Partial Differential Equations
Chapter 5 Homework

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Exercise #4. Assume $n = 1$ and $u \in W^{1,p}(0,1)$ for some $1 \leq p < \infty$.

(a) Show that u is equal a.e. to an absolutely continuous function and u' (which exists a.e.) belongs to $L^p(0,1)$.

(b) Prove that if $1 < p < \infty$, then

$$|u(x) - u(y)| \leq |x-y|^{1-\frac{1}{p}} \left(\int_0^1 |u'|^p \, dt \right)^{1/p}$$

for a.e. $x, y \in [0,1]$.

Proof.

(a) Given that $U = (0,1)$ is bounded, we can employ Theorem 3 [Evans, 266], letting $u_m \in C^\infty([0,1])$ be such that

$$u_m \to u \text{ in } W^{1,p}(U).$$

I.e., $u_m \to u$ a.e. in $(0,1)$. Now, for each u_m, we have that

$$u_m(x) = u_m(0) + \int_0^x u_m'(s) \, ds,$$

and letting $n \to \infty$ in the above gives that

$$u(x) =_{\text{a.e.}} u(0) + \int_0^x u'(s) \, ds$$

and since u' is L^1 (by definition of $W^{1,p}$), it follows that $u(x)$ is equal a.e. to the function $v(x) = u(0) + \int_0^x u'(s) \, ds$. The fact that u' is $L^p(0,1)$ follows from the fact that $u_m' \to u'$ and each $u_m' \in W^{1,p}(U)$, and is hence in L^p.

(b) Using the above sequence $\{u_m\}$ that converges to u, we have

$$\begin{aligned}
|u(x) - u(y)| &= \lim_{m\to\infty} |u_m(x) - u_m(y)| \\
&= \lim_{m\to\infty} \left| \int_x^y u_m'(s) \, ds \right| \\
&\leq \lim_{m\to\infty} \left(\int_x^y |u_m'|^p \, ds \right)^{1/p} \left(\int_x^y \, ds \right)^{1-1/p} \\
&\leq |x-y|^{1-1/p} \lim_{m\to\infty} \left(\int_0^1 |u_m'|^p \, ds \right)^{1/p} \\
&= |x-y|^{1-1/p} \left(\int_0^1 |u'|^p \, ds \right)^{1/p}.
\end{aligned}$$

□

Exercise #9. Integrate by parts to prove the interpolation inequality:

$$\|Du\|_{L^2} \le C\|u\|_{L^2}^{1/2}\left\|D^2u\right\|_{L^2}^{1/2}$$

for all $u \in C_c^\infty(U)$. Assume U is bounded, ∂U is smooth, and prove this inequality if $u \in H^2(U) \cap H_0^1(U)$.

Proof. If $u \in C_c^\infty(U)$, then we have

$$\begin{aligned}
\|Du\|_{L^2}^2 &= \int_U |Du|^2\,dx \\
&= \left|\sum_{i=1}^n \int_U u_{x_i}u_{x_i}\,dx\right| \\
&= \left|\sum_{i=1}^n \int_U uu_{x_ix_i}\,dx\right| \\
&\le \sum_{i=1}^n \int_U |u||u_{x_ix_i}|\,dx \\
&= \int_U |u||D^2u|\,dx \\
&\le \|u\|_{L^2}\left\|D^2u\right\|_{L^2}
\end{aligned}$$

and the desired inequality follows by taking square roots:

$$\|Du\|_{L^2} \le \|u\|_{L^2}^{1/2}\left\|D^2u\right\|_{L^2}^{1/2},$$

i.e., when $u \in C_c^\infty(U)$, the inequality above holds with $C = 1$.

Now, for $u \in H^2(U) \cap H_0^1(U)$, let $\{v_k\}_{k=1}^\infty \subset C_c^\infty(U)$ be a sequence converging to u in $H_0^1(U)$, $\{w_k\}_{k=1}^\infty \subset C^\infty(\bar{U})$ converging to u in $H^2(U)$. Then, we have

$$\begin{aligned}
\|Du\|_{L^2}^2 = \int_U |Du|^2\,dx &= \lim_{k\to\infty}\left|\int_U Dv_k \cdot Dw_k\,dx\right| \\
&= \lim_{k\to\infty}\left|\int_U v_k\Delta w_k + \int_{\partial U}\frac{\partial w_k}{\partial \nu}v_k dS\right|
\end{aligned}$$

