in the present context.

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We will now consider interpretations of the model that are based on (3.1.2) and (3.1.3). For ease of presentation, at first suppose X is an $N \times 1$ vector so that there is only one exogenous variable. Let x_1 be the first element of X. The extension of our analysis to the case in which X is an $N \times k$ matrix is considered below.

If there were no spatial effects in the sense that $\rho_1 = 0$, the effect of a one unit change in x_1 on either y_1 or on its mean, $E(y_1)$, would be β, which in this particular case would be a scalar. In the context of a spatial model, this effect can be thought of as a **direct effect** of x_1 on y_1 in the sense that it does not account for spatial spillovers which would take place if $\rho_1 \neq 0$. Clearly, if $\rho_1 = 0$ a change in x_1 would not have an effect on any of the other values of y_j, $j = 2, ..., N$.

Consider now the case in which $\rho_1 \neq 0$. Let

$$G = (I_N - \rho_1 W)^{-1} \tag{3.1.4}$$

and let G_{ji} be the (j, i)th element of G. From (3.1.3) it should be clear that the expected effect of a change in x_1 on all of the other elements of $E(y)$ can be expressed as

$$\frac{\partial E(y_j)}{\partial x_1} = G_{j1}\beta, \ \ j = 2, ..., N. \tag{3.1.5}$$

Two things should be noted here. For most weighting matrices, the effects described in (3.1.5) would not all be the same. The second remark is less obvious but, nonetheless, important. Suppose units 1 and j are not neighbors and so $w_{1j} = 0$. Despite this, for most weighting matrices it will still be the case that $\frac{\partial E(y_j)}{\partial x_1} \neq 0$. On an intuitive level, the reason for this is that, e.g., unit 1 may be a neighbor to some units which are neighbors to the jth unit. Therefore, a change in x_1 would be expected to effect its neighboring units, which in turn would effect the jth unit.

The effects described in (3.1.5) have been referred to in the literature as **emanating effects** (see, e.g., Hondroyiannis et al., 2009).[2] These effects describe how a change in a regressor relating to a given unit, in our illustration unit 1, "fan out" to all the units. This is illustrated in Fig. 3.1.1.

Of course, these emanating effects can also be described in terms of elasticities. Using evident notation, the elasticity corresponding to (3.1.5) can be expressed as

$$\eta_{j1} = G_{j1}\beta\frac{x_1}{y_j}, \ \ j = 2, ..., N. \tag{3.1.6}$$

2. Other authors refer to the same issue as indirect effects (e.g., LeSage and Pace, 2009).

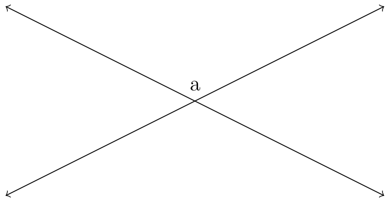

FIGURE 3.1.1 The emanating effect

A final comment may be evident. If N is large, one would not typically give tables describing the emanating effect of each unit on all the other units. In such cases a researcher would typically describe a selection of emanating effects that may be of particular importance, or selected measures (e.g., averages) of effects.

A closely related concept is that of an "**own spillover effect**." This relates to a situation where a change in a regressor (e.g., corresponding to unit one) results in a change in the dependent variables corresponding to the other units. In turn, those changes in other units "feed back" via (3.1.3) to the dependent variable in unit one. We indicated above that in the absence of spillovers, the effect of a change in x_1 on $E(y_1)$ is just β. In the presence of spillovers, that effect is

$$\frac{\partial E(y_1)}{\partial x_1} = G_{11}\beta. \tag{3.1.7}$$

Therefore, a measure of the effect of a change in x_1 on y_1, above and beyond the direct effect of x_1 on y_1, called the "**own spillover effect**", is from (3.1.5) and (3.1.7) given by

$$(G_{11} - 1)\beta. \tag{3.1.8}$$

Of course, this effect can also be expressed in terms of elasticities.

