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On the generating fields of Kloosterman sums

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Let $f(x) \in \mathbb{F}_q[x]$ be a polynomial over a finite field with $q = p^d$ elements, where p is a rational prime. Define the exponential sum

$$S_1(f) := \sum_{x \in \mathbb{F}_q} \zeta_p^{\text{Tr}(f(x))} \in \mathbb{Z}[\zeta_p].$$

A basic problem is

- (1) as a complex number, $|S_1(f)| = ?$
- (2) as a p -adic number, $|S_1(f)|_p = ?$
- (3) as an algebraic number, $\deg S_1(f) = ?$

The first two questions have been studied extensively in the literature. Define

$$L(t, f) := \prod_{x \in \overline{\mathbb{F}}_p} \left(1 - \text{Tr}_{\mathbb{F}_{q^k}(x)/\mathbb{F}_p}(f(x))t^{\deg x}\right)^{-1} = \exp\left(\sum_k S_k(f) \frac{t^k}{k}\right)$$

where $S_k(f) := \sum_{x \in \mathbb{F}_{q^k}} \zeta_p^{\text{Tr}(f(x))} \in \mathbb{Z}[\zeta_p]$.

Theorem (Dwork-Bombieri-Grothendick)

$L(t, f)$ is a rational function.

Write

$$L(t, f) = \frac{\prod_j (1 - \beta_j t)}{\prod_i (1 - \alpha_i t)}.$$

Then

$$S_k(f) = \sum_i \alpha_i^k - \sum_j \beta_j^k.$$

How to estimate the characteristic roots α_i and β_j ? We need ℓ -adic method. To describe it, let's recall the definition of sheaves.

Given a topological space X , there is a site $\text{Top}(X)$ with

- (1) objects: the open subsets of X ;
- (2) morphisms: the injection of open sets;
- (3) coverings: normal open coverings.

A sheaf \mathcal{F} on a topological space X over a field E is a contravariant functor $\text{Top}(X)^{\text{op}} \rightarrow \text{Vect}/E$, which can be uniquely glued locally. That's to say, for any open covering $U = \cup_i U_i$,

$$\mathcal{F}(U) \rightarrow \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

is exact.

Let X be a scheme. Denote by $X_{\text{ét}}$ the site with

- (1) objects: étale scheme $X' \rightarrow X$;
- (2) morphisms: étale morphisms;
- (3) coverings: $\{\varphi_i : X'_i \rightarrow X'\}$ with $X' = \cup \varphi_i(X'_i)$.

Fix a prime $\ell \neq p$ and let E be a finite extension of \mathbb{Q}_ℓ . An ℓ -adic sheaf is a sheaf on $X_{\text{ét}}$ over E (which is constructible at every finite level).

Let K be c.d.v.f, with higher ramification groups $I^{(r)}, r \geq 0$. For any E -representation M of P , we have a decomposition $M = \oplus M(x)$, such that

$$M(0) = M^P, \quad M(x)^{I^{(x)}} = 0, \quad M(x)^{I^{(y)}} = M(x), \quad y > x > 0.$$

We call x a break if $M(x) \neq 0$. Define

$$\text{Sw}(M) = \sum x \dim M(x).$$

Let C be a proper smooth geometrically connected curve over a perfect field \mathbb{F} , with function field $K = \mathbb{F}(C)$. For any closed point $x \in C(\mathbb{F})$, we have the completion K_x .

For any non-empty open $U \subset C$, we have an equivalence of abelian categories

$$\begin{aligned} \{\text{lisse } E\text{-sheaves on } U\} &\longrightarrow \text{Rep}_E^c \pi_1(U, \bar{\eta}) \\ \mathcal{F} &\longmapsto \mathcal{F}_{\bar{\eta}}. \end{aligned}$$

Since $\pi_1(U, \bar{\eta})$ is a quotient of $\text{Gal}(\bar{K}/K)$, the decomposition group $D_x \subset \text{Gal}(\bar{K}/K)$ acts on $\mathcal{F}_{\bar{\eta}}$. We can define Swan conductor of \mathcal{F} at x . If $x \in U$, the action of I_x is trivial.

