习题课

- 习题3-1
- (A)1.(1) 正确, 因为

$$\lim_{\Delta x\to 0}\frac{f(x_0)-f(x_0-\Delta x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(x_0-\Delta x)-f(x_0)}{-\Delta x}$$
$$=\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}.$$

- (2) 错误, 在一点可导和这一点取值一定相关.
- 反例: $f(x)=|x|, x_0=0$, 则 $f'_+(0)=1, f'_-(0)=-1, A=0.$

- 2. 由于

$$f'(x_0)=\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(x_0+2\Delta x)-f(x_0)}{2\Delta x}$$
$$=\lim_{\Delta x\to 0}\frac{f(x_0-\Delta x)-f(x_0)}{-\Delta x},$$

- 因此

$$\lim_{\Delta x\to 0}\frac{f(x_0+2\Delta x)-f(x_0-\Delta x)}{2\Delta x}=\lim_{\Delta x\to 0}\left[\frac{f(x_0+2\Delta x)-f(x_0)}{2\Delta x}+\frac{1}{2}\cdot\frac{f(x_0-\Delta x)-f(x_0)}{-\Delta x}\right]$$
$$=f'(x_0)+-\frac{1}{2}f'(x_0)=\frac{3}{2}f'(x_0).$$

- 我们也可以用一阶近似公式来解.

$$f(x)=f(x_0)+f'(x_0)\Delta x+o(\Delta x),\qquad \Delta x=x-x_0.$$

- 因此

$$f(x_0+2\Delta x)=f(x_0)+f'(x_0)\cdot 2\Delta x+o(\Delta x),$$

$$f(x_0-\Delta x)=f(x_0)-f'(x_0)\Delta x+o(\Delta x),$$

$$\lim_{\Delta x\to 0}\frac{f(x_0+2\Delta x)-f(x_0-\Delta x)}{2\Delta x}=\lim_{\Delta x\to 0}\left[\frac{3}{2}f'(x_0)+\frac{o(\Delta x)}{2\Delta x}\right]=\frac{3}{2}f'(x_0).$$

- 3. 从定义出发.

$$(\cos x)' = \lim_{\Delta x \to 0} \frac{\cos(x + \Delta x) - \cos x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{-\sin\left(x + \frac{\Delta x}{2}\right)\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}} = -\sin x\,.$$

- 在学习了求导的运算法则后,

$$(\cos x)' = \left[\sin\left(\frac{\pi}{2} - x\right)\right]' = \cos\left(\frac{\pi}{2} - x\right)\cdot(-1) = -\sin x\,.$$

- 4. 当时间为 $t+\Delta t$ 时温度为 $T(t+\Delta t)$, 于是时间 $[t, t+\Delta t)$ 内的平均温度差为 $T(t+\Delta t)-T(t)$. 令 $\Delta t \to 0$, 则 t 时刻温度变化速度为

$$\lim_{\Delta x \to 0} \frac{T(t+\Delta t)-T(t)}{\Delta t} = T'(t).$$

- 5. 由于 $\lim\limits_{\Delta x \to 0} \frac{f(\Delta x)-f(0)}{\Delta x} = \lim\limits_{\Delta x \to 0} \frac{\sqrt[3]{\Delta x}}{\Delta x} = \lim\limits_{\Delta x \to 0} \Delta x^{-\frac{2}{3}} = \infty$ 不存在, 因此 $f(x)$ 在 0 处不可导.
- 由于 $f'(0)=\infty$, 因此切线为 $x=0$.
- 一般地, 若曲线 $y=f(x)$ 在点 $[x_0, f(x_0)]$ 处存在切线, 则 $f'(x_0)$ 存在或者为无穷大.

- 6. 由于 $y' = -\sin x\,, y'\left(\frac{4\pi}{3}\right) = \frac{\sqrt{3}}{2}$, 因此切线方程为

$$y + \frac{1}{2} = \frac{\sqrt{3}}{2}\left(x - \frac{4\pi}{3}\right), \qquad \sqrt{3}x - 2y = 1 + \frac{4\sqrt{3}\pi}{3}.$$

- 7.(1) 由于

$$f'_{+}(0) = \lim_{\Delta x \to 0^{+}} \frac{|\sin \Delta x|}{\Delta x} = \lim_{\Delta x \to 0^{+}} \frac{\sin \Delta x}{\Delta x} = 1,$$

$$f'_{-}(0) = \lim_{\Delta x \to 0^{-}} \frac{|\sin \Delta x|}{\Delta x} = \lim_{\Delta x \to 0^{+}} \frac{-\sin \Delta x}{\Delta x} = -1,$$

- 因此在 0 处不可导.

- (2) 由于

$$\lim_{\Delta x\to 0^+}\frac{(\Delta x)^2\sin\dfrac{1}{\Delta x}}{\Delta x}=\lim_{\Delta x\to 0^+}\Delta x\sin\frac{1}{\Delta x}=0,$$

- 因此在 0 处可导且 $f'(0)=0$.
- 8. 首先 $f(x)$ 在 1 处连续, 因此 $f(1^-)=f(1)=f(1^+)$, $e=a+b$.
- 由于 $f(x)$ 在 1 处连续, 因此

$$f'_+(1)=(ax+b)'\Big|_{x=1}=a,\qquad f'_-(1)=(e^x)'\Big|_{x=1}=e.$$

- 从而 $f'(1)=a=e, b=0$.

- 当 $x<1$ 时, $f'(x)=(e^x)'=e^x$.
- 当 $x>1$ 时, $f'(x)=(ex)'=e$. 故

$$f'(x)=\begin{cases} e^x, & x\le 1 \\ e, & x>1 \end{cases}.$$

- (B)1. 由于 $x\to 0$ 时 $\cos x-1\to 0$, 因此

$$f'(x_0)=\lim_{x\to 0}\frac{f(x_0+\cos x-1)-f(x_0)}{\cos x-1}=\lim_{x\to 0}\frac{f(x_0+\cos x-1)-f(x_0)}{-\frac{1}{2}x^2},$$

- 原极限为 $-\frac{1}{2}f'(x_0)$.

