Inverse Learning Notes

FEM notes: $-\nabla\cdot(c\otimes\nabla d)+ad=g$, $d=(u,v)$. $\nabla d=\begin{pmatrix}\frac{\partial u}{\partial x}\\ \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x}\\ \frac{\partial v}{\partial y}\end{pmatrix}$

$$\begin{pmatrix}\sigma_{xx}\\ \sigma_{xy}\\ \sigma_{yy}\end{pmatrix}=\underbrace{\begin{bmatrix}\alpha & 0 & \beta\\ 0 & \omega & 0\\ \beta & 0 & \gamma\end{bmatrix}}_{\mathbb{C}}\begin{pmatrix}\tilde{e}_{xx}\\ \tilde{e}_{xy}\\ \tilde{e}_{yy}\end{pmatrix}=\mathbb{C}\left(\underbrace{\begin{bmatrix}1&0&0&0\\0&\frac{1}{2}&\frac{1}{2}&0\\0&0&0&1\end{bmatrix}}_{A}\nabla d-\underbrace{\begin{pmatrix}f_x(x_m)\\0\\f_y(x_m)\end{pmatrix}}_{f(x_m)}\right)$$

$x_{\xi x}$ $x_{\xi x \xi y}$

$$\mathbb{C}A=\begin{bmatrix}\alpha&0&0&\beta\\0&\frac{\omega}{2}&\frac{\omega}{2}&0\\ \beta&0&0&\gamma\end{bmatrix},\qquad \mathbb{C}f(x)=\begin{bmatrix}\alpha f_x(x_m)+\beta f_y(x_m)\\0\\ \beta f_x(x_m)+\gamma f_y(x_m)\end{bmatrix}$$

① $-\frac{\partial}{\partial x_k}\left(c_{ijkl}\frac{\partial d_j}{\partial x_l}\right)+a_{ij}d_j=f_j$. $N=2$.

r_1-r_2: $\nabla\cdot(\mathbb{C}A\nabla d-\mathbb{C}f(x_m))=0 \Rightarrow \nabla\cdot(\mathbb{C}A\nabla d)=\nabla\cdot\mathbb{C}f(x)$. 1-2

r_2-r_3: $\nabla\cdot(\mathbb{C}A\nabla d-\mathbb{C}f(x_m))=0 \Rightarrow \nabla\cdot(\mathbb{C}A\nabla d)=\nabla\cdot\mathbb{C}f(x)$ 2-3.

$$c=\left(\begin{array}{cccc}\alpha&0&0&\beta\\0&\frac{\omega}{2}&\frac{\omega}{2}&0\\ \hline 0&\frac{\omega}{2}&\frac{\omega}{2}&0\\ \beta&0&0&\gamma\end{array}\right)$$

$$\nabla d=\begin{bmatrix}\frac{\partial u}{\partial x}\\ \frac{\partial u}{\partial y}\\ \hline \frac{\partial v}{\partial x}\\ \frac{\partial v}{\partial y}\end{bmatrix},\quad c=\begin{bmatrix}\alpha&0&0&\beta\\0&\frac{\omega}{2}&\frac{\omega}{2}&0\\0&\frac{\omega}{2}&\frac{\omega}{2}&0\\ \beta&0&0&\gamma\end{bmatrix},\quad a=0,\quad g=\begin{bmatrix}(\alpha f'_x(x_m)+\beta f'_y(x_m))\nabla_x x_m\\ (\beta f'_x(x_m)+\gamma f'_y(x_m))\nabla_y x_m\end{bmatrix}$$

Stage 0 : Assume $\nabla X_m = (0,0)$, then, the PDE is reduced to:

$$-\nabla\cdot \overbrace{\begin{bmatrix} \alpha & 0 & 0 & \beta \\ 0 & \frac{w}{2} & \frac{w}{2} & 0 \\ 0 & \frac{w}{2} & \frac{w}{2} & 0 \\ \beta & 0 & 0 & \gamma \end{bmatrix}}^{C} \nabla d = 0$$

① B.C. $\hat{n}\cdot(C\nabla d) = 0$

② B.C.: $\hat{n}\cdot(C\nabla d) = 1$,

Forward Model:

