Conformality and Invariance

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Goal: Define a notion of distance that is preserved under holomorphic & conformal maps. Let's see why the regular notion of distance doesn't give us what we want.

The Jacobian of a Holomorphic Function

Let $U \subseteq \mathbb{C}$ be an open set, $P \in U$ a fixed point, and $f : U \to \mathbb{C}$ a holomorphic function on U. For $f(x + iy) = u(x, y) + iv(x, y)$, we can consider f as a mapping $(x, y) \to (u, v)$, where we get the real Jacobian matrix of f at P:

$$J(P) = \begin{pmatrix} u_x(P) & u_y(P) \\ v_x(P) & v_y(P) \end{pmatrix} \tag{1}$$

Since f is holomorphic, by the Cauchy-Riemann equations we know that

$$\boxed{u_x = v_y, \qquad u_y = -v_x} \tag{2}$$

Hence, we can simplify (1) in the following way:

$$\begin{aligned} J(P) &= \begin{pmatrix} u_x(P) & u_y(P) \\ v_x(P) & v_y(P) \end{pmatrix} \\ &= \begin{pmatrix} u_x(P) & u_y(P) \\ -u_y(P) & u_x(P) \end{pmatrix} \\ &= \underbrace{\sqrt{u_x(P)^2 + u_y(P)^2}}_{=:h(P)} \cdot \underbrace{\begin{pmatrix} \frac{u_x(P)}{\sqrt{u_x(P)^2+u_y(P)^2}} & \frac{u_y(P)}{\sqrt{u_x(P)^2+u_y(P)^2}} \\ \frac{-u_y(P)}{\sqrt{u_x(P)^2+u_y(P)^2}} & \frac{u_x(P)}{\sqrt{u_x(P)^2+u_y(P)^2}} \end{pmatrix}}_{=:\mathcal{J}(P)} \end{aligned}$$

Then,

$$J(P) \equiv h(P) \cdot \mathcal{J}(\mathcal{P}). \tag{3}$$

We now make the following observations:

Observation 1.

(1) $\mathcal{J}(P)$ *is an orthogonal matrix.*

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(2) *The rows of* $\mathcal{J}(P)$ *form an orthonormal basis for* $\mathbb{R}^2$ *with positive orientation.*

(3) *For* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$,

$$\|J(P)\mathbf{x} - J(P)\mathbf{y}\| = h(P)\|\mathbf{x} - \mathbf{y}\|.$$

(4) *If* $\angle(\mathbf{x}, \mathbf{y})$ *denotes the angle between two vectors* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, *then*

$$\angle(\mathbf{x}, \mathbf{y}) = \angle(J(P)\mathbf{x}, J(P)\mathbf{y}).$$

Proof.

(1) We have,

$$\begin{aligned}\left[\mathcal{J}(P) \cdot \mathcal{J}(P)^T\right]_{ij} &= [\mathcal{J}(P)]_{i1}[\mathcal{J}(P)^T]_{1j} + [\mathcal{J}(P)]_{i2}[\mathcal{J}(P)^T]_{2j} \\ &= [\mathcal{J}(P)]_{i1}[\mathcal{J}(P)]_{j1} + [\mathcal{J}(P)]_{i2}[\mathcal{J}(P)]_{j2}\end{aligned}$$

So, by direct computation,

$$\begin{aligned}\left[\mathcal{J}(P) \cdot \mathcal{J}(P)^T\right]_{11} &= [\mathcal{J}(P)]_{11}^2 + [\mathcal{J}(P)]_{12}^2 \\ &= \left(\frac{u_x(P)}{\sqrt{u_x(P)^2 + u_y(P)^2}}\right)^2 + \left(\frac{u_y(P)}{\sqrt{u_x(P)^2 + u_y(P)^2}}\right)^2 \\ &= \frac{u_x(P)^2}{u_x(P)^2 + u_y(P)^2} + \frac{u_y(P)^2}{u_x(P)^2 + u_y(P)^2} \\ &= \frac{u_x(P)^2 + u_y(P)^2}{u_x(P)^2 + u_y(P)^2} \\ &= 1\end{aligned}$$

