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On the generating fields of Kloosterman sums

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Outline

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Let $f(x) \in \mathbb{F}_q[x]$ be a polynomial over a finite field with $q = p^d$ elements, where p is a rational prime. Define the exponential sum

$$S_1(f) := \sum_{x \in \mathbb{F}_q} \zeta_p^{\mathrm{Tr}(f(x))} \in \mathbb{Z}[\zeta_p].$$

A basic problem is

(1) as a complex number, $|S_1(f)| =?$

(2) as a p-adic number, $|S_1(f)|_p =?$

(3) as an algebraic number, $\deg S_1(f) =?$

L-function

The first two questions have been studied extensively in the literature. Define

$$L(t,f) := \prod_{x\in\overline{\mathbb{F}}_p}\left(1-\mathrm{Tr}_{\mathbb{F}_q(x)/\mathbb{F}_p}(f(x))t^{\deg x}\right)^{-1} = \exp\Big(\sum_k S_k(f)\frac{t^k}{k}\Big)$$

where $S_k(f) := \sum_{x\in\mathbb{F}_{q^k}} \zeta_p^{\mathrm{Tr}(f(x))} \in \mathbb{Z}[\zeta_p]$.

Theorem (Dwork-Bombieri-Grothendick)

$L(t,f)$ is a rational function.

Write

$$L(t,f) = \frac{\prod_j(1-\beta_j t)}{\prod_i(1-\alpha_i t)}.$$

Then

$$S_k(f) = \sum_i \alpha_i^k - \sum_j \beta_j^k.$$

Sheaf

How to estimate the characteristic roots α_i and β_j? We need ℓ-adic method. To describe it, let's recall the definition of sheaves.

Given a topological space X, there is a site Top(X) with

(1) objects: the open subsets of X;

(2) morphisms: the injection of open sets;

(3) coverings: normal open coverings.

A sheaf $\mathcal{F}$ on a topological space X over a field E is a contravariant functor Top$(X)^{\mathrm{op}} \to$ Vect$/E$, which can be uniquely glued locally. That's to say, for any open covering $U = \cup_i U_i$,

$$\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

is exact.

Étale site

Let X be a scheme. Denote by $X_{\text{ét}}$ the site with

(1) objects: étale scheme $X' \to X$;

(2) morphisms: étale morphisms;

(3) coverings: $\{\varphi_i : X'_i \to X'\}$ with $X' = \cup\varphi_i(X'_i)$.

Fix a prime $\ell \neq p$ and let E be a finite extension of $\mathbb{Q}_\ell$. An ℓ-adic sheaf is a sheaf on $X_{\text{ét}}$ over E (which is constructible at every finite level).

Swan conductor

Let K be c.d.v.f, with higher ramification groups $I^{(r)}, r \geqslant 0$. For any E-representation M of P, we have a decomposition $M = \oplus M(x)$, such that

$$M(0) = M^P, \quad M(x)^{I^{(x)}} = 0, \quad M(x)^{I(y)} = M(x),\ y > x > 0.$$

We call x a break if $M(x) \neq 0$. Define

$$\mathrm{Sw}(M) = \sum x \dim M(x).$$

Curves

Let C be a proper smooth geometrically connected curve over a perfect field $\mathbb{F}$, with function field $K = \mathbb{F}(C)$. For any closed point $x \in C(\mathbb{F})$, we have the completion K_x.

For any non-empty open $U \subset C$, we have an equivalence of abelian categories

$$\begin{aligned} \{\text{lisse } E\text{-sheaves on } U\} &\longrightarrow \mathrm{Rep}^c_E \pi_1(U, \overline{\eta}) \\ \mathcal{F} &\longmapsto \mathcal{F}_{\overline{\eta}}. \end{aligned}$$

Since $\pi_1(U, \overline{\eta})$ is a quotient of $\mathrm{Gal}(\overline{K}/K)$, the decomposition group $D_x \subset \mathrm{Gal}(\overline{K}/K)$ acts on $\mathcal{F}_{\overline{\eta}}$. We can define Swan conductor of $\mathcal{F}$ at x. If $x \in U$, the action of I_x is trivial.

