Lecture 1 – Vectors, what are they?

① Vectors
- Physics : arrow(s) in space
- CS perspective : Ordered list of numbers e.g. $\begin{bmatrix}-3\\1\end{bmatrix}$ means $x=-3,\ y=1$
- Mathematician : operations of vectors e.g. $\vec{v}+\vec{w}$ $\quad \underset{\uparrow\ \text{scalar}}{2}\vec{w}$
  (mainly adding & scaling)

The interchange of above 3 views is the essence of studying linear algebra

Lecture 2 – Linear combinations, span and basis vectors

① The "span" of $\vec{v}$ and $\vec{w}$ is the set of all their linear combinations
$a\vec{v}+b\vec{w}$ where a, b vary over all real numbers

② "Span" can (Two vectors)
- spread over the 2-D space
- grow only a single line (if $\vec{v}$ and $\vec{w}$ line up)
- a single point only (if $\vec{v}$ and $\vec{w}$ same points)

③ There can be over 2 vectors. e.g.: $a\vec{u}+b\vec{v}+c\vec{w}$

④ ※
- Linearly dependent → e.g. $\vec{u}=a\vec{v}+b\vec{w}$ for some values of a and b
- Linearly independent → e.g. $\vec{u}\neq a\vec{v}+b\vec{w}$ for All values of a, b

⑤ Basis : The basis of a vector space is a set of linearly independent vectors that span the full space. $\begin{bmatrix}1&0\\0&1\end{bmatrix}$

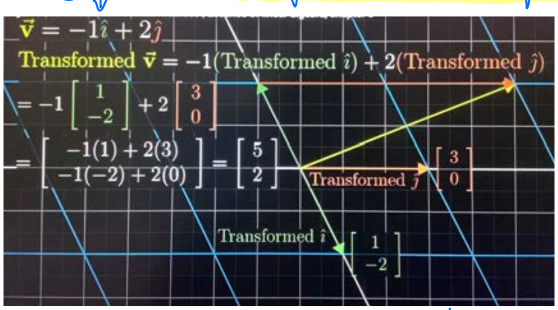
Lecture 3 – Linear transformations and matrices

① "Linear transformation" → after the movement
- Lines remain lines (including diagonals !!)
- Origin remains fixed

⇒ ※ Grid lines remain parallel and evenly spaced

② To build computer acceptable model, consider $\vec{v}=a\hat{\imath}+b\hat{\jmath}$ ※ $\left(\begin{bmatrix}1&0\\0&1\end{bmatrix}\right.$ is the original 2D-state$\left.\right)$

$\vec{v}:\begin{bmatrix}-1\\2\end{bmatrix}$ Then Transformed $\vec{v}$ = a (Transformed $\hat{\imath}$) + b (Tranformed $\hat{\jmath}$)



③ "2X2 Matrix" to represent set of basis

$\begin{bmatrix}a&b\\c&d\end{bmatrix}$ where $\hat{\imath}$ lands, where $\hat{\jmath}$ lands
- a, b x-axis value
- c, d y-axis value

each column is a description of 1 vector.

The values of a–d describe the transformation behavior.

④ 2-D linear transformation $\begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=x\begin{bmatrix}a\\c\end{bmatrix}+y\begin{bmatrix}b\\d\end{bmatrix}=\begin{bmatrix}ax+by\\cx+dy\end{bmatrix}$

⑤ Technically, a transformation L is linear if it satisfies these two properties:
$L(\vec{v}+\vec{w})=L(\vec{v})+L(\vec{w})$ "Additivity"
$L(c\vec{v})=c\,L(\vec{v})$ "Scaling"

Lecture 4 - Matrix multiplication as composition

① "Composition" of a rotation and a shear

※ Read right to left — like $f(g(x))$

The Right matrix is the Base one

$$\underbrace{\begin{bmatrix}1&1\\0&1\end{bmatrix}}_{\substack{\text{shear}\\M_2}}\underbrace{\begin{bmatrix}0&-1\\1&0\end{bmatrix}}_{\substack{\text{rotation}\\M_1}}=\underbrace{\begin{bmatrix}1&-1\\1&0\end{bmatrix}}_{\text{composition}}$$

② General "Composition" — Matrix multiplication

this first

$$※\ \underbrace{\begin{bmatrix}a&b\\c&d\end{bmatrix}}_{M_2}\underbrace{\begin{bmatrix}e&f\\g&h\end{bmatrix}}_{M_1}=\underbrace{\left[\begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}e\\g\end{bmatrix}\ \Big|\ \begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}f\\h\end{bmatrix}\right]}_{\text{break } M_1}=\begin{bmatrix}ae+bg&af+bh\\ce+dg&cf+dh\end{bmatrix}$$

$\hat{i}$ $\hat{j}$

③ M_1 & M_2 are different, order Does matter.

