THE GENERATING FIELDS OF TWISTED KLOOSTERMAN SUMS

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ABSTRACT. We use the Kloosterman sheaves constructed by Fisher to show when two twisted Kloosterman sums differ a $(q-1)$-th root of unity, and use p-adic analysis to prove the non-vanishing of twisted Kloosterman sums. Then we can determine the generating fields of twisted Kloosterman sums by these results.

1. INTRODUCTION

1.1. **Background.** Let p be a prime number, $q = p^d$ a power of p, and $\mathbb{F}_q$ the field with q elements. Denote by $\mu_n \subseteq \overline{\mathbb{Q}}^\times$ the group of n-th roots of unity. Let $\psi : \mathbb{F}_p \to \mu_p$ be a fixed non-trivial additive character. For $\boldsymbol{\chi} = \{\chi_1, \dots, \chi_n\}$ an unordered n-tuple of multiplicative characters $\chi_i : \mathbb{F}_q^\times \to \mu_{q-1}$ and $a \in \mathbb{F}_q^\times$, define the *Kloosterman sum* as

$$\mathrm{Kl}_n(\psi, \boldsymbol{\chi}, q, a) = \sum_{\substack{x_1 \cdots x_n = a \\ x_i \in \mathbb{F}_q^\times}} \chi_1(x_1) \cdots \chi_n(x_n) \psi\big(\mathrm{Tr}(x_1 + \cdots + x_n)\big),$$

where $\mathrm{Tr} = \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}$. Clearly it lies in $\mathbb{Z}[\mu_{p(q-1)}]$.

When $\boldsymbol{\chi} = \mathbf{1} = \{1, \cdots, 1\}$ is trivial, the distinctness of Kloosterman sums is studied by many peoples. It's easy to see that

$$a, b \text{ conjugate} \implies \mathrm{Kl}_n(\psi, \mathbf{1}, q, a) = \mathrm{Kl}_n(\psi, \mathbf{1}, q, b).$$

Fisher in [Fis92, Remark 4.28(2)] conjectured that the converse

$$\mathrm{Kl}_n(\psi, \mathbf{1}, q, a) = \mathrm{Kl}_n(\psi, \mathbf{1}, q, b) \implies a, b \text{ conjugate} \tag{1.1}$$

is also true if $p \geq nd$. It's known that (1.1) holds when $p > (2n^{2d} + 1)^2$ in [Fis92], or $p \geq (d-1)n + 2$ and p does not divide a certain integer in [Wan95, Theorem 1.3]. Once (1.1) holds, one can obtain that $\mathrm{Kl}_n(\psi, \mathbf{1}, q, a)$ generates $\mathbb{Q}(\mu_p)^H$, where

$$H = \Big\{ t \in \mathrm{Gal}\big(\mathbb{Q}(\mu_p)/\mathbb{Q}\big) \,\Big|\, \exists k \in \mathbb{Z} \text{ such that } t^n = a^{p^k - 1} \Big\}.$$

1.2. **Notations and main results.** In this article, we will study the *generating fields* of twisted Kloosterman sums.

We need the following notations:

- $c = c(\boldsymbol{\chi}) \mid (q-1)$ the minimal positive integer such that $\chi_i^c = 1, i = 1, \dots, n$, i.e., the least common multiplier of orders of χ_i.
- $\boldsymbol{\chi}^w := \{\chi_1^w, \cdots, \chi_n^w\}$, where $w \in \mathbb{Z}$ or $\mathbb{Z}/c\mathbb{Z}$.

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- $\boldsymbol{\chi}\eta := \{\chi_1\eta, \cdots, \chi_n\eta\}$, where η is a multiplicative character.
- $\boldsymbol{\chi}\circ\sigma := \{\chi_1\circ\sigma, \cdots, \chi_n\circ\sigma\}$, where $\sigma \in \mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p)$.
- $\prod\boldsymbol{\chi} := \chi_1\cdots\chi_n$.

Clearly, the Galois group

$$\mathrm{Gal}\big(\mathbb{Q}(\mu_{pc})/\mathbb{Q}\big) = \Big\{\sigma_t\tau_w \,\Big|\, t \in (\mathbb{Z}/p\mathbb{Z})^\times, w \in \big(\mathbb{Z}/c\mathbb{Z}\big)^\times\Big\},$$

where

$$\sigma_t(\zeta_p) = \zeta_p^t, \sigma_t(\zeta_c) = \zeta_c, \qquad \tau_w(\zeta_p) = \zeta_p, \tau_w(\zeta_c) = \zeta_c^w$$

for primitive $\zeta_p \in \mu_p, \zeta_c \in \mu_c$.

Theorem 1.1. *Assume that* $p > \max\big\{(2n^{2d}+1)^2, (3n-1)c-n\big\}$ *and for any* i, j, $\chi_i = \chi_j$ *if* $\chi_i^n = \chi_j^n$. *Then* $\mathrm{Kl}_n(\psi, \boldsymbol{\chi}, q, a)$ *generates* $\mathbb{Q}(\mu_{pc})^H$, *where* H *consists of those* $\sigma_t\tau_w$ *such that there exists an integer* k *and a character* η *satisfying*

$$t^n = a^{1-p^k}, \quad \boldsymbol{\chi}^w = \boldsymbol{\chi}^{p^k}\eta, \quad \eta(a) = \prod\boldsymbol{\chi}^w(t).$$

A basic observation tells that

$$\sigma_t\tau_w \,\mathrm{Kl}_n(\psi, \boldsymbol{\chi}, q, a) = \prod\boldsymbol{\chi}(t)^{-w}\,\mathrm{Kl}_n(\psi, \boldsymbol{\chi}^w, q, at^n).$$

To study the generating fields, we need to know when two Kloosterman sums differ a $(q-1)$-th roots of unity λ. In § 2, we will recall the construction of Kloosterman sheaves by Fisher. Then we will show when two twisted Kloosterman sums differ λ for sufficiently large p, see Theorem 2.4. We also need the non-vanishingness of Kloosterman sums, which will be proved by p-adic analysis in § 3. Then we finish the proof in § 4. We will end this paper with several examples in § 5.

2. Kloosterman sheaves and Fisher's descent

2.1. **Kloosterman sheaves.** Let $\ell \neq p$ be a prime and fix an embedding $\overline{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C}$. Then the additive and multiplicative characters ψ, χ_i can take value both in $\overline{\mathbb{Q}}_\ell$ or $\mathbb{C}$.

