The Hausdorff Measure

In geometric measure theory, we like to work with outer measures so much that we just call them measures.

Definition 1. *A(n) (outer)* ***measure*** μ *on* $\mathbb{R}^n$ *is a function* $\mu : \mathcal{P}(\mathbb{R}^n) \to [0, +\infty]$ *such that, for* $\{A_i\}_{i\in\mathbb{N}}$ *a countable collection of subsets of* $\mathbb{R}^n$ *and any* $A \subseteq \bigcup_{i\in\mathbb{N}} A_i$*, we have*

$$\mu(A) \leq \sum_{i\in\mathbb{N}} \mu(A_i).$$

A set $A \subseteq \mathbb{R}^n$ *is* ***measurable*** *if, for all* $E \subset \mathbb{R}^n$*,*

$$\mu(E \cap A) + \mu(E \cap A^C) = \mu(E).$$

The Hausdorff Measure

Some notation:

- The ***diameter*** of a set S is denoted by $\operatorname{diam}(S)$ and is given by the following formula:

$$\operatorname{diam}(S) = \sup\{|x - y| : x, y \in S\}.$$

- The Lebesgue measure of the closed unit ball $\mathbb{B}^m(\mathbf{0}, 1) \subseteq \mathbb{R}^m$ is denoted by α_m and is given by the following formula

$$\alpha_m = \frac{\pi^{m/2}}{\Gamma\left(\frac{m}{2} + 1\right)}.$$

Definition 2. *For any* $A \subseteq \mathbb{R}^n$ *define the* (δ, m)*-****Hausdorff measure*** $\mathcal{H}^m_\delta(A)$ *as*

$$\mathcal{H}^m_\delta(A) = \inf_{\substack{A \subset \bigcup S_j \\ \operatorname{diam}(S_j) \leq \delta}} \sum \alpha_m \left(\frac{\operatorname{diam}(S_j)}{2}\right)^m.$$

The m*-****dimensional Hausdorff measure*** *is defined as* $\mathcal{H}^m(A) = \lim_{\delta\to 0} \mathcal{H}^m_\delta(A)$.

Observation 1. *Let $\mathcal{H}^m$ be the m-dimensional Hausdorff measure over $\mathbb{R}^n$.*

(1) $\mathcal{H}^m$ is countably subadditive.

(2) All Borel sets of $\mathbb{R}^n$ are $\mathcal{H}^m$-measurable.

(3) For all $A \subset \mathbb{R}^n$ there exists a Borel subset $B \subset \mathbb{R}^n$ such that

$$\mathcal{H}^m(A) = \mathcal{H}^m(B).$$

Note: Conditions (2) and (3) give that $\mathcal{H}^m$ is a ***Borel regular measure***.

Proof.

(1) Let $\{A_i\}_{i\in\mathbb{N}}$ be a countable collection of subsets of $\mathbb{R}^n$, and A any subset of $\bigcup_{i\in\mathbb{N}} A_i$. Then,

$$\mathcal{H}^m_\delta(A) = \inf_{\substack{A\subseteq\bigcup S_j \\ \operatorname{diam}(S_j)\le\delta}} \sum \alpha_m \left(\frac{\operatorname{diam}(S_j)}{2}\right)^m$$
$$\mathcal{H}^m_\delta(\bigcup A_i) = \inf_{\substack{\bigcup A_i\subseteq\bigcup S_j \\ \operatorname{diam}(S_j)\le\delta}} \sum \alpha_m \left(\frac{\operatorname{diam}(S_j)}{2}\right)^m$$

Now, any countable covering of $\bigcup_{i\in\mathbb{N}} A_i$ is also countable covering of A. Therefore,

$$\left\{\{S_j\}_{j\in\mathbb{N}} : \bigcup_{i\in\mathbb{N}} A_i \subseteq \bigcup_{j\in\mathbb{N}} S_j \text{ and } \operatorname{diam}(S_j)\le\delta\right\} \subseteq \left\{\{S_j\}_{j\in\mathbb{N}} : A \subseteq \bigcup_{j\in\mathbb{N}} S_j \text{ and } \operatorname{diam}(S_j)\le\delta\right\}.$$

Recall that $A \subseteq B \implies \inf A \ge \inf B$. Hence,

$$\mathcal{H}^m_\delta\left(\bigcup A_i\right) \ge \mathcal{H}^m_\delta(A).$$

Passing to the limit as $\delta \to 0$ gives the result.

