Investigations into the Graham Pollak Theorem

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1 Notation

We shall refer to complete bipartite graphs as "bicliques".
The set $\{1, 2, \ldots, n\}$ will be denoted as $[n]$, where $n \in \mathbb{N} = \{1, 2, \ldots\}$ is a natural number.
$\mathcal{B}(X, Y)$ denotes the complete bipartite graph between the two (disjoint) sets X and Y. To state it more precisely, $\mathcal{B}(X, Y) = G(X \cup Y, \{\{x, y\} : x \in X, y \in Y\})$.

2 Preface

Under the guidance of my advisor Prof. Sundar Vishwanathan, I began investigating some problems related to the Graham-Pollak theorem. The reasons those problems attracted me were because of the inherent beauty of the Graham-Pollak theorem, as well as the peculiar fact that all known proofs of the theorem are linear algebraic, despite the theorem itself being very combinatorial in its statement, as was noted by Aigner and Ziegler in [1].
Throughout my research, I read up on various papers related to the topic and tried to bring them to bear on the problems I was interested in. Although mostly unsuccessful, I saw many interesting theorems and proofs throughout my journey, which I reproduce here.
Thus this report straddles between a survey, and a research progress report: Although not comprehensive enough to be designated a survey, it nevertheless touches on quite a few different aspects of the theorem, at the same time also underlining what the trajectory of my thought process was.

3 The Graham Pollak Theorem

Theorem 3.1. *The complete graph on $n \geq 2$ vertices can not be partitioned into $< n - 1$ complete bipartite graphs.*

We shall present 3 different proofs of this theorem: The first two have a similar flavor, while the third one differs significantly, in that it uses the polynomial method.

Proof. This first proof is due to Tverberg [7].
To every $i \in [n]$ associate a real variable $x_i \in \mathbb{R}$. Now, consider a bipartite graph with its left partition denoted by $L \subseteq [n]$, and right partition denoted by $R \subseteq [n] \setminus L$. Then, the edges of the biclique between L and R can be represented by the expression $\left(\sum_{i \in L} x_i\right)\left(\sum_{j \in R} x_j\right)$: Indeed, for every edge $\{i, j\}$ in the bipartite graph created by L and R, the aforementioned expression has the term '$x_i x_j$', and vice versa.
Thus, if m bipartite graphs $(L_k, R_k)_{1 \leq k \leq m}$ partition K_n, then we have

$$\sum_{k=1}^{m} \left(\sum_{i \in L_k} x_i\right)\left(\sum_{j \in R_k} x_j\right) = \sum_{1 \leq i < j \leq n} x_i x_j \tag{1}$$

Now, consider the homogenous system of $m+1$ linear equations:

$$\sum_{i\in[n]} x_i = 0$$
$$\sum_{i\in L_1} x_i = 0$$
$$\sum_{i\in L_2} x_i = 0$$
$$\vdots$$
$$\sum_{i\in L_m} x_i = 0$$

Since $\sum_{i\in L_k} x_i = 0$ for every $k \in [m]$, we have that $\sum_{1\leq i<j\leq n} x_i x_j$ is also 0. But, since $\sum_{i\in[n]} x_i = 0$, we get that $\sum_{i=1}^{n} x_i^2 + 2\sum_{1\leq i<j\leq n} x_i x_j = 0$, which along with the fact that $\sum_{1\leq i<j\leq n} x_i x_j = 0$, implies that $\sum_{1\leq i\leq n} x_i^2 = 0$, which implies that $x_i = 0$ for every $i \in [n]$.
Thus the above system of equations has only the trivial solution. Since a linear homogenous system of equations has non-trivial solutions if the number of equations is lesser than the number of indeterminates, we get that $m+1 \geq n \implies m \geq n-1$, as desired. $\square$

Here is another proof, which also uses linear algebra, but in a slightly different manner.
Before we get to the proof though, we present some very elementary lemmata regarding matrices.

Claim 3.2. *Rank is subadditive, ie:- for any two matrices A, B,* $\operatorname{rank}(A+B) \leq \operatorname{rank}(A) + \operatorname{rank}(B)$.

