exercise1 (Score: 17.0 / 17.0)

1. Test cell (Score: 1.0 / 1.0)
2. Task (Score: 5.0 / 5.0)
3. Test cell (Score: 1.0 / 1.0)
4. Task (Score: 2.0 / 2.0)
5. Task (Score: 5.0 / 5.0)
6. Test cell (Score: 1.0 / 1.0)
7. Task (Score: 2.0 / 2.0)

Lab 3

1. 提交作業之前，建議可以先點選上方工具列的**Kernel**，再選擇**Restart & Run All**，檢查一下是否程式跑起來都沒有問題，最後記得儲存。
2. 請先填上下方的姓名(name)及學號(stduent_id)再開始作答，例如：

```
name = "我的名字"
student_id= "B06201000"
```

3. 演算法的實作可以參考[lab-3 (https://yuanyuyuan.github.io/itcm/lab-3.html)](https://yuanyuyuan.github.io/itcm/lab-3.html), 有任何問題歡迎找助教詢問。
4. **Deadline: 10/30(Wed.)**

In [1]:

```
name = "鄭如芳"
student_id = "B05602020"
```

Exercise 1

Let $g(x) = \ln(4 + x - x^2)$ and α is a fixed point of $g(x)$ i.e. $\alpha = g(\alpha)$.

- ### Part A. Implement your fixed-point algorithm and solve it with initial guess $x_0 = 2$ within tolerance 10^{-10}, and answer the questions of error behavior analysis below.
- ### Part B. Redo Part A. by applying Aitken's acceleration.

Import libraries

In [2]:

```
import numpy as np
import matplotlib.pyplot as plt
```

Implement the target function $g(x) = \ln(4 + x - x^2)$

In [3]:

(Top)

```
def g(x):
    # ===== 請實做程式 =====
    return np.log(4+x-x*x)
    # ===================
```

In [4]:

```
        cell-c0f08330aec65e17                                           (Top)

assert round(g(0), 4) == 1.3863
### BEGIN HIDDEN TESTS
import random
x = random.random()
assert g(x) == np.log(4 + x - x**2), 'Failed on x = %f' % x
### END HIDDEN TESTS
```

Run built-in [fixed-point method (https://docs.scipy.org/doc/scipy/reference/generated/scipy.optimize.fixed_point.html#rf(1)](https://docs.scipy.org/doc/scipy/reference/generated/scipy.optimize.fixed_point.html) with Python SciPy, and use this accurate value as the fixed point α

In [5]:

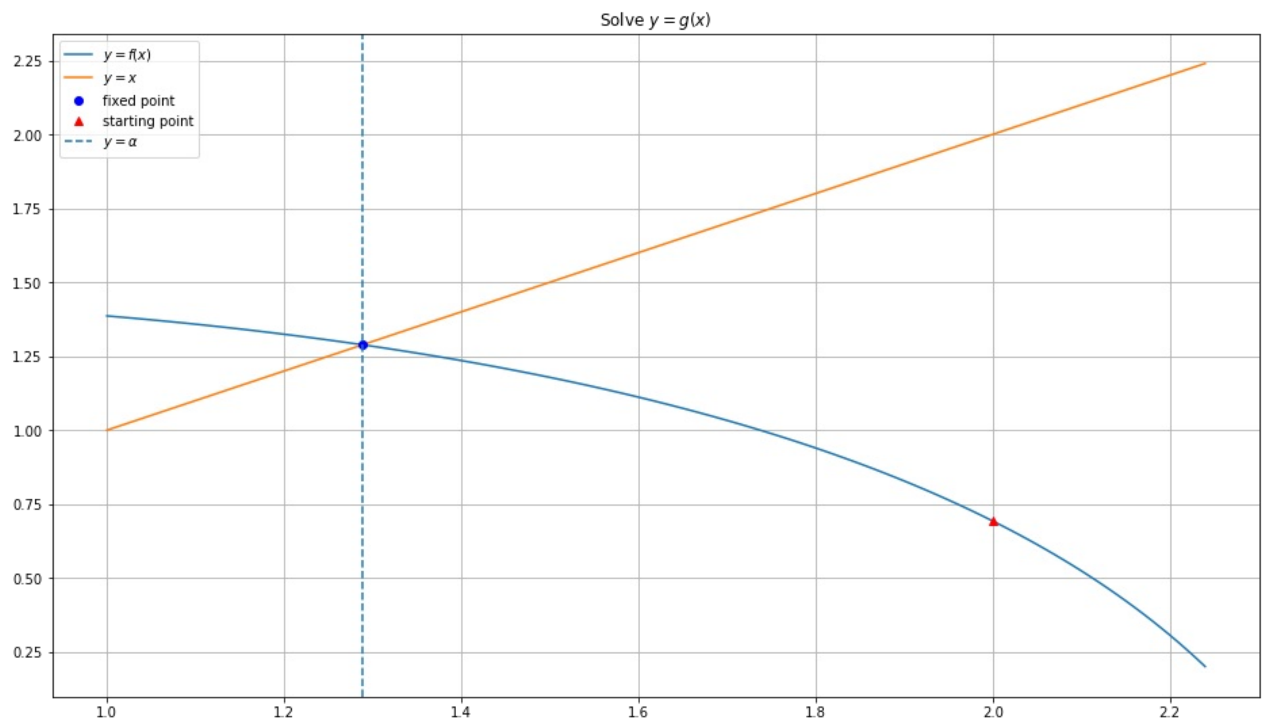
```
from scipy import optimize
alpha = optimize.fixed_point(g, x0=2, xtol=1e-12)
print('The fixed point is', alpha)
```

```
The fixed point is 1.2886779668238684
```

