Bézout's Identity

Theorem:
GCD of 2 ints x, $y = 1 \Leftrightarrow$
$\exists$ a, b s.t. $ax + by = 1$.

a and b are called the Bézout coefficients of x and y.

A pair of Bézout coefficients of 10, 3 are
1, -3 because
$1\times10 + -3\times3=1$

Bézout coefficients are not unique. -2, 7 work too:
$-2\times10 + 7\times3=1$

Since $ax + by = 1$ we could multiply both sides of this equation by any arbitrary integer n to get:

$(na)x + (nb)y = n$ *(note that na, nb are also integers)*

That means a pair of integers with no common factors (i.e. GCD=1) can be used to construct *any* integers with some integer multiples of them.

Coin Problem

For coins, however, we are not supposed to have -ve multiples of them!

Therefore some smaller sums are impossible…

10x→ 3x ↓	-2	-1	0	1	2	3	4	5	6	7	8
-3	-29	-19	-9	1	11	21	31	41	51	61	71
-2	-26	-16	-6	4	14	24	34	44	54	64	74
-1	-23	-13	-3	7	17	27	37	47	57	67	77
0	-20	-10	0	10	20	30	40	50	60	70	80
1	-17	-7	3	13	23	33	43	53	63	73	83
2	-14	-4	6	16	26	36	46	56	66	76	86
3	-11	-1	9	19	29	39	49	59	69	79	89
4	-8	2	12	22	32	42	52	62	72	82	92
5	-5	5	15	25	35	45	55	65	75	85	95
6	-2	8	18	28	38	48	58	68	78	88	98
7	1	11	21	31	41	51	61	71	81	91	101
8	4	14	24	34	44	54	64	74	84	94	104

Solution

Theorem: With enough coins, any sum $\geq (x-1)(y-1)$ can be made.

Enumerate smaller possible sums:
3, 6, 9, 10, 12, 13, 15, 16
I.e., a total 9 impossible sums: 1, 2, 4, 5, 7, 8, 11, 14, 17

However, with a limit of 10 coins each, impossible sums towards the maximum sum also have to be discounted – cannot take away an impossible sum from the maximum sum (130).
$\therefore$ Total # of possible sums = 130 − 9 − 9 = 112

Proof of the 2-coin theorem

$\mathrm{GCD}(x, y) = 1$

$ax + by = c$ for some int c (a, b are multiples of Bézout coeffs)

$b = kx + r$ for some k, r, $0 \leq r < x$

$\Rightarrow c = ax + (kx + r)y$

$\quad = x(a + ky) + ry$

$\quad = nx + ry$ for some n

$r, x, y \geq 0$. k, a can be < 0

$\therefore$ For an invalid sum, n must be < 0.

Largest c for an invalid sum: $n = -1$, $r = x - 1$ (maximise r-value)

$c = -x + (x - 1)y$

$= -x + xy - y$

$= (x - 1)(y - 1) - 1$

$\therefore$ any sum $\geq (x - 1)(y - 1)$ is always possible