Partial Differential Equations
Chapter 2 Homework

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As Evans states on the exercises page for chapter 2, all functions are assumed to be smooth (C^∞) unless otherwise stated. All sources referenced in this document are cited on the last page.

Exercise #1. Write down an explicit formula for a function u solving the initial value problem

$$\begin{cases} u_t + b \cdot Du + cu = 0 & \text{in } \mathbf{R}^n \times (0, \infty) \\ u = g & \text{on } \mathbf{R}^n \times \{t = 0\} \end{cases} \tag{1}$$

Here $c \in \mathbf{R}$ and $b \in \mathbf{R}^n$ are constants.

Solution. We start by following the procedures in [Evans, 19]. Let $(x, t) \in \mathbf{R}^{n+1}$ and set

$$z(s) := u(x + sb, t + s) \qquad (s \in \mathbf{R}).$$

Since $u_t + b \cdot Du = -cu$ by (1),

$$\dot{z}(s) = Du(x + sb, t + s) \cdot b + u_t(x + sb, t + s) = -cu(x + sb, t + s) = -cz(s).$$

Since $\dot{z} + cz = 0$, we get that $z(s) = \alpha e^{-cs}$ as the solution to this classical ODE, where $\alpha \in \mathbf{R}$ is some constant. To solve for α, note that $z(-t) = u(x - tb, 0) = g(x - tb)$. Hence,

$$z(-t) = g(x - tb) = \alpha e^{ct} \implies \alpha = e^{-ct} \cdot g(x - tb).$$

As $z(0) = u(x, t) = \alpha$, it follows that

$$\boxed{u(x, t) = e^{-ct} \cdot g(x - tb)}$$

is the solution to (1). $\square$

Exercise #3. Modify the proof of the mean value formulas to show for $n \geq 3$ that

$$u(0) = \fint_{\partial B(0,r)} g \, dS + \frac{1}{n(n-2)\alpha(n)} \int_{B(0,r)} \left(\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}} \right) f \, dx$$

provided

$$\begin{cases} -\Delta u = f & \text{in } B^0(0,r) \\ u = g & \text{on } \partial B(0,r). \end{cases}$$

Proof. Similar to the process in [Evans, 25; Theorem 2], define

$$\phi(t) := \fint_{\partial B(0,t)} u \, dS + \frac{1}{n(n-2)\alpha(n)} \int_{B(0,t)} \left(\frac{1}{|x|^{n-2}} - \frac{1}{t^{n-2}} \right) f \, dx.$$

We wish to show that $\phi'(t) = 0$. However, to differentiate ϕ, and specifcally the right-most term, we will need to recall the multi-dimensional Leibniz rule. The following is a result from [Flanders, 624] for the general Leibniz rule:

$$\frac{d}{dt} \int_{\Omega(t)} \omega = \int_{\Omega(t)} i_{\mathbf{v}}(d_x \omega) + \int_{\partial\Omega(t)} i_{\mathbf{v}} \omega + \int_{\Omega(t)} \dot{\omega}. \tag{2}$$

Here,

- $\Omega(t)$ is a time-varying domain (e.g., $\Omega(t) = B(0,t)$),
- ω is a p-form,
- $\mathbf{v} = \frac{\partial \mathbf{x}}{\partial t}$ is a vector field of the velocity of the domain,
- i_v denotes the **interior product** with v,
- $d_x \omega$ denotes the **exterior derivative** of ω with respect to the space variables, and
- $\dot{\omega}$ is the time derivative of ω.

Then, let $\Omega(t) := B(0,t)$ and $\omega = \left(\frac{1}{|x|^{n-2}} - \frac{1}{t^{n-2}} \right) f(x) \, dx$. It can be seen that any point in $\Omega(t)$ changes as t increases according to the vector field

$$\mathbf{v}(x) = \frac{x}{|x|} = \frac{\langle x^1, x^2, \ldots, x^n \rangle}{\sqrt{(x^1)^2 + (x^2)^2 + \cdots + (x^n)^2}}.$$

