Geometric Measure Theory
Need 2 Get Good at Tensor Products

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Problem 1. Prove that the tensor product of vector spaces $V_1 \times \cdots \times V_n$ exists and is unique.

Proof.

- EXISTENCE: Let F be the vector space of all functions $f : V_1 \times \cdots \times V_n \to \mathbf{R}$ such that for all $f \in F$, there exists a finite set $S_f \subseteq V_1 \times \cdots \times V_n$ such that
$$f \equiv 0 \text{ outside } S_f.$$
Further, define $\phi : V_1 \times \cdots \times V_n \to F$ by
$$\phi(v_1, \ldots, v_n) = \begin{cases} 1 & \text{at } (v_1, \ldots, v_n) \\ 0 & \text{elsewhere} \end{cases}$$
Then, the set G generated by all elements of two types:
$$\phi(v_1, \ldots, v_{i-1}, x, v_{i+1}, \ldots, v_n) + \phi(v_1, \ldots, v_{i-1}, y, v_{i+1}, \ldots, v_n)$$
$$-\phi(v_1, \ldots, v_{i-1}, x + y, v_{i+1}, \ldots, v_n)$$
and
$$\phi(v_1, \ldots, v_{i-1}, cv_i, v_{i+1}, \ldots, v_n) - c\, \phi(v_1, \ldots, v_{i-1}, v_i, v_{i+1}, \ldots, v_n)$$
with $c \in \mathbf{R}$ defines a subspace of F. Then, define
$$V_1 \otimes \cdots \otimes V_n = F/G$$
where
$$\mu = \pi \circ \phi$$
where π is the canonical map of F onto F/G.
Next, we'll check that μ is multilinear. We have,
$$\begin{aligned} \mu(v_1, \ldots, v_{i-1}, x + y, v_{i+1}, \ldots, v_n) &= \pi(\phi(v_1, \ldots, v_{i-1}, x + y, v_{i+1}, \ldots, v_n)) \\ &= \pi\left(\begin{cases} 1 & \text{at } (v_1, \ldots, v_{i-1}, x + y, v_{i+1}, \ldots, v_n) \\ 0 & \text{elsewhere} \end{cases}\right) \end{aligned}$$
The equivalence class of the above function has the two elements below:
$$[\phi(v_1, \ldots, v_{i-1}, x + y, v_{i+1}, \ldots, v_n)]$$
$$= \{\phi(v_1, \ldots, v_{i-1}, x, v_{i+1}, \ldots, v_n) + \phi(v_1, \ldots, v_{i-1}, y, v_{i+1}, \ldots, v_n), \phi(v_1, \ldots, v_{i-1}, x + y, v_{i+1}, \ldots, v_n)\}$$
Hence,
$$\begin{aligned} \pi(\phi(v_1, \ldots, v_{i-1}, x + y, v_{i+1}, \ldots, v_n)) &= \pi[\phi((v_1, \ldots, v_{i-1}, x, v_{i+1}, \ldots, v_n)) + \phi((v_1, \ldots, v_{i-1}, y, v_{i+1}, \ldots, v_n))] \\ &= \pi(\phi(v_1, \ldots, v_{i-1}, x, v_{i+1}, \ldots, v_n)) + \pi(\phi(v_1, \ldots, v_{i-1}, y, v_{i+1}, \ldots, v_n)) \\ &= \mu(v_1, \ldots, v_{i-1}, x, v_{i+1}, \ldots, v_n) + \mu(v_1, \ldots, v_{i-1}, y, v_{i+1}, \ldots, v_n) \end{aligned}$$

which establishes additivity in every slot. For the scalar property, with $c \in \mathbf{R}$, we have,

$$\begin{aligned}
\mu(v_1, \ldots, v_{i-1}, cx, v_{i+1}, \ldots, v_n) &= \pi(\phi(v_1, \ldots, v_{i-1}, cx, v_{i+1}, \ldots, v_n)) \\
&= \pi(c\phi(v_1, \ldots, v_{i-1}, x, v_{i+1}, \ldots, v_n)) \\
&= c\pi(\phi(v_1, \ldots, v_{i-1}, x, v_{i+1}, \ldots, v_n)) \\
&= c\mu(v_1, \ldots, v_{i-1}, x, v_{i+1}, \ldots, v_n)
\end{aligned}$$

and hence μ is n-linear. Now, we check that $(F/G, \mu)$ defines a tensor product of $V_1, \ldots, V_n$. Let W be another vector space, and $L : V_1 \times \cdots \times V_n \to W$ some n-linear map. Since each $f \in F$ is uniquely determined by its values on S_f, we may write

$$f(v_1, \ldots, v_n) = \sum_{V_1 \times \cdots \times V_n} c_f(v_1, \ldots, v_n)\chi_{(v_1, \ldots, v_n)}$$

