LINEAR ALGEBRA HOMEWORK 1

LECTURER: BONG H. LIAN

This homework offers warm-up exercises on sets: you will work with various set notions like memberships, subsets, intersections, unions, disjointness, and partitions. You will also work with maps of various kinds: injective, surjective, and bijective.

Exercise 0.1. *WRITE UP This exercise will show that 'fractions' is a way to break up a particular set $\mathcal{P}$ into smaller sets – namely fractions. Two fractions will either be disjoint or the same. For this reason, we say that the fractions form a* **partition** *of $\mathcal{P}$.*
Recall that for $a, b \in \mathbb{Z}$, $a \neq 0$, the fraction b/a is defined to be the set $\mathcal{P}$ of all pairs

$$(n, m), \quad n, m \in \mathbb{Z}, \quad n \neq 0, \text{ such that } am = bn.$$

(Recall that you can also think of this set as the set of all equations $m = nx$, $n, m \in \mathbb{Z}$, $n \neq 0$, s.t. $am = bn$, which obviously contains the same information as the pairs above.)

(1) *Prove that two such sets $b/a = b'/a'$ iff $ab' = ba'$.*
(2) *Prove that $b/a \cap b'/a' = \emptyset$ (i.e. the two sets have no members in common) iff $ab' \neq ba'$.*

(3) *Prove that the map $\iota : \mathbb{Z} \to \mathbb{Q}$, $n \mapsto n/1$ is injective. Therefore, we can treat $\mathbb{Z}$ as a subset of $\mathbb{Q}$ by treating set $n/1$ as the integer n.*
(4) *Prove that this map is not surjective, i.e. there is a fraction $b/a \in \mathbb{Q}$ which is not $\iota(c)$ for any $c \in \mathbb{Z}$.*

Exercise 0.2. *Verify that the multiplication rule given by*

$$\times : \mathbb{Q}^2 \to \mathbb{Q}, \ (b/a, b'/a') \mapsto b/a \times b'/a' := (bb')/(aa')$$

is well-defined. Note that this rule generalizes the usual rule for multiplying integers, i.e. grouping apples. Therefore, the multiplication rule for $\mathbb{Z}$ remains the same after we treat it as a subset of $\mathbb{Q}$. (See next exercise.)

(1) *Prove using this multiplication rule, that b/a is a solution to the equation $ax = b$. That is, $a \times b/a = b$.)*
(2) *Prove that this is the only solution. That is, if b'/a' is another solution, then $b'/a' = b/a$.*

Exercise 0.3. *Write For $n \in \mathbb{Z}$, put $\iota(n) = n/1$. Verify the identities*

$$\iota(1) = 1/1, \quad \iota(nn') = \iota(n)\iota(n'), \quad n, n' \in \mathbb{Z}.$$

Also express $\iota(n + n')$ in terms of $\iota(n)$, $\iota(n')$.

Exercise 0.4. *Let F be a field. For $X, Y, Z \in F^2$ and $\lambda, \mu \in F$, verify that*
V1. $(X+Y)+Z = X+(Y+Z)$
V2. $X+Y = Y+X$
V3. $X+0 = X$
V4. $X+(-X) = 0$
V5. $\lambda(X+Y) = \lambda X + \lambda Y$
V6. $(\lambda+\mu)X = \lambda X + \mu X$
V7. $(\lambda\mu)X = \lambda(\mu X)$
V8. $1X = X$.
The same holds true for F^n.

Suggestion: *Think about exactly what facts about F you need to use to prove each of these statements.*

Exercise 0.5. *WRITE UP Recall that for a given field F, we have a* **characteristic map** *defined by*

$$\iota_F : \mathbb{Z} \to F, \quad n \mapsto n \cdot 1_F.$$

We say that F has characteristics p if there exists a smallest positive integer p such that $\iota_F(p) = 0_F$. If such a p does not exists, we say that F has characteristics 0.

