Advanced Methods in Text Analytics

State Space Models

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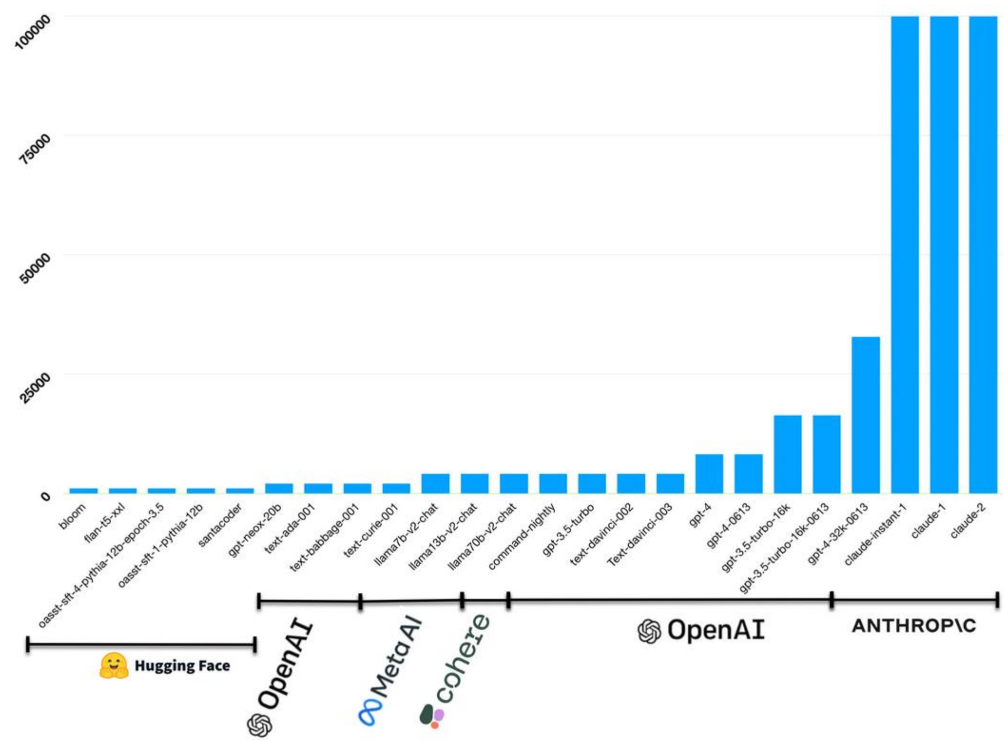
An Alternative to Transformers (1)

- Transformer-based models dominate NLP
  - Most applications tackled with learned representations
  - Learned representations obtained with transformers
- Great, right?
  - Except for quadratic cost on input sequence length
  - Severely limits size of context fed to models
  - Many efforts to reduce cost, e.g. FlashAttention (2022)
  - Recent efforts to get infinite context, e.g. TransformerFam (2024), Leave No Context Behind (2024)



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An Alternative to Transformers (2)

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- Enter: state-space models
  - Approach from control theory
  - They are sequence models that do not use self-attention
  - Great at modeling very long-range sequences (e.g. 1M tokens)
- Today:
  - **State space models**, including S4 model (precursor to Mamba), introduced structure to state space models for efficiency, reached SOTA in long range dependency tasks
  - Relation to **RNNs** (you already know more about this than you think)
  - The **Mamba architecture**, recent state space model, showed competitive performance in causal language modeling when compared with LLMs twice its size (using less memory and compute)
- **Warning:**
  - Much of this comes from a **research area** that most NLP researchers (myself included) are **likely unfamiliar** with
  - **Some things** will be discussed in **detail**, **others** will be kept at **high-level**
  - As in the literature, many things illustrated with equations, not images