$$\begin{aligned}\left[\mathcal{J}(P) \cdot \mathcal{J}(P)^T\right]_{12} &= [\mathcal{J}(P)]_{11} \underbrace{[\mathcal{J}(P)]_{21}}_{=-[\mathcal{J}(P)]_{12}} + [\mathcal{J}(P)]_{12} \underbrace{[\mathcal{J}(P)]_{22}}_{=[\mathcal{J}(P)]_{11}} \\ &= -[\mathcal{J}(\mathcal{P})]_{11}[\mathcal{J}(P)]_{12} + [\mathcal{J}(\mathcal{P})]_{11}[\mathcal{J}(P)]_{12} \\ &= 0\end{aligned}$$

$$\begin{aligned}\left[\mathcal{J}(P) \cdot \mathcal{J}(P)^T\right]_{21} &= [\mathcal{J}(P)]_{21}[\mathcal{J}(P)]_{11} + [\mathcal{J}(P)]_{22}[\mathcal{J}(P)]_{12} \\ &= 0\end{aligned}$$

$$\begin{aligned}\left[\mathcal{J}(P) \cdot \mathcal{J}(P)^T\right]_{22} &= [\mathcal{J}(P)]_{21}^2 + [\mathcal{J}(P)]_{22}^2 \\ &= (-[\mathcal{J}(P)]_{12})^2 + [\mathcal{J}(P)]_{22}^2 \\ &= 1.\end{aligned}$$

Hence,

$$\mathcal{J}(P) \cdot \mathcal{J}(P)^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(2) Just view the above computation, as we have shown that

$$[\mathcal{J}(P)]_{11}[\mathcal{J}(P)]_{21} + [\mathcal{J}(P)]_{12}[\mathcal{J}(P)]_{22} = 0.$$

To show that the rows form an orthonormal basis with *positive* orientation, we have

$$\begin{aligned}
([\mathcal{J}(P)]_{11}\mathbf{i} + [\mathcal{J}(P)]_{12}\mathbf{j}) \times ([\mathcal{J}(P)]_{21}\mathbf{i} + [\mathcal{J}(P)]_{22}\mathbf{j}) &= \begin{vmatrix} [\mathcal{J}(P)]_{11} & [\mathcal{J}(P)]_{12} \\ [\mathcal{J}(P)]_{21} & [\mathcal{J}(P)]_{22} \end{vmatrix} \mathbf{k} \\
&= ([\mathcal{J}(P)]_{11}[\mathcal{J}(P)]_{22} - [\mathcal{J}(P)]_{12}[\mathcal{J}(P)]_{21})\mathbf{k} \\
&= ([\mathcal{J}(P)]_{11}^2 + [\mathcal{J}(P)]_{12}^2)\mathbf{k} \\
&= +\mathbf{k}
\end{aligned}$$

(3) - (4) Since $\mathcal{J}(P)$ is an orthogonal matrix, it preserves lengths of vectors and angles between them, and thus (3) follows. A scaling transformation, which is what $h(P)$ is, obviously preserves angles, and so (4) follows.

□

Conformal Mappings of the Unit Disk

Since the author of this book says that ***conformal mappings*** are characterized by the fact that they infinitesimally

(i) preserve angles, and

(ii) preserve length (up to a scalar factor)

we have just shown that

Theorem 1. *The Jacobian $J(P)$ of a holomorphic map $f : U \subseteq \mathbb{C} \to \mathbb{C}$ at a point $P \in U$ is a conformal map.*

Despite this, $J(P)$ fails to preserve distances. So we have an example of a conformal map that doesn't preserve Euclidean distance. Let's reduce our view to conformal mappings of the unit disk for the time being.

When U is the unit disk in $\mathbb{C}$, we have a nice classification theorem for all conformal mappings on U.

Theorem 2 (The Conformal Mappings of the Unit Disk)**.** *Let $D = D(0,1)$ denote the unit disk in $\mathbb{C}$. Then a conformal mapping on D is either*

(i) A rotation $\rho_\lambda : z \mapsto e^{i\lambda} \cdot z$, $0 \leq \lambda < 2\pi$;

(ii) A Möbius transformation of the form $\varphi_a : z \mapsto [z - a]/[1 - \overline{a}z]$, $a \in \mathbb{C}$, $|a| < 1$; or

(iii) A composition of mappings of type (i) and (ii).

Constructing the Poincaré Metric

To "discover" the Poincaré metric, we start with what we want (invariance of the metric under conformal mappings) and go from there.