Observe that, since each $v_k \in C_c^\infty(U)$, v_k vanishes on ∂U. Hence, we continue with the Cauchy-Schwarz inequality:

$$\|Du\|_{L^2}^2 = \lim_{k\to\infty}\left|\int_U v_k\Delta w_k\right| \le \lim_{k\to\infty}\left(\int_U v_k^2\,dx\right)^{1/2}\left(\int_U (\Delta w_k)^2\right)^{1/2} \le \lim_{k\to\infty}\left(\int_U v_k^2\,dx\right)^{1/2}\left(\int_U |Dw_k|^2\right)^{1/2} = \|u\|_{L_2}\left\|D^2u\right\|_{L^2}$$

as $|\Delta w_k|^2 < |Dw_k|^2$. Taking square roots of the above chain of inequalities yields the desired result. □

Exercise #10.

(a) Integrate by parts to prove

$$\|Du\|_{L^p} \le C\|u\|_{L^p}^{1/2}\left\|D^2u\right\|_{L^p}^{1/2}$$

for $2 \le p < \infty$ and all $u \in C_c^\infty(U)$.

(b) Prove

$$\|Du\|_{L^{2p}} \le C\|u\|_{L^\infty}^{1/2}\left\|D^2u\right\|_{L^p}^{1/2}$$

for $1 \le p < \infty$ and all $u \in C_c^\infty(U)$.

Proof.

(a) First, observe that

$$\int_U |Du|^p\,dx = \int_U |Du|^2|Du|^{p-2}\,dx = \sum_{i=1}^n \int_U u_{x_i}^2|Du|^{p-2}\,dx$$

Now, using multivariable integration by parts, and noting that $u \in C_c^\infty(U)$ so that any integral of u over ∂U vanishes, we have,

$$\begin{aligned}
\sum_{i=1}^n \int_U u_{x_i}^2 |Du|^{p-2}\,dx &= \sum_{i=1}^n \left[\int_U uu_{x_ix_i}|Du|^{p-2} + (p-2)Du\cdot Du_{x_i}|Du|^{p-4}\,dx\right]\\
&\le \sum_{i=1}^n \left[\int_U |u||u_{x_ix_i}||Du|^{p-2} + (p-2)|u||u_{x_i}||Du|^{p-4}|Du||Du_{x_i}|\,dx\right]\\
&\le \sum_{i=1}^n \left[\int_U |u||D^2u||Du|^{p-2} + (p-2)|u||Du|^{p-2}|D^2u|\right]\\
&= \sum_{i=1}^n \left[\int_U (p-2+1)\,|u||D^2u||Du|^{p-2}\right]\\
&= n(p-1)\int_U |u||Du|^{p-2}|D^2u|\,dx
\end{aligned}$$

By the Generalized Holder Inequality [Evans, 707], this above integral satisfies the following inequality:

$$n(p-1)\int_U |u||Du|^{p-2}|D^2u|\,dx \le n(p-1)\|u\|_{L^p}\left\|D^2u\right\|_{L^p}\left\||Du|^{p-2}\right\|_{L^{1-2/p}}.$$

Now, observe that

$$\left\||Du|^{p-2}\right\|_{L^{1-2/p}} = \left(\int_U (|Du|^{p-2})^{\frac{1}{1-2/p}}\right)^{1-2/p} = \left(\int_U |Du|^p\right)^{1-2/p}$$

and therefore as a result of all of the above inequalities, we land at

$$\int_U |Du|^p\,dx \le n(p-1)\|u\|_{L^p}\left\|D^2u\right\|_{L^p}\left(\int_U |Du|^p\right)^{1-2/p}$$

and dividing by the integral on the right gives

$$\left(\int_U |Du|^p\,dx\right)^{2/p} \le n(p-1)\|u\|_{L^p}\left\|D^2u\right\|_{L^p}.$$

Let $C = \sqrt{n(p-1)}$, whereby it follows from taking square roots of the above equation that

$$\|Du\|_{L^p} \le C\|u\|_{L^p}\left\|D^2u\right\|_{L^p}$$

with C depending only on n and p, as desired.

