exercise1 (Score: 10.0 / 20.0)

1. Task (Score: 4.0 / 4.0)
2. Test cell (Score: 2.0 / 2.0)
3. Test cell (Score: 0.0 / 4.0)
4. Test cell (Score: 2.0 / 2.0)
5. Task (Score: 2.0 / 4.0)
6. Task (Score: 0.0 / 4.0)

# Lab 4

1. 提交作業之前，建議可以先點選上方工具列的**Kernel**，再選擇**Restart & Run All**，檢查一下是否程式跑起來都沒有問題，最後記得儲存。
2. 請先填上下方的姓名(name)及學號(stduent_id)再開始作答，例如：

```
name = "我的名字"
student_id= "B06201000"
```

3. 演算法的實作可以參考[lab-4 (https://yuanyuyuan.github.io/itcm/lab-4.html)](https://yuanyuyuan.github.io/itcm/lab-4.html), 有任何問題歡迎找助教詢問。
4. **Deadline: 11/20(Wed.)**

In [1]:

```python
name = "鄭如芳"
student_id = "B05602020"
```

---

# Exercise 1. Finite Difference

---

## Part 0.

#### Import necessary libraries. Note that diags library from scipy is used to construct the differentiation matrix below.

In [2]:

```python
import numpy as np
import matplotlib.pyplot as plt
from scipy.sparse import diags
```

## Part 1.

**Given a function $u(x)$ which we want to find its derivative with numerical methods.**

**Consider a uniform grid partitioning $x$ into $\{x_1, x_2, \dots, x_n\}$ with grid size $\Delta x = x_{j+1} - x_j, j \in \{1, 2, \dots, n\}$, and a set of corresponding data values $U = \{U_1, U_2, \dots, U_n\}$, where**

$$U_{j+k} = u(x_j + k\Delta x) = u(x_{j+k}), j \in \{1, 2, \dots, n\}.$$

**We want to use one-sided finite-difference formula**

$$\alpha_1 U_j + \alpha_2 U_{j+1} + \alpha_3 U_{j+2}$$

**to approximate the derivative of $u$ at all the points $x_j, j \in \{1, 2, \dots, n\}$, that is**

$$u'(x_j) \approx W_j \triangleq \alpha_1 U_j + \alpha_2 U_{j+1} + \alpha_3 U_{j+2}.$$

(Top)

**Part 1.1**

Find the coefficients $\alpha_j$ for $j = 1, 2, 3$ which make the stencil above accurate for as high degree polynomials as possible.

Write down your derivation in detail with Markdown/LaTeX.

$$By\,Taylor\,expansion U_j = u(x_j) U_j + 1 = u(x_j + \Delta x) \approx u(x_j) + \Delta x u'(x_j) + \frac{(\Delta x)^2}{2} u''(x) U_j + 2 = u(x_j + 2\Delta x) \approx u(x_j) + 2\Delta x u'(x_j) + \frac{4\Delta x^2)}{2} u''(x_j) Thus\, \alpha_1 = \frac{-3}{2\Delta x}$$

**Part 1.2**

Fill in the tuple variable `alpha` of lenght 3 with your answer above. (Suppose $\Delta x = 1$)

In [3]:

(Top)

```
# Hint: alpha = [value of alpha_1, value of alpha_2, value of alpha_3]
# ===== 請實做程式 =====
alpha=[-3/2,2,-1/2]
# ====================
```

In [4]:

cell-e7c9469885bebc80 (Top)

```
print('My alpha =', alpha)
### BEGIN HIDDEN TESTS
assert alpha == [-1.5, 2, -0.5] or alpha == (-1.5, 2, -0.5)
### END HIDDEN TESTS
```

```
My alpha = [-1.5, 2, -0.5]
```

## Part 2.

**Suppose we use the finite-difference formula above to approximate and assume the problem is periodic, i.e. take $U_0 = U_n$, $U_1 = U_{n+1}$, and so on.**

**Find the differentiation matrix $D$ so that the numerical differentiation problem can be represented as a matrix-vector multiplication $W \triangleq DU$, where $D \in \mathbb{R}^{n \times n}$, $U \in \mathbb{R}^n$, and $W \in \mathbb{R}^n$.**

---

### Part 2.1

Complete the following function to construct the desired differentiation matrix under the **periodic boundary condition** with given number of partition $n$, coefficients of 3-point finite-difference formula $\alpha$, and mesh size $\Delta x$.

