The Basics of Bergman Spaces

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The Space $A^2(\Omega)$

Definition 1. *Let $\Omega \subseteq \mathbb{C}$ be a **domain** (i.e., an open, connected set). Define*

$$A^2(\Omega) = \left\{ f \text{ holomorphic on } \Omega : \int_\Omega |f(z)|^2 \, dA(z) < \infty \right\} \subseteq L^2(\Omega)$$

*where dA is the ordinary two-dimensional area measure. Then, $A^2(\Omega)$ is a complex vector space, called the **Bergman space**.*

- *The **Bergman norm** is given by*

$$\|f\|_{A^2(\Omega)} = \left[\int_\Omega |f(z)|^2 \, dA(z) \right]^{1/2}.$$

- *The standard inner product on $A^2(\Omega)$ is defined by*

$$\langle f, g \rangle = \int_\Omega f(z)\overline{g(z)} \, dA(z).$$

The next lemma provides an important estimate for functions in $A^2(\Omega)$.

Lemma 1. *Let $K \subseteq \Omega$ be compact. There is a constant, $C_K > 0$, depending on K, such that*

$$\boxed{\sup_{z \in K} |f(z)| \leq C_K \|f\|_{A^2(\Omega)}}$$

for all $f \in A^2(\Omega)$.

Before we prove it, we'll review a mean value result for holomorphic functions.

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Observation 1 (Mean Value Property of Holomorphic Functions)**.** *Let* $\Omega \subseteq \mathbb{C}$ *be a domain. If* f *is holomorphic over* D*, then for all* $z \in D$*,* $r > 0$ *with* $B(z, r) \subseteq D$*, we have*

$$f(z) = \frac{1}{A(B(z,r))} \int_{B(z,r)} f(t)\, dA(t).$$

Proof. As f is holomorphic, we can apply Cauchy's integral formula:

$$f(z) = \frac{1}{2\pi i} \int_{\partial B(z,r)} \frac{f(t)}{t - z}\, dt$$

□