Deligne in [Del77, Theorem 7.8] and Katz in [Kat88, Theorem 4.11] defined the Kloosterman sheaf of $\overline{\mathbb{Q}}_\ell$-modules

$$\mathcal{K}\ell = \mathcal{K}\ell_{n,q}(\psi, \boldsymbol{\chi})$$

on $\mathbb{G}_{m/\mathbb{F}_q}$, with the following properties:

- $\mathcal{K}\ell$ is lisse of rank n and pure of weight $n-1$.
- For any $a \in \mathbb{F}_q^\times$, $\mathrm{Tr}(\mathrm{Frob}_a, \mathcal{K}\ell_{\overline{a}}) = (-1)^{n-1}\,\mathrm{Kl}_n(\psi, \boldsymbol{\chi}, q, a)$.
- $\mathcal{K}\ell$ is tame at 0.
- $\mathcal{K}\ell$ is totally wild with Swan conductor 1 at ∞. So all ∞-breaks are $1/n$.

Here Frob_a is the geometric Frobenius at a.

Definition 2.1. The n-tuple $\boldsymbol{\chi}$ is called *Kummer-induced* if there exists a non-trivial character Λ such that $\boldsymbol{\chi} = \boldsymbol{\chi}\Lambda$ as unordered n-tuples. In this case, $\prod\boldsymbol{\chi} = \prod(\boldsymbol{\chi}\Lambda) = \Lambda^n\prod\boldsymbol{\chi}$ and thus $\Lambda^n = 1$.

When $\boldsymbol{\chi}$ is not Kummer-induced, $\mathcal{K}\ell$ is not *geometrically Kummer-induced*. That's to say, $\mathcal{K}\ell \mid \mathbb{G}_{m/\overline{\mathbb{F}}_p}$ is not of type $(t \mapsto t^N)_*\mathcal{F}$ for some positive integer $N > 1$ and some lisse sheaf $\mathcal{F}$ on $\mathbb{G}_{m/\overline{\mathbb{F}}_p}$. See [Fis92, Theorem 2.9].

2.2. **Fisher's descent.** In [Fis92, §3], Fisher gave a descent of Kloosterman sheaves along an extension of finite fields. For any $a \in \mathbb{F}_q^\times$, he defined a lisse sheaf $\mathcal{F}_a(\boldsymbol{\chi})$ on $\mathbb{G}_m = \mathbb{G}_{m/\mathbb{F}_p}$, such that

- $\mathcal{F}_a(\boldsymbol{\chi}) \mid \mathbb{G}_{m/\mathbb{F}_q} = \bigotimes_{\sigma \in \mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p)} \big(t \mapsto \sigma(a)t^n\big)^* \mathcal{K}\ell_{n,q}(\psi \circ \sigma^{-1}, \boldsymbol{\chi} \circ \sigma^{-1})$.
- $\mathcal{F}_a(\boldsymbol{\chi})$ is lisse of rank n^d and pure of weight $d(n-1)$.
- For any $t \in \mathbb{F}_p^\times$, $\mathrm{Tr}\big(\mathrm{Frob}_t, \mathcal{F}_a(\boldsymbol{\chi})_{\bar{t}}\big) = (-1)^{(n-1)d} \,\mathrm{Kl}_n(\psi, \boldsymbol{\chi}, q, at^n)$.
- $\mathcal{F}_a(\boldsymbol{\chi})$ is tame at 0 and its ∞-breaks are at most 1.

Assume that $p > 2n+1$ and $\boldsymbol{\chi}$ is not Kummer-induced. Then $\mathcal{F}_a(\boldsymbol{\chi})$ has a highest weight with multiplicity one. Thus it has a subsheaf $\mathcal{G}_a(\boldsymbol{\chi})$ such that, as representations of the Lie algebra of the connected geometric monodromy group $G_{\mathrm{geom}}\big(\mathcal{F}_a(\boldsymbol{\chi})\big)^\circ$, $\mathcal{G}_a(\boldsymbol{\chi})$ is the irreducible sub-representation with highest weight. Moreover, it is geometrically irreducible and occurs exactly once in $\mathcal{F}_a(\boldsymbol{\chi})$ over $\mathbb{G}_{m/\overline{\mathbb{F}}_p}$. See [Fis92, Proposition 4.18].

The multiplicative character χ can be viewed as a character on $\mathbb{F}_p$-points of $\mathbb{B}^\times = \mathrm{Res}_{\mathbb{F}_q/\mathbb{F}_p} \mathbb{G}_m$. It gives a rank one lisse sheaf on $\mathbb{B}^\times$ constructed from the Lang torsor as in [Kat88, §4.3]. Denote by $\mathcal{L}_\psi$ its restriction on $\mathbb{G}_m$. Similarly, the additive character ψ gives a rank one lisse sheaf on $\mathbb{G}_{a/\mathbb{F}_p}$. Denote by $\mathcal{L}_\psi$ its restriction on $\mathbb{G}_m$. For any $t \in \mathbb{F}_p^\times$,

$$\mathrm{Tr}\big(\mathrm{Frob}_t, (\mathcal{L}_\chi)_{\bar{t}}\big) = \chi(t), \quad \mathrm{Tr}\big(\mathrm{Frob}_t, (\mathcal{L}_\psi)_{\bar{t}}\big) = \psi(t).$$

2.3. **Distinctness.** We will consider when

$$\mathrm{Kl}_n(\psi, \boldsymbol{\chi}, q, a) = \lambda \,\mathrm{Kl}_n(\psi, \boldsymbol{\rho}, q, b)$$

for some $\lambda \in \mu_{q-1}$. The argument is almost the same as in [Fis92], while $\lambda = 1$ in his paper. So we will only show the difference.

Proposition 2.2. *Let $a, b \in \mathbb{F}_q^\times$ and let $\boldsymbol{\chi}, \boldsymbol{\rho}$ be n-tuples of multiplicative characters $\chi_i, \rho_j : \mathbb{F}_q^\times \to \overline{\mathbb{Q}}_\ell^\times$ respectively. Assume that $p > (2n^{2d}+1)^2$, $\boldsymbol{\chi}$ is not Kummer-induced and there is $\lambda \in \mu_{q-1}$ such that*

$$\mathrm{Kl}_n(\psi, \boldsymbol{\chi}, q, a) = \lambda \,\mathrm{Kl}_n(\psi, \boldsymbol{\rho}, q, b).$$

Then $\mathcal{G}_a(\boldsymbol{\chi}) \otimes \mathcal{L}_{\prod \overline{\boldsymbol{\chi}}} \mid \mathbb{G}_{m/\overline{\mathbb{F}}_p}$ occurs at least once in $\mathcal{F}_b(\boldsymbol{\rho}) \otimes \mathcal{L}_{\prod \overline{\boldsymbol{\rho}}} \mid \mathbb{G}_{m/\overline{\mathbb{F}}_p}$.