$M_1M_2 \neq M_2M_1$ $\qquad$ $M_1M_2M_3 = M_1(M_2M_3) = (M_1M_2)M_3$ (Associativity)

Lecture 5 — Three-dimentional liear transformations

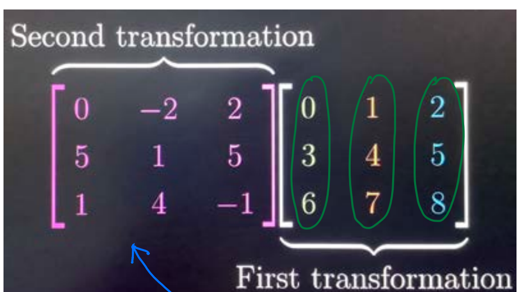
① $\begin{bmatrix}a\\b\\c\end{bmatrix}$ input $\quad L(\vec{v})\quad$ $\begin{bmatrix}d\\e\\f\end{bmatrix}$ output

② "3X3 Matrix" to represent set of basis

$$\underbrace{\hat{i}\rightarrow\begin{bmatrix}0\\0\\-1\end{bmatrix}\quad \hat{j}\rightarrow\begin{bmatrix}0\\1\\0\end{bmatrix}\quad \hat{k}\rightarrow\begin{bmatrix}1\\0\\0\end{bmatrix}}\ \Rightarrow\ \begin{bmatrix}0&0&1\\0&1&0\\-1&0&0\end{bmatrix}$$

③ $\vec{v}=\begin{bmatrix}x\\y\\z\end{bmatrix}=x\vec{i}+y\vec{j}+z\vec{k}$ $\quad$ so $\begin{bmatrix}a_1&b_1&c_1\\a_2&b_2&c_2\\a_3&b_3&c_3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=x\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix}+y\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}+z\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}$

$$=\begin{bmatrix}a_1x+b_1y+c_1z\\a_2x+b_2y+c_2z\\a_3x+b_3y+c_3z\end{bmatrix}$$

④ 

$$=\begin{bmatrix}6&6&6\\33&44&55\\6&10&14\end{bmatrix}$$

Lecture 6 — The determinant

Focus on the change of area/volume during linear transformation

$$\begin{bmatrix}a&b\\c&d\end{bmatrix}$$

rotation $\quad$ shear

shear $\quad$ rotation

① 2-D Determinant : the Scaling Factor by which linear transformation changes Any area.

② 3-D Determinant → similar scaling factor changes Any Volume. (think of parallelpiped 平行六面体)

③ Determinant can be Negative figures → Feels like flipping space

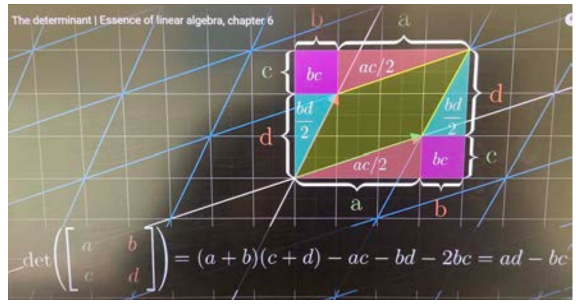
④ $\det\left(\underbrace{\begin{bmatrix}1.0&0.0&1.0\\0.5&1.0&1.5\\1.0&0.0&1.0\end{bmatrix}}\right)=0$ $\quad\begin{cases}a=1\\b=0\\c=1\end{cases}$

Columns must be linearly dependent

fulfil the guess

⑤ 2D Computation: $\det\left(\begin{bmatrix}a&b\\c&d\end{bmatrix}\right)=ad-bc$

⑥ 3D computation $\det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = a\det\left(\begin{bmatrix} e & f \\ h & i \end{bmatrix}\right) - b\det\left(\begin{bmatrix} d & f \\ g & i \end{bmatrix}\right) + c\det\left(\begin{bmatrix} d & e \\ g & h \end{bmatrix}\right)$

⑧ 

⑦ $\det(M_1M_2) = \det(M_1)\det(M_2)$

Lecture 7 – Inverse matrices, column space and null space

① Linear system of equations

$$\begin{aligned} 2x + 5y + 3z &= -3 \\ 4x + 0y + 8z &= 0 \\ 1x + 3y + 0z &= 2 \end{aligned} \rightarrow \underbrace{\begin{bmatrix} 2 & 5 & 3 \\ 4 & 0 & 8 \\ 1 & 3 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}}_{\vec{v}}$$

determinant — variables — constants

$A\vec{x} = \vec{v}$

② $\det(A) \neq 0$

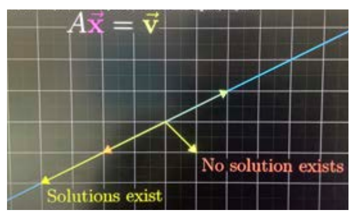
- 2D A^{-1}: Inverse transformation $A^{-1}A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ← the transformation that does nothing. (Identity transformation)
- 3D A^{-1}: $A^{-1}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\vec{x} = A^{-1}\vec{v}$ is the solution

③ if $\det(A) = 0$ → NO A^{-1} exists ;

$\vec{x}$ may or may not exist.

