Conformality and Invariance

Goal: Define a notion of distance that is preserved under holomorphic & conformal maps. Let's see why the regular notion of distance doesn't give us what we want.

The Jacobian of a Holomorphic Function

Let $U \subseteq \mathbb{C}$ be an open set, $P \in U$ a fixed point, and $f : U \to \mathbb{C}$ a holomorphic function on U. For $f(x+iy) = u(x,y) + iv(x,y)$, we can consider f as a mapping $(x,y) \to (u,v)$, where we get the real Jacobian matrix of f at P:

$$J(P) = \begin{pmatrix} u_x(P) & u_y(P) \\ v_x(P) & v_y(P) \end{pmatrix} \tag{1}$$

Since f is holomorphic, by the Cauchy-Riemann equations we know that

$$\boxed{u_x = v_y, \qquad u_y = -v_x} \tag{2}$$

Hence, we can simplify (1) in the following way:

$$\begin{aligned} J(P) &= \begin{pmatrix} u_x(P) & u_y(P) \\ v_x(P) & v_y(P) \end{pmatrix} \\ &= \begin{pmatrix} u_x(P) & u_y(P) \\ -u_y(P) & u_x(P) \end{pmatrix} \\ &= \underbrace{\sqrt{u_x(P)^2 + u_y(P)^2}}_{=:h(P)} \cdot \underbrace{\begin{pmatrix} \frac{u_x(P)}{\sqrt{u_x(P)^2+u_y(P)^2}} & \frac{u_y(P)}{\sqrt{u_x(P)^2+u_y(P)^2}} \\ \frac{-u_y(P)}{\sqrt{u_x(P)^2+u_y(P)^2}} & \frac{u_x(P)}{\sqrt{u_x(P)^2+u_y(P)^2}} \end{pmatrix}}_{=:\mathcal{J}(P)} \end{aligned}$$

Then,

$$J(P) \equiv h(P) \cdot \mathcal{J}(\mathcal{P}). \tag{3}$$

We now make the following observations:

Observation 1.

(1) $\mathcal{J}(P)$ *is an orthogonal matrix.*

(2) The rows of $\mathcal{J}(P)$ *form an orthonormal basis for* $\mathbb{R}^2$ *with positive orientation.*

(3) For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$,

$$\|J(P)\mathbf{x} - J(P)\mathbf{y}\| = h(P)\|\mathbf{x} - \mathbf{y}\|.$$

(4) If $\angle(\mathbf{x}, \mathbf{y})$ denotes the angle between two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, then

$$\angle(\mathbf{x}, \mathbf{y}) = \angle(J(P)\mathbf{x}, J(P)\mathbf{y}).$$

Proof.

(1) We have,

$$\begin{aligned}\left[\mathcal{J}(P) \cdot \mathcal{J}(P)^T\right]_{ij} &= [\mathcal{J}(P)]_{i1}[\mathcal{J}(P)^T]_{1j} + [\mathcal{J}(P)]_{i2}[\mathcal{J}(P)^T]_{2j} \\ &= [\mathcal{J}(P)]_{i1}[\mathcal{J}(P)]_{j1} + [\mathcal{J}(P)]_{i2}[\mathcal{J}(P)]_{j2}\end{aligned}$$

So, by direct computation,

$$\begin{aligned}\left[\mathcal{J}(P) \cdot \mathcal{J}(P)^T\right]_{11} &= [\mathcal{J}(P)]_{11}^2 + [\mathcal{J}(P)]_{12}^2 \\ &= \left(\frac{u_x(P)}{\sqrt{u_x(P)^2 + u_y(P)^2}}\right)^2 + \left(\frac{u_y(P)}{\sqrt{u_x(P)^2 + u_y(P)^2}}\right)^2 \\ &= \frac{u_x(P)^2}{u_x(P)^2 + u_y(P)^2} + \frac{u_y(P)^2}{u_x(P)^2 + u_y(P)^2} \\ &= \frac{u_x(P)^2 + u_y(P)^2}{u_x(P)^2 + u_y(P)^2} \\ &= 1\end{aligned}$$

$$\begin{aligned}\left[\mathcal{J}(P) \cdot \mathcal{J}(P)^T\right]_{12} &= [\mathcal{J}(P)]_{11} \underbrace{[\mathcal{J}(P)]_{21}}_{=-[\mathcal{J}(P)]_{12}} + [\mathcal{J}(P)]_{12} \underbrace{[\mathcal{J}(P)]_{22}}_{=[\mathcal{J}(P)]_{11}} \\ &= -[\mathcal{J}(\mathcal{P})]_{11}[\mathcal{J}(P)]_{12} + [\mathcal{J}(\mathcal{P})]_{11}[\mathcal{J}(P)]_{12} \\ &= 0\end{aligned}$$

$$\begin{aligned}\left[\mathcal{J}(P) \cdot \mathcal{J}(P)^T\right]_{21} &= [\mathcal{J}(P)]_{21}[\mathcal{J}(P)]_{11} + [\mathcal{J}(P)]_{22}[\mathcal{J}(P)]_{12} \\ &= 0\end{aligned}$$

$$\begin{aligned}\left[\mathcal{J}(P) \cdot \mathcal{J}(P)^T\right]_{22} &= [\mathcal{J}(P)]_{21}^2 + [\mathcal{J}(P)]_{22}^2 \\ &= (-[\mathcal{J}(P)]_{12})^2 + [\mathcal{J}(P)]_{22}^2 \\ &= 1.\end{aligned}$$

Hence,

$$\mathcal{J}(P) \cdot \mathcal{J}(P)^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(2) Just view the above computation, as we have shown that

$$[\mathcal{J}(P)]_{11}[\mathcal{J}(P)]_{21} + [\mathcal{J}(P)]_{12}[\mathcal{J}(P)]_{22} = 0.$$

