The pursuit of more accurate asymptotics [1] via clusters [2] (the three-year milestone).

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Supervisors: Nick Wormald and Mikhail Isaev.

[1] Asymptotics are describing limiting behaviours as $n \to \infty$.
[2] Clusters are connected structures.

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- The asymptotic enumeration:
  - asymptotic enumeration of regular tournaments, Eulerian digraphs, and Eulerian oriented graphs via cumulants (with Isaev, McKay).

Dependency graphs

A graph G is a dependency graph for random variables $\{X_i\}_{i \in V(G)}$ if variables $\{X_i\}_{i \in I}$ and $\{X_j\}_{j \in J}$ are independent for any disjoint non-adjacent $I, J \subset V(G)$.

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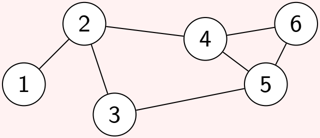
Example

Let $\{X_i\}_{i\in[6]}$ be random variables with the following dependency graph G.



Variables $\{X_1, X_2\}$ and $\{X_5, X_6\}$ are independent,
since disjoint vertex sets $\{1,2\}$ and $\{5,6\}$ are not adjacent in G.

The probability of non-occurrences

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$$\mu(S) = \mathbb{E}\left[\prod_{i\in S} X_i\right],$$

for $S \subseteq V(G)$. By the principle of inclusion–exclusion and the factorization, we have

$$\mathbb{P}(X=0) = \mathbb{P}\left(\sum_i X_i = 0\right) = \sum_{S\subseteq V(G)} (-1)^{|S|}\mu(S) = \sum_{S\subseteq V(G)} (-1)^{|S|}\mu(C_1)\mu(C_2)\dots\mu(C_k),$$

where $C_1 \cup \dots \cup C_k = S$ and $C_1, \dots, C_k$ induce maximal pairwise non-adjacent 'polymers' (that is, connected subgraphs; also called 'animals', suggested by Dobrushin, 1996).

The exponential formula reduces the sum to connected subgraphs
(idea: any simple graph is the vertex-disjoint union of connected simple graphs), we have

Formal cluster expansion (Z. 2022)

$$\log(\mathbb{P}(X=0)) \overset{\text{formally}}{=} \sum_{k\geqslant 1} \sum_{(C_1,\ldots,C_k)} \phi(C_1,\ldots,C_k) \prod_{C\in(C_1,\ldots,C_k)} (-1)^{|C|}\mu(C),$$

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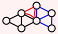
- Clusters: $\ldots,$  $,\ldots$.
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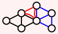
- Clusters: $\ldots,$  $,\ldots.$
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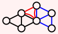
- Clusters: $\ldots,$  $,\ldots.$
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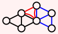
- Clusters: $\dots,$  $,\dots$.
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  - the tight instance for lower bound is when every polymer is K_1 (also by Shearer, 1985).

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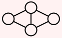
$\exp\left(-\dfrac{\mathbb{E}\big[\quad\big]}{1-p^3}\right)$ (numerator: $\mathbb{E}\big[$  $\big]$) $\overset{\text{FKG}}{\leqslant} \mathbb{P}(X=0) \overset{\text{Janson's inequality}}{\leqslant} \exp\Big(-\mathbb{E}\big[$  $\big] + \mathbb{E}\big[$  $\big]\Big),$

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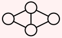
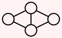
$\exp\Bigg(-\dfrac{\mathbb{E}\Big[$  $\Big]}{1-p^3}\Bigg) \overset{\text{FKG}}{\leqslant} \mathbb{P}(X=0) \overset{\text{Janson's inequality}}{\leqslant} \exp\Big(-\mathbb{E}\Big[$  $\Big] + \mathbb{E}\Big[$  $\Big]\Big),$

if $\mathbb{E}\Big[$  $\Big] = o(1)$, then

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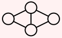
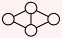
$\exp\left(-\frac{\mathbb{E}\big[\quad\big]}{1-p^3}\right)$ (fraction numerator: $\mathbb{E}\big[$  $\big]$) $\overset{\text{FKG}}{\leqslant} \mathbb{P}(X=0) \overset{\text{Janson's inequality}}{\leqslant} \exp\Big(-\mathbb{E}\Big[$  $\Big] + \mathbb{E}\Big[$  $\Big]\Big)$,

if $\mathbb{E}\Big[$  $\Big] = o(1)$, then

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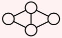
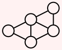
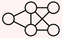
$\mathbb{P}(X=0) = \exp\Big(-\mathbb{E}\Big[$  $\Big] + \mathbb{E}\Big[$  $\Big] - \mathbb{E}\Big[$  $\Big] - \mathbb{E}\Big[$  $\Big] + o(1)\Big)$.

