Lecture 3

Big-O notation, more recurrences!!

Announcements!

- HW1 is posted! (Due Friday)
- See Piazza for a list of HW clarifications

- First recitation section was this morning, there's another tomorrow (same material). (These are optional, it's a chance for TAs to go over more examples than we can get to in class).

FAQ

- How rigorous do I need to be on my homework?
  - See our example HW solution online
  - In general, we are shooting for:

    You should be able to give a friend your solution and they should be able to turn it into a rigorous proof without much thought.
  - This is a delicate line to walk, and there's no easy answer. Think of it like more like writing a good essay than "correctly" solving a math problem.
- What's with the array bounds in pseudocode?
  - SORRY! I'm trying to match CLRS and this causes me to make mistakes sometimes. In this class, I'm *trying* to do:
    - Arrays are 1-indexed
    - A[1..n] is all entries between 1 and n, **inclusive**
    - I will also use A[1:n] (python notation) to mean the same thing (not python notation).
  - Please call me out when I mess up.

Last time….

- Sorting: InsertionSort and MergeSort
- Analyzing correctness of iterative + recursive algs
  - Via “loop invariant” and induction
- Analyzing running time of recursive algorithms
  - By writing out a tree and adding up all the work done.

Today

- How do we measure the runtime of an algorithm?
    - Worst-case analysis
    - Asymptotic Analysis
- Recurrence relations:
    - Integer Multiplication and MergeSort again
- The “Master Method” for solving recurrences.

Recall from last time…

- We analyzed INSERTION SORT and MERGESORT.
- They were both correct!
- INSERTION SORT took time about n^2
- MERGESORT took time about n log(n).

A few reasons to be grumpy

- Sorting

1	2	3	4	5	6	7	8

should take zero steps…why nlog(n)??

- What's with this $T(\text{MERGE}) < 2 + 4n <= 6n$?

Analysis

T(n) = time to run MERGESORT on a list of size n

This is called a recurrence relation: it describes the running time of a problem of size n in terms of the running time of smaller problems.

$$T(n) = T(n/2) + T(n/2) + T(\text{MERGE}) = 2T(n/2) + 6n$$

T(MERGE two lists of size n/2) is the time to do:

- 3 variable assignments (counters ← 1)
- n comparisons
- n more assignments
- 2n counter increments

So that's

$$2T(\text{assign}) + n\,T(\text{compare}) + n\,T(\text{assign}) + 2n\,T(\text{increment})$$

or ~~4n + 2~~ operations

Or 4n + 3…

Let's say T(MERGE of size n/2) ≤ 6n operations

Plucky the pedantic penguin

Lucky the lackadaisical lemur

SLIDE FROM LAST TIME

We will see later how ... relations like these automagically...bu ... first principles.

A few reasons to be grumpy

- Sorting

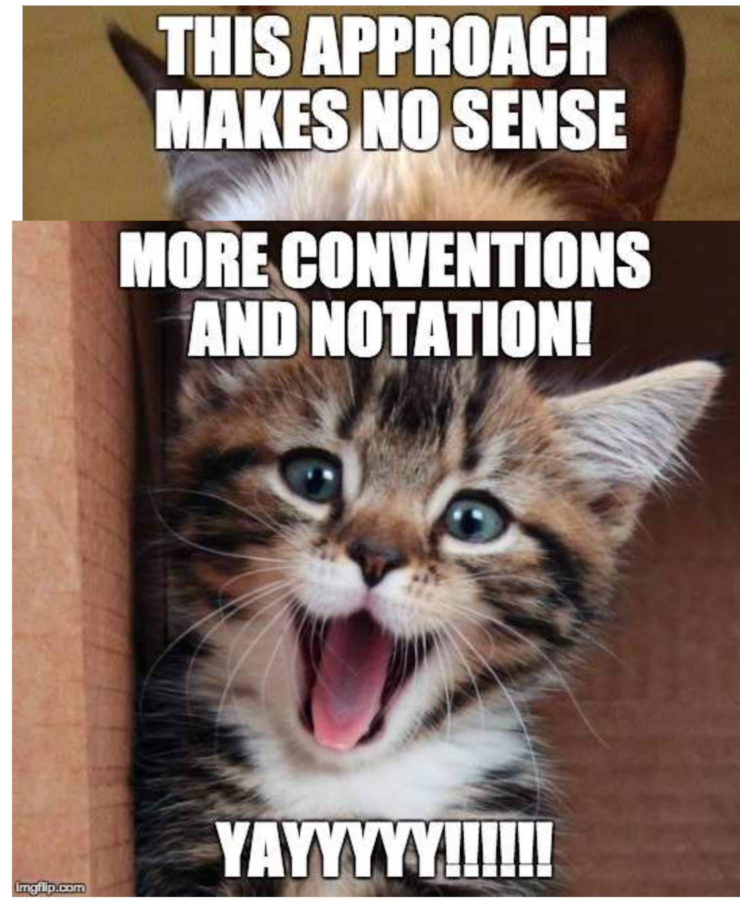
1	2	3	4	5	6	7	8

should take zero steps…why nlog(n)??

- What's with this T(MERGE) < 2 + 4n <= 6n?
  - The "2 + 4n" operations thing doesn't even make sense. Different operations take different amounts of time!
  - We bounded 2 + 4n <= 6n. I guess that's true, but that seems pretty dumb.

How we will deal with grumpiness

- Take a deep breath…
- Worst case analysis
- Asymptotic notation

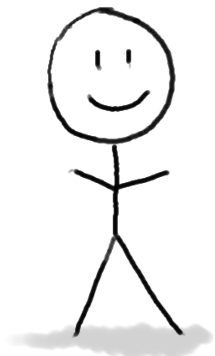


Worst-case analysis

Sorting a sorted list should be fast!!

1	2	3	4	5	6	7	8

- In this class, we will focus on worst-case analysis



Here is my algorithm!

```
Algorithm:
   Do the thing
   Do the stuff
   Return the answer
```

Algorithm designer

HERE IS AN INPUT!