We will take $\mathbb{F} = \mathbb{F}_p$, $C = \mathbb{P}^1$ and $U = \mathbb{G}_m$.

Assume that $\mu_p \subseteq E$. Deligne constructed a certain locally free of rank one ℓ -adic sheaf $\mathcal{F}_\ell(f)$ over E on $\mathbb{G}_{a, \overline{\mathbb{F}}_p} = \text{Spec } \overline{\mathbb{F}}_p[X]$, such that

$$L(t, f) = \prod_i \det(1 - t\text{Frob}, H_c^i)^{(-1)^{i+1}}$$

and

$$S_k(f) = \sum_i (-1)^i \text{Tr}(\text{Frob}^k, H_c^i).$$

Here, Frob is the geometric Frobenius (inverse of $\alpha \mapsto \alpha^p$), $H_c^i = H_c^i(\mathbb{G}_{a, \overline{\mathbb{F}}_p}, \mathcal{F}_\ell(f))$ is the compact cohomology.

Denote by ω_{ij} the eigenvalues of Frob on H_c^i , then

$$S_k(f) = \sum_{ij} (-1)^i \omega_{ij}^k.$$

Denote by $B_i = \dim_E H_c^i$ the Betti number.

Theorem (Deligne)

ω_{ij} is an algebraic integer and all its conjugates over \mathbb{Q} has same absolute value $q^{r_{ij}/2}$, where the weight $0 \leq r_{ij} \leq i$ are integers.

Thus

$$|S_k| \leq \sum_i B_i q^{ki/2}.$$

General case

In general,

- (1) V a closed variety over \mathbb{F}_q of \mathbb{A}^N ,
- (2) ψ a non-trivial additive character on \mathbb{F}_q , $\psi_k = \psi \circ \text{Tr}_{\mathbb{F}_{q^k}/\mathbb{F}_q}$,
- (3) f a regular function on V defined over \mathbb{F}_q ,
- (4) χ a multiplicative character on \mathbb{F}_q^\times , $\chi_k = \chi \circ N_{\mathbb{F}_{q^n}/\mathbb{F}_q}$,
- (5) g an invertible regular function on V .

Define

$$S_k = \sum_{x \in V(\mathbb{F}_{q^k})} \psi_k(f(x)) \chi_k(g(x)).$$

Then Deligne's results still hold in this case. Moreover, Bombieri proved that the number of characteristic roots is at most

$$(4 \max \{ \deg V + 1, \deg f \} + 5)^{2N+1}.$$

Now we will consider

$$V = V(X_1 \cdots X_n - a), \quad f = X_1 + \cdots + X_n.$$

Let $\chi = \{\chi_1, \dots, \chi_n\}$ be an unordered n -tuple of multiplicative characters $\chi_i : \mathbb{F}_q^\times \rightarrow \mu_{q-1}$. Define the Kloosterman sum as

$$\mathrm{Kl}_n(\psi, \chi, q, a) = \sum_{\substack{x_1 \cdots x_n = a \\ x_i \in \mathbb{F}_q}} \chi_1(x_1) \cdots \chi_n(x_n) \psi(\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(x_1 + \cdots + x_n)).$$

In this case, there are n characteristic roots with same weight $n - 1$. Hence $|\mathrm{Kl}_n| \leq nq^{(n-1)/2}$.

Clearly, $\text{Kl}_n \in \mathbb{Z}[\mu_{pc}]$, where

$$c = \text{lcm}_i \{\text{ord}(\chi_i)\}$$

divides $q - 1$. Write

$$\text{Gal}(\mathbb{Q}(\mu_{pc})/\mathbb{Q}) = \{\sigma_t \tau_w \mid t \in (\mathbb{Z}/p\mathbb{Z})^\times, w \in (\mathbb{Z}/c\mathbb{Z})^\times\},$$

where

$$\begin{aligned}\sigma_t(\zeta_p) &= \zeta_p^t, & \sigma_t(\zeta_c) &= \zeta_c, \\ \tau_w(\zeta_p) &= \zeta_p, & \tau_w(\zeta_c) &= \zeta_c^w.\end{aligned}$$

A basic observation tells

$$\sigma_t \tau_w \text{Kl}_n(\psi, \chi, q, a) = \prod \chi(t)^{-w} \text{Kl}_n(\psi, \chi^w, q, at^n).$$

To study the generating fields of Kl_n , we need to consider the distinctness of different Kloosterman sums.