(i.e. there may or may not have solution)



← 2D $\det(A) = 0$ example

④ "Rank" → No. of dimensions in the output of a transformation (in the column space)

"Rank 1" → All output of transformation is a line.

"Rank 2" → If all the vectors land on some 2-dimensional plain

"Collapsed" → e.g. "Rank 2" 3X3 matrices

"Full Rank" → When Rank = No. of columns of a matrix (The Rank is as its highest as it can be)

"Column space" → Span of columns of a matrix e.g. $\begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}$ $\hat{i}$ $\hat{j}$

$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is always included in the column space

(zero vector) (∵ linear transformation must ensure fixed origin)

⑤ $\begin{bmatrix}0\\0\end{bmatrix}$ <
- Full Rank matrices → Only $\begin{bmatrix}0\\0\end{bmatrix}$ lands on $\begin{bmatrix}0\\0\end{bmatrix}$
- Non-full rank matrices → Bunch of vectors land on $\begin{bmatrix}0\\0\end{bmatrix}$ $det(A) = 0$

"Null space"/"kernel" of matrices → space of all vectors that become "null"
(must $det(A) = 0$; i.e. non-full rank)   e.g. $A\vec{x} = \begin{bmatrix}0\\0\end{bmatrix}$ solve $\vec{x}$

Lecture 8 – Non-square matrices as transformations between dimensions

① Non-square matrices transform input to output between dimensions

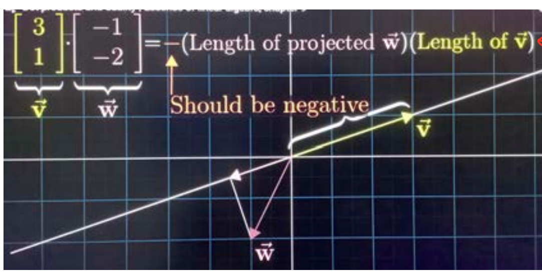
② e.g. $\underbrace{\begin{bmatrix}2\\7\end{bmatrix}}_{\substack{\text{2D input}\\ 2\times1 \text{ matrix}}} \rightarrow L(\vec{v}) \rightarrow \underbrace{\begin{bmatrix}1\\8\\7\end{bmatrix}}_{\substack{\text{3D output}\\ 3\times1 \text{ matrix}}}$

③ In reality, Non-square matrices can be transformed to Square matrice by filling 0s.

Lecture 9 – Dot products and duality

① e.g. $\begin{bmatrix}2\\7\\1\end{bmatrix}\cdot\begin{bmatrix}8\\2\\3\end{bmatrix} = [2\cdot8+7\cdot2+1\cdot3] = [33]$ $\equiv [2\ 7\ 1]\cdot\begin{bmatrix}8\\2\\3\end{bmatrix}$

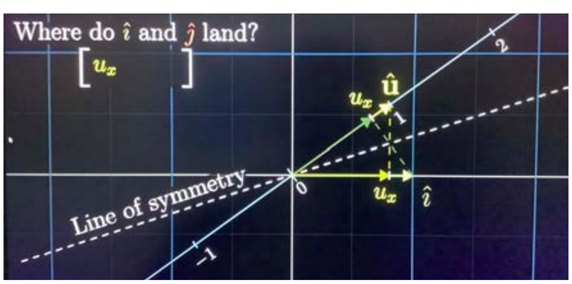
② Dot products : $n\times1$ matrix $\times$ $n\times1$ matrix (same n) = 1×1 matrix (single number)



(when $\vec{v}$ and $\vec{w}$ in different direction)
← = −(length of projected $\vec{v}$)(Length of $\vec{w}$)
this implies Symmetry of the Dot Products.

