The Basics of Bergman Spaces

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The Space $A^2(\Omega)$

Definition 1. *Let $\Omega \subseteq \mathbb{C}$ be a **domain** (i.e., an open, connected set). Define*

$$A^2(\Omega) = \left\{ f \text{ holomorphic on } \Omega : \int_\Omega |f(z)|^2 \, dA(z) < \infty \right\} \subseteq L^2(\Omega)$$

*where dA is the ordinary two-dimensional area measure. Then, $A^2(\Omega)$ is a complex vector space, called the **Bergman space**.*

- *The **Bergman norm** is given by*

$$\|f\|_{A^2(\Omega)} = \left[\int_\Omega |f(z)|^2 \, dA(z) \right]^{1/2}.$$

- *The standard inner product on $A^2(\Omega)$ is defined by*

$$\langle f, g \rangle = \int_\Omega f(z)\overline{g(z)} \, dA(z).$$

The next lemma provides an important estimate for functions in $A^2(\Omega)$.

Lemma 1. *Let $K \subseteq \Omega$ be compact. There is a constant, $C_K > 0$, depending on K, such that*

$$\boxed{\sup_{z \in K} |f(z)| \leq C_K \|f\|_{A^2(\Omega)}}$$

for all $f \in A^2(\Omega)$.

Before we prove it, we'll review a mean value result for holomorphic functions.

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Observation 1 (Mean Value Property of Holomorphic Functions)**.** *Let $\Omega \subseteq \mathbb{C}$ be a domain. If f is holomorphic over D, then for all $z \in D$, $r > 0$ with $B(z, r) \subseteq D$, we have*

$$f(z) = \frac{1}{A(B(z,r))} \int_{B(z,r)} f(t)\, dA(t).$$

Proof. As f is holomorphic, we have by Cauchy's integral formula:

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{B(z,r)} \frac{f(t)}{t-z}\, dt \\ &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z+re^{i\theta})}{re^{i\theta}} \cdot r\, d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(z+re^{i\theta})\, d\theta \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{1}{A(B(z,r))} \int_{B(z,r)} f(t)\, dA(t) &= \frac{1}{\pi r^2} \int_0^r \int_0^{2\pi} s f(z+se^{i\theta})\, d\theta\, ds \\ &= \frac{1}{\pi r^2} \int_0^r s(2\pi f(z))\, ds \qquad \text{by C.I.F.} \\ &= \frac{2}{r^2} \left(\frac{r^2}{2} f(z) \right) \\ &= f(z) \end{aligned}$$

as desired. ∎

We can now prove Lemma 1.

Proof of Lemma 1. Since K is compact, there exists $r > 0$, depending on K, such that for any $z \in K$, $D(z, r) \subseteq \Omega$. Therefore, for all $z \in K$ and $f \in A^2(\Omega)$, we have by the MVP (Observation 1) that

$$\begin{aligned} |f(z)| &= \frac{1}{A(B(z,r))} \left| \int_{B(z,r)} f(t)\, dA(t) \right| \\ &\le \frac{1}{A(B(z,r))} \int_{\mathbb{C}} \chi_{B(z,r)}(t) \cdot |f(t)|\, dA(t). \end{aligned}$$

where $\chi_{B(z,r)}(t)$ is the characteristic function over $B(z, r)$. Using the Cauchy-Schwarz inequality,

$$\begin{aligned} \int_{\mathbb{C}} \chi_{B(z,r)}(t) \cdot |f(t)|\, dA(t) &\le \frac{1}{A(B(z,r))} \cdot \int_{\mathbb{C}} |\chi_{B(z,r)}(t)|^2\, dA^{1/2} \cdot \int_{\mathbb{C}} |f(t)|^2 dA^{1/2} \\ &= \frac{1}{\sqrt{A(B(z,r))}} \int_{\mathbb{C}} |f(t)|^2\, dA^{1/2} \\ &= \frac{\|f\|_{A^2(B(z,r))}}{\sqrt{A(B(z,r))}} \le \frac{\|f\|_{A^2(\Omega)}}{\sqrt{\pi} r} = C_K \|f\|_{A^2(\Omega)}. \end{aligned}$$

The result follows by passing to the supremum. ∎

It follows from Lemma 1 that $A^2(\Omega)$ is complete, and is therefore a Hilbert space. But it is also a separable Hilbert space. We'll pack all of these results into one theorem.

Theorem 1. $A^2(\Omega)$ *is a separable Hilbert space.*

Proof.

- We'll first prove that $A^2(\Omega)$ is complete. Let $\{f_n\}$ be a Cauchy sequence in $A^2(\Omega)$. Obviously, by how $\|\cdot\|_{A^2(\Omega)}$ is defined (I mean, just look at it), $\{f_n\}$ is a Cauchy sequence in $L^2(\Omega)$, which we know from elementary measure theory to be complete. Let $f \in L^2(\Omega)$ be such that $\lim_{n\to\infty} \|f_n - f\|_{L^2(\Omega)} = \|f_n - f\|_{A^2(\Omega)} = 0$. By Lemma 1, for $K \subseteq \Omega$ compact, we have

$$\lim_{n\to\infty} \sup_{z\in K} |f(z) - f_n(z)| \leq \lim_{n\to\infty} C_K \|f(z) - f_n(z)\|_{A^2(\Omega)} = 0$$

  whereby it follows that $f \in A^2(\Omega)$. So, $A^2(\Omega)$ is a Hilbert space.

- We can show that $A^2(\Omega)$ is separable by exhibiting a countable orthonormal basis. Let

$$\varphi_n(z) = \sqrt{\frac{n+1}{\pi}} z^n, \qquad n = 0, 1, 2, \dots.$$

  For n, m distinct non-negative integers, we have

$$\langle \varphi_n, \varphi_m \rangle = \int_\Omega \varphi_n(z) \overline{\varphi_m}(z)\, dA(z)$$

$\square$