We will take $\mathbb{F} = \mathbb{F}_p, C = \mathbb{P}^1$ and $U = \mathbb{G}_m$.

Assume that $\mu_p \subseteq E$. Deligne constructed a certain locally free of rank one ℓ-adic sheaf $\mathcal{F}_\ell(f)$ over E on $\mathbb{G}_{a,\overline{\mathbb{F}}_p} = \operatorname{Spec} \overline{\mathbb{F}}_p[X]$, such that

$$L(t,f) = \prod_i \det(1 - t\mathrm{Frob}, \mathrm{H}_c^i)^{(-1)^{i+1}}$$

and

$$S_k(f) = \sum_i (-1)^i \mathrm{Tr}(\mathrm{Frob}^k, \mathrm{H}_c^i).$$

Here, Frob is the geometric Frobenius (inverse of $\alpha \mapsto \alpha^p$), $\mathrm{H}_c^i = \mathrm{H}_c^i(\mathbb{G}_{a,\overline{\mathbb{F}}_p}, \mathcal{F}_\ell(f))$ is the compact cohomology.

Denote by ω_{ij} the eigenvalues of Frob on H_c^i, then

$$S_k(f) = \sum_{ij} (-1)^i \omega_{ij}^k.$$

Denote by $B_i = \dim_E \mathrm{H}_c^i$ the Betti number.

Theorem (Deligne)

ω_{ij} is an algebraic integer and all its conjugates over $\mathbb{Q}$ has same absolute value $q^{r_{ij}/2}$, where the weight $0 \leqslant r_{ij} \leqslant i$ are integers.

Thus

$$|S_k| \leqslant \sum_i B_i q^{ki/2}.$$

In general,

(1) V a closed variety over $\mathbb{F}_q$ of $\mathbb{A}^N$,

(2) ψ a non-trivial additive character on $\mathbb{F}_q$, $\psi_k = \psi \circ \mathrm{Tr}_{\mathbb{F}_{q^k}/\mathbb{F}_q}$,

(3) f a regular function on V defined over $\mathbb{F}_q$,

(4) χ a multiplicative character on $\mathbb{F}_q^\times$, $\chi_k = \chi \circ \boldsymbol{N}_{\mathbb{F}_{q^n}/\mathbb{F}_q}$,

(5) g an invertible regular function on V.

Define

$$S_k = \sum_{x \in V(\mathbb{F}_{q^k})} \psi_k(f(x))\chi_k(g(x)).$$

Then Deligne's results still hold in this case. Moreover, Bombieri proved that the number of characteristic roots is at most

$$(4 \max \{\deg V + 1, \deg f\} + 5)^{2N+1}.$$

Now we will consider

$$V = V(X_1 \cdots X_n - a), \quad f = X_1 + \cdots + X_n.$$

Let $\boldsymbol{\chi} = \{\chi_1, \ldots, \chi_n\}$ be an unordered n-tuple of multiplicative characters $\chi_i : \mathbb{F}_q^\times \to \mu_{q-1}$. Define the Kloosterman sum as

$$\mathrm{Kl}_n(\psi, \boldsymbol{\chi}, q, a) = \sum_{\substack{x_1 \cdots x_n = a \\ x_i \in \mathbb{F}_q}} \chi_1(x_1) \cdots \chi_n(x_n) \psi(\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(x_1 + \cdots + x_n)).$$

In this case, there are n characteristic roots with same weight $n-1$. Hence $|\mathrm{Kl}_n| \leqslant nq^{(n-1)/2}$.

