The recent development of dependency graphs

Rui-Ray Zhang
rui.zhang@monash.edu
School of Mathematics, Monash University

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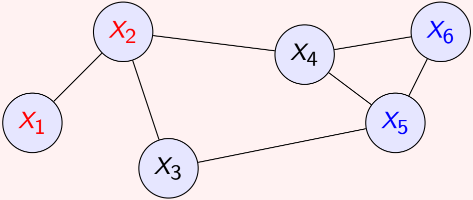

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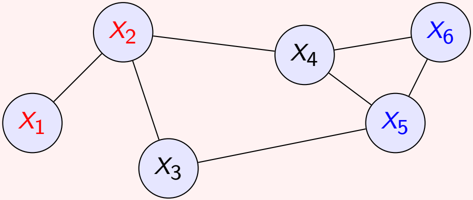

- If disjoint subsets $I, J \subset [n]$ are non-adjacent in G, then $\{X_i\}_{i \in I}$ and $\{X_j\}_{j \in J}$ are independent.
  - In the above example, $\{X_1, X_2\}$ and $\{X_5, X_6\}$ are independent.

- There are ones with weaker assumptions, such as the one used in Lovász local lemma.
- The dependency graph for variables may not necessarily be unique, and the sparser ones are the more interesting ones.

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- Limiting distribution via dependency graph.
  - Janson's normality criterion via cumulants (1988)

Concentration bounds

A McDiarmid-type bounded difference inequality for graph-dependent variables.

Theorem (Z., Liu, Wang, Wang, 2019)

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then for every $t > 0$,

$$\mathbb{P}(f(\boldsymbol{X}) - \mathbb{E}[f(\boldsymbol{X})] \geq t) \leq \exp\left(-\frac{2t^2}{\sum_{i=1}^{k} c_{\min,i}^2 + \sum_{\{i,j\} \in E(F)} (c_i + c_j)^2}\right),$$

where $c_{\min,i} = \min\{c_j : j \in V(T_i)\}$ is the minimum entry of $\mathbf{c}$ in each tree T_i.

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Let random variables $\{X_i\}_{i \in V(G)}$ be G-dependent such that every X_i takes values in an interval of length $c_i \geqslant 0$. Then, for every $t > 0$,

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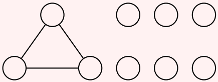
- It is no worse than Janson's, better when G is sparse.
- It generalises to certain decomposable Lipschitz functions, extending McDiarmid's.

Example

Let $\{X_i\}_{i\in[9]}$ be random indicators with the dependency graph G, and $X = \sum_{i\in[9]} X_i$.



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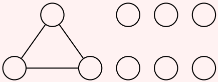

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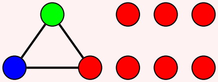

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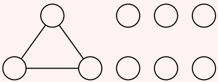

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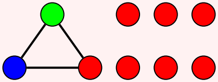

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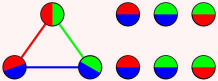
- New idea: to cover vertices with weighted induced forests such that the sum of weights for each vertex equals 1



$$\mathbb{P}(X - \mathbb{E}[X] \geq t) \leq \exp\left(-\frac{8t^2}{81}\right)$$

The probability of non-occurrences

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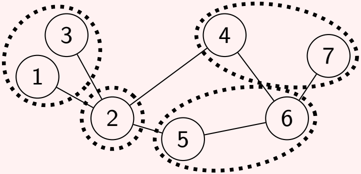
Then we have a factorization property



$$\mu(1,3)\mu(2)\mu(4,7)\mu(5,6)$$
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where $C_1 \cup \ldots \cup C_k = S$ and $C_1, \ldots, C_k$ induce maximal pairwise non-adjacent 'polymers' (that is, connected subgraphs; also called 'animals', suggested by Dobrushin, 1996).

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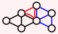
- Clusters: $\dots,$  $,\dots$.

