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Solution for problem 511

We claim that the minima is 94 which is attained when $(x_1, x_2, \cdots, x_{40}) = (\underbrace{1, 1, \cdots 1, 1}_{22}, \underbrace{2, 2 \cdots 2, 2}_{18})$.
If the minima occurs for some other tuplet where r twos have become one and k ones have increased , like

$$(\underbrace{1, 1, \cdots 1, 1}_{k}, \underbrace{2, 2 \cdots 2, 2}_{r}) \longrightarrow (1 + x_1, 1 + x_2, \cdots 1 + x_k, \underbrace{1, 1 \cdots 1, 1}_{r})$$

where $x_i > 0$ and $\sum_{i=1}^{k} x_i = r$ then:
Original sum(S_1) : $k \cdot 1^2 + r \cdot 2^2 = 4r + k$

Changed Sum(S_2): $\sum_{i=1}^{k}(1 + x_i)^2 + r \cdot 1^2 = k + 2\sum_{i=1}^{k} x_i + \sum_{i=1}^{k} x_i^2 + r$

$$\begin{aligned}
\therefore S_2 - S_1 &= k + 3r + (\sum_{i=1}^{k} x_i^2) - 4r - k \\
&= (\sum_{i=1}^{k} x_i^2) - r \qquad (1) \\
&= (\sum_{i=1}^{k} x_i^2) - (\sum_{i=1}^{k} x_i) \\
&\geq 0
\end{aligned}$$

In the penultimate step we have replaced r by $\sum_{i=1}^{k} x_i$.

$$\boxed{\therefore 94 \text{ is indeed the minima.}}$$

We claim that maxima is 400 which is attained when $(x_1, x_2, \cdots x_{40}) = (\underbrace{1, 1, \cdots 1, 1}_{39}, 19)$
If any other tuplet achieves maxima like

$$(1 + x_1, 1 + x_2, \cdots 1 + x_k, 19 - r)$$

where $\sum_{i=1}^{k} x_i = r$ and $r \leq 18$ then:

Original sum(S_1) : $k \cdot 1^2 + 19^2 = k + 361$

Changed Sum(S_2): $\sum_{i=1}^{k}(1 + x_i)^2 + (19 - r)^2 = k + \sum_{i=1}^{k} x_i^2 + 2\sum_{i=1}^{k} x_i + 361 + r^2 - 38r$
$\therefore S_2 - S_1 = 361 + k + \sum_{i=1}^{k} x_i^2 + r^2 - 36r - 361 - k = \sum_{i=1}^{k} x_i^2 + r^2 - 36r = \sum_{i=1}^{k} x_i^2 + (\sum_{i=1}^{k} x_i)^2 - 36r < 2(\sum_{i=1}^{k} x_i)^2 - 36r = 2r(r - 18) \leq 0$

$$\boxed{\therefore 400 \text{ is indeed the maxima.}}$$