

The Hausdorff Measure

In geometric measure theory, we like to work with outer measures so much that we just call them measures.

Definition 1. A(n) (outer) **measure** μ on \mathbb{R}^n is a function $\mu : \mathcal{P}(\mathbb{R}^n) \rightarrow [0, +\infty]$ such that, for $\{A_i\}_{i \in \mathbb{N}}$ a countable collection of subsets of \mathbb{R}^n and any $A \subseteq \bigcup_{i \in \mathbb{N}} A_i$, we have

$$\mu(A) \leq \sum_{i \in \mathbb{N}} \mu(A_i).$$

A set $A \subseteq \mathbb{R}^n$ is **measurable** if, for all $E \subset \mathbb{R}^n$,

$$\mu(E \cap A) + \mu(E \cap A^C) = \mu(E).$$

The Hausdorff Measure

Some notation:

- The **diameter** of a set S is denoted by $\text{diam}(S)$ and is given by the following formula:

$$\text{diam}(S) = \sup\{|x - y| : x, y \in S\}.$$

- The Lebesgue measure of the closed unit ball $\mathbb{B}^m(0, 1) \subseteq \mathbb{R}^m$ is denoted by α_m and is given by the following formula

$$\alpha_m = \frac{\pi^{m/2}}{\Gamma\left(\frac{m}{2} + 1\right)}.$$

Definition 2. For any $A \subseteq \mathbb{R}^n$ define the (δ, m) -**Hausdorff measure** $\mathcal{H}_\delta^m(A)$ as

$$\mathcal{H}_\delta^m(A) = \inf_{\substack{A \subset \bigcup S_j \\ \text{diam}(S_j) \leq \delta}} \sum \alpha_m \left(\frac{\text{diam}(S_j)}{2} \right)^m.$$

The **m -dimensional Hausdorff measure** is defined as $\mathcal{H}^m(A) = \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^m(A)$.

Observation 1. Let \mathcal{H}^m be the m -dimensional Hausdorff measure over \mathbb{R}^n .

- (1) \mathcal{H}^m is countably subadditive.
- (2) All Borel sets of \mathbb{R}^n are \mathcal{H}^m -measurable.
- (3) For all $A \subset \mathbb{R}^n$ there exists a Borel subset $B \subset \mathbb{R}^n$ such that

$$\mathcal{H}^m(A) = \mathcal{H}^m(B).$$

Note: Conditions (2) and (3) give that \mathcal{H}^m is a **Borel regular measure**.

Proof.

- (1) Let $\{A_i\}_{i \in \mathbb{N}}$ be a countable collection of subsets of \mathbb{R}^n , and A any subset of $\bigcup_{i \in \mathbb{N}} A_i$. Then,

$$\begin{aligned}\mathcal{H}_\delta^m(A) &= \inf_{\substack{A \subseteq \bigcup S_j \\ \text{diam}(S_j) \leq \delta}} \sum \alpha_m \left(\frac{\text{diam}(S_j)}{2} \right)^m \\ \mathcal{H}_\delta^m\left(\bigcup A_i\right) &= \inf_{\substack{\bigcup A_i \subseteq \bigcup S_j \\ \text{diam}(S_j) \leq \delta}} \sum \alpha_m \left(\frac{\text{diam}(S_j)}{2} \right)^m\end{aligned}$$

Now, any countable covering of $\bigcup_{i \in \mathbb{N}} A_i$ is also countable covering of A . Therefore,

$$\left\{ \{S_j\}_{j \in \mathbb{N}} : \bigcup_{i \in \mathbb{N}} A_i \subseteq \bigcup_{j \in \mathbb{N}} S_j \text{ and } \text{diam}(S_j) \leq \delta \right\} \subseteq \left\{ \{S_j\}_{j \in \mathbb{N}} : A \subseteq \bigcup_{j \in \mathbb{N}} S_j \text{ and } \text{diam}(S_j) \leq \delta \right\}.$$

Recall that $A \subseteq B \implies \inf A \geq \inf B$. Hence,

$$\mathcal{H}_\delta^m\left(\bigcup A_i\right) \geq \mathcal{H}_\delta^m(A).$$

Passing to the limit as $\delta \rightarrow 0$ gives the result.

