

Some Theory of Outer Measures

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In geometric measure theory, we like to work with outer measures so much that we just call them measures.

Definition 1. A set function $\mu : \{A : A \subset X\} \rightarrow [0, \infty] = \{t : 0 \leq t \leq \infty\}$ is called a(n) (outer) **measure** if

- (1) $\mu(\emptyset) = 0$
- (2) $\mu(A) \leq \mu(B)$ whenever $A \subset B \subset X$, and
- (3) $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i)$ whenever $A_1, A_2, \dots \subset X$.

Any countably additive non-negative set function on a σ -algebra \mathcal{A} of subsets of X produces a measure. Consider the following:

Proposition 1. Let ν be a countably additive non-negative set function on a σ -algebra \mathcal{A} of subsets of X . Then,

$$\nu^*(A) = \inf\{\nu(B) : A \subset B \in \mathcal{A}\} \quad (1)$$

defines a measure over X .

Proof. Since $\emptyset \in \mathcal{A}$, $\nu^*(\emptyset) = \nu(\emptyset)$. Now, by the finite additivity of ν (which follows from its countable additivity), we have

$$\nu(\emptyset) = \nu(\emptyset \cup \emptyset) = 2\nu(\emptyset) \implies \nu^*(\emptyset) = \nu(\emptyset) = 0.$$

Now suppose that $A \subset B \subset X$, since

$$\{\nu(C) : B \subset C \subset X\} \subseteq \{\nu(C) : A \subset C \subset X\}$$

it follows that¹

$$\nu^*(A) = \inf\{\nu(C) : A \subset C \subset X\} \leq \inf\{\nu(C) : B \subset C \subset X\} = \nu^*(B).$$

□

¹By the property of infimum where $A \subseteq B \subseteq \mathbb{R}$ implies $\inf B \leq \inf A$