# The Hausdorff Measure

In geometric measure theory, we like to work with outer measures so much that we just call them measures.

**Definition 1.** *A(n) (outer)* ***measure*** $\mu$ *on* $\mathbb{R}^n$ *is a function* $\mu : \mathcal{P}(\mathbb{R}^n) \to [0, +\infty]$ *such that, for* $\{A_i\}_{i\in\mathbb{N}}$ *a countable collection of subsets of* $\mathbb{R}^n$ *and any* $A \subseteq \bigcup_{i\in\mathbb{N}} A_i$*, we have*

$$\mu(A) \leq \sum_{i\in\mathbb{N}} \mu(A_i).$$

*A set* $A \subseteq \mathbb{R}^n$ *is* ***measurable*** *if, for all* $E \subset \mathbb{R}^n$,

$$\mu(E \cap A) + \mu(E \cap A^C) = \mu(E).$$

## The Hausdorff Measure

Some notation:

- The ***diameter*** of a set $S$ is denoted by $\operatorname{diam}(S)$ and is given by the following formula:

$$\operatorname{diam}(S) = \sup\{|x - y| : x, y \in S\}.$$

- The Lebesgue measure of the closed unit ball $\mathbb{B}^m(0, 1) \subseteq \mathbb{R}^m$ is denoted by $\alpha_m$ and is given by the following formula

$$\alpha_m = \frac{\pi^{m/2}}{\Gamma\left(\frac{m}{2} + 1\right)}.$$

**Definition 2.** *For any* $A \subseteq \mathbb{R}^n$ *define the* $(\delta, m)$*-****Hausdorff measure*** $\mathcal{H}^m_\delta(A)$ *as*

$$\mathcal{H}^m_\delta(A) = \inf_{\substack{A\subset\bigcup S_j \\ \operatorname{diam}(S_j)\leq\delta}} \sum \alpha_m \left(\frac{\operatorname{diam}(S_j)}{2}\right)^m.$$

*The* $m$*-****dimensional Hausdorff measure*** *is defined as* $\mathcal{H}^m(A) = \lim_{\delta\to 0} \mathcal{H}^m_\delta(A)$.

**Observation 1.** *Let* $\mathcal{H}^m$ *be the* $m$*-dimensional Hausdorff measure over* $\mathbb{R}^n$.

*(1)* $\mathcal{H}^m$ *is countably subadditive.*

*(2) All Borel sets of* $\mathbb{R}^n$ *are* $\mathcal{H}^m$*-measurable.*

*(3) For all* $A \subset \mathbb{R}^n$ *there exists a Borel subset* $B \subset \mathbb{R}^n$ *such that*

$$\mathcal{H}^m(A) = \mathcal{H}^m(B).$$

## The Hausdorff Dimension

**Definition 3.** *Let $A \subseteq \mathbb{R}^n$ be nonempty. The* ***Hausdorff dimension*** *of $A$ is defined as*

$$\begin{aligned} \inf\{m \geq 0 : \mathcal{H}^m(A) < \infty\} &= \inf\{m : \mathcal{H}^m(A) = 0\} \\ &= \sup\{m : \mathcal{H}^m(A) > 0\} \\ &= \sup\{m : \mathcal{H}^m(A) = \infty\}. \end{aligned}$$

**Observation 2.** *All four definitions of the Hausdorff dimension above are equivalent.*

## Numerical Implementations [Big WIP]

- For diameter...

1. Let $S = f(D)$ for some $D \subseteq \mathbb{R}^n$ and some function $f$.
2. Sample random points (the more points sampled, the more accurate the method) and take their distances. A stored array would probably look like (for defining a surface)

| $\mathbf{x}$ | $\mathbf{y}$ | $\lvert(\mathbf{x}, f(\mathbf{x}) - (\mathbf{y}, f(\mathbf{y}))\rvert$ |
|---|---|---|
| (1, 2) | (3, 4) | 0.1234 |
| (4, 3) | (1, 8) | 4.6512 |
| $\vdots$ | $\vdots$ | $\vdots$ |

3. Then, $\operatorname{diam}(S)$ is reported as the max in the third column. Let's do it.

**MATLAB 1.** $\operatorname{diam}(S)$ *numerical implementation for surfaces.*

*Implementation.* See the folder, will improve it later. ☐