Galois action

Clearly, $\mathrm{Kl}_n \in \mathbb{Z}[\mu_{pc}]$, where

$$c = \mathrm{lcm}_i\{\mathrm{ord}(\chi_i)\}$$

divides $q-1$. Write

$$\mathrm{Gal}(\mathbb{Q}(\mu_{pc})/\mathbb{Q}) = \{\sigma_t\tau_w \mid t \in (\mathbb{Z}/p\mathbb{Z})^\times, w \in (\mathbb{Z}/c\mathbb{Z})^\times\},$$

where

$$\sigma_t(\zeta_p) = \zeta_p^t, \quad \sigma_t(\zeta_c) = \zeta_c,$$
$$\tau_w(\zeta_p) = \zeta_p, \quad \tau_w(\zeta_c) = \zeta_c^w.$$

A basic observation tells

$$\sigma_t\tau_w \mathrm{Kl}_n(\psi, \boldsymbol{\chi}, q, a) = \prod \boldsymbol{\chi}(t)^{-w}\mathrm{Kl}_n(\psi, \boldsymbol{\chi}^w, q, at^n).$$

To study the generating fields of Kl_n, we need to consider the distinctness of different Kloosterman sums.

Trivial character

When $\boldsymbol{\chi} = \mathbf{1} = \{1, \ldots, 1\}$ is trivial, it's easy to see that

$$a, b \text{ conjugate } \implies \mathrm{Kl}_n(\psi, \mathbf{1}, q, a) = \mathrm{Kl}_n(\psi, \mathbf{1}, q, b).$$

When $p > (2n^{2d} + 1)^2$ (Fisher), or $p \geqslant (d-1)n + 2$ and p does not divide a certain integer (Wan), this is necessary. In general, it's conjectured that it's true when $p \geqslant nd$. Thus

$$\deg \mathrm{Kl}_n(\psi, \mathbf{1}, q, a) = \frac{p-1}{(p-1, n)}$$

under these conditions.

For our purpose, we need a different sheaf. Deligne and Katz defined the Kloosterman sheaf

$$\mathcal{K}\mathrm{l} = \mathcal{K}\mathrm{l}_{n,q}(\psi, \boldsymbol{\chi})$$

on $\mathbb{G}_m \otimes \mathbb{F}_q = \operatorname{Spec} \mathbb{F}_q[X, X^{-1}]$, with the following properties:

(1) $\mathcal{K}\mathrm{l}$ is lisse (locally constant at every finite level) of rank n and pure of weight $n-1$.

(2) For any $a \in \mathbb{F}_q^{\times}$, $\mathrm{Tr}(\mathrm{Frob}_a, \mathcal{K}\mathrm{l}_{\overline{a}}) = (-1)^{n-1}\mathrm{Kl}_n(\psi, \boldsymbol{\chi}, q, a)$.

(3) $\mathcal{K}\mathrm{l}$ is tame at 0 (Swan$= 0$).

(4) $\mathcal{K}\mathrm{l}$ is totally wild with Swan conductor 1 at ∞. So all ∞-breaks are $1/n$.

Fisher gave a descent of Kloosterman sheaves along an extension of finite fields. For any $a \in \mathbb{F}_q^\times$, he defined a lisse sheaf $\mathcal{F}_a(\boldsymbol{\chi})$ on $\mathbb{G}_m \otimes \mathbb{F}_p$, such that $\mathcal{F}_a(\boldsymbol{\chi})|\mathbb{G}_m \otimes \mathbb{F}_q = \bigotimes_{\sigma \in \mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p)} (t \mapsto \sigma(a)t^n)^* \mathcal{K}\mathrm{l}_n(\psi \circ \sigma^{-1}, \boldsymbol{\chi} \circ \sigma^{-1})$.

(1) $\mathcal{F}_a(\boldsymbol{\chi})$ is lisse of rank n^d and pure of weight $d(n-1)$.

(2) For any $t \in \mathbb{F}_p^\times$, $\mathrm{Tr}(\mathrm{Frob}_t, \mathcal{F}_a(\boldsymbol{\chi})_{\bar{t}}) = (-1)^{(n-1)d} \mathrm{Kl}_n(\psi, \boldsymbol{\chi}, q, at^n)$.

