Partial Differential Equations
Chapter 4 Homework

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Exercise #2. Consider Laplace's equation $\Delta u = 0$ in $\mathbb{R}^2$, taken with the Cauchy data

$$u = 0, \quad \frac{\partial u}{\partial x_2} = \frac{1}{n}\sin(nx_1) \quad \text{on } \{x_2 = 0\}.$$

Employ separation of variables to derive the solution

$$u = \frac{1}{n^2}\sin(nx_1)\sinh(nx_2).$$

What happens to u as $n \to \infty$? Is the Cauchy problem for Laplace's equation well-posed?

Proof. Let $u(x_1, x_2) = v(x_1)w(x_2)$ so that

$$\Delta u = \frac{\partial^2 v}{\partial x_1^2}w + \frac{\partial^2 w}{\partial x_2^2}v = 0 \implies \frac{\partial^2 v}{\partial x_1^2}\frac{1}{v} = -\frac{\partial^2 w}{\partial x_2^2}\frac{1}{w} = \mu$$

for some constant $\mu \in \mathbb{R}$. Since $\frac{\partial u}{\partial x_2} = \frac{\partial w}{\partial x_2}(0)v(x_1) = \frac{1}{n}\sin(nx_1)$, suppose that $\frac{\partial w}{\partial x_2}(0) = \frac{1}{n}$ and $v(x_1) = \sin(nx_1)$ so that

$$\frac{\partial^2 v}{\partial x_1^2}\frac{1}{v(x_1)} = -n^2\sin(nx_1)\frac{1}{\sin(nx_1)} = -n^2 = \mu.$$

Therefore,

$$\frac{\partial^2 w}{\partial x_2^2} = n^2 w \implies w(x_2) = c_1 e^{nx_2} + c_2 e^{-nx_2}.$$

Solving for c_1 and c_2, one has

$$\frac{\partial w}{\partial x_2}(0) = \frac{1}{n} \implies c_1 n - c_2 n = \frac{1}{n}$$

giving

$$c_1 - c_2 = \frac{1}{n^2}.$$

Next, $u = 0$ for $x_2 = 0$ gives

$$\sin(nx_1)(c_1 + c_2) = 0 \implies c_1 + c_2 = 0.$$

The solution of the resulting system

$$\begin{cases} c_1 - c_2 = \frac{1}{n^2} \\ c_1 + c_2 = 0 \end{cases} \tag{1}$$

is $c_1 = \frac{1}{2n^2}, c_2 = -\frac{1}{2n^2}$. Hence,

$$w(x_2) = \frac{1}{n^2}\frac{e^{nx_2} - e^{-nx_2}}{2} = \frac{1}{n^2}\sinh(nx_2)$$

and therefore we obtain the solution

$$u = v(x_1)w(x_2) = \frac{1}{n^2}\sin(nx_1)\sinh(nx_2).$$

Next, we determine if this problem is well-posed. Recall from [Evans, 7] that this means:

(i) the problem in fact has a solution,

(ii) this solution is unique, and

(iii) the solution depends continuously on the data given in the problem.

Condition (i) has just been checked, but we'll prove that the given PDE does not satisfy (iii). The given data is a function of n, where we have

$$D(n) = \left\{u = 0, \frac{\partial u}{\partial x_2} = \frac{1}{n}\sin(nx_1) \quad \text{for } x_2 = 0\right. .$$

As $n \to \infty$, the solution to the system (1) is $c_1 = c_2 = 0$, giving that the solution u_∞ depending on the Cauchy data as $n \to \infty$ should be $u_\infty \equiv 0$. However,

$$\begin{aligned}
\lim_{n\to\infty} u_n &= \lim_{n\to\infty} \frac{1}{n^2}\sin(nx_1)\sinh(nx_2) \\
&= \lim_{n\to\infty} \frac{1}{n^2}\sin(nx_1)\left(\sum_{k=0}^{\infty} \frac{(nx_2)^{2k+1}}{(2k+1)!}\right) \\
&= \lim_{n\to\infty} \sin(nx_1)\left(\sum_{k=0}^{\infty} \frac{n^{2k-1}x_2^{2k+1}}{(2k+1)!}\right)
\end{aligned}$$

and depending on x_1, x_2, this limit may not be zero. For example, take $x_1 = \frac{\pi}{2n}$ so that $\sin(nx_1) = 1$ and $x_2 = 1$, so that