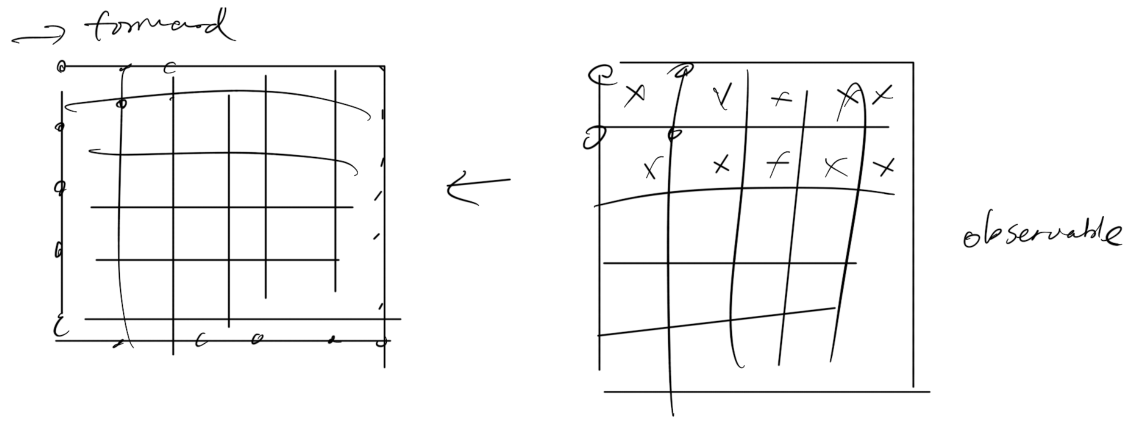
Given C, x, we can find $d=(u,v)$, ∇d. from FEM

Now given x, d, can we find C =?

1°. Numerical gradient descent, initialize $C=C_0$, solve $e_{xx}(C_0)$, $e_{xy}(C_0)$, $e_{yy}(C_0)$

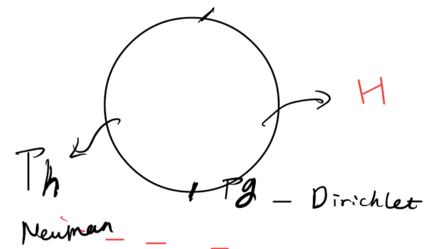
$$F = \sum_{i,j} \| e_{xy} - e_{xy}(C_0)\|^2 + \| e_{xx} - e_{xx}(C_0)\|^2 + \| e_{yy} - e_{yy}(C_0)\|^2$$

$$\left.\frac{\partial F}{\partial C_0}\right|_{C_0} = \frac{F[C+\delta C] - F[C]}{\delta C}$$


forward
observable

$u = B * Kc \backslash Fc + Ud$

$Kd = F_f + F_h + F_g$



$$\underbrace{m\frac{\partial^2 u}{\partial t^2} + d\frac{\partial u}{\partial t}}_{M = 0} - \underbrace{\nabla(c\cdot\nabla u)}_{K} + \underbrace{au}_{A = 0} = \underbrace{f}_{F} \qquad \hat{n}\cdot(c\nabla u) + \underbrace{qu}_{Q} = \underbrace{g}_{G,\ R}.$$

$$\mathbb{K}(c)\cdot d = F + G. \qquad Hu = R.$$

"$\wedge$" observable

$$\min_{\mathbb{C}} f = \sum_{ij} \| \hat{e}_{xx} - e_{xx} \|^2 + \| \hat{e}_{xy} - e_{xy} \|^2 + \| \hat{e}_{yy} - e_{yy} \|^2$$

s.t. $e_{xx} = \frac{\partial u}{\partial x}$, $e_{yy} = \frac{\partial v}{\partial y}$, $e_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$

$\nabla \cdot \sigma = 0$, $\hat{n} \cdot \sigma = 0$

$$\Leftrightarrow \quad \min_{\alpha,\beta,\gamma,\omega} f = \sum \| \hat{e}_{xx} - e_{xx} \|^2 + \| \hat{e}_{yy} - e_{yy} \|^2 + \| \hat{e}_{xy} - e_{xy} \|^2$$

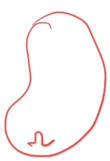
s.t. $\begin{bmatrix} e_{xx} \\ e_{xy} \\ e_{yy} \end{bmatrix} = Ad$

$\mathbb{K}(\alpha,\beta,\gamma,\omega) \cdot \nabla d = F(x_m)$, x_m Given.

$$\Leftrightarrow \quad \min_{\alpha,\beta,\gamma,\omega} f = \sum \| \hat{e} - A \nabla d \|_2^2$$

s.t. $\mathbb{K}(\alpha,\beta,\gamma,\omega) \cdot \nabla d = F(x_m)$.