(2) Note that all Borel sets are measurable if and only if Cartheodory's criterion holds:

$$\mathcal{H}^m(A_1 \cup A_2) = \mathcal{H}^m(A_1) + \mathcal{H}^m(A_2)$$

for all $A_1, A_2 \subseteq \mathbb{R}^n$ with

$$\operatorname{dist}(A_1, A_2) := \inf\{|x-y| : x \in A_1, y \in A_2\} > 0.$$

So, we check Cartheodory's criterion for $\mathcal{H}^m$ and pack our bags. So, let $A_1, A_2 \subseteq \mathbb{R}^n$ with $\operatorname{dist}(A_1, A_2) > 0$. Since we already know that

$$\mathcal{H}^m(A_1 \cup A_2) \le \mathcal{H}^m(A_1) + \mathcal{H}^m(A_2)$$

by countable (which implies finite) subadditivity, it suffices to prove

$$\mathcal{H}^m(A_1) + \mathcal{H}^m(A_2) \le \mathcal{H}^m(A_1 \cup A_2).$$

We have,

$$
\begin{aligned}
\mathcal{H}^m_\delta(A_1)+\mathcal{H}^m_\delta(A_2) &= \inf_{\substack{A_1\subseteq\bigcup S^1_j\\ \operatorname{diam}(S^1_j)\le\delta}} \sum \alpha_m \left(\frac{\operatorname{diam}(S^1_j)}{2}\right)^m + \inf_{\substack{A_2\subseteq\bigcup S^2_j\\ \operatorname{diam}(S^2_k)\le\delta}} \sum \alpha_m \left(\frac{\operatorname{diam}(S^2_k)}{2}\right)^m \\
&= \inf_{\substack{A_1\subseteq\bigcup S^1_j, A_2\subseteq\bigcup S^2_k\\ \operatorname{diam}(S^1_j)\le\delta, \operatorname{diam}(S^2_k)\le\delta}} \sum_{j,k} \alpha_m \left[\left(\frac{\operatorname{diam}(S^1_j)}{2}\right)^m + \left(\frac{\operatorname{diam}(S^2_k)}{2}\right)^m\right] \\
&\le \inf_{\substack{A_1\subseteq\bigcup S^1_j, A_2\subseteq\bigcup S^2_k\\ \operatorname{diam}(S^1_j)\le\delta, \operatorname{diam}(S^2_k)\le\delta}} \sum_{j,k} \alpha_m \left(\frac{\operatorname{diam}(S^1_j)+\operatorname{diam}(S^2_k)}{2}\right)^m \\
&\le \inf_{\substack{A_1\subseteq\bigcup S^1_j, A_2\subseteq\bigcup S^2_k\\ \operatorname{diam}(S^1_j)\le\delta, \operatorname{diam}(S^2_k)\le\delta\\ S^1_j\cap S^2_k=\emptyset}} \sum_{j,k} \alpha_m \left(\frac{\operatorname{diam}(S^1_j\cup S^2_k)}{2}\right)^m \\
&\le \inf_{\substack{A_1\subseteq\bigcup S^1_j, A_2\subseteq\bigcup S^2_k\\ \boxed{\operatorname{diam}(S^1_j\cup S^2_k)\le\delta}\\ S^1_j\cap S^2_k=\emptyset}} \sum_{j,k} \alpha_m \left(\frac{\operatorname{diam}(S^1_j\cup S^2_k)}{2}\right)^m
\end{aligned}
$$

For any covering $\{S_j\}_{j\in\mathbb{N}}$ of $A_1\cup A_2$, observe that by virtue of the fact $\operatorname{dist}(A_1,A_2)>0$, we have

$$\operatorname{diam}(S_j)\ge\operatorname{diam}(S_j\cap A_1)+\operatorname{diam}(S_j\cap A_2).$$