- Pros: very strong guarantee
- Cons: very strong guarantee

Big-O notation

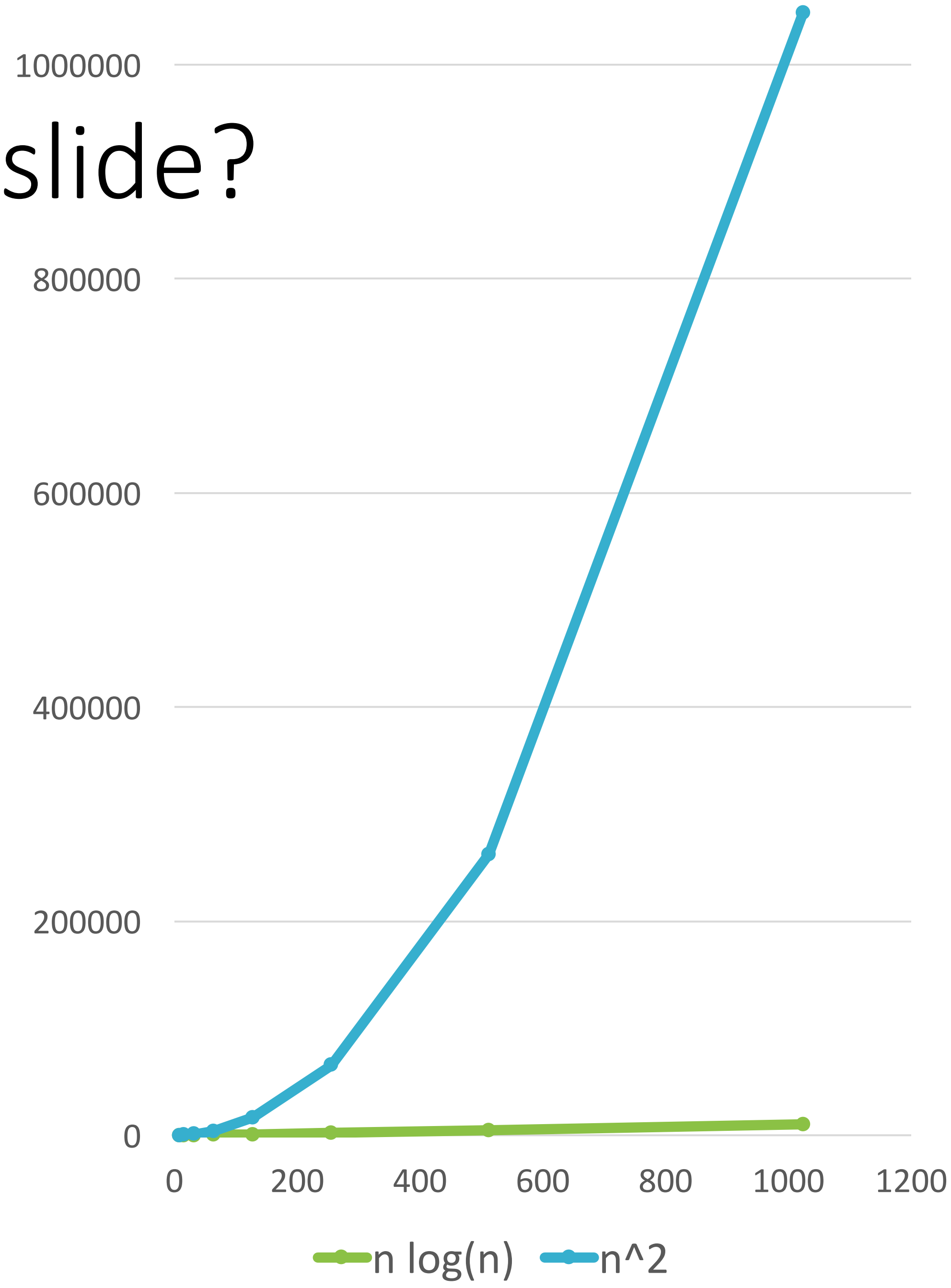
How long does an operation take? Why are we being so sloppy about that "6"?

- What do we mean when we measure runtime?
  - We probably care about wall time: how long does it take to solve the problem, in seconds or minutes or hours?

- This is heavily dependent on the programming language, architecture, etc.
- These things are very important, but are not the point of this class.
- We want a way to talk about the running time of an algorithm, independent of these considerations.