- 我们也可以用一阶近似公式 $f(x)=f(x_0)+f'(x_0)\Delta x+o(\Delta x)$，$\Delta x=x-x_0$.

$$\begin{aligned} f(x_0+\cos x-1) &= f(x_0)+f'(x_0)\cdot(\cos x-1)+o(\cos x-1) \\ &= f(x_0)+f'(x_0)\cdot(\cos x-1)+o\left(x^2\right), \end{aligned}$$

$$\begin{aligned} \lim_{x\to 0}\frac{f(x_0+\cos x-1)-f(x_0)}{x^2} &= \lim_{x\to 0}\left[\frac{f'(x_0)(\cos x-1)}{x^2}+\frac{o\left(x^2\right)}{x^2}\right] \\ &= \lim_{x\to 0}\left[\frac{f'(x_0)\left(-\frac{1}{2}x^2\right)}{x^2}\right]=-\frac{1}{2}f'(x_0). \end{aligned}$$

- 2. (A) $f(0)=\lim\limits_{x\to 0}f(x)=\left[\lim\limits_{x\to 0}\frac{f(x)}{x}\right]\cdot\left(\lim\limits_{x\to 0}x\right)=0$，因此 $f(0)=0$.

- (B) $2f(0)=\lim\limits_{x\to 0}[f(x)+f(-x)]=\left[\lim\limits_{x\to 0}\frac{f(x)+f(-x)}{x}\right]\cdot\left(\lim\limits_{x\to 0}x\right)=0$，因此 $f(0)=0$.
- (C) 由于 $f(0)=0$，因此 $f'(0)=\lim\limits_{x\to 0}\frac{f(x)}{x}$ 存在.
- (D) 错误，例如 $f(x)=|x|$.
- 3. $\lim\limits_{x\to\frac{1}{2}}[f(x)+1]=\left[\lim\limits_{x\to\frac{1}{2}}\frac{f(x)+1}{2x-1}\right]\cdot\left[\lim\limits_{x\to\frac{1}{2}}(2x-1)\right]=3\cdot 0=0.$
- 由于 $f(x)$ 在 $x=\frac{1}{2}$ 处可导，从而连续，$f\left(\frac{1}{2}\right)+1=0, f\left(\frac{1}{2}\right)=-1.$
- 所以 $f'\left(\frac{1}{2}\right)=\lim\limits_{x\to\frac{1}{2}}\frac{f(x)+1}{x-\frac{1}{2}}=2\lim\limits_{x\to\frac{1}{2}}\frac{f(x)+1}{2x-1}=6.$

- 4.(1) $f'(a)=\lim\limits_{x\to a}\frac{(x-a)\varphi(x)-0}{x-a}=\lim\limits_{x\to a}\varphi(x)=\varphi(a)$ 存在.
- (2) $f'_+(a)=\lim\limits_{x\to a^+}\frac{|x-a|\varphi(x)-0}{x-a}=\lim\limits_{x\to a^+}\varphi(x)=\varphi(a)$, 类似地 $f'_-(a)=-\varphi(a)$.
- 因此当且仅当 $\varphi(a)=0$ 时极限存在且 $f'(a)=0$.
- 5. 由于

$$f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x}\overset{\text{偶函数}}{=}\lim_{x\to 0}\frac{f(-x)-f(0)}{x}$$

$$=\lim_{t\to 0}\frac{f(t)-f(0)}{-t}=-f'(0),$$

- 因此 $f'(0)=0$.

- 习题3-2
- (A)1.(1) 正确, 因为

$$v = (u+v) - u = u - (u - v),$$

- 如果 $u \pm v$ 和 u 均可导, 则 v 也可导.
- (2)错误, 例如 $u(x) = 0$.
- 2(1) $y' = 3(1+\sin x)^2(1+\sin x)' = 3\cos x\,(1+\sin x)^2$.
- (2) $y' = (e^x)'(\sin x + \cos x) + e^x(\sin x + \cos x)'$

$$= e^x(\sin x + \cos x) + e^x(\cos x - \sin x) = 2e^x \cos x\,.$$

- (3) $y' = \dfrac{(\sec x + \tan x)'}{\sec x + \tan x} = \dfrac{\dfrac{\sin x}{\cos^2 x} + \dfrac{1}{\cos^2 x}}{\sec x + \tan x} = \dfrac{1}{\cos x} = \sec x$.

- (4) $y' = \sqrt{a^2 - x^2} + x \cdot \dfrac{-2x}{2\sqrt{a^2 - x^2}} + \dfrac{a^2}{\sqrt{1 - \dfrac{x^2}{a^2}}} \cdot \dfrac{1}{a}$

$$= \frac{1}{\sqrt{a^2 - x^2}}\left(a^2 - x^2 - x^2 + a^2\right) = 2\sqrt{a^2 - x^2}.$$

- (5) $$y' = \frac{1}{1+\left(\frac{x+1}{x-1}\right)^2}\cdot\left(\frac{x+1}{x-1}\right)' = \frac{(x-1)^2}{2x^2+2}\cdot\left(1+\frac{2}{x-1}\right)'$$

$$= \frac{(x-1)^2}{2x^2+2}\cdot\frac{-2}{(x-1)^2} = -\frac{1}{1+x^2}.$$

- (6) $$y' = n\sin^{n-1}x\cdot\cos x\cdot\cos nx + \sin^n x\cdot(-n\sin nx)$$

$$= n\sin^{n-1}x\,(\cos x\cdot\cos nx - \sin x\cdot\sin nx)$$

$$= n\sin^{n-1}x\cos(n+1)x.$$

- 3

$$y = e^{u(x)\ln v(x)} = e^{u(x)\ln v(x)} \cdot [u(x)\ln v(x)]'$$

$$= v(x)^{u(x)} \cdot \left[u'(x)\ln v(x) + u(x)\cdot\frac{v'(x)}{v(x)}\right].$$

- 也可以用对数求导法. 因此

$$\left(x^{\sin x}\right)' = x^{\sin x}\left(\cos x\ln x + \frac{\sin x}{x}\right).$$

- (B)1.(1) 错误, 例如 $u = x^2, y = |u| = x^2$.
- (2) 错误, 例如 $f(x) = 0$.