1. Given any vector, sourcing from a point P in the direction of $\mathbf{v}$, denote its *length* by $|\mathbf{v}|_P$.

2. Declare that the length of $\mathbf{e} = (1, 0)$ is 1. So, $|\mathbf{e}|_0 = 1$.

3. If ϕ is a conformal self-map of the disk, then we claim that the condition

$$|\mathbf{v}|_P = |\phi_*(P)\mathbf{v}|_{\phi(P)}, \tag{4}$$

   where $\phi_*(P)\mathbf{v} := \phi'(P) \cdot \mathbf{v}$ denotes the ***push-forward*** by ϕ of the vector $\mathbf{v}$, is equivalent to invariance under this new metric we're defining. Why? Since $\phi'(P)$ is the Jacobian of ϕ at P, then by our previous work, this action amounts to a rotation and a scaling. We're saying the above condition encapsulates that, after that rotation and scaling, the length of our vector stays the same. Now, we are *requiring* that $|\cdot|$ is such that (4) works for any conformal self-map of the disk ϕ. We're using this condition to derive an explicit formula for $|\cdot|$.

4. Let $\phi(z) = e^{i\lambda} \cdot z$. Then, first of all

$$\phi(z) = (\cos\lambda + i \sin\lambda)(x + iy) = x\cos\lambda - y\sin\lambda + i(x\sin\lambda + y\cos\lambda)$$

   so then,

$$\phi'(z) = \begin{bmatrix} \cos\lambda & -\sin\lambda \\ \sin\lambda & \cos\lambda \end{bmatrix}$$

   Condition (4) says that we must have

$$1 = |\mathbf{e}|_0 = |\phi_*(0)\mathbf{e}|_{\phi(0)} = |e^{i\lambda} \cdot \mathbf{e}|_0 = |e^{i\lambda}|_0$$

   Therefore, we conclude the length of any Euclidean unit vector based at the origin is 1 in this new invariant metric.

5. Let $\phi(z)$ be the Möbius transformation

$$\psi(z) = \frac{z + a}{1 + \overline{a}z}$$

   with $a \in \mathbb{C}$, $|a| < 1$. Let $\mathbf{v} = e^{i\lambda}$ be a unit vector at the origin. Then, with

$$\psi'(0) = 1 - |a|^2$$

   and (4), we have

$$1 = |\mathbf{v}|_0 = |\psi_*(0)\mathbf{v}|_{\psi(0)} = |(1 - |a|^2) \cdot \mathbf{v}|_a$$

   Hence,

$$|\mathbf{v}|_a = \frac{1}{1 - |a|^2}.$$

   We are ready.

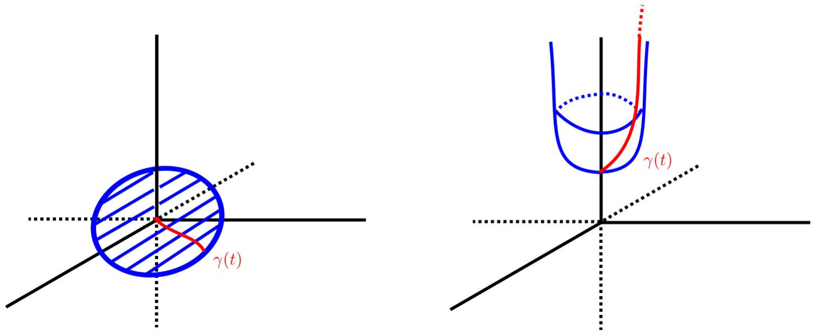

Figure 1: A visualization of how the Poincaré metric warps distances. On the left is the disk $D = \{(x, y) : x^2+y^2 < 1\}$ and on the right is the image of that disk under the map $f(x, y) = \frac{1}{1-|x^2+y^2|}$. Note that when working with the Poincaré metric, we don't "see" this surface. We are like an ant living on it and can only observe the effects. To us, it is like we are still living on a disk, but the distance between two points on the disk is different: take $p_0 = \mathbf{0}$, the origin, and take a point p on $\gamma(t)$ close to ∂D. When the disk is equipped with the standard Euclidean metric, the distance between p_0 and such p is bounded above by 1. With the Poincaré metric, as f has a ring asymptote for (x, y) close to ∂D, the distance between p and p_0 becomes unbounded.

