Technical Report: Merton Structural Credit Model
with Exponential Smoothing Improvement

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December 25, 2025

1 Introduction

This report presents an implementation and improvement of the Merton (1974) structural credit model. After implementing and diagnosing the baseline model on API-fetched 2020 data for five U.S. firms (Apple, JPMorgan Chase, Tesla, ExxonMobil, and Ford), we identified default probability (PD) instability as a primary weakness: all firms exhibit coefficient of variation (CV) > 1.0, making the model unsuitable for daily credit risk monitoring.

To approach this issue we implement exponential smoothing as a minimal improvement that reduces PD volatility by 15–20% and daily changes by 25–50% while preserving the model's ability to properly capture credit deterioration during crisis periods.

2 Model Formulation

The Merton model views a firm's equity as a European call option on its assets, with the strike price being the face value of its debt. The key equations and assumptions of the baseline Merton model are outlined below.

2.1 Baseline Merton Model

2.1.1 Key Assumptions

1. Firm asset value V_t follows geometric Brownian motion:

$$dV_t = \mu V_t \, dt + \sigma_V V_t \, dW_t \tag{1}$$

2. Firm has zero-coupon debt with face value D maturing at time T

3. Default occurs only at maturity T if $V_T < D$

4. Equity is a European call option on firm assets: $E = \max(V_T - D, 0)$

5. Markets are frictionless (no transaction costs), trading is continuous, no arbitrage(no riskless profit opportunities)

2.1.2 Key Equations/ Mathematical Formulation

Equity Value Under the Merton model, equity value is given by the Black-Scholes call option formula:

$$E = V \cdot \Phi(d_1) - D \cdot e^{-r(T-t)} \cdot \Phi(d_2) \tag{2}$$

where

$$d_1 = \frac{\ln(V/D) + (r + \sigma_V^2/2)(T-t)}{\sigma_V \sqrt{T-t}} \tag{3}$$

$$d_2 = d_1 - \sigma_V \sqrt{T-t} \tag{4}$$

and $\Phi(\cdot)$ denotes the standard normal cumulative distribution function.

Equity Volatility The relationship between equity volatility σ_E and asset volatility σ_V is:

$$\sigma_E \cdot E = \Phi(d_1) \cdot \sigma_V \cdot V \tag{5}$$

where $\Phi(d_1)$ is the option delta, representing the sensitivity of equity value to changes in asset value.

Default Probability The risk-neutral probability of default at maturity is:

$$\mathrm{PD} = \Phi(-d_2) = \Phi\left(-\frac{\ln(V/D) + (r - \sigma_V^2/2)T}{\sigma_V \sqrt{T}}\right) \tag{6}$$

Distance-to-Default The distance-to-default metric is:

$$\mathrm{DD} = \frac{\ln(V/D) + (r - \sigma_V^2/2)T}{\sigma_V \sqrt{T}} \tag{7}$$

2.1.3 Calibration Approach

The Merton model calibration involves solving the following system of equations for the unobservable asset value V and asset volatility σ_V, given the observable equity value E, equity volatility σ_E, debt face value D, time to maturity $T - t$, and risk-free rate r:

$$E = \mathrm{BlackScholes}(V, D, T-t, r, \sigma_V) \tag{8}$$

$$\sigma_E \cdot E = \Phi(d_1) \cdot \sigma_V \cdot V \tag{9}$$

2.2 Improved Model

2.2.1 Modification

We apply post-processing exponential smoothing to default probabilities. For each firm's time series of raw PDs $\{\mathrm{PD}_t^{\mathrm{raw}}\}$ we applied the following smoothing formula:

$$\mathrm{PD}_t^{\mathrm{smooth}} = \alpha \cdot \mathrm{PD}_t^{\mathrm{raw}} + (1-\alpha) \cdot \mathrm{PD}_{t-1}^{\mathrm{smooth}} \tag{10}$$

where $\alpha = 0.1$ is the smoothing parameter.

2.2.2 Justification

This improvement is justified on two grounds. Theoretically speaking, the baseline model assumes you can trade infinitely with no transaction costs. Whereas in reality, trading is discrete and every trade has costs leading to noise in observed prices that isn't properly captured in the model. Additionally, daily equity volatility contains high-frequency noise (changes in company value, as well as market microstructure noise) which is further amplified through the nonlinear Black-Scholes model into asset volatility thus into PD estimates. Practically speaking, PD estimates play a key role in credit risk management for portfolio monitoring, capital alllocation, and risk limits. When daily PD estimates oscillate wildly (for example 5-20 percent points as seen in the baseline model), they become operationally unusable. This is why smoothing techniques are applied in practice to reduce noise while preserving signal.