$\mathbb{C} = \begin{bmatrix} \alpha & 0 & \beta \\ \beta & 0 & \omega \\ 0 & \gamma & 0 \end{bmatrix}$ $\quad C = \begin{bmatrix} \alpha & 0 & 0 & \beta \\ 0 & \frac{\omega}{2} & \frac{\omega}{2} & 0 \\ \beta & 0 & 0 & \gamma \end{bmatrix}$



Convex problem.

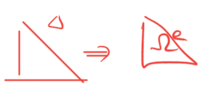
Here: $\nabla(C\nabla d) = \underset{G_\Omega}{g}$, $\hat{n}(C\nabla d) = \underset{G_{\partial\Omega}}{g}$, $g = (\nabla x_m)^T \begin{pmatrix} \alpha f_1'(x_m) + \beta f_2'(x_m) \\ \beta f_1'(x_m) + \gamma f_2'(x_m) \end{pmatrix}$

$$\Rightarrow \mathbb{K}(\alpha,\beta,\gamma,\omega)\nabla d = G_\Omega + G_{\partial\Omega}$$

$$\mathbb{K} = \mathop{A}_{e=1}^{n_{elem}} K^e, \quad K^e = \int_{\Omega^e} [B_d^e]^T [\mathbb{C}^e][B_d^e]\, d\Omega^e = \int_{\Omega^e} [B_d^e]\, \mathbb{C}\, [B_d^e]\, d\Omega^e$$

for 3-node triangle:

$$\varepsilon(d^h) = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{bmatrix} = \sum_{b=1}^{3} \begin{bmatrix} N_{b,1}^e & 0 \\ 0 & N_{b,2}^e \\ N_{b,2}^e & N_{b,1}^e \end{bmatrix} \begin{bmatrix} d_{b1}^e \\ d_{b2}^e \end{bmatrix}$$



$$\varepsilon = \sum_{b=1}^{3} \begin{bmatrix} N_{b,1}^e & 0 \\ 0 & N_{b,2}^e \\ N_{b,2}^e & N_{b,1}^e \end{bmatrix} \begin{pmatrix} u_b^e \\ v_b^e \end{pmatrix}$$

here $\frac{\partial N_b^e}{\partial(\xi,\eta)} = [J^e]^T \frac{\partial N_b^e}{\partial(x,y)}$

$$\begin{cases} N_1^e(\xi,\eta) = \xi \\ N_2^e(\xi,\eta) = \eta \\ N_3^e(\xi,\eta) = 1 - \xi - \eta \end{cases}$$

let: $\begin{cases} x = x(\xi,\eta) \\ y = y(\xi,\eta) \end{cases}$ then: $\frac{\partial N_b^e}{\partial(\xi,\eta)} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{bmatrix} \frac{\partial N_b^e}{\partial x} \\ \frac{\partial N_b^e}{\partial y} \end{bmatrix}$

$$[J^e] = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1^e & y_1^e \\ x_2^e & y_2^e \\ x_3^e & y_3^e \end{bmatrix} = \begin{bmatrix} x_1^e - x_3^e & y_1^e - y_3^e \\ x_2^e - x_3^e & y_2^e - y_3^e \end{bmatrix}$$

A^e Area of element

$$\det(J^e) = 2A^e \quad \begin{bmatrix} \frac{\partial N_b^e}{\partial x} \\ \frac{\partial N_b^e}{\partial y} \end{bmatrix} = \frac{1}{A^e} \begin{bmatrix} y_2^e - y_3^e & -(y_1^e - y_3^e) \\ -(x_2^e - x_3^e) & x_1^e - x_3^e \end{bmatrix} \begin{bmatrix} \frac{\partial N_b^e}{\partial \xi} \\ \frac{\partial N_b^e}{\partial \eta} \end{bmatrix}$$

$$\mathbb{K}^e = \int_{\Omega^e} B^{eT}\, \mathbb{C}\, B \, d\Omega^e = \frac{\partial N^e}{\partial(x^e, y^e)}^T \mathbb{C}\, \frac{\partial N^e}{\partial(x^e, y^e)} = \frac{1}{4A^e} [\,*\,]^T \mathbb{C} [\,*\,]$$

$$[\,*\,] = \begin{bmatrix} y_3^e - y_1^e & 0 \\ 0 & x_3^e - x_2^e \\ x_3^e - x_2^e & y_2^e - y_3^e \end{bmatrix} \sim \text{function of position.}$$

$\Rightarrow$ Each entry is only at most linear functions of $\mathbb{C}_{ijkl}$, a.k.a. $\alpha, \beta, \gamma, \omega$.

