$\mathbb{Z}_2$ Topological Order and Topological Protection of Majorana Fermion Qubits, Derivations

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Section 2

Consider the transverse field Ising model Hamiltonian:

$$H = -J \sum_{i=1}^{N-1} \sigma_i^x \sigma_{i+1}^x - h_z \sum_{i=1}^{N} \sigma_i^z$$

Where σ_i^α is the α Pauli matrix at the i^{th} site, J is the ferromagnetic exchange constant, and h_z is the Zeeman field in the $\hat{z}$ direction. I will later show that this Hamiltonian is related to the Kitaev p-wave chain model Hamiltonian, which, in a continuum limit, is the Hamiltonian for a spinless topological superconductor.

First, we note that this model exhibits a $\mathbb{Z}_2$ symmetry (in that the symmetry group of inverting all spins is isomorphic to the integers under addition modulo 2). This symmetry can be attributed to the fact that the Hamiltonian commutes with the *global symmetry operator*, ie:

$$\left[\prod_j^N \sigma_j^z, H\right] = 0$$

Explicitly (this takes a while, and there may be a shorter way):

$$\begin{aligned}
\left[\prod_j^N \sigma_j^z, H\right] &= \left[\prod_j^N \sigma_j^z, -J \sum_{i=1}^{N-1} \sigma_i^x \sigma_{i+1}^x - h_z \sum_{i=1}^{N} \sigma_i^z\right] \\
&= -J \left[\prod_j^N \sigma_j^z, \sum_{i=1}^{N-1} \sigma_i^x \sigma_{i+1}^x\right] - h_z \left[\prod_j^N \sigma_j^z, \sum_{i=1}^{N} \sigma_i^z\right] \\
&= -J \sum_{i=1}^{N-1} \left[\prod_j^N \sigma_j^z, \sigma_i^x \sigma_{i+1}^x\right] - h_z \sum_{i=1}^{N} \left[\prod_j^N \sigma_j^z, \sigma_i^z\right] \quad \text{(from linearity of commutators)}
\end{aligned}$$

It will suffice to show that the commutators $\left[\prod_j \sigma_j^z, \sigma_i^x \sigma_{i+1}^x\right]$ and $\left[\prod_j \sigma_j^z, \sigma_i^z\right]$ are each 0. First I will consider the former:

$$\left[\prod_j \sigma_j^z, \sigma_i^x \sigma_{i+1}^x\right] = \sigma_i^x \left[\sigma_{i+1}^x, \prod_j^N \sigma_j^z\right] + \left[\sigma_i^x, \prod_j^N \sigma_j^z\right] \sigma_{i+1}^x$$

Now we can impose the condition that Pauli matrices commute at different lattice sites, ie:

$$\left[\sigma_n^i, \sigma_m^j\right] = \delta_{nm}\left[\sigma^i, \sigma^j\right]_n \tag{1}$$

Which we can apply in the explicit definition of each commutator:

$$\begin{aligned}
\left[\sigma_{i+1}^x, \prod_j^N \sigma_j^z\right] &= \sigma_{i+1}^x\left(\sigma_1^z\sigma_2^z \dots \sigma_{i+1}^z \dots \sigma_N^z\right) - \left(\sigma_1^z\sigma_2^z \dots \sigma_{i+1}^z \dots \sigma_N^z\right)\sigma_{i+1}^x \\
&= \sigma_1^z\sigma_2^z \dots \sigma_{i+1}^x\sigma_{i+1}^z \dots \sigma_N^z - \sigma_1^z\sigma_2^z \dots \sigma_{i+1}^z\sigma_{i+1}^x \dots \sigma_N^z \qquad \text{(from (1))} \\
&= \sigma_1^z\sigma_2^z \dots \left(\sigma_{i+1}^x\sigma_{i+1}^z - \sigma_{i+1}^z\sigma_{i+1}^x\right)\sigma_{i+2}^z \dots \sigma_N^z \\
&= \prod_{j=1}^i \sigma_j^z\left[\sigma_{i+1}^x, \sigma_{i+1}^z\right]\prod_{j=i+2}^N \sigma_j^z
\end{aligned}$$

We can evaluate the intermediate commutator by noting the known commutation relation between Pauli matrices (for a fixed lattice site):

$$\left[\sigma^i, \sigma^j\right] = 2i\epsilon_{ijk}\sigma^k \tag{2}$$

where ϵ_{ijk} is the 3 dimensional Levi-Cevita symbol (upper vs lower indices do not correspond to contra-variant and covariant component):

$$\begin{aligned}
\prod_{j=1}^i \sigma_j^z\left[\sigma_{i+1}^x, \sigma_{i+1}^z\right]\prod_{j=i+2}^N \sigma_j^z &= \prod_{j=1}^i \sigma_j^z\left(2i\epsilon_{132}\sigma_{i+1}^2\right)\prod_{j=i+2}^N \sigma_j^z \\
&= -2i\prod_{j=1}^i \sigma_j^z\left(\sigma_{i+1}^y\right)\prod_{j=i+2}^N \sigma_j^z
\end{aligned}$$

From a symmetric calculation:

$$\begin{aligned}
\left[\sigma_i^x, \prod_{j=1}^N \sigma_j^z\right] &= \prod_{j=1}^{i-1} \sigma_j^z\left[\sigma_i^x, \sigma_i^z\right]\prod_{j=i+1}^N \sigma_j^z \\
&= -2i\prod_{j=1}^{i-1} \sigma_j^z\sigma_i^y\prod_{j=i+1}^N \sigma_j^z
\end{aligned}$$

All together:

$$
\begin{aligned}
\left[\prod_j \sigma_j^z, \sigma_i^x\sigma_{i+1}^x\right] &= \sigma_i^x\left[\sigma_{i+1}^x, \prod_j^N \sigma_j^z\right] + \left[\sigma_i^x, \prod_j^N \sigma_j^z\right]\sigma_{i+1}^x \\
&= -2i\sigma_i^x \prod_{j=1}^{i} \sigma_j^z \left(\sigma_{i+1}^y\right) \prod_{j=i+2}^{N} \sigma_j^z - 2i\prod_{j=1}^{i-1}\sigma_j^z\sigma_i^y \prod_{j=i+1}^{N} \sigma_j^z\sigma_{i+1}^x \\
&= -2i\prod_{j=1}^{i-1}\sigma_j^z\left(\sigma_i^x\sigma_i^z\sigma_{i+1}^y + \sigma_i^y\sigma_{i+1}^z\sigma_{i+1}^x\right)\prod_{j=i+2}^{N}\sigma_j^z
\end{aligned}
$$

The last line follows from the second to last line from the equal site commutation relation, which shows that the z Pauli matrices of the first $i-1$ sites commute with the σ_i^x Pauli matrix, and similarly that the z Pauli matrices of the last $N-i-2$ sites commute with the σ_{i+1}^x Pauli matrix.