- 2(1) $y' = e^{\sin^2\frac{1}{x}} \cdot 2\sin\frac{1}{x} \cdot \cos\frac{1}{x} \cdot \left(-\frac{1}{x^2}\right) = -\frac{1}{x^2}\sin\frac{2}{x}e^{\sin^2\frac{1}{x}}$.

- (2) $y = \ln\left(\sqrt{1+e^x}-1\right) - \ln\left(\sqrt{1+e^x}+1\right)$,

$$y' = \left(\frac{1}{\sqrt{1+e^x}-1} - \frac{1}{\sqrt{1+e^x}+1}\right)\left(\sqrt{1+e^x}\right)'$$

$$= \frac{2}{e^x} \cdot \frac{1}{2\sqrt{1+e^x}} \cdot e^x = \frac{1}{\sqrt{1+e^x}}.$$

- （3） $y' = \dfrac{\left(x+\sqrt{x+\sqrt{x}}\right)'}{2\sqrt{x+\sqrt{x+\sqrt{x}}}} = \dfrac{1}{2\sqrt{x+\sqrt{x+\sqrt{x}}}} \cdot \left[1 + \dfrac{\left(x+\sqrt{x}\right)'}{2\sqrt{x+\sqrt{x}}}\right]$

$$= \frac{1}{2\sqrt{x+\sqrt{x+\sqrt{x}}}} \cdot \left(1 + \frac{1+\frac{1}{2\sqrt{x}}}{2\sqrt{x+\sqrt{x}}}\right)$$

$$= \frac{1+2\sqrt{x}+4\sqrt{x^2+x\sqrt{x}}}{8\sqrt{x^2+x\sqrt{x}} \cdot \sqrt{x+\sqrt{x+\sqrt{x}}}}.$$

- (4) $y' = (\sin\ln x + \cos\ln x) + x\left(\cos\ln x \cdot \frac{1}{x} - \sin\ln x \cdot \frac{1}{x}\right)$

  $= 2\cos\ln x\,.$

- 3 $y' = \dfrac{[f^2(x) + g^2(x)]'}{2\sqrt{f^2(x) + g^2(x)}} = \dfrac{f(x)f'(x) + g(x)g'(x)}{\sqrt{f^2(x) + g^2(x)}}.$

- 习题3-3

- (A)1.(1) $f'''(x) = 8 \cdot 7 \cdot 6(x-10)^5,\ f'''(11) = 336.$

- (2) $y' = \dfrac{1}{\cos x} \cdot (-\sin x) = -\tan x\,,\qquad y'' = -\dfrac{1}{\cos^2 x}.$

- 2. $y' = e^x \sin x + e^x \cos x = e^x(\sin x + \cos x)$,

  $y'' = e^x(\sin x + \cos x) + e^x(\cos x - \sin x) = 2e^x \cos x$,

- 因此

$$y'' - 2y' + 2y = e^x[2\cos x - 2(\sin x + \cos x) + 2\sin x] = 0.$$

- 3(1) $y' = e^x\left(\frac{1}{x} - \frac{1}{x^2}\right)$,

$$y'' = e^x\left(\frac{1}{x} - \frac{1}{x^2}\right) + e^x\left(-\frac{1}{x^2} + \frac{2}{x^3}\right) = \frac{e^x\left(x^2 - 2x + 2\right)}{x^3}.$$

- 也可以 $y'' = (e^x)''x^{-1} + 2(e^x)'\left(x^{-1}\right)' + e^x\left(x^{-1}\right)'' = e^x\left(\frac{1}{x} - \frac{2}{x^2} + \frac{2}{x^3}\right)$.

- (2) $y' = e^x \cos x - e^x \sin x = e^x(\cos x - \sin x)$,

  $y'' = e^x(\cos x - \sin x) + e^x(-\sin x - \cos x) = -2e^x \sin x$,

  $y''' = -2e^x(\sin x + \cos x)$,

  $y^{(4)} = -4e^x \cos x$.

- 也可以 $y^{(4)} = e^x \cos x - 4e^x \sin x - 6e^x \cos x + 4e^x \sin x + e^x \cos x = -4e^x \cos x$.

- 4. $y' = f'(x^2) \cdot 2x$,

  $y'' = f''(x^2) \cdot 2x \cdot 2x + f'(x^2) \cdot 2 = 4x^2 f''(x^2) + 2f'(x^2)$.

- (B)1.(1) $y=\frac{1}{2}[\ln(1-x)-\ln(1+x)]$,

$$y'=\frac{1}{2}\left(-\frac{1}{1-x}-\frac{1}{1+x}\right)=\frac{1}{x^2-1},\qquad y''=-\frac{2x}{(1-x^2)^2},\qquad y''\Big|_{x=0}=0.$$

- 实际上 $y^{(n)}=\frac{(-1)^{n+1}(n-1)!}{2}\left[\frac{1}{(x-1)^n}-\frac{1}{(x+1)^n}\right]$.
- (2) $x\geq 0$ 时, $f(x)=x^3, f_+'(x)=3x^2, f_+''(x)=6x, f_+'''(x)=6$;
- $x\leq 0$ 时, $f(x)=-x^3, f_-'(x)=-3x^2, f_-''(x)=-6x, f_-'''(x)=6$.
- 因此 $f'(0)=0, f''(0)=0, f''(0)$ 不存在, $n=2$.