Visualization

In [6]:

```
x_range = np.arange(1, 2.25, 0.01)
plt.figure(figsize=(16, 9))
plt.title(r'Solve $y=g(x)$')
plt.plot(x_range, g(x_range), label=r'$y=f(x)$')
plt.plot(x_range, x_range, label=r'$y=x$')
plt.plot(alpha, g(alpha), 'bo', label='fixed point')
plt.plot(2.0, g(2.0), 'r^', label='starting point')
plt.axvline(x=alpha, linestyle='--', label=r'$y=\alpha$')
plt.gca().legend()
plt.grid()
plt.show()
```



Part A.

(Top)

1. Find the fixed point of $g(x)$ using your fixed-point iteration to within tolerance 10^{-10} with initial guess $x_0 = 2$.

1-1. Implement the fixed point method

In [7]:

(Top)

```
def fixed_point(
    func,
    x_0,
    tolerance=1e-7,
    max_iterations=100,
):
    '''Find the fixed point of the given function func

    Parameters
    ----------
    func : function
        The target function.
    x_0 : float
        Initial guess point for a solution func(x)=x.
    tolerance: float
        One of the termination conditions. Error tolerance.
    max_iterations : (positive) integer
        One of the termination conditions. The amount of iterations allowed.

    Returns
    -------
    solution : float
        Approximation of the root.
    history: dict
        Return history of the solving process
        history: {'x_n': list}
    '''

    # ===== 請實做程式 =====
    x_n=x_0
    num_iterations=0
    history={'x_n':[]}

    while True:
        f_of_x_n=func(x_n)
        num_iterations+=1
        error=abs(f_of_x_n-x_n)
        history['x_n'].append(x_n)

        if error<tolerance:
            print('find solution after',num_iterations,'iterations')
            return x_n,history
        if num_iterations<max_iterations:
            print("reach the max iteration")
            x_n=f_of_x_n
        else:
            return x_n,history

    # ====================
```

1-2. Find the root

In [8]:

(Top)

```
solution, history = fixed_point(
    # ===== 請實做程式 =====
    g,
    x_0=2.0,
    tolerance=1e-10,
    max_iterations=100
    # ====================
)
```

```
reach the max iteration
reach the max iteration
reach the max iteration
reach the max iteration
reach the max iteration
reach the max iteration
reach the max iteration
reach the max iteration
reach the max iteration
reach the max iteration
reach the max iteration
reach the max iteration
reach the max iteration
reach the max iteration
reach the max iteration
reach the max iteration
reach the max iteration
reach the max iteration
reach the max iteration
reach the max iteration
reach the max iteration
reach the max iteration
reach the max iteration
reach the max iteration
reach the max iteration
reach the max iteration
reach the max iteration
reach the max iteration
find solution after 29 iterations
```