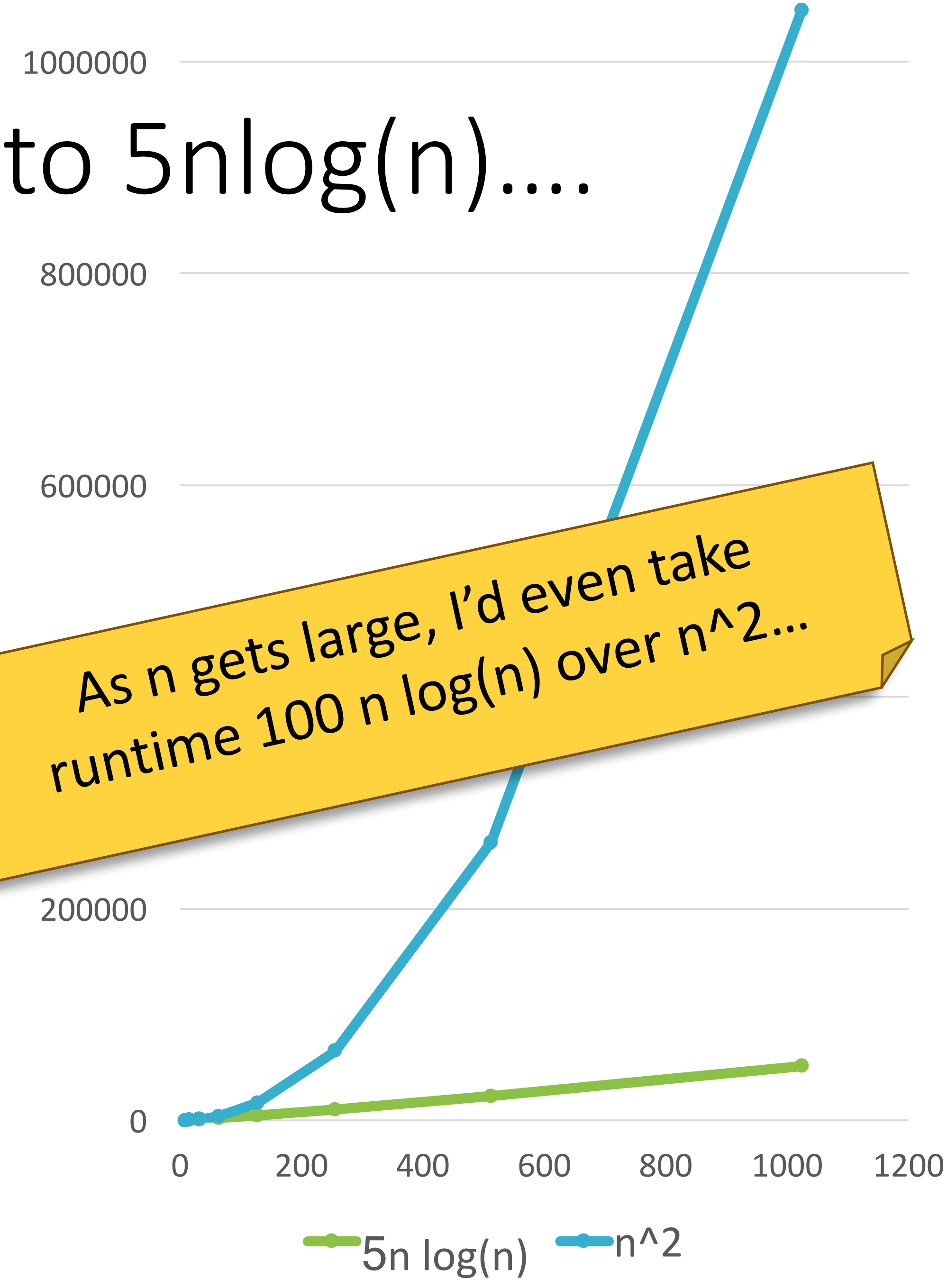
Remember this slide?

n	n log(n)	n^2
8	24	64
16	64	256
32	160	1024
64	384	4096
128	896	16384
256	2048	65536
512	4608	262144
1024	10240	1048576



Change nlog(n) to 5nlog(n)....

n	5n log(n)	n^2
8	120	64
16	320	256
32	800	1024
64	1920	4096
128	4480	16384
256	10240	65536
512	23040	262144
1024	51200	1048576



As n gets large, I'd even take runtime 100 n log(n) over n^2...

Asymptotic Analysis

How does the running time scale as n gets large?

One algorithm is "faster" than another if its runtime grows more "slowly" as n gets large.

This is especially relevant now, as data get bigger and bigger and bigger and bigger…

This will provide a formal way of saying that n^2 is "worse" than $100 \, n \log(n)$.

Pros:

- Abstracts away from hardware- and language-specific issues.
- Makes algorithm analysis much more tractable.

Without making Plucky grumpy!

Cons:

- Only makes sense if n is large (compared to the constant factors).

$2^{100000000000000} \, n$ is "better" than n^2 ?!?!

Now for some definitions…

- Quick reminders:
  - ∃: “There exists”
  - ∀: ”For all”
  - Example: ∀ students in CS161, ∃ an algorithms problem that really excites the student.
  - Much stronger statement: ∃ an algorithms problem so that, ∀ students in CS161, the student is excited by the problem.
- We’re going to formally define an upper bound:
  - “T(n) grows no faster than f(n)”

pronounced “big-oh of …” or sometimes “oh of …”