In [5]:

(Top)

```
def construct_differentiation_matrix(n, alpha, delta_x):
    ''' Construct
    Parameters
    ----------
    n : int
        number of partition
    alpha : tuple of length 3
       alpha = (α1, α2, α3)
    delta_x : float
        mesh size

    Returns
    -------
    D : scipy.sparse.diags
    '''
    # ===== 請實做程式 =====
    diagonals =
        [alpha[0] * np.ones(n),
         alpha[1] * np.ones(n-1),
         alpha[2] * np.ones(n-2),
         alpha[1] * np.ones(1),
         alpha[2] * np.ones(2)]
    D=diags(diagonals, offsets=[0, 1, 2,-n+1,-n+2])
    D/=delta_x
    '''
    I originally want to use method below
    D1=alpha[0]*np.eye(n)
    D2=alpha[1]*np.eye(n,k=1)
    D3=alpha[2]*np.eye(n,k=2)
    D4=alpha[2]*np.eye(n,k=-(n-2))
    D5=alpha[1]*np.eye(n,k=-(n-1))
    D=D1+D2+D3+D4+D5
    D=D/delta_x
    but I don't know why if I do so,
    then Part2.2 get error with
    'numpy.ndarray' object has no attribute 'toarray' lol
    '''
    # ====================
    return D
```

```
  File "<ipython-input-5-1e355d022927>", line 17
    diagonals =
               ^
SyntaxError: invalid syntax
```

### Part 2.2

Print and check your implementation.

In [6]:

cell-2ca00ba5ff115302 (Top)

```
print("For n = 8 and mesh size 1, D in dense form is")
sparse_D = construct_differentiation_matrix(8, alpha, 1)
dense_D = sparse_D.toarray()
print(dense_D)
### BEGIN HIDDEN TESTS
answer = np.array([
    [-1.5,  2.,  -0.5,  0.,   0.,   0.,   0.,   0. ],
    [ 0.,  -1.5,  2.,  -0.5,  0.,   0.,   0.,   0. ],
    [ 0.,   0.,  -1.5,  2.,  -0.5,  0.,   0.,   0. ],
    [ 0.,   0.,   0.,  -1.5,  2.,  -0.5,  0.,   0. ],
    [ 0.,   0.,   0.,   0.,  -1.5,  2.,  -0.5,  0. ],
    [ 0.,   0.,   0.,   0.,   0.,  -1.5,  2.,  -0.5],
    [-0.5,  0.,   0.,   0.,   0.,   0.,  -1.5,  2. ],
    [ 2.,  -0.5,  0.,   0.,   0.,   0.,   0.,  -1.5]
])
assert np.linalg.norm(dense_D - answer) < 1e-7
### END HIDDEN TESTS
```

```
For n = 8 and mesh size 1, D in dense form is

---------------------------------------------------------------------------
NameError                                 Traceback (most recent call last)
<ipython-input-6-54d8c64a4da6> in <module>
      1 print("For n = 8 and mesh size 1, D in dense form is")
----> 2 sparse_D = construct_differentiation_matrix(8, alpha, 1)
      3 dense_D = sparse_D.toarray()
      4 print(dense_D)
      5 ### BEGIN HIDDEN TESTS

NameError: name 'construct_differentiation_matrix' is not defined
```

## Part 3.

**Take $u(x) = e^{\sin x}$ on the domain $[-\pi, \pi]$. Find the finite difference approximation $W$ for $\{u'(x_j)\}_{j=1}^{n}$ for various values of $n = 2^k$, $k = 3, 4, \ldots, 10$, and analyze the errors.**

### Part 3.1

Define the functinos $u$ and $u'(x)$.