Outline

1. State Space Models
2. The Mamba Architecture

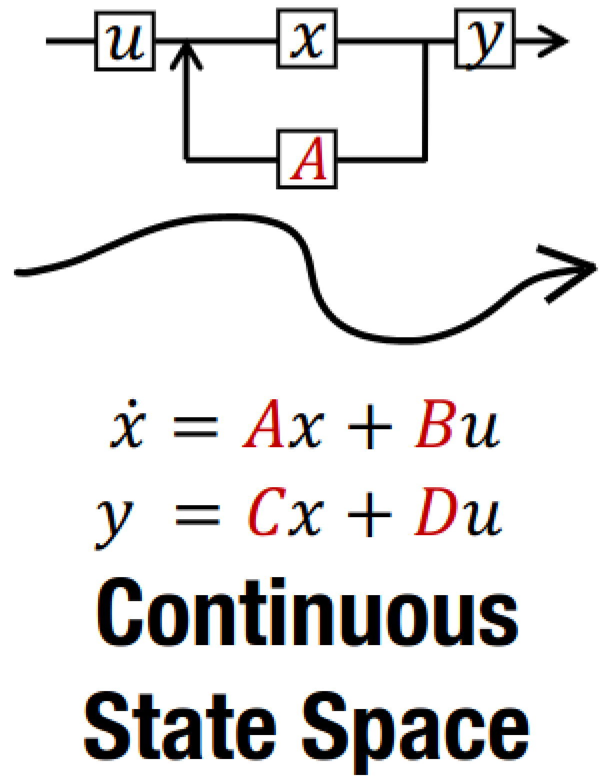
State Space Models

Continuous State Space Models (1)

- **Long range dependencies**: big challenge in sequence modelling
  - Materialized by **Long Range Arena** Benchmark (Tay et al. 2021)
  - Wide range of tasks requiring processing between 1K and 16K tokens
- Most families of sequence models have variants to tackle this challenge
  - E.g Lipschitz RNNs (Erichson et al. 2021), FlashAttention-2 (Dao et al. 2023)
- Family of models that excel at this: **state space models (SSMs)**
- A ***continuous-time*** **latent state model** does the following:
  - 1. Maps 1-dimensional signal $x(t)$ to n-dimensional latent state $\mathbf{h}(t)$
  - 2. Projects latent state (plus input signal) to 1-dimensional output signal $y(t)$
- Concretely:
  - 1. $d\mathbf{h}(t)/dt = \mathbf{Ah}(t) + \mathbf{B}x(t)$
  - 2. $y(t) = \mathbf{Ch}(t) + \mathbf{D}x(t)$
- Here, $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ are learned parameters and t is time (usually $\mathbf{D} = 0$ for simplicity because term $\mathbf{D}x(t)$ is seen as "skip connection" between input and output, easy to compute)

Continuous State Space Models (2)

- Let's note a few things about SSMs:
  - $d\mathbf{h}(t)/dt = \mathbf{Ah}(t) + \mathbf{B}x(t)$
  - $y(t) \quad = \mathbf{Ch}(t) + \mathbf{D}x(t)$
- The 1st equation is differential
  - Describes how ***h*** changes over time
- The 2nd equation is **signal-to-signal model**
- Both *d**h**(t)/dy* and *y(t)* fully described by current input signal and hidden state
  - I.e. a **Markov decision process**
  - Ideally, state encodes all past history, enables us to determine future output *y(t)*
- Great, but this is NLP!
  - **What are we supposed to do we these *continuous* SSMs?**
  - We are all about processing discrete sequences here, right?
- Yes, so **let's discretize SSMs**



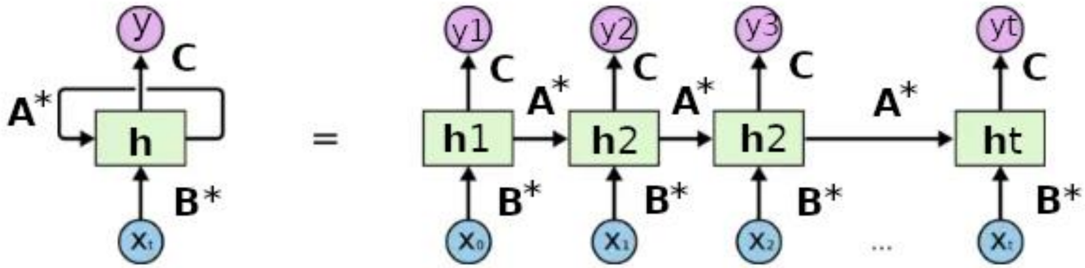
Gu et al. 2022

Discrete State Space Models (1)