$$f = \min_{\alpha,\beta,\gamma,\omega \,/\, g(x_m)} \sum_{i,j} \| \hat{e} - Bd \|_2^2 \qquad B \text{ is a matrix independent of } \alpha,\beta,\gamma,\omega$$

$$\mathbb{K}(\alpha,\beta,\gamma,\omega)\, d = G_\Omega + G_{\partial\Omega} = G$$

$$\mathbb{K}(\alpha,\beta,\gamma,\omega) \succ 0, \quad G = G(x,y) \text{ only a function of position.}$$

1° if we parameterize $g(x_m) = \sum_{n=1}^{10} a_n L_n(x_m)$, then we have:

Given $\alpha, \beta, \gamma, \omega$, $\mathbb{K}$ is know, $\Rightarrow d = \mathbb{K}^{-1} G$.

$$G = \mathop{\mathrm{A}}_{e=1}^{nelem} G^e_{\partial\Omega} + \mathop{\mathrm{A}}_{e=1}^{nelem} G^e_{\Omega} \qquad N^e = [\underbrace{N_1^e(x,y)}_{\text{coln vector}}, N_2^e(x,y), N_3^e(x,y)]$$

$$G^e_{\partial\Omega} = \int_{\partial\Omega^e} N^{eT} \begin{bmatrix} g_1^e \\ g_2^e \end{bmatrix} d\Omega^e = \int_{\partial\Omega^e} N^{eT} \begin{bmatrix} \sum_n a_n L_n(x_m) \\ \sum_n b_n L_n(x_m) \end{bmatrix} d\Omega^e \qquad x_m = x_m(x,y) \text{ known.}$$

$$= \sum_n \begin{bmatrix} a_n \\ b_n \end{bmatrix} \underbrace{\int_{\partial\Omega^e} N^{eT} L_n(x_m)\, d\Omega^e}_{D_n}$$

Similarly: $$G^e_\Omega = \int_{\Omega^e} N^{eT} \begin{bmatrix} g_1^e \\ g_2^e \end{bmatrix} d\Omega^e = \sum_n \begin{bmatrix} a_n \\ b_n \end{bmatrix} \underline{\int_{\Omega^e} N^{eT} L_n(x_m)\, d\Omega^e}$$

$$= \begin{pmatrix} D_1 & 0 & D_2 & 0 & \cdots & D_n & 0 \\ 0 & D_1 & 0 & D_2 & \cdots & 0 & D_n \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \\ a_2 \\ \vdots \\ a_n \\ b_n \end{pmatrix} = \mathbb{D} \underbrace{\begin{pmatrix} a_1 \\ b_1 \\ \vdots \\ a_n \\ b_n \end{pmatrix}}_{\text{Let it be } \begin{pmatrix} a \\ b \end{pmatrix}}$$

$\Rightarrow$

$$\mathbb{K}(\alpha,\beta,\gamma,\omega)\, d = (a\ b)^T$$

$$\left.\begin{aligned} &\min_{a\,b} \sum_{i,j} \| \hat{e} - Bd \|_2^2 \\ &\text{s.t. } d = \mathbb{K}^{-1} \end{aligned}\right\} \Rightarrow \min_{a,b} \sum_{i,j} \| \hat{e} - B\, \mathbb{K}^{-1} D^T (a,b) \|_2^2$$

non constrainted convex.

2° if we know don't know $\alpha,\beta,\gamma,\omega$. but know g. harder problem;

$$\min_{\alpha,\beta,\omega,\gamma} \|\hat{e} - Bd\|_2^2$$

s.t. $\mathbb{K}(\alpha,\beta,\gamma,\omega)\, d = G$ $\qquad d \in \mathbb{R}^{2n}$, n points

$d = \mathbb{K}(\alpha,\beta,\gamma,\omega)^{-1} G$. $\qquad \mathbb{K}(\alpha,\beta,\gamma,\omega) \succ 0 \qquad \mathbb{K} \in \mathbb{R}^{2n\times 2n}$

ple case: $\quad K^e = \int_{\Omega^e} [B]_e^T \mathbb{C} [B]_e \, d\Omega^e \quad \mathbb{K} = \mathop{A}_{e=1}^{nelem}$, $\mathbb{K}(\alpha,\beta,\gamma,\omega)$ is linear in $\alpha,\beta,\gamma,\omega$, & $\mathbb{K} \succ 0$

* Convexity Analysis:?

$$d = \mathbb{K}^{-1}(\alpha,\beta,\gamma,\omega)\, G$$

Assume 2 parameter sets: $\underbrace{\alpha_1,\beta_1,\gamma_1,\omega_1}_{\downarrow\, d_1}$ & $\underbrace{\alpha_2,\beta_2,\gamma_2,\omega_2}_{\downarrow\, d_2}$

Then $f_1 = \|\hat{e} - Bd_1\|_2^2$. $\quad f_2 = \|\hat{e} - Bd_2\|^2$. $\quad f = f(d) \quad f$ is convex in d.