Proof. Let $\mathcal{C}$ be the span of the column vectors of A and B. Then $\dim(\mathcal{C}) \leq \operatorname{rank}(A) + \operatorname{rank}(B)$, with equality iff $\mathcal{C}(A) \perp \mathcal{C}(B)$. Now, note that the columns of $A+B$ belong to $\mathcal{C}$.
Hence proved. $\square$

Claim 3.3. *If $A + A^\mathsf{T} = J - I$, then* $\operatorname{rank}(A) \geq n-1$.

Proof. We claim that if $\operatorname{rank}(A) \leq n-2$, then there exists a non-zero $\boldsymbol{x}$ such that $A\boldsymbol{x} = J\boldsymbol{x} = \mathbf{0}$: Indeed, $\operatorname{rank}(A) \leq n-2$ implies $\operatorname{nullity}(A) \geq 2$, by the Rank-Nullity Theorem. Thus choose two linearly independent vectors v_1 and v_2 in the kernel of A, and let α_1 be the sum of entries in v_1, and similarly define α_2. If either α_1 or α_2 are zero, then choose $\boldsymbol{x} = v_1$ or v_2 respectively. Otherwise, note that $\boldsymbol{x} = \alpha_1 v_2 - \alpha_2 v_1 \neq \mathbf{0}$ satisfies the given property.
Thus if $\operatorname{rank}(A) < n-1$ then we can choose $\boldsymbol{x} \neq \mathbf{0}$ such that $A\boldsymbol{x} = J\boldsymbol{x} = \mathbf{0}$. Then multiplying $\boldsymbol{x}$ on both sides in the equation $A + A^\mathsf{T} = J - I$ yields $A^\mathsf{T}\boldsymbol{x} = -\boldsymbol{x}$ implying $\boldsymbol{x}^\mathsf{T} A = -\boldsymbol{x}^\mathsf{T}$ implying $\boldsymbol{x}^\mathsf{T} A\boldsymbol{x} = -\boldsymbol{x}^\mathsf{T}\boldsymbol{x} = -\|\boldsymbol{x}\|^2 \neq 0$, but $\boldsymbol{x}^\mathsf{T} A\boldsymbol{x}$ is clearly zero, leading to a contradiction. $\square$

Finally, we remind the reader that any subset $S \subseteq [n]$ can be represented as a vector $v \in \{0,1\}^n$, where $v_i = 0$ if and only if $i \in S$. This vector is known as the characteristic vector of S. Depending on the needs of the problem at hand, we treat the characteristic vector as an element of $\mathbb{F}_2^n$, where 0 and 1 are identified as the elements of $\mathbb{F}_2$. We, alternatively, also treat the characteristic vector as a member of $\mathbb{R}^n$, where 0 and 1 are treated as real numbers.
We can finally state our second proof of the Graham-Pollak theorem.

Proof. Let there be m bipartite graphs, and let $\boldsymbol{x_i}$ and $\boldsymbol{y_i}$ denote the characteristic vectors of the left and right vertex sets of the i^{th} bipartite graph. Then the adjacency graph of the i^{th} bipartite graph is given by $\boldsymbol{x_i y_i^\intercal} + \boldsymbol{y_i x_i^\intercal}$, and since the bipartite graphs cover all the edges, we get that the adjacency matrix of the complete graph (which is $J - I$) is a sum of all these adjacency matrices, ie:-

$$J - I = \sum_{i=1}^{m} (\boldsymbol{x_i y_i^\intercal} + \boldsymbol{y_i x_i^\intercal})$$

If we define $A := \sum_{i=1}^{m} \boldsymbol{x_i y_i^\intercal}$, then $\operatorname{rank}(A) \leq \sum_{i=1}^{m} \operatorname{rank}(\boldsymbol{x_i y_i^\intercal}) = m$, where the inequality follows since rank is subadditive, and one may note that $\operatorname{rank}(\boldsymbol{x_i y_i^\intercal}) = 1$. But then $A + A^\intercal = J - I$, and thus $\operatorname{rank}(A) \geq n - 1$, and thus $n - 1 \leq \operatorname{rank}(A) \leq m$ implying $n - 1 \leq m$, as desired. □

Finally, note that equality, ie:- $m = n - 1$, can be achieved for the Graham-Pollak inequality: Indeed, consider the series of bipartite graphs given by $\mathcal{B}_k := \mathcal{B}(\{k\}, \{i : n \geq i > k\})$, where $k \in [n-1]$. Note that these bipartite graphs are actually "stars", so we shall sometimes refer to this construction as the "star-construction".
We shall now investigate a particular generalization of the Graham-Pollak theorem.