- There are several ways to **discretize a continuous signal**
  - Think of it as **sampling a continuous signal at fixed time intervals**
  - Different SSMs use different approaches, each with different properties
- Simple way to get an *intuition for discretization*: the Euler method
- Let's start with our continuous model:
  - $d\boldsymbol{h}(t)/dt = \boldsymbol{A}\boldsymbol{h}(t) + \boldsymbol{B}x(t)$ (1)
  - $y(t) = \boldsymbol{C}\boldsymbol{h}(t) + \boldsymbol{D}x(t)$ (2)
- 1st, we clean up notation a bit: $x_t = x(t)$, $\boldsymbol{h}_t = \boldsymbol{h}(t)$ and $\boldsymbol{h}_{t+1} = \boldsymbol{h}(t + \Delta)$
- 2nd, by definition of a derivative: $d\boldsymbol{h}(t)/dy = (\boldsymbol{h}(t + \Delta) - \boldsymbol{h}(t)) / \Delta$ (3)
- Then, by plugging (3) in (1), we have:
  - $\boldsymbol{h}_{t+1} - \boldsymbol{h}_t = \Delta(\boldsymbol{A}h_t + \boldsymbol{B}x_t)$
  - $\boldsymbol{h}_{t+1} = \Delta\boldsymbol{A}h_t + \Delta\boldsymbol{B}x_t + \boldsymbol{h}_t$
  - $\boldsymbol{h}_{t+1} = \boldsymbol{h}_t(\Delta\boldsymbol{A} + I) + \Delta\boldsymbol{B}x_t$
  - $\boldsymbol{h}_{t+1} = (I + \Delta\boldsymbol{A})\boldsymbol{h}_t + (\Delta\boldsymbol{B})x_t$

Discrete State Space Models (2)

- If we now set $\boldsymbol{A}^* = (I + \Delta \boldsymbol{A})$, $\boldsymbol{B}^* = \Delta \boldsymbol{B}$ and set $\boldsymbol{D} = 0$, we have:
  - $\boldsymbol{h}_{t+1} = \boldsymbol{A}^* \boldsymbol{h}_t + \boldsymbol{B}^* x_t \qquad (1)$
  - $y_t = \boldsymbol{C}\boldsymbol{h}_t \qquad (2)$
- Does that look familiar?
  - I hope we haven't forgotten! It looks like an RNN



An unrolled recurrent neural network.

Image source

- Discrete SSMs are essentially linear RNNs (i.e. seq2seq models)
  - I.e. RNN with activation $f = I$ when computing $\boldsymbol{h}$, i.e. $\boldsymbol{h}_{t+1} = f(\boldsymbol{A}^* \boldsymbol{h}_t + \boldsymbol{B}^* x_t)$
  - Computing **A***, **B*** (discretization) can be seen as first step in compute graph

Discrete State Space Models (3)

- As mentioned, different SSMs use different approaches for discretization
  - None actually uses the Euler method
- E.g. S4 uses a bilinear method:
  - $\boldsymbol{A}^* = (I - \Delta/2\ \boldsymbol{A})^{-1}(I + \Delta/2\ \boldsymbol{A})$
  - $\boldsymbol{B}^* = (I - \Delta/2\ \boldsymbol{A})^{-1}\ \Delta\boldsymbol{B}$
- Mamba uses a process called zero-order hold:
  - $\boldsymbol{A}^* = exp(\Delta\boldsymbol{A})$
  - $\boldsymbol{B}^* = (\Delta\boldsymbol{A})^{-1}(exp(\Delta\boldsymbol{A}) - I)\Delta\boldsymbol{B}$
- Such models are thus parameterized by $\boldsymbol{\theta} = [\Delta, \boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}]$
- Ok, fine, so we are back to (linear) RNNs. Is that it?
- Not quite, as **RNNs fell out of favor for two specific reasons**
  - 1. Difficulty modeling long range dependencies
  - 2. Slow to train due to sequential computation of hidden states
- **SSMs need to deal with these issues** to be successful
  - Let's have a high-level look at how SSMs addressed these "RNN challenges"

