Control Theory Homework 3

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Given non-linear system:

$$\begin{cases} (M+m)\ddot{x} - mlcos(\theta)\ddot{\theta} + mlsin(\theta)\dot{\theta}^2 = F \\ -cos(\theta)\ddot{x} + l\ddot{\theta} - gsin(\theta) = 0 \end{cases}$$

I have chosen next coefficients:

$$M = 4, m = 1, l = 0.8$$

Solution:

- We can write equations of motion of the system in manipulator form, where $q = \begin{bmatrix} x & \theta \end{bmatrix}^T$, $u = F$:

$$H(q)\ddot{q} + n(q,\dot{q}) = Bu$$

$$\begin{bmatrix} (M+m) & -mlcos(\theta) \\ -cos(\theta) & l \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} mlsin(\theta)\dot{\theta}^2 \\ -gsin(\theta) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} F$$

- We can write dynamics of the system in control affine nonlinear form:

$$z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix}, u = F$$

$$\dot{z} = f(z) + g(z)u$$

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = \begin{bmatrix} z_3 \\ z_4 \\ \begin{bmatrix} (M+m) & -mlcos(z_2) \\ -cos(z_2) & l \end{bmatrix}^{-1} \begin{bmatrix} -mlsin(z_2)z_4^2 \\ gsin(z_2) \end{bmatrix} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \begin{bmatrix} (M+m) & -mlcos(z_2) \\ -cos(z_2) & l \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix} F$$

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = \begin{bmatrix} z_3 \\ z_4 \\ \frac{-mlsin(z_2)z_4^2 + mgcos(z_2)sin(z_2)}{M+m-mcos^2(z_2)} \\ \frac{-mlcos(z_2)sin(z_2)z_4^2 + (M+m)gsin(z_2)}{(M+m-mcos^2(z_2))l} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{M+m-mcos^2(z_2)} \\ \frac{cos(z_2)}{(M+m-mcos^2(z_2))l} \end{bmatrix} F$$

- We can linearize nonlinear dynamics of the systems around equilibrium point $\bar{z} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T$:

$$\delta\dot{z} = A\delta z + B\delta u$$

$$A = \frac{\partial f(z)}{\partial z} = \begin{bmatrix} \frac{\partial f_1}{\partial z_1} & \frac{\partial f_1}{\partial z_2} & \frac{\partial f_1}{\partial z_3} & \frac{\partial f_1}{\partial z_4} \\ \frac{\partial f_2}{\partial z_1} & \frac{\partial f_2}{\partial z_2} & \frac{\partial f_2}{\partial z_3} & \frac{\partial f_2}{\partial z_4} \\ \frac{\partial f_3}{\partial z_1} & \frac{\partial f_3}{\partial z_2} & \frac{\partial f_3}{\partial z_3} & \frac{\partial f_3}{\partial z_4} \\ \frac{\partial f_4}{\partial z_1} & \frac{\partial f_4}{\partial z_2} & \frac{\partial f_4}{\partial z_3} & \frac{\partial f_4}{\partial z_4} \end{bmatrix}\Bigg|_{z=\bar{z}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{mg}{M} & 0 & 0 \\ 0 & \frac{(M+m)g}{Ml} & 0 & 0 \end{bmatrix}$$

$$B = \frac{\partial g(z)u}{\partial u} = g(z)\big|_{z=\bar{z}} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{M} \\ \frac{1}{Ml} \end{bmatrix}$$

$$\delta \dot{z} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{mg}{M} & 0 & 0 \\ 0 & \frac{(M+m)g}{Ml} & 0 & 0 \end{bmatrix} \delta z + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{M} \\ \frac{1}{Ml} \end{bmatrix} \delta u$$

- To check the stability of the linearized system, we need to check the eigenvalues of matrix A from previous step:

  $$\text{Characteristic equation: } \lambda^2(\lambda^2 - \frac{(M+m)g}{Ml}) = 0$$

  $$\text{Eigenvalues: } \lambda_1 = 0, \lambda_2 = 0, \lambda_3 = \sqrt{\frac{(M+m)g}{Ml}} > 0, \lambda_4 = -\sqrt{\frac{(M+m)g}{Ml}} < 0$$

  As we can see the system is **unstable**.

- To check whether the system is controllable or not we need to check the rank of matrix $\zeta = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix}$

  $rank(\zeta) = 4$=> Linearized system is **controllable**.

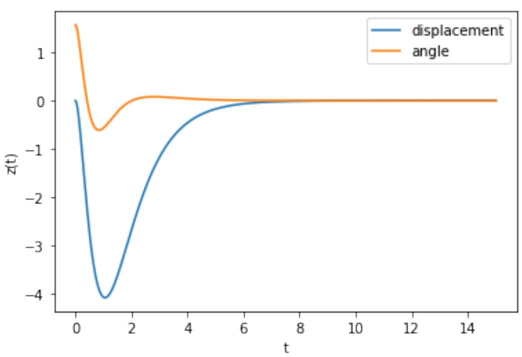
- State feedback controller u = -Kx was designed using pole placement method:

  Poles are -1, -2, -3, -4.