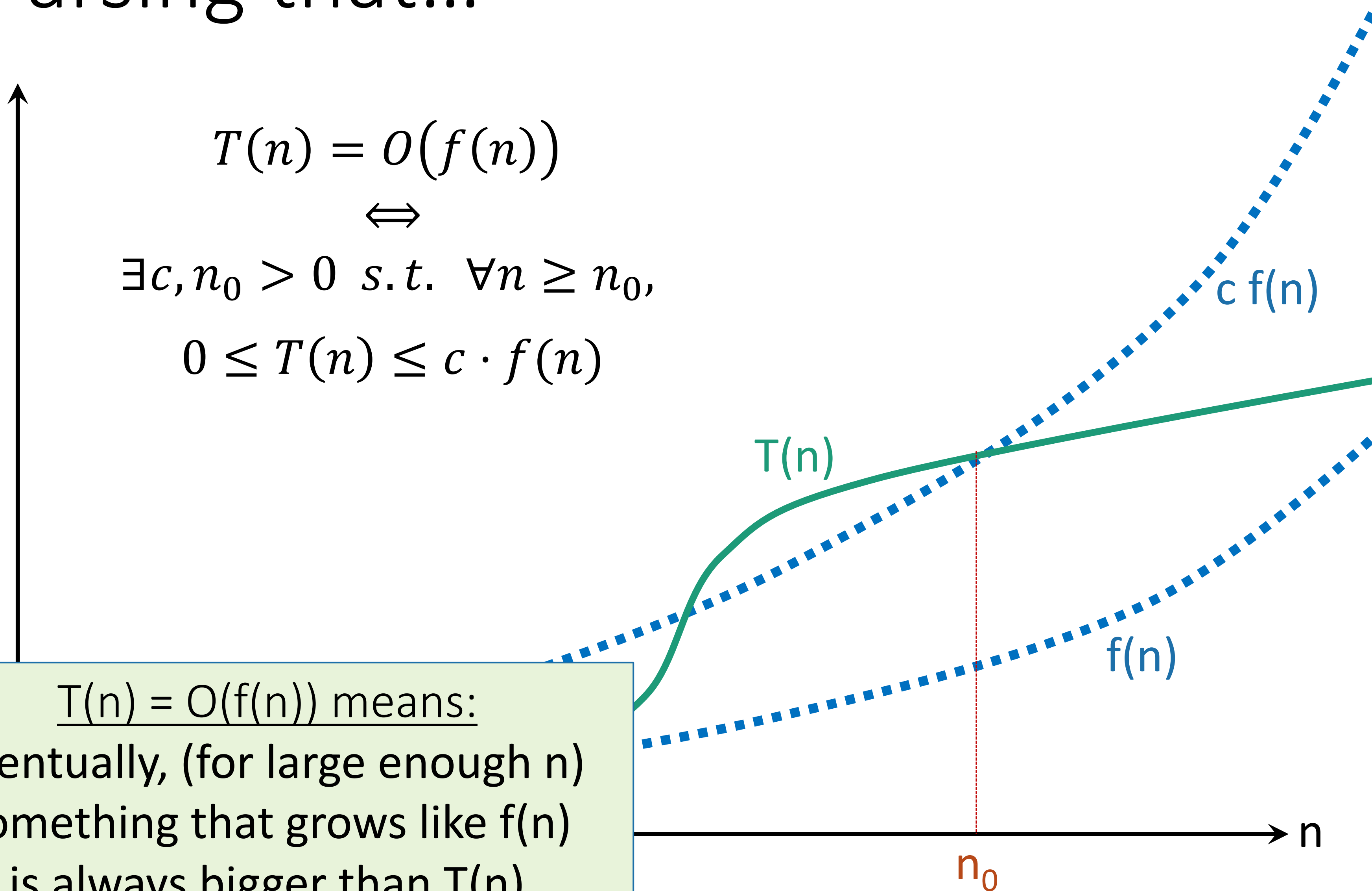
O(…) means an upper bound

- Let T(n), f(n) be functions of positive integers.
    - Think of T(n) as being a runtime: positive and increasing in n.
- We say “T(n) is O(f(n))” if f(n) grows at least as fast as T(n) as n gets large.
- Formally,

$$T(n) = O\big(f(n)\big)$$
$$\Longleftrightarrow$$
$$\exists c, n_0 > 0 \;\; s.t. \;\; \forall n \geq n_0,$$
$$0 \leq T(n) \leq c \cdot f(n)$$

Parsing that…

$$T(n) = O\big(f(n)\big)$$
$$\Leftrightarrow$$
$$\exists c, n_0 > 0 \;\; s.t. \;\; \forall n \geq n_0,$$
$$0 \leq T(n) \leq c \cdot f(n)$$

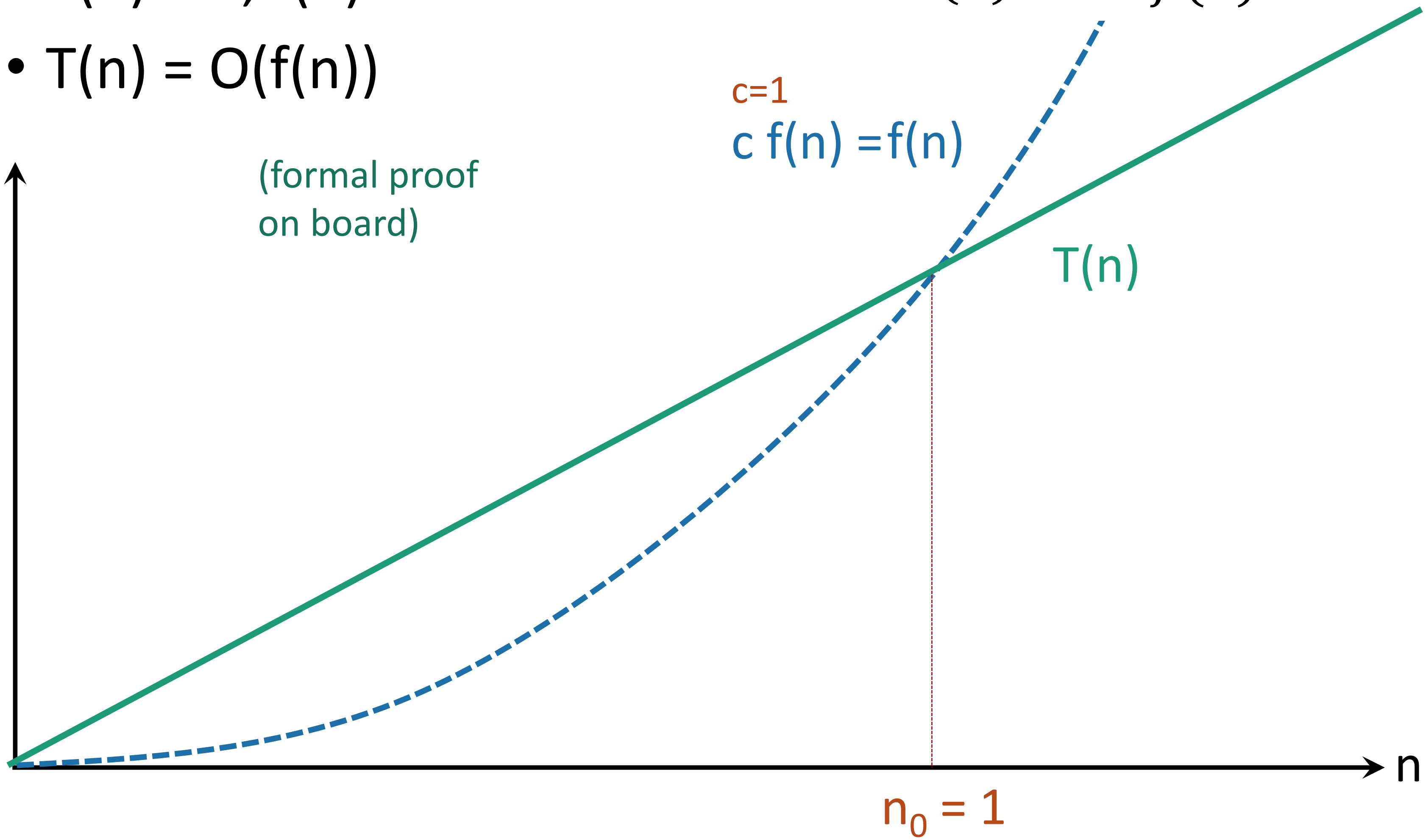


<u>T(n) = O(f(n)) means:</u>
Eventually, (for large enough n) something that grows like f(n) is always bigger than T(n).