Modeling Long Sequences with SSMs

- **HiPPO framework** introduced by Gu et al. 2020
  - **Hi**gh-order **P**olynomial **P**rojection **O**perators
  - Math-heavy work (described as "a bit of magic" by some sources)
- Essentially, framework that proposes a **principled way of initializing matrix *A*** so state ***h(t)*** can memorize input sequence ***x***
  - Specifically, a complex matrix in the following upper triangular form:

$$\textbf{(HiPPO Matrix)} \qquad \boldsymbol{A}_{nk} = -\begin{cases} (2n+1)^{1/2}(2k+1)^{1/2} & \text{if } n > k \\ n+1 & \text{if } n = k \\ 0 & \text{if } n < k \end{cases}$$

- More generally, they formalized notion of *memory as online function approximation* (based on approximation theory and signal processing)
  - The framework derived GRUs and LMUs from first principles
- Their results put SSMs on the map, reported **massive improvements**
  - E.g. from 68% to 98% on sequential MNIST benchmark
- Still, SSMs were expensive to train compared to Transformers
  - One RNN challenge down, one to go

Training SSMs Efficiently

- Important property of SSMs: **linear time invariance** (LTI)
  - Well known connection with convolutions (yet no citation in papers!)
- In short, because computing $\boldsymbol{h}_{t+1}$ is linear, we can unroll SSMs as follows:
  - $\boldsymbol{h}_0 = \boldsymbol{B}^* x_0$ $\boldsymbol{h}_1 = \boldsymbol{A}^* \boldsymbol{B}^* x_0 + \boldsymbol{B}^* x_1$ $\boldsymbol{h}_2 = \boldsymbol{A}^{*2} \boldsymbol{B}^* x_0 + \boldsymbol{A}^* \boldsymbol{B}^* x_1 + \boldsymbol{B}^* x_2$
  - $y_0 = \boldsymbol{CB}^* x_0$ $y_1 = \boldsymbol{CA}^* \boldsymbol{B}^* x_0 + \boldsymbol{CB}^* x_1$ $y_2 = \boldsymbol{CA}^{*2} \boldsymbol{B}^* x_0 + \boldsymbol{CA}^* \boldsymbol{B}^* x_1 + \boldsymbol{CB}^* x_2$

  and so on, where initial state $\boldsymbol{h}_{-1} = \boldsymbol{0}$.
- Thus, for input sequence $\boldsymbol{x}$, the entire output sequence is $\boldsymbol{y} = \boldsymbol{K} \odot \boldsymbol{x}$ where $\odot$ is the (discrete) **convolution** operation and $\boldsymbol{K}$ is the following (giant) convolution filter:
  - $\mathbf{K} = (\mathbf{CB^*}, \mathbf{CA^*B^*}, \mathbf{CA^{*2}B^*}, \ldots, \mathbf{CA^{*|x|-1}B^*})$
- I.e., **we can compute entire forward pass as single global convolution!**
  - Can be computed efficiently with Fast-Fourier Transforms (FFTs) once $\boldsymbol{K}$ is known
  - But computing **K** is expensive (repeated matrix multiplication by **A***)
- Achieving this in linear time was a main contribution of the S4 model

