On the linearity of the periods of subtraction games

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Abstract

A subtraction game is an impartial combinatorial game involving a finite set S of positive integers. The nim-sequence $\mathcal{G}_S$ associated with this game is ultimately periodic. In this paper, we study the nim-sequence $\mathcal{G}_{S\cup\{c\}}$ where S is fixed and c varies. We conjecture that there is a multiple q of the period of $\mathcal{G}_S$, such that for sufficiently large c, the pre-period and period of $\mathcal{G}_{S\cup\{c\}}$ are linear in c if c modulo q is fixed. We prove it in several cases.

We also give new examples with period 2 inspired by this conjecture.

1. Introduction

Let S be a finite set of positive integers. The *(finite) subtraction game* $\mathrm{SUB}(S)$ is a two-player game involving a heap of $n \geq 0$ counters. The two players move alternately, subtracting some $s \in S$ counters. The player who cannot make a move loses.

We always write the subtraction set as $S = \{s_1, \dots, s_k\}$ with an order $s_1 < s_2 < \cdots < s_k$. Denote by $\mathcal{G}(n) = \mathcal{G}_S(n)$ the *nim-value* (or *Grundy-value*), i.e.,

$$\mathcal{G}(n) = \mathrm{mex}\,\{\mathcal{G}(n-s) : s \in S, s \leq n\}, \quad \forall n \geq 0,$$

where mex means the minimal non-negative integer not in the set. The sequence $\mathcal{G} = \mathcal{G}_S = \{\mathcal{G}(n)\}_{n\geq 0}$ is called the *nim-sequence*.

If $d = \gcd(S) = \gcd\{s : s \in S\} > 1$ and $S' = \{s/d : s \in S\}$, then $\mathcal{G}_S(n) = \mathcal{G}_{S'}(m)$, where $md \leq n < (m+1)d$. Hence we may assume that $\gcd(S) = 1$ if necessary.

Definition 1. A subtraction game $\mathrm{SUB}(S)$ (or its nim-sequence $\mathcal{G}$) is called *ultimately periodic*, if there exist integers $p \geq 1$ and $\ell \geq 0$ such that $\mathcal{G}(n+p) = \mathcal{G}(n)$ for all $n \geq \ell$. The minimal p is called the *period* and the minimal ℓ is called the *pre-period*.

Since $\mathcal{G}(n) \leq k$, one can show that $\mathcal{G}$ is ultimately periodic by the pigeonhole principle, see [? , Theorem 7.22]. We have the following lemma to determine the period and pre-period.

Lemma 1.1 ([? , Corollary 7.34]). *The minimal integers $\ell \geq 0, p \geq 1$ such that $\mathcal{G}(n) = \mathcal{G}(n+p)$ for $\ell \leq n < \ell + s_k$ are the pre-period and period of $\mathrm{SUB}(S)$ respectively.*

In this paper, we will propose a conjecture (Conjecture 5.5) on $\mathrm{SUB}(S \cup \{c\})$ where S is fixed and c varies. More precisely, there is a positive integer q which is a multiple of the period of $\mathrm{SUB}(S)$, such that for each fixed $0 \leq r < q-1$, the pre-period and period of $\mathrm{SUB}(S \cup \{c\})$ are linear functions of c respectively if c is large enough and $c \equiv r \bmod q$. We will prove it in several cases. We also give new nim-sequences with period 2 inspired by this conjecture.

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Let t, a be non-negative integers and $\mathcal{H} = (h_1 \cdots h_k)$ a sequence of integers with finite length. As usual, we denote by a^t the sequence $\underbrace{a \cdots a}_{t \text{ copies}}$ and $\mathcal{H}^t$ the sequence $\underbrace{\mathcal{H} \cdots \mathcal{H}}_{t \text{ copies}}$. Denote by $\underline{\mathcal{H}}$ the infinite-length sequence with periodic sequence $\mathcal{H}$, i.e., $\underline{\mathcal{H}} = \mathcal{H}\mathcal{H}\cdots$. For example, if ℓ and p is the pre-period and period of a nim-sequence $\mathcal{G}$ respectively, then we can write

$$\mathcal{G} = \mathcal{G}(0)\mathcal{G}(1)\mathcal{G}(2)\cdots = \mathcal{G}(0)\cdots\mathcal{G}(\ell-1)\underline{\mathcal{G}(\ell)\cdots\mathcal{G}(\ell+p-1)}.$$

We will not give detailed proofs of all nim-sequences, since these proofs tend to involve lengthy and tedious inductions.

2. The case $S = \{1, b, c\}$

In this section, we will consider nim-sequence when $S = \{1, b, c\}$, where $1 < b < c$. Let's recall some classical cases first.