4 L-coverings of graphs

Partitioning the edges of a graph into bicliques may be an excessive requirement: As we shall see soon, we can significantly decrease the number of bicliques needed to cover the edges of a graph, provided that we allow some edges to be covered more than once.
Motivated by the above discussion, we make the following definition:

Definition 1. *Let G be a simple undirected graph, and let $L \subseteq \mathbb{N}$ be a non-empty set of natural numbers. We denote by $\operatorname{bp}_L(G)$ the minimum number of bicliques needed to cover G such that the number of times any edge is covered, lies in L.*

We refer the reader to [6] for further exploration of this topic.
We shall ourselves be concerned about only very few values of $\operatorname{bp}_L(G)$. For starters, note that the Graham-Pollak theorem can be interpreted as saying

that $\mathrm{bp}_{\{1\}}(K_n) \geq n-1$. Combined with the example mentioned above, we can say that $\mathrm{bp}_{\{1\}}(K_n) = n-1$.
For any $\lambda \in \mathbb{N}$, observe that $\mathrm{bp}_{\{\lambda\}}(K_n) \geq n-1$: Indeed, in Eq. (1), we can simply replace $\sum_{1\leq i<j\leq n} x_i x_j$ by $\lambda \sum_{1\leq i<j\leq n} x_i x_j$, and the rest of the proof goes through verbatim. The exact value of $\mathrm{bp}_{\{\lambda\}}(K_n)$, for every $\lambda \in \mathbb{N}$, is unknown: It was conjectured by de Caen in 1993, that for every $\lambda \in \mathbb{N}$, there exists some $N_\lambda \in \mathbb{N}$ such that $\mathrm{bp}_{\{\lambda\}}(K_n) = n-1$ for all $n \geq N_\lambda$.
We discuss another interesting case when $L = \mathbb{N}$.

Theorem 4.1. $\mathrm{bp}_{\mathbb{N}}(K_n) = \lceil \log_2(n) \rceil$.

Proof. This proof is due to [4].
We prove the theorem by induction. The base case of $n = 1$ is easy to verify.
Consider the largest biclique in any optimal biclique covering of K_n, $n \geq 2$. WLOG we can assume that all n vertices are present in that biclique, because if not, then we can simply add the remaining vertices in any manner to that biclique without changing the number of bicliques in that covering.
By the pigeonhole principle, one of the partitions of our biclique is of size at least $\lceil \frac{n}{2} \rceil$, and thus, by the induction hypothesis that partition requires at least $\lceil \log_2 \lceil \frac{n}{2} \rceil \rceil \geq \lceil \log_2 n \rceil - 1$ bicliques to cover. Combined with the biclique under consideration, we thus need at least $\lceil \log_2 n \rceil$ bicliques to cover K_n, as desired.
Finally, equality can be seen to hold by the following construction: Consider $\lceil \log_2 n \rceil$ graphs, where the k^{th} bipartite graph is $\mathcal{B}_k := \mathcal{B}(L_k, R_k)$, where $k \in [\lceil \log_2 n \rceil]$, $L_k \subseteq [n]$, is the set of all numbers whose k^{th} bits, in their binary expansions, is 0. Similarly, R_k is the set of numbers whose k^{th} binary bits are 1.
These bipartite graphs cover every edge $\{i,j\}, i \neq j$, since $i \neq j$ implies that their binary expansions must differ in some bit. $\square$

We now move on to a very important class of L-coverings, namely the case for which $L = \{1, 2, \ldots, t\}$ for some $t \in \mathbb{N}$.

4.1 $[t]$-biclique coverings of K_n

The results in this section are from this [2] paper by N. Alon.
In this section, we are interested in calculating the minimum number of bicliques needed to cover the complete graph provided we are allowed to cover every edge at most t times.

Theorem 4.2. $\mathrm{bp}_{\{1,2,\ldots,t\}}(K_n) \leq t\left(\left\lceil n^{\frac{1}{t}} \right\rceil - 1\right)$.