In passing we note that the above discussion relates to the effect of a change in a single regressor corresponding to a given unit, i.e., the first unit. Of course, one can also calculate the effects on each given unit of a change in a subset of the regressors corresponding to that given unit, as well as to one or more of the other units. For example, consider the model in (3.1.1) and assume now that X is an $N \times k$ matrix. Then, the change in $E(y)$ as given in (3.1.3) in response to a change in a subset of the variables in X is

$$\Delta E(y) = (I_N - \rho_1 W)^{-1}[X^H - X^0]\beta \tag{3.1.9}$$

where X^H is the hypothesized value of the regressor matrix in response to a change in one or more of the elements of X, and X^0 is the value of the regressor matrix, X, before the change. One application of this is given in

Kelejian et al. (2013) in their study relating to spillovers which are involved in institutional development, i.e., how institutions in one country are influenced by corresponding levels in neighboring countries. Another study dealing with similar issues is Seldadyo et al. (2010).

Illustration 3.1.1: The murder rate: Own spillover effects

Let us consider again a slightly modified version of the murder rate model in Illustration 2.2.1.2, where we add a spatial lag of the dependent variable, but we delete the spatial lag of the execution variable, *wexec*. We are going to use this model to compute the own spillover effects as described in (3.1.8). The first step towards the calculation of these effects consists in estimating the model to obtain an estimate of the coefficients. Recall that the spatial lag of the dependent variable is endogenous. Using two stages least squares, the estimate of ρ_1 turns out to be 0.273, while the slope corresponding to the execution variable turns out to be 0.121. The second step is using the estimate of ρ_1 to estimate $G = (I_N - \rho_1 W)^{-1}$. The own spillover effect of a change in prisoners executions on the murder rate for Alabama (i.e., the first observation in the data) turns out to be 0.0020. The average of the own spillover effects over all the 49 US States is 0.0023, which is quite similar to that for Alabama. In this case these own spillover effects are quite small, but still positive.

3.2 EMANATING EFFECTS OF A UNIFORM WORSENING OF FUNDAMENTALS

In their study, Kelejian et al. (2006) used a spatial model to study contagion problems in foreign exchange markets. In this scenario, the currency of a given country experiences a "run" and it depreciates due to speculative activity. Depreciations then fan out to other related countries in the system. Hondroyiannis et al. (2009) considered a variant of the emanating effects described above. In particular, instead of calculating the effect of a given variable (or a set of variables) in one country on the other countries, they considered a "uniform worsening" of the exogenous variables of that originating country on the other countries involved.

In more detail, let X be an $N \times k$ regressor matrix, and let X_i be its ith column, e.g., X_i is the $N \times 1$ vector of N values of the ith regressor for the N countries. Write X as $X = (X_1, ..., X_k)$. Given this notation, express the conditional mean in (3.1.3) as, using evident notation,

$$\begin{aligned} E(y) &= (I - \rho_1 W)^{-1} X\beta \\ &= G[X_1\beta_1 + ... + X_k\beta_k] \end{aligned} \tag{3.2.1}$$

where $\beta' = (\beta_1, ..., \beta_k)$. Suppose that high values of the dependent variable are associated with more severe exchange problems. In this case if the coefficient of a regressor is negative, a worsening of the corresponding variable in the originating country would relate to a decrease in that variable, i.e., the change in that variable multiplied by the corresponding coefficient is positive. Similarly, if a coefficient of a regressor is positive, a worsening of that variable would be an increase in the value of the corresponding regressor and so again the product would be positive.

Let the first value of X_i be $x_{1,i}$, $i = 1, ..., k$, and, in order to avoid unnecessary tediousness, assume that the values of all of the regressors are positive. The implication of the above is that the response of $E(y_j)$, $j = 2, ..., N$ with respect to a worsening of all k of the regressors of country 1 would be

$$\begin{aligned}\Delta E(y_j) &= G_{j1}(\Delta x_{1,1}\beta_1) + ... + G_{j1}(\Delta x_{1,k}\beta_k) \\ &= G_{j1}|\Delta x_{1,1}\beta_1| + ... + G_{j1}|\Delta x_{1,k}\beta_k|, \;\; j = 2, ..., N\end{aligned} \tag{3.2.2}$$