Lemma 2.1. *Let p be the period of* $\mathrm{SUB}(S)$. *Let $S' = S \cup \{x + pt\}$ for some $x \in S$ and $t \geq 1$. If the pre-period of* $\mathrm{SUB}(S)$ *is zero, then $\mathcal{G}_{S'} = \mathcal{G}_S$.*

Proof. Certainly $\mathcal{G}_{S'}(0) = \mathcal{G}_S(0) = 0$. Suppose that $\mathcal{G}_{S'}(i) = \mathcal{G}_S(i)$ for $0 \leq i \leq n-1$. If $n < x + pt$, then

$$\mathcal{G}_{S'}(n) = \operatorname{mex}\{\mathcal{G}(n-s) : s \in S, s \leq n\} = \mathcal{G}(n).$$

If $n \geq x + pt$, then

$$\begin{aligned}\mathcal{G}_{S'}(n) &= \operatorname{mex}\{\mathcal{G}(n-x-pt), \mathcal{G}(n-s) : s \in S, s \leq n\}\\ &= \operatorname{mex}\{\mathcal{G}(n-x), \mathcal{G}(n-s) : s \in S, s \leq n\} = \mathcal{G}(n).\end{aligned}$$

The lemma then follows by induction.

Example 2.2. *Certainly, $\mathcal{G}_{\{1\}} = \underline{01}$. If $1 \in S$ and all elements of S are odd, then $\mathcal{G}_S = \underline{01}$ by applying Lemma 2.1 several times. This condition is also necessary for $\mathcal{G}_S = \underline{01}$, see [**?**].*

Example 2.3. *Let $S = \{a, c\}$ with $1 < a < c$. Write $c = at + r, 0 \leq r < a$. Then*

$$\mathcal{G} = \begin{cases} \underline{(0^a1^a)^{t/2}0^r2^{a-r}1^r}, & \text{if } t \text{ is even;}\\ \underline{(0^a1^a)^{(t+1)/2}2^r}, & \text{if } t \text{ is odd,}\end{cases}$$

*$\ell = 0$ and $p = c + a$ or $2a$. See [**?**].*

Example 2.4. 1. *Let $S = \{1, b, c\}$ with odd b, $1 < b < c$. Then*

c	$\mathcal{G}$	ℓ	p
odd	$\underline{01}$	0	2
even	$\underline{(01)^{c/2}(23)^{(b-1)/2}2}$	0	$c+b$

2. *Let $S = \{1, 2, c\}$, $2 < c$. Note that $\mathcal{G}_{\{1,2\}} = \underline{012}$ with period 3. Write $c = 3t + r, 0 \leq r < 3$. Then*

r	$\mathcal{G}$	ℓ	p
0	$\underline{(012)^t3}$	0	$c+1$
1, 2	$\underline{012}$	0	3

3. *Let $S = \{1, 4, c\}$, $4 < c$. Denote by $\mathcal{H} = 01012$, then $\mathcal{G}_{\{1,4\}} = \underline{\mathcal{H}}$ with period 5. Write $c = 5t + r, 0 \le r < 5$. Then*

r, c	$\mathcal{G}$	ℓ	p
$r = 0, c = 5$	$\underline{\mathcal{H}\,323}$	0	8
$r = 0, c > 5$	$\mathcal{H}^t\,323013\underline{\mathcal{H}^{t-1}012012}$	$c+6$	$c+1$
$r = 1, 4$	$\underline{\mathcal{H}}$	0	5
$r = 2$	$\underline{\mathcal{H}^t\,012}$	0	$c+1$
$r = 3$	$\underline{\mathcal{H}^{t+1}\,32}$	0	$c+4$

Proposition 2.5. *Let $S = \{1, b, c\}$, where $b = 2k \ge 6$ is even. Write $c = t(b+1) + r$ with $0 \le r \le b$.*

1. *If $r = 1, b$, then $\ell = 0$ and $p = b + 1$.*
2. *If $3 \le r \le b - 1$ is odd, then $\ell = 0$ and $p = c + b$.*
3. *If $r = b - 2$, then $\ell = 0$ and $p = c + 1$.*
4. *If $c = b + 1$, then $\ell = 0, p = 2b$;*
5. *If $c > b+1$, $0 \le r \le b-4$ is even and $t + r/2 \ge k$, then $\ell = \left(\frac{b-r}{2} - 1\right)(c + b + 2) - b$ and $p = c + 1$.*
6. *If $c > b+1$, $0 \le r \le b-4$ is even and $t + r/2 \le k - 1$, then $\ell = t(c + b + 2) - b$. If $t + r/2 < k - 1$, then $p = c + b$; if $t + r/2 = k - 1$, then $p = b - 1$.*

Proof. Denote by $\mathcal{H} = (01)^k 2$, then $\mathcal{G}_{\{1,b\}} = \underline{\mathcal{H}}$ with period $b + 1$.