Proof. Consider the cartesian product $\left[\left\lceil n^{\frac{1}{t}} \right\rceil\right]^t$: Clearly its size is $\left(\left\lceil n^{\frac{1}{t}} \right\rceil\right)^t \geq n$, and thus let $f : [n] \mapsto \left[\left\lceil n^{\frac{1}{t}} \right\rceil\right]^t$ be an injection.
Now, let our bipartite graphs be defined as

$$\mathcal{B}_{j,k} := \mathcal{B}\left(\{x \in [n] : f(x)_j = k\}, \{x \in [n] : f(x)_j > k\}\right)$$

for every $j \in [t]$ and $k \in \left[\left\lceil n^{\frac{1}{t}} \right\rceil - 1\right]$, where $f(x)_j$ denotes the j^{th} coordinate of $f(x)$.
Our result follows since every edge $\{i,j\}$ is covered exactly $\Delta(f(i), f(j)) \leq t$ times, where $\Delta(\cdot,\cdot)$ denotes the Hamming distance between two tuples. $\square$

Remark: Note that assigning coordinates to every number in $[n]$ basically creates equivalence classes: For example, one equivalence class could be of vertices all of whose first coordinates were equal to 1.
Turns out that we can give a lower bound of the same asymptotic order too, using the so-called polynomial method.

Theorem 4.3. *Suppose K_n is covered with d bicliques such that each edge is covered at most t times. Then $n \leq \sum_{i=0}^{t} 2^i \binom{d}{i} < 2\left(\frac{2ed}{t}\right)^t$.*
Thus $d > \frac{t}{2e}(\frac{n}{2})^{\frac{1}{t}}$, implying that $\mathrm{bp}_{\{1,2,\ldots,t\}}(K_n) = \Theta(n^{\frac{1}{t}})$.

Proof. Suppose our bicliques are $\mathcal{B}_k = \mathcal{B}(L_k, R_k)$, where $k \in [d]$ and $L_k, R_k \subseteq [n]$.
Consider the following sequence of n polynomials on $\mathbb{R}^{2d}$:

$$p_i(x_1, \ldots, x_d, y_1, \ldots, y_d) := \prod_{j=1}^{t} \left(\sum_{p: i \in L_p} x_p + \sum_{q: i \in R_q} y_q - j \right), i \in [n]$$

For each $i \in [n]$, we define $e_i := (b_{i1}, \ldots, b_{id}, a_{i1}, \ldots, a_{id}) \in \{0,1\}^{2d}$, where $a_{ip} := \mathbb{1}_{i \in L_p}, b_{iq} := \mathbb{1}_{i \in R_q}$.
The main point is that if $i \neq j$, then $\left(\sum_{p:i\in L_p} x_p + \sum_{q:i \in R_q} y_q\right)$, evaluated at e_j, counts the number of bicliques in which i and j lie in different partitions, or in other words, counts the number of times the edge $\{i,j\}$ is covered by the bicliques. Since this number lies between 1 and t, $p_i(e_j) = 0$ for $i \neq j$. Similarly, $p_i(e_i) = (-1)^t t! \neq 0$.
Now, we claim that the polynomials $p_1, \ldots, p_n$ are linearly independent: Indeed, if $\sum_{i=1}^{n} \lambda_i p_i(x) = 0$, then substituting $x = e_i$ yields $\lambda_i p_i(e_i) = 0 \implies \lambda_i = 0$, since $p_i(e_i) \neq 0$.
Finally, expand $p_i(x_1, \ldots, x_d, y_1, \ldots, y_d)$, and replace every monomial $\prod x_s^{\delta_s} \prod y_\ell^{\eta_\ell}$ by $\prod x_s \prod y_\ell$: For example, $x_1^2 x_3 y_2^4$ would be replaced by $x_1 x_3 y_2$. This process is known as *multilinearization*, and let $\widetilde{p_i}$ be the polynomial produced by multilinearizing p_i. Note that since $0^\delta = 0, 1^\delta = 1$ for every $\delta > 0$, $\widetilde{p_i}(e_j) = p_i(e_j)$ for every $i,j \in [n]$. Consequently, $\widetilde{p_1}, \ldots, \widetilde{p_n}$ are linearly independent.
Now, note that these polynomials lie in the real vector space spanned by $\mathcal{V} := \mathrm{span}\{\prod_{s \in S} x_s \prod_{m \in M} y_m : S, M \subseteq [n], |S| + |M| \leq t, S \cap M = \emptyset\}$, where the $|S| + |M| \leq t$ condition has been imposed because all the polynomials $\widetilde{p_i}$ are of degree at most t. Finally, note that $\dim(\mathcal{V}) = \#\{S, M \subseteq [n], |S| + |M| \leq t, S \cap M = \emptyset\} = \sum_{k=0}^{t} 2^k \binom{d}{k}$.
Since $\widetilde{p_1}, \ldots, \widetilde{p_n}$ are linearly independent in $\mathcal{V}$, we have that $n \leq \dim(\mathcal{V})$, as desired. $\square$