- *(1) What is the characteristics of the field $\mathbb{Q}$? Prove your answer.*
- *(2) Prove that if F is finite then p exists and it must be a prime number.*
- *(3) Fix a prime number p. For each integer, put*
  $$\bar{a} = a + p\mathbb{Z} := \{a + pn | n \in \mathbb{Z}\} = \{..., a-p, a, a+p, a+2p, ...\}.$$
  Define the set
  $$\mathbb{Z}/p := \{a + p\mathbb{Z} | a \in \mathbb{Z}\}.$$
  Consider the map
  $$f_p : [p] := \{0, 1, ..., p-1\} \to \mathbb{Z}/p, \quad a \mapsto \bar{a} := a + p\mathbb{Z}.$$
  Show that f_p is a bijection, hence $\#\mathbb{Z}/p = p$.
- *(4) Show that the set $\mathbb{Z}/p$ can be made a field with distinguished members $\bar{0}, \bar{1}$, by giving it 4 operations $+, \times, -, 1/\cdot$. Therefore, for every prime number p, you have constructed a* **finite field** *$\mathbb{F}_p$ with p members.*

LINEAR ALGEBRA HOMEWORK 2

LECTURER: BONG H. LIAN

While you should always try all assigned problems, you should write up only the ones you are asked to write up.

Assume U, V, W are F-vector spaces.

Exercise 0.1. *If $f : U \to V$ and $g : V \to W$ are linear maps, verify that their composition $gf \equiv g \circ f : U \to W$ is also linear.*

Exercise 0.2. *Let $\mathrm{Iso}(V, V)$ be the set of isomorphisms (i.e. linear bijections) $\phi : V \to V$, and $\mathrm{Iso}(V, W)$ the set of isomorphisms $f : V \to W$. Suppose $f_0 \in \mathrm{Iso}(V, W)$. Show that the map*

$$T : \mathrm{Iso}(V, V) \to \mathrm{Iso}(V, W), \;\; \phi \mapsto f_0 \circ \phi$$

bijects, by writing the inverse map.

Exercise 0.3. *Describe sol(E) to the following system E in $\mathbb{R}^4$, using row reduction and then giving an isomorphism $f : \mathbb{R}^\ell \to \ker L_A$ (including specifying the appropriate ℓ), where A is the coefficient matrix of the system:*

$$E_0 : \qquad \begin{aligned} & x + y + z + t = 0 \\ & x + y + 2z + 2t = 0 \\ & x + y + 2z - t = 0. \end{aligned}$$

Replace the 0 on the right side of first equation by 1 to get a new inhomogeneous system E_1, and then describe its solution set sol(E_1) by writing down an explicit translation map.

Exercise 0.4. *WRITE UP* Let *E be an n-variable F-linear system. Prove that*

$$\begin{aligned} & E \textit{ is homogenous} \\ \Leftrightarrow \;\; & 0 \in sol(E) \\ \Leftrightarrow \;\; & sol(E) \textit{ is } F\textit{-subspace of } F^n. \end{aligned}$$

Try to make your proof as simple as possible, say less than half a page.

Exercise 0.5. *WRITE UP Let $F[x]_d$ be the F-subspace of $F[x]$ consisting of all polynomials $p(x)$ of degree at most d, i.e. the highest power*

x^n appearing in $p(x)$ is at most x^d. Consider the map

$$L_{n,d} := (1-x^2)(\frac{d}{dx})^2 - 2x\frac{d}{dx} + n(n+1)id : F[x]_d \to F[x]_d$$

for integer $n \geq 0$. Verify that $L_{n,d}$ is F-linear. Describe $\ker L_{n,d}$ *by solving the linear equation*

$$L_{n,d}(f) = 0$$

for $n = 0, 1, 2$, say by giving a basis of $\ker L_{n,d}$*. Equivalently, find $k \in \mathbb{Z}_{\geq 0}$ (which can depend on n, d) such that you can construct an F-isomorphism*

$$f : F^k \to \ker L_{n,d}.$$

LINEAR ALGEBRA HOMEWORK 3

LECTURER: BONG H. LIAN

F denotes a field. Assume U, V, W are F-vector spaces, and all dimensions are F-dimensions.