To show that the rows form an orthonormal basis with *positive* orientation, we have

$$\begin{aligned}([\mathcal{J}(P)]_{11}\mathbf{i}+[\mathcal{J}(P)]_{12}\mathbf{j})\times([\mathcal{J}(P)]_{21}\mathbf{i}+[\mathcal{J}(P)]_{22}\mathbf{j}) &= \begin{vmatrix}[\mathcal{J}(P)]_{11} & [\mathcal{J}(P)]_{12}\\ [\mathcal{J}(P)]_{21} & [\mathcal{J}(P)]_{22}\end{vmatrix}\mathbf{k}\\ &= ([\mathcal{J}(P)]_{11}[\mathcal{J}(P)]_{22}-[\mathcal{J}(P)]_{12}[\mathcal{J}(P)]_{21})\mathbf{k}\\ &= ([\mathcal{J}(P)]_{11}^2+[\mathcal{J}(P)]_{12}^2)\mathbf{k}\\ &= +\mathbf{k}\end{aligned}$$

(3) - (4) Since $\mathcal{J}(P)$ is an orthogonal matrix, it preserves lengths of vectors and angles between them, and thus (3) follows. A scaling transformation, which is what $h(P)$ is, obviously preserves angles, and so (4) follows.

$\square$

Conformal Mappings of the Unit Disk

Since the author of this book says that ***conformal mappings*** are characterized by the fact that they infinitesimally

(i) preserve angles, and

(ii) preserve length (up to a scalar factor)

we have just shown that

Theorem 1. *The Jacobian $J(P)$ of a holomorphic map $f : U \subseteq \mathbb{C} \to \mathbb{C}$ at a point $P \in U$ is a conformal map.*

Despite this, $J(P)$ fails to preserve distances. So we have an example of a conformal map that doesn't preserve Euclidean distance. Let's reduce our view to conformal mappings of the unit disk for the time being.

When U is the unit disk in $\mathbb{C}$, we have a nice classification theorem for all conformal mappings on U.

Theorem 2 (The Conformal Mappings of the Unit Disk)**.** *Let $D = D(0,1)$ denote the unit disk in $\mathbb{C}$. Then a conformal mapping on D is either*

(i) A rotation $\rho_\lambda : z \mapsto e^{i\lambda}\cdot z$, $0 \le \lambda < 2\pi$;

(ii) A Möbius transformation of the form $\varphi_a : z \mapsto [z-a]/[1-\overline{a}z]$, $a \in \mathbb{C}$, $|a| < 1$; or

(iii) A composition of mappings of type (i) and (ii).

Constructing the Poincaré Metric

To "discover" the Poincaré metric, we start with what we want (invariance of the metric under conformal mappings) and go from there.

1. Given any vector, sourcing from a point P in the direction of $\mathbf{v}$, denote its *length* by $|\mathbf{v}|_P$.

2. Declare that the length of $\mathbf{e} = (1,0)$ is 1. So, $|\mathbf{e}|_0 = 1$.

3. If ϕ is a conformal self-map of the disk, then we claim that the condition

$$|\mathbf{v}|_P = |\phi_*(P)\mathbf{v}|_{\phi(P)}, \tag{4}$$

where $\phi_*(P)\mathbf{v} := \phi'(P) \cdot \mathbf{v}$ denotes the ***push-forward*** by ϕ of the vector $\mathbf{v}$, is equivalent to invariance under this new metric we're defining. Why? Since $\phi'(P)$ is the Jacobian of ϕ at P, then by our previous work, this action amounts to a rotation and a scaling. We're saying the above condition encapsulates that, after that rotation and scaling, the length of our vector stays the same. Now, we are *requiring* that $|\cdot|$ is such that (4) works for any conformal self-map of the disk ϕ. We're using this condition to derive an explicit formula for $|\cdot|$.

4. Let $\phi(z) = e^{i\lambda} \cdot z$. Then, first of all

$$\phi(z) = (\cos\lambda + i\sin\lambda)(x+iy) = x\cos\lambda - y\sin\lambda + i(x\sin\lambda + y\cos\lambda)$$

so then,

$$\phi'(z) = \begin{bmatrix} \cos\lambda & -\sin\lambda \\ \sin\lambda & \cos\lambda \end{bmatrix}$$

Condition (4) says that we must have

$$1 = |\mathbf{e}|_0 = |\phi_*(0)\mathbf{e}|_{\phi(0)} = |e^{i\lambda} \cdot \mathbf{e}|_0 = |e^{i\lambda}|_0$$

Therefore, we conclude the length of any Euclidean unit vector based at the origin is 1 in this new invariant metric.

5. Let $\phi(z)$ be the Möbius transformation

$$\psi(z) = \frac{z+a}{1+\overline{a}z}$$

with $a \in \mathbb{C}$, $|a| < 1$. Let $\mathbf{v} = e^{i\lambda}$ be a unit vector at the origin. Then, with

$$\psi'(0) = 1 - |a|^2$$

and (4), we have

$$1 = |\mathbf{v}|_0 = |\psi_*(0)\mathbf{v}|_{\psi(0)} = |(1-|a|^2) \cdot \mathbf{v}|_a$$

Hence,

$$|\mathbf{v}|_a = \frac{1}{1-|a|^2}.$$

We are ready.

Definition 1 (The Poincaré Metric)**.** *Let P be a point of the unit disk D and let $\mathbf{v}$ be any vector based at that point. Then,*

$$\boxed{|\mathbf{v}|_P = \frac{\|\mathbf{v}\|}{1-|P|^2}}$$

*is called the **Poincaré metric**.*

Note that, with the above definition, $|\mathbf{v}|_P$ as P approaches ∂D.