Proof. See [Fis92, Corollary 4.20]. Denote by

$$\mathcal{F} = \mathcal{F}_a(\boldsymbol{\chi}) \otimes \mathcal{L}_{\prod \overline{\boldsymbol{\chi}}}, \quad \mathcal{F}' = \mathcal{F}_b(\boldsymbol{\rho}) \otimes \mathcal{L}_{\prod \overline{\boldsymbol{\rho}}}, \quad \mathcal{G} = \mathcal{G}_a(\boldsymbol{\chi}) \otimes \mathcal{L}_{\prod \overline{\boldsymbol{\chi}}}.$$

For any $t \in \mathbb{F}_p^\times$, we have $\sigma_t \lambda = \lambda$. Since

$$\sigma_t\big(\mathrm{Kl}_n(\psi, \boldsymbol{\chi}, q, a)\big) = \prod \overline{\boldsymbol{\chi}}(t) \cdot \mathrm{Kl}_n(\psi, \boldsymbol{\chi}, q, at^n) = (-1)^{(n-1)d} \,\mathrm{Tr}\big(\mathrm{Frob}_t, \mathcal{F}_{\bar{t}}\big),$$
$$\sigma_t\big(\mathrm{Kl}_n(\psi, \boldsymbol{\rho}, q, b)\big) = \prod \overline{\boldsymbol{\rho}}(t) \cdot \mathrm{Kl}_n(\psi, \boldsymbol{\rho}, q, bt^n) = (-1)^{(n-1)d} \,\mathrm{Tr}\big(\mathrm{Frob}_t, \mathcal{F}'_{\bar{t}}\big),$$

we have $\mathrm{Tr}\big(\mathrm{Frob}_t, \mathcal{F}_{\bar{t}}\big) = \lambda \,\mathrm{Tr}\big(\mathrm{Frob}_t, \mathcal{F}'_{\bar{t}}\big)$.

Let $V = \overline{\mathbb{Q}}_\ell \cdot e$ with $\mathrm{Frob}_p \cdot e = \lambda e$, where $\mathrm{Frob}_p \in \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ is the geometric Frobenius. Denote by $\mathcal{L}_0$ the sheaf on $\mathrm{Spec}\, \mathbb{F}_p$ corresponding to this module and let $\mathcal{L}$ be its pulling-back along $\mathbb{G}_m \to \mathrm{Spec}\, \mathbb{F}_p$. Then for any $t \in \mathbb{F}_p^\times$,

$$\mathrm{Tr}(\mathrm{Frob}_t, \mathcal{L}_{\bar{t}}) = \mathrm{Tr}(\mathrm{Frob}_p, \mathcal{L}_0) = \lambda, \quad \mathrm{Tr}\big(\mathrm{Frob}_t, (\mathcal{F}' \otimes \mathcal{L})_{\bar{t}}\big) = \mathrm{Tr}(\mathrm{Frob}_t, \mathcal{F}_{\bar{t}}).$$

Since $\mathcal{L} \mid \mathbb{G}_{m/\overline{\mathbb{F}}_p}$ is trivial, the result then follows by applying Lemma 2.3 to sheaves $\mathcal{F}, \mathcal{F}' \otimes \mathcal{L}, \mathcal{G}$ with $r = s = n^d, M_0 = 0$ and $M_\infty \le 1$. ☐

Lemma 2.3 ([Fis92, Lemma 4.9])**.** *Let $\mathcal{F}, \mathcal{F}'$ be lisse sheaves on $\mathbb{G}_m$ of same rank r and pure of the same weight w. Assume that for any $t \in \mathbb{F}_p^\times$,*

$$\operatorname{Tr}\bigl(\operatorname{Frob}_t, \mathcal{F}_{\bar{t}}\bigr) = \operatorname{Tr}\bigl(\operatorname{Frob}_t, \mathcal{F}'_{\bar{t}}\bigr).$$

Let $\mathcal{G}$ be a geometrically irreducible sheaf of rank s on $\mathbb{G}_m$, pure of weight w, such that $\mathcal{G} \mid \mathbb{G}_{m/\overline{\mathbb{F}}_p}$ occurs exactly once in $\mathcal{F} \mid \mathbb{G}_{m/\overline{\mathbb{F}}_p}$. Then $\mathcal{G} \mid \mathbb{G}_{m/\overline{\mathbb{F}}_p}$ occurs at least once in $\mathcal{F}' \mid \mathbb{G}_{m/\overline{\mathbb{F}}_p}$, provided that $p > \bigl(2rs(M_0 + M_\infty) + 1\bigr)^2$, where M_η is the largest η-break of $\mathcal{F} \oplus \mathcal{F}'$.

Theorem 2.4. *Let $a, b \in \mathbb{F}_q^\times$ and let $\boldsymbol{\chi}, \boldsymbol{\rho}$ be n-tuples of multiplicative characters. Assume that $\boldsymbol{\chi}, \boldsymbol{\rho}$ are not Kummer-induced and neither of them is of type $\bigl\{\xi_1, \xi_1^{-1}, 1, \Lambda_2\bigr\}\xi_2$. If $p > (2n^{2d} + 1)^2$ and*

$$\operatorname{Kl}_n(\psi, \boldsymbol{\chi}, q, a) = \lambda \operatorname{Kl}_n(\psi, \boldsymbol{\rho}, q, b)$$

for some $\lambda \in \mu_{q-1}$, then there exists $\sigma \in \operatorname{Gal}(\mathbb{F}_q/\mathbb{F}_p)$ and a multiplicative character η, such that $b = \sigma(a)$ and $\boldsymbol{\rho} = (\boldsymbol{\chi} \circ \sigma^{-1})\eta$ as unordered tuples. Moreover, either both Kloosterman sums vanish or $\eta(b) = \lambda^{-1}$.

