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Introduction

Einstein's Equations¹

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}$$

Einstein's equations describe how matter and energy curve space time. The left-hand side is the Einstein tensor which describes curvature, and the right-hand side is the stress-energy tensor, which describes matter/energy.

"Spacetime tells matter how to move; matter tells spacetime how to curve."

The Weak Energy Condition²

The weak energy condition is a condition imposed upon the stress-energy tensor to ensure that negative energy density cannot exist.

$$T_{ab}v^a v^b \geq 0$$

If this condition is violated, Einstein's equations can produce "exotic spacetimes", including closed timelike curves, which can lead to structures like warp drives.

Negative Energy Density²

Classically, it is impossible to have negative energy density.

In quantum mechanics, it is possible to introduce negative energy locally, which can be accomplished through Quantum Energy Teleportation (QET).

The aim of this project is to optimize the negative energy density produced by QET applied to a lattice of quantum harmonic oscillators.

Previous Work⁵

QET has been applied in the context of the light-matter interaction. It was used to generate negative energy densities on quantum field states.

The **quantum interest conjecture**³ (QEC) places restrictions on the amount of negative energy density that can be produced in a given quantum system.

The **optimal QET protocol** found in (5) saturated the restrictions imposed by the QEC. This project aims to replicate these results in a different context.

Setup: A Lattice of Oscillators

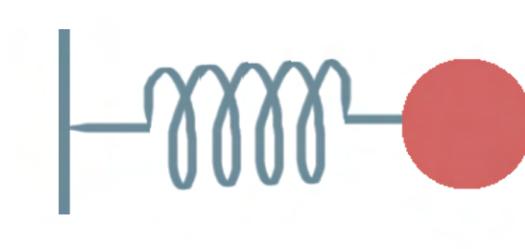
Why use a lattice of oscillators?

Quantization of classical space time gives a quantum field. Generating negative energy densities on spacetime using QET then amounts to applying QET on a quantum field. However, quantum fields are almost always perturbatively manipulated; exact solutions are difficult and often impossible to compute.

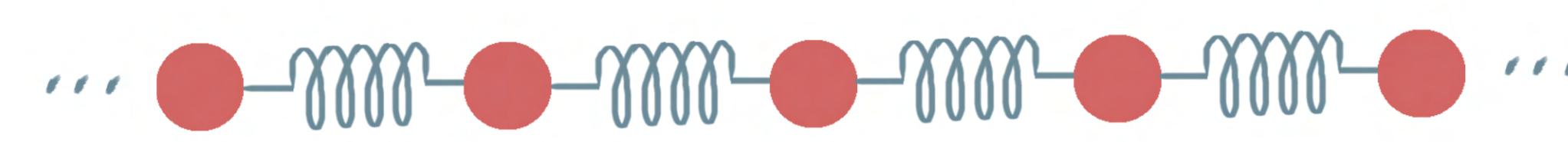
A quantum field is described as the continuum limit of a finite lattice of quantum harmonic oscillators (QHOs/oscillators). Although quantum fields are difficult to work with non-perturbatively, it is possible to use quantum mechanics to work with finite lattices of QHOs exactly. Applying QET on a finite lattice of QHOs can give us insight on the case with quantum fields, with the added benefit of non-perturbative solutions.

What is a lattice of oscillators?

A QHO is the solution to Schrodinger's equation for a particle in a quadratic potential. It is the quantum analog of a mass on a spring.



In this project, the system of interest is a one-dimensional lattice of these oscillators, each with some coupling between them. This is the quantum analog of several springs attached to each other.



The Hamiltonian for this system is:

$$(1) \quad H = \frac{\omega}{2} \sum_i (p_i^2 + q_i^2) + \lambda \sum_{i,j, i \neq j} q_i q_j$$

Here, the q_i and p_i represent the position and momenta of each oscillator respectively, ω represents the oscillating frequency, and λ is the coupling strength between each oscillator. We assume periodic boundary conditions, i.e. the first oscillator is coupled to the last.