Let $\tilde{f} = \lambda f_1 + (1-\lambda) f_2$, with $\tilde{d}$. then:

$\lambda f(d_1) + (1-\lambda) f(d_2) \geq f(\tilde{d})$. $\tilde{d} = \lambda d_1 + (1-\lambda) d_2$. $\forall \lambda$.

$$\tilde{d} = \mathbb{K}(\tilde{\alpha},\tilde{\beta},\tilde{\gamma},\tilde{\omega})^{-1} G = [\lambda \mathbb{K}_1^{-1} + (1-\lambda)\mathbb{K}_2^{-1}] G = (\mathbb{K}_{\frac{c_1}{\lambda}}^{-1} + \mathbb{K}_{\frac{c_2}{1-\lambda}}^{-1}) G$$

$\neq \mathbb{K}^{-1}_{(\lambda c_1 + (1-\lambda) c_2)} G$

$d_1 = \mathbb{K}(\alpha_1,\beta_1,\gamma_1,\omega_1)^{-1} G = \mathbb{K}_{c_1}^{-1} G$

$d_2 = \mathbb{K}(\alpha_2,\beta_2,\gamma_2,\omega_2)^{-1} G = \mathbb{K}_{c_2}^{-1} G$

We want to know when. $\lambda f(\overbrace{\underline{\alpha_1,\dots,\omega_1}}^{c_1}) + (1-\lambda) f(\overbrace{\underline{\alpha_2,\dots,\omega_2}}^{c_2}) \overset{?}{\geq} f(\lambda c_1 + (1-\lambda) c_2)$, $\forall \lambda$.

$\mathbb{K}_{c_1} d_1 = G = \mathbb{K}_{c_2} d_2 \qquad \mathbb{K}_{\lambda c_1 + (1-\lambda) c_2}\, d = G$, $\quad d$

$\mathbb{K}(\alpha_1,\beta_1,c,\gamma_1) + \mathbb{K}$

$\mathbb{K}_{c_1} \underline{\underline{\lambda d}} + \mathbb{K}_{c_2} \underline{\underline{(1-\lambda) d}} = G$

Implementation:

N nodes, nth order basis

$\mathbf{a} = (a_1; \dots; a_n)$

$\mathbf{b} = (b_1; \dots; b_n)$

$$\min_{\mathbf{a},\mathbf{b}} \| \underset{3N\times 1}{\hat{e}} - e_{cal} \|^2$$

$$\mathbb{K} d = G = \sum_{i=1}^{n} D_i \begin{bmatrix} a_i \\ b_i \end{bmatrix} \quad D_i \in \mathbb{R}^{2N\times 2}, \; a_i \in \mathbb{R}$$

$$\mathbb{K} \in \mathbb{R}^{2N\times 2N}$$

$$d \in \mathbb{R}^{2n\times 1}$$

$$d = \sum_{i=1}^{n} \mathbb{K}^{-1} D_i a_i$$

$$e_{cal} = \underset{3N\times 2N}{B} \overset{2N\times 1}{d} = \sum_{i=1}^{n} B\mathbb{K}^{-1} D_i a_i = \overset{3N\times 2N}{B\mathbb{K}^{-1}} \overset{2N\times 2n}{D} \overset{2n\times 1}{\begin{bmatrix} a_1 \\ b_1 \\ a_2 \\ b_2 \\ \vdots \\ a_n \\ b_n \end{bmatrix}}$$

$$\min_{a_1,b_1,\dots,a_n,b_n} \| \hat{e} - \underbrace{\boxed{B\mathbb{K}^{-1} D}}_{\Downarrow \; E} \begin{bmatrix} a_1 \\ b_1 \\ \vdots \\ a_n \\ b_n \end{bmatrix} \|^2 \Rightarrow \begin{bmatrix} a_1 \\ b_1 \\ \vdots \\ a_n \\ b_n \end{bmatrix} = E \setminus \hat{e}$$

$$= (E^T E)^{-1} E^T \hat{e}$$

$$f = \| \overset{3N\times 1}{\hat{e}} - \overset{3N\times 1}{\underline{e_{cal}}} \|_2^2 \quad . \quad e_{cal} = E a$$