(3) $\mathcal{F}_a(\boldsymbol{\chi})$ is tame at 0 and its ∞-breaks are at most 1.

Key lemma

Lemma

Let $\mathcal{F}, \mathcal{F}'$ be lisse sheaves on $\mathbb{G}_m \otimes \mathbb{F}_p$ of same rank r and pure of the same weight w. Assume that there is a root of unity λ such that for any $t \in \mathbb{F}_p^\times$, we have

$$\mathrm{Tr}(\mathrm{Frob}_t, \mathcal{F}_{\bar{t}}) = \lambda \mathrm{Tr}(\mathrm{Frob}_t, \mathcal{F}'_{\bar{t}}).$$

Let $\mathcal{G}$ be a geometrically irreducible sheaf of rank s on $\mathbb{G}_m \otimes \mathbb{F}_p$, pure of weight w, such that $\mathcal{G} \mid \mathbb{G}_m \otimes \overline{\mathbb{F}}_p$ occurs exactly once in $\mathcal{F} \mid \mathbb{G}_m \otimes \overline{\mathbb{F}}_p$. Then $\mathcal{G} \mid \mathbb{G}_m \otimes \overline{\mathbb{F}}_p$ occurs at least once in $\mathcal{F}' \mid \mathbb{G}_m \otimes \overline{\mathbb{F}}_p$, provided that $p > [2rs(M_0 + M_\infty) + 1]^2$, where M_η is the largest η-break of $\mathcal{F} \oplus \mathcal{F}'$.

Key lemma, proof

Assume not. Applying the Lefschetz Trace Formula to $\mathcal{G}^\vee \otimes \mathcal{F}$ and $\mathcal{G}^\vee \otimes \mathcal{F}'$, we have

$$\sum_{i=0}^{2}(-1)^i \mathrm{Tr}\big(\mathrm{Frob}, \mathrm{H}_c^i(\mathcal{G}^\vee \otimes \mathcal{F})\big) = \lambda \sum_{i=0}^{2}(-1)^i \mathrm{Tr}\big(\mathrm{Frob}, \mathrm{H}_c^i(\mathcal{G}^\vee \otimes \mathcal{F}')\big).$$

Apply Euler-Poincaré formula

$$\begin{aligned} & h_c^0(\mathcal{F}) - h_c^1(\mathcal{F}) + h_c^2(\mathcal{F}) \\ = & \operatorname{rank} \mathcal{F} \cdot \chi_c(\mathbb{G}_m \otimes \mathbb{F}_p) - \mathrm{Sw}_0(\mathcal{F}) - \mathrm{Sw}_\infty(\mathcal{F}) \end{aligned}$$

to estimate $\mathrm{Tr}(\mathrm{Frob}, \mathrm{H}_c^1)$ (weight $\leqslant 1$ by Weil II).

Corollary

The n-tuple $\boldsymbol{\chi}$ is called *Kummer-induced* if there exsists a non-trivial character Λ such that $\boldsymbol{\chi} = \boldsymbol{\chi}\Lambda := \{\chi_1\Lambda, \ldots, \chi_n\Lambda\}$ as unordered n-tuples. In this case, $\prod \boldsymbol{\chi} = \prod(\boldsymbol{\chi}\Lambda) = \Lambda^n \prod \boldsymbol{\chi}$ and thus $\Lambda^n = 1$.

Assume that $p > 2n+1$ and $\boldsymbol{\chi}$ is not Kummer-induced. Then $\mathcal{F}_a(\boldsymbol{\chi})$ has a highest weight with multiplicity one. Thus it has a subsheaf $\mathcal{G}_a(\boldsymbol{\chi})$ such that, as representations of the Lie algebra $\mathfrak{g}(\mathcal{F}_a(\boldsymbol{\chi}))$, $\mathcal{G}_a(\boldsymbol{\chi})$ is the irreducible sub-representation with highest weight. Moreover, it is geometrically irreducible and occurs exactly once in $\mathcal{F}_a(\boldsymbol{\chi})$ over $\mathbb{G}_m \otimes \overline{\mathbb{F}}_p$.