- 2.(1) $\dfrac{d^2x}{dy^2}=\dfrac{d}{dy}\left(\dfrac{dx}{dy}\right)=\dfrac{d}{dy}\left(\dfrac{1}{y'}\right)$

$$=\frac{1}{y'}\frac{d}{dx}\left(\frac{1}{y'}\right)=\frac{1}{y'}\cdot\left(-\frac{y''}{(y')^2}\right)=-\frac{y''}{(y')^3}.$$

- (2) $\dfrac{d^3x}{dy^3}=\dfrac{d}{dy}\left(\dfrac{d^2x}{dy^2}\right)=-\dfrac{1}{y'}\dfrac{d}{dx}\left[\dfrac{y''}{(y')^3}\right]$

$$=-\frac{1}{y'}\cdot\left[\frac{y'''}{(y')^3}-\frac{y''\cdot 3y''}{(y')^4}\right]=\frac{3(y'')^2-y'y'''}{(y')^5}.$$

- 3.(1) $y^{(20)} = x^2(\cos x)^{(20)} + 20 \cdot 2x(\cos x)^{(19)} + 190 \cdot 2(\cos x)^{(18)}$

$$= x^2 \cos\left(x + 20 \cdot \frac{\pi}{2}\right) + 40x \cos\left(x + 19 \cdot \frac{\pi}{2}\right) + 380 \cos\left(x + 18 \cdot \frac{\pi}{2}\right)$$

$$= x^2 \cos x + 40x \sin x - 380 \cos x\,.$$

- (2) $y = \frac{1}{2} - \frac{1}{2}\cos 2x$, 因此 $y^{(n)} = -2^{n-1}\cos\left(2x + \frac{n\pi}{2}\right)$.

- 4. $y'' = 2[f(e^{-x})]' + x[f(e^{-x})]''$

$$= 2f'(e^{-x})(-e^{-x}) + x[f'(e^{-x}) \cdot (-e^{-x})]'$$

$$= -2f'(e^{-x})e^{-x} + x\left[f''(e^{-x}) \cdot (-e^{-x})^2 + f'(e^{-x}) \cdot e^{-x}\right]$$

$$= xe^{-2x}f''(e^{-x}) + (x-2)e^{-x}f'(e^{-x}).$$

- 5. $f(x) = -1 + \frac{2}{1+x}$, 因此 $f^{(n)}(x) = \frac{2(-1)^n n!}{(x+1)^{n+1}}$.
- 6. 归纳法. $n = 1$ 已经成立.
- 假设 $f^{(n)}(x) = n!\,[f(x)]^{n+1}$, 则

$$f^{(n+1)}(x) = n! \cdot (n+1) f(x)^n f'(x) = (n+1)!\,[f(x)]^{n+2}.$$

- 习题3-4
- (A)1.(1) $2x + 2y + 2xy' - 2yy' = 2$,

$$y' = \frac{x+y-1}{y-x} = 1 + \frac{2x-1}{y-x}.$$

- (2) $y + xy' = e^{x+y}(1 + y')$. 此时我们不必解出 y'.
- 当 $x = 0$ 时, $e = e^{y(0)}, y(0) = 1, 1 = e\big(1 + y'(0)\big), y'(0) = \frac{1-e}{e}$.
- (3) $2x - y' = e^y y'$, $y' = \frac{2x}{1+e^y}$,

$$y'' = \frac{2}{1 + e^y} - \frac{2x}{(1 + e^y)^2} \cdot e^y \cdot y' = \frac{2}{1 + e^y} - \frac{4x^2 e^y}{(1 + e^y)^3}.$$

- 当 $x = 0$ 时, $-y(0) + 1 = e^{y(0)}, y(0) = 0, y'(0) = 0, y''(0) = 1.$
- 也可以直接: $-y'(0) = e^{y(0)} y'(0), y'(0) = 0,$
- $2 - y'' = e^y (y')^2 + e^y y''$, $2 - y''(0) = y''(0)$, $y''(0) = 1.$

- 2. $3x^2+3y^2y'-3y-3xy'=0.$
- 将 $x=\sqrt[3]{2}, y=\sqrt[3]{4}$ 代入得到 $3\sqrt[3]{4}+3\left(\sqrt[3]{4}\right)^2y'-3\sqrt[3]{4}-3\sqrt[3]{2}y'=0, y'=0.$
- 因此切线方程为 $y=\sqrt[3]{4}$, 法线方程为 $x=\sqrt[3]{2}$.

- 3.(1) $y'=\dfrac{dy}{dx}=\dfrac{dy/dt}{dx/dt}=\dfrac{-\sin t}{3t^2}=-\dfrac{\sin t}{3t^2},$

$$\frac{dy'}{dt}=-\frac{\cos t}{3t^2}-\sin t\left(-2\cdot\frac{1}{3}t^{-3}\right)=\frac{2\sin t-t\cos t}{3t^3},$$

$$y''=\frac{d^2y}{dx^2}=\frac{dy'}{dx}=\frac{dy'/dt}{dx/dt}=\frac{2\sin t-t\cos t}{9t^5}.$$

- (2) $y' = \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{\sin t + t\cos t}{\cos t - t\sin t}$,

$$\frac{dy'}{dt} = \frac{(\cos t + \cos t - t\sin t)(\cos t - t\sin t) - (\sin t + t\cos t)(-\sin t - \sin t - t\cos t)}{(\cos t - t\sin t)^2}$$

$$= \frac{2 + t^2}{(\cos t - t\sin t)^2},$$

$$y'' = \frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{dy'/dt}{dx/dt} = \frac{2 + t^2}{(\cos t - t\sin t)^3}.$$

