GENERAL NETWORK EFFECTS AND WELFARE

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Abstract. (In)direct network effects arise frequently in economic models but, for reasons of analytical tractability, are often assumed to be linear. Here, we examine the general non-linear case with two platforms and show that the assumption of linearity is not 'innocent' but constrains equilibria and welfare outcomes in important ways. We establish the general conditions characterising equilibria and show that welfare changes can be related in a simple, intuitive way to the degree of diminishing returns of the network effects function.

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1. Introduction

Since the early work of Katz and Shapiro (1985), Farrell and Saloner (1986), and Arthur (1989) there has been substantial work on models which involve 'network effects' whether direct or indirect. Furthermore, many of the recent models of platform and two-sided markets display reduced form indirect network effects – see e.g. Church et al. (2003).[1] Much of this literature has focused, for reasons of analytical tractability, on the case linear network effects (and heterogeneity).[2] It is natural to wonder whether such a restriction is 'innocent'.

In this paper we analyse the case of general network effects. Focusing on the classic case of a Hotelling line of consumers choosing between two platforms/networks, we characterise

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[1]See also e.g. Church and Gandal (1992), Caillaud and Jullien (2003). For examples of work on two-sided markets see e.g. Armstrong (2006); Nocke et al. (2007).

[2]Where this is not the case, such as in Church et al. (2003), only one specific functional form is considered.

equilibria in terms of the 'advantage' function and use this to sign welfare changes in the neighbourhood of an equilibrium as a function of a) a simple change in network size and b) a change in an 'exogenous' variable such as a platform owner's price. It is shown that these changes are closely related to the degree of diminishing returns in the network effects function with two distinct categories. Linear network effects exhibit no diminishing returns and fall clearly within one of these two. As such, the choice of linearity is not 'innocent'.

Finally, we note that our work complements the (limited) literature on the functional form of network effects. For example, Swann (2002) (who summarizes most of the existing, largely informal, work) examines the case of a telephone network and shows that network effects obey Metcalfe's law (linearity) only under fairly restrictive conditions. When these conditions are not satisfied, he argues that network effects will be S-shaped.[3] This takes on a new relevance in the context of this paper as our results show that the equilibria and welfare outcomes may differ markedly when on the 'upward' (increasing returns) and 'downward' (decreasing returns) portion of such an S-shaped network effects function. We would also point out that, in two-sided models, standard (e.g. spatial or monopolistically competitive) models of imperfect competition on the 'seller/service' side result in indirect non-linear network effects with sharply diminishing returns.[4]

2. The Model

We start with the standard (in)direct networks model. There are two platforms/networks: $X = A, B$ located at either end of a Hotelling line of consumers (buyers) modelled by the interval $[0, 1]$. The index, $t \in [0, 1]$, is used to label consumers and the measure of consumers on platform X is denoted by n_X. All consumers are assumed to purchase from at most one platform and for simplicity we assume all consumers purchase so $n_B = 1 - n_A$.[5] Where no confusion will result we drop the subscript on n_A and just write n for A's market share. A consumer who purchases from platform $X = A, B$ has a utility function which

[3]Arguing along more informal lines, Odlyzko and Tilly (2005) also reject Metcalfe's law and propose the logarithmic form as a replacement.
[4]See Church et al. (2003); Pollock (2007).
[5]This assumption is standard to the literature. It simplifies the analysis and places no great restriction on the results.

displays (reduced form) network effects:[6]

$$u_X(t, n_X, P, Z) = g_X(P) - h_X(t) + \nu_X(n_X^e, Z, P)$$

P here represents a set of variables controlled by the platform owners – for example platform price – while Z is some set of exogenous variables outside the control of any participant (e.g. production costs). In most work, the ownership of platforms and the associated competitive structure would be the focus of interest. Here we wish to leave this largely to one side and simply assume that control variables enter the utility function in some general way as described. h_X represents consumer heterogeneity ('travel costs'), ν_X the indirect network effects function. All functions are assumed to be continuous and differentiable. The full sequence of actions in this model is then as follows:

(1) Platform 'Owners' move (setting P).
(2) Consumers form (common) expectations of platform sizes. Given these, each consumer joins the platform which maximizes expected utility.
(3) Payoffs are realized.