Corollary

Let $a, b \in \mathbb{F}_q^\times$ **and let** χ **and** ρ **be** n**-tuples of multiplicative characters** $\chi_i, \rho_j : \mathbb{F}_q^\times \to \overline{\mathbb{Q}}_\ell^\times$**. Assume that** $p > (2n^{2d}+1)^2$**,** χ **is not Kummer-induced and**

$$\mathrm{Kl}_n(\psi, \boldsymbol{\chi}, q, a) = \lambda \mathrm{Kl}_n(\psi, \boldsymbol{\rho}, q, b)$$

for a fixed root of unity $\lambda \in \mu_{q-1}$**. Then** $\mathcal{G}_a(\boldsymbol{\chi}) \otimes \mathcal{L}_{\prod \overline{\boldsymbol{\chi}}} \mid \mathbb{G}_m \otimes \overline{\mathbb{F}}_p$ **occurs at least once in** $\mathcal{F}_b(\boldsymbol{\rho}) \otimes \mathcal{L}_{\prod \overline{\boldsymbol{\rho}}} \mid \mathbb{G}_m \otimes \overline{\mathbb{F}}_p$**.**

Here $\mathcal{L}_\chi$ is a rank one lisse sheaf on $\mathbb{G}_m \otimes \mathbb{F}_p$ such that for $t \in \mathbb{F}_p^\times$,

$$\mathrm{Tr}(\mathrm{Frob}_t, (\mathcal{L}_\chi)_{\overline{t}}) = \chi(t).$$

Denote by

$$\mathcal{F} = \mathcal{F}_a(\boldsymbol{\chi}) \otimes \mathcal{L}_{\prod \overline{\chi}}, \ \mathcal{F}' = \mathcal{F}_b(\boldsymbol{\rho}) \otimes \mathcal{L}_{\prod \overline{\rho}}, \ \mathcal{G} = \mathcal{G}_a(\boldsymbol{\chi}) \otimes \mathcal{L}_{\prod \overline{\chi}}.$$

For $t \in \mathbb{F}_p^\times$, we have $\sigma_t \lambda = \lambda$ and thus

$$\begin{aligned}
&(-1)^{(n-1)d}\mathrm{Tr}(\mathrm{Frob}_t, \mathcal{F}_{\overline{t}}) = \prod \overline{\boldsymbol{\chi}}(t) \cdot \mathrm{Kl}_n(\psi, \boldsymbol{\chi}, q, at^n) \\
=&\sigma_t(\mathrm{Kl}_n(\psi, \boldsymbol{\chi}, q, a)) = \lambda \sigma_t(\mathrm{Kl}_n(\psi, \boldsymbol{\rho}, q, b)) \\
=&\lambda \prod \overline{\boldsymbol{\rho}}(t) \cdot \mathrm{Kl}_n(\psi, \boldsymbol{\rho}, q, bt^n) = (-1)^{(n-1)d}\lambda \mathrm{Tr}(\mathrm{Frob}_t, \mathcal{F}'_{\overline{t}}).
\end{aligned}$$

Apply Lemma to $r = s = n^d, M_0 = 0, M_\infty \leqslant 1.$

Now

$$\mathcal{G}_a(\boldsymbol{\chi}) \otimes \mathcal{L}_{\prod \overline{\boldsymbol{\chi}}} \hookrightarrow \mathcal{F}_b(\boldsymbol{\rho}) \otimes \mathcal{L}_{\prod \overline{\boldsymbol{\rho}}}, \quad \mathcal{G}_b(\boldsymbol{\rho}) \otimes \mathcal{L}_{\prod \overline{\boldsymbol{\rho}}} \hookrightarrow \mathcal{F}_a(\boldsymbol{\chi}) \otimes \mathcal{L}_{\prod \overline{\boldsymbol{\chi}}}.$$

Thus the highest weight $\lambda_a(\boldsymbol{\chi}) = \lambda_b(\boldsymbol{\rho})$. Derived from this, and combining Fisher's arguments, we have:

Theorem (Z.)