Here, Λ_2 denotes the non-trivial quadratic character on $\mathbb{F}_q^\times$.

Proof. Denote by

$$\mathcal{H} = \mathcal{K}\ell_{n,q}(\psi, \boldsymbol{\chi}) \mid \mathbb{G}_{m/\overline{\mathbb{F}}_p} \quad \text{and} \quad \mathcal{K} = \mathcal{K}\ell_{n,q}(\psi, \boldsymbol{\rho}) \mid \mathbb{G}_{m/\overline{\mathbb{F}}_p}.$$

By our assumptions, $\mathcal{H}$ and $\mathcal{K}$ are not Kummer-induced by [Fis92, Theorem 2.9]. If $G_{\text{geom}}(\mathcal{H})^\circ = \mathrm{SO}(4)$, then $n = 4$ and there is a multiplicative character η such that $\overline{\boldsymbol{\chi}} = \boldsymbol{\chi}\eta$ as unordered 4-tuples and $\prod \boldsymbol{\chi} = \Lambda_2\eta^{-2}$ by [Fis92, Proposition 2.10]. Since $\boldsymbol{\chi}$ is not Kummer-induced, we have $\boldsymbol{\chi} = (\xi_1, \xi_1^{-1}, 1, \Lambda_2)\xi_2$ for some ξ_1, ξ_2. This contradicts to our assumptions. Thus $G_{\text{geom}}(\mathcal{H})^\circ \neq \mathrm{SO}(4)$. Similarly, $G_{\text{geom}}(\mathcal{K})^\circ \neq \mathrm{SO}(4)$.

Let $\mathfrak{g}$ be the Lie algebra of the connected geometric monodromy group of

$$\bigoplus_{\sigma \in \operatorname{Gal}(\mathbb{F}_q/\mathbb{F}_p)} T^*_{\sigma(a)} \mathcal{K}\ell_{n,q}(\psi \circ \sigma^{-1}, \boldsymbol{\chi} \circ \sigma^{-1}) \oplus \bigoplus_{\tau \in \operatorname{Gal}(\mathbb{F}_q/\mathbb{F}_p)} T^*_{\tau(a)} \mathcal{K}\ell_{n,q}(\psi \circ \tau^{-1}, \boldsymbol{\rho} \circ \tau^{-1}),$$

where T is the translation. As showned in [Fis92, Theorem 4.22], we have

$$\mathcal{G}_a(\boldsymbol{\chi}) \hookrightarrow \mathcal{F}_b(\boldsymbol{\rho}), \quad \mathcal{G}_b(\boldsymbol{\rho}) \hookrightarrow \mathcal{F}_a(\boldsymbol{\chi})$$

as representations of $\mathfrak{g}$ by applying Corollary 2.2 and [Fis92, Lemma 4.19] twice. By following Fisher's argument step by step, there are $\sigma \in \operatorname{Gal}(\mathbb{F}_q/\mathbb{F}_p)$ and a multiplicative character η, such that $b = \sigma(a)$ and $\boldsymbol{\rho} = (\boldsymbol{\chi} \circ \sigma^{-1})\eta$ as unordered tuples. This implies that

$$\operatorname{Kl}_n(\psi, \boldsymbol{\rho}, q, b) = \eta(b) \operatorname{Kl}_n(\psi, \boldsymbol{\chi}, q, a).$$

Hence both Kloosterman sums vanish or $\eta(b) = \lambda^{-1}$. $\square$

Remark 2.5. In [Fis92, Corollary 4.27], Fisher showed that if $p > (2n^{4d} + 1)^2$ and

$$\left|\operatorname{Kl}_n(\psi, \boldsymbol{\chi}, q, a)\right| = \left|\operatorname{Kl}_n(\psi, \boldsymbol{\rho}, q, b)\right|,$$

then $b = \sigma(a), \boldsymbol{\rho} = (\boldsymbol{\chi} \circ \sigma^{-1})\eta$, or $b = (-1)^n\sigma(a), \boldsymbol{\rho} = (\boldsymbol{\chi}^{-1} \circ \sigma^{-1})\eta$.

Corollary 2.6. *Keeping the hypotheses of Theorem 2.4. Assume that $\boldsymbol{\chi}$ is defined over $\mathbb{F}_p$, that's to say, $\boldsymbol{\chi} = \boldsymbol{\chi}_0 \circ \mathbf{N}_{\mathbb{F}_q/\mathbb{F}_p}$ for some n-tuple $\boldsymbol{\chi}_0$ of characters on $\mathbb{F}_p^\times$. If*

$$\mathrm{Kl}_n(\psi, \boldsymbol{\chi}, q, a) = \lambda \, \mathrm{Kl}_n(\psi, \boldsymbol{\chi}, q, b), \quad \lambda \in \mu_{q-1},$$

then $b = \sigma(a)$ for some $\sigma \in \mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p)$, and $\mathrm{Kl}_n(\psi, \boldsymbol{\chi}, q, a) = \mathrm{Kl}_n(\psi, \boldsymbol{\chi}, q, b)$.

Proof. In this case, we have $\boldsymbol{\chi} = \eta\boldsymbol{\chi}$ and then $\eta = 1$. The result then follows easily. □

3. The non-vanishing of Kloosterman sums

The case $n = 1$ is trivial. We will assume that $n \geq 2$ in this section.

Theorem 3.1. *Assume that $p > (3n-1)c - n$ and for any i, j, $\chi_i = \chi_j$ if $\chi_i^n = \chi_j^n$. Then $\mathrm{Kl}_n(\psi, \boldsymbol{\chi}, q, a)$ is nonzero.*

Proof. Let $\mathfrak{p}$ be a prime above p in $\mathbb{Q}(\mu_{q-1})$ and $\mathfrak{P}$ the unique prime above $\mathfrak{p}$ in $\mathbb{Q}(\mu_{(q-1)p})$. Let v be the normalized $\mathfrak{P}$-adic valuation. Once we fix an isomorphism from $\mathbb{F}_q$ to the residue field of $\mathfrak{p}$, the Teichmüller lifting of the residue map at $\mathfrak{p}$ gives a primitive character ω of $\mathbb{F}_q^\times$. Denote by

$$g(m) := \sum_{t \in \mathbb{F}_q^\times} \omega^{-m}(t)\psi\big(\mathrm{Tr}(t)\big)$$

the Gauss sum. Then the Stickelberger's congruence theorem tells that

$$v\big(g(m)\big) = \sum_{j=0}^{d-1} m_j, \tag{3.1}$$

where

$$0 \leq m \leq q-2, \quad m = \sum_{j=0}^{d-1} m_j p^j, \ 0 \leq m_j \leq p-1,$$

see [Sti90] or [Was97, Chap. 6].