Note that in equilibrium the resulting platform sizes must be consistent with rational expectations. That is: actual and expected platform sizes must be equal. In this paper we will be focusing only on the solution of the game from step 2 onwards taking P as given.

3. Solving the Model

We proceed by the usual method based on finding the marginal consumer indifferent between the two platforms though introducing some new terminology and a novel 'advantage function' approach that will be of benefit when we come to consider welfare.

Define the *conditional utility advantage* of platform A over platform B for consumer t when (expected) platform size is n_A (suppressing Z, P variables for conciseness):

$$\hat{A}(t, n_A) = u_A(t, n_A) - u_B(t, 1 - n_A)$$

[6]One can derive this functional form from a more complex two-sided platform model in which consumers care not about a platform per-se but about obtaining 'software/services' ('sellers') associated with the platform. See Church et al. (2003); Pollock (2007).

and the *utility advantage (function)*, as the utility advantage of platform A over B if t is the marginal consumer (so $t = n_A$):

$$A(t) = \hat{A}(t,t) = \nu_A(t) - \nu_B(1-t) - (h_A(t) - h_B(t)) + (g_A - g_B)$$

Define $E_0 = \{t : A(t) = 0\}$ (the interior equilibria), $E_{-0} = \{0 : A(0) < 0\} \cup \{1 : A(1) > 0\}$ (the 'standardization' equilibria), and $E = E_0 \cup E_{-0}$.

Proposition 1. *The equilibria of the subgame are E. An equilibrium $t_e \in E_0$ is stable (with respect to perturbations in expectations) if $A'(t_e) < 0$. All $t_e \in E_{-0}$ are stable.*

Proof. See appendix. □

Note that the advantage function implicitly depends on P, Z and therefore so does the set of equilibria $E = E(P, Z)$. As such we can also think of a given equilibria being a function of P, Z (at least for small changes in their value): $t_e = t_e(P, Z)$.[7] As such t_e can be seen as a reduced-form demand function for platform A.

4. Welfare

Total welfare, W, equals the sum of consumer welfare and profits: $W = W^C + \Pi$ where W^C is consumer welfare and $\Pi = \Pi_A + \Pi_B$ are profits accruing to those associated with the two platforms.[8] We shall not specify the detail of the profits function other than to make the weak assumption hat it is a function of choice variables and market share alone $\Pi = \Pi(P, t(P, Z))$. Consumer welfare can be derived directly from the utility function and is given by (suppressing all variables other than n):

$$W^C(n,P,Z) = -g_B + n(g_A - g_B) + n\nu_A(n) + (1-n)\nu_B(1-n) - \int_0^n h_A(x)dx - \int_n^1 h_B(x)dx$$

Consider a change in one of the P or Z variables, denoted by p:

$$\frac{dW^C}{dp} = \frac{\partial W^C}{\partial p} + \frac{dn}{dp}\frac{dW^C}{dn}$$

[7]Pick initial values P^0, Z^0 and a particular equilibrium $t_e^0 \in E(P^0, Z^0)$. Then for small changes in P, Z the solution of $A(t, P, Z)$ will vary continuously with P, Z and this is then $t_e(P, Z)$. The reason we must restrict ourselves to small changes is the possibility, due to the effect of the network 'externality', of a sudden tipping from one equilibrium to another.

[8]Implicitly equal weight are being given to profits and consumer welfare.

There are clearly two distinct effects: a direct one (the first term) and an indirect one (the second) which operates via a change in platform sizes. We consider the indirect effect first.

Proposition 2. *When current platform size is n_0 the marginal change in consumer welfare with respect to platform size n is:*

$$\frac{dW^C}{dn} = A(n_0) + \mu(n_0)$$

where $A(n)$ is the utility advantage function and $\mu(n) \equiv n\nu'_A(n) - (1-n)\nu'_B(1-n)$.