- (2) Note that all Borel sets are measurable if and only if Cartheodory's criterion holds:

$$\mathcal{H}^m(A_1 \cup A_2) = \mathcal{H}^m(A_1) + \mathcal{H}^m(A_2)$$

for all $A_1, A_2 \subseteq \mathbb{R}^n$ with

$$\text{dist}(A_1, A_2) := \inf\{|x - y| : x \in A_1, y \in A_2\} > 0.$$

So, we check Cartheodory's criterion for \mathcal{H}^m and pack our bags. So, let $A_1, A_2 \subseteq \mathbb{R}^n$ with $\text{dist}(A_1, A_2) > 0$. Since we already know that

$$\mathcal{H}^m(A_1 \cup A_2) \leq \mathcal{H}^m(A_1) + \mathcal{H}^m(A_2)$$

by countable (which implies finite) subadditivity, it suffices to prove

$$\mathcal{H}^m(A_1) + \mathcal{H}^m(A_2) \leq \mathcal{H}^m(A_1 \cup A_2).$$

We have,

$$\begin{aligned}
\mathcal{H}_\delta^m(A_1) + \mathcal{H}_\delta^m(A_2) &= \inf_{\substack{A_1 \subseteq \bigcup S_j^1 \\ \text{diam}(S_j^1) \leq \delta}} \sum \alpha_m \left(\frac{\text{diam}(S_j^1)}{2} \right)^m + \inf_{\substack{A_2 \subseteq \bigcup S_j^2 \\ \text{diam}(S_j^2) \leq \delta}} \sum \alpha_m \left(\frac{\text{diam}(S_j^2)}{2} \right)^m \\
&= \inf_{\substack{A_1 \subseteq \bigcup S_j^1, A_2 \subseteq \bigcup S_k^2 \\ \text{diam}(S_j^1) \leq \delta, \text{diam}(S_k^2) \leq \delta}} \sum_{j,k} \alpha_m \left[\left(\frac{\text{diam}(S_j^1)}{2} \right)^m + \left(\frac{\text{diam}(S_k^2)}{2} \right)^m \right] \\
&\leq \inf_{\substack{A_1 \subseteq \bigcup S_j^1, A_2 \subseteq \bigcup S_k^2 \\ \text{diam}(S_j^1) \leq \delta, \text{diam}(S_k^2) \leq \delta}} \sum_{j,k} \alpha_m \left(\frac{\text{diam}(S_j^1) + \text{diam}(S_k^2)}{2} \right)^m \\
&\leq \inf_{\substack{A_1 \subseteq \bigcup S_j^1, A_2 \subseteq \bigcup S_k^2 \\ \text{diam}(S_j^1) \leq \delta, \text{diam}(S_k^2) \leq \delta \\ S_j^1 \cap S_k^2 = \emptyset}} \sum_{j,k} \alpha_m \left(\frac{\text{diam}(S_j^1 \cup S_k^2)}{2} \right)^m \\
&\leq \inf_{\substack{A_1 \subseteq \bigcup S_j^1, A_2 \subseteq \bigcup S_k^2 \\ \boxed{\text{diam}(S_j^1 \cup S_k^2) \leq \delta} \\ S_j^1 \cap S_k^2 = \emptyset}} \sum_{j,k} \alpha_m \left(\frac{\text{diam}(S_j^1 \cup S_k^2)}{2} \right)^m
\end{aligned}$$

For any covering $\{S_j\}_{j \in \mathbb{N}}$ of $A_1 \cup A_2$, observe that by virtue of the fact $\text{dist}(A_1, A_2) > 0$, we have

$$\text{diam}(S_j) \geq \text{diam}(S_j \cap A_1) + \text{diam}(S_j \cap A_2).$$

The collections $\{S_j \cap A_1\}_{j \in \mathbb{N}}$ and $\{S_j \cap A_2\}_{j \in \mathbb{N}}$ form disjoint covers of A_1, A_2 respectively. So, when considering $\mathcal{H}_\delta^m(A_1 \cup A_2)$, it suffices to take the infimum over disjoint covers of A_1 and A_2 ... in fact, these infima are equal since we have reduced the size of our cover, and we use the fact that $A \subseteq B \implies \inf(A) \geq \inf(B)$, which forces the following equality:

$$\inf_{\substack{A_1 \subseteq \bigcup S_j^1, A_2 \subseteq \bigcup S_k^2 \\ \boxed{\text{diam}(S_j^1 \cup S_k^2) \leq \delta} \\ S_j^1 \cap S_k^2 = \emptyset}} \sum_{j,k} \alpha_m \left(\frac{\text{diam}(S_j^1 \cup S_k^2)}{2} \right)^m = \inf_{\substack{(A_1 \cup A_2) \subseteq \bigcup S_j \\ \text{diam}(S_j) \leq \delta}} \sum \alpha_m \left(\frac{\text{diam}(S_j)}{2} \right)^m.$$

The result follows by passing to the limit as $\delta \rightarrow 0$.

