Advanced Methods in Text Analytics
Exercise 3: Language Models - Part 1
Solutions

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1 Language Modeling Basics

(a) Let x_i be word in position i in a sequence of n words. Then $p(x_{n+1}|x_1, x_2, \ldots, x_{n-1}, x_n)$ is the probability of predicting word $n+1$ given previous n words.

(b) We use the chain rule to compute the joint distribution of the words in a sequence. That is, $p(x_1, x_2, \ldots, x_n) = p(x_1)p(x_2|x_1)\ldots p(x_n|x_1, x_2, \ldots, x_{n-1})$.

(c) N-gram models assume that predicting a given word x_n only depends on the $n-1$ words that precede it. This is known as a Markov assumption, which often refers to the assumption that we can predict future probabilities without looking too much into the past. A bigram model predicts a word given only the single previous word. So using the chain rule, we can compute the probability of a given sequence as follows:

$$p(x_n|x_1, \ldots, x_{n-1}) = p(x_1)p(x_2|x_1)p(x_3|x_2)\ldots p(x_n|x_{n-1}).$$

Similarly, for trigram models, we have:

$$p(x_n|x_1, \ldots, x_{n-1}) = p(x_1)p(x_2|x_1)p(x_3|x_1, x_2)\ldots p(x_n|x_{n-1}, x_{n-2}).$$

In practice, special tags <s> and </s> are used to mark the beginning and end of sentences. Thus, a bigram model's prediction would be as follows:

$$p(x_n|x_1, \ldots, x_{n-1}) = p(x_1|\text{<s>})p(x_2|x_1)p(x_3|x_2)\ldots p(x_n|x_{n-1}).$$

Similarly, if a model predicts </s>, then we know the generated sentence is over.

(d) The probability for predicting word x_n is given by comparing the probability of the sequence of n words, i.e. the $n-1$ words followed by x_n, against the probability of the sequence of previous $n-1$ words. These probabilities are estimated by counting these sequences over a large corpus. That is:

$$p(x_n|x_1, \ldots, x_{n-1}) = \frac{\text{count}(x_1, x_2, \ldots, x_n)}{\text{count}(x_1, x_2, \ldots, x_{n-1})}.$$

It can be shown that this corresponds to performing MLE.

(e) Counting becomes expensive as n grows larger, because there are many more possible sequences. In addition, storing all counts is expensive, and this count matrix is often sparse, as most possible sequences are not semantically relevant and thus never observed in a corpus.

(f) The proposed architecture was a feed-forward neural network, with an embedding layer as input, a single hidden layer with a *tanh* activation, and a softmax layer as output. The input sequence was represented by concatenating all embeddings of the given sequence. The probabilities were computed using a softmax layer, where the target was the correct word to predict.

As parameters, let $\boldsymbol{W}_w \in \mathbb{R}^{|V| \times d}$ be the word embedding matrix in the input embedding layer, $\boldsymbol{W}_h \in \mathbb{R}^{(n \cdot d) \times h}$ be the weight matrix for the hidden layer with n being the (maximum) number of input tokens, and $\boldsymbol{W}_s \in \mathbb{R}^{h \times |V|}$ be the weight matrix for the softmax layer. Then, the model was parameterized by $\boldsymbol{\theta} = \{\boldsymbol{W}_w, \boldsymbol{W}_h, \boldsymbol{W}_s\}$.

(g) Self-supervision refers to the process of constructing labeled examples by hiding part of an input example and asking the model to predict the hidden part given the rest of it. For images, we can hide part of it and ask the model to predict it. For video, we can ask a model to predict a frame given previous $n-1$ frames. In the context of language models, any sequence of length n in a corpus can be used to construct an example for predicting the n-th word given previous $n-1$ words. This is what Bengio et al. did.