Exercise 0.1. *Let V be a F-subspace of F^n. Decide whether each of the following is TRUE of FALSE. Justify your answer. For (a)-(e), assume that* $\dim V = 3$.
(a) Any 4-tuple of V is linearly dependent.
(b) Any 2-tuple of V is linearly independent.
(c) Any 3-tuple of V is a basis.
(d) Some 3-tuple of V is a basis.
(e) V contains a linear subspace W with $\dim\ W = 2$.
(f) $(1, \pi)$, $(\pi, 1)$ *form a basis of* $\mathbb{R}^2$. *You can assume that* $|\pi - 3.14| < 0.01$.
(g) $(1, 0, 0)$, $(0, 1, 0)$ *do not form a basis of the plane* $x - y - z = 0$.
(h) $(1, 1, 0)$, $(1, 0, 1)$ *form a basis of the plane* $x - y - z = 0$.
(i) If A is a 3×4 matrix, then the subspace V of F^4 generated by the rows of A is at most 3 dimensional.
(j) If A is a 4×3 matrix, then the subspace V of F^3 generated by the rows of A is at most 3 dimensional.

Exercise 0.2. WRITE UP *Let*

$$V = \{(a + b, a, c, b + c) | a, b, c \in F\} \subset F^4.$$

Verify that V is an F-subspace of F^4. Find a basis of V.

Exercise 0.3. *Fix $0 < k < n$ and consider the decomposition*

$$F^n \equiv F^k \oplus F^{n-k}, \quad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \equiv \begin{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix} \\ \begin{bmatrix} x_{k+1} \\ \vdots \\ x_n \end{bmatrix} \end{bmatrix} = \begin{bmatrix} u_k \\ \ell_{n-k} \end{bmatrix}.$$

Show if $A \in M_{n,n}$, then A also 'decomposes' into a corresponding block form

$$A \equiv \begin{bmatrix} P_{k,k} & Q_{k,n-k} \\ R_{n-k,k} & S_{n-k,n-k} \end{bmatrix}$$

so that the column vector Ax can be expressed in terms of the column vectors $Pu, Q\ell, Ru, S\ell$. If you are confused, do the special case $n = 3, k = 1$ first.

Exercise 0.4. WRITE UP *In 1 line, prove that every matrix $A \in M_{n,n}$ satisfies a nontrivial polynomial equation of the form*

$$a_0 I_n + a_1 A + \cdots + a_k A^k = 0.$$

Exercise 0.5. *Find a basis of sol*(E) *in* F^4 *for*

$$E: \quad x - y + 2z + t = 0.$$

Exercise 0.6. *Find a basis for each of the subspaces* $\ker L_A$ *and* $\operatorname{im} L_A$ *of* F^4*, where* A *is the matrix*

$$\begin{bmatrix} -2 & -3 & 4 & 1 \\ 0 & -2 & 4 & 2 \\ 1 & 0 & 1 & 1 \\ 3 & 4 & -5 & -1 \end{bmatrix}.$$

Exercise 0.7. *We know that* $V^2 = V \times V$ *form a vector space. Define an* F*-vector space structure on* $U \times V$ *a vector space. Let's call it the* **direct sum** $U \oplus V$ *of* U, V*. If* $\dim U = k$ *and* $\dim V = n$*, what is* $\dim(U \oplus V)$*? Prove your assertion in 5 lines.*