- 4. $\dfrac{dx}{dt}=\dfrac{1+t^2-t\cdot 2t}{(1+t^2)^2}=\dfrac{1-t^2}{(1+t^2)^2},$

$$\frac{dy}{dt}=\frac{d}{dt}\left(1-\frac{1}{1+t^2}\right)=\frac{2t}{(1+t^2)^2},$$

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{2t}{1-t^2},$$

$$x\Big|_{t=2}=\frac{2}{5},\qquad y\Big|_{t=2}=\frac{4}{5},\qquad y'\Big|_{t=2}=-\frac{4}{3}.$$

- 因此切线方程为 $y=-\frac{4}{3}\left(x-\frac{2}{5}\right)+\frac{4}{5}=-\frac{4}{3}x+\frac{4}{3},$
- 法线方程为 $y=\frac{3}{4}\left(x-\frac{2}{5}\right)+\frac{4}{5}=\frac{3}{4}x+\frac{1}{2}.$

- (B)1.(1) $\dfrac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(\dfrac{y'}{x}-\dfrac{y}{x^2}\right)=\dfrac{1}{2}\cdot\dfrac{1}{x^2+y^2}\cdot(2x+2yy'),$

$$xy'-y=x+yy',y'=\frac{x+y}{x-y}.$$

$$y'+xy''-y'=1+(y')^2+yy'',$$

$$y''=\frac{1+(y')^2}{x-y}=\frac{2\left(x^2+y^2\right)}{(x-y)^3}.$$

- (2) $y' = e^{xy} + xe^{xy}(y + xy') = e^{xy}\left(1 + xy + x^2y'\right)$,

$$y' = \frac{1 + xy}{e^{-xy} - x^2}.$$

$$y(0) = 1, \qquad y'(0) = 1.$$

$$y'' = e^{xy}(y + xy')\left(1 + xy + x^2y'\right) + e^{xy}\left(y + xy' + 2xy' + x^2y''\right),$$

$$y''(0) = 1 + 1 = 2.$$

- 2.(1) $\frac{dx}{dt}=\frac{2t}{1+t^2}$, $\frac{dy}{dt}=1-\frac{1}{1+t^2}=\frac{t^2}{1+t^2}$, $y'=\frac{dy/dt}{dx/dt}=\frac{t}{2}$,

$$\frac{dy'}{dt}=\frac{1}{2},\qquad \frac{d^2y}{dx^2}=\frac{dy'/dt}{dx/dt}=\frac{1+t^2}{4t}.$$

- (2) $\frac{dx}{dt}=f''(t)$, $\frac{dy}{dt}=f'(t)+tf''(t)-f'(t)=tf''(t)$,

$$y'=\frac{dy/dt}{dx/dt}=t,\qquad \frac{dy'}{dt}=1,\qquad \frac{d^2y}{dx^2}=\frac{dy'/dt}{dx/dt}=\frac{1}{f''(t)}.$$

- 3.(1) $x = e^{\theta}\cos\theta, y = e^{\theta}\sin\theta.$

$$\frac{dx}{d\theta} = e^{\theta}\cos\theta - e^{\theta}\sin\theta, \qquad \frac{dy}{d\theta} = e^{\theta}\sin\theta + e^{\theta}\cos\theta,$$

$$\frac{dy}{dx} = \frac{\sin\theta + \cos\theta}{\cos\theta - \sin\theta}, \qquad \left.\frac{dy}{dx}\right|_{\theta=\pi} = 1, x_0 = -e^{\pi}, y_0 = 0.$$

- 切线方程为 $y = x + e^{\pi}$, 法线方程为 $y = -x - e^{\pi}$.

- 习题3-5

- (A)1.(1) $\Delta y = y(1+\Delta x) - y(1)$

$$= (1+\Delta x)^2 + 2(1+\Delta x) - 3$$

$$= 4\Delta x + (\Delta x)^2,$$

$$dy = 4\Delta x,$$

$$\Delta y\Big|_{\Delta x=1} = 5, \qquad dy\Big|_{\Delta x=1} = 4,$$

$$\Delta y\Big|_{\Delta x=0.1} = 0.41, \qquad dy\Big|_{\Delta x=0.1} = 0.4,$$

$$\Delta y\Big|_{\Delta x=0.01} = 0.0401, \qquad dy\Big|_{\Delta x=0.01} = 0.04.$$

- 2.(1) $d\left(\frac{1}{2}x^2\right) = xdx.$
- (2) $d(-\sin x) = \cos x\, dx.$
- (3) $d(\ln|1+x|) = \frac{1}{1+x}dx.$
- (4) $d(e^{-x}) = -e^{-x}dx,\quad d(\cot x) = \csc^2 x dx,$
- $d(-e^{-x} - \cot x) = \left(e^{-x} - \csc^2 x\right)dx.$
- 3.(1) $dy = \left(2x\sin 2x + x^2\cos 2x\cdot 2\right)dx = \left(2x\sin 2x + 2x^2\cos 2x\right)dx.$

- (2) $dy = \left[\frac{1}{2a}\left(\frac{1}{x+a} - \frac{1}{x-a}\right)\right]dx = \frac{dx}{a^2 - x^2}.$

- (3) $dy = \left[\arcsin\frac{x}{2} + x \cdot \frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}} \cdot \frac{1}{2} + \frac{1}{2\sqrt{4-x^2}} \cdot (-2x)\right]dx$