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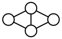
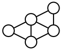
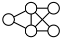
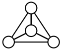
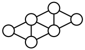
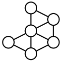
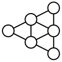
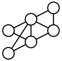
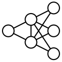
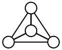
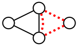
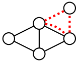
$\log \mathbb{P}(X=0) \overset{\text{formally}}{=} -\mathbb{E}\Big[$  $\Big] + \mathbb{E}\Big[$  $\Big] - \mathbb{E}\Big[$  $\Big] - \mathbb{E}\Big[$  $\Big] - \mathbb{E}\Big[$  $\Big]$

$+\,\mathbb{E}\Big[$  $\Big] + \mathbb{E}\Big[$  $\Big] + \mathbb{E}\Big[$  $\Big] + \mathbb{E}\Big[$  $\Big] + \mathbb{E}\Big[$  $\Big] + \mathbb{E}\Big[$  $\Big]$

$-\,\widetilde{\mathbb{E}}\Big[$  $\Big] - \widetilde{\mathbb{E}}\Big[$  $\Big] + \widetilde{\mathbb{E}}\Big[$  $\Big] + \dots$

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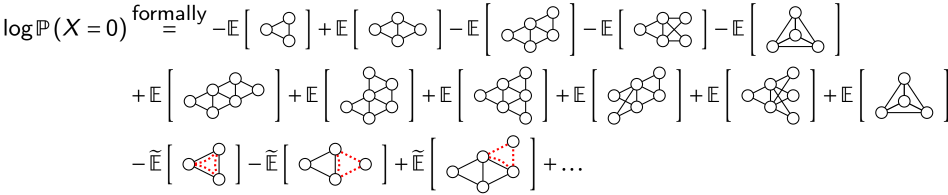

Stark and Wormald, 2018; Mousset, Noever, Panagiotou, Samotij, 2020

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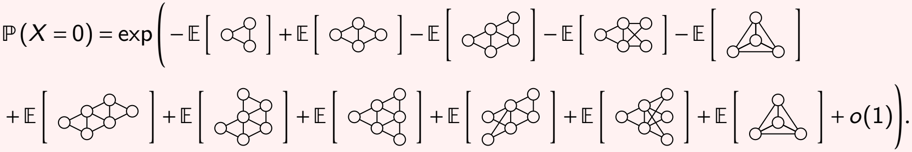

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- Stark and Wormald (2018) also have results for $\mathscr{G}_{n,m}$:
  if $d = m\binom{n}{2}^{-1} = o\left(n^{-7/11}\right)$, then

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- In Z. 2022, the truncation error is by Mousset, Noever, Panagiotou, Samotij's results.
- Coefficients match McKay's conjecture based upon numerical simulation.

A recent finding:

Corollary

$$\kappa(S) = \sum_{\{C_1,\ldots,C_n\}\in\Pi(S)} (-1)^{n-1}|\mathscr{T}| \prod_{i\in[n]} \mu(C_i) 1_{\{C_i \text{ is connected}\}}$$
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where $\mathscr{T} = \mathscr{T}(C_1,\ldots,C_n)$ denotes a special set of spanning trees by Penrose (1967), who used it to obtain the first convergence criterion of the cluster expansion.

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- This implies Janson's bound (1988) on cumulants, thus gives asymptotic normality.

A recent finding:

Corollary

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- This implies Janson's bound (1988) on cumulants, thus gives asymptotic normality.
- This relates to 'weighted dependency graph' by Feray (2018), who also bounds cumulants via spanning trees and obtains asymptotic normality under weak dependence (e.g., small subgraphs in $\mathscr{G}(n,m)$).

Asymptotic linearity of hypergraphs

McKay and Tian, 2020

If $d = \binom{n}{3}^{-1} m = o\left(n^{-3/2}\right)$, then via switching method,

$$\mathbb{P}\left(H_3(n,m) \text{ is linear}\right) = \exp\left(-\frac{1}{4}n^4d^2 - \frac{1}{12}n^5d^3 + o(1)\right).$$

Wormald and Z. 2022+

If $d = \binom{n}{3}^{-1} m = o\left(n^{-7/5}\right)$, then via perturbation method,

$$\mathbb{P}\left(H_3(n,m) \text{ is linear}\right) = \exp\left(-\frac{1}{4}n^4d^2 - \frac{1}{12}n^5d^3 - \frac{1}{24}n^6d^4 + \frac{3}{2}n^3d^2 + o(1)\right).$$

Dependency digraphs

Dependency digraphs

A digraph $\mathrm{D} = ([d], E)$ is a dependency digraph for the events $\{A_i\}_{i \in [d]}$ if A_i is mutually independent of all the non-adjacent $\{A_j : (i,j) \notin E\}$ for every $i \in [d]$.