Definition 1 (The Poincaré Metric)**.** *Let P be a point of the unit disk D and let $\mathbf{v}$ be any vector based at that point. Then,*

$$\boxed{|\mathbf{v}|_P = \frac{\|\mathbf{v}\|}{1 - |P|^2}}$$

*is called the **Poincaré metric**.*

Note that, with the above definition, as P approaches ∂D, my ramen noodles get cold. Just kidding. Actually, $|\mathbf{v}|_P \to \infty$ is what happens, but at the same time my ramen noodles do indeed get cold.

What does space "look like" if you were living under the Poincaré metric? With regards to Definition 1, simply look at the image of D under the map

$$f(x, y) = \frac{1}{1 - |x^2 + y^2|}, \qquad x^2 + y^2 < 1.$$

As in Figure 1 above.

Lengths and Distances under the Poincaré Metric

We now define the length of a curve under the Poincaré metric using geodesics.

Definition 2. *Let $\gamma : [0, 1] \to D$ be a continuously differentiable curve. The **length** of γ in the Poincaré metric is given by*

$$\ell(\gamma) = \int_0^1 |\gamma'(t)|_{\gamma(t)}\, dt = \int_0^1 \frac{\|\gamma'(t)\|}{1 - |\gamma(t)|}\, dt.$$

Example 1. *Let $\epsilon > 0$ and consider the curve $\gamma(t) = (1-\epsilon)t$, $0 \le t \le 1$. Then, the length of γ is given by*

$$\begin{aligned}
\ell(\gamma) &= \int_0^1 |\gamma'(t)|_{\gamma(t)} = \int_0^1 \frac{(1-\epsilon)}{1-|\gamma(t)|^2}\,dt \\
&= \int_0^1 \frac{(1-\epsilon)}{1-[(1-\epsilon)t]^2}\,dt = \frac{1}{2}\int_{u(0)}^{u(1)} \frac{du}{1-u} \qquad u = (1-\epsilon)t^2 \\
&= \frac{1}{2}\left[\log\left(\left|\frac{u(1)+1}{u(1)-1}\right|\right) - \log\left(\left|\frac{u(0)-1}{u(0)+1}\right|\right)\right] = \frac{1}{2}\left[\log\left(\frac{2-\epsilon}{\epsilon}\right) - \log\left(\left|\frac{-1}{1}\right|\right)\right] \\
&= \frac{1}{2}\log\left(\frac{2-\epsilon}{\epsilon}\right)
\end{aligned}$$

The use of this example is to give a computational understanding of Figure 1. As $\epsilon \to 0^+$, $\ell(\gamma) \to +\infty$. As $\epsilon \to 0^+$, $\ell(\gamma)$ is becoming the line connecting 0 to the boundary point $(\cos(\pi/4), \sin(\pi/4))$, in some sense, the distance from 0 to the boundary (along this linear path) is infinite.

Definition 3. *Let $P, Q \in D$. Then, the* ***Poincaré distance*** *$d(P,Q)$ between P and Q is given by*

$$d(P,Q) = \inf_{\gamma \in E} \ell(\gamma)$$

where E is the collection of all piecewise continuously differentiable curves connecting P to Q. (Note that the length of a piecewise differentiable curve is the sum of the lengths of its continuously differentiable pieces.)

Proposition 1.

(1) The disk D equipped with the Poincaré distance $d(\cdot,\cdot)$ is a metric space.

(2) Let $P \in D$. Then, the Poincaré distance of 0 to P is equal to

$$d(0,P) = \frac{1}{2} \cdot \log \frac{1+|P|}{1-|P|}.$$

Proof.

(1)

- ($d(x,x) = 0$) Let $x \in D$. By definition 2, first take note that

$$0 \le \ell(\gamma) = \int_0^1 \frac{\|\gamma'(t)\|}{1-|\gamma(t)|}\,dt.$$

  Now, $\gamma(t) \equiv_{t \in [0,1]} x$ is a continuously differentiable curve connecting x to itself, we have

$$\ell(\gamma) = \int_0^1 \frac{0}{1-x}\,dt = 0$$

  and it follows that $d(x,x) = 0$.