Then, define a map $g^* : F \to W$ by

$$g^*\left(\sum_{V_1 \times \cdots \times V_n} c_f(v_1, \ldots, v_n)\chi_{(v_1, \ldots, v_n)}\right) = \sum_{V_1 \times \cdots \times V_n} c_f(v_1, \ldots, v_n)L(v_1, \ldots, v_n)$$

It is easy to see that g^* is linear, as for $k \in \mathbf{R}$,

$$\begin{aligned}
&g^*\left(\sum_{V_1 \times \cdots \times V_n} c_f(v_1, \ldots, v_n)\chi_{(v_1, \ldots, v_n)} + k\sum_{V_1 \times \cdots \times V_n} c_{f'}(v_1, \ldots, v_n)\chi_{(v_1, \ldots, v_n)}\right) \\
&= g^*\left(\sum_{V_1 \times \cdots \times V_n} (c_f(v_1, \ldots, v_n) + (kc_{f'}(v_1, \ldots, v_n))\chi_{(v_1, \ldots, v_n)}\right) \\
&= \sum_{V_1 \times \cdots \times V_n} (c_f(v_1, \ldots, v_n) + kc_{f'}(v_1, \ldots, v_n))L(v_1, \ldots, v_n) \\
&= \sum_{V_1 \times \cdots \times V_n} c_f(v_1, \ldots, v_n)L(v_1, \ldots, v_n) + k\sum_{V_1 \times \cdots \times V_n} c_{f'}(v_1, \ldots, v_n)L(v_1, \ldots, v_n) \\
&= g^*\left(\sum_{V_1 \times \cdots \times V_n} c_f(v_1, \ldots, v_n)\chi_{(v_1, \ldots, v_n)}\right) + kg^*\left(\sum_{V_1 \times \cdots \times V_n} c_{f'}(v_1, \ldots, v_n)\chi_{(v_1, \ldots, v_n)}\right)
\end{aligned}$$

By virtue of this, g^* descends to a linear map $F/G \to W$, which we call g. More formally, we define

$$g : F/G \to W \qquad \text{where} \qquad [\phi] \mapsto g^*(\phi).$$

Uniqueness of g is trivial.

- Uniqueness: Suppose that (T, μ) and (T', μ') are both tensor products of $V_1, \ldots, V_n$. Then we have two factorizations:

$$\begin{tikzcd}
V_1 \times \cdots \times V_n \arrow[r, "\mu"] \arrow[d, "\mu'"'] & T \arrow[dl, dashed, "\exists!\, g"] \\
T' &
\end{tikzcd}
\qquad
\begin{tikzcd}
V_1 \times \cdots \times V_n \arrow[r, "\mu'"] \arrow[d, "\mu"'] & T' \arrow[dl, dashed, "\exists!\, g'"] \\
T &
\end{tikzcd}$$

Then, we may demonstrate that T and T are isomorphic by demonstrating that the following diagram

$$\begin{tikzcd}
& T \arrow[d, "g"] \arrow[dd, bend left=60, "\mathrm{Id}_T"] \\
V_1 \times \cdots \times V_n \arrow[ur, "\mu"] \arrow[r, "\mu'"] \arrow[dr, "\mu"'] & T' \arrow[d, "g'"] \\
& T
\end{tikzcd}$$

commutes. It must, for with $\mu : V_1 \times \cdots \times V_n \to T$ has unique factorization through T, that's just the identity map. This lends the result.

□

Problem 2. Prove that for any linear maps

$$f_1 : V_1 \to V_1', \dots, f_n : V_n \to V_n'$$

there exists a unique linear map

$$f_1 \otimes \cdots \otimes f_n : V_1 \otimes \cdots \otimes V_n \to V_1' \otimes \cdots \otimes V_n'$$

such that

$$(f_1 \otimes \cdots \otimes f_n)(v_1 \otimes \cdots \otimes v_n) = f_1(v_1) \otimes \cdots \otimes f_n(v_n)$$

whenever $v_j \in V_j$ for $j = 1, \dots, n$.

Proof. Let

$$f : V_1 \times \cdots \times V_n \to V_1' \times \cdots \times V_n', \qquad (v_1, \dots, v_n) \mapsto (f_1(v_1), \dots, f_n(v_n))$$

Given that each f_i is linear for $1 \le i \le n$, it is clear that f is n linear. Then, we seek to find φ such that the following diagram commutes:

$$\begin{tikzcd} V_1 \times \cdots \times V_n \arrow[r, "\mu"] \arrow[d, "f"'] & V_1 \otimes \cdots \otimes V_n \arrow[d, dashed, "\exists!\ \varphi"] \\ V_1' \times \cdots \times V_n' \arrow[r, "\mu'"'] & V_1' \otimes \cdots \otimes V_n' \end{tikzcd}$$