For any $1 \leq i \leq n$, there is s_i such that $\chi_i = \omega^{-s_i}$. Take $x = x_1 \cdots x_n a^{-1}$ in the identity

$$\sum_{m=0}^{q-2} \omega^{-m}(x) = \begin{cases} q-1, & \text{if } x = 1; \\ 0, & \text{if } x \neq 1, \end{cases}$$

we get

$$(q-1)\,\mathrm{Kl}_n(\psi, \boldsymbol{\chi}, q, a) = \sum_{m=0}^{q-2} \omega^m(a) \prod_{i=1}^n g(m + s_i).$$

There is a unique m such that $v\big(\prod_{i=1}^n g(m+s_i)\big)$ is minimal by Proposition 3.2. Hence the Kloosterman sum has a finite valuation and then is nonzero. □

We may assume that $1 \leq s_i \leq q-1$ (notice the bound). Write

$$s_i = \sum_{j=0}^{d-1} s_{ij} p^j$$

with $0 \leq s_{ij} \leq p-1$.

Proposition 3.2. *Assume that $p > (3n-1)c - n$ and for any i, j, $\chi_i = \chi_j$ if $\chi_i^n = \chi_j^n$. Then there is a unique $0 \le m \le q-2$ such that $v\big(\prod_{i=1}^n g(m+s_i)\big)$ is minimal.*

Proof. Since $c(\boldsymbol{\chi}\chi_1^{-1}) \le c(\boldsymbol{\chi})$, we may assume that $\chi_1 = 1, s_1 = q-1$ for simplicity. Write

$$m + s_i - (q-1)\epsilon_{i,-1} = \sum_{j=0}^{d-1} m_{ij} p^j,\ 1 \le i \le n$$

where $\epsilon_{i,-1} \in \{0,1\}$ is the integer part of $(m+s_i)/(q-1)$ and $0 \le m_{ij} \le p-1$. Then

$$m_{ij} = m_j + s_{ij} + \epsilon_{i,j-1} - p\epsilon_{ij}, \quad \epsilon_{ij} \in \{0,1\}, \quad \epsilon_{i,d-1} = \epsilon_{i,-1}$$

and

$$v\Big(\prod_{i=1}^n g(m+s_i)\Big) = \sum_{i=1}^n \sum_{j=0}^{d-1} m_{ij} \tag{3.2}$$

by the Stickelberger's congruence theorem (3.1).

There exsits a permutation $\sigma_j \in S_n$ such that

$$s_{\sigma_j(1),j} \ge s_{\sigma_j(2),j} \ge \cdots \ge s_{\sigma_j(n),j}. \tag{3.3}$$

If $s_{ij} = s_{i'j}$, then $\chi_i^n = \chi_{i'}^n$, $\chi_i = \chi_{i'}$ and $\epsilon_{ij} = \epsilon_{i'j}$ by Lemma 3.3. If $s_{ij} > s_{i'j}$, then

$$s_{ij} + \epsilon_{i,j-1} \ge s_{i'j} + \epsilon_{i',j-1} \quad \text{and} \quad \epsilon_{ij} \ge \epsilon_{i'j}.$$

In other words, $\{\epsilon_{ij}\}_i$ and $\{s_{ij} + \epsilon_{i,j-1}\}_i$ have the same orderings as (3.3). Therefore, there exists $0 \le u_j \le n$ such that

$$\epsilon_{\sigma_j(1),j} = \cdots = \epsilon_{\sigma_j(u_j),j} = 1, \quad \epsilon_{\sigma_j(u_j+1),j} = \cdots = \epsilon_{\sigma_j(n),j} = 0,$$

$$m_{\sigma_j(1),j} \ge \cdots \ge m_{\sigma_j(u_j),j}, \quad m_{\sigma_j(u_j+1),j} \ge \cdots \ge m_{\sigma_j(n),j}.$$

Note that $s_1 = q-1, \epsilon_{1,-1} = 1$. Since $s_{1j} = p-1$, one can show that $\epsilon_{1,j} = 1$ inductively, which means $u_j \ne 0$. If $u_j \ne n$ but $m_{\sigma_j(u_j),j} = m_{\sigma_j(n),j}$, then

$$s_{\sigma_j(u_j),j} = p-1, \epsilon_{\sigma_j(u_j),j} = 1, \quad s_{\sigma_j(n),j} = \epsilon_{\sigma_j(n),j} = 0.$$

By Lemma 3.3, this implies that $\chi_{\sigma_j(u_j)} = \chi_{\sigma_j(n)}$ and then $\epsilon_{\sigma_j(u_j),j} = \epsilon_{\sigma_j(n),j}$, which is impossible. Hence

$$m_j' := m_{\sigma_j(u_j),j} = m_j + s_{\sigma_j(u_j),j} + \epsilon_{\sigma_j(u_j),j-1} - p$$

is the unique minimum among $\{m_{ij}\}_i$. Therefore, the valuation (3.2) becomes

$$\begin{aligned} \sum_{i,j} m_{ij} &= \sum_{i,j} \big[m_j' + p - s_{\sigma_j(u_j),j} - \epsilon_{\sigma_j(u_j),j-1} + s_{ij} + \epsilon_{i,j-1} - p\epsilon_{ij}\big] \\ &= ndp + \sum_{i,j} s_{ij} + \sum_j \Big[nm_j' - ns_{\sigma_j(u_j),j} - n\epsilon_{\sigma_j(u_j),j-1} + (p-1)u_j\Big]. \end{aligned} \tag{3.4}$$

By Lemma 3.3, there exists a unique U_j such that

$$s_{\sigma_j(U_j),j} + \frac{p-1}{n} U_j = \max_{1 \le i \le n} \Big\{ s_{\sigma_j(i),j} + \frac{p-1}{n} i \Big\}.$$

Moreover,

$$s_{\sigma_j(U_j),j} + \frac{p-1}{n} U_j > s_{\sigma_j(i),j} + \frac{p-1}{n} i + 1 \tag{3.5}$$

for any $i \neq U_j$. This follows from Lemma 3.3 if $\chi_{\sigma_j(U_j)} \neq \chi_{\sigma_j(i)}$. If $\chi_{\sigma_j(U_j)} = \chi_{\sigma_j(i)}$, this follows from $(p-1)/n > 1$.