(h) Neural language models such as the one described above did not need to store counts of n-grams, thus reducing the memory costs significantly. In addition, the sparsity problem is no longer relevant, as the model focuses on observed sequences only. However, a persisting problem is that the context window is still too small, and it can't be enlarged easily in such models. Why? Note that the input sequence is represented with a vector of size $d \cdot n$, where d is the size of the word embeddings and n the size of the input sequence. The hidden layer that takes this vector as input must be parameterized by matrix $\boldsymbol{W}_h$ of size $(n \cdot d) \times h$, where n is the size of the input sequence. Thus, by increasing the window size, we increase the size of $\boldsymbol{W}_h$, thus making the model more difficult to train as the window grows larger. And this still does not solve the problem that given any fixed window size, we will never be able to input sequences longer than that.

2 Language Model Evaluation

(a) Machine translation: a LM may determine which of n predicted phrases is more likely. Speech recognition: given $n-1$ recognized spoken words, an LM may determine which of a small set of possible words is mostly likely to follow. As we can see, LMs are very general tools in NLP.

(b) Extrinsic: evaluating model based on performance on a specific task, e.g. machine translation. This downstream task must use a *downstream* model that uses a language model as part of it, i.e. a machine translation model. PROs: we rely on meaningful performance on a task. CONs: reported performance dependent on details of task-specific downstream model and evaluation, may not rely so much on language model performance. Also, performance may not translate to different tasks.

Intrinsic: task-neutral evaluation of a model. PROs: no need for possibly expensive downstream pipeline. CONs: performance on intrinsic task may not translate to some/all downstream tasks.

(c) We could split our data into training and test, use training data to learn our LM, then evaluate it by computing the probability of entire test corpus, i.e. computing the loss over by taking the entire test split as a single sequence.

(d) We can split into train and test, estimates LM probabilities with training, then evaluate by computing probability of entire test corpus.

(e) The higher the likelihood, the lower perplexity. Thus, we want to minimize perplexity.

(f) For simplicity, let $V = |V|$. Being unigram-based and uniform, this model assigns a probability of $\frac{1}{V}$ to each word. It's likelihood is given by:

$$\begin{aligned} l(w_1, w_2, \ldots, w_n) &= p(w_1, w_2, \ldots, w_n) \\ &= p(w_1)p(w_2)\ldots p(w_n) \\ &= \left(\frac{1}{V}\right)^n \end{aligned}$$

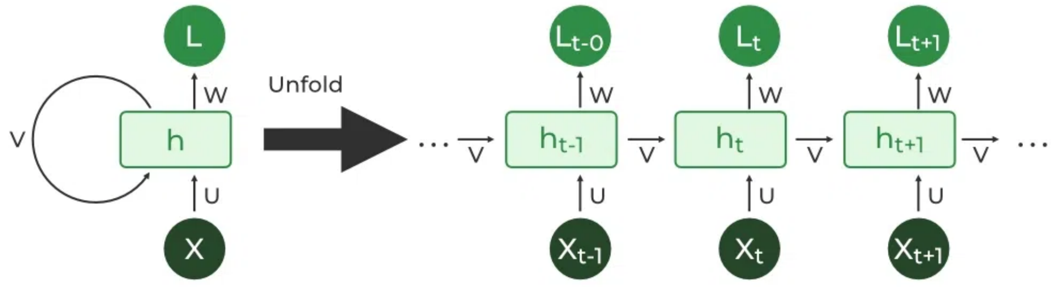
Its perplexity is then:

$$\begin{aligned} ppl(w_1, w_2, \ldots, w_n) &= l(w_1, w_2, \ldots, w_n)^{-\frac{1}{n}} \\ &= \left(\left(\frac{1}{V}\right)^n\right)^{-\frac{1}{n}} \\ &= \left(\frac{1}{V}\right)^{-1} \\ &= V. \end{aligned}$$

Thus, the perplexity of LMs is often seen as proportional to the size of the vocabulary, where V (the size of the vocabulary) is informally seen as an lower-bound for the performance of a model, similar to comparing a classifier to a model that predicts a random class. In other words, for some vocabulary V, if the perplexity of a model is greater than $|V|$, we say it performs worse than a unigram model that assigns $1/|V|$ probability to each word. For example, on the *Penn Treebank* dataset, a 5-gram model achieves a perplexity of 141, but this can be improved to 80 using an RNNs with LSTMs. In both cases, this is a much lower number that the size of the vocabulary.