The collections $\{S_j\cap A_1\}_{j\in\mathbb{N}}$ and $\{S_j\cap A_2\}_{j\in\mathbb{N}}$ form disjoint covers of A_1, A_2 respectively. So, when considering $\mathcal{H}^m_\delta(A_1\cup A_2)$, it suffices to take the infimum over disjoint covers of A_1 and A_2... in fact, these infima are equal since we have reduced the size of our cover, and we use the fact that $A\subseteq B \implies \inf(A)\ge\inf B$, which forces the following equality:

$$\inf_{\substack{A_1\subseteq\bigcup S^1_j, A_2\subseteq\bigcup S^2_k\\ \boxed{\operatorname{diam}(S^1_j\cup S^2_k)\le\delta}\\ S^1_j\cap S^2_k=\emptyset}} \sum_{j,k} \alpha_m \left(\frac{\operatorname{diam}(S^1_j\cup S^2_k)}{2}\right)^m = \inf_{\substack{(A_1\cup A_2)\subseteq\bigcup S_j\\ \operatorname{diam}(S_j)\le\delta}} \sum \alpha_m \left(\frac{\operatorname{diam}(S_j)}{2}\right)^m.$$

The result follows by passing to the limit as $\delta\to 0$.

(3) Let $A\subseteq\mathbb{R}^n$ be arbitrary, and note that for any S_j in a covering of A, we have

$$\operatorname{diam}(S_j)=\operatorname{diam}(\overline{S_j}).$$

Since $\overline{S_j}$ is closed, it is Borel. We can write

$$\mathcal{H}^m(A)=\lim_{\delta\to0}\inf_{\substack{A\subseteq\bigcup S_j\\ \operatorname{diam}(S_j)<\delta}} \sum \alpha_m \left(\frac{\operatorname{diam}(S_j)}{2}\right)^m = \lim_{\delta\to0}\inf_{\substack{A\subseteq\bigcup \overline{S_j}\\ \operatorname{diam}(\overline{S_j})<\delta}} \sum \alpha_m \left(\frac{\operatorname{diam}(\overline{S_j})}{2}\right)^m.$$

By a property of infimum, there exists a countable sequence of such Borel coverings $\{S_j^{(k)}\}_{k\in\mathbb{N}}$ defining $\mathcal{H}^m(A)$ (in the sense that their values when plugged into the summation above forms a non-increasing sequence of positive numbers tending to the value of $\mathcal{H}^m(A)$). Hence, let

$$B = \bigcap_k \bigcup_j S_j^{(k)}$$

which has the same Hausdorff measure of A. Cool beans dude.

$\square$

Hausdorff Measure Exercises

Exercise 1. Let I be the unit interval $[0,1]$ in $\mathbb{R}$. Prove that $\mathcal{H}^1(I) = 1$.

Proof. To compute $\mathcal{H}^1(I) = 1$, we simply show two inequalities

$$\mathcal{H}^1(I) \geq 1 \qquad \text{and} \qquad \mathcal{H}^1(I) \leq 1.$$

Let's start by writing the formula for $\mathcal{H}^1(I)$:

$$\mathcal{H}^1(I) = \lim_{\delta\to 0} \inf_{\substack{I\subset\bigcup S_j \\ \operatorname{diam}(S_j)\leq\delta}} \sum \alpha_1 \left(\frac{\operatorname{diam}(S_j)}{2}\right). \tag{1}$$

Note that $\alpha_1 = \frac{\sqrt{\pi}}{\Gamma(3/2)} = \frac{\sqrt{\pi}}{\frac{\sqrt{\pi}}{2}} = 2$ and so (1) reduces to

$$\mathcal{H}^1(I) = \lim_{\delta\to 0} \inf_{\substack{I\subset\bigcup S_j \\ \operatorname{diam}(S_j)\leq\delta}} \sum \operatorname{diam}(S_j).$$

For any given δ, there exists n large enough such that

$$\frac{1}{2^n} \leq \delta.$$

Then, we can cover I by taking

$$I \subset [0, 2^{-n}] \cup [2^{-n}, 2\cdot 2^{-n}] \cup \cdots [(k-1)2^{-n}, k2^{-n}] \cup \cdots \cup [1-2^{-n}, 1].$$

Each element of the cover has diameter $\leq \delta$. This is a partition of $[0,1]$, and so the sum of the diameters of the members of this partition is clearly 1. Hence, we have so far that $\mathcal{H}^1(I) \leq 1$.