Structured State Space Models

- **S4**: **s**tructured **s**tate **s**pace **s**equence models ([Gu et al. 2022](#))
  - Again, math-heavy paper, only high-level discussion here
- **Main contribution**:
  - Computing global kernel ***K*** in linear time (normally quadratic)
- They **enforce structure** to parameter matrix ***A*** in SSMs
  - ***A*** is special form of **diagonal matrix**: diagonalizable by normal matrices
  - Work far from trivial, as matrix ***A*** had to retain the properties of the HiPPO matrix that accounts for the success of SSMs at long range dependencies
- An S4 layer was proposed to construct deep S4 models
  - **S4 block**: SSM seq2seq model + dropout + non-linearity + linear projection
  - Since S4 processes single scalar, **S4 block contains *H* copies of S4 model**
  - Then, each block processes inputs of size (batch size, input length, hidden size), same as RNNs, Transformers (but S4 is linear in input length!)
  - Can be **computed as convolution** (training) **or autoregressively** (inference) (nicely shown in code [here](#))
- So, with **S4** we get **efficient SSM training without performance loss**
  - Now let's recap SSMs vs RNNs before we look at Mamba

SSMs vs RNNs

- While similar, there are **important differences between SSMs and RNNs**
  - 1. SSMs (and linear RNNs) can be efficiently parallelized (global convolution)
  - 2. SSMs are "linear RNNs" but with special requirements for how to compute some parameters (discretization)
  - 3. SSMs are complex valued and initialized according to the HiPPO theory
- Also, note that while SSM matrices **A***, **B*** are equivalent to RNN parameters, they actually share parameters themselves (usually Δ and **A**)
- In short, **these differences are important**, account for success of SSMs!
- After S4's success, **recent work "resurrected RNNs"** ([Orvieto et al. 2023](#))
  - Started with a linear RNN, "ablated" their way to reach S4's performance
  - Specifically, they introduced **diagonalization**, **special initialization** and **parameterization**, as well as **normalization** to RNNs
  - Insightful in terms of where success of SSMs comes from
- Having "covered" how both RNN challenges were addressed by SSMs, **we can now discuss Mamba!**
  - This because the model is an extension of the S4 architecture

The Mamba Architecture

Motivation

- Perspective proposed by authors:
  - *"A fundamental problem of sequence models is compressing context into a small state"*
- For example, **self-attention** is:
  - **Effective:** does not compress context at all, stores it entirely (KV cache in causal LMs)
  - **Inefficient:** KV cache takes linear space in input sequence length
- Conversely, **RNNs** are:
  - **Efficient:** compress context in hidden state ***h*** (size usually much smaller than the long range sequences we want to model with SSMs)
  - **Ineffective:** especially for long sequences, as performance depends on how well they compress context (though the "RNN resurrection" paper improved considerably on this thanks to research in SSMs)
- In short, **efficient vs effective trade-off** summarized by how well a model compresses their state
  - Proposed **fundamental principle** for building sequence models: **selectivity**

Improving SSMs with Selection

- **Selectivity:** context-aware ability to focus on or filter out inputs into a sequential state
  - Sounds awfully familiar, right?
- In other words, an **attention-like mechanism** that **controls how information propagates** or **interacts** between elements of a sequence
  - Is it really an attention-like mechanism?
- Yes! The **main Mamba design innovation** is allowing **more access to input sequence** by letting the **SSM parameters** be **input-dependent**
  - Recall attention on RNNs was basically: *Hey, why don't we just access the input at each inference step instead of just interacting with the hidden state?*
  - Mamba is the SSM that asks the same question (likely inspired by the success of the Transformer).
- Specifically, some parameters became functions of input sequence $\boldsymbol{x}$
  - Matrix $\boldsymbol{B} = \boldsymbol{W}_B\boldsymbol{x}$, matrix $\boldsymbol{C} = \boldsymbol{W}_C\boldsymbol{x}$ where $\boldsymbol{W}_B$, $\boldsymbol{W}_C$ are learned projections
- **Problem:** this change means we lose the ability to train SSMs efficiently
  - Let's see why

Improving SSMs with Selection?