The polynomial method is a highly powerful method in combinatorics: Among many other things, we can use it to re-prove the Graham-Pollak theorem. Thus we digress a bit from L-coverings and present this beautiful polynomial-space proof of (a slight generalization of) the Graham-Pollak theorem, due to Vishwanathan [8].

Theorem 4.4 (Generalized Graham Pollak Theorem)**.** *For any simple undirected graph G, let $\overline{G}$ denote its complement. Also, let $A(G)$ denote the adjacency matrix of G.*
Suppose some graph $G = G(V, E)$ has n vertices, and consider a partition of the edges of G by m bicliques.
Then $m \geq n - 1 - \operatorname{rank}(A(\overline{G}))$.

Proof. As before, we identify the vertices of G with $[n]$.
Let our bipartite graphs be $\mathcal{B}_1 = \mathcal{B}(L_1, R_1), \dots, \mathcal{B}_m = \mathcal{B}(L_m, R_m)$. Define, for every $i \in [n]$,

$$A_i := \{k \in [m] : i \in L_k\} \subseteq [m]$$

$$B_i := \{k \in [m] : i \in R_k\} \subseteq [m]$$

$$S_i := A_i \cup B_i$$

Clearly, $A_i \cap B_i = \emptyset$. Furthermore, note that $|A_i \cap B_j| + |B_i \cap A_j| = 1$ if and only if i and j are adjacent in G.
Consider the following sequence of $n + m$ polynomials on $\mathbb{R}^{2m}$:

$$p_i(x_1, \dots, x_m, y_1, \dots, y_m) := \sum_{k \in A_i} x_k + \sum_{r \in B_i} y_r - 1, i \in [n]$$

$$q_k(x_1, \dots, x_m, y_1, \dots, y_m) := x_k + y_k, k \in [m]$$

We now investigate the nullspace of the polynomials $p_1, \dots, p_n, q_1, \dots, q_m$. Indeed, let

$$\sum_{i=1}^{n} \alpha_i p_i + \sum_{k=1}^{m} \beta_k q_k = 0 \tag{2}$$

Consider $e_i := (\mathbb{1}_{1 \in B_i}, \dots, \mathbb{1}_{m \in B_i}, \mathbb{1}_{1 \in A_i}, \dots, \mathbb{1}_{m \in A_i})$ for every $i \in [n]$. Then $p_j(e_i) = |A_j \cap B_i| + |B_j \cap A_i| - 1$ for any $i, j \in [n]$.
Thus substituting $x = e_i$ in Eq. (2), we get

$$-\alpha_i - \sum_{\{i,j\} \notin E} \alpha_j + \sum_{k \in S_i} \beta_k = 0 \tag{3}$$

Further consider $\widetilde{e_k} := (\mathbb{1}_{1=k}, \dots, \mathbb{1}_{m=k}, \mathbb{1}_{1=k}, \dots, \mathbb{1}_{m=k})$.
Substituting $x = \widetilde{e_k}$ in Eq. (2), we get

$$-\sum_{k \notin S_j} \alpha_j + 2\beta_k = 0 \tag{4}$$

Substituting Eq. (4) in Eq. (3) yields

$$-\alpha_i - \sum_{\{i,j\}\notin E} \alpha_j + \frac{1}{2}\sum_{k\in S_i}\sum_{k\notin S_j} \alpha_j = 0$$

Simplifying the above equation yields

$$-\alpha_i - \sum_{\{i,j\}\notin E} \alpha_j + \frac{1}{2}\sum_{j\in[n]} \alpha_j |S_i \cap \overline{S_j}| = 0 \tag{5}$$