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} \frac{n^{2k-1}}{(2k+1)!} = \infty.$$

I.e., $\lim_{n\to\infty} u_n(1,1) \neq 0$, even though the Cauchy data demands that $u_n \equiv 0$ as $n \to \infty$. This means that the solution to the given PDE does not depend continuously on the data, and so **the problem is not well-posed**. □

Exercise #4. If we look for a radial solution $u(x) = v(r)$ of the nonlinear elliptic equation

$$-\Delta u = u^p \qquad \text{in } \mathbb{R}^n,$$

where $r = |x|$ and $p > 1$, we are led to the nonautonomous ODE

$$v'' + \frac{n-1}{r}v' + v^p = 0. \tag{$*$}$$

Show that the *Emden-Fowler transformation*

$$t := \log r, \quad x(t) := e^{\frac{2t}{p-1}} v(e^t)$$

converts $(*)$ into an autonomous ODE for the new unknown $x = x(t)$.

Proof. With the Emden-Fowler transformation, if $x(t) = e^{\frac{2t}{p-1}} v(e^t)$, then

$$v(r) = r^{\frac{2}{1-p}} x(\log r)$$

and therefore, we compute (omitting intermediary computations, writing $t = \log r$ for brevity)

$$v' = r^{\frac{1+p}{1-p}}\left(x'(t) + \left(\frac{2}{1-p}\right)x(t)\right)$$

$$v'' = r^{\frac{2p}{1-p}}\left(x''(t) + \left(\frac{2}{1-p}\right)x'(t)\right) + \left(\frac{1+p}{1-p}\right)r^{\frac{2p}{1-p}}\left(x'(t) + \left(\frac{2}{1-p}\right)x(t)\right)$$

and so the nonautonomous ODE becomes

$$r^{\frac{2p}{1-p}}\left[\left(x''(t) + \left(\frac{2}{1-p}\right)x'(t)\right) + \left(\frac{1+p}{1-p}\right)\left(x'(t) + \left(\frac{2}{1-p}\right)x(t)\right)\right] + (n-1)r^{\frac{2p}{1-p}}\left(x'(t) + \left(\frac{2}{1-p}\right)x(t)\right) + r^{\frac{2p}{1-p}}x^p(t) = 0.$$

Now the $r^{(2p)/(1-p)}$ terms are eliminated, and we obtain the following second-order autonomous ODE after simplifying:

$$\boxed{x''(t) + x'(t)\left(\frac{2}{1-p} + \frac{1+p}{1-p} + n - 1\right) + x(t)\left(\frac{2}{1-p} + \frac{2(n-1)}{1-p}\right) + x^p(t) = 0}.$$

□

Exercise #6. Find a solution of
$$-\Delta u + u^{\frac{n+2}{n-2}} = 0 \qquad \text{in } B(0,1)$$
having the form $u = \alpha(1-|x|^2)^{-\beta}$ for positive constants α, β. This example shows that a solution of a nonlinear PDE can be finite within a region and yet approach infinity everywhere on its boundary.

Proof. We first start with some basic computations. If $u = \alpha(1-|x|^2)^{-\beta}$, then

$$\begin{aligned}
\frac{\partial u}{\partial x_i} &= 2\alpha\beta x_i(1-|x|^2)^{-\beta-1}\\
\frac{\partial^2 u}{\partial x_i^2} &= 4\alpha\beta(\beta+1)x_i^2(1-|x|^2)^{-\beta-2} + 2\alpha\beta(1-|x|^2)^{-\beta-1}\\
\Delta u &= 4\alpha\beta(\beta+1)|x|^2(1-|x|^2)^{-\beta-2} + 2\alpha\beta n(1-|x|^2)^{-\beta-1}
\end{aligned}$$