2.2.3 Mathematical Formulation of Improvement

The exponential weighted moving average (EWMA), the smoothing formula we are using, has the following properties.

- Parameter α controls reactivity: lower α yields more smoothing
- No look-ahead bias (uses only past observations)
- Preserves long-term trends while filtering short-term noise
- Weights decay exponentially: observation k periods ago has weight $\alpha(1-\alpha)^k$

Diving into the math, property 1 of the EWMA smoothing formula which is equation (10) can be demonstrated by unrolling the recursion:

$$\mathrm{PD}_t^{\mathrm{smooth}} = \alpha \sum_{k=0}^{\infty} (1-\alpha)^k \cdot \mathrm{PD}_{t-k}^{\mathrm{raw}} \tag{11}$$

This shows how α controls the weight decay rate, thus the reactivity. For example, with $\alpha = 0.9$ the decay factor $(1-\alpha) = 0.1$ which means today would be 90%, yesterday 9%, two days ago 0.9%, etc. Showing that the higher alpha has high reactivity to recent changes since recent data dominates. Conversely, with $\alpha = 0.1$ the decay factor $(1-\alpha) = 0.9$ which means today would be 10%, yesterday 9%, two days ago 8.1%, etc. This shows that lower alpha has low reactivity since weights are more evenly spread out over time.

Property 2 is evident since the formula only uses $\mathrm{PD}_t^{\mathrm{raw}}$ and $\mathrm{PD}_{t-1}^{\mathrm{smooth}}$ which is computed from past raw PDs. Additionally the code implementation processes data chronologically to avoid look-ahead bias.

Property 3 is evident since the EWMA formula averages over past raw PDs, thus filtering out high-frequency noise while preserving long-term trends.

Property 4 is shown in equation (11) where the weight for observation k periods ago is $\alpha(1-\alpha)^k$, demonstrating the exponential decay.

2.2.4 Calibration Changes

Smoothing is applied post-calibration. The algorithm is:

1. Calibrate (V, σ_V) using baseline approach
2. Compute PD_t^{raw} from equation (6)
3. Apply EWMA smoothing to obtain $\text{PD}_t^{\text{smooth}}$

Note that asset values V, asset volatilities σ_V, and distance-to-default remain unchanged from the baseline model.

3 Calibration Methodology

3.1 Numerical Method

We solve the system of two non-linear equations for two unknowns using scipy's `fsolve` optimizer, which implements a modified Powell hybrid method—a quasi-Newton approach with adaptive damping.

System of Equations

$$F_1(V, \sigma_V) = \text{BlackScholes}(V, D, T, r, \sigma_V) - E = 0 \tag{12}$$
$$F_2(V, \sigma_V) = \Phi(d_1) \cdot \sigma_V \cdot V - \sigma_E \cdot E = 0 \tag{13}$$

Algorithm We use scipy's `fsolve` optimizer with the following parameters:

1. **Initialize:**

$$V_0 = E + D \quad \text{(balance sheet approximation)} \tag{14}$$
$$\sigma_{V,0} = \sigma_E \cdot \frac{E}{E + D} \quad \text{(de-levered volatility)} \tag{15}$$

2. **Solve:** Call `fsolve(equations, [V_0, σ_V,0], xtol=1e-6)`

   `fsolve` implements a modified Powell hybrid method that approximates the Jacobian via finite differences, uses a trust region approach with adaptive damping, and converges when the relative error in the solution update (Δx) is $||\Delta x|| < 10^{-6}$.

3. **Validate:** After convergence, check the convergence flag utilizing ier (integer flag that indicates sucess or failure of solver) `ier == 1`, the economic validity (asset value should be larger than equity value by a small margin) $V \geq 1.01E$, and reasonable range $0.01\% \leq \sigma_V \leq 200\%$.

3.2 Edge Case Handling

1. **Invalid inputs**: ($E \leq 0$, $\sigma_E \leq 0$, $D < 0$): reject observation
2. **Jacobian singularity**: reject observation (caught through code exception handling)
3. **Non-convergence**: reject observation (caught through `fsolve` ier flag)
4. **Economic Validity**: $V > 1.01E$ (covered in validation above)