(b) Similarly, we have

$$\begin{aligned}
\int_U |Du|^{2p}\,dx &= \sum_{i=1}^n \int_U u_{x_i}^2(|Du|^2)^{p-1}\,dx\\
&= \sum_{i=1}^n \left[\int_U uu_{x_ix_i}(|Du|^2)^{p-1} + 2(p-1)uu_{x_i}(|Du|^2)^{p-2}Du\cdot Du_{x_i}\,dx\right]\\
&\le \sum_{i=1}^\infty \left[|u||u_{x_ix_i}|(|Du|^2)^{p-1} + (2p-2)|u|(|Du|^2)^{p-1}|D^2u|dx\right]\\
&= n(2p-1)\int_U |u||D^2u|(|Du|^2)^{p-1}\,dx\\
&\le n(2p-1)\left\|D^2u\right\|_{L^\infty}\int_U |u|(|Du|^2)^{p-1}\,dx \le n(2p-1)\left\|D^2u\right\|_{L^\infty}\|u\|_{L^p}\left\||Du|^{2(p-1)}\right\|_{L^{\frac{1}{1-1/p}}}
\end{aligned}$$

To sum up the results so far, we have deduced

$$\int_U |Du|^{2p}\,dx \le n(2p-1)\|u\|_{L^p}\left\|D^2u\right\|_{L^\infty}\left\||Du|^{2(p-1)}\right\|_{L^{\frac{1}{1-1/p}}} = n(2p-1)\|u\|_{L^p}\left\|D^2u\right\|_{L^\infty}\left(\int_U |Du|^{2p}\right)^{1-1/p}.$$

As before, we divide by the furthest right integral to obtain

$$\int_U |Du|^{2p}\, dx \le n(2p-1)\|u\|_{L^p} \left\|D^2 u\right\|_{L^\infty}.$$

With $C = \sqrt{n(2p-1)}$, if we take the square root of both sides, we get

$$\|Du\|_{L^{2p}} \le C\|u\|_{L^p}^{1/2} \left\|D^2 u\right\|_{L^\infty}^{1/2}$$

as desired. □

Exercise #11. Suppose U is connected and $u \in W^{1,p}(U)$ satisfies

$$Du = 0 \qquad \text{a.e. in } U.$$

Prove u is constant a.e. in U.

Proof. Let $u \in W^{1,p}(U)$ be given. By referring to Theorem 1 [Evans, 264], let $u^\varepsilon := \eta_\varepsilon * u$, where η_ε is defined as in [Evans, 713-714]. Now, u^ε is defined on

$$U_\varepsilon = \{x \in U : \operatorname{dist}(x, \partial U) > \varepsilon\}.$$

Then, by equation (1) in [Evans, 264], we know that

$$Du^\varepsilon = \eta_\varepsilon * Du = 0.$$

Since u^ε is smooth on U_ε (see [Evans, 714: Theorem 7(i)]), and $Du^\varepsilon = 0$, then by a corollary of the multivariable mean value theorem u^ε is constant on $V \subseteq U^\varepsilon$, where V is any open connected subset of U^ε (connectedness is required, see [Rudin *Principles of Mathematical Analysis* (PMCA), 239; Problem 9]).

To sum up the above discussion, each u^ε is locally constant on U^ε. By [Evans, 714; Theorem 7(ii)], since $u^\varepsilon \to u$ almost everywhere, u is also locally constant almost-everywhere on U. But U is connected, and thus u must be constant a.e. in U. Indeed, if one supposes for contradiction that u is not constant a.e. in U, let $x \in U$ be any point. Since u is locally constant, there is a neighborhood $V \ni x$ on which u is constant a.e.. Then, we can write the disjoint union

$$U = \{y : u(y) = u(x)\} \cup \{y : u(y) \neq u(x)\}$$

with the latter set in the union non-empty the fact we are supposing u to be non-constant. If $z \in \{y : u(y) = u(x)\}$, then $u(z) = u(x)$, and indeed there is a neighborhood V_z on which u is constant. But since $u(z) = u(x)$, then $u \equiv u(x)$ a.e. on this neighborhood V_z. Hence, $\{y : u(y) = u(x)\}$ is open, and an identical argument shows $\{y : u(y) \neq u(x)\}$ is open. Therefore, we have represented U as the disjoint union of open sets, which contradicts the connectedness of U. Thus, u must be constant almost-everywhere. □

Exercise #15. Fix $\alpha > 0$ and let $U = B^0(0,1)$. Show there exists a constant C, depending only on n and α, such that

$$\int_U u^2 dx \le C \int_U |Du|^2\, dx,$$

provided

$$|\{x \in U : u(x) = 0\}| \ge \alpha, \qquad u \in H^1(U).$$

Proof. Let $A = \{x \in U : u(x) = 0\}$. Recall that $(u)_U$ denotes the average of u over the set U. First, we have

$$\|u\|_{L^2} = \|u - (u)_U + (u)_U\|_{L^2} = \|u - (u)_U\|_{L^2} + \|(u)_U\|_{L^2}.$$