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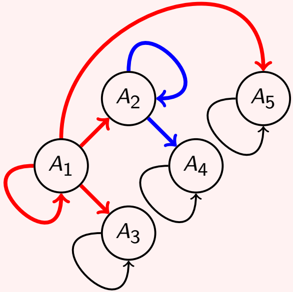
Example

Let $\{A_i\}_{i \in [5]}$ be events with the following dependency digraph.

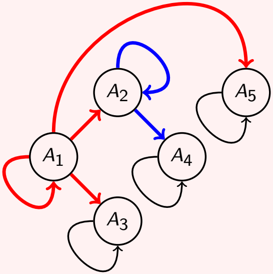


A_1 is independent of A_4.

A_2 is independent of $\{A_1, A_3, A_5\}$.

A natural refinement is

- adjacent vertices → the pairs of 'strongly dependent' events, $D_i \subseteq [d]$ denotes the 'strongly dependent neighbours' of event A_i.
- non-adjacent vertices → the pairs of 'weakly dependent' events.



$\mathbb{P}(A_4 \mid A_1) \approx \mathbb{P}(A_4)$

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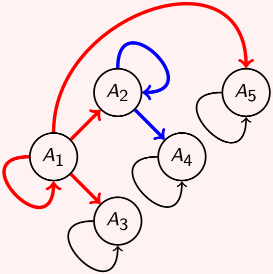

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If we change '$\approx$' to '$\geqslant$', then it is the 'negative dependency graph' by Erdős and Spencer (1991) to obtain lopsided Lovász local lemma.

New dependency digraphs

For any events $\{A_i\}_{i\in[d]}$ and digraph D on $[d]$:

- φ-mixing coefficient is used to measure 'weak dependencies':

$$\varphi = \max_{i\in[d]} \left| \mathbb{P}\left(\bigcup_{j\in[i-1]\setminus D_i} A_j \mid A_i \right) - \mathbb{P}\left(\bigcup_{j\in[i-1]\setminus D_i} A_j \right) \right|.$$

- The influence of 'strongly dependent' events is

$$\Delta_1 = \sum_{i\in[d]} \mathbb{P}\left(A_i \cap \bigcup_{j\in[i-1]\cap D_i} A_j \right) \prod_{k\in[d]\setminus[i]} \mathbb{P}\left(\overline{A_k}\right)$$

$$\Delta_2 = \sum_{i\in[d]} \mathbb{P}(A_i)\, \mathbb{P}\left(\bigcup_{j\in[i-1]\cap D_i} A_j \right) \prod_{k\in[d]\setminus[i]} \mathbb{P}\left(\overline{A_k}\right).$$

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$$\Delta_1 \leqslant \Delta_1' = \sum_{i\in[d]} \sum_{j\in[i-1]\cap D_i} \mathbb{P}(A_i \cap A_j),$$

$$\Delta_2 \leqslant \Delta_2' = \sum_{i\in[d]} \sum_{j\in[i-1]\cap D_i} \mathbb{P}(A_i)\mathbb{P}(A_j).$$

- Chen–Stein Poisson approximation involves similar terms as φ, Δ_1', and Δ_2'. Chen (1975) also considered the φ-mixing condition.
- Suen's inequality (1990) also involves Δ_1' and Δ_2'.

Theorem (Isaev, Rodionov, Z., Zhukovskii, 2020, 2022+)

For any events $\{A_i\}_{i\in[d]}$ and digraph D with vertex set $[d]$, we have

$$\left|\mathbb{P}\left(\bigcap_{i\in[d]}\overline{A_i}\right)-\prod_{i\in[d]}\mathbb{P}\left(\overline{A_i}\right)\right|\leqslant\left(1-\prod_{i\in[d]}\mathbb{P}\left(\overline{A_i}\right)\right)\varphi+\max(\Delta_1,\Delta_2).$$

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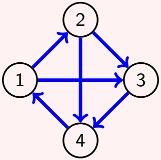
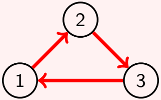
- Max codegrees, max clique-extension counts of random (hyper)graphs, etc. are asymptotically Gumbel.
- Potential application: asymptotic enumeration of rainbow matchings (Latin transversals).
- Lu and Székely (2009+?) also introduced a notion of 'positive' dependency graph to obtain upper bounds for Poisson approximation for random matchings.

Regular tournaments

A tournament is a digraph obtained by assigning a direction for each edge in an undirected complete graph.