When $\chi = \mathbf{1} = \{1, \dots, 1\}$ is trivial, it's easy to see that

$$a, b \text{ conjugate} \implies \mathrm{Kl}_n(\psi, \mathbf{1}, q, a) = \mathrm{Kl}_n(\psi, \mathbf{1}, q, b).$$

When $p > (2n^{2d} + 1)^2$ (Fisher), or $p \geq (d-1)n + 2$ and p does not divide a certain integer (Wan), this is necessary. In general, it's conjectured that it's true when $p \geq nd$. Thus

$$\deg \mathrm{Kl}_n(\psi, \mathbf{1}, q, a) = \frac{p-1}{(p-1, n)}$$

under these conditions.

For our purpose, we need a different sheaf. Deligne and Katz defined the Kloosterman sheaf

$$\mathcal{Kl} = \mathcal{Kl}_{n,q}(\psi, \chi)$$

on $\mathbb{G}_m \otimes \mathbb{F}_q = \operatorname{Spec} \mathbb{F}_q[X, X^{-1}]$, with the following properties:

- (1) \mathcal{Kl} is lisse (locally constant at every finite level) of rank n and pure of weight $n - 1$.
- (2) For any $a \in \mathbb{F}_q^\times$, $\operatorname{Tr}(\operatorname{Frob}_a, \mathcal{Kl}_{\bar{a}}) = (-1)^{n-1} \mathcal{Kl}_n(\psi, \chi, q, a)$.
- (3) \mathcal{Kl} is tame at 0 (Swan = 0).
- (4) \mathcal{Kl} is totally wild with Swan conductor 1 at ∞ . So all ∞ -breaks are $1/n$.

Fisher gave a descent of Kloosterman sheaves along an extension of finite fields. For any $a \in \mathbb{F}_q^\times$, he defined a lisse sheaf $\mathcal{F}_a(\chi)$ on $\mathbb{G}_m \otimes \mathbb{F}_p$, such that

$$\mathcal{F}_a(\chi)|_{\mathbb{G}_m \otimes \mathbb{F}_q} = \bigotimes_{\sigma \in \text{Gal}(\mathbb{F}_q/\mathbb{F}_p)} (t \mapsto \sigma(a)t^n)^* \text{Kl}_n(\psi \circ \sigma^{-1}, \chi \circ \sigma^{-1}).$$

- (1) $\mathcal{F}_a(\chi)$ is lisse of rank n^d and pure of weight $d(n-1)$.
- (2) For any $t \in \mathbb{F}_p^\times$, $\text{Tr}(\text{Frob}_t, \mathcal{F}_a(\chi)_{\bar{t}}) = (-1)^{(n-1)d} \text{Kl}_n(\psi, \chi, q, at^n)$.
- (3) $\mathcal{F}_a(\chi)$ is tame at 0 and its ∞ -breaks are at most 1.

Lemma

Let $\mathcal{F}, \mathcal{F}'$ be lisse sheaves on $\mathbb{G}_m \otimes \mathbb{F}_p$ of same rank r and pure of the same weight w . Assume that there is a root of unity λ such that for any $t \in \mathbb{F}_p^\times$, we have

$$\mathrm{Tr}(\mathrm{Frob}_t, \mathcal{F}_{\bar{t}}) = \lambda \mathrm{Tr}(\mathrm{Frob}_t, \mathcal{F}'_{\bar{t}}).$$

Let \mathcal{G} be a geometrically irreducible sheaf of rank s on $\mathbb{G}_m \otimes \mathbb{F}_p$, pure of weight w , such that $\mathcal{G} \mid \mathbb{G}_m \otimes \bar{\mathbb{F}}_p$ occurs exactly once in $\mathcal{F} \mid \mathbb{G}_m \otimes \bar{\mathbb{F}}_p$. Then $\mathcal{G} \mid \mathbb{G}_m \otimes \bar{\mathbb{F}}_p$ occurs at least once in $\mathcal{F}' \mid \mathbb{G}_m \otimes \bar{\mathbb{F}}_p$, provided that $p > [2rs(M_0 + M_\infty) + 1]^2$, where M_η is the largest η -break of $\mathcal{F} \oplus \mathcal{F}'$.

