Markov Chain

When we study a system that can change over time, we need a way to keep track of those changes. A Markov chain is a particular model for keeping track of systems that change according to given probabilities. A Markov chain may allow one to predict future events, but the predictions become less useful for events farther into the future (much like predictions of the stock market or weather).

A **state** is any particular situation that is possible in the system. For example, if we are studying rainy days, then there are two states:

1. It's raining today.
2. It's not raining today.

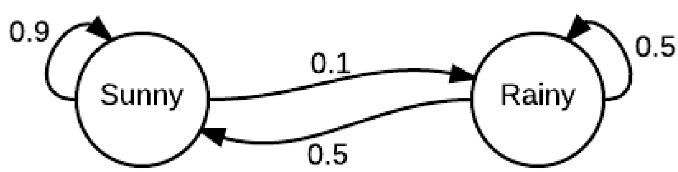
The system could have many more than two states, but we will stick to two for this small example.

The term **Markov chain** refers to any system in which there are a certain number of states and given probabilities that the system changes from any state to another state.

That's a lot to take in at once, so let's illustrate using our rainy days example. The probabilities for our system might be:

- If it is sunny today, then there is a 90% chance it will sunny tomorrow, and 10% chance of rainy.
- If it doesn't sunny today, then there is a 50% chance it will sunny tomorrow and 50% chance of rainy.

It may help to organize this data in what we call a **state diagram** by using the probability in decimal form, rather than percentage.



Left circle represents sunny, and the right represents rainy. Arrows indicate the probability to change state.

The Transition Matrix

In mathematics, a stochastic matrix is a square matrix used to describe the transitions of a Markov chain. Each of its entries is a non-negative real number representing a probability. It is also called a *probability matrix, transition matrix, substitution matrix,* or *Markov matrix.*

If a Markov chain consists of k states, the transition matrix is the $k \times k$ matrix, whose entries record the probability of moving from each state to another state. Let's build a transition matrix for the state diagram defined above.

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At first, the probabilities demonstrated in the state diagram are shown in table form:

	Sunny (S)	Rainy (R)
Sunny (S)	0.9	0.1
Rainy (R)	0.5	0.5

Therefore, the transition matrix (T) will be:

$$T = \begin{vmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{vmatrix}$$

State Vector:
At the start of the process, we might know exactly which state the system is in (sunny or rainy), but starting with the second state and further, we only know probabilities of the system being in any given state. To keep track of these probabilities, we use a state vector.

A **state vector** is a vector (list) that records the probabilities that the system is in any given state at a particular step of the process.

For example, if we know for sure that it is sunny today, then the state vector for *todays* weather will be $x^{(0)} = |1 \quad 0|$. But tomorrow is another day!

By using transition matrix (T), the state vector, $x^{(1)}$ of *tomorrow's* weather can be predicted by:

$$\begin{aligned} x^{(1)} &= x^{(0)}T \\ &= |1 \quad 0| \begin{vmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{vmatrix} \\ &= |0.9 \quad 0.1| \quad \text{(using matrix multiplication)} \end{aligned}$$

Similarly, the state vector, $x^{(2)}$ of *day after tomorrow's* weather can be predicted by:

$$\begin{aligned} x^{(2)} &= x^{(1)}T \\ &= |0.9 \quad 0.1| \begin{vmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{vmatrix} \\ &= |0.86 \quad 0.14| \quad \text{(using matrix multiplication)} \end{aligned}$$

After continuing such numerous steps, the state vector eventually reached in a equilibrium state when the prediction value will not change anymore. Such vector is named as the steady state vector.

However, in general rules to obtain the state vector of day n will be:

$$x^{(n)} = x^{(n-1)}T$$

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Steady State Vector:
The steady-state vector of the transition matrix (T) is the unique probability vector, (V) that satisfies the following equation: .

$$VT = V \quad .... (1)$$

Now, we are going to estimate the steady state vector of our previous weather example, where the transition matrix is given as follows:

$$T = \begin{vmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{vmatrix}$$

From equation (1), we will get:

$$VT - V = 0$$

$$V(T - 1) = 0$$

$V(T - I) = 0$ (In matrix multiplication, an identity matrix, I is equal to 1)

$$V\left(\begin{vmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{vmatrix} - \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}\right) = 0$$

$$V\left(\begin{vmatrix} -0.1 & 0.1 \\ 0.5 & -0.5 \end{vmatrix}\right) = 0 \quad .... (2)$$

Since, there are two states in this problem (sunny and rainy), the steady state is composed of the probability of sunny (S) and rainy (R).

