17 Option pricing with proportional transaction cost

Suppose that the price of the stock follows a GBM

$$dS_t = \alpha S_t dt + \sigma S_t dB_t. \tag{31}$$

In the presence of proportional transaction costs, the investor pays $0 < \zeta < 1$ and $0 < \mu < 1$ of the dollar value transacted on purchase and sale of the underlying stock, respectively. Let x_t denote the number of shares held in stock and y_t the dollar value of the investment in bond which pays a fixed risk-free rate r. The investor's position (x_t, y_t) in stock and bond is driven by:

$$\begin{aligned} dx_t &= dL_t - dM_t, && (32)\\ dy_t &= ry_t dt - aS_t dL_t + bS_t dM_t, && (33) \end{aligned}$$

with $a = 1 + \zeta$ and $b = 1 - \mu$, where L_t and M_t are nondecreasing and non-anticipating processes and represent the cumulative numbers of shares of stock bought or sold, respectively, within the time interval $[0, t]$, $0 \le t \le T$. Denote the terminal settlement value of the stock, when there is no option position, by $Z(S_T, x_T)$, which is given by

$$Z(S, x) = xS(aI_{\{x<0\}} + bI_{\{x\geqslant 0\}}). \tag{34}$$

The investor's terminal wealth in terms of the total hedging cost is

$$W_T := y_t\, e^{r(T-t)} - \left(a \int_{[t,T)} e^{r(T-u)} S_u dL_u - b \int_{[t,T)} e^{r(T-u)} S_u dM_u\right) + Z(S_T, x_T).$$

Let $U : \mathbb{R} \to \mathbb{R}$ be the writer's utility, which is concave and increasing with $U(0) = 0$. We assume that the investor's goal is to maximize the expected utility of terminal wealth under the market model (31)-(33)

$$V(t, S, x, y) = \sup_{(L_t, M_t)\in\mathcal{T}(y_0)} E[W_T | S_t = S, x_t = x, y_t = y], \tag{35}$$

Suppose that the writer's utility $U(z)$ has the negative exponential utility function $U(z) = 1 - e^{-\gamma z}$, in which γ is the constant absolute risk aversion parameter. Then

$$V(t, S, x, y) = 1 - \exp\{-\gamma y e^{r(T-t)}\} H(t, S, x), \tag{36}$$

and $H(t, S, x)$ solves the following free-boundary problem

$$\begin{cases} \dfrac{\partial H}{\partial t} + \alpha S \dfrac{\partial H}{\partial S} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 H}{\partial S^2} = 0 & x \in [X_b(t, S), X_s(t, S)], \\ \dfrac{\partial H}{\partial x} + a\gamma S e^{r(T-t)} H = 0 & x \le X_b(t, S), \\ \dfrac{\partial H}{\partial x} + b\gamma S e^{r(T-t)} H = 0 & x \ge X_s(t, S), \\ H(T, S, x) = e^{-\gamma Z(S,x)}, & \end{cases} \tag{37}$$

in which $X_b(t, S)$ and $X_s(t, S)$ are the unknown buy and sell boundaries, respectively.