By virtue of the Poincare inequality [Evans, 290; Theorem 1], we then have

$$\|u - (u)_U\|_{L^2} + \|(u)_U\|_{L^2} \le C\|Du\|_{L^2} + \|(u)_U\|_{L^2},$$

where C is said to only depend on n and p (it would also depend on U, but in our case U is fixed). For $\|(u)_U\|_{L^2}$, observe that

$$\|(u)_U\|_{L^2} = \frac{1}{|U|^{1/2}} \sqrt{\int_U \left|\left(\int_U u\, dy\right)\right|^2 dy} \le \frac{1}{|U|^{1/2}} \sqrt{\int_{U-A} \left(\int_{U-A} u dy\right)^2 dy} = \frac{1}{|U|^{1/2}} \|u\|_{L^2(U-A)} |U - A|^{1/2}.$$

Now, $|U - A|$ is precisely $|U| - \alpha$, and so far we have that

$$\|(u)_U\|_{L^2} \le \sqrt{\frac{|U| - \alpha}{|U|}} \|u\|_{L^2}.$$

By virtue of this result, we have

$$\begin{aligned}\|u\|_{L^2} &= \|u-(u)_U\|_{L^2} + \|(u)_U\|_{L^2} \leq C\|Du\|_{L^2} + \|(u)_U\|_{L^2}\\ &\leq C\|Du\|_{L^2} + \sqrt{\frac{|U|-\alpha}{|U|}}\|u\|_{L^2}\end{aligned}$$

which gives

$$\left(1-\sqrt{\frac{|U|-\alpha}{|U|}}\right)\|u\|_{L^2} \leq C\|Du\|_{L^2}.$$

Dividing both sides by $\sqrt{\frac{|U|-\alpha}{|U|}}$ and letting C absorb this constant, we obtain the desired inequality

$$\|u\|_{L^2} \leq C\|Du\|_{L^2}$$

with C depending on p, n, and α. □

Exercise #20. Use the Fourier transform to prove that if $u \in H^s(\mathbb{R}^n)$ for $s > n/2$ then $u \in L^\infty(\mathbb{R}^n)$, with the bound

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leq C\|u\|_{H^s(\mathbb{R}^n)}$$

for a constant C depending only on s and n.

Proof. Let $u \in H^s(\mathbb{R}^n)$ for $s > n/2$. Since, by definition, $u \in L^2(\mathbb{R}^n)$, we consider $\|u\|_{L^2}$. Considering its Fourier transform $\hat{u}$, we have by Plancherel's Theorem,

$$\|u\|_{L^1(\mathbb{R}^n)} \leq \|u\|_{L^2(\mathbb{R}^n)} = \|\hat{u}\|_{L^2(\mathbb{R}^n)} = \left\|\frac{1+|y|^s}{1+|y|^s}\hat{u}\right\|_{L^2(\mathbb{R}^n)} \leq \left\|(1+|y|^s)\hat{u}\right\|_{L^2}\left\|\frac{1}{1+|y|^s}\right\|_{L^2} = \left\|\frac{1}{1+|y|^s}\right\|_{L^2}\|u\|_{H^s}.$$

Now, with regards to the norm of the first term,

$$\begin{aligned}\left\|\frac{1}{1+|y|^s}\right\|_{L^2}^2 &= \int_{\mathbb{R}^n}\frac{1}{(1+|y|^s)^2}\,dy = \int_0^\infty\int_{\partial B(0,r)}\frac{1}{(1+|y|^s)^2}\,dS\,dr = \int_0^\infty\frac{n\alpha(n)r^{n-1}}{(1+r^s)^2}\,dr\\ &= n\alpha(n)\int_0^\infty\frac{r^{n-1}}{r^{2s}+r^s+1}\,ds\end{aligned}$$

Since $s > n/2$, the degree of the denominator is greater than the degree of the numerator, and hence the above improper integral converges. Let $C = \left\|(1+|y|^s)^{-1}\right\|_{L^2}$, and thus so far we have

$$\|u\|_{L^1(\mathbb{R}^n)} \leq C\|u\|_{H^s}.$$

By the Fourier inversion formula on [Evans, 189], we have

$$|u(x)| = \left|\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}e^{ix\cdot y}\hat{u}(y)\,dy\right| \leq \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}|e^{ix\cdot y}||\hat{u(y)}|\,dy = \frac{1}{(2\pi)^{n/2}}\|\hat{u}\|_{L^1} \leq \frac{C}{(2\pi)^{n/2}}\|u\|_{H^s}.$$

Letting C absorb the extra constant term and passing the above inequality to the essential supremum lends the final result

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leq C\|u\|_{H^s}.$$

□