Some Theory of Outer Measures

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In geometric measure theory, we like to work with outer measures so much that we just call them measures.

Definition 1. *A set function $\mu : \{A : A \subset X\} \to [0, \infty] = \{t : 0 \leq t \leq \infty\}$ is called a(n) (outer)* ***measure*** *if*

(1) $\mu(\emptyset) = 0$

(2) $\mu(A) \leq \mu(B)$ *whenever* $A \subset B \subset X$*, and*

(3) $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i)$ *whenever* $A_1, A_2, \cdots \subset X$.

Any countably additive non-negative set function on a σ algebra $\mathcal{A}$ of subsets of X produces a measure. Consider the following:

Proposition 1. *Let ν be a countably additive non-negative set function on a σ-algebra $\mathcal{A}$ of subsets of X. Then,*

$$\nu^*(A) = \inf\{\nu(B) : A \subset B \in \mathcal{A}\} \tag{1}$$

defines a measure over X.

Proof. Since $\emptyset \in \mathcal{A}$, $\nu^*(\emptyset) = \nu(\emptyset)$. Now, by the finite additivity of ν (which follows from its countable additivity), we have

$$\nu(\emptyset) = \nu(\emptyset \cup \emptyset) = 2\nu(\emptyset) \implies \nu^*(\emptyset) = \nu(\emptyset) = 0.$$

Now suppose that $A \subset B \in \mathcal{A}$, since

$$\{\nu(C) : B \subset C \in \mathcal{A}\} \subseteq \{\nu(C) : A \subset C \in \mathcal{A}\}$$

it follows that[1]

$$\nu^*(A) = \inf\{\nu(C) : A \subset C \in \mathcal{A}\} \leq \inf\{\nu(C) : B \subset C \in \mathcal{A}\} = \nu^*(B).$$

To show the countable subadditivity of ν^*, for a sequence $\{A_i\}_{i\in\mathbb{N}} \in \mathcal{P}(X)$ and for another sequence $\{B_{i,j}\}_{j\in\mathbb{N}} \in \mathcal{P}(X)$ such that $A_i \subset \bigcup_{i=j}^{n} B_{i,j}$ for all $i \in \mathbb{N}$, let $\epsilon > 0$ such that

$$\nu^*(A_i) + \frac{\epsilon}{2^i} \geq$$

$\square$

[1] By the property of infimum where $A \subseteq B \subseteq \mathbb{R}$ implies $\inf B \leq \inf A$