Exercise 0.8. *(Revisit MMC) We specialize to the case* $V = F^2$*. Let* $(v_1, v_2) \in V^2$*, put* $A = [v_1, v_2] \in M_{2,2}$*, and write* $v_1 = \begin{bmatrix} a_{11} \\ a_{12} \end{bmatrix}$*,* $v_2 = \begin{bmatrix} a_{21} \\ a_{22} \end{bmatrix}$*.*
(a) (A numerical test for isomorphism) Show that v_1, v_2 *are 'parallel', i.e. one a scalar multiple of the other, iff they are dependent, iff*

$$a_{11}a_{22} - a_{12}a_{21} = 0$$

iff L_A *is not an isomorphism, iff* (v_1, v_2) *is not a basis of* V*.*
(b) Now suppose L_A *is an isomorphism. Can you find the matrix* B *corresponding to (under MMC) to the inverse isomorphism* $L_A^{-1} : F^2 \to F^2$*?*

LINEAR ALGEBRA HOMEWORK 4

LECTURER: BONG H. LIAN

Assume U, V, W are F-vector spaces. Put $\operatorname{End} V = \operatorname{Hom}(V, V)$.

Exercise 0.1. *Find the dimension of $M_{2,2}$ by giving a basis of this vector space. Generalize your result to $M_{k,l}$.*

Exercise 0.2. *Let $f, g : V \to V$ be two given maps such that $f \circ g = id_V$.*
(a) Show that g is injective and f is surjective.
(b) Assume in addition that $\dim V < +\infty$ and f is linear. Show that f is injective, hence g is surjective. (Hint: Use COD.)
(c) Conclude that g is bijective, and that $f = g^{-1}$ and $g \circ f = id_V$.
(d) Let $A, B \in M_{n,n}$. Show that if $AB = I$, then $BA = I$.

Exercise 0.3. *Another proof. Show that if $\ker(BA) = (0)$ then $\ker A = (0)$, hence A is an isomorphism. Conclude that $B = A^{-1}$. (Hint: COD.)*

Exercise 0.4. WRITE UP *Prove that for $A \in M_{n,n}$, $\det A^t = \det A$. You will need the fact that $\operatorname{sgn} \sigma^{-1} = \operatorname{sgn} \sigma$ for any bijection of $\{1, 2, .., n\}$.*

Exercise 0.5. *Decide if $A = [e_3, e_1 + e_2, e_2] \in M_{3,3}$ is invertible. If so, compute A^{-1}. Here e_i are the standard unit vectors if F^3.*

Exercise 0.6. WRITE UP *Let $U \subset V$ be a subspace and $x \in \operatorname{End} V$ such that $xU \subset U$. In 5 lines, prove that there is a canonical map*

$$\bar{x} : V/U \to V/U, \quad v + U \mapsto xv + U.$$

That is check that this is well-defined. Show it satisfies the following: if $p(t) \in F[t]$, and $p(x) = 0$ in $\operatorname{End} V$ then $p(\bar{x}) = 0$ in $\operatorname{End} V/U$.

Exercise 0.7. *By row reduction, compute*

$$\det[e_3 + e_2 + e_1, e_1 + e_2, e_2].$$

Redo this it by using linearity of $\det$ in each column.

Exercise 0.8. *Assume that $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in M_{2\times 2}$ is invertible. Find a formula for A^{-1}. That is to say, find each entry of A^{-1} in terms of the 4 entries a_{ij} of A. Be sure to check that you do get $AA^{-1} = A^{-1}A = I$. From this, can you guess the answer for 3×3 matrices.*

Exercise 0.9. WRITE UP *Prove that the minimal polynomial of a matrix $A \in M_{n,n}$ is conjugation invariant, i.e. $\mu_{g^{-1}Ag}(x) = \mu_A(x)$ for all $g \in \operatorname{Aut}_n$. Conclude that the algebra $F[x]/\mu_A(x)F[x]$ does not change under conjugations of A.*

Exercise 0.10. *Compute the* $\mu_A(x)$ *for* $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$.