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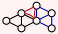
- Clusters: $\ldots,$  $,\ldots.$
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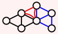
- Clusters: …,  ,….
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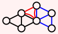
- Clusters: $\ldots,$  $,\ldots.$
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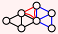
- Clusters: $\ldots,$  $,\ldots$.
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  - the tight instance for lower bound is when every polymer is K_1 (also by Shearer, 1985).

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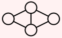
$\exp\Bigg(-\dfrac{\mathbb{E}\Big[$  $\Big]}{1-p^3}\Bigg) \overset{\text{FKG}}{\leqslant} \mathbb{P}(X=0) \overset{\text{Janson's inequality}}{\leqslant} \exp\Big(-\mathbb{E}\Big[$  $\Big] + \mathbb{E}\Big[$  $\Big]\Big),$

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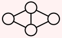
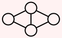
$\exp\left(-\dfrac{\mathbb{E}[\,$  $\,]}{1-p^3}\right) \overset{\text{FKG}}{\leqslant} \mathbb{P}(X=0) \overset{\text{Janson's inequality}}{\leqslant} \exp\Big(-\mathbb{E}\big[$  $\big] + \mathbb{E}\big[$  $\big]\Big)$,

if $\mathbb{E}\big[$  $\big] = o(1)$, then

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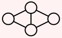
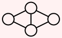
$\exp\Big(-\dfrac{\mathbb{E}[}{}$  $\dfrac{]}{1-p^3}\Big) \overset{\text{FKG}}{\leqslant} \mathbb{P}(X=0) \overset{\text{Janson's inequality}}{\leqslant} \exp\Big(-\mathbb{E}[$  $] + \mathbb{E}[$  $]\Big)$,

if $\mathbb{E}[$  $] = o(1)$, then

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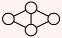
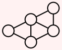
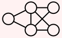
$\mathbb{P}(X=0) = \exp\Big(-\mathbb{E}[$  $] + \mathbb{E}[$  $] - \mathbb{E}[$  $] - \mathbb{E}[$  $] + o(1)\Big)$.

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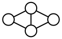
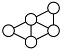
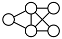
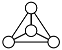
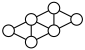
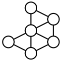
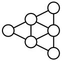
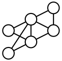
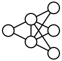
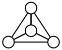
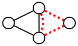
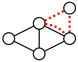
$\log \mathbb{P}(X=0) \overset{\text{formally}}{=} -\mathbb{E}\Big[$  $\Big] + \mathbb{E}\Big[$  $\Big] - \mathbb{E}\Big[$  $\Big] - \mathbb{E}\Big[$  $\Big] - \mathbb{E}\Big[$  $\Big]$

$+\mathbb{E}\Big[$  $\Big] + \mathbb{E}\Big[$  $\Big] + \mathbb{E}\Big[$  $\Big] + \mathbb{E}\Big[$  $\Big] + \mathbb{E}\Big[$  $\Big] + \mathbb{E}\Big[$  $\Big]$

$-\widetilde{\mathbb{E}}\Big[$  $\Big] - \widetilde{\mathbb{E}}\Big[$  $\Big] + \widetilde{\mathbb{E}}\Big[$  $\Big] + \dots$

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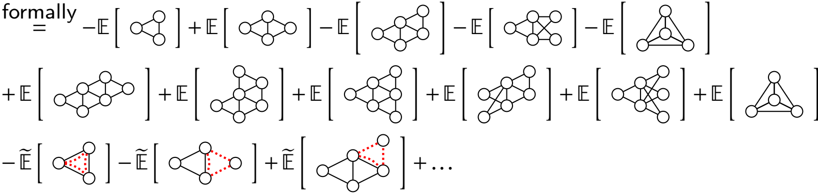
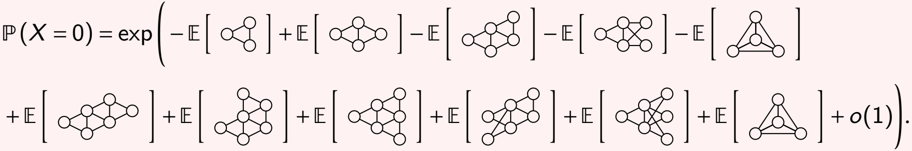