Assume not. Applying the Lefschetz Trace Formula to $\mathcal{G}^\vee \otimes \mathcal{F}$ and $\mathcal{G}^\vee \otimes \mathcal{F}'$, we have

$$\sum_{i=0}^2 (-1)^i \mathrm{Tr}(\mathrm{Frob}, H_c^i(\mathcal{G}^\vee \otimes \mathcal{F})) = \lambda \sum_{i=0}^2 (-1)^i \mathrm{Tr}(\mathrm{Frob}, H_c^i(\mathcal{G}^\vee \otimes \mathcal{F}')).$$

Apply Euler-Poincaré formula

$$\begin{aligned} & h_c^0(\mathcal{F}) - h_c^1(\mathcal{F}) + h_c^2(\mathcal{F}) \\ &= \mathrm{rank} \mathcal{F} \cdot \chi_c(\mathbb{G}_m \otimes \mathbb{F}_p) - \mathrm{Sw}_0(\mathcal{F}) - \mathrm{Sw}_\infty(\mathcal{F}) \end{aligned}$$

to estimate $\mathrm{Tr}(\mathrm{Frob}, H_c^1)$ (weight ≤ 1 by Weil II).

The n -tuple χ is called *Kummer-induced* if there exists a non-trivial character Λ such that $\chi = \chi\Lambda := \{\chi_1\Lambda, \dots, \chi_n\Lambda\}$ as unordered n -tuples. In this case, $\prod \chi = \prod(\chi\Lambda) = \Lambda^n \prod \chi$ and thus $\Lambda^n = 1$.

Assume that $p > 2n + 1$ and χ is not Kummer-induced. Then $\mathcal{F}_a(\chi)$ has a highest weight with multiplicity one. Thus it has a subsheaf $\mathcal{G}_a(\chi)$ such that, as representations of the Lie algebra $\mathfrak{g}(\mathcal{F}_a(\chi))$, $\mathcal{G}_a(\chi)$ is the irreducible sub-representation with highest weight. Moreover, it is geometrically irreducible and occurs exactly once in $\mathcal{F}_a(\chi)$ over $\mathbb{G}_m \otimes \overline{\mathbb{F}}_p$.

Corollary

Let $a, b \in \mathbb{F}_q^\times$ and let χ and ρ be n -tuples of multiplicative characters $\chi_i, \rho_j : \mathbb{F}_q^\times \rightarrow \overline{\mathbb{Q}}_\ell^\times$. Assume that $p > (2n^{2d} + 1)^2$, χ is not Kummer-induced and

$$\mathrm{Kl}_n(\psi, \chi, q, a) = \lambda \mathrm{Kl}_n(\psi, \rho, q, b)$$

for a fixed root of unity $\lambda \in \mu_{q-1}$. Then $\mathcal{G}_a(\chi) \otimes \mathcal{L}_{\prod \bar{\chi}} \mid \mathbb{G}_m \otimes \overline{\mathbb{F}}_p$ occurs at least once in $\mathcal{F}_b(\rho) \otimes \mathcal{L}_{\prod \bar{\rho}} \mid \mathbb{G}_m \otimes \overline{\mathbb{F}}_p$.

Here \mathcal{L}_χ is a rank one lisse sheaf on $\mathbb{G}_m \otimes \mathbb{F}_p$ such that for $t \in \mathbb{F}_p^\times$,

$$\mathrm{Tr}(\mathrm{Frob}_t, (\mathcal{L}_\chi)_{\bar{t}}) = \chi(t).$$

Denote by

$$\mathcal{F} = \mathcal{F}_a(\chi) \otimes \mathcal{L}_{\prod \bar{\chi}}, \quad \mathcal{F}' = \mathcal{F}_b(\rho) \otimes \mathcal{L}_{\prod \bar{\rho}}, \quad \mathcal{G} = \mathcal{G}_a(\chi) \otimes \mathcal{L}_{\prod \bar{\chi}}.$$