In [7]:

(Top)

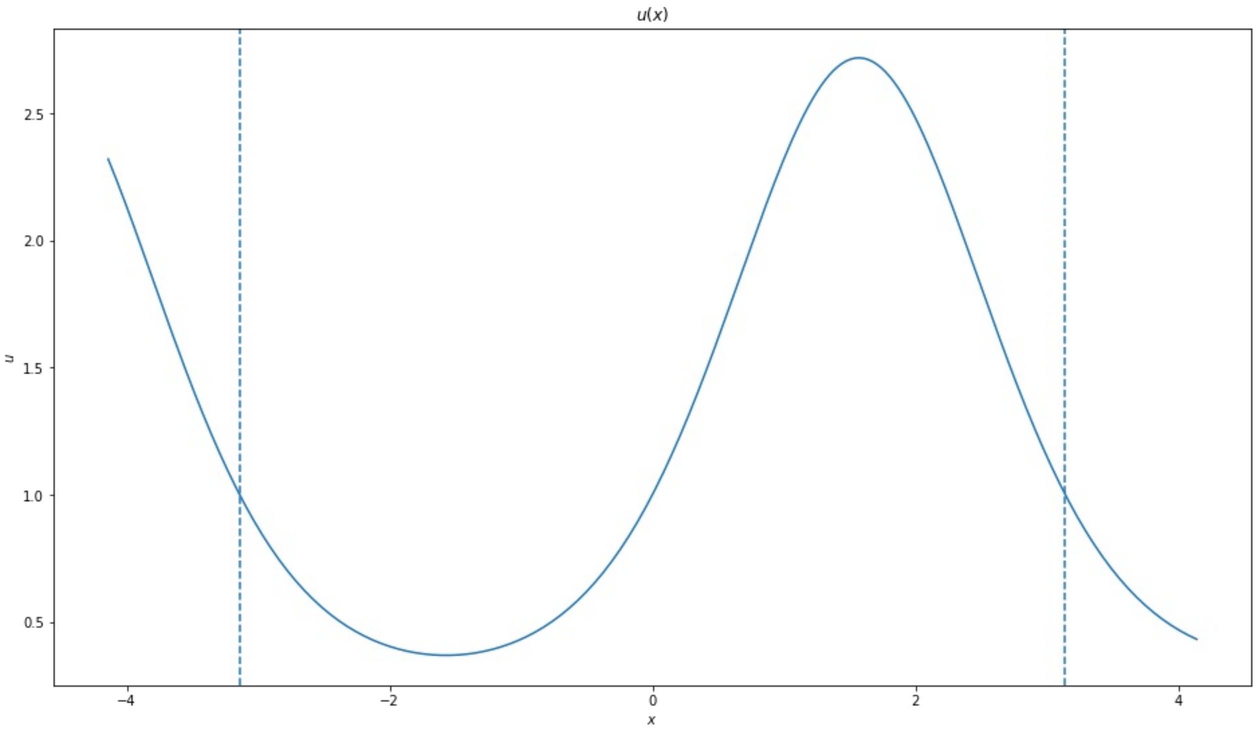
```
def u(x):
    # ===== 請實做程式 =====
    return np.exp(np.sin(x))
    # ====================

def d_u(x):
    # ===== 請實做程式 =====
    return np.cos(x)*np.exp(np.sin(x))
    # ====================
```

Plot and check the functions

In [8]:

cell-f97d6fb0842a6055 (Top)

```
x_range = np.linspace(-np.pi-1, np.pi+1, 2**8)
plt.figure(figsize=(16, 9))
plt.plot(x_range, u(x_range))
plt.axvline(x=np.pi, linestyle='--')
plt.axvline(x=-np.pi, linestyle='--')
plt.ylabel(r'$u$')
plt.xlabel(r'$x$')
plt.title(r'$u(x)$')
plt.show()
### BEGIN HIDDEN TESTS
assert u(1) == np.exp(np.sin(1))
assert u(3.14) == np.exp(np.sin(3.14))
assert d_u(1) == np.cos(1) * np.exp(np.sin(1))
assert d_u(0) == np.cos(0) * np.exp(np.sin(0))
### END HIDDEN TESTS
```




(Top)