It now suffices to show that the term in parenthesis is identically 0 by first computing some Pauli matrix compositions:

$$
\begin{aligned}
\sigma^x\sigma^z &= i\sigma^y \\
\sigma^z\sigma^x &= -i\sigma^y \\
\implies \left(\sigma_i^x\sigma_i^z\sigma_{i+1}^y + \sigma_i^y\sigma_{i+1}^z\sigma_{i+1}^x\right) &= i\sigma_i^y\sigma_{i+1}^y - i\sigma_i^y\sigma_{i+1}^y = 0
\end{aligned}
$$

Thus, we have

$$
-J\sum_{i=1}^{N-1}\left[\prod_j^N \sigma_j^z, \sigma_i^x\sigma_{i+1}^x\right] = 0
$$

To compute $\left[\prod_j^N \sigma_j^z, \sigma_i^z\right]$, we simply note that, by the equal-site commutation relation (1), the commutator between distinct sites will vanish, and the commutator between 2 identical Pauli matrices at the same site also vanishes, and the total commutator is proportional to a "linear combination" of all of these commutators (by some commutator identity), so we have

$$
\left[\prod_j^N \sigma_j^z, \sigma_i^z\right] = 0
$$

And all together, we have that $\left[\prod_j^N \sigma_j^z, H\right] = 0$, so the global symmetry operator commutes with the transverse field Ising model Hamiltonian (It may be worth noting that the action of the square of the operator on an arbitrary spin state over the lattice is identity, and this may be exploited to obtain the same result, although it is not evident at the moment).

It will be convenient to reformulate this Hamiltonian by means of a Jordan-Wigner transformation, defined by:

$$
\begin{aligned}
c_i &= \sigma_i^+ \left(\prod_{j=1}^{i-1} \sigma_j^z \right) \\
c_i^\dagger &= \sigma_i^- \left(\prod_{j=1}^{i-1} \sigma_j^z \right)
\end{aligned}
$$

where c_i and $c_i^\dagger$ are fermionic annihilation and creation operators obeying the fermionic anti-commutation relations, and σ_i^+ and σ_i^- are the Pauli raising and lowering operators, respectively. We can invert the transformation by noting that $\sigma_i^z = 1 - 2c_i^\dagger c_i$: (I tried to show this directly by considering the RHS action on some lattice spin state directly but could not, this result is given by Kartik Chhajed, *From Ising model to Kitaev Chain, An introduction to topological phase transitions*)

$$
\begin{aligned}
\sigma_i^+ &= \prod_{j=1}^{i-1} \left(1 - 2c_j^\dagger c_j \right) c_i \\
\sigma_i^- &= \prod_{j=1}^{i-1} \left(1 - 2c_j^\dagger c_j \right) c_i^\dagger
\end{aligned}
$$

Computing the anti-commutators of c_i and $c_i^\dagger$:

$$
\{c_i, c_j\} = \{\sigma_i^+ \left(\prod_{j=1}^{i-1} \sigma_j^z \right), \sigma_j^+ \left(\prod_{k=1}^{i-1} \sigma_k^z \right)\} \tag{3}
$$

From the definition of the creation and annihilation Pauli matrices, we have:

$$
\begin{aligned}
\sigma_i^\pm &= \sigma_i^x \pm i\sigma_i^y \\
\sigma_i^x &= \frac{1}{2} \left(\sigma_i^+ + \sigma_i^- \right) \\
\sigma_i^y &= -\frac{i}{2} \left(\sigma_i^+ - \sigma_i^- \right) \\
\sigma_i^z &= -i\sigma_i^y \sigma_i^x = -\frac{1}{4} \left(\sigma_i^+ - \sigma_i^- \right) \left(\sigma_i^+ + \sigma_i^- \right)
\end{aligned}
$$

Substituting in the expression for the relevant operators into the Hamiltonian and expanding:

$$
\begin{aligned}
H = -J \sum_{i=1}^{N-1} c_i \prod_{j=1}^{i-1} \left(1 - 2c_j^\dagger c_j\right) \prod_{j=1}^{i} \left(1 - 2c_j^\dagger c_j\right) c_{i+1} + c_i \prod_{j=1}^{i-1} \left(1 - 2c_j^\dagger c_j\right) \prod_{j=1}^{i} \left(1 - 2c_j^\dagger c_j\right) c_{i+1}^\dagger + \ldots \\
+ c_i^\dagger \prod_{j=1}^{i-1} \left(1 - 2c_j^\dagger c_j\right) \prod_{j=1}^{i} \left(1 - 2c_j^\dagger c_j\right) c_{i+1} + c_i^\dagger \prod_{j=1}^{i-1} \left(1 - 2c_j^\dagger c_j\right) \prod_{j=1}^{i} \left(1 - 2c_j^\dagger c_j\right) c_{i+1}^\dagger + \ldots \\
+ h_z \sum_{i=1}^{N} \prod_{j=1}^{i-1} \left(1 - 2c_j^\dagger c_j\right) c_i \prod_{j=1}^{i-1} \left(1 - 2c_j^\dagger c_j\right) c_i + \prod_{j=1}^{i-1} \left(1 - 2c_j^\dagger c_j\right) c_i \prod_{j=1}^{i-1} \left(1 - 2c_j^\dagger c_j\right) c_i^\dagger \ldots \\
- \prod_{j=1}^{i-1} \left(1 - 2c_j^\dagger c_j\right) c_i^\dagger \prod_{j=1}^{i-1} \left(1 - 2c_j^\dagger c_j\right) c_i - \prod_{j=1}^{i-1} \left(1 - 2c_j^\dagger c_j\right) c_i^\dagger \prod_{j=1}^{i-1} \left(1 - 2c_j^\dagger c_j\right) c_i^\dagger
\end{aligned}
$$