Write

$$E_{\sigma_j(1),j} = \cdots = E_{\sigma_j(U_j),j} = 1, \quad E_{\sigma_j(U_j+1),j} = \cdots = E_{\sigma_j(n),j} = 0.$$

If m is

$$M = \sum_{j=0}^{d-1} M_j p^j, \text{ where } M_j = p - s_{\sigma_j(U_j),j} - E_{\sigma_j(U_j),j-1},$$

then $m'_j = 0, \epsilon_{ij} = E_{ij}$ and $u_j = U_j$. Denote by V the corresponding valuation (3.2) for $m = M$.

If all $u_j = U_j$, then $\epsilon_{ij} = E_{ij}$ and

$$\sum_{i,j} m_{ij} = V + n \sum_j m'_j \geq V.$$

The equality holds if and only if all $m'_j = 0$, i.e., $m = M$. If there exists j such that $u_j \neq U_j$, then by (3.4) and (3.5), we have

$$\begin{aligned}
&\sum_{i,j} m_{ij} - V \\
=&\sum_j \Big[n m'_j - n s_{\sigma_j(u_j),j} - n\epsilon_{\sigma_j(u_j),j-1} + (p-1)u_j \Big] \\
&\quad - \sum_j \Big[-n s_{\sigma_j(U_j),j} - nE_{\sigma_j(U_j),j-1} + (p-1)U_j \Big] \\
\geq& \sum_j \Big[n(s_{\sigma_j(U_j),j} - s_{\sigma_j(u_j),j}) + n(E_{\sigma_j(U_j),j-1} - \epsilon_{\sigma_j(u_j),j-1}) + (p-1)(U_j - u_j) \Big] \\
\geq& n \sum_j \Big[s_{\sigma_j(U_j),j} + \frac{p-1}{n} U_j - s_{\sigma_j(u_j),j} - \frac{p-1}{n} u_j + E_{\sigma_j(U_j),j-1} - \epsilon_{\sigma_j(u_j),j-1} \Big] \\
\geq& n \sum_{u_j \neq U_j} \Big[s_{\sigma_j(U_j),j} + \frac{p-1}{n} U_j - s_{\sigma_j(u_j),j} - \frac{p-1}{n} u_j - 1 \Big] > 0.
\end{aligned}$$

Hence the valuation (3.2) is minimal if and only if $m = M$. $\square$

Lemma 3.3. *Assume that $p > (3n-1)c - n$. If $\chi_i^n \neq \chi_{i'}^n$, then there is no integer $0 \leq \alpha \leq n$ such that $|s_{ij} - s_{i'j} - \frac{p-1}{n}\alpha| \leq 1$.*

Proof. There exists r, r' such that

$$s_i = \frac{(q-1)r}{c}, \quad s_{i'} = \frac{(q-1)r'}{c}$$

and

$$s_{ij} = \frac{a_{j+1}p - a_j}{c}, \quad s_{i'j} = \frac{a'_{j+1}p - a'_j}{c},$$

where $a_j \equiv rp^{-j}, a'_j \equiv r'p^{-j} \bmod c$ with $1 \leq a_j, a'_j \leq c$. Let $a''_j := a_j - a'_j$. Then $|a''_j| \leq c-1$.

If

$$\frac{p-1}{n}\alpha + t = s_{ij} - s_{i'j} = \frac{a''_{j+1}p - a''_j}{c}$$

for some $0 \le \alpha \le n$ and $|t| \le 1$, then

$$\big(na''_{j+1} - \alpha c\big)p = na''_j - \alpha c + nct.$$

There are three cases:

- If $na''_{j+1} - \alpha c \neq 0$ and $\alpha = n$, then
$$p \le |(a''_{j+1} - c)p| = |a''_j - c + ct| \le 3c - 1 \le (3n-1)c - n$$
since $n \ge 2$.
- If $na''_{j+1} - \alpha c \neq 0$ and $\alpha < n$, then
$$p \le \big|na''_j - \alpha c + nct\big| \le n(c-1) + c(n-1) + nc \le (3n-1)c - n.$$
- If $na''_{j+1} - \alpha c = 0$, then $n(r - r') \equiv na''_{j+1}p^{j+1} \equiv 0 \bmod c$ and then $\chi_i^n = \chi_{i'}^n$.

The result then follows. □

Remark 3.4. When $n = 2$, $p > 3c - 2$ is enough by a careful estimation, see [Zha21, Lemma 3.4, Proposition 3.6].

4. Proof of the main theorem

Theorem 4.1. *Assume that* $p > \max\big\{(2n^{2d}+1)^2, (3n-1)c - n\big\}$ *and for any* i, j, $\chi_i = \chi_j$ *if* $\chi_i^n = \chi_j^n$. *Then* $\mathrm{Kl}_n(\psi, \boldsymbol{\chi}, q, a)$ *generates* $\mathbb{Q}(\mu_{pc})^H$, *where* H *consists of those* $\sigma_t\tau_w$ *such that there exists an integer* k *and a character* η *satisfying*

$$t^n = a^{1-p^k}, \quad \boldsymbol{\chi}^w = \boldsymbol{\chi}^{p^k}\eta, \quad \eta(a) = \prod \boldsymbol{\chi}^w(t).$$

Proof. Note that if $\boldsymbol{\chi}$ is Kummer-induced, then there is a non-trivial character Λ such that $\boldsymbol{\chi} = \boldsymbol{\chi}\Lambda$ and $\Lambda^n = 1$. Thus there exists $i \neq j$ such that $\chi_i = \chi_j\Lambda$ and $\chi_i^n = \chi_j^n$, which contradicts to our assumptions. Certainly, $\boldsymbol{\chi} = (\xi_1, \xi_1^{-1}, 1, \Lambda_2)\xi_2$ is also impossible.