To prove that $\mathcal{H}^1(I) \geq 1$, we'll show that given a cover $\{S_j\}$ of I, $\sum \operatorname{diam}(S_j) \geq 1$. This is obvious, since it is easy to see as a property of diameters that if $A \subseteq B$, $\operatorname{diam}(A) \leq \operatorname{diam}(B)$. Hence,

$$1 = \operatorname{diam}(I) \leq \operatorname{diam}\left(\bigcup S_j\right) \leq \sum \operatorname{diam}(S_j)$$

and so we get that $\mathcal{H}^1(I) \geq 1$, and the equality follows. $\square$

Exercise 2. Prove that $\mathcal{H}^n(\mathbb{B}^n(\mathbf{0},1)) < \infty$ just using the definition of Hausdorff measure.

Proof. Let's just write the definition of the Hausdorff measure

$$\mathcal{H}^n(\mathbb{B}(\mathbf{0},1)) = \lim_{\delta\to 0} \inf_{\substack{\mathbb{B}(\mathbf{0},1)\subset\bigcup S_j \\ \operatorname{diam}(S_j)\le\delta}} \sum \alpha_n \left(\frac{\operatorname{diam}(S_j)}{2}\right)^n$$

where we see that it suffices to show that $\mathcal{H}^n_\delta(\mathbb{B}(\mathbf{0},1))$ is uniformly bounded with respect to δ. We're going to use the fact that

$$\mathbb{B}(\mathbf{0},1) \subseteq \underbrace{[0,1]\times[0,1]\times\cdots\times[0,1]}_{n \text{ times}} = [0,1]^n$$

and follow a procedure similar to Exercise 1. For all $\delta \ge \sqrt{n}$, we have that $\mathcal{H}^n(\mathbb{B}(\mathbf{0},1)) \le \alpha_n \operatorname{diam}([0,1]^n) = \alpha_n\sqrt{n}$. We partition the hypercube into boxes of side length $\frac{1}{2^m}$, for m large enough such that the diameter of one of these boxes, which is $\frac{\sqrt{n}}{2^m} \le \delta$. It follows that $\mathcal{H}^n_\delta(\mathbb{B}(\mathbf{0},1))$ is uniformly bounded by $\alpha_n\sqrt{n}$ for all δ. $\square$

Exercise 3. Let A be a nonempty subset of $\mathbb{R}^n$. First, prove that if $0 \le m < k$ and $\mathcal{H}^m(A) < \infty$, then $\mathcal{H}^k(A) = 0$.

Exercise 4. Define a set $A \subset \mathbb{R}^2$ as follows: Let A_0 be a closed equilateral triangle of side 1. Let A_1 be the three equilateral triangular regions of side 1/3 in the corners of A_0. In general, let A_{j+1} be the triangular regions, a third of the size, in the corners of the triangles of A_j. Let $A = \bigcap A_j$. Prove that $\mathcal{H}^1(A) = 1$.

The Hausdorff Dimension

Definition 3. *Let $A \subseteq \mathbb{R}^n$ be nonempty. The **Hausdorff dimension** of A is defined as*

$$\begin{aligned}\inf\{m \geq 0 : \mathcal{H}^m(A) < \infty\} &= \inf\{m : \mathcal{H}^m(A) = 0\} \\ &= \sup\{m : \mathcal{H}^m(A) > 0\} \\ &= \sup\{m : \mathcal{H}^m(A) = \infty\}.\end{aligned}$$

Observation 2. *All four definitions of the Hausdorff dimension above are equivalent.*

Numerical Implementations [Big WIP]

- For diameter...

  1. Let $S = f(D)$ for some $D \subseteq \mathbb{R}^n$ and some function f.
  2. Sample random points (the more points sampled, the more accurate the method) and take their distances. A stored array would probably look like (for defining a surface)

$\mathbf{x}$	$\mathbf{y}$	$\lvert(\mathbf{x}, f(\mathbf{x}) - (\mathbf{y}, f(\mathbf{y}))\rvert$
(1, 2)	(3, 4)	0.1234
(4, 3)	(1, 8)	4.6512
$\vdots$	$\vdots$	$\vdots$

  3. Then, $\operatorname{diam}(S)$ is reported as the max in the third column. Let's do it.

MATLAB 1. $\operatorname{diam}(S)$ *numerical implementation for surfaces.*

Implementation. See the folder, will improve it later. □