Let $a, b \in \mathbb{F}_q^\times$. Assume that $\boldsymbol{\chi}, \boldsymbol{\rho}$ are not Kummer-induced and neither of them is of type $(\xi_1, \xi_1^{-1}, 1, \Lambda_2)\xi_2$. If $p > (2n^{2d} + 1)^2$ and

$$\mathrm{Kl}_n(\psi, \boldsymbol{\chi}, q, a) = \lambda \mathrm{Kl}_n(\psi, \boldsymbol{\rho}, q, b)$$

for some $\lambda \in \mu_{q-1}$, then there exists $\sigma \in \mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p)$ and a multiplicative character η, such that $b = \sigma(a)$ and $\boldsymbol{\rho} = \eta \cdot (\boldsymbol{\chi} \circ \sigma^{-1})$ as unordered tuples. Moreover, either both Kloosterman sums vanish or $\eta(b) = \lambda^{-1}$.

The last step is to show the non-vanishingness.

Theorem

If $p > (3n-1)C_{\boldsymbol{\chi}} - n$ and for any i, j, $\chi_i = \chi_j$ if $\chi_i^n = \chi_j^n$, then $\mathrm{Kl}_n(\psi, \boldsymbol{\chi}, q, a)$ is nonzero. Here

$$C_{\boldsymbol{\chi}} = \max_{i,j} \operatorname{lcm}(\operatorname{ord}(\chi_i), \operatorname{ord}(\chi_j)) \tag{1}$$

is the supremum of least common multipliers of the orders of any two characters in $\boldsymbol{\chi}$.

Non-vanishingness, continue

We can express Kl_n as Gauss sums

$$(q-1)\mathrm{Kl}_n(\psi, \boldsymbol{\chi}, q, a) = \sum_{m=0}^{q-2} \omega^m(a) \prod_{i=1}^{n} g(m+s_i)$$

by Fourier transform on $\mathbb{F}_q^\times$, where $\chi_i = \omega^{s_i}$ for a Teichmüller character. What we need to do is to proof there is a unique m such that the valuation of $\prod_{i=1}^n g(m+s_i)$ is minimal.

Main result

Theorem (Z.)

If $p > \max\left\{(2n^{2d}+1)^2, (3n-1)C_{\chi} - n\right\}$ **and for any** i, j, $\chi_i = \chi_j$ **if** $\chi_i^n = \chi_j^n$**, then** $\mathrm{Kl}_n(\psi, \boldsymbol{\chi}, q, a)$ **generates** $\mathbb{Q}(\mu_{pc})^H$**, where** H **consists of those** $\sigma_t\tau_w$ **such that there exists an integer** β **and a character** η **satisfying**

$$t = \lambda a_1^{\beta}, \lambda^{n_1} = 1,\ \boldsymbol{\chi}^w = \eta\boldsymbol{\chi}^{q_1^{\beta}},\ \eta(a) = \prod \boldsymbol{\chi}^w(t).$$

Here $n_1 = (n, p-1)$**,** $q_1 = \#\mathbb{F}_p(a^{(p-1)/n_1})$ **and** $a_1 \in \mathbb{F}_p^{\times}$ **such that** $a_1^{n/n_1} = \boldsymbol{N}_{\mathbb{F}_{q_1}/\mathbb{F}_p}(a^{(1-p)/n_1}) = a^{(1-q_1)/n_1}$**.**