O: order

$a \in \mathbb{R}^{2O}$

$$\text{grad} f = 2 \sum \overset{3N\times 1}{\left[\underset{3N\times 1}{\hat{e} - e_{cal}} \right]} \cdot \underset{3N\times 2O}{\frac{\partial e_{cal}}{\partial a}}$$

get D & B.

$$\begin{pmatrix} C_{1111} & C_{1112} & C_{1211} & C_{1212} \\ 0 & C_{1122} & C_{1221} & C_{1222} \\ 0 & 0 & C_{2211} & C_{2212} \\ 0 & 0 & 0 & C_{2222} \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{pmatrix}$$

$$\begin{array}{ccc} & C_{11} & C_{12} \\ & \left(\begin{array}{cc|cc} \alpha & 0 & 0 & \beta \\ & \frac{w}{2} & \frac{w}{2} & \\ & \frac{w}{2} & \frac{w}{2} & \\ \beta & 0 & 0 & \gamma \end{array}\right) & C_{22} \end{array}$$

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \\ \beta & \gamma & \\ & & \frac{w}{2} \end{pmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{pmatrix}$$

$$C_{11} = (11 \quad 13 \quad 33) = [\alpha \quad 0 \quad \tfrac{w}{2}]$$

$$C_{12} = (13 \quad 12 \quad 33 \quad 23) = (0 \quad \beta \quad \tfrac{w}{2} \quad 0)$$

$$C_{22} = (33 \quad 23 \quad 22) = (\tfrac{w}{2} \quad 0 \quad \gamma)$$

$$\cancel{A} \overset{u}{\begin{pmatrix} \\ \\ \\ \end{pmatrix}} = A \begin{pmatrix} u \end{pmatrix} = \begin{pmatrix} d_{qp} \end{pmatrix} \begin{pmatrix} \\ \end{pmatrix}$$

$$A = D \cdot U \backslash I$$

$$\cancel{K} d = G$$

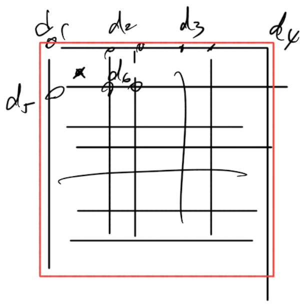
$$K = \alpha A_1 + A_0$$

forward model:

$$\nabla \cdot \sigma = 0, \quad \hat{n} \cdot \sigma = 0$$

find d: $\quad K(\alpha)\, d = G$. $\quad d = (x, y) \in \mathbb{R}^2$

$G = \text{const}$



$$d = \begin{bmatrix} \vdots \\ \vdots \end{bmatrix}$$

$K(\alpha) \succ 0$.

$$d = K \backslash G.$$

Discuss & Simplify: $K(\alpha) \succ 0$.

$$d = K(\alpha)^{-1} G. \qquad K(\alpha) = \alpha A + B.$$

A, B is const. defined by geometry & querypoints.

Objective: $\min\limits_{\alpha} f = \| \hat{d} - d(\alpha) \|_2^2$.

$$f = (\hat{d} - d(\alpha))^T (\hat{d} - d(\alpha)) =$$

$$f = \hat{d}^2 - 2\, d(\alpha)\, \hat{d} + d(\alpha)^2.$$

$$\frac{\partial f}{\partial \alpha} = -2\, \hat{d} \cdot \frac{\partial d(\alpha)}{\partial \alpha} + 2\, d(\alpha) \cdot \frac{\partial d(\alpha)}{\partial \alpha}.$$

$$d = (\alpha A + B)^{-1} G \qquad \frac{\partial K^{-1}}{\partial p} = -K^{-1} \frac{\partial K}{\partial p} K^{-1}$$

$$\frac{\partial d}{\partial \alpha} = \frac{\partial (\alpha A + B)^{-1}}{\partial \alpha} \cdot G = -(\alpha A + B)^{-1} \cdot \frac{\partial (\alpha A + B)}{\partial \alpha} \cdot (\alpha A + B)^{-1}$$

$$= -(\alpha A + B)^{-1} \cdot A\, (\alpha A + B)^{-1} G$$

$$\frac{\partial d(\alpha)}{\partial \alpha} = -\,\underline{K(\alpha)^{-1}}\ \underset{\triangle}{A}\ K(\alpha)^{-1}\, G \qquad\qquad \nabla^2 C(\alpha)\, d = 0$$