1. In this case, $\mathcal{G} = \underline{\mathcal{H}}$, $\ell = 0$ and $p = b + 1$ by Lemma 2.1.
2. In this case, $\mathcal{G} = \underline{\mathcal{H}^{t+1}\,(32)^{(r-1)/2}}$, $\ell = 0$ and $p = c + b$.
3. In this case, $\mathcal{G} = \underline{\mathcal{H}^t\,(01)^{k-1}2}$, $\ell = 0$ and $p = c + 1$.
4. In this case, $\mathcal{G} = \underline{(01)^k(23)^k} = \underline{\mathcal{H}3(23)^{k-1}}$, $\ell = 0$ and $p = 2b$.
5. Write $r = 2v$. If $1 \le v \le k - 2$, the leading $(c+1)(k-v+1)$ terms of $\mathcal{G}$ are (the bold part is the first periodic nim-sequence)

i	$\mathcal{G}\big((c+1)i + j\big),\ 0 \le j \le c$
0	$\mathcal{H}^t, (01)^v 2$
1	$(32)^{k-v-1}(01)^{v+1}2, \mathcal{H}^{t-1}, (01)^v 0$
2	$1(01)^{k-v-2}2(01)^{v+1}2, (32)^{k-v-2}(01)^{v+2}2, \mathcal{H}^{t-2}, (01)^v 0$
i	$1(01)^{k-v-2}2(01)^{v+1}0, \dots, 1(01)^{k-v-i+1}2(01)^{v+i-2}0,$ $1(01)^{k-v-i}2(01)^{v+i-1}2, (32)^{k-v-i}(01)^{v+i}2, \mathcal{H}^{t-i}, (01)^v 0$
$k-v-1$	$1(01)^{k-v-2}2(01)^{v+1}0, \dots, 1(01)^2 2(01)^{k-3}0, 1(01)2(01)^{k-2}2,$ $(32)^{\mathbf{1}}\mathbf{(01)^{k-1}2}, \boldsymbol{\mathcal{H}^{t-k+v+1}}, \mathbf{(01)^{v}0}$
$k-v$	$\mathbf{1(01)^{k-v-2}2(01)^{v+1}0, \dots, 1(01)2(01)^{k-2}0, 1}2(01)^{k-1}2,$ $\mathcal{H}^{t-k+v-1}, (01)^v 0.$

If $v = 0$, the leading $(c+1)(k+1)$ terms of $\mathcal{G}$ are

i	$\mathcal{G}\big((c+1)i+j\big),\ 0 \le j \le c$
0	$\mathcal{H}^t 3$
1	$(23)^{k-1}013, \mathcal{H}^{t-1}0$
2	$1(01)^{k-2}2(01)2, (32)^{k-2}(01)^2 2, \mathcal{H}^{t-2}0$
i	$1(01)^{k-2}2(01)0, \cdots, 1(01)^{k-i+1}2(01)^{i-2}0, 1(01)^{k-i}2(01)^{i-1}2,$ $(32)^{k-i}(01)^i 2, \mathcal{H}^{t-i}0$
$k-1$	$1(01)^{k-2}2(01)0, \cdots, 1(01)^2 2(01)^{k-3}0, 1(01)^1 2(01)^{k-2}2,$ $(3\mathbf{2})^{\mathbf{1}}(\mathbf{01})^{\mathbf{k-1}}\mathbf{2}, \mathcal{H}^{\mathbf{t-k+1}}\mathbf{0}$
k	$\mathbf{1(01)^{k-2}2(01)0}, \cdots, \mathbf{1(01)^{1}2(01)^{k-2}0}, \mathbf{1}2(01)^{k-1}2, \mathcal{H}^{t-k+1}0.$

In both cases, we have $\ell = \left(\frac{b-r}{2} - 1\right)(c+b+2) - b$, $p = c+1$ and

$$\mathcal{G} = \cdots \underline{2(01)^{k-1}\big(2(01)^k\big)^{t-k+v+1}\big(2(01)^{k-1}\big)^{k-v-1}}.$$