We now compute $i_{\mathbf{v}}(d_x \omega), i_{\mathbf{v}} \omega$, and $\dot{\omega}$. First of all,

$$i_{\mathbf{v}}(d_x \omega) = 0 \tag{3}$$

because ω is an n-form on $\mathbb{R}^n$, and since the exterior derivative increases the degree of a differential form, $d_x \omega$ will be an $n+1$ form. However, the only $n+1$ form on $\mathbb{R}^n$ is 0, and so we are left with $i_{\mathbf{v}}(0) = 0$. Next,v

$$i_{\mathbf{v}} \omega = f(x) \mathbf{v} \cdot d\mathbf{\Sigma} \tag{4}$$

$$\begin{aligned} \dot{\omega} &= \frac{\partial}{\partial t} \left(\frac{1}{|x|^{n-2}} - \frac{1}{t^{n-2}} \right) f(x) \, dx \\ &= (n-2) t^{1-n} f(x) \, dx \end{aligned}$$

Then, (2) becomes

$$\frac{d}{dt} \int_{\Omega(t)} \omega = \int_{\partial\Omega(t)} f(x) \mathbf{v} \cdot d\mathbf{\Sigma} + (n-2) t^{1-n} \int_{\Omega(t)} f(x) \, dx$$

and

$$\begin{aligned} \int_{\partial\Omega(t)} f(x) \mathbf{v} \cdot d\mathbf{\Sigma} &= \int_{\partial\Omega(t)} f(x) \mathbf{v} \cdot \mathbf{n} \, dS(x) \\ &= \int_{\partial\Omega(t)} f(x) |v|^2 \, dS(x) \\ &= \int_{\partial\Omega(t)} f(x) \, dS(x) \end{aligned}$$

finally yielding

$$\frac{d}{dt}\int_{\Omega(t)} \omega = \int_{\partial\Omega(t)} f(x)\ dS(x) + (n-2)t^{1-n}\int_{\Omega(t)} f(x)\ dx$$

and then

$$\begin{aligned}\frac{1}{n(n-2)\alpha(n)}\left(\frac{d}{dt}\int_{\Omega(t)}\omega\right) &= \frac{1}{n(n-2)\alpha(n)}\int_{\partial\Omega(t)} f(x)\ dS(x) + \frac{(n-2)t^{1-n}}{n(n-2)\alpha(n)}\int_{\Omega(t)} f(x)\ dx \\ &= \frac{t^{n-1}}{n-2}⨍_{\partial\Omega(t)} f\ dS(x) + \frac{t}{n}⨍_{\Omega(t)} f(x)\ dx\end{aligned}$$

So far, we have

$$\phi'(t) = \frac{d}{dt}⨍_{\partial B(0,t)} u\ dS + \frac{t^{n-1}}{n-2}⨍_{\partial B(0,t)} f\ dS(x) + \frac{t}{n}⨍_{B(0,t)} f\ dx.$$

By the same process of differentiating on [Evans, 26], differentiating the leftmost term ultimately gives

$$\frac{d}{dt}⨍_{\partial B(0,t)} u\ dS = \frac{t}{n}⨍_{B(0,t)} \Delta u\ dx = -\frac{t}{n}⨍_{B(0,t)} f\ dx$$

which, when substituted back into the expression for ϕ' above, cancels with the other term to give

$$\phi'(t) = \frac{t^{n-1}}{n-2}⨍_{\partial B(0,t)} f\ dS(x).$$

Now, given that the following integrand showing up in ϕ:

$$\int_{B(0,t)}\left(\frac{1}{|x|^{n-2}} - \frac{1}{t^{n-2}}\right) f\ dx$$

is zero on $\partial B(0,t)$, the values of f on $\partial B(0,t)$ don't matter to the representation of u, and hence we may safely let $f \equiv 0$ on $\partial B(0,t)$, giving that $\phi'(t) = 0$ whence ϕ is constant. Next, we observe that

$$u(0) = \lim_{t\downarrow 0}\phi(t)$$

since $\lim_{t\downarrow 0}⨍_{\partial B(0,t)} u\ dS = u(0)$ and $\lim_{t\downarrow 0}\frac{1}{n(n-2)\alpha(n)}\int_{B(0,t)}\left(\frac{1}{|x|^{n-2}} - \frac{1}{t^{n-2}}\right) f\ dx \le \lim_{t\downarrow 0}\mu(B(0,t))||I||_{L^1} = 0$ where I is the given integrable integrand. Hence, $\phi \equiv u(0)$ and therefore,