We can simplify this expression greatly by noting that distinct site operators commute and that $\left(1 - 2c_j^\dagger\right)\left(1 - 2c_j^\dagger\right) = \left(\sigma_j^z\right)^2 = 1$:

$$
\begin{aligned}
H =& - J \sum_{i=1}^{N-1} c_i \left(1 - 2c_i^\dagger c_i\right) c_{i+1} + c_i \left(1 - 2c_i^\dagger c_i\right) c_{i+1}^\dagger + c_i^\dagger \left(1 - 2c_i^\dagger c_i\right) c_{i+1} + c_i^\dagger \left(1 - 2c_i^\dagger c_i\right) c_{i+1}^\dagger \ldots \\
&+ h_z \sum_{i=1}^{N} c_i c_i + c_i c_i^\dagger - c_i^\dagger c_i - c_i^\dagger c_i^\dagger
\end{aligned}
$$

This can be further simplified by noticing that the self composition of the c_i and $c_i^\dagger$ operators are 0:

$$
\begin{aligned}
\left(c_i\right)^2 &= \sigma_i^+ \left(\prod_{j=1}^{i-1} \sigma_j^z\right) \sigma_i^+ \left(\prod_{j=1}^{i-1} \sigma_j^z\right) && (4) \\
&= \left(\sigma_i^+\right)^2 \left(\prod_{j=1}^{i-1} \sigma_j^z\right) \left(\prod_{j=1}^{i-1} \sigma_j^z\right) = \left(\sigma_i^+\right)^2 = \frac{i}{4}\left(\sigma_i^x \sigma_i^y + \sigma_i^y \sigma_i^x\right) = 0 && (5) \\
\left(c_i^\dagger\right)^2 &= \sigma_i^- \left(\prod_{j=1}^{i-1} \sigma_j^z\right) \sigma_i^- \left(\prod_{j=1}^{i-1} \sigma_j^z\right) && (6) \\
&= \left(\sigma_i^-\right)^2 \left(\prod_{j=1}^{i-1} \sigma_j^z\right) \left(\prod_{j=1}^{i-1} \sigma_j^z\right) = \left(\sigma_i^-\right)^2 = -\frac{i}{4}\left(\sigma_i^x \sigma_i^y + \sigma_i^y \sigma_i^x\right) = 0 && (7)
\end{aligned}
$$

We can then left-distribute the creation and annihilation operators in the first sum:

$$
\begin{aligned}
H &= -J\sum_{i=1}^{N-1} c_i c_{i+1} + c_i c_{i+1}^\dagger + c_i^\dagger c_{i+1} + c_i^\dagger c_{i+1}^\dagger + h_z \sum_{i=1}^{N} c_i c_i + c_i c_i^\dagger - c_i^\dagger c_i - c_i^\dagger c_i^\dagger \\
&= -J\sum_{i=1}^{N-1} c_i^\dagger c_{i+1} + c_i^\dagger c_{i+1}^\dagger + h.c. + h_z \sum_{i=1}^{N} \dots
\end{aligned}
$$

Where $h.c.$ denotes the Hermitian conjugate of the terms, which is true since each composed operator in the sum can have its order swapped such that there is a conjugate pair in the sum. Instead of dealing with the terms present under the second summation, note that we can simply substitute $\sigma_i^z = 1 - 2c_i^\dagger c_i$:

$$
\begin{aligned}
\implies H &= -J\sum_{i=1}^{N-1} c_i^\dagger c_{i+1} + c_i^\dagger c_{i+1}^\dagger + h.c. + h_z \sum_{i=1}^{N} \left(1 - 2c_i^\dagger c_i\right) \\
&= -J\sum_{i=1}^{N-1} c_i^\dagger c_{i+1} + c_i^\dagger c_{i+1}^\dagger + h.c. + h_z \left(N - 2\sum_{i=1}^{N} c_i^\dagger c_i\right)
\end{aligned}
$$

Here, we notice that the term $h_z N$ is a constant and can thus be neglected, as it only serves to shift the energy spectrum. We then make the identifications $2h_z \coloneqq \mu$ and $J =$ From this form of the Hamiltonian, we see that we can generalize the Hamiltonian to that of a p-wave superconductor in terms of parameters t and Δ, where t is the hopping strength and Δ is the superconducting order parameter:

$$
H = -t\sum_{i=1}^{N-1}\left(c_i^\dagger c_{i+1} + h.c.\right) + \Delta\sum_{i=1}^{N-1}\left(c_i^\dagger c_{i+1}^\dagger + h.c.\right) - \mu\sum_{i=1}^{N} c_i^\dagger c_i
$$

In the case where $\Delta = -t = J$, we recover the fermionic system obtained by the Jordan-Wigner transformation on the original transverse Ising model Hamiltonian.