Proof. Differentiate with respect to n and use the definition of $A(n)$ and $\mu(n)$. □

Corollary 3. *At an interior equilibrium $n_e \in (0,1), A(n_e) = 0$, and hence $\frac{dW^C}{dn} = \mu(n_e)$. Thus, at an interior equilibrium, the change in consumer welfare, with respect to platform size, is a function of 'network effects' alone.*

This is a significant result: even though both network effects and heterogeneity are fully general, at an equilibrium we may reduce the welfare impact of a change in platform size purely to the marginal impact on network benefits on the two platforms (encapsulated in μ).

Let us examine μ a little further. First, restrict to the case where the network effects function on the two platforms is the same: $\nu_A = \nu_B = \nu$. Then μ is anti-symmetric with regard to swapping the roles of the two platforms ($n \to 1-n$) and thus, WLOG, we may restrict to $n_e \in [0.5, 1]$. There are two basic possibilities: consumer welfare is increasing ($\mu(n_e) > 0$) with platform size – this corresponds to 'standardization' being preferable (all consumers on one platform); or, it is decreasing ($\mu(n_e) < 0$) with platform size – more symmetrical platform shares are preferable.

The sign of $\mu(n_e)$ is itself determined by the degree of curvature of the indirect network effects function, ν which corresponds to the degree of diminishing returns to network effects in platform size (how rapidly the benefit of a new user falls as the number of users on the platform grows). Simple examination indicates that the dividing line between the two cases is given by the natural logarithm: $\nu(n) = C + ln(n)$. When marginal network effects fall with platform size more gradually (e.g. ν linear) then $\mu > 0$ and so standardization is preferable. When marginal network effects fall more strongly (e.g. $\nu(n) \propto -1/\sqrt{n})$[9] then $\mu < 0$ and symmetry is preferable.

Returning to overall consumer welfare we can summarize the situation in a table 1. As this shows, depending on the signs of the different effects, the overall impact is positive, negative or ambiguous and the patterns of these outcomes varies sharply with the degree of diminishing returns in the networks effect function with what is ambiguous in one case being certain in the other and vice-versa.

To proceed further we need to impose some restrictions. To this end focus upon the case where the variable being modified p comes from the set of choice variables P. Additionally, let us suppose that the p variable is under the control of the A platform (the following discussion applies with the obvious modifications to the case where p is controlled by B).

Network Effects	$\frac{\partial W^C}{\partial p}$	$\frac{dn}{dp}$	$\frac{dW^C}{dn}$	**Consumer Welfare**
Weakly Diminishing	+	+	+	+
	+	-	+	?
	-	+	+	?
	-	-	+	-
Strongly Diminishing	+	+	-	?
	+	-	-	+
	-	+	-	-
	-	-	-	?

Table 1. Consumer Welfare Changes. '?' indicates the effect is ambiguous.

First, observe that almost always the interests of consumers and platform owners are anti-aligned. That is, whatever benefits a platform owner (for example a higher price) reduces consumer utility. Assuming, WLOG, that increases in p benefit the platform this statement implies that direct effect is negative: $\frac{\partial W^C}{\partial p} < 0$. This effect can operate via two distinct routes: an increase in p reduces either u_A or u_B. The first case corresponds, for example, to the platform raising its price to increase revenue from its own users, while the second to a situation where the platform is reducing the attractiveness of the competitor in some way (e.g. by raising costs). Let us refer to these two cases as the 'own platform' and 'other platform' cases. In terms of market share n the 'own platform' case corresponds to $\frac{dn}{dp} < 0$ while the 'other platform' case corresponds to $\frac{dn}{dp} > 0$. Putting this together

[9]This form can be obtained from a standard spatial product differentiation (circular city) model of indirect network effects – see Pollock (2007).

we have table 2.[10] This clearly shows the central role played by the form of the network effects function. In particular, the weak and strong cases are 'mirror images' of each other and thus the conclusions reached in models with weakly diminishing returns may differ markedly from those with strongly diminishing ones.