$$u(0) = \phi(r) = \frac{1}{n\alpha(n)r^{n-1}}\int_{\partial B(0,r)} u\ dS(x) + \frac{1}{n(n-2)\alpha(n)}\int_{B(0,r)}\left(\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}}\right) f\ dx$$

and the result follows by applying the appropriate boundary conditions. □

Exercise #5. We say $v \in C^2(\overline{U})$ is *subharmonic* if

$$-\Delta v \leq 0 \qquad \text{in } U.$$

(a) Prove for subharmonic v that

$$v(x) \leq \fint_{B(x,r)} v \, dy \qquad \text{for all } B(x,r) \subset U.$$

(b) Prove that therefore $\max_{\overline{U}} v = \max_{\partial U} v$.

(c) Let $\phi : \mathbf{R} \to \mathbf{R}$ be smooth and convex. Assume u is harmonic and $v := \phi(u)$. Prove v is subharmonic.

(d) Prove $v := |Du|^2$ is subharmonic, whenever u is harmonic.

Proof.

(a) In the manner of [Evans, 26], set

$$\phi(r) = \fint_{\partial B(x,r)} u(y) \, dS(y) = \fint_{\partial B(0,1)} u(x + rz) \, dS(z)$$

whereby we have

$$\begin{aligned}
\phi'(r) &= \fint_{\partial B(0,1)} Du(x + rz) \cdot z \, dS(z) = \fint_{\partial B(x,r)} Du(y) \cdot \frac{y - x}{r} \, dS(y) \\
&= \fint_{\partial B(x,r)} \frac{\partial u}{\partial \nu} \, dS(y) \\
&= \frac{r}{n} \int_{B(x,r)} \Delta u(y) dy \geq 0 \qquad \text{(since } u \text{ is subharmonic).}
\end{aligned}$$

That is, $\phi' \geq 0$ and so ϕ is nondecreasing for all $r \geq 0$. Hence, for any $x \in U$,

$$v(x) = \lim_{r \downarrow 0} \phi(r) \leq \phi(r) \implies v(x) \leq \fint_{\partial B(x,r)} u(y) \, dS(y). \tag{5}$$

Then, (5) yields the folowing chain of implications:

$$\begin{aligned}
v(x) \leq \fint_{\partial B(x,r)} u(y) \, dS(y) &\implies n\alpha(n) r^{n-1} v(x) \leq \int_{\partial B(x,r)} u(y) \, dS(y) \\
&\implies \int_0^r n\alpha(n) s^{n-1} v(x) \, ds \leq \int_0^r \int_{\partial B(x,s)} u(y) \, dS(y) \, ds \\
&\implies n\alpha(n) \frac{s^n}{n} v(x) \leq \int_{B(x,r)} u(y) \, dy \\
&\implies \alpha(n) s^n v(x) \leq \int_{B(x,r)} u(y) \, dy \\
&\implies v(x) \leq \frac{1}{n\alpha(n)} \int_{B(x,r)} u(y) \, dy
\end{aligned}$$

and the result follows.

(b) We'll first suppose U is connected, and that there exists a point $x_0 \in U$ with $v(x_0) = M := \max_{\overline{U}} v$. The goal is to show that $E = \{x \in U : v(x) = M\}$ is clopen in U, so that $E = U$. Then, for $0 < r < \operatorname{dist}(x_0, \partial U)$, one has

$$M = v(x_0) \leq \fint_{B(x_0,r)} v \, dy \leq M$$

and hence $v \equiv M$ on $B(x_0, r)$. Since x_0 is really just an arbitrary point in E, this proves that E is open. Since $E = v^{-1}\{M\}$, a closed set, and v is C^2, we also have that E is closed. Hence, $E = U$ by the connectivity of U. Since v is constant in U, extending v by continuity gives that v is constant on $\overline{U}$, giving that

$$\max_{\overline{U}} v = \max_{\partial U} v.$$

If U is just any open set, recalling some point-set topology we get

$$U = \bigsqcup_{n \in \mathbb{N}} U_n$$

where each U_n is a connected open subset of U. Then,

$$\begin{aligned}\max_{\overline{U}} v &= \max_{n\in\mathbb{N}}(\max_{\overline{U_n}} v)\\ &= \max_{n\in\mathbb{N}}(\max_{\partial U_n} v)\\ &= \max_{\partial U} v.\end{aligned}$$