Define $g : V_1 \times \cdots V_n \to V_1' \otimes \cdots \otimes V_n'$ by

$$g(v_1, \dots, v_n) = f(v_1) \otimes \cdots \otimes f(v_n).$$

Then g is clearly n-linear, so there exists a unique map φ such that the diagram

$$\begin{tikzcd} V_1 \times \cdots \times V_n \arrow[r, "\mu"] \arrow[d, "f"'] \arrow[dr, "g"'] & V_1 \otimes \cdots \otimes V_n \arrow[d, dashed, "\exists!\ \varphi"] \\ V_1' \times \cdots \times V_n' \arrow[r, "\mu'"'] & V_1' \otimes \cdots \otimes V_n' \end{tikzcd}$$

commutes. That is, there exists a unique map φ such that

$$g(v_1, \dots, v_n) = \varphi \circ \mu(v_1, \dots, v_n)$$

That is,

$$\varphi \circ \mu(v_1, \dots, v_n) = \varphi(v_1 \otimes \cdots \otimes v_n) = f_1(v_1) \otimes \cdots \otimes f_n(v_n)$$

Therefore, set $\varphi = f_1 \otimes \cdots \otimes f_n$. □

Problem 3. Prove the following isomorphisms:

(a) For each permutation λ of $\{1, \dots, n\}$,

$$V_1 \otimes \cdots \otimes V_n \simeq V_{\lambda(1)} \otimes \cdots \otimes V_{\lambda(n)}$$

(b) For $m < n$,

$$(V_1 \otimes \cdots \otimes V_m) \otimes (V_{m+1} \otimes \cdots \otimes V_n) \simeq V_1 \otimes \cdots \otimes V_n$$

(c) If $V \simeq P \oplus Q$, then

$$V \otimes W \simeq (P \otimes W) \oplus (Q \otimes W)$$

Proof.

(a) Let $f : V_1 \times \cdots \times V_n \to V_{\lambda(1)} \otimes \cdots \otimes V_{\lambda(n)}$ where

$$(v_1, \dots, v_n) \mapsto v_{\lambda(1)} \otimes \cdots \otimes v_{\lambda(n)}.$$

Then, we have the following commutative diagram

$$\begin{tikzcd} V_1 \times \cdots \times V_n \arrow[r, "\mu"] \arrow[d, "f"'] & V_1 \otimes \cdots \otimes V_n \arrow[dl, "\exists!\ \Phi"] \\ V_{\lambda(1)} \otimes \cdots \otimes V_{\lambda(n)} & \end{tikzcd}$$

This lends that $\Phi(v_1 \otimes \cdots \otimes v_n) = v_{\lambda(1)} \otimes \cdots \otimes v_{\lambda(n)}$, which is a linear map. Defining $\Phi^{-1}(v_1 \otimes \cdots \otimes v_n) = v_{\lambda^{-1}(1)} \otimes \cdots \otimes v_{\lambda^{-1}(n)}$, we get that $\Phi \circ \Phi^{-1}$ and $\Phi^{-1} \circ \Phi$ are identity maps, whereby the two spaces are isomorphic.

(b) We try another method by demonstrating that

$$(V_1 \otimes \cdots \otimes V_m) \otimes (V_{m+1} \otimes \cdots \otimes V_n)$$

is a tensor product of $V_1, \ldots, V_n$. We have the following commutative diagram

$$\begin{tikzcd}
V_1 \times \cdots \times V_m \arrow[r, "\mu_1"] \arrow[dr, "f \circ \mathrm{pr}_1"] & V_1 \otimes \cdots \otimes V_m \arrow[d, dotted, "\exists!\, g_1"] \\
V_1 \times \cdots \times V_n \arrow[u, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"] \arrow[r, "f"] & W \\
V_{m+1} \times \cdots \times V_n \arrow[r, "\mu_2"] \arrow[ur, "f \circ \mathrm{pr}_2"] & V_{m+1} \otimes \cdots \otimes V_n \arrow[u, dotted, "\exists!\, g_2"]
\end{tikzcd}$$

Then, the solution to the universal mapping problem

$$\begin{tikzcd}
V_1 \times \cdots \times V_n \arrow[r, "\mu"] \arrow[d, "f"] & (V_1 \otimes \cdots \otimes V_m) \otimes (V_{m+1} \otimes \cdots \otimes V_n) \arrow[dl, "\exists!\, g"] \\
W &
\end{tikzcd}$$

is given by setting $\mu = \mu_1 \otimes \mu_2$ and $g = g_1 \otimes g_2$.