6. If $1 \le v \le k-2$, the leading $(c+1)(t+2)$ terms of $\mathcal{G}$ are

i	$\mathcal{G}\big((c+1)i+j\big), 0 \le j \le c$
0	$\mathcal{H}^t\,(01)^v 2$
1	$(32)^{k-v-1}(01)^{v+1}2, \mathcal{H}^{t-1}(01)^v 0$
2	$1(01)^{k-v-2}2(01)^{v+1}2, (32)^{k-v-2}(01)^{v+2}2, \mathcal{H}^{t-2}(01)^v 0$
i	$1(01)^{k-v-2}2(01)^{v+1}0, \ldots, 1(01)^{k-v-i+1}2(01)^{v+i-2}0,$ $1(01)^{k-v-i}2(01)^{v+i-1}2, (32)^{k-v-i}(01)^{v+i}2, \mathcal{H}^{t-i}(01)^v 0$
$t-1$	$1(01)^{k-v-2}2(01)^{v+1}0, \ldots, 1(01)^{k-v-t+2}2(01)^{v+t-3}0,$ $1(01)^{k-v-t+1}2(01)^{v+t-2}2, (32)^{k-v-t+1}(01)^{v+t-1}2, \mathcal{H}^1(01)^v 0$
t	$1(01)^{k-v-2}2(01)^{v+1}0, \ldots, 1(01)^{k-v-t+1}2(01)^{v+t-2}0,$ $1(01)^{k-v-t}2(01)^{v+t-1}2, (3\mathbf{2})^{\mathbf{k-v-t}}(\mathbf{01})^{\mathbf{v+t}}\mathbf{2}, (\mathbf{01})^{\mathbf{v}}\mathbf{0}$
$t+1$	$\mathbf{1(01)^{k-v-2}2(01)^{v+1}0}, \ldots, \mathbf{1(01)^{k-v-t+1}2(01)^{v+t-2}0},$ $\mathbf{1(01)^{k-v-t}2(01)^{v+t-1}0}, \mathbf{1(01)^{k-v-t-1}2(01)^{v+t}}2, (32)^{k-v-t-1}01\cdots$

Therefore, $\ell = t(c+b+2) - b$. If $t+v < k-1$, then $p = c+b$ and

$$\mathcal{G} = \cdots \underline{2(32)^{k-v-t-1}(01)^{v+t}2\big((01)^{k-1}2\big)^t(01)^{v+t}}.$$

If $t+v = k-1$, then $p = b-1$ and $\mathcal{G} = \cdots \underline{2(01)^{k-1}}$.

If $v = 0$, the leading $(c+1)(t+2)$ terms of $\mathcal{G}$ are

i	$\mathcal{G}\big((c+1)i+j\big), 0 \le j \le c$
0	$\mathcal{H}^t\,3$
1	$(23)^{k-1}013, \mathcal{H}^{t-1}0$
2	$1(01)^{k-2}2(01)2, (32)^{k-2}(01)^2 2, \mathcal{H}^{t-2}0$
i	$1(01)^{k-2}2(01)0, \cdots, 1(01)^{k-i+1}2(01)^{i-2}0, 1(01)^{k-i}2(01)^{i-1}2,$ $(32)^{k-i}(01)^i 2, \mathcal{H}^{t-i}0$
$t-1$	$1(01)^{k-2}2(01)0, \cdots, 1(01)^{k-t+2}2(01)^{t-3}0, 1(01)^{k-t+1}2(01)^{t-2}2,$ $(32)^{k-t+1}(01)^{t-1}2, \mathcal{H}^1 0$
t	$1(01)^{k-2}2(01)0, \cdots, 1(01)^{k-t+1}2(01)^{t-2}0, 1(01)^{k-t}2(01)^{t-1}2,$ $(3\mathbf{2})^{\mathbf{k-t}}(\mathbf{01})^{\mathbf{t}}\mathbf{20}$
$t+1$	$\mathbf{1(01)^{k-2}2(01)0}, \cdots, \mathbf{1(01)^{k-t}2(01)^{t-1}0}, \mathbf{1(01)^{k-t-1}2(01)^{t}0},$ $\mathbf{1(01)^{k-t-1}2(01)^{t}}2, (32)^{k-t-1}01\cdots$

Therefore, $\ell = t(c+b+2) - b$. If $t < k-1$, then $p = c+b$ and

$$\mathcal{G} = \cdots \underline{2(32)^{k-t-1}(01)^t 2\big((01)^{k-1}2\big)^t (01)^t}.$$

If $t = k-1$, then $p = b-1$ and $\mathcal{G} = \cdots \underline{2(01)^{k-1}}$.

Remark 1. The case $c < 4b$ is studied in [**?**], but there are some incorrect data. In Table 1, $p = a-1$ if $r = a-3 \geq 3$. In Table B.11, $n_0 = a+2b+4$ if $2 \leq r \leq a-4$. In Table B.12, $n_0 = 2a+3b+6$ if $3 \leq r \leq a-5$. The corresponding pre-period nim-values also need to be modified.