and thus we analyze the following equation:

$$-4\alpha\beta(\beta+1)|x|^2(1-|x|^2)^{-\beta-2} - 2\alpha\beta n(1-|x|^2)^{-\beta-1} + \alpha^{\frac{n+2}{n-2}}(1-|x|^2)^{-\beta\left(\frac{n+2}{n-2}\right)} = 0$$

letting $\omega = 1-|x|^2$, the above becomes

$$-4\alpha\beta(\beta+1)(1-\omega)\omega^{-\beta-2} - 2\alpha\beta n\omega^{-\beta-1} + \alpha^{\frac{n+2}{n-2}}\omega^{-\beta\left(\frac{n+2}{n-2}\right)} = 0$$

and multiplying both sides by $\omega^{\beta+2}$ lends

$$-4\alpha\beta(\beta+1)(1-\omega) - 2\alpha\beta n\omega + \alpha^{\frac{n+2}{n-2}}\omega^{-\beta\left(\frac{n+2}{n-2}\right)+\beta+2} = 0$$

we can also factor out an α to get

$$-4\beta(\beta+1)(1-\omega) - 2\beta n\omega + \alpha^{\frac{4}{n-2}}\omega^{-\beta\left(\frac{n+2}{n-2}\right)+\beta+2} = 0.$$

Then,

$$-4\beta(\beta+1) + \omega(4\beta(\beta+1) - 2\beta n) + \alpha^{\frac{4}{n-2}}\omega^{-\beta\left(\frac{n+2}{n-2}\right)+\beta+2} = 0.$$

In particular,

$$\omega(4\beta(\beta+1) - 2\beta n) + \alpha^{\frac{4}{n-2}}\omega^{-\beta\left(\frac{n+2}{n-2}\right)+\beta+2} = 4\beta(\beta+1).$$

From the above we see that the left hand side is constant in ω. Differentiating with respect to ω, we obtain that

$$4\beta(\beta+1) - 2\beta n + \alpha^{\frac{4}{n-2}}\left(-\beta\frac{n+2}{n-2} + \beta + 2\right)\omega^{-\beta\left(\frac{n+2}{n-2}\right)+\beta+1} = 0.$$

Then, we must have

$$-\beta\left(\frac{n+2}{n-2}\right) + \beta + 1 = 0, \qquad 4\beta(\beta+1) - 2\beta n + \alpha^{\frac{4}{n-2}}\left(-\beta\frac{n+2}{n-2} + \beta + 2\right) = 0.$$

Solving the first equation for β, we get that

$$\beta = \frac{-1}{1-\frac{n+2}{n-2}} = \frac{n-2}{4}.$$

In the second equation, we have

$$\alpha = \left[\frac{2\beta n - 4\beta(\beta+1)}{\beta + 2 - \beta\frac{n+2}{n-2}}\right]^{\frac{n-2}{4}} = [\frac{n^2}{4} - n + 1]^{\frac{n-2}{4}}.$$

The term on the right makes sense since the inside quadratic is positive for all $n \neq 2$. In summary,

$$\boxed{\beta = \frac{n-2}{4}, \qquad \alpha = \left[\frac{n^2}{4} - n + 1\right]^{\frac{n-2}{4}}}$$

□

Exercise #7. Consider the viscous conservation law

$$u_t + F(u)_x - au_{xx} = 0 \qquad \text{in } \mathbb{R} \times (0, \infty), \tag{$*$}$$

where $a > 0$ and F is uniformly convex.

(a) Show u solves ($*$) if $u(t) = v(x - \sigma t)$ and v is defined implicitly by the formula

$$s = \int_c^{v(s)} \frac{a}{F(z) - \sigma z + b} dz \qquad (s \in \mathbb{R})$$

where b and c are constants.