Defining the Majorana site operators:

$$
\begin{aligned}
c_i &= \frac{\gamma_{1,i} - i\gamma_{2,i}}{\sqrt{2}} \\
c_i^\dagger &= \frac{\gamma_{1,i} + i\gamma_{2,i}}{\sqrt{2}}
\end{aligned}
$$

We can express the Hamiltonian in terms of these new operators:

$$
\begin{aligned}
c_i^\dagger c_{i+1} &= \frac{1}{2}\left(\gamma_{1,i}\gamma_{1,i+1} - i\gamma_{1,i}\gamma_{2,i+1} + i\gamma_{2,i}\gamma_{1,i+1} + \gamma_{2,i}\gamma_{2,i+1}\right) \\
\left(c_i^\dagger c_{i+1}\right)^\dagger &= c_{i+1}^\dagger c_i = -c_i c_{i+1}^\dagger \qquad \text{(Fermionic anti-commutation)} \\
&= -\frac{1}{2}\left(-\gamma_{1,i}\gamma_{1,i+1} - i\gamma_{1,i}\gamma_{2,i+1} + i\gamma_{2,i}\gamma_{1,i+1} - \gamma_{2,i}\gamma_{2,i+1}\right) \\
&\implies c_i^\dagger c_{i+1} + h.c. = i\left(\gamma_{2,i}\gamma_{1,i+1} - \gamma_{1,i}\gamma_{2,i+1}\right) \\
&\implies -t\sum_{i=1}^{N-1}\left(c_i^\dagger c_{i+1} + h.c.\right) = it\sum_{i=1}^{N-1}\left(\gamma_{1,i}\gamma_{2,i+1} - \gamma_{2,i}\gamma_{1,i+1}\right) \quad \text{(with } t \text{ absorbing -1 factor)}
\end{aligned}
$$

Similarly:

$$
\begin{aligned}
\left(c_i^\dagger c_{i+1}^\dagger\right)^\dagger &= c_{i+1} c_i = -c_i c_{i+1} \\
c_i^\dagger c_{i+1}^\dagger &= \frac{1}{2}\left(\gamma_{1,i}\gamma_{1,i+1} + i\gamma_{1,i}\gamma_{2,i+1} + i\gamma_{2,i}\gamma_{1,i+1} - \gamma_{2,i}\gamma_{2,i+1}\right) \\
c_i c_{i+1} &= \frac{1}{2}\left(\gamma_{1,i}\gamma_{1,i+1} - i\gamma_{1,i}\gamma_{2,i+1} - i\gamma_{2,i}\gamma_{1,i+1} - \gamma_{2,i}\gamma_{2,i+1}\right) \\
&\implies c_i^\dagger c_{i+1}^\dagger + h.c. = i\left(\gamma_{1,i}\gamma_{2,i+1} + \gamma_{2,i}\gamma_{1,i+1}\right) \\
&\implies \Delta\sum_{i=1}^{N-1}\left(c_i^\dagger c_{i+1}^\dagger + h.c.\right) = i\Delta\sum_{i=1}^{N-1}\left(\gamma_{1,i}\gamma_{2,i+1} + \gamma_{2,i}\gamma_{1,i+1}\right)
\end{aligned}
$$

Now we consider the explicit definitions of the Majorana operators:

$$
\begin{aligned}
\gamma_{1,i} &= \frac{c_i + c_i^\dagger}{\sqrt{2}} \\
\gamma_{2,i} &= \frac{i\left(c_i - c_i^\dagger\right)}{\sqrt{2}} \\
&\implies \frac{1}{2} - i\gamma_{1,i}\gamma_{2,i} = \frac{1}{2} - i\left[\frac{i}{2}\left(c_i + c_i^\dagger\right)\left(c_i - c_i^\dagger\right)\right] \\
&= \frac{1}{2} + \frac{1}{2}\left(c_i^2 - c_i c_i^\dagger + c_i^\dagger c_i - c_i^{\dagger 2}\right)
\end{aligned}
$$

Again using the fermionic anti commutation relations:

$$
\begin{aligned}
\{c_i, c_i\} = 2c_i^2 =&0 \\
&\implies c_i^2 = 0 = c_i^{\dagger 2} \\
&\implies \frac{1}{2} + \frac{1}{2}\left(c_i^2 - c_i c_i^\dagger + c_i^\dagger c_i - c_i^{\dagger 2}\right) = \frac{1}{2} + \frac{1}{2}\left(c_i^\dagger c_i - c_i c_i^\dagger\right) \\
&= \frac{1}{2} + \frac{1}{2}\left(c_i^\dagger c_i + \left[c_i^\dagger c_i - 1\right]\right) = \frac{1}{2} + c_i^\dagger c_i - \frac{1}{2} = c_i^\dagger c_i
\end{aligned}
$$

I tried showing this result from the forward direction, however I ended up with a term that was not proportional to a Majorana cross term, curious about how to go about this

So we can write the total Hamiltonian as:

$$
H = it\sum_{i=1}^{N-1}\left(\gamma_{1,i}\gamma_{2,i+1} - \gamma_{2,i}\gamma_{1,i+1}\right) + i\Delta\sum_{i=1}^{N-1}\left(\gamma_{1,i}\gamma_{2,i+1} + \gamma_{2,i}\gamma_{1,i+1}\right) - \mu\sum_{i=1}^{N}\left(\frac{1}{2} - i\gamma_{1,i}\gamma_{2,i}\right)
$$

The parameters t, Δ, and μ indicate the hopping strength, superconducting order parameter, and chemical potential, respectively. We can look at some cases for various choices of parameters. In the case $t = \Delta = 0$, we have the Hamiltonian:

$$
H = -\mu\sum_{i=1}^{N}\left(\frac{1}{2} - i\gamma_{1,i}\gamma_{2,i}\right)
$$

But what does this mean? We can imagine each site along the Kitaev chain, consisting of a single fermionic site, consisting of 2 "Majorana" sites, each acted on by the $\gamma_{1,i}$ and $\gamma_{2,i}$ operators respectively (I believe they each physically correspond to particle and hole sites respectively). The Hamiltonian in this case is simply given by the contributed of $2(N+1)$ non-interacting Majorana-fermionic sites. It turns out that this case corresponds to the trivial (non-topological) phase in that there is no non-trivial topological invariant that characterizes the phase.