  $= \arcsin\frac{x}{2}dx.$

- (4) $dy = [-e^{-x}\cos(x-3) + e^{-x} \cdot (-\sin(x-3))]dx$

  $= -e^{-x}[\cos(x-3) + \sin(x-3)]dx.$

- 4.(1) 由于 $\sqrt[3]{1000}=10$, $997=10^3(1-0.003)$, 所以

$$\sqrt[3]{997}=10(1-0.003)^{\frac{1}{3}}\approx 10\left(1-\frac{1}{3}\cdot 0.003\right)=9.99.$$

- 换种写法, 令 $f(x)=10\sqrt[3]{x}$, 则

$$f'(x)=\frac{10}{3}x^{-\frac{2}{3}},\qquad f'(1)=\frac{10}{3},$$

$$\sqrt[3]{997}=f(0.997)\approx f(1)+f'(1)(0.997-1)$$

$$=10-\frac{10}{3}\cdot 0.003=9.99.$$

- (2) 令 $f(x)=\arctan x$, $f'(x)=\frac{1}{1+x^2}$, $f'(1)=\frac{1}{2}$,

$$\arctan 1.05=f(1.05)\approx f(1)+f'(1)(1.05-1)=\frac{\pi}{4}+0.025\approx 0.8104.$$

- (3) 令 $f(x)=\ln x$, $f'(x)=\frac{1}{x}$, $f'(1)=1$,

$$\ln 1.01=f(1.01)\approx f(1)+f'(1)(1.01-1)=0.01.$$

- 5. 球体的体积为 $\frac{4\pi D^3}{3}$, 因此球壳体积

$$V=\frac{4\pi}{3}\left[(D+h)^3-D^3\right]\approx\left(\frac{4\pi D^3}{3}\right)'h=4\pi D^2 h.$$

- (B)1. $dy = f'(x_0)\Delta x + o(\Delta x)^2 = \frac{1}{2}\Delta x + o(\Delta x)^2$, 选 B.
- 2. $e^{xy\ln 2} = x + y$, $\quad e^{xy\ln 2}\ln 2\,(y + xy') = 1 + y'$.

$$y(0) = 1, \qquad \ln 2 = 1 + y'(0),$$

$$y'(0) = \ln 2 - 1, \qquad dy = (\ln 2 - 1)dx.$$

- 3. $2y' - 1 = (1 - y')\ln(x - y) + (x - y)\cdot\frac{1}{x-y}\cdot(1 - y')$,

$$y' = 1 - \frac{1}{3 + \ln(x - y)}, \qquad dy = \left[1 - \frac{1}{3 + \ln(x - y)}\right]dx.$$

- 4. $$y' = f'(\arcsin x^2) \cdot \frac{1}{\sqrt{1-x^4}} \cdot 2x - \sin f(x) \cdot f'(x),$$

$$dy = \left[f'(\arcsin x^2) \cdot \frac{2x}{\sqrt{1-x^4}} - \sin f(x) \cdot f'(x) \right] dx.$$

- 总复习题三

- 1.(1) $$f(t) = \lim_{x\to\infty} t\left(\frac{x+t}{x-t}\right)^x = te^{\lim\limits_{x\to\infty}\left(\frac{x+t}{x-t}-1\right)x} = te^{2t},$$

$$f'(t) = e^{2t} + 2te^{2t} = (2t+1)e^{2t}.$$

- (2) $\cos xy \cdot (y + xy') + \dfrac{1}{y - x} \cdot (y' - 1) = 1,$

$$1 + y'(0) - 1 = 1, \qquad y'(0) = 1,$$

- 切线方程为 $y = x + 1.$

- (3) $\dfrac{dx}{dt} = \cos t\,, \qquad \dfrac{dy}{dt} = \sin t + t\cos t - \sin t = t\cos t\,,$

$$y' = \frac{dy/dt}{dx/dt} = t, \qquad \frac{dy'}{dt} = 1, \qquad y'' = \frac{dy'/dt}{dx/dt} = \frac{1}{\cos t}, \qquad y''\Big|_{t=\frac{\pi}{4}} = \sqrt{2}.$$

- (4) $f^{(n)}(x) = x^2 \cdot (\ln 2)^n \cdot 2^x + n \cdot 2x \cdot (\ln 2)^{n-1} \cdot 2^x$

$$+\frac{n(n-1)}{2} \cdot 2 \cdot (\ln 2)^{n-2} \cdot 2^x, \qquad f^{(n)}(0) = n(n-1)(\ln 2)^{n-2}.$$

- 2.(1) $\lim\limits_{x\to 0}\frac{xf(x)-2f\left(x^2\right)}{x^2} = \lim\limits_{x\to 0}\frac{f(x)-f(0)}{x-0} - 2 \cdot \lim\limits_{x\to 0}\frac{f\left(x^2\right)-f(0)}{x^2-0} = f'(0) - 2f'(0) = -f'(0)$, 选 C.
- 也可以代入 $f(x) = x$ 用排除法.

- 也可以 $f(x)=f(0)+f'(0)x+o(x)=f'(0)x+o(x)$,

$$xf(x)-2f\left(x^2\right)=f'(0)x^2+o\left(x^2\right)-2f'(0)x^2+o\left(x^2\right)$$

$$=-f'(0)x^2+o\left(x^2\right).$$

$$\lim_{x\to 0}\frac{xf(x)-2f\left(x^2\right)}{x^2}=\lim_{x\to 0}\frac{xf(x)-2f\left(x^2\right)}{x^2}=-f'(0)+\lim_{x\to 0}\frac{o\left(x^2\right)}{x^2}=-f'(0).$$

- (2) $\left[f^2(x)\right]'=2f(x)f'(x)>0$, 因此 $f^2(x)$ 单增, $f^2(1)>f^2(-1)$, 选 C.