3 Recurrent Neural Networks

(a) RNNs are composed of (possibly parameterized) units that are shared across time-steps, i.e. across elements of a sequence. Let's separate these components down into input units, hidden units, and output units. The image below (source here) shows these components.



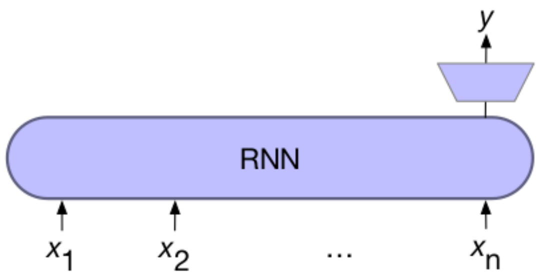
Input units: At time-step t, a RNN takes as input $\boldsymbol{x}_t$, which is a representation of the t-th element of a given sequence, e.g. a word embedding. It then transforms it via matrix $\boldsymbol{U}$, and passes this transformed input to hidden state $\boldsymbol{h}_t$.

Hidden units: In addition to the transformed input $\boldsymbol{x}_t$, a hidden state $\boldsymbol{h}_t$ takes as additional input $\boldsymbol{h}_{t-1}$, i.e. the hidden state at previous time-step $t-1$, but transformed by matrix $\boldsymbol{V}$. This connection between the same hidden state across time-steps is the recurrent connection that gives the network its name (which comes from recurrent functions). Thus, $\boldsymbol{h}_t$ takes as input $\boldsymbol{U}\boldsymbol{x}_t$ and $\boldsymbol{V}\boldsymbol{h}_{t-1}$, combines them via some operation (usually addition), and then applies a (possibly non-linear) activation function f, before passing this result to the hidden state in time-step $t+1$, i.e. $\boldsymbol{h}_{t+1}$. In other words, $\boldsymbol{h}_t = f(\boldsymbol{U}\boldsymbol{x}_t + \boldsymbol{V}\boldsymbol{h}_{t-1})$. By sharing parameters $\boldsymbol{U}, \boldsymbol{V}$ and passing hidden state $\boldsymbol{h}_i$ forward, the intuition is that, at any given time-step t, $\boldsymbol{h}_t$ encodes what the model has "seen" in the sequence so far, i.e. all elements up until t.

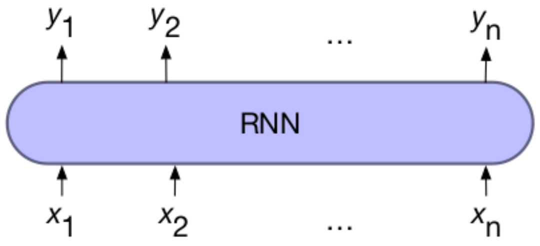
Output units: Optionally, the hidden state may also be passed to an output unit $\boldsymbol{L}_t$, but transformed via some matrix $\boldsymbol{W}$ and a (potentially non-linear) activation function g. That is, $\boldsymbol{L}_t = g(\boldsymbol{W}\boldsymbol{h}_t)$. Whether we have an output unit per time-step or not depends on the task at hand (more in the next question).

RNNs are different from FNNs in that they share parameters across time-steps, and are thus deep in time. This is different from fully-connected networks in that those typically use different weight matrices in each hidden layer. However, RNNs can also be stacked to become deep in the same sense as fully-connected neural networks. That is, an RNN layer may produce an output for each input element, thus producing an output sequence, which can in turn be used as input for another RNN layer. In such cases, parameters are not shared across layers.

(b) In *sequence classification*, we require a single prediction per sequence. E.g. in sentiment classification, we may predict whether a given sentence describes a positive or negative sentiment, i.e. binary classification. Thus, we do not need an output unit per element in the input unit. Instead, we may produce a single output input based on the hidden state of the final time-step. This output may be passed by a softmax function that produces a probability vector of as many components as there are target classes, or in the case of binary classification, a logistic function. The following figure illustrates such an architecture.