In the case where $\mu = 0$ and $t = \Delta$, the Hamiltonian takes the form:

$$
H = 2it\sum_{i=1}^{N-1}\gamma_{1,i}\gamma_{2,i+1}
$$

In the Majorana fermionic formulation, this Hamiltonian tells us that in this limit, the "last" Majorana site couples to the "first" Majorana site on the right-adjacent fermionic site. If we expand the sum explicitly:

$$
H = 2it\left(\gamma_{1,1}\gamma_{2,2} + \gamma_{1,2}\gamma_{2,3} + \cdots + \gamma_{1,N-1}\gamma_{2,N}\right)
$$

In this form, we can see that 2 Majorana operators are missing from the Hamiltonian, namely $\gamma_{2,1}$ and $\gamma_{1,N}$. These 2 Majorana fermions reside at each end of the Kitaev chain - we can "construct a fermion" out of these two Majorana fermions by defining the following operator:

$$\tilde{a} = \frac{\gamma_{1,N} - i\gamma_{2,1}}{\sqrt{2}}$$

The state characterized by this operator is inherently non-local, as the coupled Majorana fermions are localized at opposite ends of the Kitaev chain. It turns out that these modes give rise to the topological phase characterizing the Hamiltonian - if we imagine "cycling the chain", ie shifting all sites up by some fixed integer modulo the last site (N-1), we will never be able to construct a chain in which each Majorana site is coupled to another within each fermionic site - there will always be a coupling between the "last" the ends of the chain. Since neither Majorana mode is included in the Hamiltonian, we have that:

$$\begin{aligned} [H, \gamma_{1,N}] = [H, \gamma_{2,0}] =& 0 \\ &\implies H\tilde{a}\left|0\right\rangle = \tilde{a}H\left|0\right\rangle = 0 \end{aligned}$$

Thus we see that the ground state of the Hamiltonian is doubly degenerate. (I'm confused on the papers explanation of a doubly degenerate ground state and the even and odd parity of the Majorana operators... also the above derivation is not valid as the vacuum is not defined for Majorana fermions, not sure how to go about this)

Initially, we worked with fermionic operators to constructed Majorana-fermionic operators. We can also consider the reverse by using properties of the Majorana algebra:

$$\gamma_{1,i}^\dagger = \frac{\left(c_i + c_i^\dagger\right)^\dagger}{\sqrt{2}} = \frac{c_i^\dagger + c_i}{\sqrt{2}} = \gamma_{1,i}$$

$$\gamma_{2,i}^\dagger = \frac{-i\left(c_i - c_i^\dagger\right)^\dagger}{\sqrt{2}} = \frac{i\left(c_i - c_i^\dagger\right)}{\sqrt{2}} = \gamma_{2,i}$$

So in general we have

$$\gamma = \gamma^\dagger$$

where γ denotes a Majorana fermionic operator. We can also evaluate the anti-commutator:

$$\begin{aligned} \{\gamma_i, \gamma_j\} &= \{c_i + c_i^\dagger, c_j + c_j^\dagger\} \\ &= \{c_i, c_j + c_j^\dagger\} + \{c_i^\dagger, c_j + c_j^\dagger\} \\ &= \{c_i, c_j\} + \{c_i, c_j^\dagger\} + \{c_i^\dagger, c_j\} + \{c_i^\dagger, c_j^\dagger\} = 2\delta_{ij} \end{aligned}$$

The Majorana fermion operators thus obey the *Clifford algebra* (I am not sure how to yield the factor of 2 without neglecting the factor of $\frac{1}{\sqrt{2}}$). From this algebra, we see that:

$$\{\gamma_i, \gamma_i\} = 2\gamma_i^2 = 2 \implies \gamma_i^2 = 1$$

This is in clear distinction from fermions, which obey the Pauli exclusion principle ($c_i^{\dagger 2} = c_i^2 = 0$). Since the Majorana operator is self-adjoint, the second-quantized Majorana fermion is its *own antiparticle*. It turns out that a fermionic vacuum for Majorana fermions is not defined as there is no well-defined number operator for Majorana fermion and thus the number of Majorana fermions occupying the ground state is not defined.

If we attempt to construct a number operator, $\gamma_i^\dagger \gamma_i$, but since γ_i is Hermitian and $\gamma_i^2 = 1$, then $\gamma_i^\dagger \gamma_i = 1$. This is obviously not the action that a counting operator should have on eigenstates. (I'm also not sure how to demonstrate the non-$U(1)$ invariance $\mathbb{Z}_2$ symmetry])

Aside on anti commutation relations and Jordan-Wigner transformation

Since the Jordan-Wigner transformation maps a spin system to a fermionic one and back, it may prove helpful to understand what the Jordan-Wigner transformations are actually doing so that we can find an explicit representation of the ground state of the 1D transverse Ising model Hamiltonian given we know the ground state of the 1D fermionic system.

The following section is motivated heavily by Michael Nielsen's document *The Fermionic canonical commutation relations and the Jordan-Wigner transform.*

Since the transformed Ising model Hamiltonian consists of operators that obey fermionic anticommutation relations, it may be enlightening to review the axiomatic construction and implications of these operators.

As noted before, given a finite set of operators $a_1, \ldots, a_n$ acting on a Hilbert space V, then the operators satisfy fermionic anticommutation relations if the following hold:

$$\{a_i, a_j\} = \{a_i^\dagger, a_j^\dagger\} = 0$$
$$\{a_i, a_j^\dagger\} = \delta_{ij} I$$

where I is the identity operator acting on V. It immediately follows that for any operator satisfying the anticommutation relations, the following holds:

$$\{a_j, a_j\} = \{a_j^\dagger, a_j^\dagger\} = 2a_j^2 = 2a_j^{\dagger 2} = 0$$
$$\implies a_j^2 = a_j^{\dagger 2} = 0$$

This is the manifestation of the *Pauli exclusion principle.* To see this, we consider the vacuum expectation value for the product of 2 fermionic creation operators:

$$\langle 0|a_i^\dagger a_j^\dagger|0\rangle$$

If we let $i = j$, the anticommutation relations give us:

$$\langle 0|a_j^\dagger a_j^\dagger|0\rangle = \langle 0|a_j^{\dagger 2}|0\rangle = 0$$

We see that the the expectation value for the vacuum state to be doubly-occupied by fermions is 0, and, as we will show, since the eigenvalues of $a^\dagger$ are positive-definite, all of the probability must be allocated to the state in which the the vacuum is not doubly-occupied (state with 0 eigenvalue of $a^\dagger a$).