Example 1

$$T(n) = O\big(f(n)\big)$$
$$\Leftrightarrow$$
$$\exists c, n_0 > 0 \;\; s.t. \;\; \forall n \geq n_0,$$
$$0 \leq T(n) \leq c \cdot f(n)$$

- $T(n) = n$, $f(n) = n^2$.
- $T(n) = O(f(n))$



Examples 2 and 3

- All degree k polynomials with positive leading coefficients are $O(n^k)$.
- For any $k \geq 1$, n^k is not $O(n^{k-1})$.

(On the board)

Take-away from examples

- To prove $T(n) = O(f(n))$, you have to come up with c and n_0 so that the definition is satisfied.

- To prove $T(n)$ is NOT $O(f(n))$, one way is by contradiction:
  - Suppose that someone gives you a c and an n_0 so that the definition is satisfied.
  - Show that this someone must by lying to you by deriving a contradiction.

O(…) means an upper bound, and
Ω(…) means a lower bound

- We say "T(n) is Ω(f(n))" if f(n) grows at most as fast as T(n) as n gets large.

- Formally,

$$T(n) = \Omega\big(f(n)\big)$$
$$\Longleftrightarrow$$
$$\exists c, n_0 > 0 \;\; s.t. \;\; \forall n \geq n_0,$$
$$0 \leq c \cdot f(n) \leq T(n)$$

Switched these!!

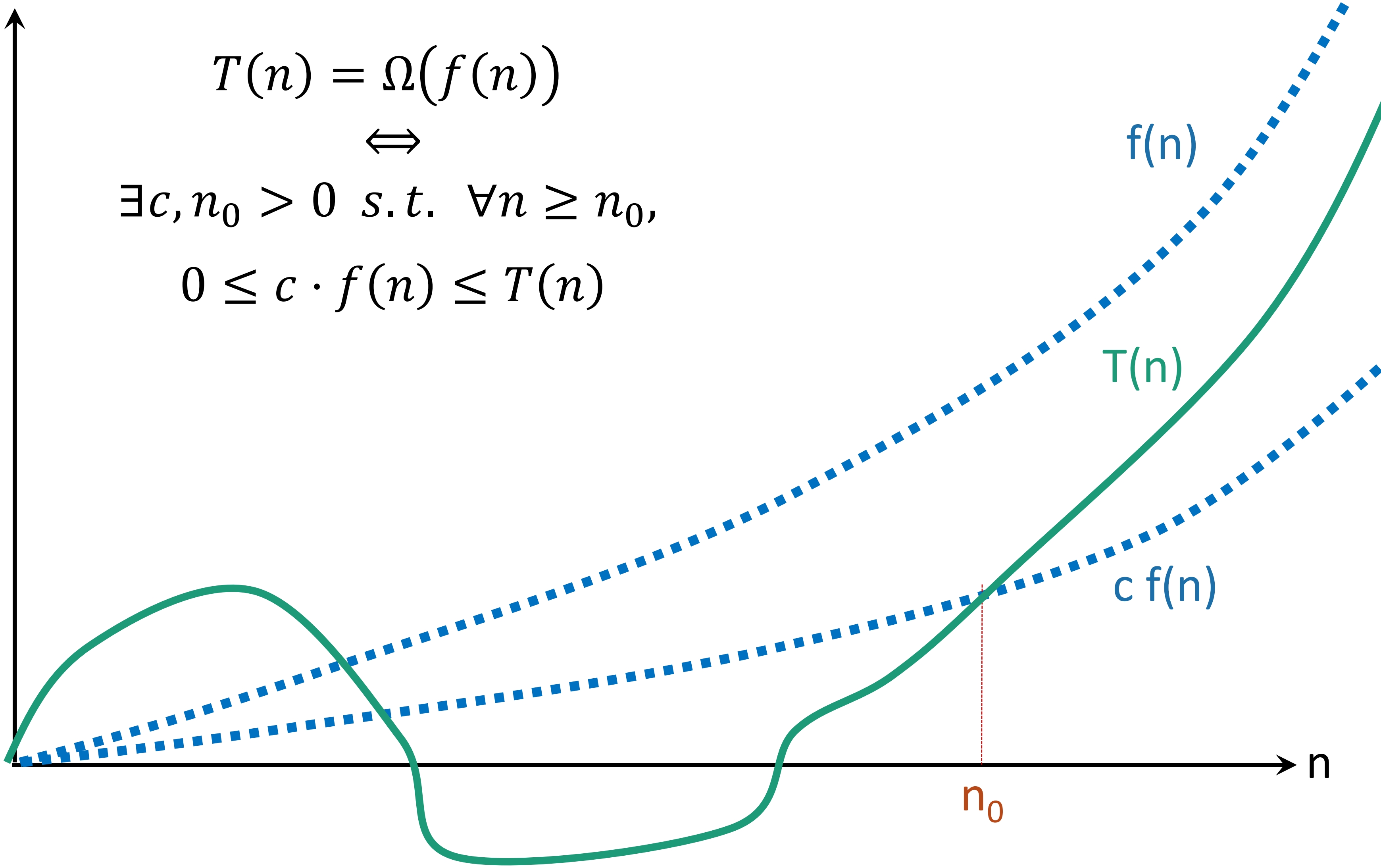
Parsing that...

$$T(n) = \Omega\big(f(n)\big)$$

$$\Leftrightarrow$$

$$\exists c, n_0 > 0 \;\; s.t. \;\; \forall n \geq n_0,$$

$$0 \leq c \cdot f(n) \leq T(n)$$



Θ(…) means both!