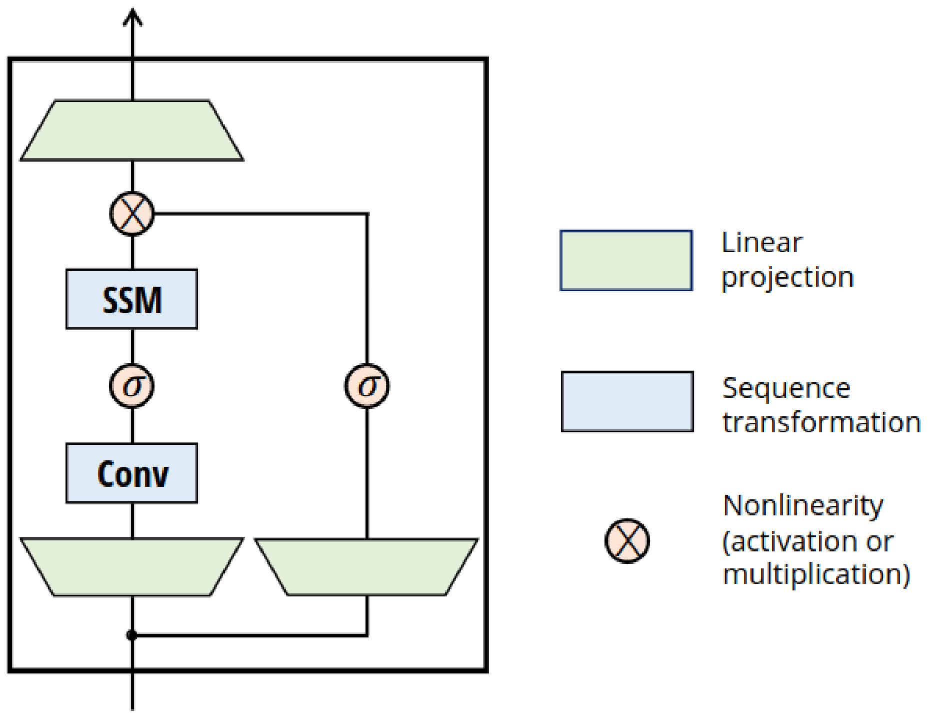
- Recall efficient training of SSMs:
  - Given input sequence ***x***, entire output sequence is ***y* = *K*** ⊙ ***x*** where ⊙ is (discrete) convolution operation and ***K*** is the following (giant) convolution kernel: $\mathbf{K} = (\mathbf{CB^*}, \mathbf{CA^*B^*}, \mathbf{CA^{*2}B^*}, \ldots, \mathbf{CA^{*|x|-1}B^*})$
  - Efficient! Compute kernel elements separately, apply [convolution theorem]()
- Now, note the following:
  - $\boldsymbol{h}_t$ depends on $\boldsymbol{h}_{t+1}$ because $\boldsymbol{h}_{t+1} = \boldsymbol{A^*h}_t + \boldsymbol{B^*}x_t$
  - Since $\boldsymbol{B} = f(x)$, $\boldsymbol{B^*} = f(x)$ because **B*** is a function (discretized form) of ***B***
  - For $t = 1$, $x = [x_1]$, for $t = 2$, $x = [x_1, x_2]$, for $t = n$, $x = [x_1, x_2, \ldots, x_n]$
- So, ***B**** is now time dependent, i.e. $\boldsymbol{B^*}_t$
  - Thus, a more accurate notation now is: $\boldsymbol{h}_{t+1} = \boldsymbol{A^*h}_t + \boldsymbol{B^*}_t x_t$ since ***B**** is a function of ***x***, which changes per time step (autoregressive models)
- In other words, **hidden states now need to be computed in sequence**
  - We lost one of the two improvements SSMs had over RNNs
  - So, another main **contribution from Mamba**: **make training efficient again**