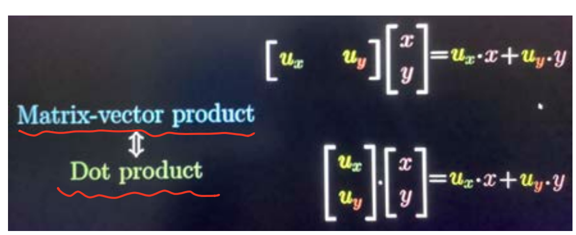
③ In 2-Dimension, assume $\vec{v} = \begin{bmatrix}u_x\\u_y\end{bmatrix}$, due to symmetry, the projecting tranformation matrix is $[u_x\ u_y]$ ← 1D basis

$\hat{i}$ → u_x, u_y ← $\hat{j}$



⇐ Why symmetry?

④ Duality of Dot products derived from ③ ⇒



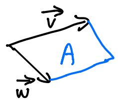
Lecture 10 – Cross Products (Standard Introduction)

① 2D Cross Products: $\vec{v} \times \vec{w}$ = Area of parallelogram

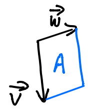
② Signs of 2D Cross Products $\vec{v} \times \vec{w}$
- Positive: $\vec{v}$ is on the Right of $\vec{w}$
- Negative: $\vec{v}$ is on the Left of $\vec{w}$

$\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$

e.g.



$\vec{v} \times \vec{w} = -|A|$;



$\vec{v} \times \vec{w} = |A|$

③ Computation of Cross Products: Using determinant → Scaling factor by which linear transformation changes the Area of basis vectors

e.g. $\vec{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ $\vec{w} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ $\vec{v} \times \vec{w} = \det\left(\begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix}\right)$

④ Scaling of Cross Products: $(a\vec{v}) \times \vec{w} = a \cdot (\vec{v} \times \vec{w})$

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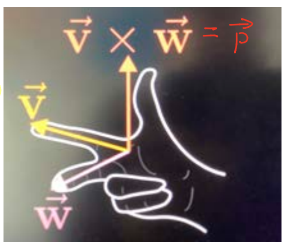
⑤ 3D Cross Products of two vectors: Result will be a vector perpendicular to the parallelogram.
(垂直的) (formed by 2 vectors)

Right Hand Rule



which direction perpendicular?

(有2个方向，用右手大拇指确定结果的垂直方向)

⑥ 3D Computation of 2 vectors' products: this cal. only equivalent with basis vectors, NOT pure numbers!!

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \times \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \det\left(\begin{bmatrix} \hat{i} & v_1 & w_1 \\ \hat{j} & v_2 & w_2 \\ \hat{k} & v_3 & w_3 \end{bmatrix}\right) = \hat{i}\det\left(\begin{bmatrix} v_2 & w_2 \\ v_3 & w_3 \end{bmatrix}\right) + \hat{j}\det\left(\begin{bmatrix} v_1 & w_1 \\ v_3 & w_3 \end{bmatrix}\right) + \hat{k}\det\left(\begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix}\right)$$

basis vectors

$$= \hat{i}(v_2w_3 - v_3w_2) + \hat{j}(v_3w_1 - v_1w_3) + \hat{k}(v_1w_2 - v_2w_1) = \begin{bmatrix} v_2 \cdot w_3 - w_2 \cdot v_3 \\ v_3 \cdot w_1 - w_3 \cdot v_1 \\ v_1 \cdot w_2 - w_1 \cdot v_2 \end{bmatrix}$$

Lecture 11 – Cross Products (Geometric Understanding) - 3D products

① 2D-to-1D Duality of "Matrix-vector product" & "Dot product"

$$\underbrace{\begin{bmatrix} m \\ n \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}}_{\text{Dot product}} = \underbrace{\begin{bmatrix} m & n \end{bmatrix}}_{\text{Transformation}}\underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\vec{v}}$$

② 3D cross product results a vector

$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \det\left(\begin{bmatrix} \underbrace{x}_{\text{variable}} & \overbrace{v_1}^{\vec{v}} & \overbrace{w_1}^{\vec{w}} \\ y & v_2 & w_2 \\ z & v_3 & w_3 \end{bmatrix}\right)$$

Remark: $\vec{v} \times \vec{w} = \vec{p}$

∵ f is linear, and it's a 3D-to-1D linear transformation, we can encode a 1X3 matrix:

$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} P_1 & P_2 & P_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix}}_{\vec{p}}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{P_1x + P_2y + P_3z = \begin{matrix} x(v_2w_3 - v_3w_2) + \\ y(v_3w_1 - v_1w_3) + \\ z(v_1w_2 - v_2w_1) \end{matrix}}$$