By Theorems 2.4 and 3.1, the fact that

$$\sigma_t\tau_w \,\mathrm{Kl}_n(\psi, \boldsymbol{\chi}, q, a) = \prod \boldsymbol{\chi}^{-w}(t)\,\mathrm{Kl}_n(\psi, \boldsymbol{\chi}^w, q, at^n),$$

and $t^p = t$, we have

$$t^n = a^{1-p^k}, \quad \boldsymbol{\chi}^w = \boldsymbol{\chi}^{p^k}\eta, \quad \eta(a) = \prod \boldsymbol{\chi}^w(t)$$

for some integer k. □

Remark 4.2. Denote by $\alpha = \gcd(k, d)$ and $\lambda := a^{p^\alpha - 1}$. Since the order of a divides

$$\gcd\big((p^k-1)(p-1), p^d - 1\big) = (p^\alpha - 1)\gcd\big(p-1, \frac{p^d-1}{p^\alpha - 1}\big) = (p^\alpha - 1)\gcd\big(p-1, \frac{d}{\alpha}\big),$$

we have $\lambda^{\frac{d}{\alpha}} = 1$. If $\lambda \neq 1$, then

$$\mathrm{Tr}(a) = \big(1 + \lambda + \cdots + \lambda^{\frac{d}{\alpha}-1}\big)\cdot\big(a + a^p + \cdots + a^{p^{\alpha-1}}\big) = 0.$$

Hence if $\mathrm{Tr}(a) \neq 0$, then $\lambda = 1, t^n = a^{1-p^k} = 1$. If moreover $\boldsymbol{\chi} = \mathbf{1}$, then

$$H = \big\{t \in \mathrm{Gal}\big(\mathbb{Q}(\mu_p)/\mathbb{Q}\big) \,\big|\, t^n = 1\big\}.$$

In fact, this holds for any p, see [Wan95]. See also [KRV11] for an attempt on a weaker condition.

Remark 4.3. Consider the Kloosterman sums

$$S_m = \mathrm{Kl}(\psi, \boldsymbol{\chi} \circ \mathbf{N}_{\mathbb{F}_{q^m}/\mathbb{F}_q}, q^m, a).$$

The L-function

$$L(T) = \exp\left(\sum_{m=1}^{\infty} \frac{T^m}{m} S_m\right)$$

is a rational function over $\mathbb{Q}(\zeta_{p(q-1)})$ by the Dwork-Bombieri-Grothendick rationality theorem. Thus the sequence $\{S_m\}_m$ is a linear recurrence sequence. As shown in [WY20, Theorem 3], the sequence $\{\mathbb{Q}(S_m)\}_{m\geq N}$ is periodic of period r for some r, N.

Assume that for any i, j, $\chi_i = \chi_j$ if $\chi_i^n = \chi_j^n$. By Theorem 1.1, if $p > \max\left\{(2n^{2dm}+1)^2, (3n-1)c-n\right\}$, then $\mathbb{Q}(S_m) = \mathbb{Q}(\mu_{pc})^H$, where H consists of those $\sigma_t\tau_w$ such that there exists an integer k and a character η on $\mathbb{F}_q^\times$ satisfying

$$t^n = a^{1-p^k}, \quad \boldsymbol{\chi}^w = \boldsymbol{\chi}^{p^k}\eta, \quad \eta(a) = \gamma \cdot \prod \boldsymbol{\chi}^w(t) \text{ with } \gamma^m = 1. \tag{4.1}$$

Hence $\mathbb{Q}(S_m) = \mathbb{Q}(S_{m-c})$ since $\gamma^c = 1$.

If $p > \max\left\{\left(2n^{2d(N+r)}+1\right)^2, (3n-1)c-n\right\}$, then the generating field of S_m is determined by (4.1) for any m. But unfortunately, we do not have a bound on N. We guess that S_m has the predicted generating field if $p > 3ndc$.

5. Examples

Denote by $n_0 := (n, p-1)$, d_0 the degree of $a^{(1-p)/n_0}$ and

$$a_0 := \mathbf{N}_{\mathbb{F}_{p^{d_0}}/\mathbb{F}_p}\left(a^{(1-p)/n_0}\right) = a^{(1-p^{d_0})/n_0}.$$

Since

$$\left(a^{(1-p)/n_0}\right)^{p^k-1} = t^{(p-1)n/n_0} = 1,$$

we have $k = d_0\beta$ for some integer β. Moreover,

$$t^n = a^{1-p^k} = a_0^{n_0(1-p^k)/(1-p^{d_0})} = a_0^{n_0\beta}.$$

5.1. An example: $n = 2$ case.

Proposition 5.1. *Let $\boldsymbol{\chi} = \{1, \chi\}$, where χ is a multiplicative character of order $c \neq 2$. If $p > \max\left\{(2^{2d+1}+1)^2, 5c-2)\right\}$, then $\mathrm{Kl}(\psi, \boldsymbol{\chi}, p^d, a)$ generates $\mathbb{Q}(\mu_{pc})^H$, where*

$$H = \begin{cases} \langle \tau_{q_0}\sigma_{a_0}, \sigma_{-1}, \tau_{-1}\rangle, & \text{if } \chi(-1) = 1, \chi(a) = 1; \\ \langle \tau_{-q_0}\sigma_{a_0}, \sigma_{-1}\rangle, & \text{if } \chi(-1) = 1, \chi(a) = \chi(a_0) = -1; \\ \langle \tau_{q_0^\alpha}\sigma_{a_0^\alpha}, \sigma_{-1}\rangle, & \text{if } \chi(-1) = 1, \chi(a)^\alpha \neq 1; \\ \langle \tau_{q_0}\sigma_{-a_0}, \tau_{-1}\sigma_{-1}\rangle, & \text{if } \chi(-1) = -1, \chi(a) = \chi(a_0) = -1; \\ \langle \tau_{q_0}\sigma_{a_0}, \tau_{-1}\rangle, & \text{if } \chi(-1) = -1, \chi(a) = 1; \\ \langle \tau_{q_0}\sigma_{a_0}, \tau_{-1}\sigma_{-1}\rangle, & \text{if } \chi(-1) = -1, \chi(a) = -1, \chi(a_0) = 1; \\ \langle \tau_{q_0^{\alpha/2}}\sigma_{-a_0^{\alpha/2}}\rangle, & \text{if } \chi(-1) = -1, 2 \mid \alpha, \chi(a) \neq \pm 1; \\ \langle \tau_{q_0^\alpha}\sigma_{a_0^\alpha}\rangle, & \text{if } \chi(-1) = -1, 2 \nmid \alpha, \chi(a) \neq \pm 1. \end{cases}$$

is a subgroup of $\mathrm{Gal}(\mathbb{Q}(\mu_{pc})/\mathbb{Q})$, $q_0 = \#\mathbb{F}_p(a^{(1-p)/2}), a_0 = a^{(1-q_0)/2} \in \mathbb{F}_p^\times$ *and α is the order of $\chi(a_0) \in \mu_{p-1}$.*