(c) We have,

$$\begin{aligned}\Delta\phi(u(x)) &= \sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}\phi(u(x))\\ &= \sum_{i=1}^{n}\frac{\partial}{\partial x_i}[\phi'(u(x))u_{x_i}(x)]\\ &= \sum_{i=1}^{n}[\phi'(u(x))u_{x_ix_i}(x)+\phi''(u(x))u_{x_i}^2(x)]\\ &= \phi'(u(x))\Delta u+\phi''(u(x))|Du|^2\\ &= \phi''(u(x))|Du|^2 \geq 0\end{aligned}$$

as ϕ is convex, its second derivative is nonnegative. Hence, $\Delta\phi(u(x)) \geq 0$ giving that $v = \phi(u)$ is subharmonic.

(d) Observe that the function $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}$ where $\phi(x) = |x|^2$ is itself a convex function. So, it suffices to demonstrate that Du is harmonic and then we can just apply the results from (c). One has,

$$\begin{aligned}\Delta(Du) &= \Delta\langle u_{x_1},\ldots,u_{x_n}\rangle\\ &= \nabla\left(\nabla\cdot\langle u_{x_1},\ldots,u_{x_n}\rangle\right)\\ &= \nabla(\Delta u)\\ &= \nabla(0) = 0.\end{aligned}$$

Since Du is harmonic, and ϕ is convex, it follows that $\phi(Du) = |Du|^2$ is subharmonic.

□

Exercise #6. Prove that there exists a constant C, depending only on n, such that

$$\max_{B(0,1)}|u| \leq C\left(\max_{\partial B(0,1)}|g| + \max_{B(0,1)}|f|\right)$$

whenever u is a smooth solution of

$$\begin{cases}-\Delta u = f & \text{in } B^0(0,1)\\ u = g & \text{on } \partial B(0,1).\end{cases}$$

Proof. Let $\lambda := \max_{B(0,1)}|f|$ and observe that

$$\begin{aligned}-\Delta(u+\frac{|x|^2}{2n}\lambda) &= -\Delta u - \frac{\lambda}{2n}\Delta|x|^2\\ &= -\frac{\lambda}{2n}\left(\sum_{i=1}^{n}2\right)\\ &= -\lambda \leq 0.\end{aligned}$$

That is, the function $v = u + \frac{|x|^2}{2n}\lambda$ is subharmonic. By part (b) of Exercise #5,

$$\begin{aligned}\max_{B(0,1)} v = \max_{\partial B(0,1)} v &= \max_{\partial B(0,1)}\left(u+\frac{|x|^2}{2n}\lambda\right)\\ &\leq \max_{\partial B(0,1)}|u| + \max_{\partial B(0,1)}\frac{|x|^2}{2n}\lambda\\ &= \max_{\partial B(0,1)}|g| + \frac{1}{2n}\max_{B(0,1)}|f|\end{aligned}$$

Then, let $C \geq \max\{1, \frac{1}{2n}\}$ whereby we have

$$\max_{B(0,1)} |u| \leq \max_{B(0,1)} |v| \leq C(\max_{\partial B(0,1)} |g| + \max_{B(0,1)} |f|)$$

giving the desired result. □

Exercise #7. Use Poisson's formula for the ball to prove

$$r^{n-2}\frac{r-|x|}{(r+|x|)^{n-1}}u(0) \leq u(x) \leq r^{n-2}\frac{r+|x|}{(r-|x|)^{n-1}}u(0)$$

whenever u is positive and harmonic in $B^0(0,r)$. This is an explicit form of Harnack's inequality.

