# Geometric Measure Theory
# Need 2 Get Good at Tensor Algebras

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**Problem 1.** Discuss the canonical grading of the polynomial ring $\mathbf{Z}[x]$.

*Proof.* Let $\mathbf{Z}[x,i]$ be the ring of *homogeneous polynomials* in $\mathbf{Z}[x]$ of degree $i$. That is, for $p(x) \in \mathbf{Z}[x,i]$, we have,

$$p(x) = cx^i \qquad c \in \mathbf{Z}$$

Then, we clearly have

$$\mathbf{Z}[x] = \bigoplus_{n=0}^{\infty} \mathbf{Z}[x,i]$$

Now, we discuss the graded structure of $\mathbf{Z}[x]$. The function $\mu : \mathbf{Z}[x] \times \mathbf{Z}[x] \to \mathbf{Z}[x]$ is defined as just standard polynomial multiplication. Since for $p \in \mathbf{Z}[x,i]$ and $q \in \mathbf{Z}[x,j]$ we have

$$p(x) = cx^i, \qquad q(x) = dx^j \qquad\qquad p(x)q(x) = cdx^{i+j} \in \mathbf{Z}[x, i+j]$$

we get

$$\mu(\mathbf{Z}[x,i], \mathbf{Z}[x,j]) \subseteq \mathbf{Z}[x, i+j]$$

whence $\mathbf{Z}[x]$ is a graded ring. □

**Problem 2.** If $A$ and $B$ are graded algebras, show that the **graded tensor product**

$$A \otimes B = \bigoplus_{m=0}^{\infty} \bigoplus_{p+q=m} A_p \otimes B_q$$

can be made a graded algebra with either of the following two standard definitions of multiplication:

(a) Using the **commutative product**:

$$(a \otimes b) \cdot (c \otimes d) = (a \cdot c) \otimes (b \cdot d)$$

whenever $a \in A$, $b \in B$, $c \in A$, and $d \in B$.

(b) Using the **anticommutative product**:

$$(a \otimes b) \cdot (c \otimes d) = (-1)^{qr} (a \cdot c) \otimes (b \cdot d)$$

whenever $a \in A_p$, $b \in B_q$, $c \in A_r$, $d \in B_s$.

*Proof.*

(a) To reiterate, we have

$$A \otimes B = \bigoplus_{m=0}^{\infty} C_m$$

where

$$C_m = \bigoplus_{p+q=m} A_p \otimes B_q.$$

Let $\mu_{\text{comm}}$ denote the commutative product. Then we have,

$$\mu_{\text{comm}}(C_m, C_n) = \mu_{\text{comm}}\left( \bigoplus_{p+q=m} A_p \otimes B_q, \bigoplus_{p+q=n} A_p \otimes B_q \right)$$

So, let $x \in C_m = \bigoplus_{p+q=m} A_p \otimes B_q$ and $y \in C_n = \bigoplus_{p+q=n} A_p \otimes B_q$ so that

$$x = (a_0 \otimes b_m, a_1 \otimes b_{m-1}, \ldots, a_m \otimes b_0)$$
$$y = (a'_0 \otimes b'_n, a'_1 \otimes b'_{n-1}, \ldots, a'_n \otimes b'_0)$$

whereby

$$\begin{aligned} x \cdot y &= (a_0 \otimes b_m, a_1 \otimes b_{m-1}, \ldots, a_m \otimes b_0) \cdot (a'_0 \otimes b'_n, a'_1 \otimes b'_{n-1}, \ldots, a'_n \otimes b'_0) \\ &= ((a_0 \cdot a'_0 \otimes b_m \cdot b'_n), (a_1 \cdot a'_1 \otimes \underbrace{b_{m-1} b'_{n-1}}_{m+n-1}) \ldots, (a_m \cdot a'_n \otimes b_0 \cdot b'_0)) \\ &\in \bigoplus_{p+q=m+n} A_p \otimes B_q \end{aligned}$$

Hence, we have shown $\mu_{\text{comm}}(C_m, C_n) \subseteq C_{n+m}$. Hence, the graded tensor product can be made a graded algebra with the commutative product.

(b) Let $\mu_{\text{acomm}}$ denote the anticommutative product. Using the notation from above, let $x \in C_m$ and $y \in C_n$, therefore,

$$\begin{aligned} x \cdot y &= (a_0 \otimes b_m, a_1 \otimes b_{m-1}, \ldots, a_m \otimes b_0) \cdot (a'_0 \otimes b'_n, a'_1 \otimes b'_{n-1}, \ldots, a'_n \otimes b'_0) \\ &= ((-1)^{m \cdot 0}(a_0 \cdot a'_0) \otimes (b_m \cdot b'_n), (-1)^{(m-1)\cdot(1)}(a_1 \cdot a'_1) \otimes (b_{m-1} b'_{n-1}), \ldots) \\ &\in \bigoplus_{p+q=m+n} A_p \otimes B_q \end{aligned}$$

as desired.

□

**Problem 3.**

(a) Show that the anticommutative products $A \otimes B$ and $B \otimes A$ are isomorphic.

(b) Show that the commutative products $A \otimes B$ and $B \otimes A$ are isomorphic.