(b) Demonstrate that we can find a traveling wave satisfying

$$\lim_{s\to-\infty} v(s) = u_l, \qquad \lim_{s\to\infty} v(s) = u_r$$

for $u_l > u_r$ if and only if

$$\sigma = \frac{F(u_l) - F(u_r)}{u_l - u_r}.$$

(c) Let u^ε denote the above traveling wave solution of ($*$) for $a = \varepsilon$, with $u^\varepsilon(0,0) = \frac{u_l+u_r}{2}$. Compute $\lim_{\varepsilon\to0} u^\varepsilon$ and explain your answer.

Proof. (a) Let $s = x - \sigma t$. Then the given PDE $u_t + F(u)_x - au_{xx}$ becomes

$$-\sigma v' + F'(v)v' = av''$$

where $' = \frac{d}{ds}$. Integrating both sides of the above with respect to s, we obtain

$$av' = \int (-\sigma + F'(v))v' \, ds$$

and by substitution, the above equation becomes

$$av' = -\sigma v + F(v) + b$$

where b is some constant of integration. In particular,

$$a\frac{dv}{ds} = F(v) - \sigma v + b \implies \frac{a}{F(v) - \sigma v + b} dv = ds.$$

Integrating both sides, one obtains

$$\int_0^s \frac{a}{F(v(s)) - \sigma v(s) + b} dv = \int_0^s ds \implies s = \int_0^s \frac{a}{F(v(s)) - \sigma v(s) + b} dv$$

and making the change of variable $z = v$, one obtains

$$s = \int_{c:=v(0)}^{v(s)} \frac{a}{F(z) - \sigma z + b} dz$$

as desired.

(b) Suppose that v is a traveling wave such that $\lim_{s\to-\infty} v(s) = u_l$, $\lim_{t\to\infty} v(s) = u_r$ for $u_l < u_r$. By (a), we found that

$$av' = F(v) - \sigma v + b.$$

Now, since $\lim_{s\to\infty} v(s) = u_r$, for all s, t, there exists $a < r < b$ with

$$v'(r) = \frac{v(t) - v(s)}{s - t}$$

by the mean value theorem. Note that, as $s, t \to \infty$, $v(t) - v(s) \to 0$ since v as a limit at infinity. Hence, $\lim_{s\to\infty} v'(s) = 0$. The same result holds (and is proven identically) for the limit of v' at $-\infty$. Therefore, we have

$$\lim_{s\to\pm\infty} av' = \lim_{s\to\pm\infty} F(v(s)) - \sigma v(s) + b \implies 0 = F(u_{l,r}) - \sigma u_{l,r} + b.$$

That is,

$$F(u_l) - \sigma u_l + b = F(u_r) - \sigma u_r + b \implies F(u_l) - F(u_r) = \sigma(u_l - u_r) \implies \sigma = \frac{F(u_l) - F(u_r)}{u_l - u_r}$$

as we wanted.

Conversely, suppose $\sigma = \frac{F(u_l)-F(u_r)}{u_l-u_r}$ with $u_l > u_r$. And observe that we have the following identity

$$F(u_l) - \sigma u_l = F(u_r) - \sigma u_r$$

because

$$\begin{aligned} F(u_l) - \sigma u_l - (F(u_r) - \sigma u_r) &= F(u_l) - F(u_r) + \sigma(u_r - u_l) \\ &= F(u_l) - F(u_r) + \left(\frac{F(u_l) - F(u_r)}{u_l - u_r}\right)(u_r - u_l) \\ &= F(u_l) - F(u_r) - (F(u_l) - F(u_r)) \\ &= 0. \end{aligned}$$

So, we let $F(u_l) - \sigma u_l = F(u_r) - \sigma u_r = \mu$ for some $\mu \in \mathbb{R}$, so that $F(u_l) - \sigma u_l - \mu = F(u_r) - \sigma u_r - \mu = 0$. Therefore, if one considers the expression $G(v) = F(v) - \sigma v - \mu = 0$, we see that by the uniformity convexity of F (i.e., $F'' \geq \theta > 0$ for some $\theta \in \mathbb{R}$), u_r, u_l are the only roots of G, as a third root of G would require F to have an inflection point (i.e., a point where $F'' = 0$), contradicting its uniform convexity. Now, since F is uniformly convex and u_l, u_r are the only roots of G,

$$\begin{cases} G(z) < 0 & \text{for } z \in (u_r, u_l) \\ G(z) > 0 & \text{for } z \in (u_r, u_l)^c \end{cases}$$