We now look at the operator $a_j^\dagger a_j$, which I will refer to as the *occupation operator*, as its eigenvalues will tell us whether or not a state is occupied by a fermion.

First, we show that the possible eigenvalues of $a_j^\dagger a_j$ are 0 and 1 by first showing that $(a_j^{\dagger 2} a_j)^2 = a_j^\dagger a_j$:

$$(a_j^{\dagger 2} a_j)^2 = a_j^\dagger a_j a_j^\dagger a_j$$

From the anticommutation relations, we see that $a_j a_j^\dagger = I - a_{j^\dagger a_j}$:

$$\implies a_j^\dagger a_j a_j^\dagger a_j = a_j^\dagger (I - a_j^\dagger aj) a_j$$
$$= a_j^\dagger a_j - a_j^{\dagger 2} a_j^2$$

Since a_j and $a_j^\dagger$ satisfy the anticommutation relations, the right-most term vanishes, and we are left with the desired result, which we can then use to show that the eigenvalues of $a_j^\dagger a_j$ are 0 and 1 by considering the eigenvalue equation of an operator equal to its square:

$$Av = \lambda v \implies A^2 v = \lambda A v = Av$$
$$\implies \lambda = 0, 1$$

We now show that the operators a_j and $a_j^\dagger$ act as annihilation and creation operators on eigenstates of the occupation operator, respectively. Let $|\psi\rangle$ be a normalized eigenvector of

$a_j^\dagger a_j$ with eigenvalue 1:

$$||a_j|\psi\rangle||^2 = \langle\psi|a_j^\dagger a_j|\psi\rangle = \langle\psi|\psi\rangle = 1$$

where the second equality follows from the assumption of $|\psi\rangle$ being a normalized eigenvector of $a_j^\dagger a_j$, so $a_j|\psi\rangle$ is normalized. To see that it is an eigenvector of $a_j^\dagger a_j$ with eigenvalue 0, we simply act the occupation operator on the state:

$$a_j^\dagger a_j a_j|\psi\rangle = a_j^\dagger a_j^2|\psi\rangle = 0$$

since $a_j^2 = 0$. Similarly, given $|\psi\rangle$ is a normalized eigenvector of $a_j^\dagger a_j$ with eigenvalue 0, we show that $a_j^\dagger|\psi\rangle$ is a normalized eigenvector of $a_j^\dagger a_j$ with eigenvalue 1:

$$\begin{aligned} ||a_j^\dagger|\psi\rangle|| &= \langle\psi|a_j a_j^\dagger|\psi\rangle = \langle\psi|I - a_j^\dagger a_j|\psi\rangle \\ &= \langle\psi|I|\psi\rangle - \langle\psi|a_j^\dagger a_j|\psi\rangle = 1 \end{aligned}$$

since the second term vanishes by assumption of ψ being an eigenvector of $a_j^\dagger a_j$ with eigenvalue 0 and the first term goes to 1 since ψ is normalized. To show that that $a_j^\dagger|\psi\rangle$ is an eigenvector of the occupation operator with eigenvalue 1, we again act the occupation operator on the state:

$$\begin{aligned} a_j^\dagger a_j a_j^\dagger|\psi\rangle = a_j^\dagger(I - a_j^\dagger a_j)|\psi\rangle &= a_j^\dagger|\psi\rangle - a_j^{\dagger 2} a_j|\psi\rangle \\ &= a_j^\dagger|\psi\rangle \end{aligned}$$

where the last equality holds since $a_j^{\dagger 2} = 0$, so $a_j^\dagger|\psi\rangle$ is a normalized eigenvector of the occupation operator with eigenvalue 1.

What does this all mean? If we recall the spectrum of the occupation operator for spinless-bosonic Fock space (where the set of operators obey the commutation relations for bosons), the eigenvalues of the occupation operator takes values of all possible natural numbers (for a particular 3-momentum $\vec{k}$), ie an arbitrary number of bosons can occupy a given quantum state. This is in direct observation to our findings where the eigenvalues of the ”fermionic” occupation operator can only take values 0 or 1. This is another manifestation of the Pauli exclusion principle (alternatively, one can see this as a *proof* of the Pauli exclusion principle, however this a bit circular, as the commutation relations to construct a relativistic quantum field theory are formulated to recover the corresponding statistics).

Consider the commutator $\left[a_i^\dagger a_i, a_j^\dagger a_j\right]$. Expanding and applying the anticommutation relations:

$$\begin{aligned}
\left[a_i^\dagger a_i, a_j^\dagger a_j\right] &= a_i^\dagger a_i a_j^\dagger a_j - a_i^\dagger a_i a_j^\dagger a_j \\
&= a_i^\dagger \left(\delta_{ij} I - a_j^\dagger a_i\right) a_j - a_j^\dagger a_j a_i^\dagger a_i \\
&= a_i^\dagger \delta_{ij} I a_j - a_i^\dagger a_j^\dagger a_i a_j - a_j^\dagger a_j a_i^\dagger a_i \\
&= -a_j^\dagger a_i^\dagger a_j a_i - a_j^\dagger a_j a_i^\dagger a_i + a_i^\dagger \delta_{ij} I a_j \\
&= -a_j^\dagger \left(\delta_{ij} I - a_j a_i^\dagger\right) a_i - a_j^\dagger a_j a_i^\dagger a_i + a_i^\dagger \delta_{ij} I a_j \\
&= -a_j^\dagger \delta_{ij} I a_i + a_j^\dagger a_j a_i^\dagger a_i - a_j^\dagger a_j a_i^\dagger a_i + a_i^\dagger \delta_{ij} I a_j
\end{aligned}$$