3. The case $S = \{a, 2a, c\}$

Proposition 3.1. *Let $S = \{a, 2a, c\}$, $2a < c$. Write $c = 3at + r$ with $0 \leq r < 3a$. Then*

$$\ell = \begin{cases} c+a-r, & 0 < r < a; \\ 0, & \textit{otherwise}, \end{cases} \qquad p = \begin{cases} 3a/2, & r = a/2; \\ 3a, & a/2 < r \leq 2a; \\ c+a, & \textit{otherwise}. \end{cases}$$

PROOF. Denote by $\mathcal{H} = 0^a 1^a 2^a$. Then $\mathcal{G}_{\{a,2a\}} = \underline{\mathcal{H}}$ with period $q = 3a$. Write $a = 2k-1$ if a is odd; $a = 2k$ if a is even.

1. If $a \leq r \leq 2a$, then $\mathcal{G} = \underline{\mathcal{H}}$, $\ell = 0$ and $p = 3a$.
2. If $r = 0$, then $\mathcal{G} = \underline{\mathcal{H}^t 3^a}$, $\ell = 0$ and $p = c+a$.
3. If $0 < r < k$, then

$$\mathcal{G} = \mathcal{H}^t 0^r 3^{a-r} \underline{(1^r 0^{a-r} 2^r 1^{a-r} 0^r 2^{a-r})^t 1^r 0^r 3^{a-2r} 2^r},$$

   $\ell = c+a-r$ and $p = c+a$.
4. If $k \leq r < a$, then

$$\mathcal{G} = \mathcal{H}^t 0^r 3^{a-r} \underline{1^r 0^{a-r} 2^r 1^{a-r} 0^r 2^{a-r}},$$

   $\ell = c+a-r$ and $p = 3a$ or $3a/2$.
5. If $r > 2a$, then $\mathcal{G} = \underline{\mathcal{H}^{t+1} 3^{r-2a}}$, $\ell = 0$ and $p = c+a$.

Remark 2. The pre-period and period of $\textsc{sub}(S)$ are not easy to determine, even if $S = \{s_1, s_2, s_3\}$ is a 3-element set. In [**?** , §4, Conjecture], Althofer and Bultermann conjectured that the period of $\textsc{sub}(S)$ is bounded by a quadratic polynomial in s_3. Ho also studied $\textsc{sub}(S)$ for 3-element set S in [**?**].

4. The case S contains successive numbers

Proposition 4.1. *Let $S = \{a, a+1, \ldots, b-1, b, c\}$, where $a < b < c$. Write $c = t(a+b) + r$ with $0 \leq r < a+b$. Then*

$$\ell = 0, \quad p = \begin{cases} a+b, & a \leq r \leq b; \\ c+a, & r = 0 \textit{ or } r > b; \\ c+b, & 0 < r < a. \end{cases}$$

PROOF. Write $b = ak + s$, $0 \leq s \leq a-1$ and denote by $\mathcal{H} = 0^a 1^a \cdots k^a (k+1)^s$, then $\mathcal{G}_{\{a,a+1,\ldots,b\}} = \underline{\mathcal{H}}$ with period $q = a+b = a(k+1)+s$.

1. If $a \le r \le b$, then $\mathcal{G} = \underline{\mathcal{H}}$, $\ell = 0$ and $p = a + b$ by Lemma 2.1.
2. If $r = 0$, then

$$\mathcal{G} = \underline{\mathcal{H}^t (k+1)^{a-s} (k+2)^s}.$$

If $r > b$ and $r + s > q$, then

$$\mathcal{G} = \underline{\mathcal{H}^{t+1} (k+1)^{a-s} (k+2)^{r+s-q}}.$$

If $r > b$ and $r + s \le q$, then

$$\mathcal{G} = \underline{\mathcal{H}^{t+1} (k+1)^{a+r-q}}.$$

In all cases, we have $\ell = 0$ and $p = c + a$.

3. If $0 < r < a - 2s$, then

$$\begin{aligned}\mathcal{G} =& \underline{\mathcal{H}^t, 0^r (k+1)^{a-s-r} (k+2)^s, 1^r (k+2)^{a-s-r} (k+3)^s, \cdots,} \\ & \underline{(k-1)^r (2k)^{a-s-r} (2k+1)^s, k^r (2k+1)^s}.\end{aligned}$$

If $a - 2s \le r < a - s$, then

$$\begin{aligned}\mathcal{G} =& \underline{\mathcal{H}^t, 0^r (k+1)^{a-s-r} (k+2)^s, 1^r (k+2)^{a-s-r} (k+3)^s, \cdots,} \\ & \underline{(k-1)^r (2k)^{a-s-r} (2k+1)^s, k^r (2k+1)^{a-s-r} (2k+2)^{2s+r-a}}.\end{aligned}$$

If $a - s \le r < a$, then

$$\mathcal{G} = \underline{\mathcal{H}^t, 0^r (k+2)^{a-r}, 1^r (k+3)^{a-r}, \cdots (k-1)^r (2k+1)^{a-r}, k^r (k+1)^s},$$

In all cases, we have $\ell = 0$ and $p = c + b$.