Moreover, G is such that

$$G'(u_l) = F'(u_l) - \sigma > 0, \qquad G'(u_r) = F'(u_r) - \sigma < 0.$$

We summarize these points in the graphic below:

notes/hw2 graphic.JPG

Therefore, when we consider the function v implicitly defined by

$$s = \int_c^{v(s)} \frac{a}{F(z) - \sigma z - \mu} dz$$

where $a, u_r < c < u_l$ are any constants, one has that

$$\lim_{s\to\infty} s = \lim_{s\to\infty} \int_c^{v(s)} \frac{a}{F(z) - \sigma z - \mu} dz = \infty.$$

Since we know that u_l is a zero of the denominator (and the only one greater than c), it follows that $\lim_{s\to\infty} v(s) = u_l$. With the dual argument, we find that since u_r is the only zero of the denominator less than c, we obtain that $\lim_{s\to-\infty} v(s) = u_r$. I.e., $v(s)$ as defined satisfies our desired requirements, and hence we obtain the result.

(c) Let $c = v(0) = u^\varepsilon(0,0) = \frac{u_l+u_r}{2}$ and we have

$$s = \int_{\frac{u_l+u_r}{2}}^{v(s)} \frac{\varepsilon}{F(z) - \sigma z - \mu} = \int_{\frac{u_l+u_r}{2}}^{u^\varepsilon(s)} \frac{\varepsilon}{F(z) - \sigma z - \mu} dz$$

where μ is as in (b). Then,

$$\frac{s}{\varepsilon} = \int_{\frac{u_l+u_r}{2}}^{u^\varepsilon(s)} \frac{1}{F(z) - \sigma z + b} dz.$$

From the previous problem, as $\varepsilon \to 0$, $\int_{\frac{u_l+u_r}{2}}^{u^\varepsilon(s)} \frac{1}{F(z)-\sigma z+b} dz \to \mathrm{sgn}(s) \cdot \infty$. By our work in (b), this requires that $u^\varepsilon(s) \to u_r$ if $\mathrm{sgn}(s) = +\infty$ and $u^\varepsilon(s) \to u_l$ if $\mathrm{sgn}(s) = -1$. In particular, by our work in (b) we see that as $\varepsilon \to 0$, u^ε converges to a traveling wave solution of the given PDE, but with the lower bound c in the integral formula set to $c = \frac{u_l+u_r}{2}$.

□

Exercise #8. Prove that if u is the solution of problem (23) for Schrodinger's equation in §4.3 given by formula (20), then

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \le \frac{1}{(4\pi|t|^{n/2})}\|g\|_{L^1(\mathbb{R}^n)}$$

for each $t \neq 0$.

Proof. First, we recall that if u solves

$$\begin{cases} iu_t + \Delta u = 0 & \text{in } \mathbb{R}^n \times (-\infty,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$$

then

$$u(x,t) = \frac{1}{(4\pi i t)^{n/2}} \int_{\mathbb{R}^n} e^{i\frac{|x-y|}{4t}} g(y)\, d(y).$$

So, given $t \neq 0$, we have

$$\begin{aligned} \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} &= \text{ess sup}_x\, |u(x,t)| = \text{ess sup}_x \left| \frac{1}{(4\pi i t)^{n/2}} \int_{\mathbb{R}^n} e^{i\frac{|x-y|}{4t}} g(y)\, d(y) \right| \\ &\le \text{ess sup}_x \left(\frac{1}{|4\pi t|^{n/2}} \int_{\mathbb{R}^n} |e^{i\frac{|x-y|}{4t}}| \cdot |g(y)|\, d(y) \right) \\ &= \text{ess sup}_x \left(\frac{1}{|4\pi t|^{n/2}} \int_{\mathbb{R}^n} |g(y)|\, d(y) \right) \end{aligned}$$