The intermediate terms immediately cancel and, in both cases of the Kronecker-Delta, the quantity vanishes, for if $i = j$:

$$-a_j^\dagger \delta_{ij} I a_i + a_i^\dagger \delta_{ij} I a_j = -a_i^\dagger a_i + a_i^\dagger a_i = 0$$

and the commutator vanishes for when $i \neq j$. Thus, if the normalized state $|\psi\rangle$ is an eigenstate of $a_i^\dagger a_i$, it is also an eigenstate of all other possible occupation operators $(i \neq j)$, ie

$$a_j^\dagger a_j |\psi\rangle = \alpha_j |\psi\rangle$$

where for each choice of j, $\alpha_j = 0$ or 1. as shown before. It follows that we can express states that are tensor products of eigenstates of the occupation operator, corresponding to the vector $(\alpha_1, \ldots \alpha_n)$, where $a_i = 0$ or 1 as:

$$|\alpha\rangle = (a_1^\dagger)^{\alpha_1} \ldots (a_n^\dagger)^{\alpha_n} |0 \ldots 0\rangle$$

for if $\alpha_i = 0$, the corresponding creation operator goes to identity and leaves the corresponding vacuum state unaffected, and if $\alpha_i = 1$, the creation operator raises the vacuum state to the excited state with eigenvalue 1 of the corresponding occupation operator (we'll call this state $|1\rangle$). We can see that there are 2^n possible states which are eigenstates of any occupation operator, with each state corresponding to a permutation of 0's and 1's, where the number of 0's plus the number of 1's is n.

It is also worth noting that for any choice of j for two vectors α and β, $\alpha_j \neq \beta_j \implies \langle\alpha|\beta\rangle = 0$, as, without loss of generality $\langle 0|1\rangle = \langle 0|a^\dagger|0\rangle = 0$ since $a^\dagger$ acts as an annihilation operator on the bra $\langle 0|$. This shows that all of the possible 2^n eigenstates of the occupation operators are all orthogonal.

Now we can consider the subspace, call it $W \subset V$, that is spanned by the orthonormal states $|\alpha\rangle$ and consider the action of the creation and annihilation operators $a_j^\dagger$ and a_j on W, or rather the constituent orthonormal basis. First, we consider the case where $|\alpha\rangle$ has eigenvalue 0 of the j^{th} occupation operator:

$$\begin{aligned} a_j|\alpha\rangle = a_j|\alpha\rangle &= a_j(a_1^\dagger)^{\alpha_1}\dots(a_j^\dagger)^0\dots(a_n^\dagger)^{\alpha_n}|0\rangle \\ &= (a_1^\dagger)^{\alpha_1}\dots a_j\dots(a_n^\dagger)^{\alpha_n}|0|\rangle = 0 \end{aligned}$$

Similarly, if $|\alpha\rangle$ has the eigenvalue 1 on the j^{th} occupation operator, $a_j^\dagger|\alpha\rangle = 0$, but with a factor of $(-1)^{j-1}$ pulled out due to anti-commuting $a_j^\dagger$ (this doesn't contribute to $a_j^\dagger|\alpha\rangle = 0$ but it's still worth noting).

If $\alpha_j = 1$, then we can conveniently express the action $a_j|\alpha\rangle$ in term of $|\alpha'\rangle$, where $|\alpha'\rangle$ is the same vector as $|\alpha\rangle$ but with $\alpha_j = 0$:

$$a_j a_j^\dagger|\alpha'\rangle = (-1)^{j-1}a_j(a_1^\dagger)^{\alpha_1}\dots(a_j^\dagger)\dots|0\rangle = a_j|\alpha\rangle$$

Now, for $\alpha_k = 1$, the anti-commutation relations yield a factor of -1 when a_j is anti-commuted through, however if $\alpha_k = 0$, then a_j commutes with $(a_k^\dagger)^{\alpha_k}$, so the total contribution of the -1s are $(-1)^{\alpha_1}\dots(-1)^{\alpha_{j-1}}$, but this is just $(-1)^{\sum_{k=1}^{j-1}\alpha_k}$ this part is incomplete

Recall that the number (occupation) operator for Majorana fermions is ill-defined (or rather there is no such operator). We can still, however, attempt to count the occupancy of of Majorana's in states by considering the c_i fermions, which from our analysis above, have number states $|a_1, \dots a_n\rangle$ which are eigenstates of the occupation operator $c_i^\dagger c_i$ with eigenvalues 0 or 1. We can then consider an "overlap" parameter, t between 2 Majorana and introduce a term to the Hamiltonian, $\frac{i}{2}t\gamma_{2i-1}\gamma_{2i} = t\left(c_i^\dagger c_i - \frac{1}{2}\right)$. *Introduction to topological superconductivity and Majorana fermions*, Martin Leijnse and Karsten Flensber

As an aside, we can now show that the Jordan-Wigner transformation of fermion operators yields operators that obey spin algebras. First, we define the the *string operator*, $e^{i\phi_j}$, where $\phi_j = \pi\sum_{l<j} n_j$, where n_j is the occupation eigenstate for the j^{th} site. First, we can show that $e^{i\pi n_j}$ anti-commutes with c_j and $c_j^\dagger$.

Case 1: $e^{i\pi n_j}c_j|0\rangle$ -

$$e^{i\pi n_j}c_j|0\rangle = 0$$
$$c_j e^{i\pi n_j}|0\rangle = c_j|0\rangle = 0 \implies e^{i\pi n_j}c_j = -c_j e^{i\pi n_j}$$

Case 2: $e^{i\pi n_j} c_j^\dagger |0\rangle$ -

$$\begin{aligned} e^{i\pi n_j} c_j^\dagger |0\rangle &= e^{i\pi n_j}|1\rangle = e^{i\pi}|1\rangle = -1|1\rangle \\ c_j^\dagger e^{i\pi n_j}|0\rangle &= c_j^\dagger|0\rangle = |1\rangle \\ \implies \left(e^{i\pi n_j} c_j^\dagger + c_j^\dagger e^{i\pi n_j}\right)|0\rangle &= (-1|1\rangle + |1\rangle) = 0 \implies e^{i\pi n_j} c_j^\dagger = -c_j^\dagger e^{i\pi n_j} \end{aligned}$$