5. Piecewise linearity of pre-periods and periods

Let S be a fixed subtraction set. Denote by ℓ_c the pre-period and p_c the period of $\textsc{sub}(S \cup \{c\})$.

Example 5.1. *Let $S = \{6, 17\}$. Then $\mathcal{G} = \underline{0^6 1^6 0^5 2 1^5}$ with period 23. For $116 \le c \le 500$, we have*

$$\ell_c = \begin{cases} (9 - 2\lambda)c + (147 - 35\lambda), & c \equiv \lambda \text{ or } \lambda + 12 \bmod 23, \lambda \in [0, 4]; \\ 0, & \text{otherwise}, \end{cases}$$

$$p_c = \begin{cases} c + 6, & c \equiv 0, 1, 2, 3, 4, 5, 12, 13, 14, 15, 16 \bmod 23; \\ c + 17, & c \equiv 7, 8, 9, 10, 11, 18, 19, 20, 21, 22 \bmod 23; \\ 23, & r = 6 \text{ or } 17. \end{cases}$$

See https://ruhuasiyu.github.io/nim/example5.1.html.

Example 5.2. *Let $S = \{3, 5, 8\}$. Then $\mathcal{G} = \underline{0^3 1^3 2^3 3^2}$ with period 11. For $13 \le c \le 500$, we have*

$$\ell_c = \begin{cases} c + 18, & c \equiv 1, 2 \bmod 11; \\ 0, & \text{otherwise}, \end{cases} \qquad p_c = \begin{cases} c + 3, & c \equiv 0, 1, 9, 10 \bmod 11; \\ c + 25, & c \equiv 2 \bmod 11; \\ 11, & \text{otherwise}. \end{cases}$$

See https://ruhuasiyu.github.io/nim/example5.2.html.

Example 5.3. *Let* $S = \{2,3,5,7\}$. *Then* $\mathcal{G} = \underline{0^2 1^2 2^2 3^2 4}$ *with period* 9. *For* $11 \le c \le 500$, *we have*

$$\ell_c = \begin{cases} 2c-4, & c \equiv 1 \bmod 18; \\ c+5, & c \equiv 10 \bmod 18; \\ 0, & \textit{otherwise}, \end{cases} \qquad p_c = \begin{cases} c+2, & c \equiv 0,8,9,10,17 \bmod 18; \\ 4, & c \equiv 1 \bmod 18; \\ 9, & \textit{otherwise}. \end{cases}$$

See https://ruhuasiyu.github.io/nim/example5.3.html.

Example 5.4. *Let* $S = \{4,11,12,14\}$. *Then* $\mathcal{G} = \cdots \underline{20^4 1^4 0^3 31^3 2^3 03^3 12}$ *with pre-period* 24 *and period* 25. *Write* $r \equiv c \bmod 25, 0 \le r < 25$. *For* $101 \le c \le 500$, *we have*

$$\ell_c = \begin{cases} 4c+91, & r=0; & 2c+8, & r=1; & 2c+34, & r=2; \\ c-6, & r=3; & 2c+16, & r=4; & 2c+36, & r=5; \\ 3c+4, & r=6; & c+26, & r=9; & c+12, & r=12; \\ 0, & r=13; & 2c+37, & r=18; & c+14, & r=19; \\ c+2, & r=20; & 12, & r=21; & 3c+5, & r=22; \\ c+52, & r=23; & 2c+33, & r=24; & 24, & \textit{otherwise}, \end{cases}$$

$$p_c = \begin{cases} c+37, & r=0,1,9,18; & c+14, & r=2,10; \\ c+11, & r=6,7,8,15,16,17; & c+12, & r=13; \\ 2c+41, & r=19; & c+4, & r=21; \\ c+28, & r=22; & 25, & \textit{otherwise}. \end{cases}$$

See https://ruhuasiyu.github.io/nim/example5.4.html.