Now, observe that since $4\pi > 1$, $(4\pi)^{n/2} \ge 4\pi$ for $n \ge 2$, and hence $|t|^{n/2}(4\pi)^{n/2} \ge 4\pi|t|^{n/2}$. This gives that

$$\text{ess sup}_x \left(\frac{1}{|4\pi t|^{n/2}} \int_{\mathbb{R}^n} |g(y)|\, d(y) \right) \le \frac{1}{4\pi|t|^{n/2}} \left\|g\right\|_{L^1(\mathbb{R}^n)}.$$

Hence, we obtain

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \le \frac{1}{4\pi|t|^{n/2}} \left\|g\right\|_{L^1(\mathbb{R}^n)}.$$

□

Additional Exercise. Find the Radon transform of

$$f(\mathbf{x}) = e^{-|\mathbf{x}|^2} = e^{-x_1^2 - x_2^2 - \cdots - x_n^2}.$$

Proof. From the handout you provided to me, the Radon transform of f is given by

$$\mathcal{R}f(s,\mathbf{v}) = \int_{\mathbb{R}^n} e^{-|\mathbf{x}|^2} \delta(s - \mathbf{v}\cdot\mathbf{x})\, dx_1 \ldots dx_n.$$

We shall prove that

$$\int_{\mathbb{R}^n} e^{-|\mathbf{x}|^2} \delta(s - \mathbf{v}\cdot\mathbf{x}) dx_1 dx_2 \ldots dx_n = (\sqrt{\pi})^{n-1} e^{-s^2}.$$

Since $\mathbf{v}$ is a unit vector, if $O(n) : \mathbb{R}^n \to \mathbb{R}^n$ denotes any orthogonal transformation, then we know

$$\mathbf{v}\cdot\mathbf{x} = O(\mathbf{v}) \cdot O(\mathbf{x}).$$

Hence, let $O : \mathbb{R}^n \to \mathbb{R}^n$ be an orthogonal transformation that takes $\mathbf{v}$ to $\mathbf{e}_n = \langle 0, \ldots, 1\rangle$, and O also preserves the norm $|\cdot|$. Therefore,

$$\int_{\mathbb{R}^n} e^{-|\mathbf{x}|^2} \delta(s - \mathbf{v}\cdot\mathbf{x}) d\mathbf{x} = \int_{\mathbb{R}^n} e^{-|O(\mathbf{x})|^2} \delta(s - \mathbf{e}_n \cdot O(\mathbf{x})) d\mathbf{x}.$$

A standard result from measure theory is that the Lebesgue measure is invariant under orthogonal transformations. Hence, $d\mathbf{x} = dO(\mathbf{x})$. To clean up notation, let $\mathbf{y} = O(\mathbf{x})$, and therefore the above turns into

$$\begin{aligned} \int_{\mathbb{R}^n} e^{-|O(\mathbf{x})|^2} \delta(s - \mathbf{e}_n \cdot O(\mathbf{x})) dO(\mathbf{x}) &= \int_{\mathbb{R}^n} e^{-|\mathbf{y}|^2} \delta(s - y_n)\, d\mathbf{y} \\ &= \int_{\mathbb{R}^n} e^{-y_1^2 - y_2^2 - \cdots - y_n^2} \delta(s - y_n)\, d\mathbf{y} \\ &= \int_{\mathbb{R}^{n-1}} e^{-y_1^2 - y_2^2 - \cdots - y_{n-1}^2 - s^2} dy_1 dy_2 \ldots y_{n-1} \\ &= e^{-s^2} \int_{\mathbb{R}^{n-1}} e^{-y_1^2 - \cdots - y_{n-1}^2} dy_1 \ldots dy_{n-1}. \end{aligned}$$

The remaining integrals amount to $n - 1$ Gaussian integrals, which are all equal to $\sqrt{\pi}$. Hence,

$$\int_{\mathbb{R}^n} e^{-|\mathbf{x}|^2} \delta(s - \mathbf{v} \cdot \mathbf{x}) \, d\mathbf{x} = e^{-s^2} (\sqrt{\pi})^{n-1}.$$

$\square$