Conversely, requiring a label for each time-step is known as *sequence labeling.* For example, in POS tagging, we want to determine whether each word in a sentence is a verb, noun, etc. Here, the RNN should produce an output at each time-step, as illustrated by the image below.



Each output unit could include a softmax function for classification.

In both cases, the inputs and hidden states are as described in answer to question (a) above.

(c) Language models predict the next word given a sequence of n previous words. This is done by producing an output at each time-step given the corresponding input element and the hidden state at the previous time-step. Thus, unlike LMs implemenented with fully-connected networks, such as the model from Bengio et al. (2003), here we do not have a maximum length of the input sequence that we accept, but can take arbitrarily long input sequence. In addition, unlike n-gram models, predictions made by RNN-based language models are not limited by any fixed number of previous words, since the hidden state can *in principle* encode all the sequence seen so far.

(d) Bidirectional RNNs use an additional hidden state to process a given sequence from right to left. These hidden states are passed "backward" from the end of the sequence to the beginning, thus encoding it in the opposite direction. This allows the model to, at time-step t, encode information about both previous and subsequent elements in the input sequence. Thus, the model has more context to make predictions at any given time-step. A disadvantage is that this is more costly to train, and more importantly, such a model cannot be used to make real-time predictions, since we do not have information about future time-steps.

Perhaps more importantly, isn't it *cheating* to use information about future time-steps? Well, that depends on the application. If the information is available at inference time, then why not use it? For example, in machine translation, we are usually given an entire sequence to translate, so a model can be trained to read the entire sequence in both directions before producing a sequence in the target language. This intuition extends to the information we give to any model during training, independent of its architecture. For example, when the goal is to learn generally useful word representations, then why not use information in both directions to make better use of the context of words in the training set? This is what the transformer-based model BERT does. Conversely, when predicting the next word in a sequence, models typically have access only to previous words. Thus, it would not make sense to train models to use information about future time-steps to make predictions, as this information is not available at inference time.

(e) The main problem with RNN-based models is that the hiddens state needs to encode the entirety of a sequence. In practice, it's often the case the the hidden states "remembers" information mostly aobut the most recently seen elements, and not those seen far into the past. This is mostly due to the vanishing gradient problem, which describes the impact that operators have over gradient information over a long period of time. Specifically, if an operator makes a gradient smaller, applying the same operator multiple times would continuously reduce this gradient.

LSTM units were designed to address this issue by allowing the model to control what to remember and what to forget about the information it has seen. This allows a model to realize during training that, in order to reduce the loss, it's more convenient to retain some information further in the past instead of using more recently seen information. This is not possible in vanilla RNNs.

LSTM units use parameterized gated mechanisms to allow the model to control the flow of information across time-steps. Specifically, it uses the following gates:

- **Forget Gate:** allows model to delete information that it has seen but no longer needs.
- **Input/Add Gate:** allows model to select what information from the current input and hidden state to use.
- **Output Gate:** allows the model to separate information needed to produce current output from what is needed to pass to future time-steps.

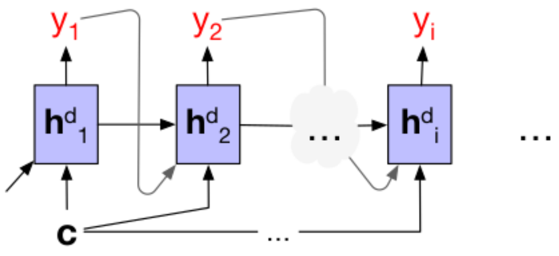
(f) An encoder-decoder architecture is made up of three components: an encoder which takes as input a given source sequence, a context vector produced by the encoder that represents the input sequence, and a decoder, which takes as input the context vector and produces an output sequence. Let's describe the encoder and decoder in more detail.

Encoder: In machine translation, the encoder takes as input the source sequence, and produces a context vector (usually the hidden state of the last time-step). This encoder produces no other outputs and general can be any architecture that makes as much use of the input data as possible. RNN-based encoders are typically stacked bidirectional RNNs with LSTM units (biLSTMs for short), but this can also be done with other types of architectures, e.g. transformers.