Now, let $D \in \mathbb{R}^{n\times n}$ be the incidence matrix of S_i's, ie:- $D_{ij} = \mathbb{1}_{j\in S_i}$. Then, if we denote $\alpha = \begin{bmatrix}\alpha_1 & \alpha_2 & \dots & \alpha_n\end{bmatrix}^\mathsf{T}$, then Eq. (5) can be written as

$$\left(-I - A(\overline{G}) + \frac{1}{2}(J-D)D^\mathsf{T}\right)\alpha = 0 \implies \left(I + \frac{1}{2}DD^\mathsf{T} + A(\overline{G}) - \frac{1}{2}JD^\mathsf{T}\right)\alpha = 0$$

Now, note that $I+\frac{1}{2}DD^\mathsf{T}$ is a full rank matrix since it is strictly positive definite (due to the presence of I). Furthermore, $\text{rank}(\frac{1}{2}JD^\mathsf{T}) = 1$. Thus, by the sub-additivity of rank, $\text{rank}\left(I + \frac{1}{2}DD^\mathsf{T} + A(\overline{G}) - \frac{1}{2}JD^\mathsf{T}\right) \geq n - 1 - \text{rank}(A(\overline{G}))$.
Consequently, the size of the nullspace of the polynomials $p_1, \dots, p_n, q_1, \dots, q_m$ is at most $1+\text{rank}(A(\overline{G}))$. Now, note that the polynomials are linear polynomials of $2m$ variables, ie:- all the polynomials lie in $\text{span}(x_1, \dots, x_m, y_1, \dots, y_m)$. Thus the "rank" of $\{p_1, \dots, p_n, q_1, \dots, q_m\}$, ie:- the size of the maximal independent set, is at most $2m$.
Thus, by the rank-nullity theorem,

$$\underbrace{\text{rank}(p_1, \dots, p_n, q_1, \dots, q_m)}_{\leq 2m} + \underbrace{\text{nullity}(p_1, \dots, p_n, q_1, \dots, q_m)}_{\leq 1+\text{rank}(A(\overline{G}))} = m + n$$

$$\implies 2m + 1 + \text{rank}(A(\overline{G})) \geq m + n \implies m \geq n - 1 - \text{rank}(A(\overline{G}))$$

as desired. □

We now explore $\{1,2\}$-covering in further detail (a detailed exposition can be found in [3]): By Theorem 4.3 and Theorem 4.2, $\sqrt{n-1} \leq \text{bp}_{\{1,2\}}(K_n) \leq 2(\lceil\sqrt{n}\rceil - 1)$.
Now, we can slightly improve the construction in Theorem 4.2 if we have some information about n: For example, if n is a perfect square, then $\text{bp}_{\{1,2\}}(K_n) \leq 2(\sqrt{n}-1)$. If $k^2 < n \leq k^2 + k$ for some k, then $\text{bp}_{\{1,2\}}(K_n) \leq 2\lfloor\sqrt{n}\rfloor - 1$, and if $k^2 + k < n < (k+1)^2$, then $\text{bp}_{\{1,2\}}(K_n) \leq 2\lfloor\sqrt{n}\rfloor$.
Further, [3] verify that the above bounds hold for all $n \in \{3,4,5,6,7,8,9,12\}$, and based on that they conjecture that these upper bounds are actually tight for all natural numbers $n \in \mathbb{N}$.
One possible approach for attacking the aforementioned conjecture was provided by Vishwanathan, which we mention now.

4.2 Triangle-free Biclique intersection graph

Consider any biclique covering of K_n, and let the bicliques be $\mathcal{B}_1 = \mathcal{B}(L_1, R_1), \ldots, \mathcal{B}_m = \mathcal{B}(L_m, R_m)$. We say that $\mathcal{B}_i$ *intersects* $\mathcal{B}_j$ if $L_i \cap L_j \neq \emptyset, L_i \cap R_j \neq \emptyset, R_i \cap L_j \neq \emptyset, R_i \cap R_j \neq \emptyset$.
We can then form a biclique intersection graph, whose vertices are $\mathcal{B}_1, \ldots, \mathcal{B}_m$, and $\mathcal{B}_i$ and $\mathcal{B}_j$ have an edge between them if they intersect.
Just to clarify matters, let's calculate the biclique intersection graph of Alon's construction for $t = 2$.
Remember that we associated $[n]$ with $[\lceil\sqrt{n}\rceil]^2$: In other words, every $i \in [n]$ was associated with a unique $(x_i, y_i) \in [\lceil\sqrt{n}\rceil]^2$. The bipartite graphs went as follows:

$$\mathcal{B}(\{i : x_i = k\}, \{i : x_i > k\}), k \in [\lceil\sqrt{n}\rceil - 1]$$
$$\mathcal{B}(\{i : y_i = k\}, \{i : y_i > k\}), k \in [\lceil\sqrt{n}\rceil - 1]$$