Example





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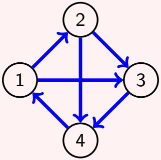

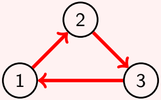

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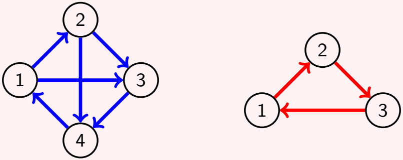

A tournament is regular if the in-degree is equal to the out-degree for each vertex. Let $\mathrm{RT}(n)$ denote the number of regular tournament on n vertices.

Theorem (McKay, 1990)

For odd $n \to \infty$ and $\varepsilon > 0$,

$$\mathrm{RT}(n) = \left(1 + O\left(n^{-1/2+\varepsilon}\right)\right)\left(\frac{n}{e}\right)^{1/2}\left(\frac{2^{n+1}}{\pi n}\right)^{(n-1)/2}.$$

Asymptotic enumeration of regular tournaments

The method was developed by McKay and Wormald (1990)
based on generating function + Cauchy's integral theorem + the saddle point method.

$$\begin{aligned}\mathrm{RT}(n) &= \frac{1}{(2\pi i)^n}\oint\cdots\oint \frac{\prod_{1\leqslant j<k\leqslant n}(x_j/x_k + x_k/x_j)}{x_1\cdots x_n}\, dx_1\cdots dx_n,\\ &= \frac{2^{n(n-1)/2}}{(2\pi)^n}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\prod_{1\leqslant j<k\leqslant n}\cos(\theta_j-\theta_k)\, d\boldsymbol{\theta}.\end{aligned}$$

The remaining is to estimate the n-dimensional integral.

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Via the complex martingale method by Isaev and McKay (2018),
and an estimate of the integral via cumulants of multivariate Gaussians, we have:

Theorem (Isaev, McKay, Z., 2022+)

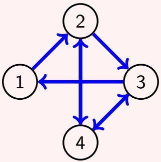
For odd $n \to \infty$,

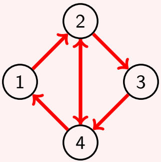
$$\mathrm{RT}(n) = n^{1/2}\left(\frac{2^{n+1}}{\pi n}\right)^{(n-1)/2} \exp\left(-\frac{1}{2} + \frac{1}{4n} + \frac{1}{4n^2} + \frac{7}{24n^3} + \frac{37}{120n^4} + \frac{31}{60n^5} + \frac{81}{28n^6} + \frac{5981}{336n^7} + \frac{22937}{240n^8} + \frac{90031}{180n^9} + \frac{1825009}{660n^{10}} + \frac{4344847}{264n^{11}} + O\left(n^{-12}\right)\right).$$

Eulerian digraphs

A simple directed graph is a directed graph having no multiple edges or loops.

Example





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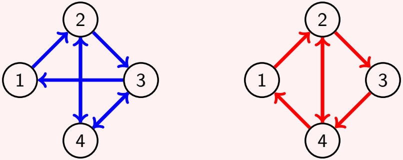

An Eulerian digraph is a digraph s.t. the in-degree equals the out-degree for each vertex.

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For $n \to \infty$ and $\varepsilon > 0$,

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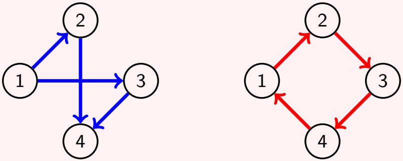
$$\left. + \frac{435581}{86016n^7} + \frac{1145941}{61440n^8} + \frac{13318871}{184320n^9} + \frac{99074137}{337920n^{10}} + \frac{1339710847}{1081344n^{11}} + O\left(n^{-12}\right)\right).$$

Eulerian oriented graphs

A directed graph having no symmetric pair of directed edges is an oriented graph (a complete oriented graph is a tournament).

Example



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Theorem (Isaev, McKay, Z., 2022+)

For $n \to \infty$,

$$\mathrm{EOG}(n) = n^{1/2} \left(\frac{3^{n+1}}{4\pi n}\right)^{(n-1)/2} \exp\Bigg(-\frac{3}{8} + \frac{11}{64n} + \frac{7}{64n^2} + \frac{233}{2048n^3} + \frac{497}{2560n^4} + \frac{27583}{61440n^5} + \frac{55463}{43008n^6} + \frac{33678923}{7340032n^7} + \frac{101414573}{5242880n^8} + \frac{1882520759}{20971520n^9} + \frac{101145677531}{230686720n^{10}} + \frac{2469157786549}{1107296256n^{11}} + O\left(n^{-12}\right)\Bigg).$$

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Some other results include

- concentration under graph-dependence (2022),
- learning under graph-dependence (with Amini),
- ....

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Thanks!