**Part 3.2**

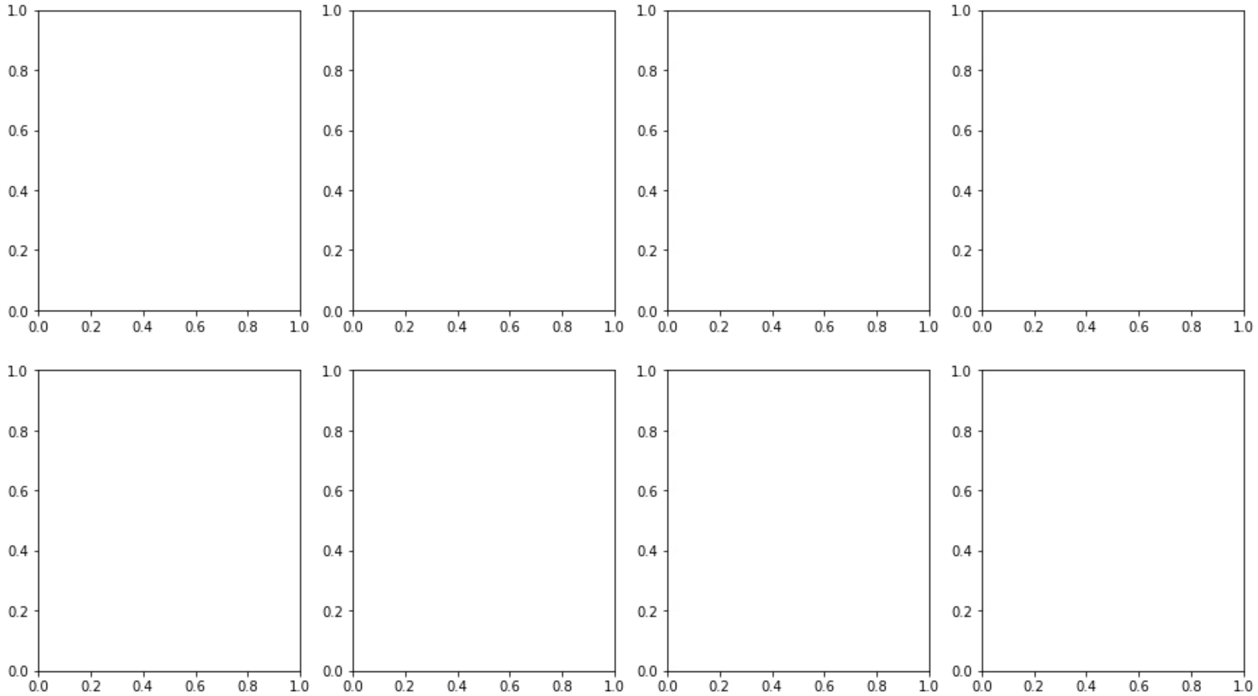
Plot the $u'$ and $W$ together for each point $x_j, j \in \{1, 2, \dots, n\}$ with $n = 2^k, k \in \{3, 4, \dots, 10\}$. Note that there're total 8 figures to be plotted. And you need to compute the error, display them in the plots, and store them into the list variable `error_list` for further analysis below.

In [9]:

(Top)

```python
error_list = []
fig, axes = plt.subplots(2, 4, figsize=(16,9))
for idx, ax in enumerate(axes.flatten()):

    '''Hints:
    For each case in this for loop, you may follow the steps below
        1. Use idx to set k and n.
        2. Prepare n partition points of the domain.
        3. Construct D.
        4. Find u', U, and W.
        5. Compute the error between u' and W.
        6. Append the error into error_list.
        7. Use ax to plot u', W with proper labels, title
        8. Enable legend to show the labels of curves.
        9. To make the plots more readable, set a consistent range of y-axis e.g. ax.set_ylim([-3, 3])
    '''
    # ===== 請實做程式 =====
    k=idx+3
    n=2**k #there are k points
    mesh_size=(2*np.pi)/n
    partition=np.linspace(-np.pi,np.pi,n)
    f_partition=[u(partition)]
    f_partition=np.transpose(f_partition)

    D = construct_differentiation_matrix(n,alpha,mesh_size)
    D = D.toarray()

    T = d_u(partition)         #Ture
    W = np.dot(D,f_partition)  #estimated

    ax.plot(partition, W,label='Estimate')
    ax.plot(partition, T,label='True')
    ax.set_title("k="+str(k))
    ax.set_ylim([-3, 3])
    ax.legend()
    # ====================
```

```
---------------------------------------------------------------------------
NameError                                 Traceback (most recent call last)
<ipython-input-9-050e23152ba4> in <module>
     23     f_partition=np.transpose(f_partition)
     24 
---> 25     D = construct_differentiation_matrix(n,alpha,mesh_size)
     26     D = D.toarray()
     27 

NameError: name 'construct_differentiation_matrix' is not defined
```



Plot the `error_list` with respect to $k = 3, 4, \ldots, 10$ in log scale to show the error behavior.