Proof. Since u satisfies the boundary-value problem

$$\begin{cases} \Delta u = 0 & \text{in } B^0(0,r) \\ u = g & \text{on } \partial B(0,r) \end{cases}$$

then u is represented in *Poisson's formula*:

$$\begin{aligned} u(x) &= \frac{r^2-|x|^2}{n\alpha(n)r}\int_{\partial B(0,r)} \frac{u(y)}{|x-y|^n}\,dS(y) \qquad (x \in B^0(0,r)) \\ &= \frac{r^2-|x|^2}{n\alpha(n)r}(n\alpha(n)r^{n-1})⨍_{\partial B(0,r)} \frac{u(y)}{|x-y|^n}\,dS(y) \\ &= r^{n-2}(r^2-|x|^2)⨍_{\partial B(0,r)} \frac{u(y)}{|x-y|^n}\,dS(y). \end{aligned}$$

Using the fact that $||x|-|y|| \leq |x-y| \leq |x|+|y|$, we have

$$⨍_{\partial B(0,r)} \frac{u(y)}{(|x|+|y|)^n}\,dS(y) \leq ⨍_{\partial B(0,r)} \frac{u(y)}{|x-y|^n}\,dS(y) \leq ⨍_{\partial B(0,r)} \frac{u(y)}{(||x|-|y||)^n}\,dS(y)$$

and since the integration is over $\partial B(0,r)$, replacing $|y|=r$ gives

$$\begin{aligned} ⨍_{\partial B(0,r)} \frac{u(y)}{(|x|+r)^n}\,dS(y) \leq \quad & ⨍_{\partial B(0,r)} \frac{u(y)}{|x-y|^n}\,dS(y) \quad & \leq ⨍_{\partial B(0,r)} \frac{u(y)}{(r-|x|)^n}\,dS(y) \\ \frac{1}{(|x|+r)^n}⨍_{\partial B(0,r)} u(y)\,dS(y) \leq \quad & ⨍_{\partial B(0,r)} \frac{u(y)}{|x-y|^n}\,dS(y) \quad & \leq \frac{1}{(r-|x|)^n}⨍_{\partial B(0,r)} u(y)\,dS(y) \\ \frac{r^{n-2}(r^2-|x|^2)}{(|x|+r)^n}⨍_{\partial B(0,r)} u(y)\,dS(y) \leq \quad & r^{n-2}(r^2-|x|^2)⨍_{\partial B(0,r)} \frac{u(y)}{|x-y|^n}\,dS(y) \quad & \leq \frac{r^{n-2}(r^2-|x|^2)}{(r-|x|)^n}⨍_{\partial B(0,r)} u(y)\,dS(y). \end{aligned}$$

So far, we have shown,

$$\frac{r^{n-2}(r^2-|x|^2)}{(|x|+r)^n}⨍_{\partial B(0,r)} u(y)\,dS(y) \leq u(x) \leq \frac{r^{n-2}(r^2-|x|^2)}{(r-|x|)^n}⨍_{\partial B(0,r)} u(y)\,dS(y) \tag{6}$$

and since, by the results of Exercise #3,

$$⨍_{\partial B(0,r)} u(y)\,dS(y) = u(0)$$

Eqn. 7 becomes

$$\frac{r^{n-2}(r^2-|x|^2)}{(|x|+r)^n}u(0) \leq u(x) \leq \frac{r^{n-2}(r^2-|x|^2)}{(r-|x|)^n}u(0)$$

and since $r^2-|x|^2 = (r+|x|)(r-|x|)$, we obtain the final (and desired) inequality

$$\frac{r^{n-2}(r-|x|)}{(r+|x|)^{n-1}}u(0) \leq u(x) \leq \frac{r^{n-2}(r+|x|)}{(r-|x|)^{n-1}}u(0).$$

□

Exercise #8. Assume $g \in C(\partial B(0,r))$ and define u by

$$u(x) = \frac{r^2 - |x|^2}{n\alpha(n)r} \int_{\partial B(0,r)} \frac{g(y)}{|x-y|^n}\, dS(y) \qquad (x \in B^0(0,r)).$$

Then,

(i) $u \in C^\infty(B^0(0,r))$,

(ii) $\Delta u = 0$ in $B^0(0,r)$, and

(iii) $\lim_{\substack{x \to x^0 \\ x \in B^0(0,r)}} u(x) = g(x^0)$ for each point $x^0 \in \partial B(0,r)$.

Proof. **Warning!** This is a very long proof, and many prerequisite results have to be established (this makes up most of the length of the proof) first that were not given in the text in order to minimize handwaving.