- (3) $f'(x)=(e^x-1)\left[(e^{2x}-2)\cdots(e^{nx}-n)\right]'+e^x\left[(e^{2x}-2)\cdots(e^{nx}-n)\right]$,
- $f'(0)=(1-2)\cdots(1-n)=(-1)^{n-1}(n-1)!$, 选 A.
- 也可以

$$f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{e^x-1}{x}(e^{2x}-2)\cdots(e^{nx}-n)$$
$$=(-1)^{(n-1)}(n-1)!.$$

- (4) 选 D.
- 3. $(e^x)'|_{x=0}=e^x|_{x=0}=1$, 因此 $f'(0)=1$. 由于 $f(0)=1$, 因此 $x\to 0$ 时, $f(x)-1\to x$ (不能这么写).

- 因此 $\frac{f(x)-1}{x} \to 1.$

$$\lim_{n\to\infty}\sqrt{n\left[f\left(\frac{2}{n}\right)-1\right]}=\lim_{t\to 0}\sqrt{\frac{f(2t)-1}{t}}=\lim_{t\to 0}\sqrt{2\cdot\frac{f(2t)-1}{2t}}=\sqrt{2}.$$

- 或者 $f(x)=1+x+o(x)$,

$$\lim_{n\to\infty}\sqrt{n\left[f\left(\frac{2}{n}\right)-1\right]}=\lim_{n\to\infty}\sqrt{2+\frac{o(1/n)}{1/n}}=\sqrt{2}.$$

- 4. $$\lim_{x\to 0}\frac{f(e^x)-f(1)}{e^x-1}=\lim_{e^x\to 1}\frac{f(e^x)-f(1)}{e^x-1}=\lim_{t\to 1}\frac{f(t)-f(1)}{t-1}=f'(1),$$
- 因此

$$\lim_{x\to 0}\frac{f(e^x)-f(1)}{x}=\lim_{x\to 0}\frac{e^x-1}{x}\cdot\lim_{x\to 0}\frac{f(e^x)-f(1)}{e^x-1}=f'(1).$$

$$\lim_{x\to 0}\frac{f(1+\sin x)-f(1)}{\sin x}=\lim_{\sin x\to 0}\frac{f(1+\sin x)-f(1)}{\sin x}=\lim_{t\to 1}\frac{f(1+t)-f(1)}{t}=f'(1),$$

- 因此

$$\lim_{x\to 0}\frac{f(1+\sin x)-f(1)}{x}=\lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{f(1+\sin x)-f(1)}{\sin x}=f'(1).$$

- 故

$$3f'(1)=\lim_{x\to 0}\frac{f(e^x)+2f(1+\sin x)-3f(1)}{x}=\lim_{x\to 0}\frac{6x-3f(1)+o(x)}{x}=6.$$

- 这迫使 $f(1)=0, f'(1)=2$, 从而 $f(-1)=0, f'(-1)=2$.
- 切线方程为 $y=2(x+1)=2x+2$.
- 也可以 $f(x)=f(1)+f'(1)(x-1)+o(x-1)$, 于是 $x\to 0$ 时,

$$f(e^x)+2f(1+\sin x)=3f(1)+f'(1)(e^x-1+2\sin x)+o(x)$$

- 由于 $e^x-1=x+o(x), \sin x=x+o(x)$, 因此

$$f(e^x)+2f(1+\sin x)=3f(1)+3f'(1)x+o(x)=6x+o(x)$$

- 所以 $f(1)=0, f'(1)=2$.

- 5. 首先注意到 f 在 x_0 处连续.
- 如果 $f(x_0) \neq 0$, 则由极限的保号性可知存在 x_0 的一个邻域 $(x_0-\delta, x_0+\delta)$, 使得对任意 $x \in (x_0-\delta, x_0+\delta)$, $f(x)$ 和 $f(x_0)$ 符号相同.
- 于是

$$\lim_{x\to x_0}\frac{|f(x)|-|f(x_0)|}{x-x_0}=\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}\operatorname{sgn}[f(x_0)]=\operatorname{sgn}[f(x_0)]\cdot f'(x_0)$$

- 存在, 因此 $|f(x)|$ 在 x_0 处可导.
- 如果 $f(x_0)=0$ 且 $f'(x_0)=0$, 则 $0 \le \left|\frac{|f(x)|-0}{x-x_0}\right| = \left|\frac{f(x)}{x-x_0}\right|$.
- 由夹逼准则, $\lim\limits_{x\to x_0}\frac{|f(x)|-0}{x-x_0}=0$, 因此 $|f(x)|$ 在 x_0 处可导且导数为 0.

- 如果 $f(x_0)=0$ 且 $f'(x_0)=A>0$, 则由极限的保号性可知存在 x_0 的一个邻域 $(x_0-\delta, x_0+\delta)$, 使得对任意 $x\in(x_0-\delta, x_0+\delta)$, $\frac{f(x)}{x-x_0}>0$.
- 于是

$$\lim_{x\to x_0^+}\frac{|f(x)|}{x-x_0}=\lim_{x\to x_0^+}\frac{f(x)}{x-x_0}=f'(x_0),$$

$$\lim_{x\to x_0^-}\frac{|f(x)|}{x-x_0}=-\lim_{x\to x_0^+}\frac{f(x)}{x-x_0}=-f'(x_0),$$

- 因此 $|f(x)|$ 在 x_0 处不可导. $f'(x_0)<0$ 情形类似.
- 综上, 如果 $f(x_0)\neq 0$, 或者 $f(x_0)=0$ 且 $f'(x_0)=0$, 则 $|f(x)|$ 在 x_0 处可导. 如果 $f(x_0)=0$ 且 $f'(x_0)\neq 0$, 则 $|f(x)|$ 在 x_0 处不可导.