How to represent a lattice of oscillators?⁶

We define the vector ξ as the vector of containing the canonical variables.

$$\xi = (q_1, q_2, \dots, q_N, p_1, p_2, \dots, p_N)$$

In the phase space picture of quantum mechanics, each state is represented with a Wigner function that is a function of ξ . For a system that has a Hamiltonian that is quadratic in all of its canonical variables, the Wigner function is Gaussian. Since the Hamiltonian (1) is quadratic in q_i and p_i , its eigenstates can be represented with a Gaussian Wigner function.

$$W(\xi) = \frac{1}{(h\pi)^n \sqrt{\det(\sigma)}} \exp(-(\xi - \xi_0)^\mu (\xi - \xi_0)^\nu (\sigma^{-1})_{\mu\nu})$$

Here, ξ_0 is the vector of means and σ is the covariance matrix.

$$\xi_0 = (\langle q_1 \rangle, \langle q_2 \rangle, \dots, \langle q_N \rangle, \langle p_1 \rangle, \langle p_2 \rangle, \dots, \langle p_N \rangle)$$

$$\sigma^{\mu\nu} := \langle \xi^\mu \xi^\nu + \xi^\nu \xi^\mu \rangle - 2 \langle \xi^\mu \rangle \langle \xi^\nu \rangle$$

For example, if there was just one oscillator, ξ_0 and σ would take the form:

$$\xi_0 = \begin{pmatrix} \langle q \rangle \\ \langle p \rangle \end{pmatrix} \quad \sigma = \begin{pmatrix} 2\langle q^2 \rangle - 2\langle q \rangle^2 & \langle qp + pq \rangle - 2\langle qp \rangle \\ \langle qp + pq \rangle - 2\langle qp \rangle & 2\langle p^2 \rangle - 2\langle p \rangle^2 \end{pmatrix}$$

Applying the QET Protocol⁵

1) Finding the Ground State

The protocol starts with a lattice of N oscillators in the ground state of the Hamiltonian (1).

The vector of means of the ground state can be set to the 0 vector. The covariance matrix of a lattice of uncoupled oscillators, σ_0 is known and can be directly computed from the density matrix.

Williamson's Theorem states that any real positive semi-definite matrix can be symplectically diagonalized.⁶

$$\sigma_D = S \sigma S^T$$

The covariance matrix in the original basis is found by undiagonalizing the diagonal covariance matrix using the inverse symplectic transform.

$$\sigma = S^{-1} \sigma_D (S^T)^{-1}$$

2) Measuring Site A

Site A and Site B on the lattice are at least two oscillators apart. Site A is measured through a detector D coupled to A via a SWAP operation.



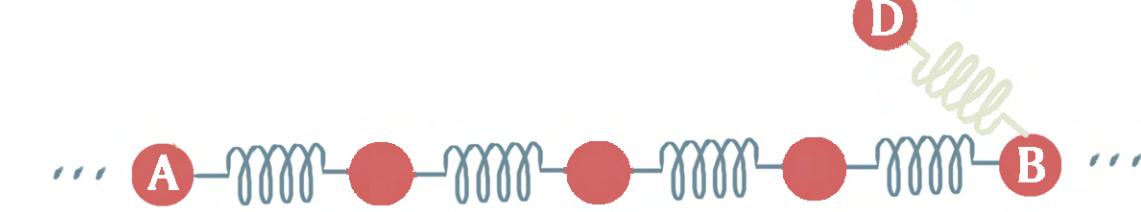
The resulting covariance matrix can be computed by applying the SWAP operation to the ground state covariance matrix.

$$S_{SW} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\sigma_A = S_{SW} \sigma S_{SW}^T$$

3) Unitary at Site B

The measurement of A is relayed to B through D.



This is done through the most general quadratic delta coupling between B and D given by the Hamiltonian:

$$H_{BD} = \lambda_{qq} q_B q_D + \lambda_{qp} q_B p_D + \lambda_{pq} p_B q_D + \lambda_{pp} p_B p_D$$

Where the coupling strengths of each interaction in the Hamiltonian are varied in order to minimize the energy at B prior to the coupling.

Optimizing the Energy at B

Computing Expectation Values of Energy Density

After the protocol is performed, the final covariance matrix is used to compute the energy density at B.

The expectation value for any quadratic observable can be directly computed from the covariance matrix using the formula below:⁶

$$\langle \xi^\mu \xi^\nu \rangle = \frac{1}{4} (\sigma^{\mu\nu} + \sigma^{\nu\mu})$$

The energy density at B is:

$$H_B = \frac{\omega}{2} (q_B^2 + p_B^2) + \frac{\lambda}{2} (q_B q_{B+1} + q_B q_{B-1})$$

The expectation value of which is easily calculated in terms of the elements of the covariance matrix.