We call the first class of graphs 'x-bicliques', and the second class 'y-bicliques'. Note that any two x and y bicliques intersect, while no two x-bicliques or y-bicliques intersect.
Thus the biclique intersection graph of Alon's construction for $t = 2$ is also a biclique, with each partition containing $\leq \lceil\sqrt{n}\rceil$ elements.
Now, as mentioned earlier, Alon's constructions are believed to be optimal for 2-covering bicliques. Thus, motivated by that, Vishwanathan conjectures that *every optimal* $\{1, 2\}$*-biclique covering of* K_n *has a triangle-free biclique intersection graph.*
Clearly, Alon's biclique intersection graphs are triangle-free, since they are bipartite. We believe that this can be shown to be true for all optimal $\{1, 2\}$-biclique coverings of K_n.

5 The Szemerédi-Katona Theorem

Before we sign off, we see a last interesting fact about biclique-coverings.

Theorem 5.1 (Szemerédi-Katona Theorem)**.** *Consider any biclique covering of* K_n *(ie:- every edge in* K_n *is covered at least once). Then the total number of vertices contained among those bicliques must be at least* $n \log_2(n)$.

Proof. This proof can be found here [5].
Let our bipartite graphs be $\mathcal{B}_1 = \mathcal{B}(L_1, R_1), \ldots, \mathcal{B}_m = \mathcal{B}(L_m, R_m)$. We have to show that $\sum_{k=1}^{m}(|L_i| + |R_i|) \geq n \log_2(n)$.
Construct a $m \times n$ matrix M such that

$$M_{ij} = \begin{cases} 0, & \text{if } j \in L_i \\ 1, & \text{if } j \in R_i \\ 2, & \text{otherwise} \end{cases}$$

Let the number of zeros and ones in the j^{th} column be denoted as h_j. Note that the total number of zeros and ones in M is $\sum_{k=1}^{m}(|L_k| + |R_k|)$. Thus

$\sum_{j=1}^{n} h_j = \sum_{k=1}^{m}(|L_k| + |R_k|)$.
Now, for every edge $\{i, j\}$, there is some bipartite matrix $\mathcal{B}(L, R)$ such that $i \in L$ and $j \in R$. In other words, for any two columns in M, there is a row such that one of the columns equals 1 on that row, and the other column equals 0 on that row.
Now, note that this means that every two columns of M are distinct since one can always find a row that distinguishes them. Now, consider two columns C_i, C_j. Replace all the twos in C_i, C_j arbitrarily by zeros and ones. Note that the two columns continue to remain distinct because the entries of their distinguishing row were not affected when we flipped the twos to zeros and ones. Note that we can generate 2^{m-h_i} distinct columns out of C_i by flipping the twos. Thus, the total number of columns that can be generated by flipping all the twos in all the columns is $\sum_{i=1}^{n} 2^{m-h_i}$. Furthermore, as argued above, all of these columns are distinct.
Now, the total number of distinct columns having only zeros and ones is 2^m. Thus

$$\sum_{i=1}^{n} 2^{m-h_i} \leq 2^m \implies \sum_{i=1}^{n} 2^{-h_i} \leq 1$$

By the AM-GM inequality,

$$2^{-\sum_{i=1}^{n} h_i} \leq \left(\frac{\sum_{i=1}^{n} 2^{-h_i}}{n}\right)^n$$

But

$$\left(\frac{\sum_{i=1}^{n} 2^{-h_i}}{n}\right)^n \leq \left(\frac{1}{n}\right)^n$$

Thus

$$2^{-\sum_{i=1}^{n} h_i} \leq \left(\frac{1}{n}\right)^n \implies \sum_{i=1}^{n} h_i \geq n \log_2(n)$$

as desired.
Furthermore, note that equality occurs if and only if all the h_i's are equal to $\log_2(n)$ [1], ie:- if every vertex is part of an equal number of bicliques. □

[1] thus equality can only occur if n is a power of 2

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