- 6. $y' = \dfrac{1}{\tan\frac{x}{2}} \cdot \dfrac{1}{\cos^2\frac{x}{2}} \cdot \dfrac{1}{2} + \sin x \cdot \ln\tan x - \cos x \cdot \dfrac{1}{\tan x} \cdot \dfrac{1}{\cos^2 x}$

$$= \frac{1}{\sin x} + \sin x \cdot \ln\tan x - \frac{1}{\sin x} = \sin x \cdot \ln\tan x\,.$$

- 7. $\tan y + x \cdot \dfrac{1}{\cos^2 y} \cdot y' = -\sin xy \cdot (y + xy'),$

$$y' = -\frac{\tan y + y\sin(xy)}{x(\sin(xy) + \sec^2 y)}, \qquad dy = -\frac{\tan y + y\sin(xy)}{x(\sin(xy) + \sec^2 y)}dx.$$

- 8. $y' = \dfrac{1}{\cos^2[f(x^2)]} \cdot f'(x^2) \cdot 2x,$

$$y'' = -\frac{2}{\cos^3[f(x^2)]} \cdot (-\sin[f(x^2)]) \cdot [f'(x^2) \cdot 2x]^2$$

$$+\frac{1}{\cos^2[f(x^2)]} \cdot f''(x^2) \cdot (2x)^2 + \frac{1}{\cos^2[f(x^2)]} \cdot f'(x^2) \cdot 2$$

$$= 2\sec^2[f(x^2)]\, f'(x^2) + 4x^2 \sec^2[f(x^2)]\, f''(x^2)$$

$$+8x^2 \sec^2[f(x^2)] \tan[f(x^2)]\, f'(x^2)^2.$$

- 9. $\ln x + f(y) = y,\qquad \frac{1}{x} + f'(y)y' = y',$

$$y' = \frac{1}{x[1 - f'(y)]},\qquad -\frac{1}{x^2} + f''(y)(y')^2 + f'(y)y'' = y'',$$

$$y'' = \frac{1}{1 - f'(y)}\left[f''(y)(y')^2 - \frac{1}{x^2}\right] = \frac{f''(y) - [1 - f'(y)]^2}{x^2[1 - f'(y)]^3}.$$

- 10. $y\big|_{t=0}=1,\quad \frac{dx}{dt}=6t+2,\quad \frac{dx}{dt}\Big|_{t=0}=2,\quad e^y\frac{dy}{dt}\sin t+e^y\cos t-\frac{dy}{dt}=0,$

$$\frac{dy}{dt}=\frac{\cos t}{e^{-y}-\sin t},\quad \frac{dy}{dt}\Big|_{t=0}=e,\quad y'=\frac{dy}{dx}=\frac{\cos t}{(6t+2)(e^{-y}-\sin t)},$$

$$\frac{dy'}{dt}=\frac{-\sin t}{(6t+2)(e^{-y}-\sin t)}$$

$$-\frac{\cos t}{[(6t+2)(e^{-y}-\sin t)]^2}\left[6(e^{-y}-\sin t)+(6t+2)\left(-e^{-y}\frac{dy}{dt}-\cos t\right)\right],$$

$$\frac{dy'}{dt}\Big|_{t=0}=-\frac{1}{4e^{-2}}(6e^{-1}-4)=\frac{2e^2-3e}{2},\quad \frac{dy'}{dx}\Big|_{t=0}=\frac{2e^2-3e}{4}.$$

- 11. $y=x(x+1)^{-\frac{1}{2}}$,

$$y'=(x+1)^{-\frac{1}{2}}-\frac{1}{2}x(x+1)^{-\frac{3}{2}}=\left(\frac{x}{2}+1\right)(x+1)^{-\frac{3}{2}},$$

$$y''=\frac{1}{2}(x+1)^{-\frac{3}{2}}-\frac{3}{2}\left(\frac{x}{2}+1\right)(x+1)^{-\frac{5}{2}}=\left(-\frac{x}{4}-1\right)(x+1)^{-\frac{5}{2}},$$

$$y'''=-\frac{1}{4}(x+1)^{-\frac{5}{2}}-\frac{5}{2}\left(-\frac{x}{4}-1\right)(x+1)^{-\frac{7}{2}}=\left(\frac{3x}{8}+\frac{9}{4}\right)(x+1)^{-\frac{7}{2}},$$

$$y^{(4)}=\frac{3}{8}(x+1)^{-\frac{7}{2}}-\frac{7}{2}\left(\frac{3x}{8}+\frac{9}{4}\right)(x+1)^{-\frac{9}{2}}$$

$$=\left(-\frac{15}{16}x-\frac{15}{2}\right)(x+1)^{-\frac{9}{2}}=-\frac{15(x+8)}{16(x+1)^{4}\sqrt{x+1}}.$$

- 另解 $y = (x+1)^{\frac{1}{2}} - (x+1)^{-\frac{1}{2}}$,
- $y^{(4)} = \frac{1}{2} \cdot \left(-\frac{1}{2}\right) \cdot \left(-\frac{3}{2}\right) \cdot \left(-\frac{5}{2}\right)(x+1)^{-\frac{7}{2}} - \left(-\frac{1}{2}\right) \cdot \left(-\frac{3}{2}\right) \cdot \left(-\frac{5}{2}\right) \cdot \left(-\frac{7}{2}\right)(x+$

- 12. $y + xy' = 0, y' = -\frac{y}{x}$, 因此在 (x_0, y_0) 处的切线方程为

$$y = -\frac{y_0}{x_0}(x - x_0) + y_0.$$

- 它与横纵坐标轴的交点分别为 $(2x_0, 0), (0,2y_0)$, 因此面积为

$$\frac{1}{2}(2x_0)(2y_0) = 2x_0y_0 = 2a^2.$$

- 13. $y' = 2ax = (\ln x)' = \frac{1}{x}$, 因此 $x = \frac{1}{\sqrt{2a}}$.
- 此时 $y = \frac{1}{2} = \ln x = -\frac{1}{2}\ln(2a)$, 因此 $a = \frac{1}{2e}$.