or, if a uniform **percentage** worsening is considered,

$$\begin{aligned}\Delta E(y_j) &= G_{j1}\frac{|\Delta x_{1,1}\beta_1|x_{1,1}}{x_{1,1}} + ... + G_{j1}\frac{|\Delta x_{1,k}\beta_k|x_{1,k}}{x_{1,k}} \\ &= G_{j1}|\beta_1|\, x_{1,1}\alpha + ... + G_{j1}|\beta_k|\, x_{1,k}\alpha\end{aligned} \tag{3.2.3}$$

where $\alpha = |\Delta x_{1,i}|/x_{1,i} > 0$ is the uniform percentage worsening, $i = 1, ..., k$. Similarly, one can calculate the emanating percentage change of $E(y_j)$ with respect to the uniform percentage worsening of all of the regressors in the originating country as

$$\frac{\Delta E(y_j)}{\alpha y_j} = G_{j1}\left[\frac{x_{1,1}}{y_j}\,|\beta_1| + ... + \frac{x_{1,k}}{y_j}\,|\beta_k|\right]\alpha. \tag{3.2.4}$$

Of course, an alternative to (3.2.4) would be to replace y_j in the denominator of (3.2.4) by $E(y_j)$ which can be estimated from (3.1.5). Incidentally, we note that this sort of analysis can easily be applied to other models, such as spatial labor market models, housing market models, and government quality indices models, just to name a few.[3] As an aside, we note that these emanating effects have been generalized to a dynamic panel framework in Kelejian and Mukerji (2011). Additionally, they have also been considered by LeSage and Fischer (2008), and LeSage and Pace (2009).

3. See LeSage and Pace (2009), or Bivand and Piras (2015) for details relating to a Monte Carlo approach to some of these issues.

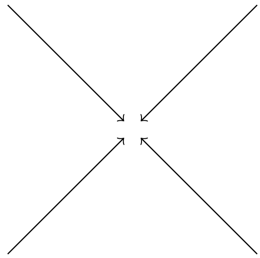

FIGURE 3.3.1 The vulnerability effect

3.3 VULNERABILITY OF A GIVEN UNIT TO SPILLOVERS

The emanating effect describes how events in one unit spill over to other units due to spatial interactions. As an example, the emanating effect would correspond to the effect that the smoking habits of a teenager might have on the smoking habits of his/her peers. Sometimes it is interesting to consider the reverse situation that we call "vulnerability." The vulnerability effect is the reverse of the emanating effect in that it describes the response of a given unit to events in neighboring units. In the teenager smoking example, the issue would be how the smoking habits of a given teenager are affected by the smoking habits of his/her peers. This index was introduced by Kelejian and Mukerji (2011), and is illustrated by Fig. 3.3.1.

More formally consider again the model in (3.1.1)

$$y = X\beta + \rho_1 W y + u$$

where X is an $N \times 1$ vector whose ith element is x_i, $G = (I_N - \rho_1 W)^{-1}$, and $E(y) = GX\beta$. Again note that in this case β is a scalar. For ease of interpretation, suppose the units are countries, X is a vector of GDPs, and y is a vector of exports. For purposes of illustration, suppose $\beta > 0$.

It should be clear that the expected effect of a change in the GDPs in all countries other than the first, namely changes in $x_2, ..., x_N$, on the exports of the first country is V_1 where

$$\begin{aligned} V_1 &= \sum_{j=2}^{N} \frac{\partial E(y_1)}{\partial x_j} \qquad (3.3.1) \\ &= \sum_{j=2}^{N} G_{1j}\beta \end{aligned}$$

The index V_1 is an example of the vulnerability effect. It illustrates the extent of vulnerability of country's exports to changes in the GDPs of all other countries.

The discrete counterpart to (3.3.1) is

$$\Delta y_1 = \sum_{j=2}^{N} G_{1j} \Delta x_j \beta$$

Consider the vulnerability of country ones exports to a uniform worsening (decrease) in the GDPs of the other countries. In this case $\Delta x_j < 0$, for all $j = 2, ..., N$ and since $\beta > 0$, $\Delta x_j \beta < 0$. Therefore the vulnerability of the first country's exports in this case would be

$$\begin{aligned} y_1 &= \sum_{j=2}^{N} G_{1j} \Delta x_j \beta \\ &= \sum_{j=2}^{N} G_{1j} \frac{\Delta x_j}{x_j} \beta x_j \\ &= \sum_{j=2}^{N} G_{1j} \beta x_j \alpha \end{aligned}$$

where $\alpha = \frac{\Delta x_j}{x_j} < 0$.

