McDiarmid-Type Inequalities for Graph-Dependent Variables and Stability Bounds

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[1] This work was done when this author was a master student at the Institute of Computing Technology, Chinese Academy of Sciences and University of Chinese Academy of Sciences.

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Concentration of Measure

- Concentration iequalities

$$\mathbf{Pr}(f(\mathbf{X}) - \mathbf{E}[f(\mathbf{X})] \geq t) \leq ?$$

- common assumption: random variables are independent
- what if r.v. are not independent
  - mixing coefficients: α-mixing [Rosenblatt, 1956], β-mixing [Volkonskii and Rozanov, 1959], ϕ-mixing [Ibragimov, 1962], η-mixing [Kontorovich, 2007], etc.
  - dependency graph: Local Lemma [Erdos and Lovász, 1975], Normal/Poisson Approximation [Chen, 1978, Janson et al., 1988, Baldi et al., 1989]
- Goal: McDiarmid-type concentration inequality for graph-dependent random variables

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Janson's Hoeffding-type inequality

Definition (Dependency Graphs)

An undirected graph G is called a dependency graph of a random vector $\mathbf{X} = (X_1, \ldots, X_n)$ if

1. $V(G) = [n]$
2. *if $I, J \subset [n]$ are non-adjacent in G, $\{X_i\}_{i \in I}$ and $\{X_j\}_{j \in J}$ are independent.*

Theorem (Janson's inequality [Janson, 2004])

$$\mathbf{Pr}\left(\sum_{i=1}^{n} X_i - \mathbf{E}\left[\sum_{i=1}^{n} X_i\right] \geq t\right) \leq \exp\left(-\frac{2t^2}{\chi^*(G)\|\mathbf{c}\|_2^2}\right)$$

- $\chi^*(G)$: fractional coloring number of a dependency graph G of random variables $\mathbf{X}$
- idea: decomposition of summation to summation over independent set

McDiarmid's inequality

Definition (c-Lipschitz, bounded differences condition)

Given a vector $\mathbf{c} = (c_1, \ldots, c_n) \in \mathbb{R}_+^n$*, a function* $f : \boldsymbol{\Omega} \to \mathbb{R}$ *is said to be* $\mathbf{c}$*-Lipschitz if for any* $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{x}' = (x_1', \ldots, x_n') \in \boldsymbol{\Omega}$*, it satisfies*

$$|f(\mathbf{x}) - f(\mathbf{x}')| \leq \sum_{i=1}^{n} c_i \mathbf{1}_{\{x_i \neq x_i'\}}$$

where c_i *is called the* i*-th Lipschitz coefficient of* f*.*

originated from Hoeffding-Azuma [Hoeffding et al., 1948, Azuma, 1967]

Theorem (McDiarmid's inequality [McDiarmid, 1989])

Suppose $f : \boldsymbol{\Omega} \to \mathbb{R}$ *is* $\mathbf{c}$*-Lipschitz, and* $\mathbf{X} = (X_1, \ldots, X_n)$ *is a vector of independent r.v. with each* X_i *taking values in* Ω_i*. Then for any* $t > 0$*,*

$$\mathbf{Pr}(f(\mathbf{X}) - \mathbf{E}[f(\mathbf{X})] \geq t) \leq \exp\left(-\frac{2t^2}{\|\mathbf{c}\|_2^2}\right) \tag{1}$$

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McDiarmid-type inequality for dependency tree

Theorem

Suppose that $f : \boldsymbol{\Omega} \to \mathbb{R}$ is a $\mathbf{c}$-Lipschitz function and T is a dependency tree of a random vector $\mathbf{X}$ that takes values in $\boldsymbol{\Omega}$. Then for any $t > 0$,

$$\mathbf{Pr}(f(\mathbf{X}) - \mathbf{E}[f(\mathbf{X})] \geq t) \leq \exp\left(- \frac{2t^2}{\sum_{\langle i,j \rangle \in E(T)} (c_i + c_j)^2 + c_{\min}^2} \right)$$

where $c_{\min}$ is the minimum entry in $\mathbf{c}$.