LINEAR ALGEBRA HOMEWORK 5

LECTURER: BONG H. LIAN

Exercise 0.1. *WRITE UP Find all conjugacy classes of solutions to the matrix equation*

$$X^3 = 0$$

in $M_3(\mathbb{C})$*.*

Exercise 0.2. *Work out the structure of the* $\mathbb{Z}$*-module* $M = \mathbb{Z}^3/A\mathbb{Z}^2$ *where*

$$A = \begin{bmatrix} -1 & 2 \\ 0 & 1 \\ 3 & 4 \end{bmatrix}.$$

Namely determine its free part and torsion part of M*.*

Exercise 0.3. *WRITE UP Take* $R = \mathbb{C}[t]$*, the polynomial algebra with complex coefficientgs. Work out the structure of the* R*-module* $M = R^3/AR^2$ *where*

$$A = \begin{bmatrix} t^2 & 2t \\ 0 & t \\ t^3 - 1 & t - 2 \end{bmatrix}.$$

Namely determine its free part and torsion part of M*.*

Exercise 0.4. *WRITE UP Show that there is a one to one correspondence between the isomorphism classes of* n*-dimensional* $F[t]$*-modules and the conjugacy classes of* $n \times n$ *matrices.*

LINEAR ALGEBRA HOMEWORK 5

LECTURER: BONG H. LIAN

Exercise 0.1. *Let* $x, y \in M_{n,n}$. *Recall that* x, y *are conjugates of each other iff there exists an invertible matrix* g *such that* $y = g^{-1}xg$. *Prove your assertions.*
(a) Suppose $\det x \neq \det y$. *Can* x, y *be translates of each other, i.e. can* $[x] = [y]$*?*
(b) Suppose $\det x = \det y$. *Does this imply that* $[x] = [y]$*?*

Exercise 0.2. *Prove that if* $x \in M_{n,n}$ *has characteristic polynomial* $p_x(t)$ *which has* n *distinct roots, then* x *is diagonalizable.*

Exercise 0.3. *WRITE UP Let* $x_0 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, *and let* $[x_0]$ *be its conjugation class in* $M_{2,2}$. *Show that there is a surjective map*

$$\pi : [x_0] \to \mathbb{P}^1 := \text{the set of all lines in } F^2$$

given by $x \mapsto \ker x$. *Here, a line in* F^2 *is a one dimensional subspace of* F^2. *Can you describe the subset*

$$\pi^{-1}(\ker x) = \{y \in [x_0] | \ker y = \ker x\}$$

for each x*? Prove your assertions.*

Exercise 0.4. *WRITE UP Let* A *be an* F*-algebra and* V *be a finite dimensional* A*-space. Show that* V *is a quotient* A*-space of a direct sum* $A^{\oplus k}$ *of* k *copies of* A*, regarded as an* A*-space. In other words, there exists a surjective* A*-space homomorphism*

$$\pi : A^{\oplus k} \twoheadrightarrow V.$$

We say that an A*-space* M *is semi-minimal if it decomposes into a independent sum of* A*-subspaces which are minimal. Show that if* A *is semi-minimal as an* A*-space, then any* A*-space* V *is semi-minimal.*

2021 MATHCAMP RESEARCH PROJECTS: LINEAR ALGEBRA

LECTURER: BONG H. LIAN

This is a preliminary version of the research projects, subject to updates later.

0. Basic Assumptions and Notations

Unless stated otherwise, we shall make the following assumptions and use the following notations. F will denote a field of characteristic zero (i.e. F contains $\mathbb{Z}$ as a subset). For simplicity, you can think about the case $F = \mathbb{R}$ or $\mathbb{C}$, the field of real or complex numbers. A vector space means a finite dimensional F-vector space, usually denoted by $U, V, W, \ldots$. Likewise a linear map means an F-linear, and an F-matrix means a matrix with entries in F. Put