Decoder: The decoder takes as input the context vector and a a "beginning of sentence" tag (in machine translation, this tag is used to separate source from target sentences pairs shown to the model). Using this, it produces a hidden state and an output word, *both of which* are then used as input for the hidden state of the decoder in the next time step. In addition, it's common to use the context vector produce by the encoder as input in each time-step of the decoder, to preserve this information. Formally, the hidden state of the decoder at time-step t is given by:

$$\boldsymbol{h}_t^d = g(\boldsymbol{y}_{t-1}, \boldsymbol{h}_{t-1}^d, \boldsymbol{c}),$$

where $\boldsymbol{y}_{t-1}$ is the embedding of word produce by decoder in the previous time-step, and $\boldsymbol{c}$ is the context vector produced by the encoder. This decoder architecture is illustrated in the image below.



4 Sampling for Text Generation

(a) (i) **Autoregressive generation:** generate sequence of word/tokens by repeatedly sampling the next word/token conditioned on previously chosen words/tokens.

(ii) **Advantages:** can capture dependencies between words/tokens in a sequence.

(iii) **Disadvantages:** can be slow to generate long sequences, then longer the range of dependencies that need to be captured, the higher the cost of sampling. In other words, non-autoregressive generation is cheaper/faster.

(b) $p(y_t|y_{<t}, X, \theta)$

(c)
```
$w \leftarrow$ sample($p(w_i| < s >)$)
out $\leftarrow [w]$
**while** $w \neq < /s >$ **do**
    $w \leftarrow$ sample($p(w_i|out)$)
    out $\leftarrow$ out $+[w]$
**end while**
**return** out
```

(d) Advantages are that it is straightforward to understand and implement. But an important disadvantage is that it can lead to generation of text that is predictable and repetitive. In fact, given a model and the same input sequence, greedy sampling should deterministically produce the same output sequence. This is convenient when we want to predict facts that do not change, e.g. the answer to "What is the capital of Germany"? But this is an undesired property when we want to generate text that is more original/creative, such as what should follow after "The title of my talk is"

(e) Random sampling can lead to more diverse and creative text, as it allows the model to sample from the entire vocabulary. However, it can also lead to less coherent text, as the model may sample words that are less likely to follow the previous words. In addition, the size of the vocabulary is often very large, and for a given input sequence, the distribution over words is very skewed. This means that the model may sample from words that are very rare, and thus produce text that is less coherent.

(f) The answer is *top-k* sampling. This is a generalization of *greedy* sampling, where we now sample from the set of k words with the most probability. When $k = 1$, we have greedy sampling, and when $k = |V|$, we have random sampling. Generally, for values of $k > 1$, we allow the model to generate words that are not so predictible, but still probable enough that the resulting text can be of high-quality.

One important disadvantage of *top-k* sampling is that k is a fixed value, but the distributions over words produced by a model change for given inputs. E.g. when the input sequence is “The capital of Germany is”, most of the probability mass in the resulting distribution would ideally be in the Berlin token (or combination of tokens), which is a (proper) noun. But when the input sequence is “The title of my talk”, the probability mass would most likely be in the token “is” or similar verbs. To address this issue, *top-p* sampling was proposed, where we sample from the words that add up to the top p probability mass in the distribution. That way, we can sample from a variable number of words given different input sequences. A good model may indeed learn to put most of the mass in the answer to a factual question, e.g. Berlin is the capital of Germany, or more generally distribute it more evenly among several probable words in other settings.

(g) As *tau* approaches zero and is less than 1, it will increase the values of all logits, e.g. $12 < 12/0.5 < 12/0.3$. This will result in increased probability mass on words that already have high mass, and decreased probability mass on words that had low mass in the first place. Thus, as we decrease the temperature, most of the probability mass in the distribution concentrates around the most probable words. Conversely, if we increase the temperature to values greater than 1 and beyond, the probability mass will be more evenly distributed across all words, as the operation will decrease values of all logits. See some animations here.