Based on these observations, we propose the following conjecture:

Conjecture 5.5. *Fix a subtraction set* S. *There is*

- *a positive integer* q*, which is a multiple of the period of* SUB(S)*;*
- *integers* $\alpha_r, \beta_r, \lambda_r, \mu_r$ *for each* $0 \le r < q$*;*
- *an integer* N,

such that if $c \ge N$ *and* $c \equiv r \bmod q$,

- *the pre-period of* SUB($S \cup \{c\}$) *is* $\ell_c = \alpha_r c + \beta_r$*;*
- *the period of* SUB($S \cup \{c\}$) *is* $p_c = \lambda_r c + \mu_r$*.*

Theorem 5.6. *Conjecture 5.5 holds in the following cases:*

1. $1 \in S$ *and the elements of* S *are all odd;*
2. $S = \{1, b\}$*;*
3. $S = \{a, 2a\}$*;*
4. $S = \{a, a+1, \ldots, b-1, b\}$.

PROOF. 1. The period of SUB(S) is $q = 2$. If c is odd, then $\mathcal{G}_{S\cup\{c\}} = \underline{01}$. If c is even, denote by s the maximal number in S. Then

$$\mathcal{G}_{S\cup\{c\}} = \underline{(01)^{c/2}(23)^{(s-1)/2}2},$$

$\ell = 0$ and $p = c + s$.

2. follows from Example 2.4 and Proposition 2.5.
3. follows from Proposition 3.1.
4. follows from Proposition 4.1.

Remark 3. Once Conjecture 5.5 holds with effective q, N, then one can get the pre-period and period of $\textsc{sub}(S \cup \{c\})$ for all c effectively. That is because we only need to calculate the pre-periods and periods of $SUB(S \cup \{c\})$ for $c \leq N + 2q$.

6. Ultimately bipartite nim-sequences

A subtraction game (or its nim-sequence) is said to be *ultimately bipartite* if the period is 2. It is known that $\mathcal{G}_S$ is ultimately bipartite with pre-period 0 if and only if $1 \in S$ and all elements in S are odd, see [**?**].

Example 6.1. *Let $a \geq 3$ be an odd integer. If S is one of the following:*

- $S = \{3, 5, 9, \ldots, 2^a + 1\}$;
- $S = \{3, 5, 2^a + 1\}$;
- $S = \{a, a + 2, 2a + 3\}$;
- $S = \{a, 2a + 1, 3a\}$,

*then $\textsc{sub}(S)$ is ultimately bipartite. See [**?** , Theorem 2] and [**?** , Theorem 5].*

Lemma 6.2. *If $\mathcal{G}_S$ is ultimately bipartite, then all elements in S are odd.*

Proof. As shown in [**?** , Theorem 3], there exists an integer n_0 such that for $n \geq n_0$, $\mathcal{G}(n) = 0$ if n is even; $\mathcal{G}(n) = 1$ if n is odd. Take an even number $n \geq n_0 + s_k$, where s_k is the maximal element in S. Then

$$0 = \mathcal{G}(n) = \operatorname{mex} \{\mathcal{G}(n - s) : s \in S\},$$

which implies that $\mathcal{G}(n - s) = 1$ for all $s \in S$. Hence all $s \in S$ are odd.

We have the following new ultimately bipartite subtraction sets inspired by our conjecture.

Theorem 6.3. *Let $a \geq 3$ be an odd integer and $t \geq 1$. The subtraction game $\textsc{sub}(S)$ is ultimately bipartite in the following cases:*

1. $S = \{a, a + 2, (2a + 2)t + 1\}$;
2. $S = \{a, 2a + 1, (3a + 1)t - 1\}$;
3. $S = \{a, 2a - 1, (3a - 1)t + a - 2\}$.

Proof. Let c be the maximal element in S. Write $a = 2k + 1$.

1. If $k \geq 2$, then the leading $(k + 1)(a + 1)(2t + 1)$ terms of $\mathcal{G}$ are

i	$\mathcal{G}\big((a+1)(2t+1)i+j\big), 0 \le j < (a+1)(2t+1) = c+a$			
0	0^a1	$[\ 1^{a-1}22$ $1^{a-1}22$	0^a1 $02^{a-3}331$	$]^{t-1},$
1	$030^{a-2}1$	$[\ 01^{a-2}21$ $01^{a-2}21$	$020^{a-2}1$ $0202^{a-5}321$	$]^{t-1},$
i	$(01)^{i-1}030^{a-2i}1$	$[(01)^{i-1}01^{a-2i}21$ $(01)^{i-1}01^{a-2i}21$	$(01)^{i-1}020^{a-2i}1$ $(01)^{i-1}0202^{a-2i-3}321$	$]^{t-1},$
$k-1$	$(01)^{k-2}030^31$	$[\ (01)^{k-2}01^321$ $(01)^{k-2}01^321$	$(01)^{k-2}020^31$ $(01)^{k-2}020321$	$]^{t-1},$
k	$[\ (01)^{k-1}0301$	$(01)^{k-1}0121]^{t-1}$ $(01)^{k-1}0101$	$(01)^{k-1}03\mathbf{01},$ $(01)^{k-1}0101.$	

Hence the pre-period is

$$\ell = (k+1)(c+a) - 2a - 4 = (k+1)c + 2k^2 - k - 5$$

and the period is $p = 2$. The case $a = 3$ will be shown in Case 3.