- We say "$T(n)$ is $\Theta(f(n))$" if:

$$T(n) = O(f(n))$$

-AND-

$$T(n) = \Omega(f(n))$$

Yet more examples

- $n^3 - n^2 + 3n = O(n^3)$
- $n^3 - n^2 + 3n = \Omega(n^3)$
- $n^3 - n^2 + 3n = \Theta(n^3)$

- 3^n is **not** $O(2^n)$
- $n \log(n) = \Omega(n)$
- $n \log(n)$ is **not** $\Theta(n)$.

Fun exercise:
check all of these
carefully!!

We'll be using lots of asymptotic notation from here on out

- This makes both Plucky and Lucky happy.
  - Plucky the Pedantic Penguin is happy because there is a precise definition.
  - Lucky the Lackadaisical Lemur is happy because we don't have to pay close attention to all those pesky constant factors like "4" or "6".
- But we should always be careful not to abuse it.
- In the course, (almost) every algorithm we see will be actually practical, without needing to take $n \geq n_0 = 2^{10000000}$.

Questions about asymptotic notation?

Back to recurrence relations

$T(n)$ = time to solve a problem of size n.

We've seen three recursive algorithms so far.

- Needlessly recursive integer multiplication
- $T(n) = 4\,T(n/2) + O(n)$
- $T(n) = O(\,n^2\,)$

(Reminders on board)

- Karatsuba integer multiplication
- $T(n) = 3\,T(n/2) + O(n)$
- $T(n) = O(\,n^{\log\_2(3)} \approx n^{1.6}\,)$

- MergeSort
- $T(n) = 2T(n/2) + O(n)$
- $T(n) = O(\,n\log(n)\,)$

What's the pattern?!?!?!?!

The master theorem

- A formula that solves recurrences when all of the sub-problems are the same size.
- (We’ll see an example Wednesday when not all problems are the same size).

A useful
formula it is.
Know why it works
you should.

Jedi master Yoda

The master theorem

- Suppose $T(n) = a \cdot T\left(\frac{n}{b}\right) + O(n^d)$. Then

$$T(n) = \begin{cases} O(n^d \log(n)) & \text{if } a = b^d \\ O(n^d) & \text{if } a < b^d \\ O(n^{\log_b(a)}) & \text{if } a > b^d \end{cases}$$

Three parameters:
a : number of subproblems
b : factor by which input size shrinks
d : need to do n^d work to create all the subproblems and combine their solutions.

Many symbols those are....

Examples

(details on board)

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + O(n^d).$$

$$T(n) = \begin{cases} O(n^d \log(n)) & \text{if } a = b^d \\ O(n^d) & \text{if } a < b^d \\ O(n^{\log_b(a)}) & \text{if } a > b^d \end{cases}$$

- Needlessly recursive integer mult.
- T(n) = 4 T(n/2) + O(n)
- T(n) = O(n^2)

a = 4
b = 2
d = 1

a > b^d ✓

- Karatsuba integer multiplication
- T(n) = 3 T(n/2) + O(n)
- T(n) = O($n^{\text{log\_2(3)}} \approx n^{1.6}$)

a = 3
b = 2
d = 1

a > b^d ✓

- MergeSort
- T(n) = 2T(n/2) + O(n)
- T(n) = O(nlog(n))

a = 2
b = 2
d = 1

a = b^d ✓

Proof of the master theorem

- We'll do the same recursion tree thing we did for MergeSort, but be more careful.
- Suppose that $T(n) = a \cdot T\left(\frac{n}{b}\right) + c \cdot n^d$.

Hang on! The hypothesis of the Master Theorem was the the extra work at each level was $O(n^d)$. That's NOT the same as work <= cn for some constant c.