Numerical vs. Analytical Optimizations

The energy at B can be numerically optimized over the different coupling parameters to be minimum. The plots showcase the numerically optimized minimum energy at B as a function of the lattice coupling λ overlayed with the leading order analytical approximation.

The two coincide for λ close to λ_{max} .

Leading Order Energy at B for N=6

The protocol is applied to a lattice of N=6 oscillators. When the inter-lattice coupling λ approaches a maximum value, the expression for the final energy at B is dominated by terms that go as:

$$\sim \sqrt{\frac{1}{\lambda_{max} - \lambda}}$$

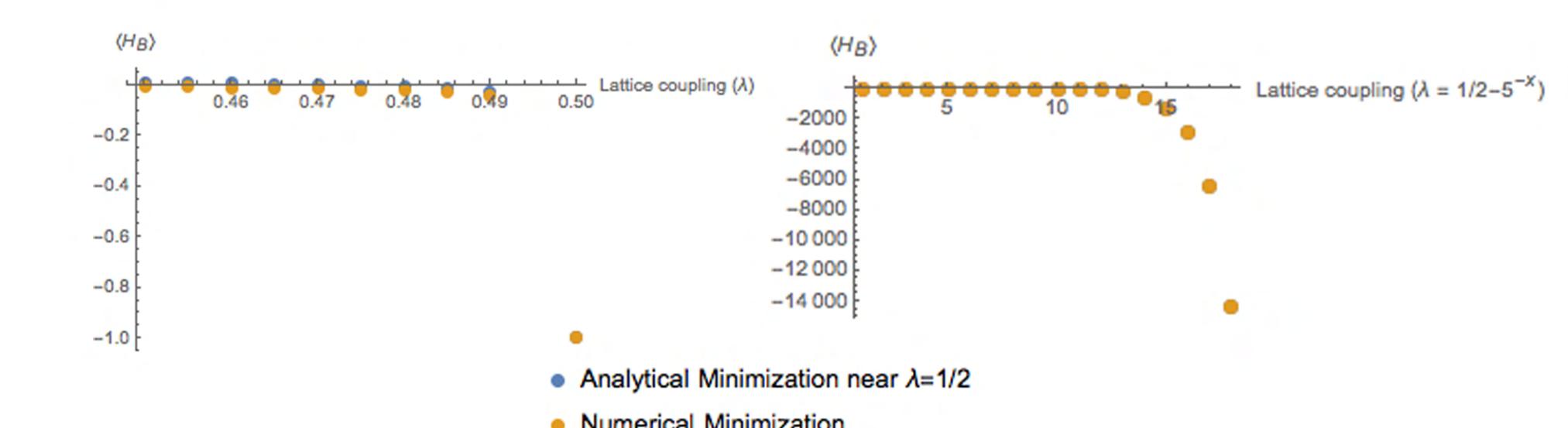
The expression for the final energy at B as a function of the arbitrary coupling parameters for λ close to $\lambda_{max} = 1/2$ is:

$$\langle H_B \rangle = \frac{1}{12\sqrt{2}} \sqrt{\frac{1}{\frac{1}{2} - \lambda}} \left(-\frac{1}{2} + \lambda - \lambda \cosh(\kappa) + \frac{1}{2} \cosh^2(\kappa) \right)$$

where κ is a function of the quadrature coupling.

$$\kappa = \sqrt{\lambda_{qp} \lambda_{pq} - \lambda_{qq} \lambda_{pp}}$$

The energy is most negative when $\kappa = i \frac{\pi}{3}$.



Conclusions

- Einstein's equations with negative energy densities can lead to exotic spacetimes (closed timelike curves, warp drives).
- QET is used to generate negative energy densities.
- Optimal QET applied to quantum field states has been shown to saturate the restrictions imposed by the quantum interest conjecture.
- QET was optimized non-perturbatively for a 1+1D finite lattice of quantum harmonic oscillators using Gaussian quantum mechanics.

Future Work

- Prove that the optimal negative energy density found obeys the quantum interest conjecture.
- Find the optimal energy for a lattice of arbitrary length.
- Implement the same process for higher dimensional lattices.
- Use a Gaussian projection for the measurement process.

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