- idea: vertex exposure ordering + Doob martingale + conditional probability coupling

McDiarmid-type inequality for dependency forest

Theorem

Suppose that $f : \mathbf{\Omega} \to \mathbb{R}$ *is a* $\mathbf{c}$*-Lipschitz function and* G *is a dependency graph of a random vector* $\mathbf{X}$ *that takes values in* $\mathbf{\Omega}$*. If* G *is a forest consisting of trees* $\{T_i\}_{i \in [k]}$*, then for any* $t > 0$,

$$\mathbf{Pr}(f(\mathbf{X}) - \mathbf{E}[f(\mathbf{X})] \geq t) \leq \exp\left(-\frac{2t^2}{\sum_{\langle i,j \rangle \in E(G)} (c_i + c_j)^2 + \sum_{i=1}^{k} c_{\min,i}^2}\right)$$

where $c_{\min,i} = \min\{c_j : j \in V(T_i)\}$

- strict generalization of the McDiarmid's inequality for i.i.d. random variables

McDiarmid-type inequality for general dependency graph

idea: transform graph to forest via merging vertices

$$\lambda_{(\phi,F)} = \sum_{\langle u,v\rangle \in E(F)} \left(|\phi^{-1}(u)| + |\phi^{-1}(v)|\right)^2 + \sum_{i=1}^{k} \min_{u \in V(T_i)} |\phi^{-1}(u)|^2$$

We call

$$\Lambda(G) = \min_{(\phi,F) \in \Phi(G)} \lambda_{(\phi,F)}$$

the Forest Complexity of the graph G

Theorem

$$\mathbf{Pr}(f(\mathbf{X}) - \mathbf{E}[f(\mathbf{X})] \geq t) \leq \exp\left(-\frac{2t^2}{\Lambda(G)\|\mathbf{c}\|_\infty^2}\right)$$

independent case: $\Lambda(G) = n$, complete graph: $\Lambda(G) = n^2$

Examples

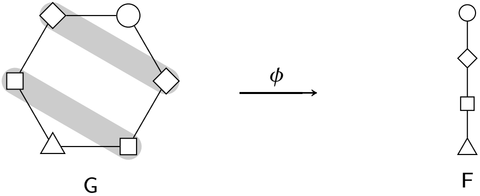

Figure: C_6: $\Lambda(G) \leq 8n - 13 = O(n)$

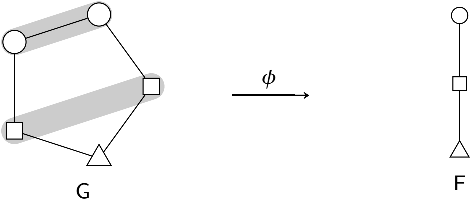

Figure: C_5: $\Lambda(G) \leq 8n - 14 = O(n)$

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Examples

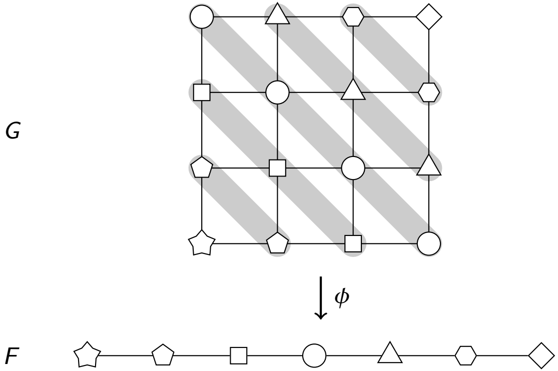

Figure: 4×4-grid $\Lambda(G) = O(m^3) = O(n^{\frac{3}{2}})$

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Uniform stability

Define $f_{\mathbf{S}}^{\mathscr{A}} : \mathscr{X} \to \mathscr{Y}$ to be the the hypothesis that $\mathscr{A}$ has learned from the sample $\mathbf{S}$

Definition (Uniform stability [Bousquet and Elisseeff, 2002])