– ($d(x,y) > 0$) Let $x, y \in D$ by distinct points. Suppose for contradiction that $d(x,y) = 0$ so that

$$\inf \ell(\gamma) = 0.$$

Therefore, there exist $\gamma_1, \gamma_2, \gamma_3, \ldots$ connecting x and y such that $\ell(\gamma_1) \geq \ell(\gamma_2) \geq \ell(\gamma_3) \geq \cdots$ and

$$\lim_{n\to\infty} \ell(\gamma_n) = \lim_{n\to\infty} \int_0^1 \frac{\|\gamma_n'(t)\|}{1 - |\gamma_n(t)|}\, dt = 0.$$

For now, suppose that the $\gamma_1, \gamma_2, \ldots$ are continuously differentiable. By the mean value theorem for vector-valued functions[1], for each $n = 1, 2, 3 \ldots$, there exists $c \in (0,1)$ such that

$$|\gamma(1) - \gamma(0)| \leq (1-0)\|\gamma_n'(c)\| \implies |y - x| \leq \|\gamma_n'(c)\|.$$

Since $\lim_{n\to\infty} \ell(\gamma_n) = 0$, it follows that as $n \to \infty$, $\ell(\gamma_n') \to 0$ and hence $|y - x| = 0$, a contradiction to our presupposition that they are distinct. Now, what if the curves in the sequence $\gamma_1, \gamma_2, \gamma_3 \ldots$ are piecewise continuously differentiable? Without loss of generality, suppose $\gamma_1, \gamma_2, \gamma_3$ are all continuously differentiable on the same segments $\{I_m = (a_m, a_{m+1})\}_{m=1}^N$ with $a_1 = 0$ and $a_{N+1} = 1$ (we can say w.l.o.g. because we can start with γ_1 and re-parameterize $\gamma_2, \gamma_3, \ldots$ to be continuously differentiable on the same intervals that γ_1 is), and so

$$\ell(\gamma_n) = \sum_{m=1}^{N} \int_{a_m}^{a_{m+1}} \frac{\|\gamma_n'(t)\|}{1 - |\gamma_n(t)|}\, dt.$$

Since all constituent parts of the sum are positive, and we know that $\lim_{n\to\infty} \ell(\gamma_n) = 0$, it follows that for all $m = 1, 2, \ldots, N$,

$$\lim_{n\to\infty} \int_{a_m}^{a_{m+1}} \frac{\|\gamma_n'(t)\|}{1 - |\gamma'(t)|}\, dt = 0.$$

Applying the same analysis as above, gives that $\lim_{n\to\infty} \gamma(a_m) = \lim_{n\to\infty} \gamma(a_{m+1})$ for all $m = 1, 2, \ldots, N$, giving

$$\lim_{n\to\infty} \gamma(a_1) = \lim_{n\to\infty} \gamma(a_{N+1}) \implies y = x$$

which is a contradiction.

– (Triangle inequality) Let $x, y, z \in D$ be distinct. We have

$$\begin{aligned} d(x,y) + d(y,z) &= \inf \ell(\gamma_{xy}) + \inf \ell(\gamma_{yz}) \\ &= \inf(\ell(\gamma_{xy}) + \ell(\gamma_{yz})). \end{aligned}$$

Let $\{(\gamma_{xy}^{(n)}, \gamma_{yz}^{(n)})\}_{n\in\mathbb{N}}$ be a sequence of pairs of piecewise differentiable curves connecting x to y and y to z respectively, such that $\ell(\gamma_{xy}^{(1)}) + \ell(\gamma_{yz}^{(1)}) \geq \ell(\gamma_{xy}^{(2)}) + \ell(\gamma_{yz}^{(2)}) \geq \cdots$ decreasing to the limit $d(x,y) + d(y,z)$. For each $n \in \mathbb{N}$, we define the following path from x to z

$$\gamma_{xz}^{(n)}(t) := \begin{cases} \gamma_{xy}^{(n)}(2t) & 0 \leq t \leq \frac{1}{2} \\ \gamma_{yz}^{(n)}(2t-1) & \frac{1}{2} < t < 1 \end{cases}$$

[1]See *Principles of Mathematical Analysis* (3rd edition) by Walter Rudin, p. 113, Theorem 5.19

and we have the following inequality for all $n \in \mathbb{N}$:

$$d(x,z) \leq \ell(\gamma_{xz}^{(n)}(t)) = \ell(\gamma_{xy}^{(n)}(t)) + \ell(\gamma_{yz}^{(n)})(t).$$

Letting $n \to \infty$ gives $d(x,z) \leq d(x,y) + d(y,z)$ as desired.

