Conformality and Invariance

Goal: Define a notion of distance that is preserved under holomorphic maps. Let's see why the regular notion of distance doesn't give us what we want.

The Jacobian of a Holomorphic Function

Let $U \subseteq \mathbb{C}$ be an open set, $P \in U$ a fixed point, and $f : U \to \mathbb{C}$ a holomorphic function on U. For $f(x+iy) = u(x,y) + iv(x,y)$, we can consider f as a mapping $(x,y) \to (u,v)$, where we get the real Jacobian matrix of f at P:

$$J(P) = \begin{pmatrix} u_x(P) & u_y(P) \\ v_x(P) & v_y(P) \end{pmatrix} \tag{1}$$

Since f is holomorphic, by the Cauchy-Riemann equations we know that

$$\boxed{u_x = v_y, \qquad u_y = -v_x} \tag{2}$$

Hence, we can simplify (1) in the following way:

$$\begin{aligned} J(P) &= \begin{pmatrix} u_x(P) & u_y(P) \\ v_x(P) & v_y(P) \end{pmatrix} \\ &= \begin{pmatrix} u_x(P) & u_y(P) \\ -u_y(P) & u_x(P) \end{pmatrix} \\ &= \underbrace{\sqrt{u_x(P)^2 + u_y(P)^2}}_{=:h(P)} \cdot \underbrace{\begin{pmatrix} \frac{u_x(P)}{\sqrt{u_x(P)^2+u_y(P)^2}} & \frac{u_y(P)}{\sqrt{u_x(P)^2+u_y(P)^2}} \\ \frac{-u_y(P)}{\sqrt{u_x(P)^2+u_y(P)^2}} & \frac{u_x(P)}{\sqrt{u_x(P)^2+u_y(P)^2}} \end{pmatrix}}_{=:\mathcal{J}(P)} \end{aligned}$$

Then,

$$J(P) \equiv h(P) \cdot \mathcal{J}(\mathcal{P}). \tag{3}$$

We now make the following observations:

Observation 1.

(1) $\mathcal{J}(P)$ *is an orthogonal matrix.*

(2) The rows of $\mathcal{J}(P)$ *form an orthonormal basis for* $\mathbb{R}^2$ *with positive orientation.*

(3) For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$*,*

$$\|J(P)\mathbf{x} - J(P)\mathbf{y}\| = h(P)\|\mathbf{x} - \mathbf{y}\|.$$

(4) If $\angle(\mathbf{x}, \mathbf{y})$ denotes the angle between two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, then

$$\angle(\mathbf{x}, \mathbf{y}) = \angle(J(P)\mathbf{x}, J(P)\mathbf{y}).$$

Proof.

(1) We have,

$$\begin{aligned} \left[\mathcal{J}(P) \cdot \mathcal{J}(P)^T\right]_{ij} &= [\mathcal{J}(P)]_{i1}[\mathcal{J}(P)^T]_{1j} + [\mathcal{J}(P)]_{i2}[\mathcal{J}(P)^T]_{2j} \\ &= [\mathcal{J}(P)]_{i1}[\mathcal{J}(P)]_{j1} + [\mathcal{J}(P)]_{i2}[\mathcal{J}(P)]_{j2} \end{aligned}$$

So, by direct computation,

$$\begin{aligned} \left[\mathcal{J}(P) \cdot \mathcal{J}(P)^T\right]_{11} &= [\mathcal{J}(P)]_{11}^2 + [\mathcal{J}(P)]_{12}^2 \\ &= \left(\frac{u_x(P)}{\sqrt{u_x(P)^2 + u_y(P)^2}}\right)^2 + \left(\frac{u_y(P)}{\sqrt{u_x(P)^2 + u_y(P)^2}}\right)^2 \\ &= \frac{u_x(P)^2}{u_x(P)^2 + u_y(P)^2} + \frac{u_y(P)^2}{u_x(P)^2 + u_y(P)^2} \\ &= \frac{u_x(P)^2 + u_y(P)^2}{u_x(P)^2 + u_y(P)^2} \\ &= 1 \end{aligned}$$