Network Effects	**Own Platform**	**Other Platform**
Weakly Diminishing	-	?
Strongly Diminishing	?	-

TABLE 2. Welfare change as p increases. '?' indicates effect is ambiguous.

5. CONCLUSION

We have examined the case of two competing platforms/networks when heterogeneity and network effects have a very general form. The welfare effects of a change in network sizes, and hence of variables such as prices, depend crucially on the degree of diminishing returns in the (in)direct network effects function. With weakly diminishing returns (e.g. linear) increasing platform size (all else being equal) increases welfare but the opposite is true when diminishing returns are strong (sub-log). Much of the literature focuses on the linear case. However there is no great theoretical or empirical justification for this choice. In many real-world cases, especially where network effects are indirect and arise from an underlying 'platform' model, diminishing returns are likely to be substantial and certainly not zero as with the linear case. In such circumstances the linearity assumption will not be an 'innocent' simplification but will have a significant effect on the results obtained.

APPENDIX A. PROOF OF 1

Proof. Fix expectations of network size n^e and suppress (for the present) n^e in $\hat{A}(t, n^e)$ to give $\hat{A}(t)$. A consumer chooses platform A over B if $\hat{A}(t) \geq 0$. Now $\hat{A}'(t) = \frac{d\hat{A}(t,n^e_A)}{dt} = -h'_A(t) - h'_B((1-t) < 0$ so $\hat{A}(t)$ is decreasing. Hence, if consumer t_m chooses platform A then all consumers $t \in [0, t_m)$ choose platform A (and vice-versa for B).

There now three possibilities. 1) $\hat{A}(t) > 0, \forall t$ in which case all consumers choose A; 2) $\hat{A}(t) < 0, \forall t$ in which case all consumers choose B; 3) $\exists t_m \in [0,1], \hat{A}(t_m) = 0$ and the resulting platform size of A is t_m (since $\hat{A}' < 0$ there is at most one such solution).

[10]Note that, at an equilibrium of the overall game, P variables are being chosen to maximize profits Π. In this case $\frac{d\Pi}{dp} \leq 0$ (own effects are zero and the impact on a competitor will be zero or negative) hence the results shown in the table will extend from consumer welfare to total welfare.

Thus one may define a function $f : [0,1] \to [0,1]$ where for a given expected platform size, n^e, $f(n^e)$ is the resulting platform size. Imposing rational expectations requires that n is an equilibrium if and only if $f(n) = n$. Now, $n \in (0,1)$ is a solution of $f(n) = n \iff \hat{A}(n,n) = 0 \iff n \in E_0$ while $n = 0, 1$ is a solution of $f(n) = n \iff n \in E_{-0}$.

Stability. Take $t_m \in E_0$ with $A'(t_m) = -2k < 0$ and consider a perturbation in expectations of network size $\epsilon > 0$. $A' < 0$ so for ϵ small enough have $\hat{A}(t_m + \epsilon, t_m + \epsilon) = A(t_m + \epsilon) = -j < -k\epsilon$. Now $\hat{A}$ is continuous and decreasing in its first argument hence $\exists \delta \in (0, \epsilon)$ (unique) such that $\hat{A}(t_m + \epsilon - \delta, t_m + \epsilon) = 0$ and all consumers with indices $x \in (t_m + \epsilon - \delta, t_m + \epsilon]$ wish to switch back to B (as $\hat{A}(x, t_m + \epsilon) < 0$). We can now replace ϵ by $\epsilon - \delta$. Repeating this process we converge back to the equilibrium t_m (were we to halt before reaching t_m at $t' \in (t_m, t_m + \epsilon)$ say, we would have $A(t') = 0$ which is impossible). The same argument can be made for negative ϵ. Thus the equilibrium is stable. For an equilibrium $t_m \in E_{-0}$ we must have $|A(t_m)| < 0$ and hence by continuity of $A, \hat{A}$ it too is stable. □

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