The Hausdorff Measure

In geometric measure theory, we like to work with outer measures so much that we just call them measures.

Definition 1. *A(n) (outer)* ***measure*** *μ on $\mathbb{R}^n$ is a function $\mu : \mathcal{P}(\mathbb{R}^n) \to [0, +\infty]$ such that, for $\{A_i\}_{i\in\mathbb{N}}$ a countable collection of subsets of $\mathbb{R}^n$ and any $A \subseteq \bigcup_{i\in\mathbb{N}} A_i$, we have*

$$\mu(A) \leq \sum_{i\in\mathbb{N}} \mu(A_i).$$

A set $A \subseteq \mathbb{R}^n$ is ***measurable*** *if, for all $E \subset \mathbb{R}^n$,*

$$\mu(E \cap A) + \mu(E \cap A^C) = \mu(E).$$

The Hausdorff Measure

Some notation:

- The ***diameter*** of a set S is denoted by $\operatorname{diam}(S)$ and is given by the following formula:

$$\operatorname{diam}(S) = \sup\{|x - y| : x, y \in S\}.$$

- The Lebesgue measure of the closed unit ball $\mathbb{B}^m(0, 1) \subseteq \mathbb{R}^m$ is denoted by α_m and is given by the following formula

$$\alpha_m = \frac{\pi^{m/2}}{\Gamma\left(\frac{m}{2} + 1\right)}.$$

Definition 2. *For any $A \subseteq \mathbb{R}^n$ define the (δ, m)-**Hausdorff measure** $\mathcal{H}^m_\delta(A)$ as*

$$\mathcal{H}^m_\delta(A) = \inf_{\substack{A\subset\bigcup S_j \\ \operatorname{diam}(S_j)<\delta}} \sum \alpha_m \left(\frac{\operatorname{diam}(S_j)}{2}\right)^m.$$

*The m-**dimensional Hausdorff measure** is just $\lim_{\delta\to 0} \mathcal{H}^m_\delta(A)$.*

Numerical Implementations [Big WIP]

- For diameter...

  1. Let $S = f(D)$ for some $D \subseteq \mathbb{R}^n$ and some function f.

2. Sample random points (the more points sampled, the more accurate the method) and take their distances. A stored array would probably look like (for defining a surface)

$\mathbf{x}$	$\mathbf{y}$	$\lvert(\mathbf{x}, f(\mathbf{x}) - (\mathbf{y}, f(\mathbf{y}))\rvert$
(1, 2)	(3, 4)	0.1234
(4, 3)	(1, 8)	4.6512
$\vdots$	$\vdots$	$\vdots$

3. Then, $\operatorname{diam}(S)$ is reported as the max in the third column. Let's do it.

MATLAB 1. $\operatorname{diam}(S)$ *numerical implementation for surfaces.*

Implementation. See the folder, will improve it later. □