Before embarking on the main proof, we will first need to compute Green's function for the ball $U = B^0(0,r)$. For any $x \in \mathbb{R}^n - \{0\}$ define

$$\tilde{x} = \frac{r^2 x}{|x|^2}$$

as the point **dual** to x with respect to $\partial B(0,r)$.

Fix $x \in B^0(0,1)$. Following the procedure in [Evans, 39], we must find a corrector function $\phi^x = \phi^x(y)$ solving

$$\begin{cases} \Delta \phi^x = 0 & \text{in } B^0(0,r) \\ \phi^x = \Phi(y-x) & \text{on } \partial B(0,r) \end{cases} \tag{7}$$

whereby the Green's function will be

$$G(x,y) = \Phi(y-x) - \phi^x(y), \tag{8}$$

where Φ is the fundamental solution of Laplace's equation. Assume for the moment that $n \geq 3$. The mapping $y \mapsto \Phi(y - \tilde{x})$ is harmonic for $y \neq \tilde{x}$ by the discussion at the bottom of [Evans, 22]. So $y \mapsto |x|^{2-n}\Phi\left(\frac{y-\tilde{x}}{r}\right)$ is also harmonic for $y \neq \tilde{x}$, and so one defines

$$\phi^x(y) := \Phi\left(|x|\left(\frac{y-\tilde{x}}{r}\right)\right) = |x|^{2-n}\Phi\left(\frac{y-\tilde{x}}{r}\right)$$

harmonic in U. Furthermore, if $y \in \partial B(0,r)$ and $x \neq 0$,

$$\begin{aligned} |x|^2|y-\tilde{x}|^2 &= |x|^2\left(|y|^2 - 2y\cdot\tilde{x} + |\tilde{x}|^2\right) \\ &= |x|^2\left(|y|^2 - 2y\cdot\frac{r^2 x}{|x|^2} + \frac{r^4}{|x|^2}\cdot\right) \\ &= r^2|x|^2 - 2r^2 y\cdot x + r^4 \\ &= r^2|x-y|^2 \end{aligned}$$

giving that $(|x||y-\tilde{x}|)^{2-n} = r^{2-n}|x-y|^{2-n}$. Consequently, for $y \in \partial B(0,r)$,

$$\begin{aligned} \phi^x(y) &= |x|^{2-n}\Phi\left(\frac{y-\tilde{x}}{r}\right) \\ &= |x|^{2-n}\left(\frac{1}{n(n-2)\alpha(n)}\left|\frac{y-\tilde{x}}{r}\right|^{2-n}\right) \\ &= \left(\frac{1}{n(n-2)\alpha(n)}\frac{r^{2-n}}{r^{2-n}}|x-y|^{2-n}\right) \\ &= \Phi(y-x). \end{aligned}$$

Therefore, the Green's function (boxed for my notes) for $B(0,r)$ is

$$\boxed{G(x,y) := \Phi(y-x) - \Phi\left(|x|\frac{y-\tilde{x}}{r}\right)}$$

with $x, y \in B(0,r)$, $x \neq y$. For any fixed x, the mapping $y \mapsto G(x,y)$ is harmonic, except for when $y = x$. Since $G(x,y) = G(y,x)$ by Theorem 13, $x \mapsto G(x,y)$ is harmonic for $x \neq y$. Thus, **in considering the map** $x \mapsto -\frac{\partial G}{\partial y_i}(x,y)$ for $1 \leq i \leq n$, which is

$$\begin{aligned}
\frac{\partial G}{\partial y_i}(x,y) &= \frac{\partial}{\partial y_i}\left(\Phi(y-x) - \Phi\left(|x|\frac{y-\tilde{x}}{r}\right)\right) \\
&= \frac{\partial}{\partial y_i}\Phi(y-x) - \frac{\partial}{\partial y_i}\Phi\left(|x|\frac{y-\tilde{x}}{r}\right) \\
&= -\frac{1}{n\alpha(n)}\frac{y_i - x_i}{|y-x|^n} + \frac{1}{n\alpha(n)}\frac{|x|^{2-n}}{r^{2-n}}\frac{y_i - \tilde{x}_i}{|y-\tilde{x}|^n} \\
&= -\frac{1}{n\alpha(n)}\left[\frac{y_i - x_i}{|y-x|^n} - \frac{|x|^{2-n}}{r^{2-n}}\frac{y_i - \tilde{x}_i}{|y-\tilde{x}|^n}\right],
\end{aligned}$$