*Proof.*

(a) Define a map $\Phi : A \otimes B \to B \otimes A$ by specifying where it maps pure tensors:

$$\Phi(a \otimes b) := (-1)^{pq} b \otimes a \qquad a \in A_p, b \in B_q.$$

Then, additivity on all of $A \otimes B$ is simply induced. It remains to check the preservation of scalar and algebraic multiplication. Let $k \in \mathbf{R}$, then for $a \in A_p$ and $b \in B_q$, we have

$$\begin{aligned} \Phi(k(a \otimes b)) &= \Phi((ka) \otimes b) \\ &= (-1)^{pq} b \otimes (ka) \\ &= k\,((-1)^{pq} b \otimes a) \\ &= k\Phi(a \otimes b) \end{aligned}$$

Next, let $c \in A_r$ and $d \in B_s$. Then,

$$\begin{aligned} \Phi((a \otimes b) \cdot (c \otimes d)) &= \Phi((-1)^{qr}(a \cdot c) \otimes (b \cdot d)) \\ &= (-1)^{qr}\Phi((a \cdot c) \otimes (b \cdot d)) \\ &= (-1)^{qr}(-1)^{(r+p)(q+s)}(b \cdot d) \otimes (a \cdot c) \\ &= (-1)^{2qr+rs+qp+ps}(b \cdot d) \otimes (a \cdot c) \\ &= (-1)^{rs+qp+ps}(b \cdot d) \otimes (a \cdot c) \end{aligned}$$

Whereas,

$$\begin{aligned} \Phi(a \otimes b) \cdot \Phi(c \otimes d) &= (-1)^{pq}(b \otimes a) \cdot (-1)^{rs}(d \otimes c) \\ &= (-1)^{pq+rs}(b \otimes a) \cdot (d \otimes c) \\ &= (-1)^{pq+rs+ps}(b \cdot d) \otimes (a \cdot c) \end{aligned}$$

and we have equality. Hence, $\Phi$ preserves multiplication and is therefore an algebraic homomorphism. Define $\Phi^{-1}$ canonically, whereby surjectivity and injectivity trivially follow.

(b) Trivial.

$\square$

**Problem 4.**

(a) Prove that for every graded algebra $A$ there is a unique linear map

$$\Phi : A \otimes A \to A$$

such that $\Phi(x \otimes y) = x \cdot y$ whenever $x, y \in A$.

(b) If $A$ is an associative commutative (anticommutative) algebra, prove that $\Phi$ is a graded algebra homomorphsim of the commutative (anticommutative) product $A \otimes A$ into $A$.

*Proof.*

(a) Apply the universal property of the tensor product to the bilinear map $\cdot : A \times A \to A$.

(b) Done above.

$\square$

**Problem 5.**

(a) Verify the associative law for tensor algebras.

(b) Show that the element 1 of $\bigotimes_0 V = T^0(V)$ is a unit element of the ring $\bigotimes_* V = T(V)$.

*Proof.*

(a) In particular, we'll demonstrate that

$$T^\ell(V) \otimes (T^m(V) \otimes T^n(V)) \simeq (T^\ell(V) \otimes T^m(V)) \otimes T^n(V)$$

This is easy. We have,

$$\begin{aligned} T^\ell(V) \otimes (T^m(V) \otimes T^n(V)) &\simeq T^\ell(V) \otimes (T^{m+n}(V)) \\ &\simeq T^{\ell+m+n}(V) \\ &\simeq T^{\ell+m}(V) \otimes T^n(V) \\ &\simeq (T^\ell(V) \otimes T^m(V)) \otimes T^n(V) \end{aligned}$$

(b) For $1 \in T^0(V) = \mathbf{R}$, and for any $v_1 \otimes \cdots \otimes v_n \in T^n(V)$, we have

$$1(v_1 \otimes \cdots \otimes v_n) = (1v_1 \otimes \cdots \otimes v_n) = \cdots = v_1 \otimes \cdots \otimes v_n.$$

$\square$

**Problem 6.** Prove that for every graded associative algebra $A_1$ with unit element (also called a **unital associative algebra**), each linear map of $V$ into $A$ can be uniquely extended to a unit preserving algebra homomorphism of $T(V)$ into $A$, carrying $T^m(V)$ into $A_m$ for each $m$.

*Proof.* We are solving the following universal mapping property:

$$\begin{tikzcd} V \arrow[r, "\iota"] \arrow[d, "f"'] & T(V) \arrow[dl, dotted, "\exists!\, g"] \\ A_1 & \end{tikzcd}$$

where $\iota$ is the inclusion of $V$ into $T(V)$, where $\iota[V] = T^1(V)$. Then, define $g$ on each $T^n(V)$ by taking

$$g(0, 0, \ldots, v_1 \otimes \cdots \otimes v_n, \ldots 0, 0) := \prod_{i=1}^{n} f(v_i) \in A_m$$

(Note that this gives $g(c, 0, \ldots, 0) = c1_{\mathbf{A}}$) Then, we have

$$g \circ \iota(v) = g(0, v, 0, \ldots, 0) = f(v).$$

Now, to prove uniqueness, suppose $g_1$ and $g_2$ both satisfy the universal mapping property, so that

$$f = g_1 \circ \iota \qquad \text{and} \qquad f = g_2 \circ \iota.$$

Then, in this case, define the inclusion

$$\iota_n : T^n(V) \to T(V)$$

so that we subsequently have $g_1 \circ \iota_n = g_2 \circ \iota_n$. Well,

$$g_1 = \sum g_1 \circ \iota_n, \qquad g_2 = \sum g_2 \circ \iota_n$$

and so since they agree for all $n$, they are precisely equal. □

**Problem 7.** Prove that each linear map $f : V \to V'$ can be uniquely extended to a unit preserving algebra homomorphism

$$Tf : T(V) \to T(V').$$

Start by showing that $f$ is the direct sum of the linear maps

$$T^m f : T^m(V) \to T^m(V')$$

*Proof.* Just define

$$T^m f(v_1 \otimes \cdots \otimes v_m) := f(v_1) \otimes \cdots \otimes f(v_m) \in T^m(V').$$

The, take $Tf = \sum T^m f$. Define $Tf$ so that the unit maps to the unit. It clearly preserves algebraic multiplication and is unit preserving. □