For $t \in \mathbb{F}_p^\times$, we have $\sigma_t \lambda = \lambda$ and thus

$$\begin{aligned} (-1)^{(n-1)d} \text{Tr}(\text{Frob}_t, \mathcal{F}_{\bar{t}}) &= \prod \bar{\chi}(t) \cdot \text{Kl}_n(\psi, \chi, q, at^n) \\ &= \sigma_t(\text{Kl}_n(\psi, \chi, q, a)) = \lambda \sigma_t(\text{Kl}_n(\psi, \rho, q, b)) \\ &= \lambda \prod \bar{\rho}(t) \cdot \text{Kl}_n(\psi, \rho, q, bt^n) = (-1)^{(n-1)d} \lambda \text{Tr}(\text{Frob}_t, \mathcal{F}'_{\bar{t}}). \end{aligned}$$

Apply Lemma to $r = s = n^d, M_0 = 0, M_\infty \leq 1$.

Now

$$\mathcal{G}_a(\chi) \otimes \mathcal{L}_{\prod \bar{\chi}} \hookrightarrow \mathcal{F}_b(\rho) \otimes \mathcal{L}_{\prod \bar{\rho}}, \quad \mathcal{G}_b(\rho) \otimes \mathcal{L}_{\prod \bar{\rho}} \hookrightarrow \mathcal{F}_a(\chi) \otimes \mathcal{L}_{\prod \bar{\chi}}.$$

Thus the highest weight $\lambda_a(\chi) = \lambda_b(\rho)$. Derived from this, and combining Fisher's arguments, we have:

Theorem (Z.)

Let $a, b \in \mathbb{F}_q^\times$. Assume that χ, ρ are not Kummer-induced and neither of them is of type $(\xi_1, \xi_1^{-1}, 1, \Lambda_2)\xi_2$. If $p > (2n^{2d} + 1)^2$ and

$$\text{Kl}_n(\psi, \chi, q, a) = \lambda \text{Kl}_n(\psi, \rho, q, b)$$

for some $\lambda \in \mu_{q-1}$, then there exists $\sigma \in \text{Gal}(\mathbb{F}_q/\mathbb{F}_p)$ and a multiplicative character η , such that $b = \sigma(a)$ and $\rho = \eta \cdot (\chi \circ \sigma^{-1})$ as unordered tuples. Moreover, either both Kloosterman sums vanish or $\eta(b) = \lambda^{-1}$.

The last step is to show the non-vanishingness.

Theorem

If $p > (3n - 1)C_\chi - n$ and for any i, j , $\chi_i = \chi_j$ if $\chi_i^n = \chi_j^n$, then $\text{Kl}_n(\psi, \chi, q, a)$ is nonzero. Here

$$C_\chi = \max_{i,j} \text{lcm}(\text{ord}(\chi_i), \text{ord}(\chi_j)) \quad (1)$$

is the supremum of least common multipliers of the orders of any two characters in χ .

We can express Kl_n as Gauss sums

$$(q-1)\text{Kl}_n(\psi, \chi, q, a) = \sum_{m=0}^{q-2} \omega^m(a) \prod_{i=1}^n g(m + s_i)$$

by Fourier transform on \mathbb{F}_q^\times , where $\chi_i = \omega^{s_i}$ for a Teichmüller character. What we need to do is to prove there is a unique m such that the valuation of $\prod_{i=1}^n g(m + s_i)$ is minimal.

Theorem (Z.)

If $p > \max \left\{ (2n^{2d} + 1)^2, (3n - 1)C_{\chi} - n \right\}$ and for any i, j , $\chi_i = \chi_j$ if $\chi_i^n = \chi_j^n$, then $\text{Kl}_n(\psi, \chi, q, a)$ generates $\mathbb{Q}(\mu_{pc})^H$, where H consists of those $\sigma_t \tau_w$ such that there exists an integer β and a character η satisfying

$$t = \lambda a_1^{\beta}, \lambda^{n_1} = 1, \chi^w = \eta \chi^{q_1^{\beta}}, \eta(a) = \prod \chi^w(t).$$

Here $n_1 = (n, p - 1)$, $q_1 = \#\mathbb{F}_p(a^{(p-1)/n_1})$ and $a_1 \in \mathbb{F}_p^{\times}$ such that $a_1^{n/n_1} = N_{\mathbb{F}_{q_1}/\mathbb{F}_p}(a^{(1-p)/n_1}) = a^{(1-q_1)/n_1}$.