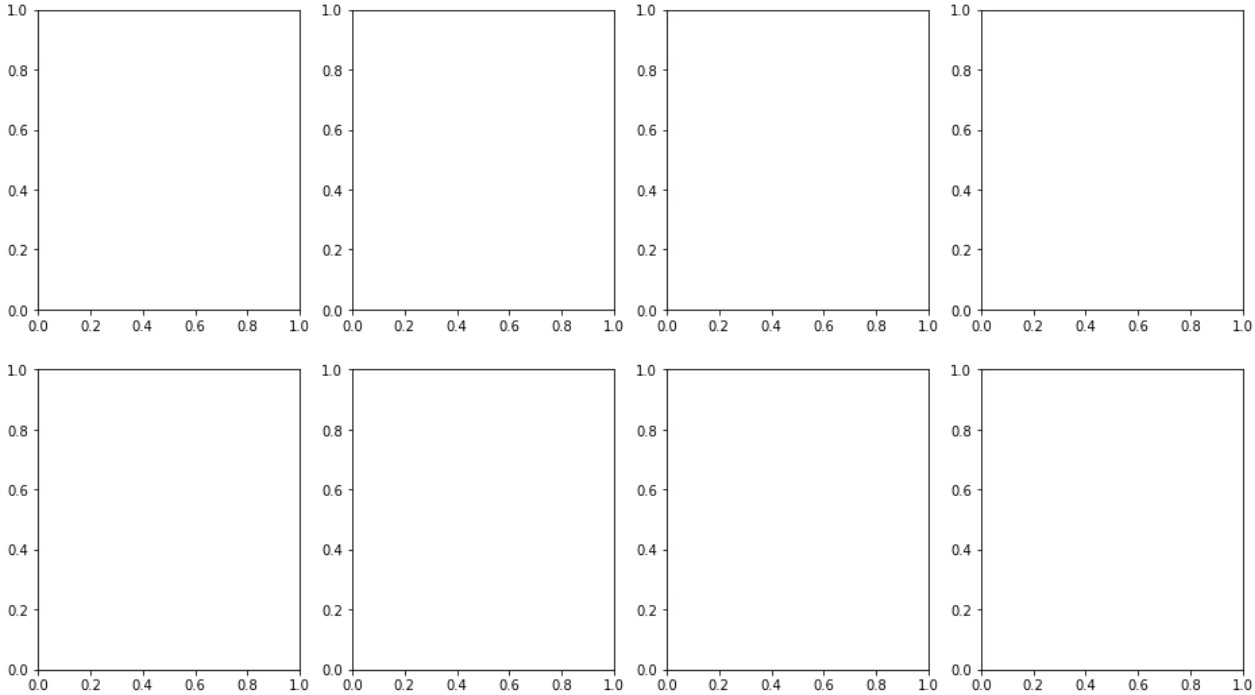
In [10]:

(Top)

```
fig, axes = plt.subplots(2, 4, figsize=(16,9))
'''
I tried
T = [T[i] for i in range(0,len(T))]
W = [W[i][0] for i in range(0,len(T))]
partiton = [partition[i] for i in range(0,len(T))]
error = [np.abs(T[i]-W[i]) for i in range(len(T))]
error_list.append(error)
but failed to plot this QQ
'''
```

Out[10]:

```
'\nI tried  \nT = [T[i] for i in range(0,len(T))]\nW = [W[i][0] for i in range(0,len(T))]\npa
rtiton = [partition[i] for i in range(0,len(T))]\nerror = [np.abs(T[i]-W[i]) for i in range(l
en(T))]\nerror_list.append(error)\nbut failed to plot this QQ\n'
```




(Top)

**Part 3.3**

From the figure above, what rates of convergence do you observe as $\Delta x \to 0$?

Please write down your answer here.