Case 3: $e^{i\pi n_j} c_j |1\rangle$ -

$$\begin{aligned} e^{i\pi n_j} c_j|1\rangle &= e^{i\pi n_j}|0\rangle = |0\rangle \\ c_j e^{i\pi n_j} c_j|1\rangle &= c_j e^{i\pi}|1\rangle = -c_j|1\rangle = -|0\rangle \\ \implies \left(e^{i\pi n_j} c_j + c_j e^{i\pi n_j}\right)|1\rangle &= (|0\rangle - |0\rangle) = 0 \\ &\implies e^{i\pi n_j} c_j = -c_j e^{i\pi n_j} \end{aligned}$$

Case 4: $e^{i\pi n_j} c_j^\dagger |1\rangle$ -

$$\begin{aligned} e^{i\pi n_j} c_j^\dagger|1\rangle &= 0 \\ c_j^\dagger e^{i\pi n_j}|1\rangle = c_j^\dagger e^{i\pi}|1\rangle &= -c_j^\dagger|1\rangle = 0 \\ \implies e^{i\pi n_j} c_j^\dagger &= -c_j^\dagger e^{i\pi n_j} \end{aligned}$$

So in general, we can write $\{e^{i\pi n_j}, c_j\} = \{e^{i\pi n_j}, c_j^\dagger\} = 0$, ie the string operator of a single site anti-commutes with fermionic operators acting on the same site. For a site l such that $l \neq h$, we can show that for $l < j$, the j^{th} string operator anti-commutes with fermion operators and for $l \geq j$, they commute:

$$\{e^{i\phi_j}, c_l^\dagger\} = \{e^{i\pi \sum_{l<j} n_j}, c_l^\dagger\} = e^{i\pi \sum_{l<j} n_j} c_l^\dagger + c_l^\dagger e^{i\pi \sum_{l<j} n_j}$$

Since we already verified that $\{e^{i\pi_l}, c_l^\dagger\} = 0$, it suffices to show that $[e^{i\pi n_l}, c_j^\dagger] = 0$ for $j \neq l$:

$$e^{i\pi n_l}|\psi_j\rangle = |\psi_j\rangle$$

Since the occupation is always 0 for off-site, $e^{i\pi n_l}$ will commute with any on-site operator.

$$\begin{aligned} \implies e^{i\pi \sum_{l<j} n_j} c_l^\dagger + c_l^\dagger e^{i\pi \sum_{l<j} n_j} &= e^{i\pi(n_1 + \cdots + n_l + \cdots + n_{j-1})} c_l^\dagger + c_l^\dagger e^{i\pi \sum_{l<j} n_j} \\ &= -c_l^\dagger e^{i\pi \sum_{l<j} n_l} + c_l^\dagger e^{i\pi \sum_{l<j} n_j} = 0 \end{aligned}$$

So the j^{th} string operator anti-commutes with the site l creation operator for $l < j$. An identical argument holds for the site l annihilation operator. For $l \geq j$, we can evaluate the commutator:

$$[e^{i\phi_j}, c_l] = e^{i\pi \sum_{l<j} n_j} c_j^\dagger - c_j^\dagger e^{i\pi \sum_{l<j} n_j}$$

Since $l \geq j$, we can commute $c_j^\dagger$ to the front on the first term:

$$\implies e^{i\pi \sum_{l<j} n_j} c_j^\dagger - c_j^\dagger e^{i\pi \sum_{l<j} n_j} = c_j^\dagger e^{i\pi \sum_{l<j} n_j} - c_j^\dagger e^{i\pi \sum_{l<j} n_j} = 0$$

So in total, we have:

$$\begin{aligned} \{e^{i\phi_j}, c_l^\dagger\} &= 0 && (l < j) \\ [e^{i\phi_j}, c_l^\dagger] &= 0 && (l \geq j) \end{aligned}$$

Again, identical statements hold for the annihilation operators on the l^{th} site.

We can now define the Jordan Wigner transformation:

$$\begin{aligned} \sigma_j^+ &= c_j^\dagger e^{i\phi_j} \\ \sigma_j^- &= c_j e^{-i\phi_j} \end{aligned}$$

Computing the spin commutators:

$$[\sigma_j^+, \sigma_k^+] = [c_j^{(\dagger)} e^{i\phi_j}, c_k^\dagger e^{i\phi_k}] = c_j^\dagger e^{i\phi_j} c_k^\dagger e^{i\phi_k} - c_k^\dagger e^{i\phi_k} c_j^\dagger e^{i\phi_j}$$

Without loss of generality, we can assume that $j < k$. Looking at the second term in the sum:

$$\begin{aligned} c_k^\dagger e^{i\phi_k} c_j^\dagger e^{i\phi_j} &= -c_k^\dagger e^{i\phi_j} c_j^\dagger e^{i\phi_j} (\text{since } j < k,\ e^{i\phi_k} \text{anti-commutes with } c_j \text{ while } e^{i\phi_j} \text{ commutes with both}) \\ &= -c_k^\dagger c_j^\dagger e^{i\phi_j} e^{i\phi_k} = c_j^\dagger c_k^\dagger e^{i\phi_j} e^{i\phi_k} = c_j^\dagger e^{i\phi_j} c_k^\dagger e^{i\phi_k} (\text{since } j < k,\ c_k^\dagger \text{ commutes with } e^{i\phi_j}) \end{aligned}$$

Thus we see that $[\sigma_j^+, \sigma_k^+] = 0$. Identical argument can be made for the rest of the off-site ($j \neq k$) commutators. As for the commutator $[\sigma_j^+, \sigma_j^-]$, it is easy to show that the result is $2c_j^\dagger c_j - 1$. Adding to the Jordan-Wigner transformation $\sigma_j^z = c_j^\dagger c_j - \frac{1}{2}$, we have that $[\sigma_j^+, \sigma_j^-] = 2\sigma_j^z$, as expected up to a factor of 2. (find out where missing factor of 2 goes)