Proof. As remarked above, $k = d_0\beta$ and $t^2 = a_0^{2\beta}$ for some integer β, where $q_0 = p^{d_0}$. Hence $t = \pm a_0^\beta$ and

$$\boldsymbol{\chi}^w = \{1, \chi^w\} = \boldsymbol{\chi}^{q_0^\beta}\eta = \left\{\eta, \eta\chi^{q_0^\beta}\right\}, \quad \eta(a) = \chi^w(t).$$

There are two cases:
(i) If $\eta = 1, \chi^w = \chi^{q_0^\beta}$, then $w \equiv q_0^\beta \bmod c$ and

$$1 = \eta(a) = \chi^w(t) = \chi(t) = \chi(\pm a_0^\beta).$$

(ii) If $\eta = \chi^w, \eta\chi^{q_0^\beta} = 1$, then $w \equiv -q_0^\beta \bmod c$. Since $\chi^w(a) = \eta(a) = \chi^w(t)$, we have $\chi(a) = \chi(t) = \chi(\pm a_0^\beta)$. Since $a_0 = a^{\frac{1-q_0}{2}} \in \mathbb{F}_p^\times$, we have

$$\chi(a_0)^2 = \chi(a)^{1-q_0} = \chi(a_0)^{(1-q_0)\beta} = 1.$$

Thus $\chi(a_0) = \pm 1$ and $\alpha = 1$ or 2.

Case $\chi(-1) = 1$: In case (i), $\beta = \alpha m$ for some m and $w \equiv q_0^{\alpha m}, t = \pm a_0^{\alpha m}$. In case (ii), if $\alpha = 1$, $\chi(a_0) = \chi(a) = 1$, then $w \equiv -q_0^m, t = \pm a_0^m$; if $\alpha = 2$, $\chi(a_0) = \chi(a) = -1$, then $w \equiv -q_0^{1+2m}, t = \pm a_0^{1+2m}$.

Case $\chi(-1) = -1$ and $2 \mid \alpha$: In case (i), $w \equiv q_0^{\alpha m}, t = a_0^{\alpha m}$ or $w \equiv q_0^{\alpha(m+1/2)}, t = -a_0^{\alpha(m+1/2)}$. In case (ii), $\alpha = 2$, $\chi(a) = \chi(a_0) = -1$. Then $w \equiv -q_0^{1+2m}, t = a_0^{1+2m}$ or $w \equiv -q_0^{2m}, t = -a_0^{2m}$.

Case $\chi(-1) = -1$ and $2 \nmid \alpha$: In case (i), $w \equiv q_0^{\alpha m}, t = a_0^{\alpha m}$. In case (ii), $\alpha = 1$ and $\chi(a_0) = 1$. If $\chi(a) = 1$, then $w \equiv -q_0^m, t = a_0^m$; if $\chi(a) = -1$, then $w \equiv -q_0^m, t = -a_0^m$. $\square$

Example 5.2. If $a \in \mathbb{F}_p^\times$, then $q_0 = p, \alpha = 1$ or 2. One can easily obtain that

$$H = \begin{cases} \langle \tau_p, \tau_{-1}, \sigma_{-1}\rangle, & \text{if } \chi(-1) = 1 \text{ and } \chi(a) = 1; \\ \langle \tau_p, \sigma_{-1}\rangle, & \text{if } \chi(-1) = 1 \text{ and } \chi(a) = -1; \\ \langle \tau_p, \tau_{-1}\rangle, & \text{if } \chi(-1) = -1 \text{ and } \chi(a) = 1; \\ \langle \tau_p, \tau_{-1}\sigma_{-1}\rangle, & \text{if } \chi(-1) = -1 \text{ and } \chi(a) = -1; \\ \langle \tau_p\rangle, & \text{if } \chi(-1) = -1 \text{ and } \chi(a) \neq \pm 1. \end{cases}$$

This drops the combinatorial condition on (p, d) and the non-vanishing condition on $\mathrm{Tr}(a)$ in [Zha21, Theorems 1.1, 1.3], while we require that p is large with respect to d.

Remark 5.3. Assume that $\chi = \Lambda_2$. If $\Lambda_2(a) \neq 1$, then the Kloosterman sum vanishes. If $\Lambda_2(a) = 1$ and $\mathrm{Tr}(\sqrt{a}) \neq 0$, then the Kloosterman sum generates $\mathbb{Q}(\mu_p)^+$ if $\chi(-1) = 1$; $\mathbb{Q}(\mu_p)$ if $\chi(-1) = -1$. See [Zha21, Theorem 1.1(1)].

5.2. **The upper bound of the generating field.** If $\eta = 1$, then $\chi_i^w = \chi_i^{q_0^\beta}$. Thus $w \equiv q_0^\beta \bmod c$. Denote by

$$\alpha := \min\left\{\alpha \in \mathbb{Z}_{>0} \,\middle|\, \exists t_0 \in \mathbb{F}_p^\times \text{ such that } t_0^n = a_0^{n_0\alpha}, \prod \boldsymbol{\chi}(t_0) = 1\right\}.$$

Write $\beta = \alpha s + r, 0 \le r < \alpha$. Then

$$(tt_0^{-s})^n = a_0^{n_0\beta - n_0\alpha s} = a_0^{n_0 r}, \quad \prod \boldsymbol{\chi}(tt_0^{-s}) = 1.$$

This forces $r = 0$ and $t = \lambda t_0^s$ with $\lambda^n = 1, \prod \boldsymbol{\chi}(\lambda) = 1$. Hence

$$H \supseteq H_0 := \langle \tau_{q_0^\alpha}\sigma_{t_0}, \sigma_\lambda \mid \lambda^n = 1, \prod \boldsymbol{\chi}(\lambda) = 1\rangle.$$

This gives an upper bound of the degree of $\mathrm{Kl}(\psi, \boldsymbol{\chi}, p^d, a)$.

Example 5.4. Denote by $m(\xi)$ the multiplicity of ξ in the n-tuple $\boldsymbol{\chi}$. Assume that there exists a character ξ such that $m(\xi) \neq m(\xi')$ for any $\xi' \neq \xi$. Then one can easily show that $\eta = 1$ and $H = H_0$.

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