Now, if $y \in \partial B(0,r)$, using the fact that $|x||y-\tilde{x}| = r|x-y|$ (see above), we obtain

$$\begin{aligned}
\frac{\partial G}{\partial y_i}(x,y) &= -\frac{1}{n\alpha(n)}\left[\frac{y_i - x_i}{|y-x|^n} - \frac{|x|^{2-n}}{r^{2-n}}\frac{y_i - \tilde{x}_i}{\frac{r^n}{|x|^n}|y-x|^n}\right] \\
&= -\frac{1}{n\alpha(n)}\frac{y_i - x_i - \frac{|x|^2}{r^2}(y_i - \tilde{x}_i)}{|y-x|^n} \\
&= -\frac{1}{n\alpha(n)}\frac{y_i - x_i - \frac{|x|^2 y_i}{r^2} + x_i}{|y-x|^n} \\
&= -\frac{1}{n\alpha(n)}\frac{y_i\left(1 - \frac{|x|^2}{r^2}\right)}{|y-x|^n}
\end{aligned}$$

Then, if $\mathbf{n} = \frac{y}{|y|}$ is the vector field normal to $\partial B(0,r)$, for $y \in \partial B(0,r)$, one has

$$\begin{aligned}
\frac{\partial G}{\partial \mathbf{n}} = D_y G \cdot \mathbf{n} &= \sum_{i=1}^{n}\left(-\frac{1}{n\alpha(n)}\frac{y_i\left(1 - \frac{|x|^2}{r^2}\right)}{|y-x|^n}\frac{y_i}{|y|}\right) \\
&= -\frac{1}{n\alpha(n)}\frac{\left(1 - \frac{|x|^2}{r^2}\right)}{|y-x|^n}\sum_{i=1}^{n}\frac{y_i^2}{|y|} \\
&= -\frac{1}{n\alpha(n)}\frac{\left(\frac{r^2 - |x|^2}{r^2}\right)}{|y-x|^n}r \\
&= -\frac{1}{n\alpha(n)r}\frac{r^2 - |x|^2}{|y-x|^n} \\
&= -\frac{1}{n\alpha(n)r}\frac{r^2 - |x|^2}{|y-x|^n}
\end{aligned}$$

and multiplying the above by the (-1) omitted from computations earlier, we obtain the fact that the kernel of the integrand $x \mapsto -\frac{\partial G}{\partial \mathbf{n}} = K(x,y)$, where

$$K(x,y) = \frac{r^2 - |x|^2}{n\alpha(n)r}\frac{1}{|x-y|^n}$$

is harmonic for $x \in B(0,r)$, $y \in \partial B(0,r)$, and hence smooth. Having shown that the kernel is a smooth function, it will be straightforward to demonstrate the smoothness of u. The first step is to evaluate the integral of $K(x,y)$. Observe that if we consider the function

$$K_y(x) = \int_{\partial B(0,r)} K(x,y)\, dS(y)$$

for y fixed, one has

$$\begin{aligned}
K_y(0) &= \int_{\partial B(0,r)} \left(\frac{r}{n\alpha(n)} \frac{1}{|y|^n}\right) \, dS(y) \\
&= \int_{\partial B(0,r)} \left(\frac{r}{n\alpha(n) r^n}\right) \, dS(y) \\
&= \frac{1}{n\alpha(n) r^{n-1}} \int_{\partial B(0,r)} \, dS(y) \\
&= ⨍_{\partial B(0,r)} \, dS(y) = 1.
\end{aligned}$$

We shall use the maximum principle to show that $K_y(x)$ is constant and equal to 1 for all $x \in B^0(0,r)$. Observe that,

$$\max_{x \in B^0(0,r)} K_y(x) = \max_{x \in B^0(0,s)} \frac{r^2 - |x|^2}{n\alpha(n) r} \int_{\partial B(0,r)} \frac{1}{|x-y|^n} \, dS(y)$$