2. The leading $(k+1)((3a+1)t + a - 1)$ terms of $\mathcal{G}$ are

i	$\mathcal{G}\big(((3a+1)t+a-1)i+j\big), 0 \le j < (3a+1)t+a-1 = c+a$			
0	$[\ 0^a$	1^a	$02^{a-1}1\ ]^t,$	3^{a-1}
1	$[\ 020^{a-2}$ 020^{a-2}	101^{a-2} 101^{a-2}	$(01)^132^{a-3}1\ ]^{t-1},$ $(01)02^{a-3}1$	 $(01)3^{a-3}$
i	$[(01)^{i-1}020^{a-2i}$ $(01)^{i-1}020^{a-2i}$	$1(01)^{i-1}01^{a-2i}$ $1(01)^{i-1}01^{a-2i}$	$(01)^i32^{a-2i-1}1]^{t-1},$ $(01)^i02^{a-2i-1}1$	 $(01)^i3^{a-2i-1}$
$k-1$	$[\ (01)^{k-2}020^3$ $(01)^{k-2}020^3$	$1(01)^{k-2}01^3$ $1(01)^{k-2}01^3$	$(01)^{k-1}32^21\ ]^{t-1},$ $(01)^{k-1}02^21$	 $(01)^{k-1}3^2$
k	$[\ (01)^{k-1}020$ $(01)^{k-1}02\mathbf{0}$	$1(01)^{k-1}01$ $\mathbf{1}(01)^{k-1}01$	$(01)^k31\ ]^{t-1},$ $(01)^k01$	 $(01)^k.$

Hence the pre-period is

$$\ell = (k+1)(c+a) - 3a - 1 = (k+1)c + 2k^2 - 3k - 3$$

and the period is $p = 2$.

(3) The leading $(k+1)(3a-1)(t+1)$ terms of $\mathcal{G}$ are

i	$\mathcal{G}\big((3a-1)(t+1)i+j\big), 0 \le j < (3a-1)(t+1) = c+2a+1$				
0	$[0^{a-1}$	01^{a-1}	$12^{a-1}\ ]^t,$	$0^{a-2}3$	$31^{a-3}(10)^1$	$2^{a-2}(01)^1$
1	$[0^{a-3}(01)^1$	$31^{a-3}(10)^1$	$2^{a-2}(01)^1]^t,$	$0^{a-4}3(01)^1$	$31^{a-5}(10)^2$	$2^{a-4}(01)^2$
i	$[0^{a-2i-1}(01)^i$	$31^{a-2i-1}(10)^i$	$2^{a-2i}(01)^i]^t,$	$0^{a-2i-2}3(01)^i$	$31^{a-2i-3}(10)^{i+1}$	$2^{a-2i-2}(01)^{i+1}$
$k-1$	$[0^2(01)^{k-1}$	$31^2(10)^{k-1}$	$2^3(01)^{k-1}]^t,$	$0^13(01)^{k-1}$	$3(10)^k$	$2^1(01)^k$
k	$[(01)^k$	$3(10)^k$	$2(\mathbf{01})^k\ ]^{t-1},(01)^{6k+2}.$			

Hence the pre-period is

$$\ell = (k+1)(c+2a+1) - 2(7k+2) = (k+1)c + 4k^2 - 7k - 1$$

and the period is $p = 2$.

Remark 4. One may expect that if $\textsc{sub}(a, b, c)$ is ultimately bipartite, then so is $\textsc{sub}(a, b, d)$ for sufficient large d with $d \equiv c \mod (a + b)$. This is not true in general. For example, $\textsc{sub}(3, 11, 13)$ is ultimately bipartite but $\textsc{sub}(3, 11, 14t + 13)$ has period $14t + 16$, $t \geq 1$.

Remark 5. Write $a = 2k + 1$. Consider the four-element subtraction set $S = \{a, 2a + 1, 3a, c\}$, $c > 3a$ is odd. For $3 \leq a \leq 25, c < 500$, we find the following phenomenon.

- If $c = 4a + 1$, then $\ell = 0$ and $p = 5a + 1$.
- If $c = (4i + 2)a - 1$ with $1 \leq i < k$, then $\ell = (8i - 1)a + 2i - 1$ and $p = 4a$.
- Otherwise, $\textsc{sub}(S)$ is ultimately bipartite.

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