∴ $P_1 = v_2w_3 - v_3w_2$; $P_2 = v_3w_1 - v_1w_3$; $P_3 = v_1w_2 - v_2w_1$

③ Geometrically, What vector $\vec{p}$ has the property that

$$\vec{p}\cdot\begin{bmatrix}x\\y\\z\end{bmatrix} = (\text{Length of projection}) \times (\text{Length of } \vec{p}) \Rightarrow \underbrace{\begin{bmatrix}p_1\\p_2\\p_3\end{bmatrix}}_{\vec{p}}\begin{bmatrix}x\\y\\z\end{bmatrix} = \det\left(\begin{array}{ccc} & \vec{v} & \vec{w} \\ \left[\begin{matrix}x\\y\\z\end{matrix}\right. & \begin{matrix}v_1\\v_2\\v_3\end{matrix} & \left.\begin{matrix}w_1\\w_2\\w_3\end{matrix}\right] \\ \text{variable} & & \end{array}\right)$$

$\vec{v}\times\vec{w} = (\text{Area of parallelogram}) \times (\text{Component of } \begin{bmatrix}x\\y\\z\end{bmatrix} \text{ perpendicular to } \vec{v} \text{ and } \vec{w})$ (平行六边形的体积)

$=$ Dot product between $\begin{bmatrix}x\\y\\z\end{bmatrix}$ and a vector $\vec{p}$ that is perpendicular to $\vec{v}$ and $\vec{w}$ with its length equal to the area of parallelogram

④ In Conclusion, let $\begin{bmatrix}x\\y\\z\end{bmatrix}$ be basis, we have $\vec{p} = \det\left(\begin{bmatrix}\hat{i} & v_1 & w_1\\ \hat{j} & v_2 & w_2\\ \hat{k} & v_3 & w_3\end{bmatrix}\right)$
(and by Duality)

Remark: Length of $\vec{p}$ = Area of the parallelogram spanned by $\vec{v}$ and $\vec{w}$.

Lecture 12 — Cramer's Rule (explained geometrically)

① Cramer's Rule

e.g. $\underbrace{\begin{bmatrix}-4 & 2 & 3\\ -1 & 0 & 2\\ -4 & 6 & -9\end{bmatrix}}_{A}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}7\\-8\\3\end{bmatrix} \Rightarrow$

"Cramer's rule"

$$x = \frac{\det\left(\begin{bmatrix}7 & 2 & 3\\ -8 & 0 & 2\\ 3 & 6 & -9\end{bmatrix}\right)}{\det\left(\begin{bmatrix}-4 & 2 & 3\\ -1 & 0 & 2\\ -4 & 6 & -9\end{bmatrix}\right)} \quad y = \frac{\det\left(\begin{bmatrix}-4 & 7 & 3\\ -1 & -8 & 2\\ -4 & 3 & -9\end{bmatrix}\right)}{\det\left(\begin{bmatrix}-4 & 2 & 3\\ -1 & 0 & 2\\ -4 & 6 & -9\end{bmatrix}\right)} \quad z = \frac{\det\left(\begin{bmatrix}-4 & 2 & 7\\ -1 & 0 & -8\\ -4 & 6 & 3\end{bmatrix}\right)}{\det\left(\begin{bmatrix}-4 & 2 & 3\\ -1 & 0 & 2\\ -4 & 6 & -9\end{bmatrix}\right)}$$

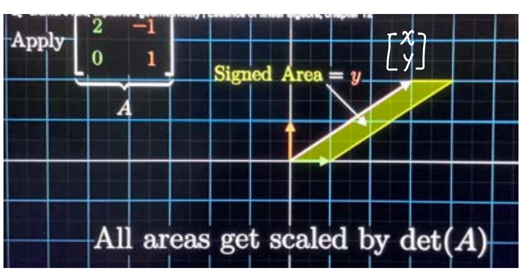
② Remark: Cramer's Rule is NOT the most efficient way to do, "Gaussian Elimination", for example, can do faster.

③ To solve linear equations, we can understand by examine Cramer's Rule.

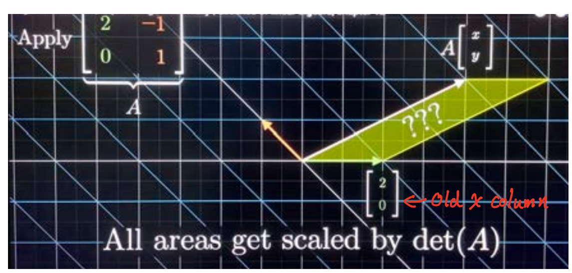
$$x = \frac{\text{Area with } x \text{ column transformed}}{\det(A)} \qquad y = \frac{\text{Area with } y \text{ column transformed}}{\det(A)} \qquad \text{Note: } \det(A) \neq 0$$

$$z = \frac{\text{Area with } z \text{ column transformed}}{\det(A)} \qquad \text{(transformed by the output vector)}$$