Hardware-Aware Selectivity

- To regain training efficiency, the authors did two things:
  - 1. Switch from convolution to a scan operation
  - 2. Implement **scan in a GPU-optimized way**
- **Scan:** otherwise known as a cumulative sum
  - E.g. given sequence $x = [3, 2, 6]$, output is sequence $y_1 = 3$, $y_2 = 5$, $y_3 = 11$
  - Can be parallelized to be computed in *log n* (nicely shown here)
  - Already used for efficiently training linear RNNs (Orvieto et al. 2023)
- So, in short: **what is Mamba?**
  - An **S4 model** (structured SSM) with **S**electivity
  - An S4 model trained with the **S**can operation
  - **Hence**, authors described it as an **S6 model**
- **Transformers vs Mamba:** overall training costs for sequence of length *n*
  - **Memory:** $O(n)$ vs $O(1)$ (KV cache vs hidden state)
  - **Runtime:** $O(n^2)$ vs $O(n)$ (self-attention vs GPU-optimized SSM)
- In other words, massive improvements!
  - If performance on par with LLMs, we may have a new king

The Mamba Block

- As with the Transformer, we have a **Mamba block**
- **Inter-token communication:**
  - Transformer: self-attention
  - Mamba: SSM
- **Intra-token computation:**
  - Transformer: MLP
  - Mamba: MLP
- **Efficiency:**
  - Transformer: linear attention
  - Mamba: linear RNN
- **Expressivity:**
  - Transformer: non-linearity between MLP projections
  - Mamba: non-linearity between MLP projections
- In practice:
  - Up-projection by a factor of 2 (compared to usual 4 in LLMs)
  - Silu/Swish activations, optional layer normalization before/after block



Gu et al. 2023

Mamba vs LLMs

- Some **results** on **downstream tasks** (with **5x faster inference** compared to transformer-based LLMs)

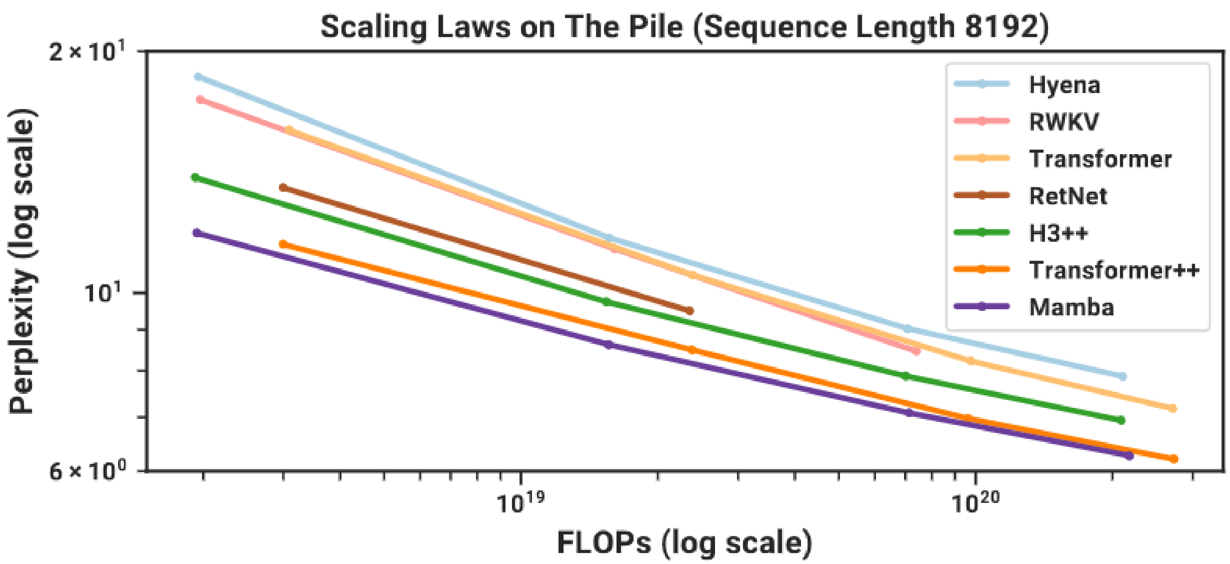
MODEL	TOKEN.	PILE PPL ↓	LAMBADA PPL ↓	LAMBADA ACC ↑	HELLASWAC ACC ↑
GPT-Neo 2.7B	GPT2	—	5.63	62.2	55.8
Hybrid H3-2.7B	GPT2	—	7.92	55.7	59.7
OPT-2.7B	OPT	—	5.12	63.6	60.6
Pythia-2.8B	NeoX	6.73	5.04	64.7	59.3
RWKV-3B	NeoX	7.00	5.24	63.9	59.6
Mamba-2.8B	NeoX	**6.22**	**4.23**	**69.2**	**66.1**
GPT-J-6B	GPT2	–	4.10	68.3	66.3
OPT-6.7B	OPT	–	4.25	67.7	67.2
Pythia-6.9B	NeoX	6.51	4.45	67.1	64.0
RWKV-7.4B	NeoX	6.31	4.38	67.2	65.5