$$\begin{aligned} \left[\mathcal{J}(P) \cdot \mathcal{J}(P)^T\right]_{12} &= [\mathcal{J}(P)]_{11} \underbrace{[\mathcal{J}(P)]_{21}}_{=-[\mathcal{J}(P)]_{12}} + [\mathcal{J}(P)]_{12} \underbrace{[\mathcal{J}(P)]_{22}}_{=[\mathcal{J}(P)]_{11}} \\ &= -[\mathcal{J}(\mathcal{P})]_{11}[\mathcal{J}(P)]_{12} + [\mathcal{J}(\mathcal{P})]_{11}[\mathcal{J}(P)]_{12} \\ &= 0 \end{aligned}$$

$$\begin{aligned} \left[\mathcal{J}(P) \cdot \mathcal{J}(P)^T\right]_{21} &= [\mathcal{J}(P)]_{21}[\mathcal{J}(P)]_{11} + [\mathcal{J}(P)]_{22}[\mathcal{J}(P)]_{12} \\ &= 0 \end{aligned}$$

$$\begin{aligned} \left[\mathcal{J}(P) \cdot \mathcal{J}(P)^T\right]_{22} &= [\mathcal{J}(P)]_{21}^2 + [\mathcal{J}(P)]_{22}^2 \\ &= (-[\mathcal{J}(P)]_{12})^2 + [\mathcal{J}(P)]_{22}^2 \\ &= 1. \end{aligned}$$

Hence,

$$\mathcal{J}(P) \cdot \mathcal{J}(P)^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(2) Just view the above computation, as we have shown that

$$[\mathcal{J}(P)]_{11}[\mathcal{J}(P)]_{21} + [\mathcal{J}(P)]_{12}[\mathcal{J}(P)]_{22} = 0.$$

To show that the rows form an orthonormal basis with *positive* orientation, we have

$$\begin{aligned}
([\mathcal{J}(P)]_{11}\mathbf{i} + [\mathcal{J}(P)]_{12}\mathbf{j}) \times ([\mathcal{J}(P)]_{21}\mathbf{i} + [\mathcal{J}(P)]_{22}\mathbf{j}) &= \begin{vmatrix} [\mathcal{J}(P)]_{11} & [\mathcal{J}(P)]_{12} \\ [\mathcal{J}(P)]_{21} & [\mathcal{J}(P)]_{22} \end{vmatrix} \mathbf{k} \\
&= ([\mathcal{J}(P)]_{11}[\mathcal{J}(P)]_{22} - [\mathcal{J}(P)]_{12}[\mathcal{J}(P)]_{21})\mathbf{k} \\
&= ([\mathcal{J}(P)]_{11}^2 + [\mathcal{J}(P)]_{12}^2)\mathbf{k} \\
&= +\mathbf{k}
\end{aligned}$$

(3) - (4) Since $\mathcal{J}(P)$ is an orthogonal matrix, it preserves lengths of vectors and angles between them, and thus (3) follows. A scaling transformation, which is what $h(P)$ is, obviously preserves angles, and so (4) follows.

$\square$

Conformal Mappings of the Unit Disk

Since the author of this book says that ***conformal mappings*** are characterized by the fact that they infinitesimally

(i) preserve angles, and

(ii) preserve length (up to a scalar factor)

we have just shown that

Theorem 1. *The Jacobian $J(P)$ of a holomorphic map $f : U \subseteq \mathbb{C} \to \mathbb{C}$ at a point $P \in U$ is a conformal map.*

Despite this, $J(P)$ fails to preserve distances. So we have an example of a conformal map that doesn't preserve Euclidean distance. Let's reduce our view to conformal mappings of the unit disk for the time being.

When U is the unit disk in $\mathbb{C}$, we have a nice classification theorem for all conformal mappings on U.

Theorem 2 (The Conformal Mappings of the Unit Disk)**.** *Let $D = D(0,1)$ denote the unit disk in $\mathbb{C}$. Then a conformal mapping on D is either*

(i) A rotation $\rho_\lambda : z \mapsto e^{i\lambda} \cdot z$, $0 \le \lambda < 2\pi$;

(ii) A Möbius transformation of the form $\varphi_a : z \mapsto [z-a]/[1-\overline{a}z]$, $a \in \mathbb{C}$, $|a| < 1$; or

(iii) A composition of mappings of type (i) and (ii).