Geometric Measure Theory Homework 3 (Exterior Algebras)

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October 2022

Problem 1.

- (a) In the associative tensor algebra $\bigotimes_* V$, demonstrate that the ideal $\mathfrak{U}V$ generated by all the elements $x \otimes x \in \bigotimes_2 V$ is homogeneous.
- (b) Explain why

$$\mathfrak{U}V = \bigoplus_{m=2}^{\infty} \left(\bigotimes_{m} V \cap \mathfrak{U}V \right)$$

and deduce

$$\bigwedge_* V = \bigoplus_{m=0}^{\infty} \bigwedge_m V$$
 where $\bigwedge_m V = \bigotimes_m V / (\bigotimes_m V \cap \mathfrak{U} V)$.

(c) Study the natural algebraic operation on $\bigwedge_* V$. Show that $x \wedge y = -y \wedge x$ and $x \wedge x = 0$.

Problem 2. (Universal Property for the Exterior Algebra) Prove that for every anticommutative associative unital algebra, each linear map of V into A_1 can be uniquely extended to a unital algebra homomorphism of $\bigwedge_* V$ into A, carrying $\bigwedge_m V$ into A_m for each m.

Problem 3. Prove that each linear map $f: V \to V'$ can be uniquely extended to a unit preserving algebra homomorphism

$$\bigwedge_* f : \bigwedge_* V \to \bigwedge_* V'$$

and that, subsequently, $\bigwedge_* f$ is the direct sum of linear maps

$$\bigwedge_{m} f: \bigwedge_{m} V \to \bigwedge_{m} V'.$$

Problem 4.

(a) Show that, if $V \simeq P \oplus Q$, then

$$\bigwedge_* V \simeq \bigwedge_* P \otimes \bigwedge_* Q.$$

(b) If *V* is a finite dimensional vector space, prove that

$$\dim \bigwedge_{m} V = \binom{n}{m}$$

for $m \ge n$, and $\bigwedge_m = \{0\}$ for m > n.

(c) Show that the basis of \bigwedge_m has a basis equipotent with the set $\Lambda(n, m)$ of all increasing maps of $\{1, \ldots, m\}$ into $\{1, \ldots, n\}$.

Problem 5. The **diagonal map** of $\bigwedge_* V$ is the unit preserving algebra homomorphism

$$\Psi: \bigwedge V \to \bigwedge V \otimes \bigwedge V$$
 (anticommutative product)

such that $\Psi(v) = v \otimes 1 + 1 \otimes v$ whenever $v \in V$.

(a) Show that

$$\Psi(v_1 \wedge \cdots \wedge v_m) = \prod_{i=1}^m (v_i \otimes 1 + 1 \otimes v_i)$$

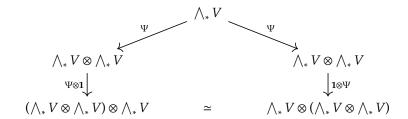
using the rules

$$(v_i \otimes 1) \cdot (1 \otimes v_i) = (v_i \otimes v_i) = -(1 \otimes v_i) \cdot (v_i \otimes 1).$$

(b) Prove that

$$\prod_{i=1}^{m} (v_i \otimes 1 + 1 \otimes v_i) = \sum_{p=0}^{m} \sum_{\sigma \in Sh(p,m-p)} = \operatorname{index}(\sigma) \cdot (v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(p)}) \otimes (v_{\sigma(p+1)} \wedge \cdots \wedge v_{\sigma(m)}).$$

(c) Demonstrate that the diagonal map Ψ is associative. That is, prove the following diagram is commutative:



- (d) Show that the diagonal map Ψ is anticommutative: If α is the automorphism of the algebra $\bigwedge_* V \otimes \bigwedge_* V$ which maps $x \otimes y$ onto $(-1)^{pq}y \otimes x$ whenever $x \in \bigwedge_p V$ and $y \in \bigwedge_q V$, then $\alpha \circ \Psi = \Psi$.
- (e) Show that the diagonal map is a natural transformation: If f is a linear map of V into V', with diagonal map Ψ' , then

$$\Psi' \circ \bigwedge_{*} f = \left(\bigwedge_{*} f \otimes \bigwedge_{*} f\right) \circ \Psi$$

Problem 6. (Determinants) Let V be a finite-dimensional vector space, dim V = n.

- (a) Explain why dim $\bigwedge_n V = 1$.
- (b) For a linear endomorphism $f: V \to V$, prove there exists a unique real number $\det(f)$ such that $(\bigwedge_n f)\xi = \det(f) \cdot \xi$ whenever $\xi \in \bigwedge_n V$.
- (c) Relative to any choice of basis vectors e_1, \ldots, e_n of V, explain why f can be described by the matrix a consisting of real coefficients $a_{i,j}$ such that

$$f(e_i) = \sum_{i=1}^{n} a_{i,j}e_j \text{ for } j = 1, \dots, n.$$

(d) Show that

$$(\wedge_n f)(e_1 \wedge \cdots \wedge e_n) = f(e_1) \wedge \cdots \wedge f(e_n) = \sum_{\lambda} \left(\prod_{i=1}^n a_{i,\lambda(i)} \right) e_{\lambda}$$

where the summation is over the set of all permutations λ of $\{1, ..., n\}$.

(e) Show that $e_{\lambda} = \operatorname{index}(\lambda)e_1 \wedge \cdots \wedge e_n$, and obtain

$$\det(f) = \sum_{\lambda} \operatorname{index}(\lambda) \prod_{i=1}^{n} a_{i,\lambda(i)}.$$

(f) If g is another endomorphism of V, prove that

$$\wedge_n(g \circ f) = (\wedge_n g) \circ (\wedge_n f)$$

hence

$$\det(g \circ f) = \det(g) \cdot \det(f).$$

(g) Using base vectors e_1, \ldots, e_n associate each permutation λ of $\{1, \ldots, n\}$ the endomorphism $\phi(\lambda)$ of V which maps e_i onto $e_{\lambda(i)}$. Show that ϕ and det are multiplicative homomorphisms, and so is index = det $\circ \phi$.