※ Signed area with new y column = $\det(A)\cdot$(original area with $\begin{bmatrix}x\\y\end{bmatrix}$ + basis x-axis) = $\det(A)\cdot y$



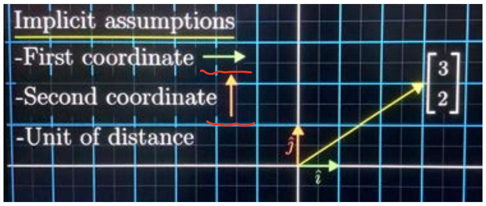
linear transf. $\Rightarrow$ with A



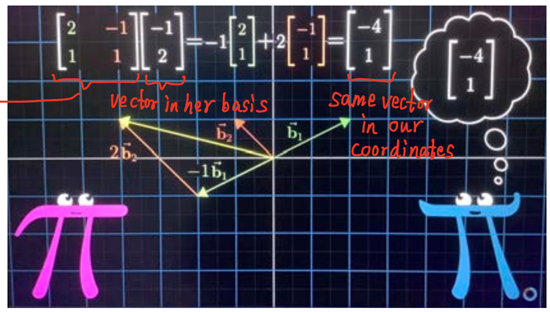
(Use 2D-vectors as an example)

Lecture 13 – Change of Basis

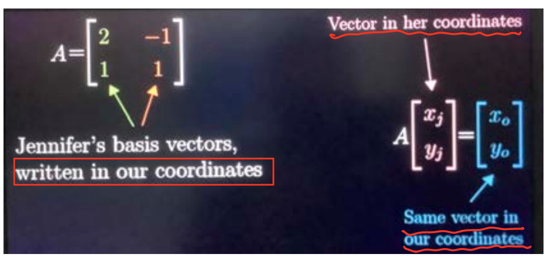
① Coordinates : think of each numbers in the system as scalars



② Alternative Basis : like Different languages (Origin still the same)



③ Translation between different basis systems



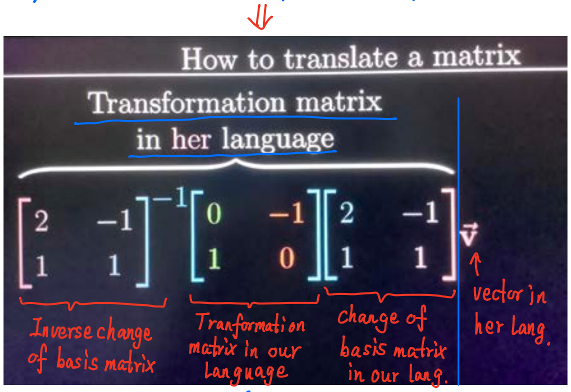
$$\begin{bmatrix} x_j \\ y_j \end{bmatrix} = A^{-1}\begin{bmatrix} x_o \\ y_o \end{bmatrix}$$

④ Find "New vector in Jennifer's language"

※

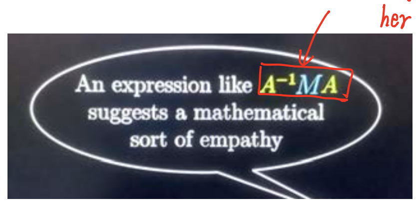
say we want to have $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ transformation in our lang.

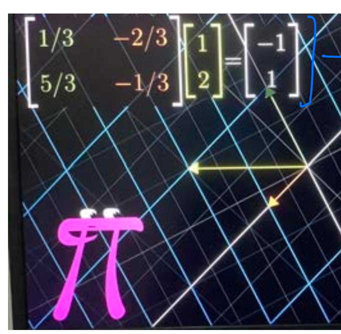
⇓



⇑
we need to know the specific transformation matrix in our lang.

the same transformation in her language



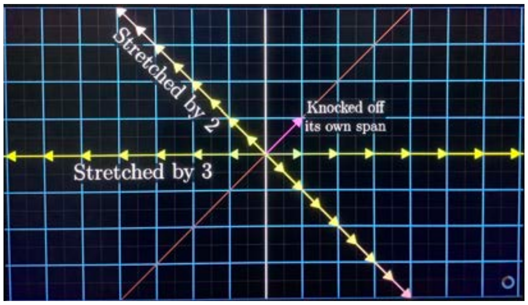


result in her basis

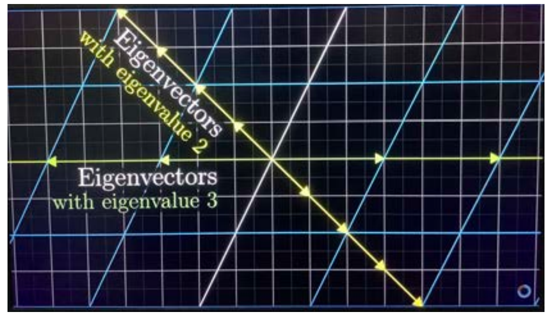
Lecture 14 — Eigenvectors and Eigenvalues

① **Eigenvectors** (/Characteristic vectors): A nonzero vector pointing in its original direction (or reversed direction) after a specific linear transformation. (Stay on the line that it's spanned out)

An example (2D case)



⇒



Eigenvalues: the scalar factor by which Eigenvectors are sketched.