An example: $n = 2$ case

Let $\chi = \{1, \chi\}$, where χ is a multiplicative character of order $c \neq 2$. If $p > \max \left\{ (2^{2d+1} + 1)^2, 5c - 2 \right\}$, then $\text{Kl}(\psi, \chi, p^d, a)$ generates $\mathbb{Q}(\mu_{pc})^H$, where

$$H = \begin{cases} \langle \tau_{q_1} \sigma_{a_1}, \sigma_{-1}, \tau_{-1} \rangle, & \text{if } \chi(-1) = 1, \chi(a) = 1; \\ \langle \tau_{-q_1} \sigma_{a_1}, \sigma_{-1} \rangle, & \text{if } \chi(-1) = 1, \chi(a) = \chi(a_1) = -1; \\ \langle \tau_{q_1^\alpha} \sigma_{a_1^\alpha}, \sigma_{-1} \rangle, & \text{if } \chi(-1) = 1, \chi(a)^\alpha \neq 1; \\ \langle \tau_{q_1} \sigma_{-a_1}, \tau_{-1} \sigma_{-1} \rangle, & \text{if } \chi(-1) = -1, \chi(a) = \chi(a_1) = -1; \\ \langle \tau_{q_1} \sigma_{a_1}, \tau_{-1} \rangle, & \text{if } \chi(-1) = -1, \chi(a) = 1; \\ \langle \tau_{q_1} \sigma_{a_1}, \tau_{-1} \sigma_{-1} \rangle, & \text{if } \chi(-1) = -1, \chi(a) = -1, \chi(a_1) = 1; \\ \langle \tau_{q_1^{\alpha/2}} \sigma_{-a_1^{\alpha/2}} \rangle, & \text{if } \chi(-1) = -1, 2 \mid \alpha, \chi(a) \neq \pm 1; \\ \langle \tau_{q_1^\alpha} \sigma_{a_1^\alpha} \rangle, & \text{if } \chi(-1) = -1, 2 \nmid \alpha, \chi(a) \neq \pm 1. \end{cases}$$

$q_1 = \#\mathbb{F}_p(a^{(1-p)/2})$, $a_1 = a^{(1-q_1)/2}$ and α is the order of $\chi(a_1) \in \mu_{p-1}$.

Consider the Kloosterman sums

$$S_k = \text{Kl}(\psi, \chi \circ N_{\mathbb{F}_{q^k}/\mathbb{F}_q}, q^k, a).$$

If $p > \max \left\{ (2n^{2dk} + 1)^2, (3n - 1)C_\chi - n \right\}$, then $\mathbb{Q}(S_k) = \mathbb{Q}(\mu_{pc})^H$, where H consists of those $\sigma_t \tau_w$ such that there exists an integer β and a character η on \mathbb{F}_q^\times satisfying

$$t = \lambda a_1^\beta, \lambda^{n_1} = 1, \quad \chi^w = \eta \chi^{q_1^\beta}, \quad \eta(a) = \gamma \cdot \prod \chi^w(t), \gamma^k = 1.$$

Thus $\mathbb{Q}(S_k) = \mathbb{Q}(S_{k-c})$ since $\gamma^c = 1$.

The L -function

$$L(T) = \exp \left(\sum_{k=1}^{\infty} \frac{T^k}{k} S_k \right)$$

is a rational function. Thus the sequence $\{S_k\}_k$ is linear recurrence sequence. The sequence $\{\mathbb{Q}(S_k)\}_{k \geq N}$ is periodic of period r for some N (Wan, Yin). Thus if $p > \max \left\{ (2n^{2d(N+r)} + 1)^2, (3n - 1)C_{\chi} - n \right\}$, the generating field of S_k is determined by the previous equations for any k . For this purpose, we need to decrease the bound $(2n^{2d} + 1)^2$ and estimate the period r and N . We conjecture that S_k has the predicted generating field if $p > 3ndc$.

Thank you!