Maximizing the term $\frac{r^2-|x|^2}{n\alpha(n)r}$ requires $x = 0$. Maximizing the integrand $\frac{1}{|x-y|^n}$ requires $|x-y|^n$ to by minimized for all $y \in B(0,r)$, which also requires $x = 0$. I.e., our maximum is indeed $K_y(0) = \max_{x \in B(0,r)} K_y(x) = 1$. To see that $K_y(x)$ is harmonic, we use the discussion on [Evans, 38] to see that the harmonicity of $K(x,y)$ yields

$$\Delta K_y(x) = \int_{\partial B(0,r)} \Delta_x K(x,y) \, dS(y) = 0$$

and by the maximum principle it follows

$$\int_{\partial B(0,r)} K(x,y) \, dS(y) = 1.$$

(i) We are now ready to discuss the smoothness of

$$u(x) = \frac{r^2 - |x|^2}{n\alpha(n) r} \int_{\partial B(0,r)} \frac{g(y)}{|x-y|^n} \, dS(y) = \int_{\partial B(0,r)} K(x,y) g(y) \, dS(y)$$

which follows from the fact that

$$D_x^\alpha u(x) = \int_{\partial B(0,r)} D_x^\alpha K(x,y) g(y) \, dS(y)$$

by virtue of $\partial B(0,r)$ being compact, $K(x,y)$ being smooth in x and the fact that g is bounded (since $g \in C(\partial B(0,r))$, the above equality is a result of the mean value theorem and dominated convergence theorem (see [Gutierrez, 2013], Theorem 1).

(ii) By the harmonicity of $K(x,y)$ in the variable x, it also follows from the above equality derived from [Gutierrez, 2013] that

$$\Delta u(x) = \int_{\partial B(0,r)} \Delta_x K(x,y) g(y) \, dS(y) = 0$$

for $x \in B^0(0,r)$.

(iii) Since g is continuous, for $x_0 \in \partial B(0,r)$ and for all $\varepsilon > 0$ there exists $\delta > 0$ with

$$|y - x_0| < \delta \implies |g(y) - g(x_0)| < \varepsilon \tag{9}$$

for $y \in \partial B(0,r)$. Then, suppose for $x \in B^0(0,r)$ that $|x - x_0| < \delta/2$ and note

$$\int_{\partial B(0,r)} g(x_0) K(x,y) \, dS(y) = g(x_0) \int_{\partial B(0,r)} K(x,y) \, dS(y) = g(x_0)$$

and therefore

$$\begin{aligned}
|u(x) - g(x_0)| &= \left| \int_{\partial B(0,r)} K(x,y) |g(y) - g(x_0)| \, dS(y) \right| \\
&\le \int_{\partial B(0,r) \cap B(x_0,\delta)} K(x,y) |g(y) - g(x_0)| \, dS(y) + \int_{\partial B(0,r) - B(x_0,\delta)} K(x,y) |g(y) - g(x_0)| \, dS(y) \\
&=: I + J.
\end{aligned}$$

Now, since we have established the fact that $\int_{\partial B(0,r)} K(x,y)\, dS(y) = 1$, together with (10) we have that

$$I \leq \varepsilon \int_{\partial B(0,r)} K(x,y)\, dS(y) = \varepsilon.$$

Turning to J, if $|x - x_0| \leq \frac{\delta}{2}$ and $|y - x_0| \geq \delta$, then

$$|y - x_0| \leq |y - x| + \frac{\delta}{2} \leq |y - x| + \frac{1}{2}|y - x_0|$$

whereby $|y - x| \geq \frac{1}{2}|y - x_0|$. Thus,

$$\begin{aligned} J &= \int_{\partial B(0,r) - B(x_0,\delta)} K(x,y)|g(y) - g(x_0)|\, dS(y) \leq 2||g||_{L^\infty} \int_{\partial B(0,r) - B(x_0,\delta)} K(x,y)\, dS(y) \\ &\leq 2^{n+2}||g||_{L^\infty} \frac{r^2 - |x|^2}{n\alpha(n)r} \int_{\partial B(0,r) - B(x_0,\delta)} \frac{1}{|y - x_0|^n}\, dS(y) \end{aligned}$$

As $x \to x_0$, $|x| \to |x_0| = r$ since $x_0 \in \partial B(0,r)$ whereby $|u(x) - g(x_0)| \to 0$ as $x \to x_0$, establishing the desired result.

$\square$

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