$\downarrow$ FEM

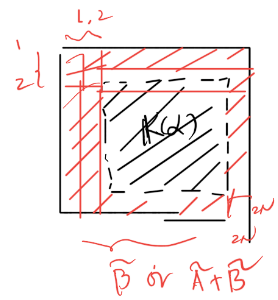
$$K(\alpha) = \alpha A + B \qquad\qquad \underline{K(\alpha)}\, d = 0$$

$K(\alpha) = \alpha A + B$

Let $\alpha = 0$. $K(\alpha) = B$ $\Rightarrow$ $\alpha = 0$, FEM.Kc $= \tilde{B}$, $B = \tilde{B}(3{:}\mathrm{end}{-}1,\ 3{:}\mathrm{end}{-}1)$

Let $\alpha = 1$, $K(\alpha) = A + B$ $\quad$ $\alpha = 1$, $\boxed{\text{FEM.Kc} = \tilde{A} + \tilde{B}}$

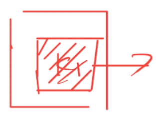
$A + B = (\tilde{A} + \tilde{B})(3{:}\mathrm{end}{-}1,\ 3{:}\mathrm{end}{-}1)$



$$\begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_{2N-1} \\ d_{2N} \end{bmatrix} = \tilde{G} = \begin{bmatrix} - \\ - \\ G_1 \\ - \end{bmatrix} \Leftarrow \text{FEM.Fc}$$

G_1

Script: PDE_ constraint:



$K_0 =$ $\rightarrow$ $K_1 \succeq 0$

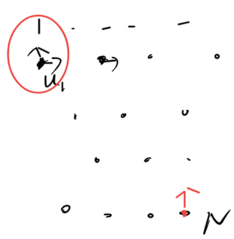
① $K_0 = \text{FEM.Kc}$ $\overset{②}{\Rightarrow}$ $\underset{\triangle}{K_1} = \text{FEM.Kc}(3{:}\mathrm{end}{-}1,\ 3{:}\mathrm{end}{-}1)$

$\therefore K_0 \succeq 0$

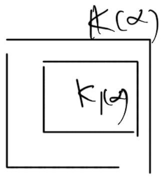
$K(\alpha) = K_1$.

$$d = \begin{bmatrix} 0 \\ 0 \\ \tilde{d}(\alpha) \\ 0 \end{bmatrix} \longrightarrow \frac{\partial d}{\partial \alpha} = \begin{bmatrix} 0 \\ 0 \\ \frac{\partial d(\alpha)}{\partial \alpha} \\ 0 \end{bmatrix} \rightarrow K_1(\alpha)^{-1} A\, K_1(\alpha)^{-1} \cdot G_1$$

$$\begin{bmatrix} d_1 \\ \vdots \\ d_N \\ d_{N+1} \\ \vdots \\ d_{2N} \end{bmatrix} \begin{matrix} u_1 \\ u_2 \\ \vdots \\ u_N \\ v_1 \\ \vdots \\ v_N \end{matrix}$$



$\Rightarrow$ So: $$\frac{\partial f}{\partial \alpha} = -2\frac{\partial d(\alpha)}{\partial \alpha}\hat{d} + 2\, d(\alpha)\cdot\frac{\partial d(\alpha)}{\partial \alpha}$$



$$d(\alpha) = \begin{bmatrix} 0 \\ 0 \\ d_1(\alpha) \\ 0 \end{bmatrix} \qquad \frac{\partial d(\alpha)}{\partial \alpha} = \begin{bmatrix} 0 \\ 0 \\ \frac{\partial d_1}{\partial \alpha} \\ 0 \end{bmatrix}$$

$$\frac{\partial d_1(\alpha)}{\partial \alpha} = -\mathbb{K}_1(\alpha)^{-1} A\, \mathbb{K}_1(\alpha)^{-1} G_1$$

$$\mathbb{K}_1(\alpha) = A\alpha + B$$

$$A = \mathbb{K}_1(1) - \mathbb{K}_1(0)$$

$\hat{d}$: known; A can be computed.

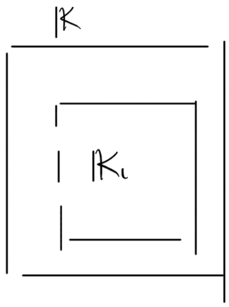
Implement this first.