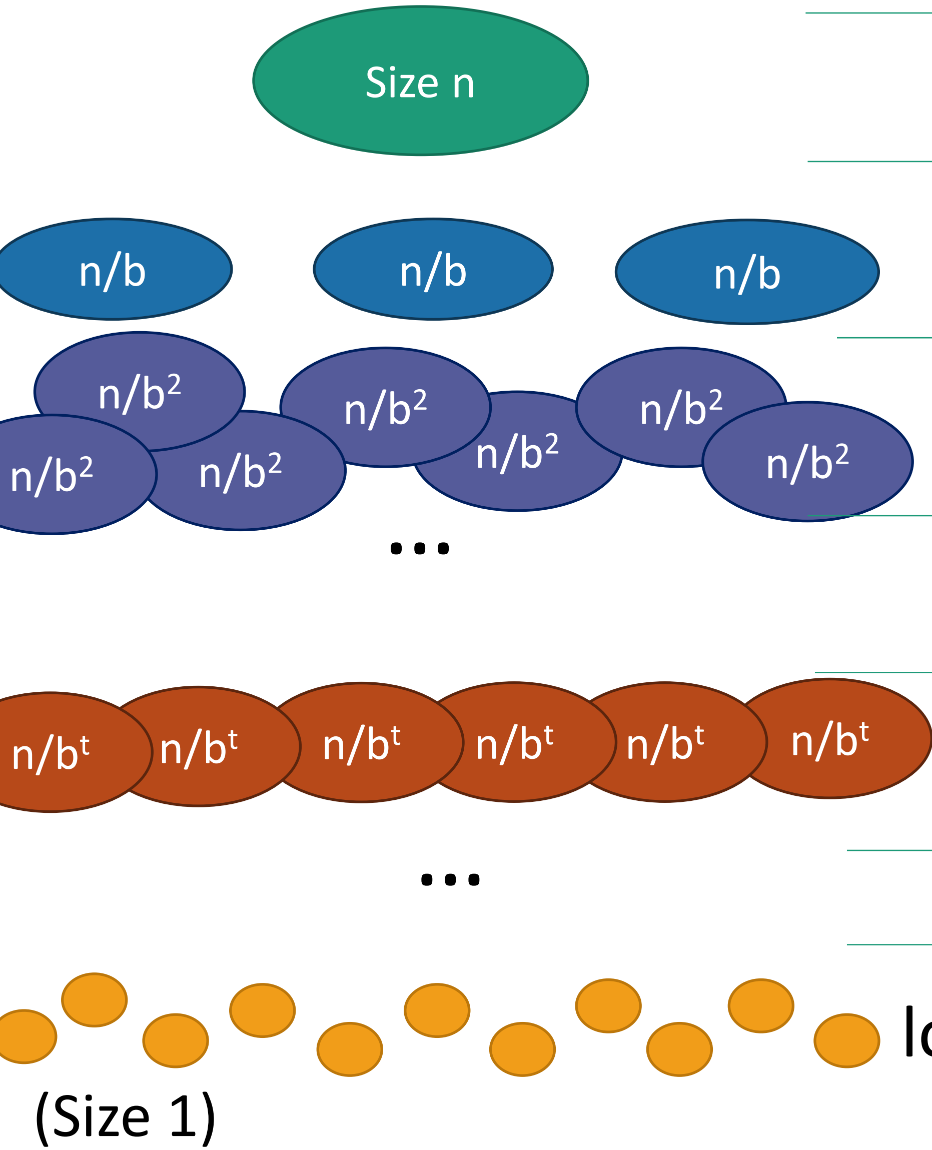
Plucky the Pedantic Penguin

That's true … we'll actually prove a weaker statement that uses this hypothesis instead of the hypothesis that $T(n) = a \cdot T\left(\frac{n}{b}\right) + O(n^d)$.
It's a good exercise try to make this proof work rigorously with the O() notation.

Lucky the lackadaisical lemur

Recursion tree

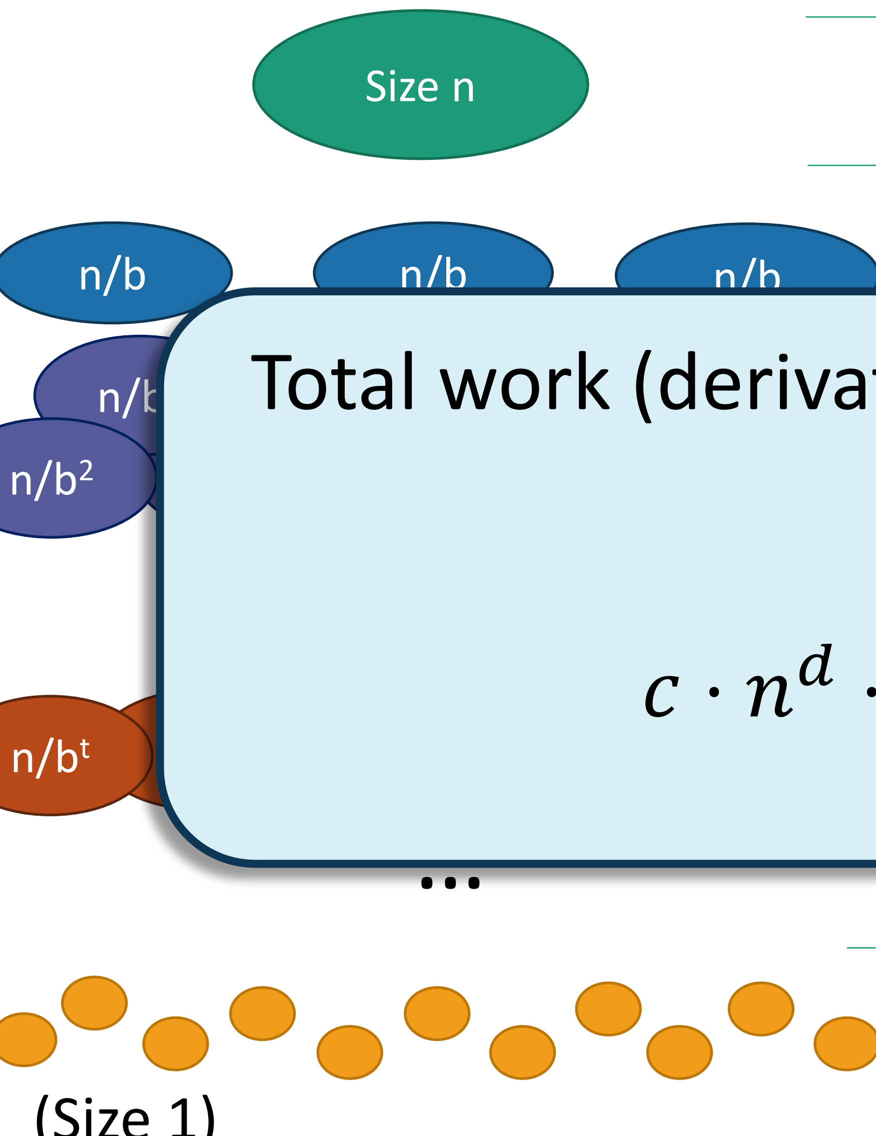
$T(n) = a \cdot T\left(\frac{n}{b}\right) + c \cdot n^d$



Level	# problems	Size of each problem	Amount of work at this level
0	1	n	$c \cdot n^d$
1	a	n/b	$ac\left(\frac{n}{b}\right)^d$
2	a^2	n/b^2	$a^2c\left(\frac{n}{b^2}\right)^d$
t	a^t	n/b^t	$a^tc\left(\frac{n}{b^t}\right)^d$
…			
$\log_b(n)$	$a^{\log\_b(n)}$	1	$a^{\log\_b(n)}c$

Recursion tree

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + c \cdot n^d$$



Level	# problems	Size of each problem	Amount of work at this level
0	1	n	$c \cdot n^d$
1	a	n/b	$ac\left(\frac{n}{b}\right)^d$
...			
$\log_b(n)$	$a^{\log\_b(n)}$	1	$a^{\log\_b(n)}c$

Total work (derivation on board) is at most:

$$c \cdot n^d \cdot \sum_{t=0}^{log_b(n)} \left(\frac{a}{b^d}\right)^t$$

Now let’s check all the cases (on board)

$$T(n) = \begin{cases} O(n^d \log(n)) & \text{if } a = b^d \\ O(n^d) & \text{if } a < b^d \\ O(n^{\log_b(a)}) & \text{if } a > b^d \end{cases}$$

Even more generally, for T(n) = aT(n/b) + f(n)...

