Geometric Measure Theory
Homework 3 (Exterior Algebras)

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Problem 1.

(a) In the associative tensor algebra $\bigotimes_* V$, demonstrate that the ideal $\mathfrak{U}V$ generated by all the elements $x \otimes x \in \bigotimes_2 V$ is homogeneous.

(b) Explain why

$$\mathfrak{U}V = \bigoplus_{m=2}^{\infty} \left(\bigotimes_m V \cap \mathfrak{U}V \right)$$

and deduce

$$\bigwedge_* V = \bigoplus_{m=0}^{\infty} \bigwedge_m V \qquad \text{where} \qquad \bigwedge_m V = \bigotimes_m V / \left(\bigotimes_m V \cap \mathfrak{U}V \right).$$

(c) Study the natural algebraic operation on $\bigwedge_* V$. Show that $x \wedge y = -y \wedge x$ and $x \wedge x = 0$.

Problem 2. (Universal Property for the Exterior Algebra) Prove that for every anticommutative associative unital algebra, each linear map of V into A_1 can be uniquely extended to a unital algebra homomorphism of $\bigwedge_* V$ into A, carrying $\bigwedge_m V$ into A_m for each m.

Problem 3. Prove that each linear map $f : V \to V'$ can be uniquely extended to a unit preserving algebra homomorphism

$$\bigwedge_* f : \bigwedge_* V \to \bigwedge_* V'$$

and that, subsequently, $\bigwedge_* f$ is the direct sum of linear maps

$$\bigwedge_m f : \bigwedge_m V \to \bigwedge_m V'.$$

Problem 4.

(a) Show that, if $V \simeq P \oplus Q$, then

$$\bigwedge_* V \simeq \bigwedge_* P \otimes \bigwedge_* Q.$$

(b) If V is a finite dimensional vector space, prove that

$$\dim \bigwedge_m V = \binom{n}{m}$$

for $m \geq n$, and $\bigwedge_m = \{0\}$ for $m > n$.

(c) Show that the basis of $\bigwedge_m$ has a basis equipotent with the set $\Lambda(n, m)$ of all increasing maps of $\{1, \ldots, m\}$ into $\{1, \ldots, n\}$.

Problem 5. The **diagonal map** of $\bigwedge_* V$ is the unit preserving algebra homomorphism

$$\Psi : \bigwedge_* V \to \bigwedge_* V \otimes \bigwedge_* V \qquad \text{(anticommutative product)}$$

such that $\Psi(v) = v \otimes 1 + 1 \otimes v$ whenever $v \in V$.

(a) Show that

$$\Psi(v_1 \wedge \cdots \wedge v_m) = \prod_{i=1}^{m} (v_i \otimes 1 + 1 \otimes v_i)$$

using the rules

$$(v_i \otimes 1) \cdot (1 \otimes v_j) = (v_i \otimes v_j) = -(1 \otimes v_j) \cdot (v_i \otimes 1).$$

(b) Prove that

$$\prod_{i=1}^{m}(v_i \otimes 1 + 1 \otimes v_i) = \sum_{p=0}^{m} \sum_{\sigma \in \mathrm{Sh}(p,m-p)} = \mathrm{index}(\sigma) \cdot (v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(p)}) \otimes (v_{\sigma(p+1)} \wedge \cdots \wedge v_{\sigma(m)}).$$

(c) Demonstrate that the diagonal map Ψ is associative. That is, prove the following diagram is commutative:

$$\begin{tikzcd}
 & \bigwedge_* V \arrow[dl, "\Psi"'] \arrow[dr, "\Psi"] & \\
\bigwedge_* V \otimes \bigwedge_* V \arrow[d, "\Psi\otimes\mathbf{1}"'] & & \bigwedge_* V \otimes \bigwedge_* V \arrow[d, "\mathbf{1}\otimes\Psi"] \\
(\bigwedge_* V \otimes \bigwedge_* V) \otimes \bigwedge_* V & \simeq & \bigwedge_* V \otimes (\bigwedge_* V \otimes \bigwedge_* V)
\end{tikzcd}$$

(d) Show that the diagonal map Ψ is anticommutative: If α is the automorphism of the algebra $\bigwedge_* V \otimes \bigwedge_* V$ which maps $x \otimes y$ onto $(-1)^{pq} y \otimes x$ whenever $x \in \bigwedge_p V$ and $y \in \bigwedge_q V$, then $\alpha \circ \Psi = \Psi$.

(e) Show that the diagonal map is a natural transformation: If f is a linear map of V into V', with diagonal map Ψ', then

$$\Psi' \circ \bigwedge_* f = \left(\bigwedge_* f \otimes \bigwedge_* f\right) \circ \Psi$$

Problem 6. (Determinants) Let V be a finite-dimensional vector space, $\dim V = n$.

(a) Explain why $\dim \bigwedge_n V = 1$.

(b) For a linear endomorphism $f : V \to V$, prove there exists a unique real number $\det(f)$ such that $(\bigwedge_n f)\xi = \det(f) \cdot \xi$ whenever $\xi \in \bigwedge_n V$.

(c) Relative to any choice of basis vectors $e_1, \ldots, e_n$ of V, explain why f can be described by the matrix a consisting of real coefficients $a_{i,j}$ such that

$$f(e_i) = \sum_{j=1}^{n} a_{i,j} e_j \text{ for } j = 1, \ldots, n.$$

(d) Show that

$$(\wedge_n f)(e_1 \wedge \cdots \wedge e_n) = f(e_1) \wedge \cdots \wedge f(e_n) = \sum_{\lambda} \left(\prod_{i=1}^{n} a_{i,\lambda(i)} \right) e_\lambda$$

where the summation is over the set of all permutations λ of $\{1, \ldots, n\}$.

(e) Show that $e_\lambda = \mathrm{index}(\lambda) e_1 \wedge \cdots \wedge e_n$, and obtain

$$\det(f) = \sum_{\lambda} \mathrm{index}(\lambda) \prod_{i=1}^{n} a_{i,\lambda(i)}.$$

(f) If g is another endomorphism of V, prove that

$$\wedge_n(g \circ f) = (\wedge_n g) \circ (\wedge_n f)$$

hence

$$\det(g \circ f) = \det(g) \cdot \det(f).$$

(g) Using base vectors $e_1, \ldots, e_n$ associate each permutation λ of $\{1, \ldots, n\}$ the endomorphism $\phi(\lambda)$ of V which maps e_i onto $e_{\lambda(i)}$. Show that ϕ and det are multiplicative homomorphisms, and so is index $= \det \circ \phi$.