- (3) Let $A \subseteq \mathbb{R}^n$ be arbitrary, and note that for any S_j in a covering of A , we have

$$\text{diam}(S_j) = \text{diam}(\overline{S_j}).$$

Since $\overline{S_j}$ is closed, it is Borel. We can write

$$\mathcal{H}^m(A) = \lim_{\delta \rightarrow 0} \inf_{\substack{A \subseteq \bigcup S_j \\ \text{diam}(S_j) < \delta}} \sum \alpha_m \left(\frac{\text{diam}(S_j)}{2} \right)^m = \lim_{\delta \rightarrow 0} \inf_{\substack{A \subseteq \bigcup \overline{S_j} \\ \text{diam}(\overline{S_j}) < \delta}} \sum \alpha_m \left(\frac{\text{diam}(\overline{S_j})}{2} \right)^m.$$

By a property of infimum, there exists a countable sequence of such Borel coverings $\{S_j^{(k)}\}_{k \in \mathbb{N}}$ defining $\mathcal{H}^m(A)$ (in the sense that their values when plugged into the summation above forms a non-increasing sequence of positive numbers tending to the value of $\mathcal{H}^m(A)$). Hence, let

$$B = \bigcap_k \bigcup_j S_j^{(k)}$$

which has the same Hausdorff measure of A . Cool beans dude.

□

Hausdorff Measure Exercises

Exercise 1. Let I be the unit interval $[0, 1]$ in \mathbb{R} . Prove that $\mathcal{H}^1(I) = 1$.

Exercise 2. Prove that $\mathcal{H}^n(\mathbb{B}^n(\mathbf{0}, 1)) < \infty$ just using the definition of Hausdorff measure.

Exercise 3. Let A be a nonempty subset of \mathbb{R}^n . First, prove that if $0 \leq m < k$ and $\mathcal{H}^m(A) < \infty$, then $\mathcal{H}^k(A) = 0$.

Exercise 4. Define a set $A \subset \mathbb{R}^2$ as follows: Let A_0 be a closed equilateral triangle of side 1. Let A_1 be the three equilateral triangular regions of side $1/3$ in the corners of A_0 . In general, let A_{j+1} be the triangular regions, a third of the size, in the corners of the triangles of A_j . Let $A = \bigcap A_j$. Prove that $\mathcal{H}^1(A) = 1$.

The Hausdorff Dimension

Definition 3. Let $A \subseteq \mathbb{R}^n$ be nonempty. The **Hausdorff dimension** of A is defined as

$$\begin{aligned} \inf\{m \geq 0 : \mathcal{H}^m(A) < \infty\} &= \inf\{m : \mathcal{H}^m(A) = 0\} \\ &= \sup\{m : \mathcal{H}^m(A) > 0\} \\ &= \sup\{m : \mathcal{H}^m(A) = \infty\}. \end{aligned}$$

Observation 2. All four definitions of the Hausdorff dimension above are equivalent.

Numerical Implementations [Big WIP]

- For diameter...

1. Let $S = f(D)$ for some $D \subseteq \mathbb{R}^n$ and some function f .
2. Sample random points (the more points sampled, the more accurate the method) and take their distances. A stored array would probably look like (for defining a surface)

\mathbf{x}	\mathbf{y}	$ \mathbf{x}, f(\mathbf{x}) - \mathbf{y}, f(\mathbf{y}) $
(1, 2)	(3, 4)	0.1234
(4, 3)	(1, 8)	4.6512
\vdots	\vdots	\vdots

3. Then, $\text{diam}(S)$ is reported as the max in the third column. Let's do it.

MATLAB 1. $\text{diam}(S)$ numerical implementation for surfaces.

Implementation. See the folder, will improve it later.

□