② **3D - Rotation** : Consider 3D rotation, the characteristic vectors are "**Axis of those specific Rotations**". In these cases, Eigenvalues are always 1.

③ **Computational ideas**

$$\underbrace{A}_{\text{Transformation Matrix}}\underbrace{\vec{v}}_{\text{Eigenvector}} = \underbrace{\lambda}_{\text{Eigenvalue}}\underbrace{\vec{v}}_{\text{Eigenvector}}$$

Matrix-vector multiplication = Scalar multiplication

Solve for λ and $\vec{v}$ to make this expression true

Define I as the identity matrix.

$\equiv A\vec{v} - \lambda I\vec{v} = \vec{0}$

$\equiv (A - \lambda I)\vec{v} = \vec{0}$; **We need $\det(A-\lambda I) = 0$**

Why? We need the transformation of $\vec{v}$ Squish the space into a lower dimension. (压扁)

$\therefore$ Squishification $\Rightarrow \det(A-\lambda I) = 0$

④ there **could be NO eigenvectors**

If there is **no Real solution λ** for $\det(A-\lambda I)=0$ then there are no eigenvectors in this case.

(anticlockwise 90° rotation)

e.g. $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $\det\left(\begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}\right) = 0$ has no Real λ solutions. $(\lambda^2 + 1 = 0)$

Remark: Eigenvalues λ which are complex numbers generally correspond to some kinds of rotation in the transformation.

⑤ there **could be only ONE fixed eigenvalue.**

e.g. $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\det\left(\begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix}\right) = (1-\lambda)^2 = 0 \Rightarrow \lambda = 1$ All such eigenvectors have eigenvalue 1

(sheer 45°)

However, **a single eigenvalue can have more than a line full of eigenvectors.**

e.g. $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ Scale everything by 2 & all vectors are eigen vectors

⑥ **Eigenbasis** — What if both of the alternative basis vectors are eigenvectors?

"**Diagonal matrix**": Matrix with all non-diagonal numbers equal to 0. e.g. $\begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

⑥ (Continued) The same linear transformation matrix in her language $A^{-1}MA$ is Guaranted to be a diagonal matrix if her basis are eigenvectors of M transformation.

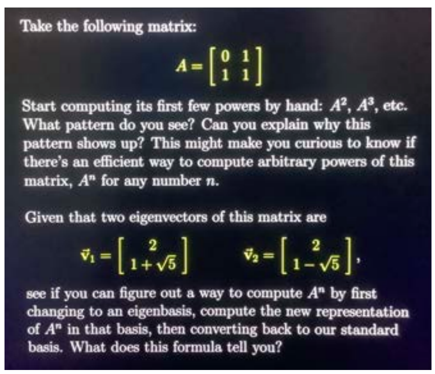
e.g. $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$

⑦ Usage of eigenbasis: Compute high level power of matrix, e.g. $\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}^{100}$ (repeat same linear transformation)

Remark: It's very easy to compute power of a diagonal matrix but hard of a non-diagonal. We can compute in her language first, and transfered back to our basis to get the result.

Limitation: Can not do this for all linear transformations, we must have ≥ 2 eigenvectors first. e.g. a sheer doesn't have enough eigenvectors to do this.

Application:



Lecture 15 - Abstract vector spaces

① Determinant & Eigenvectors do NOT care about the coordinate system. (fixed underlining values)

② Relationship between Functions and Vectors:

Addition — $(f+g)(x) = f(x) + g(x) \iff \begin{bmatrix} x_1 \\ y_1 \\ \vdots \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} x_1+x_2 \\ y_1+y_2 \\ \vdots \end{bmatrix}$

Scaling — $(2f)(x) = 2f(x)$

③ Linear transformation of functions (= linear operations)

e.g. derivative $\frac{d}{dx}\left(\frac{1}{9}x^3 - x\right) = \frac{1}{3}x^2 - 1$

④ Formal definition of linearity $\Rightarrow$ (Abstractness)
$\begin{cases} \text{Additivity: } L(\vec{v}+\vec{w}) = L(\vec{v}) + L(\vec{w}) \\ \text{Scaling: } L(c\vec{v}) = cL(\vec{v}) \end{cases}$