An example: $n=2$ case

Let $\boldsymbol{\chi}=\{1,\chi\}$, where χ is a multiplicative character of order $c\neq 2$. If $p>\max\left\{(2^{2d+1}+1)^2, 5c-2\right\}$, then $\mathrm{Kl}(\psi,\boldsymbol{\chi},p^d,a)$ generates $\mathbb{Q}(\mu_{pc})^H$, where

$$H=\begin{cases}\langle \tau_{q_1}\sigma_{a_1},\sigma_{-1},\tau_{-1}\rangle, & \text{if } \chi(-1)=1,\chi(a)=1;\\ \langle \tau_{-q_1}\sigma_{a_1},\sigma_{-1}\rangle, & \text{if } \chi(-1)=1,\chi(a)=\chi(a_1)=-1;\\ \langle \tau_{q_1^\alpha}\sigma_{a_1^\alpha},\sigma_{-1}\rangle, & \text{if } \chi(-1)=1,\chi(a)^\alpha\neq 1;\\ \langle \tau_{q_1}\sigma_{-a_1},\tau_{-1}\sigma_{-1}\rangle, & \text{if } \chi(-1)=-1,\chi(a)=\chi(a_1)=-1;\\ \langle \tau_{q_1}\sigma_{a_1},\tau_{-1}\rangle & \text{if } \chi(-1)=-1,\chi(a)=1;\\ \langle \tau_{q_1}\sigma_{a_1},\tau_{-1}\sigma_{-1}\rangle, & \text{if } \chi(-1)=-1,\chi(a)=-1,\chi(a_1)=1;\\ \langle \tau_{q_1^{\alpha/2}}\sigma_{-a_1^{\alpha/2}}\rangle, & \text{if } \chi(-1)=-1,2\mid\alpha,\chi(a)\neq\pm1;\\ \langle \tau_{q_1^\alpha}\sigma_{a_1^\alpha}\rangle, & \text{if } \chi(-1)=-1,2\nmid\alpha,\chi(a)\neq\pm1.\end{cases}$$

$q_1=\#\mathbb{F}_p(a^{(1-p)/2}), a_1=a^{(1-q_1)/2}$ and α is the order of $\chi(a_1)\in\mu_{p-1}$.

Remark

Consider the Kloosterman sums

$$S_k = \mathrm{Kl}(\psi, \boldsymbol{\chi} \circ \boldsymbol{N}_{\mathbb{F}_{q^k}/\mathbb{F}_q}, q^k, a).$$

If $p > \max\left\{(2n^{2dk}+1)^2, (3n-1)C_{\boldsymbol{\chi}} - n\right\}$, then $\mathbb{Q}(S_k) = \mathbb{Q}(\mu_{pc})^H$, where H consists of those $\sigma_t \tau_w$ such that there exists an integer β and a character η on $\mathbb{F}_q^\times$ satisfying

$$t = \lambda a_1^\beta, \lambda^{n_1} = 1, \quad \boldsymbol{\chi}^w = \eta \boldsymbol{\chi}^{q_1^\beta}, \quad \eta(a) = \gamma \cdot \prod \boldsymbol{\chi}^w(t), \gamma^k = 1.$$

Thus $\mathbb{Q}(S_k) = \mathbb{Q}(S_{k-c})$ since $\gamma^c = 1$.

Remark, continue

The L-function

$$L(T) = \exp\left(\sum_{k=1}^{\infty} \frac{T^k}{k} S_k\right)$$

is a rational function. Thus the sequence $\{S_k\}_k$ is linear recurrence sequence. The sequence $\{\mathbb{Q}(S_k)\}_{k \geqslant N}$ is periodic of period r for some N (Wan, Yin). Thus if $p > \max\left\{\left(2n^{2d(N+r)} + 1\right)^2, (3n-1)C_\chi - n\right\}$, the generating field of S_k is determined by the previous equations for any k. For this purpose, we need to decrease the bound $(2n^{2d} + 1)^2$ and estimate the period r and N. We conjecture that S_k has the predicted generating field if $p > 3ndc$.

Thank you!