(c) Let $V \simeq_\Phi P \oplus Q$. We'll show that $(P \otimes W) \oplus (Q \otimes W)$ is a tensor product of $V \times W$:

$$\begin{tikzcd}
V \times W \arrow[r, dotted, "\exists\, \mu"] \arrow[d, "f"] & \boxed{(P \otimes W) \oplus (Q \otimes W)} \arrow[dl, dotted, "\exists!\, g"] \\
W &
\end{tikzcd}$$

First, we have $V \simeq_\Phi P \oplus Q$. This lends the linear isomorphism $\Phi \times \mathbf{1}_W$ between $V \times W$ and $(P \oplus Q) \times W$. From this, we have the standard tensor product mapping $\mu_1 : (P \oplus Q) \times W \to (P \oplus Q) \otimes W$ where $((u, v), w) \mapsto (u, v) \otimes w$. Finally, we define a mapping

$$\Psi : (P \oplus Q) \otimes W \to (P \otimes W) \oplus (Q \otimes W)$$

by

$$(u, v) \otimes w \mapsto (u \otimes w, v \otimes w).$$

First we check that this mapping is well-defined. If the projection maps $\Psi_1 : (P \oplus Q) \otimes W \to P \otimes W$ and $\Psi_2 : (P \oplus Q) \otimes W \to Q \otimes W$ are well defined, then so is $\Psi = \Psi_1 \oplus \Psi_2$. Now, let $\mathrm{pr}_1 : P \oplus Q \to P$ where $(u, v) \mapsto u$. Then

$$\Psi_1 = \mathrm{pr}_1 \otimes \mathbf{1}_W$$

and a similar argument shows Ψ_2 is well defined, whereby Ψ is well defined and we have an isomorphism between $(P \oplus Q) \otimes W$ and $(PW) \oplus (Q \otimes W)$, lending the result.

$$V \times W \xrightarrow{\Phi \times \mathbf{1}_W} (P \oplus Q) \times W \xrightarrow{\mu_1} (P \oplus Q) \otimes W \xrightarrow{\Psi} (P \otimes W) \oplus (Q \otimes W)$$

□

Problem 4. Prove that if B_j is a basis for V_j for each j, then the elements $b_1 \otimes \cdots \otimes b_n$ with $b_j \in B_j$, form a basis of $V_1 \otimes \cdots \otimes V_n$. Therefore,

$$\dim(V_1 \otimes \cdots \otimes V_n) = \prod_{j=1}^{n} \dim V_j$$

Proof. Since B_j is a basis for V_j, we have that $V_j \simeq \bigoplus_{b \in B_j} \mathrm{span}(b)$. Therefore, by **Problem 3**,

$$\begin{aligned}
\bigotimes_{j=1}^{n} V_j &\simeq \bigotimes_{j=1}^{n} \left(\bigoplus_{b \in B_j} \mathrm{span}(b) \right) \\
&\simeq \bigoplus_{(b_1, \ldots, b_j) \in B_1 \times \cdots \times B_j} \mathrm{span}(b_1 \otimes \cdots \otimes b_j)
\end{aligned}$$

□

Problem 5. Let $V_1, V_2, \ldots, V_n$ be vector spaces. For $v_1 \otimes \cdots \otimes v_n \in V_1 \otimes \cdots \otimes V_n$ and $v_1' \otimes \cdots \otimes v_n' \in V_1 \otimes \cdots \otimes V_n$, prove that

$$v_1 \otimes \cdots \otimes v_n = v_1' \otimes \cdots \otimes v_n'$$

if and only if for all n-linear $f : V_1 \times \cdots \times V_n \to W$, and for all W, where W is some other vector space,

$$f(v_1, \ldots, v_n) = f(v_1', \ldots, v_n').$$

Proof. Suppose $v_1 \otimes \cdots \otimes v_n = v_1' \otimes \cdots \otimes v_n'$, and let $f : V_1 \times \cdots \times V_n \to W$ be n-linear. Then, we have the following commutative diagram.

$$\begin{tikzcd} V_1 \times \cdots \times V_n \arrow[r, "\mu"] \arrow[d, "f"'] & V_1 \otimes \cdots \otimes V_n \arrow[dl, "\exists!\, g"] \\ W & \end{tikzcd}$$

Since $v_1 \otimes \cdots \otimes v_n = v_1' \otimes \cdots \otimes v_n'$,

$$f(v_1, \ldots, v_n) = g(v_1 \otimes \cdots \otimes v_n) = g(v_1' \otimes \cdots \otimes v_n') = f(v_1', \ldots, v_n').$$

Conversely, suppose that for all n-linear $f : V_1 \times \cdots \times V_n \to W$,

$$f(v_1, \ldots, v_n) = f(v_1', \ldots, v_n').$$

We must show

$$\mu(v_1, \ldots, v_n) = v_1 \otimes \cdots \otimes v_n = v_1' \otimes \cdots \otimes v_n' = \mu(v_1', \ldots, v_n').$$

Simply pick $W = V_1 \otimes \cdots \otimes V_n$, and we are done. $\square$