(2) We first make the note that the definition of length is rotationally invariant since $\|\cdot\|$ and $|\cdot|$ are. Hence, it suffices to let P be a positive real number $P = (1-\epsilon) + i0 = (1-\epsilon, 0)$. Without loss of generality[2], we consider only curves of the form $\gamma(t) = (t, g(t))$ for $0 \leq t \leq 1-\epsilon$. Then

$$\begin{aligned}\ell(\gamma) &= \int_0^{1-\epsilon} \frac{\|\gamma'(t)\|}{1-|\gamma(t)|^2}\,dt = \int_0^{1-\epsilon} \frac{\sqrt{1^2+|g'(t)|^2}}{1-t^2-|g^2(t)|}\,dt \\ &\geq \int_0^{1-\epsilon} \frac{1}{1-t^2}\,dt = \frac{1}{2}\cdot\log\left(\frac{2-\epsilon}{\epsilon}\right) \\ &= \frac{1}{2}\log\left(\frac{1+|P|}{1-|P|}\right)\end{aligned}$$

and it follows from the exercise we did earlier that $\mu(t) = (t,0)$ is the shortest curve from 0 to P.

□

Invariance and Completeness of the Poincaré Distance

To prove the completeness of D under the Poincaré distance, which we have proved is a metric, we need some preliminary results.

Lemma 1. *Let $d(\cdot,\cdot)$ denote the Poincaré distance on D, and $\varphi : D \to D$ any conformal self-map of the disk. Then,*

(1) If $\gamma : [0,1] \to D$ is continuously differentiable[3], then $\ell(\gamma) = \ell(\varphi \circ \gamma)$.

(2) For any $z, w \in D$, $d(z,w) = d(\varphi(z), \varphi(w))$.

Proof.

(1) From how we went through constructing the Poincaré metric, we already know it's conformally invariant. Hence, it's quite easy to see that

$$\ell(\varphi\circ\gamma) = \int_0^1 |(\varphi\circ\gamma)'(t)|_{\varphi\circ\gamma}\,dt = \int_0^1 |\gamma'(t)|_{\gamma(t)}\,dt = \ell(\gamma).$$

[2]We don't lose generality because for any path $\gamma = \gamma_1 + i\gamma_2$, $\gamma_1(0) = 0, \gamma_1(1) = 1-\epsilon$, the most efficient choice of γ_1 is a reparametrization of t so that γ's time domain is $[0, 1-\epsilon]$ and $\gamma_1(t) = t$... most efficient in the sense that this is the best choice of $\gamma_1(t)$ to minimize $\|\gamma'(t)\| = \sqrt{1+|\gamma_2'(t)|^2}$. (I think I need to explain this WLOG argument better, or it can be avoided. **Edit referencing notes on Anosov flows later to avoid WLOG!**)

[3]The result for piecewise continuously differentiable γ follows immediately.

(2) This follows directly from (1): We have for any continuously differentiable γ from z to w

$$\begin{aligned}
d(\varphi(z),\varphi(w)) &= \inf_{\substack{\gamma \text{ c.d.} \\ \gamma(0)=\varphi(z),\gamma(1)=\varphi(w)}} \int_0^1 |\gamma'(t)|_{\gamma(t)}\, dt \\
&= \inf_{\substack{\gamma \text{ c.d.} \\ \gamma(0)=\varphi(z),\gamma(1)=\varphi(w)}} \int_0^1 |(\varphi^{-1}\circ\gamma)'(t)|_{(\varphi^{-1}\circ\gamma)(t)}\, dt \\
&\geq \inf_{\substack{\gamma \text{ c.d.} \\ \gamma(0)=z,\gamma(1)=w}} \int_0^1 |\gamma'(t)|_{\gamma(t)}\, dt \\
&= d(z,w).
\end{aligned}$$

We now go in the other direction,

$$\begin{aligned}
d(z,w) &= \inf_{\substack{\gamma \text{ c.d.} \\ \gamma(0)=z,\gamma(1)=w}} \int_0^1 |\gamma'(t)|_{\gamma(t)}\, dt \\
&= \inf_{\substack{\gamma \text{ c.d.} \\ \gamma(0)=z,\gamma(1)=w}} \int_0^1 |(\varphi\circ\gamma)'(t)|_{(\varphi\circ\gamma)(t)}\, dt \\
&\geq \inf_{\substack{\gamma \text{ c.d.} \\ \gamma(0)=\varphi(z),\gamma(1)=\varphi(w)}} \int_0^1 |\gamma'(t)|_{\gamma(t)}\, dt \\
&= d(\varphi(z),\varphi(w)).
\end{aligned}$$

Hence, we get $d(z,w) = d(\varphi(z),\varphi(w))$ as desired.