Therefore,

$$V = \begin{vmatrix} S & R \end{vmatrix}$$

and $S + R = 1$ (Since sum of the probability of all states is 1) .... (3)

Equation (2) can be rewritten as:

$$\begin{vmatrix} S & R \end{vmatrix}\left(\begin{vmatrix} -0.1 & 0.1 \\ 0.5 & -0.5 \end{vmatrix}\right) = \begin{vmatrix} 0 & 0 \end{vmatrix}$$

Hence, we get the following two equations:

$-0.1S + 0.5R = 0$ ....(4)

$0.1S - 0.5R = 0$ ....(5)

By using and solving equation (3) and (4), we will get the steady state probability distribution:

$$S = 0.833$$
$$R = 0.167$$

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Therefore, the steady state vector, V will be:

$$V = |S \quad R| = |0.833 \quad 0.167|$$

From this, we can conclude that in the long term, about 83.3% of days are sunny.

Problem Formulation that Containing Three States:

For the following problem, I just illustrate the computation. Please follow the presentation style as described in previous section.

Consider the following transition matrix,

$$T = \begin{vmatrix} 0.6 & 0.3 & 0.1 \\ 0.2 & 0.7 & 0.1 \\ 0.3 & 0.2 & 0.5 \end{vmatrix}$$

$$VT = V$$

$$VT - V = 0$$

$$V(T-1) = 0$$

$V(T - I) = 0$ (In matrix multiplication, an identity matrix, I is equal to 1)

$$V\left(\begin{vmatrix} 0.6 & 0.3 & 0.1 \\ 0.2 & 0.7 & 0.1 \\ 0.3 & 0.2 & 0.5 \end{vmatrix} - \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}\right) = 0$$

$$V\left(\begin{vmatrix} -0.4 & 0.3 & 0.1 \\ 0.2 & -0.3 & 0.1 \\ 0.3 & 0.2 & -0.5 \end{vmatrix}\right) = 0 \quad \text{.............. (6)}$$

Since, there are three states in this problem, let, the steady state is composed of the probability of X, Y, and Z.

Therefore,

$$V = |X \quad Y \quad Z|$$

and $X + Y + Z = 1$ (Since sum of the probability of all the states is 1) .... (7)

Equation (6) can be rewritten as:

$$|X \quad Y \quad Z|\left(\begin{vmatrix} -0.4 & 0.3 & 0.1 \\ 0.2 & -0.3 & 0.1 \\ 0.3 & 0.2 & -0.5 \end{vmatrix}\right) = |0 \quad 0 \quad 0|$$

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Hence, we get the following three equations:

$$-0.4X + 0.2Y + 0.3Z = 0 \quad ....(8)$$

$$0.3X - 0.3Y + 0.2Z = 0 \quad ....(9)$$

$$0.1X + 0.1Y - 0.5Z = 0 \quad ....(10)$$

(4 × Equation (10)) + Equation (8) will produce:

$$0.4X + 0.4Y - 2Z - 0.4X + 0.2Y + 0.3Z = 0$$

$$0.6Y - 1.7Z = 0$$

$$Y = 2.83Z \quad ....(11)$$

(3 × Equation (10)) + Equation (9) will produce:

$$0.3X + 0.3Y - 1.5Z + 0.3X - 0.3Y + 0.2Z = 0$$

$$0.6X - 1.3Z = 0$$

$$X = 2.17Z \text{ .... } (12)$$

Put the value of X and Y from equation (12) and (11) into equation (7)

$$2.17Z + 2.83Z + Z = 1$$

$$Z = 0.17$$

Put the value of Z into equation (12)

$$X = 2.17 \times 0.17$$

$$X = 0.37$$

Put the value of Z into equation (11)

$$Y = 2.83 \times 0.17$$

$$Y = 0.48$$

Therefore, the steady state vector, V will be:

$$V = |X \quad Y \quad Z| = |0.37 \quad 0.48 \quad 0.17|$$

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