[Gu et al. 2023]

- Some **concerns** from the community:
  - Are baselines strong enough? E.g. no latest OpenAI models
  - More importantly, **why no comparison with larger LLMs?**

Scaling Laws

- Authors also provided scaling laws on all tested tasks (here only LM)
  - They followed the Chinchilla protocol
  - Roughly, statistical model that measures loss as function of model size



Gu et al. 2023

- **"First attention-free model to match performance of very strong Transformer recipe (Transformer++)"** (with cheaper memory/compute!)
  - Transformer++ recipe: Rotary, SwiGLU, MLP, RMSProp, no linear bias, higher learning rate (PaLM, LLaMA)

A Different Paradigm (1)

- Nice discussion on a potential paradigm shift by Kola Ayonrinde
  - Specifically about using pre-trained causal language models (CLMs)
- **Sources of information** in transformer-based CLMs **at inference time?**
  - **Training data**, i.e. pre-trained weights (long-term memory, but compressed)
  - **In-context data**, i.e. prompt (short-term memory, must be read every time)
- Do we have **selectivity when using such CLMs?**
  - Yes, via **prompting** (what to leave in or out), **RAG**, etc.
  - This because entire prompt is accessed (attention) to predict next tokens
- What about when using **SSM-based CLMs like Mamba**?
  - Training data is there in the same way, i.e. compressed long-term memory
  - Selectivity is a core design of the system as it reads any input
  - But **in-context data is also long-term now**! Why?
- Because **we can store the last hidden state, plug it in later as needed**!
  - Transformers not designed to have a state or representation plugged in
  - They must instead "read" the entire input to produce contextualized vectors

A Different Paradigm (2)

- This is **essentially a new form of prompting**
  - Swapping hidden states as needed
  - A modularity akin to swapping LoRA modules
- Remember: **SSMs are very good at long range dependencies**
  - We could potentially read entire set of textbooks, store hidden state $\boldsymbol{h}_n$
  - Then, at inference time, use $\boldsymbol{h}_n$ as initial state of our prompt
- **Question:** can transformers' contextual embeddings also act as states?
  - There is already some work on this, so maybe...
- And with that, our discussion on SMMs comes to an end
  - So, let's summarize!

Summary

- **State Space Models (SSMs):**
  - **Sequence models similar to linear RNNs**, but with **key differences**
  - Differences make them **very good at long range dependencies**
  - This ability comes from **principled design choices** (HiPPO framework)
- **Mamba Architecture:**
  - **SSM with attention-like mechanism** for selectively using input sequence
  - Performs **competitively with LLMs twice its size**
  - Scaling laws show promise for scaling up the model
- SSMs vs Transformers
  - **SSMs much more efficient**, memory: $O(1)$ vs $O(n)$, compute: $O(n)$ vs $O(n^2)$
  - But **no results on Mamba vs largest LLMs yet**, e.g. 70B or larger

References

- Efficiently Modeling Long Sequences with Structured State Spaces by Gu et al, 2022
- The Annotated S4 by Sasha Rush and Sidd Karamcheti
- Resurrecting Recurrent Neural Networks for Long Sequences by Orvieto et al. 2023
- Mamba: Linear-Time Sequence Modeling with Selective State Spaces by Gu and Dao, 2023
- Mamba Explained by Kola Ayonrinde
- Mamba from Scratch by Algorithmic Simplicity
- References linked in corresponding slides