Theorem 3.2 (Master Theorem). *Let $T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n)$ be a recurrence where $a \geq 1$, $b > 1$. Then,*

- *If $f(n) = O\left(n^{\log_b a - \epsilon}\right)$ for some constant $\epsilon > 0$, $T(n) = \Theta\left(n^{\log_b a}\right)$.*
- *If $f(n) = \Theta\left(n^{\log_b a}\right)$, $T(n) = \Theta\left(n^{\log_b a} \log n\right)$.*
- *If $f(n) = \Omega\left(n^{\log_b a + \epsilon}\right)$ for some constant $\epsilon > 0$ and if $af(n/b) \leq cf(n)$ for $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$.*

Recap

- O() notation makes our lives easier.
- The ”Master Method” also make our lives easier.

Next time:

- What if the sub-problems are different sizes?
- And when might that happen?

Extra slides…

Some brainteasers

- Are there functions f, g so that NEITHER $f = O(g)$ nor $f = \Omega(g)$?
- Are there non-decreasing functions f, g so that the above is true?
- Define the n'th fibonacci number by $F(0) = 1$, $F(1) = 1$, $F(n) = F(n-1) + F(n-2)$ for $n > 2$.
  - 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, …

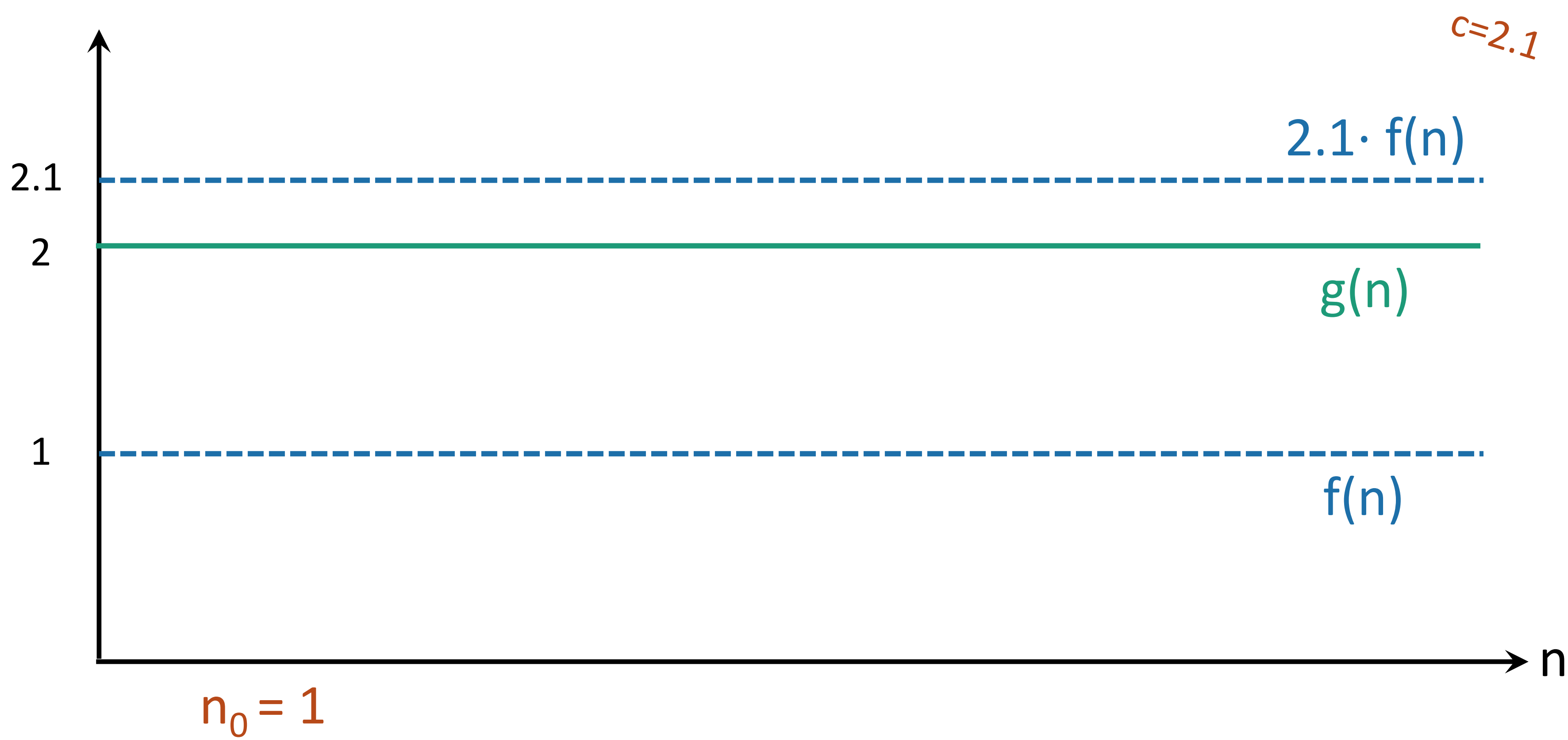
  True or false:
  - $F(n) = O(2^n)$
  - $F(n) = \Omega(2^n)$

A few more O() examples

Example A

$$T(n) = O\big(f(n)\big)$$
$$\Leftrightarrow$$
$$\exists c, n_0 > 0 \;\; s.t. \;\; \forall n \geq n_0,$$
$$0 \leq T(n) \leq c \cdot f(n)$$

- g(n) = 2, f(n) = 1.
- g(n) = O(f(n)) (and also f(n) = O(g(n)))



Example B

$$T(n) = O\big(f(n)\big)$$
$$\Leftrightarrow$$
$$\exists c, n_0 > 0 \;\; s.t. \;\; \forall n \geq n_0,$$
$$0 \leq T(n) \leq c \cdot f(n)$$

- f(n) = 1, g(n) as below.
- g(n) = O(f(n)) (and also f(n) = O(g(n)))