In [9]:

cell-2d72f68109ee500c (Top)

```
print('My estimation is', solution)
### BEGIN HIDDEN TESTS
assert np.round(solution, 9) == np.round(alpha, 9), 'Wrong answer!'
### END HIDDEN TESTS
```

```
My estimation is 1.2886779668876651
```

(Top)

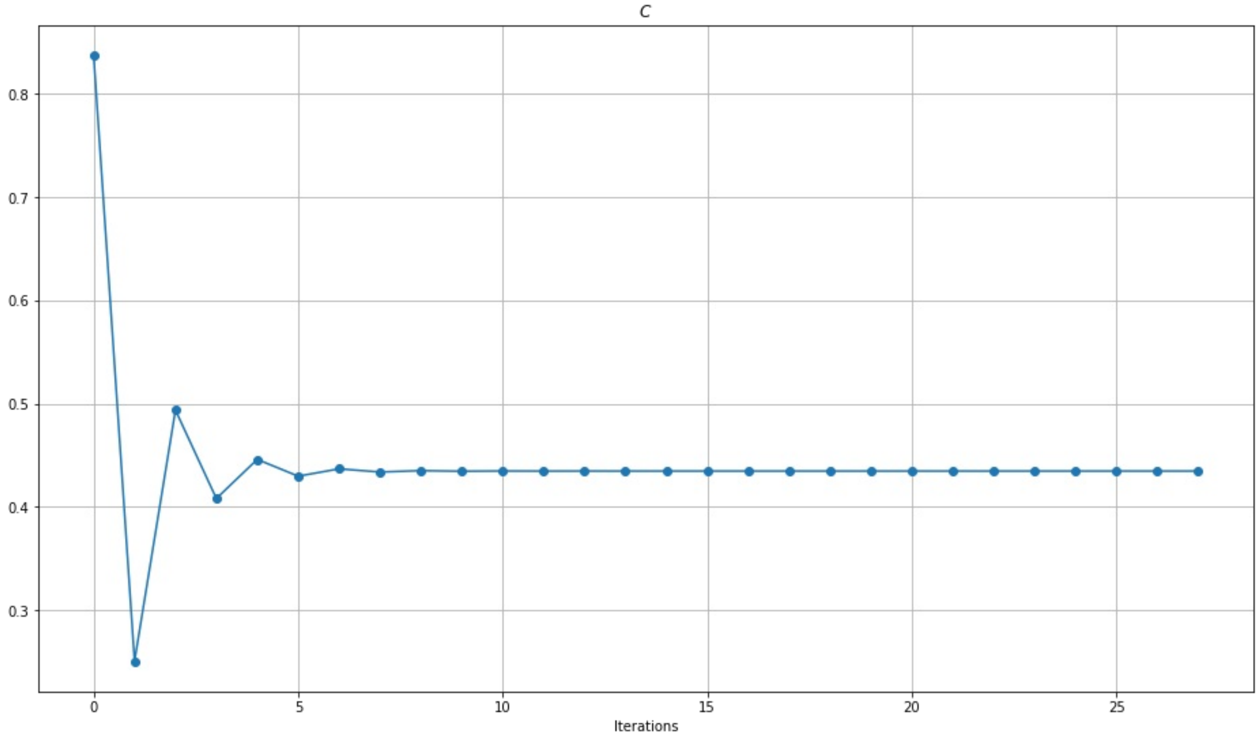
2. Estimate graphically the asymptotic error constant C

$$\lim_{n \to \infty} \frac{|x_{n+1} - \alpha|}{|x_n - \alpha|} = C$$

In [10]:

(Top)

```
'''
Hint:
    1. Prepare the sequences: x_n(from the history of algorithm)
    2. Compute the error of sequence: e_n
    3. Compute the sequence: e_{n+1}/e_{n}
    4. Plot the curve
    5. Fill in the name of x,y axes
    6. Show the plot
'''
# ===== 請實做程式 =====
x_n = history['x_n']
e_n = abs(alpha - x_n)
plt.figure(figsize=(16, 9))
plt.plot(e_n[1:] / e_n[:-1], 'o-')
plt.title(r'$C$')
plt.xlabel('Iterations')
plt.grid()
plt.show()
# ====================
```



In [11]:

```
e_n
```

Out[11]:

```
array([7.11322033e-01, 5.95530786e-01, 1.49424421e-01, 7.37759316e-02,
       3.01175170e-02, 1.34341803e-02, 5.77438806e-03, 2.52292671e-03,
       1.09455027e-03, 4.76326095e-04, 2.07010431e-04, 9.00187013e-05,
       3.91348283e-05, 1.70153888e-05, 7.39774845e-06, 3.21637197e-06,
       1.39839239e-06, 6.07985951e-07, 2.64336596e-07, 1.14926813e-07,
       4.99672326e-08, 2.17244749e-08, 9.44524570e-09, 4.10655110e-09,
       1.78542336e-09, 7.76256170e-10, 3.37496475e-10, 1.46734846e-10,
       6.37967457e-11])
```

In [12]:

```
e_n[1:]
```

Out[12]:

```
array([5.95530786e-01, 1.49424421e-01, 7.37759316e-02, 3.01175170e-02,
       1.34341803e-02, 5.77438806e-03, 2.52292671e-03, 1.09455027e-03,
       4.76326095e-04, 2.07010431e-04, 9.00187013e-05, 3.91348283e-05,
       1.70153888e-05, 7.39774845e-06, 3.21637197e-06, 1.39839239e-06,
       6.07985951e-07, 2.64336596e-07, 1.14926813e-07, 4.99672326e-08,
       2.17244749e-08, 9.44524570e-09, 4.10655110e-09, 1.78542336e-09,
       7.76256170e-10, 3.37496475e-10, 1.46734846e-10, 6.37967457e-11])
```

In [13]:

```
e_n[:-1]
```

Out[13]:

```
array([7.11322033e-01, 5.95530786e-01, 1.49424421e-01, 7.37759316e-02,
       3.01175170e-02, 1.34341803e-02, 5.77438806e-03, 2.52292671e-03,
       1.09455027e-03, 4.76326095e-04, 2.07010431e-04, 9.00187013e-05,
       3.91348283e-05, 1.70153888e-05, 7.39774845e-06, 3.21637197e-06,
       1.39839239e-06, 6.07985951e-07, 2.64336596e-07, 1.14926813e-07,
       4.99672326e-08, 2.17244749e-08, 9.44524570e-09, 4.10655110e-09,
       1.78542336e-09, 7.76256170e-10, 3.37496475e-10, 1.46734846e-10])
```

Part B.

(Top)

1. Accelerate the convergence of the sequence $\{x_n\}$ obtained in *Part A.* using Aitken's Δ^2 method, yielding sequence $\{\hat{x}_n\}$.

1-1. Introduce Aitken's acceleration into the original method.

In [14]:

(Top)

```python
def aitken(
    func,
    x_0,
    tolerance=1e-7,
    max_iterations=100,
):
    '''Approximate solution of f(x)=0 on interval [a,b] by the secant method.

    Parameters
    ----------
    func : function
        The target function.
    x_0 : float
        Initial guess point for a solution f(x)=x.
    tolerance: float
        One of the termination conditions. Error tolerance.
    max_iterations : (positive) integer
        One of the termination conditions. The amount of iterations allowed.

    Returns
    -------
    solution : float
        Approximation of the root.
    history: dict
        Return history of the solving process
        history: {'x_n': list}
    '''

    # ===== 請實做程式 =====
    x_n=x_0
    num_iterations=0
    history_hat={'x_n':[]}

    while True:
        x_1=func(x_0)
        x_2=func(x_1)
        x_n=x_2-((x_2-x_1)**2)/((x_2-x_1)-(x_1-x_0))

        num_iterations+=1
        error=abs(x_n-x_0)
        history_hat['x_n'].append(x_n)

        if error<tolerance:
            print('find solution after',num_iterations,'iterations')
            return x_n,history_hat
        if num_iterations<max_iterations:
            print("reach the max iteration")
            x_0=x_n
        else:
            return x_n,history_hat
    # ====================
```