Illustration 3.3.1: The murder rate: The vulnerability effect

In this illustration we calculate the vulnerability effect based on the results which were obtained for the model that was used in Illustration 3.1.1. The vulnerability of the murder rate in Alabama to changes in executions in all of the other US States turns out to be 0.0435. This figure is only given for illustrative purposes concerning the calculations involved because the estimated coefficient of the execution variable, upon which it is based, is not significant. Therefore, one would not conclude that the murder rate in Alabama would increase in response to an increase in executions in other states. All of this suggests that the model's specification may not be correct. One possible specification error is that the execution variable was treated as exogenous in the estimation of the model, but it may, in fact, be endogenous. This endogeneity may arise because of a simultaneous relationship between these two variables: executions may indeed have a negative effect on the murder rate, but the crime of murder is, sometimes, punishable by execution. That suggests that if one were to explain the number of executions, one of the explanatory variables would be the number of murders, and its coefficient would be expected to be positive. The proper estimation of models some of whose explanatory variables are endogenous is explained in Chapter 6.

An Illustration of the Vulnerability Effect

Again, assume that high values of the dependent variable are associated with more severe problems, and suppose now that X is an $N \times 2$ matrix and let the jth row of X be $x_{j.}$. Let the two variables of X be a credit variable, $Credit$, and a national debt variable, $NatD$, so, that using evident notation, $x_{j.} = (Credit_j, NatD_j)$ and $\beta_0' = (\beta_{CR}, \beta_{ND})$. Let $\Delta Credit_j$ and $\Delta NatD_j$ be the hypothesized changes in the credit and national debt variables. Then, the vulnerability effect index with respect to country one can be expressed as

$$\Delta y_1 = \sum_{j=2}^{N} G_{1j}[\Delta Credit_j \beta_{CR} + \Delta NatD_j \beta_{ND}] \tag{3.3.2}$$

or, in percentage terms,

$$\upsilon_{y_1} = \sum_{j=2}^{N} G_{1j}[\upsilon_{j,CR}\frac{Credit_j}{y_1}\beta_{CR} + \upsilon_{j,ND}\frac{NatD_j}{y_1}\beta_{ND}]$$

where

$$\begin{aligned} \upsilon_{j,CR} &= 100\frac{\Delta Credit_j}{Credit_j}, \quad \upsilon_{ND} = 100\frac{\Delta NatD_j}{NatD_j}, \\ \upsilon_{y_1} &= 100\frac{\Delta y_1}{y_1}. \end{aligned} \tag{3.3.3}$$

Clearly, it could be of interest to apply this index to markets of labor, housing, exchange, etc.

SUGGESTED PROBLEMS

1. Consider the model

$$(P.1) \quad y = X\beta + \rho_1 W_1 y + \rho_2 W_2 y + u$$

where W_1 and W_2 are weighting matrices. Assuming typical conditions, give expressions for the emanating and vulnerability effects that relate to unit 1.

2. Consider the model

$$\begin{aligned}(P.2) \quad y_1 &= X_1\beta_1 + \lambda_1 W_1 y_1 + y_2\alpha_1 + u_1,\\ y_2 &= X_2\beta_2 + y_1\alpha_2 + u_2\end{aligned}$$

where y_1 and y_2 are $N \times 1$ vectors on dependent variables (e.g., wages and productivity), and the remaining notation should be evident. To simplify notation assume that X_1 and X_2 are $N \times 1$ vectors. Give expressions for the emanating and vulnerability effects as they relate to the first unit of y_1.

3. Consider the model

$$(P.3) \quad y = X\beta_1 + WX\beta_2 + u$$

where the notation should be evident. Describe the emanating and vulnerability effects for this model as they relate to unit 1. Again, assume that X is an $N \times 1$ vector.

4. Consider the model $(P.1)$ above. Suppose X is an $N \times k$ matrix, and large values of the elements of y are not desired. Give and expression for the emanating effect of a uniform percentage worsening of the exogenous variables in unit 1.