Stark and Wormald, 2018; Mousset, Noever, Panagiotou, Samotij, 2020

If $p = o\left(n^{-7/11}\right)$, then



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Joint cumulant (1929)

For any multiset of random variables $\{X_i\}_{i\in S}$,

$$\kappa(S) = \sum_{\pi\in\Pi(S)} (-1)^{|\pi|-1}(|\pi|-1)! \prod_{P\in\pi} \mu(P).$$

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  if $d = m\binom{n}{2}^{-1} = o\left(n^{-7/11}\right)$, then

$$\mathbb{P}(X=0) = \exp\left(-\frac{1}{6}n^3d^3 - \frac{1}{8}n^4d^6 - \frac{1}{2}n^2d^3 + o(1)\right).$$

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If $p = o\left(n^{-3/2}\right)$, then

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Z. 2022; Wormald and Z., 2022+

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- In Z. 2022, the truncation error is by Mousset, Noever, Panagiotou, Samotij's results.
- Coefficients match McKay's conjecture based upon numerical simulation.

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Dependency digraphs

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A digraph $\mathbf{D} = ([d], E)$ is a dependency digraph for the events $\{A_i\}_{i \in [d]}$ if A_i is mutually independent of all the non-adjacent $\{A_j : (i,j) \notin E\}$ for every $i \in [d]$.

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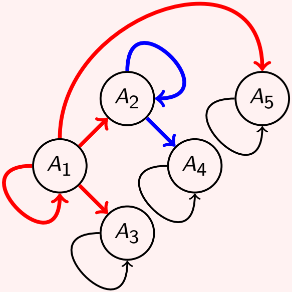
Example

Let $\{A_i\}_{i\in[5]}$ be events with the following dependency digraph.

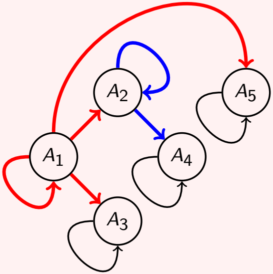


A_1 is independent of A_4.

A_2 is independent of $\{A_1, A_3, A_5\}$.

A natural refinement is

- adjacent vertices → the pairs of 'strongly dependent' events, $D_i \subseteq [d]$ denotes the 'strongly dependent neighbours' of event A_i.
- non-adjacent vertices → the pairs of 'weakly dependent' events.



$\mathbb{P}(A_4 \mid A_1) \approx \mathbb{P}(A_4)$

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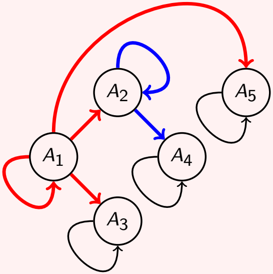

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If we change '$\approx$' to '$\geqslant$', then it is the 'negative dependency graph' by Erdős and Spencer (1991) to obtain lopsided Lovász local lemma.

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$$\Delta_2 = \sum_{i\in[d]} \mathbb{P}(A_i)\, \mathbb{P}\left(\bigcup_{j\in[i-1]\cap D_i} A_j \right) \prod_{k\in[d]\setminus[i]} \mathbb{P}\left(\overline{A_k}\right).$$

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- Chen–Stein Poisson approximation involves similar terms as φ, Δ_1', and Δ_2'. Chen (1975) also considered the φ-mixing condition.
- Suen's inequality (1990) also involves Δ_1' and Δ_2'.

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- Max codegrees, max clique-extension counts of random (hyper)graphs, etc. are asymptotically Gumbel.
- Lu and Székely (2009) also introduced a notion of 'positive' dependency graph to obtain upper bounds for Poisson approximation for random matchings.

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- To design a suitable model for $\mathscr{G}(n, m)$;
  e.g., Féray's weighted dependency graphs for normality of subgraph counts.

Thanks!