$$\begin{aligned}
\mathrm{Hom}(U,V) &:= \text{the set of all linear maps } U \to V \\
\mathrm{End}\, V &:= \mathrm{Hom}(V,V), \ \text{ the algebra of linear maps } V \to V \\
\mathrm{Aut}\, V &:= \{f \in \mathrm{End}\, V | f \text{ is bijective}\} \\
\mathrm{Aut}\,_n F &:= \mathrm{Aut}\, F^n \quad \text{the set of all isomorphisms } F^n \to F^n. \\
(M_n, \times) \equiv M_n \equiv M_{n,n}(F) &:= \text{the associative algebra of } n \times n \ F\text{-matrices} \\
& \text{ with the usual matrix product} \\
I &\equiv I_n := [e_1, .., e_n], \ \text{ the identity matrix in } M_n
\end{aligned}$$

We usually denote composition of maps as $fg \equiv f \circ g$.

These objects will be quite thoroughly studied in class during the first two weeks.

1. Statements of Problems in Project 1

We say that two matrices $x, x' \in M_n(F)$ are **conjugate** if

$$\boxed{x' = gxg^{-1}}$$

for some $g \in \mathrm{Aut}_n F$.

Problem 1.1.

Solve the matrix equation

$$\boxed{x^2 = I}$$

in the 2×2 matrix algebra $M_2(F)$ up to conjugation. In other words, classify solutions to the equations up to conjugation by $\mathrm{Aut}_2 F$. Thus two solutions are considered equivalent if they are conjugate of each other. How would you describe a 'nice' matrix x that represent each equivalence class of solutions to the equation? This same notion of solving a matrix equation shall apply to the next two problems as well.

Do the same for the matrix equation

$$\boxed{x^2 = 0}$$

Problem 1.2.

Generalize Problem 1.1 by solving each of the equations

$$\boxed{x^k = I, \quad k = 2, 3, ...}$$

in the matrix algebra $M_n(F)$. Do the same for

$$\boxed{x^k = 0, \quad k = 2, 3, ...}$$

Can you say any thing more in these problems when F is assumed to be a finite field of prime characteristic p?

Problem 1.3.

For $F = \mathbb{C}$, solve the matrix equation

$$\boxed{\exp(x) = I}$$

in the 2×2 matrix algebra $M_2(F)$ up to conjugation. You may assume that $exp(x)$ is the limit (in the sense of calculus) of the sequences of matrices:

$$I, \quad I + x, \quad I + x + \frac{x^2}{2!}, \quad \cdots$$

with respective to the length function $||A|| = \max_{ij} |a_{ij}|$ on matrices.

Problem 1.4.

Generalize Problem 1.3 to the case of matrix algebra $M_n(F)$ for $F = \mathbb{C}$.

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$$
\begin{aligned}
\mathrm{Hom}(U,V) &:= \text{the set of all linear maps } U \to V \\
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\mathrm{Aut}_n F &:= \mathrm{Aut}\, F^n \;\; \text{the set of all isomorphisms } F^n \to F^n. \\
(M_n, \times) \equiv M_n \equiv M_{n,n}(F) &:= \text{the associative algebra of } n \times n \; F\text{-matrices} \\
& \text{ with the usual matrix product} \\
U_n &:= \{A = (a_{ij}) \in M_n \mid A \text{ is upper triangular, i.e., } a_{ij} = 0 \text{ for } i > j\} \\
I \equiv I_n &:= [e_1, \cdots, e_n] \text{ the identity matrix in } M_n
\end{aligned}
$$

These objects will be quite thoroughly studied in class during the first two weeks.

1. Statements of Problems in Project II

Problem 1.1. *Classify all F-algebra homomorphisms $M_n \to F[x]$.*

Problem 1.2. *Classify all F-algebra homomorphisms $U_n \to F[x]$.*

Recall the definition of rings and ring homomorphisms. Do the following problems.

Problem 1.3. *Classify all ring homomorphisms $M_n \to F[x]$.*

Problem 1.4. *Classify all ring homomorphisms $U_n \to F[x]$.*

Problem 1.5. *What can you say about these problems when F is a finite field of prime characteristic p?*