□

Lemma 2 (Explicit formula for the Poincaré distance)**.** *Given* $z, w \in D$*,*

$$\boxed{d(z,w) = \frac{1}{2}\log\left(\frac{|1-z\overline{w}| + |z-w|}{|1-z\overline{w}| - |z-w|}\right)}$$

Proof. For any $z, w \in D$, let φ be the conformal self-map of the disk defined by

$$\varphi(\zeta) = \frac{\zeta - w}{1 - \zeta\overline{w}}.$$

Observe that $\varphi(w) = 0$. Then, using Lemma 1, item 2, together with Proposition 1, item 2,

$$\begin{aligned} d(z,w) = d(\varphi(z), \varphi(w)) = d(0, \varphi(z)) &= \frac{1}{2}\log\left(\frac{1+|\varphi(z)|}{1-|\varphi(z)|}\right) \\ &= \frac{1}{2}\log\left(\frac{1+\left|\frac{z-w}{1-z\overline{w}}\right|}{1-\left|\frac{z-w}{1-z\overline{w}}\right|}\right) \\ &= \frac{1}{2}\log\left(\frac{\frac{|1-z\overline{w}|+|z-w|}{|1-z\overline{w}|}}{\frac{|1-z\overline{w}|-|z-w|}{|1-z\overline{w}|}}\right) \\ &= \frac{1}{2}\log\left(\frac{|1-z\overline{w}|+|z-w|}{|1-z\overline{w}|-|z-w|}\right) \end{aligned}$$

as desired. □

Lemma 3 (NEEDS WORK[4]). *Let*

$$\beta(P,r) := \{z \in D : d(z,P) < r\}.$$

Then,

(1) $\beta(P,r)$ is a Euclidean disk, and

(2) $\beta(P,r)$ has compact closure in D.

Proof.

(1) Let's go backwards by taking a Euclidean disk $D(C,R)$ and turning it into a metric disk. For $z \in D(C,R)$,

$$\begin{aligned} d(z,C) &= \frac{1}{2}\log\left(\frac{|1-z\overline{C}|+|z-C|}{|1-z\overline{C}|-|z-C|}\right) \\ &\leq \frac{1}{2}\log\left(\frac{|1-z\overline{C}|+R}{|1-z\overline{C}|-R}\right). \end{aligned}$$

For $C = 0$ (which suffices, because we can turn this disk into any other disk using a disk-preserving conformal automorphism), this gives

$$d(z,C) \leq \frac{1}{2}\log\left(\frac{1+R}{1-R}\right).$$

So, $D(0,R)$ is a metric disk centered at the metric origin with metric radius $r = \frac{1}{2}\log\left(\frac{1+R}{1-R}\right)$. Solving for R gives the Euclidean radius

$$R = \frac{e^{2r}-1}{e^{2r}+1} < 1 \qquad (0 \leq r < 1).$$

[4]Reference *Ergodic Theory, Symbolic Dynamics, and Hyperbolic Spaces* by Beadford and Keane

We can now go back and forth between metric disks $\beta(0, r)$ and Euclidean disks $D(0, R)$. Now, since the Poincaré metric is conformally invariant, this stuff always applies regardless of if we change the center. Let $\varphi(\zeta) = \frac{\zeta - P}{1 - \zeta \overline{P}}$, where it follows that $\varphi(\beta(P, r))$ is a Euclidean circle and so $\beta(P, r)$ is by our preceding discussion. I think the center does not change.

(2) Since $\beta(P, r)$ is a Euclidean disk that is a subset of D, it follows that $\overline{\beta(P, r)} \subseteq D$ and so $\beta(P, r)$ is compactly contained in D.

$\square$

Theorem 3. *The metric space* $(D, d(\cdot, \cdot))$ *is complete.*

Proof. By the above discussions, it follows that the topology induced by the Poincaré distance coincides with the Euclidean topology on the disk. Since that Euclidean topology is complete, so is $(D, d(\cdot, \cdot))$. $\square$

This closes out our discussion for the section.