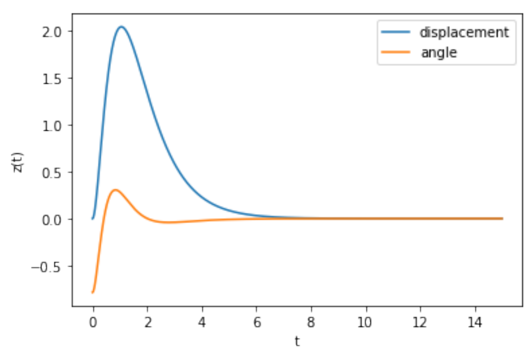
  $$K = \begin{bmatrix} -7.83673469 & 167.26938776 & -16.32653061 & 45.06122449 \end{bmatrix}$$

  Without initial displacement, only angle is varying:

$x = 0,\ \theta = 90°,\ \dot{x} = 0,\ \dot{\theta} = 0$

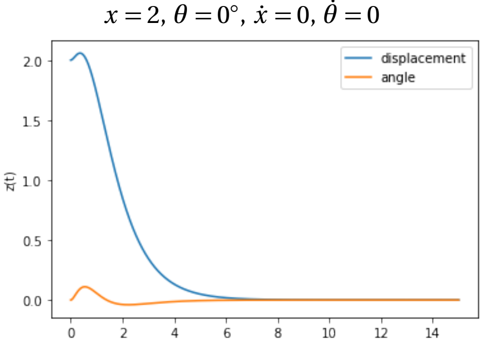


$x = 0,\ \theta = -45°,\ \dot{x} = 0,\ \dot{\theta} = 0$

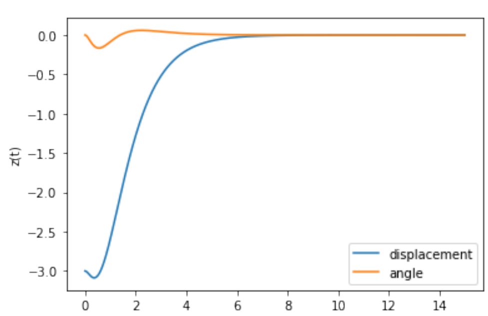


Without initial angle, only initial displacement is varying:

$x = 2,\ \theta = 0°,\ \dot{x} = 0,\ \dot{\theta} = 0$

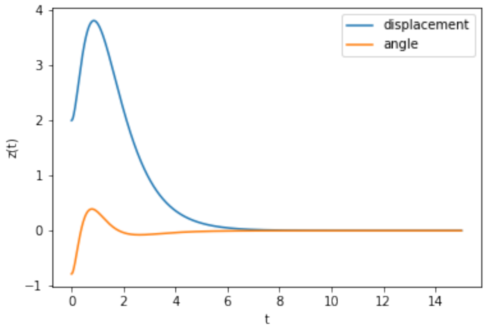


$x = -3,\ \theta = 0°,\ \dot{x} = 0,\ \dot{\theta} = 0$

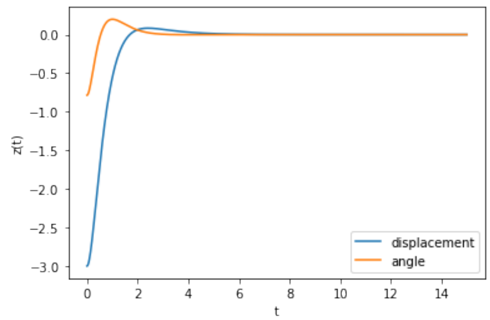


With initial displacement and angle:

$x = 2,\ \theta = -45°,\ \dot{x} = 0,\ \dot{\theta} = 0$

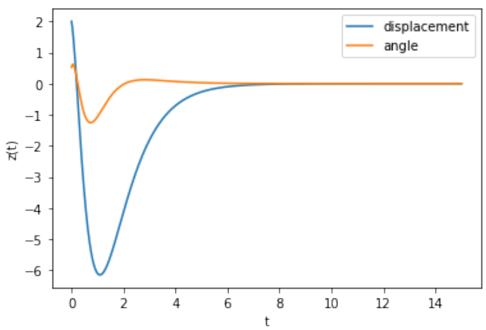


$x = -3,\ \theta = -45°,\ \dot{x} = 0,\ \dot{\theta} = 0$

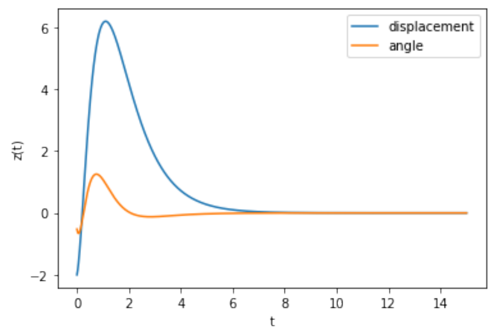


With initial displacement, angle, velocity, angular velocity:

$x = 2,\ \theta = 30°,\ \dot{x} = -5,\ \dot{\theta} = 4$

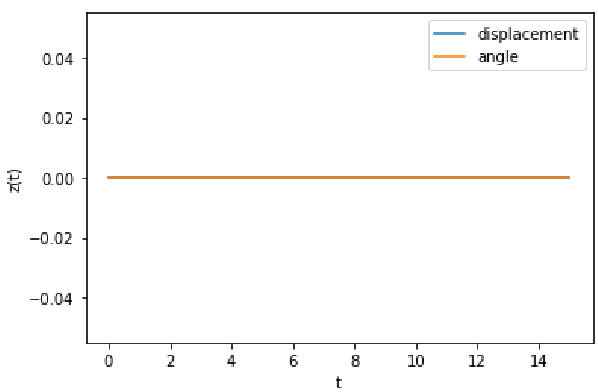


$x = -2,\ \theta = -30°,\ \dot{x} = 4,\ \dot{\theta} = -5$



Without initial displacement, angle, velocity, angular velocity:

$x = 0,\ \theta = 0°,\ \dot{x} = 0,\ \dot{\theta} = 0$



As we can see performance of controller is pretty good, it reaches equilibrium state in 6 seconds approximately and stays there without oscillations.

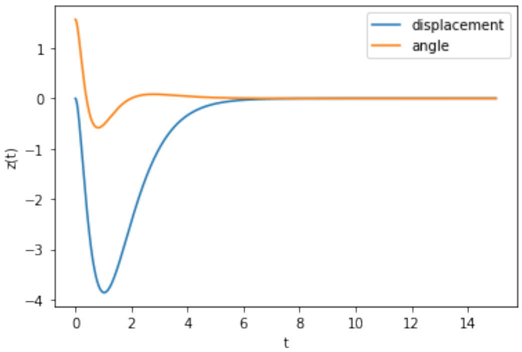
- Linear quadratic regulator was designed, with Q and R chosen:

$$Q = 100 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ R = \begin{bmatrix} 1 \end{bmatrix}$$

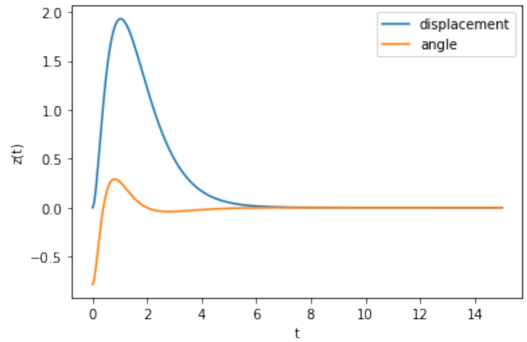
$$K = \begin{bmatrix} -10 & 189.41311337 & -19.66106239 & 52.13000272 \end{bmatrix}$$

Without initial displacement, only angle is varying:

$x = 0,\ \theta = 90°,\ \dot{x} = 0,\ \dot{\theta} = 0$

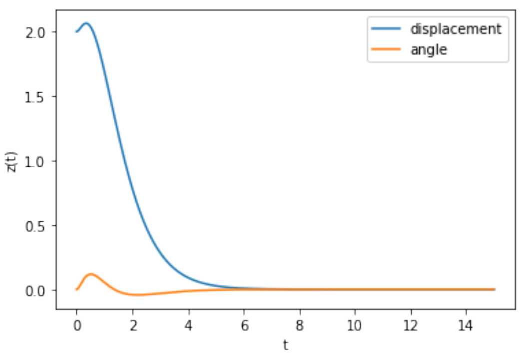


$x = 0,\ \theta = -45°,\ \dot{x} = 0,\ \dot{\theta} = 0$

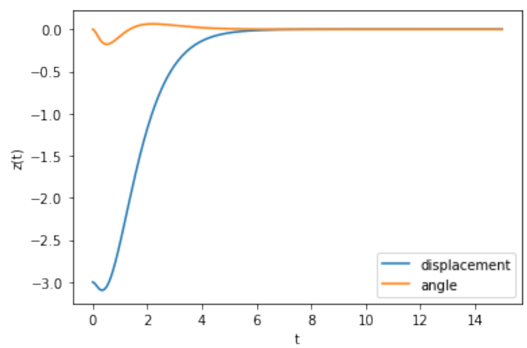


Without initial angle, only initial displacement is varying:

$x = 2,\ \theta = 0°,\ \dot{x} = 0,\ \dot{\theta} = 0$

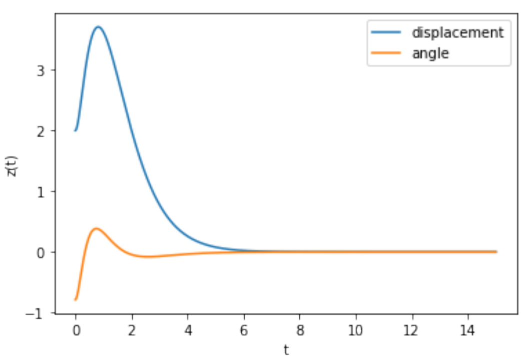


$x = -3,\ \theta = 0°,\ \dot{x} = 0,\ \dot{\theta} = 0$

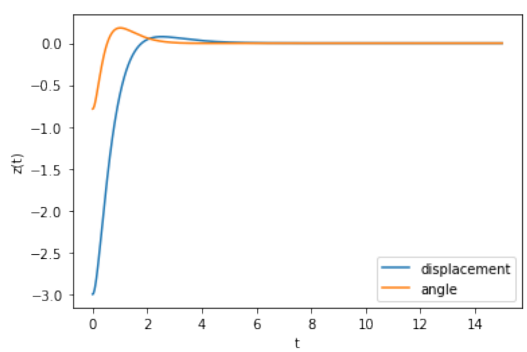


With initial displacement and angle:

$x = 2,\ \theta = -45°,\ \dot{x} = 0,\ \dot{\theta} = 0$

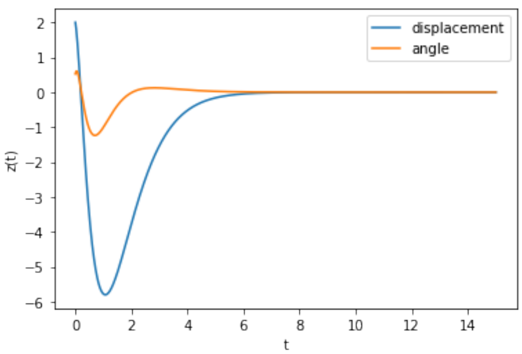


$x = -3,\ \theta = -45°,\ \dot{x} = 0,\ \dot{\theta} = 0$

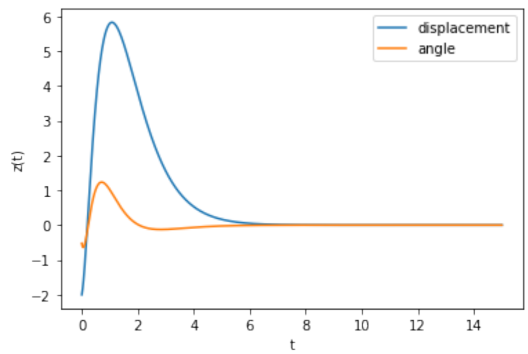


With initial displacement, angle, velocity, angular velocity:

$x = 2,\ \theta = 30°,\ \dot{x} = -5,\ \dot{\theta} = 4$

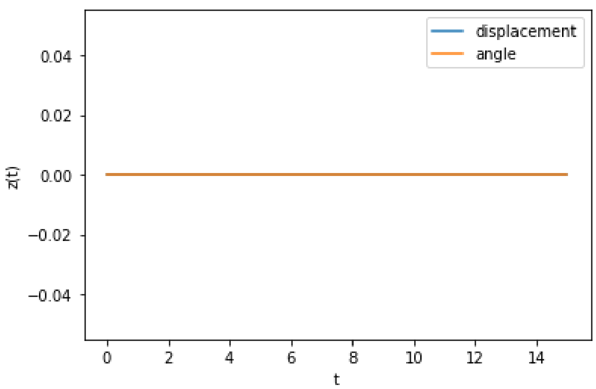


$x = -2,\ \theta = -30°,\ \dot{x} = 4,\ \dot{\theta} = -5$



Without initial displacement, angle, velocity, angular velocity:

$x = 0,\ \theta = 0°,\ \dot{x} = 0,\ \dot{\theta} = 0$



As we can see performance of controller is a little bit better than pole placement method, because this is optimal controller, it also reaches equilibrium state in 6 seconds and stays there without oscillations.

- Initial conditions:
  First two initial conditions for verifying how controller behaves only with initial angle
  Second two initial conditions for verifying how controller behaves only with initial displacement
  Third two initial conditions for verifying how controller behaves only with initial displacement and angle
  Fourth two initial conditions for verifying how controller behaves with all initial components
  Last one for verifying how controller behaves in equilibrium state as initial condition.

  All displacement, velocity, angular velocity were chosen at random, but angle was chosen from $[-90°; 90°]$