Given integer $n > 0$, the learning algorithm $\mathscr{A}$ is called β_n-uniformly stable with respect to the loss function ℓ, if for any $i \in [n]$, $\mathbf{S} \in (\mathscr{X} \times \mathscr{Y})^n$, and $(x, y) \in \mathscr{X} \times \mathscr{Y}$, it holds that

$$|\ell(y, f_{\mathbf{S}}^{\mathscr{A}}(x)) - \ell(y, f_{\mathbf{S}^{\setminus i}}^{\mathscr{A}}(x))| \le \beta_n.$$

define $\Phi_{\mathscr{A}}(\mathbf{S}) = R(f_{\mathbf{S}}^{\mathscr{A}}) - \widehat{R}(f_{\mathbf{S}}^{\mathscr{A}})$

Lemma

$$\mathbf{E}[\Phi_{\mathscr{A}}(\mathbf{S})] \le 2\beta_{n,\Delta}(\Delta + 1).$$

Stability Bound for Learning Graph-dependent Data

Theorem

Given a sample $\mathbf{S}$ *of size* n *with dependency graph* G*, assume that the learning algorithm* $\mathscr{A}$ *is* β_i*-uniformly stable for any* $i \leq n$*. Suppose the maximum degree* G *is* Δ*, and the loss function* ℓ *is bounded by* M*. Let* $\beta_{n,\Delta} = \max_{i \in [0,\Delta]} \beta_{n-i}$*. For any* $\delta \in (0,1)$*, with probability at least* $1-\delta$*, it holds that*

$$R(f_{\mathbf{S}}^{\mathscr{A}}) \leq \widehat{R}(f_{\mathbf{S}}^{\mathscr{A}}) + 2\beta_{n,\Delta}(\Delta+1) + \frac{4n\beta_n + M}{n}\sqrt{\frac{\Lambda(G)\ln(1/\delta)}{2}}.$$

Applications

Example (Spatial Poisson point process)

Consider a Poisson point process on $\mathbb{R}^2$. The number of points in each finite region follows a Poisson distribution, and the number of points in disjoint regions are independent. Given a finite set $\mathscr{I} = \{I_i\}_{i=1}^n$ of regions in $\mathbb{R}^2$, let X_i be the number of points in region I_i, $1 \le i \le n$. Then the graph

$$G\left([n], \left\{\langle i,j\rangle : I_i \cap I_j \neq \emptyset\right\}\right)$$

is a dependency graph of the random variables $\{X_i\}_{i=1}^n$.

Example (m-dependence [Hoeffding et al., 1948])

For some $m, n \in \mathbb{N}_+$, a sequence of random variables $\{X_i\}_{i=1}^n$ is called m-dependent if for any $i \in [n-m-1]$, $\{X_j\}_{j=1}^i$ is independent of $\{X_j\}_{j=i+m+1}^n$.

$$\Lambda(G) \leq \left(\left\lceil \frac{n}{m} \right\rceil - 1\right)(m+m)^2 + m^2 \leq 4mn = O(mn)$$

$$R(f_{\mathbf{S}}^{\mathscr{A}}) \leq \widehat{R}(f_{\mathbf{S}}^{\mathscr{A}}) + 2\beta_{n,2m}(2m+1) + (4n\beta_n + M)\sqrt{\frac{2m\ln(1/\delta)}{n}}.$$

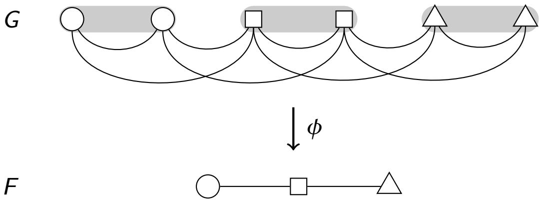
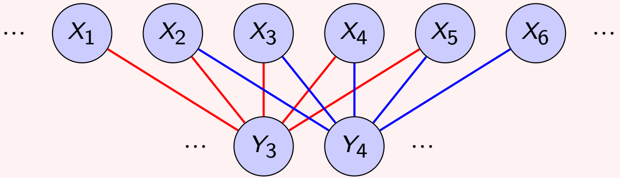


Figure: 2-dependent sequence

Example

Let y_i be the observation at location i, e.g., the house price, and x_i stand for the random variable modeling geographical effect at location i.

Example



$(\mathbf{X}_i, Y_i)$: geographical effect, house price;
$\{((X_1, X_2, X_3, X_4, X_5), Y_3), ((X_2, X_3, X_4, X_5, X_6), Y_4)\}$

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