Linear transformation preserve the addition and scalar multiplication.

e.g. Derivative is linear
$\frac{d}{dx}(x^3+x^2) = \frac{d}{dx}(x^3) + \frac{d}{dx}(x^2)$
$\frac{d}{dx}(4x^3) = 4\frac{d}{dx}(x^3)$

⑤ Transfor working spaces

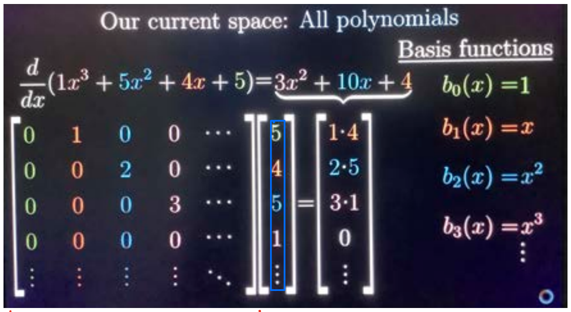
e.g. Derivatives of Polynomials : Our current space: All polynomials

Basis functions: $1, x, x^2, x^3, \ldots$

i.e. $b_0(x)=1$ $b_1(x)=x$ $b_2(x)=x^2, \ldots$

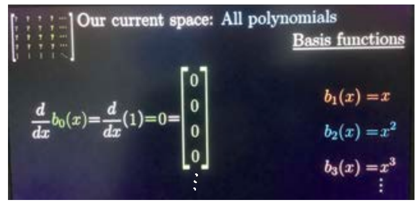
Format: already written as a linear combination, say $1x^2+3x+5\cdot 1$

we can write as vectors $1x^2+3x+5\cdot 1 = \begin{bmatrix} 5 \\ 3 \\ 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$] infinite many "0"s

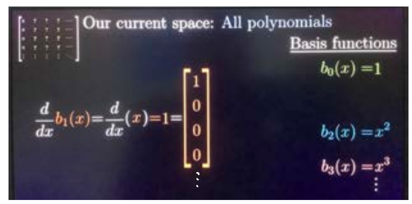


How this matrix constructs?

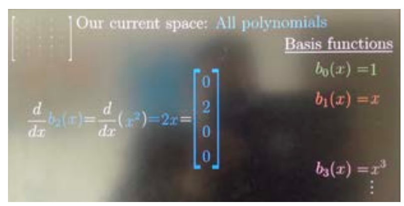
step 1



step 2



Step 3



⋮

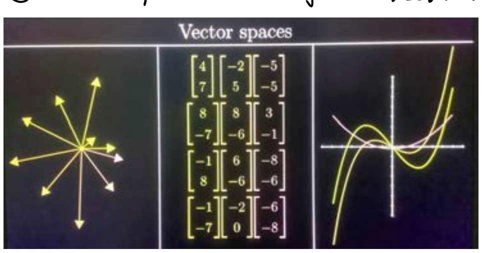
⑥ Concepts Interchangeable between "Linear Algebra" & "Functions"

Linear algebra concepts	Alternate names when applied to functions
Linear transformations	Linear operators
Dot products	Inner products
Eigenvectors	Eigenfunctions

⑦ there are many different "vector-ish" things in maths

- Arrows in the spaces
- Matrices
- Functions

⑧ Vector Spaces – Set of all "vector-ish" things



⑨ Axioms for all vector spaces (for generality)

Rules for vectors addition and scaling

1. $\vec{u}+(\vec{v}+\vec{w})=(\vec{u}+\vec{v})+\vec{w}$
2. $\vec{v}+\vec{w}=\vec{w}+\vec{v}$
3. There is a vector $\mathbf{0}$ such that $\mathbf{0}+\vec{v}=\vec{v}$ for all $\vec{v}$
4. For every vector $\vec{v}$ there is a vector $-\vec{v}$ so that $\vec{v}+(-\vec{v})=0$
5. $a(b\vec{v})=(ab)\vec{v}$
6. $1\vec{v}=\vec{v}$
7. $a(\vec{v}+\vec{w})=a\vec{v}+a\vec{w}$
8. $(a+b)\vec{v}=a\vec{v}+b\vec{v}$

"Axioms"

⑩ Axioms are an interface (NOT rules of ※ nature)

Axioms give a "duck-like" definition & make the tool useful and can be utilised widely.

⑪ Abstractness is the price of generality

In mordern mathematical theory, the form the vectors take does NOT really matter.
Just follow the abstract rules ("Axioms") → Maths abstracts all into a single, intangible notion.
But thinking concretely is also important, especially for beginners.