1-2. Find the root

In [15]:

(Top)

```python
solution, history_hat = aitken(
    # ===== 請實做程式 =====
    g,
    x_0=2.0,
    tolerance=1e-7,
    max_iterations=100
    # ====================
)
```

```
reach the max iteration
reach the max iteration
reach the max iteration
reach the max iteration
find solution after 5 iterations
```

In [16]:

cell-5c862e35ba0aa7d9

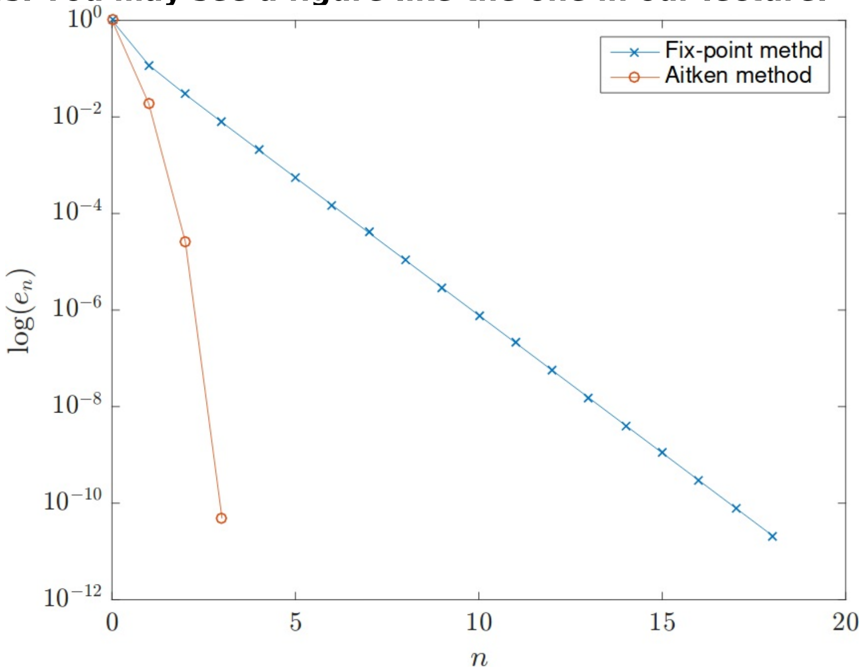
(Top)

```
print('My estimation is', solution)
### BEGIN HIDDEN TESTS
assert np.round(solution, 9) == np.round(alpha, 9), 'Wrong answer!'
### END HIDDEN TESTS
```

My estimation is 1.2886779668238684

(Top)

2. Plot the error curves of each algorithm w.r.t iterations n in log scale to compare the convergence rates. You may see a figure like the one in our lecture.



Ref. Page15 of [cmath2019_note1_aitken.pdf (https://ceiba.ntu.edu.tw/course/7a770d/content/cmath2019_note1_aitken.pdf)](https://ceiba.ntu.edu.tw/course/7a770d/content/cmath2019_note1_aitken.pdf)

In [17]:

(Top)

```python
'''
Hint:
    1. Prepare the sequences: x_n, x_n_hat(from the history of each algorithm)
    2. Compute the error of sequences: e_n, e_n_hat
    3. Plot the curves of e_n, e_n_hat respectively
    4. Change scale into log
    5. Fill in the name of x,y axes
    6. Enable legend(show curve names)
    7. Show the plot
'''
# ===== 請實做程式 =====
plt.figure(figsize=(11,7))
x_n = history['x_n']
e_n = abs(alpha - x_n)
x_n_hat=history_hat['x_n']
e_n_hat=abs(alpha-x_n_hat)
plt.plot(e_n,'-x',label='Fix-point method')
plt.plot(e_n_hat,'-o',label='Aitken method')
plt.legend(loc='upper right')
plt.ylabel('log(en)')
plt.xlabel('n')
plt.yscale('log')
plt.show()
# ====================
```