Next step:

When $\mathbb{C}$ is $\mathbb{C}(\alpha, \beta, \gamma, \frac{w}{2})$ $\quad d \in \mathbb{R}^{2N\times 1}$, $d_1 \in \mathbb{R}^{2N-3}$.

forward Model: $\nabla \cdot \mathbb{C}(\alpha, \beta, \gamma, \omega) \nabla d = 0.$

$\hookrightarrow$ FEM: $K(\alpha, \beta, \gamma, \omega) \cdot d = 0 \qquad d = \begin{bmatrix} 0 \\ 0 \\ d_1 \\ 0 \end{bmatrix}$



$\mathbb{K} \succeq 0. \quad \mathbb{K}_1 \succ 0 \qquad \mathbb{K}_1 = K(3{:}\text{end}-1, 3{:}\text{end}-1)$

$\mathbb{K}_1 = \alpha A + \beta B + \gamma C + \omega D + F$ $\quad 0 \quad$ $K_0 = K_1(0,0,0,0) = 0$

$K_1(0,0,0,0) = F$

$\mathbb{K}_1(1,0,0,0) = A + F$

$\Rightarrow$ $A = K_1(1,0,0,0).$

$B = K_1(0,1,0,0).$

$C = K_1(0,0,1,0).$

$D = K_1(0,0,0,1)$

Backward: $\min\limits_{G = \alpha, \beta, \gamma, \omega} f = (\hat{d} - d)^2 = \hat{d}^2 - 2d\hat{d} + d^2$

s.t. $\mathbb{K}(\alpha, \beta, \gamma, \omega)\, d = G$

KKT: $\frac{\partial f}{\partial \alpha} = 0,$

$\frac{\partial f}{\partial \beta} = 0$

$\frac{\partial f}{\partial \gamma} = 0$

$\frac{\partial f}{\partial \omega} = 0$

$\frac{\partial f}{\partial \alpha} = -2\hat{d}\frac{\partial d}{\partial \alpha} + 2d\frac{\partial d}{\partial \alpha}.$

$\frac{\partial d}{\partial \alpha} = \frac{\partial\, \mathbb{K}^{-1} G}{\partial \alpha} = \begin{bmatrix} 0 \\ 0 \\ \frac{\partial K_1^{-1} G_1}{\partial \alpha} \\ 0 \end{bmatrix}$

$\frac{\partial (K_1^{-1} G_1)}{\partial \alpha} = -K_1^{-1} A K_1^{-1} G_1$

Similarly: $\frac{\partial K_1^{-1} G_1}{\partial \beta} = \underline{-K_1^{-1} B K_1^{-1} G_1}$

$\frac{\partial K_1^{-1} G_1}{\partial \gamma} = -K_1^{-1} C K_1^{-1} G_1$

$\frac{\partial K_1 G_1}{\partial \omega} = -K_1 D K_1^{-1} G_1$

$$\frac{du}{dx} = e_{xx}$$

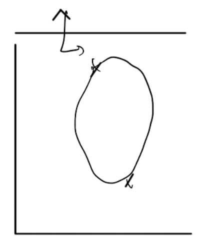
$$\left\{ \begin{array}{l} \dfrac{dv}{dy} = e_{yy} \\ \dfrac{du}{dy} + \dfrac{dv}{dx} = 2e_{xy} \end{array} \right.$$

$$\frac{du}{dy} - \frac{dv}{dx} = 2\theta$$

$$\boxed{\begin{array}{ll} \dfrac{du}{dx} = e_{xx} & \dfrac{dv}{dx} = e_{xy} - \theta \\ \dfrac{du}{dy} = e_{xy} + \theta & \dfrac{dv}{dy} = e_{yy} \end{array}}$$

$$u = u_0 + Ax + By + Cxy$$

$$v = v_0 + Ax + By + Cxy$$

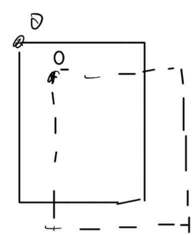


$$\frac{du}{dx} = f_1$$

$$\frac{du}{dy} = f_2$$

$$u = \int f_1 dx + \int f_2 dy + C_1$$

$$v = \int f_1 dx + \int f_2 dy + C_2$$



$$x = X\cos\theta - Y\sin\theta$$

$$y = X\sin\theta + Y\sin\theta$$

$$u = u - u_1$$

$$v = u - v_1$$

$$u_N \sin\theta + v_N \cos\theta = 0 \Rightarrow \tan\theta = -\frac{v_N}{u_N} \Rightarrow \begin{cases} \sin\theta = \dfrac{-v_N}{\sqrt{v_N^2 + u_N^2}} \\ \cos\theta = \dfrac{u_N}{\sqrt{u_N^2 + v_N^2}} \end{cases}$$

$$u' =$$

$$v' =$$

basis

$$\begin{array}{c|ccc|} & f_1 & f_2 \cdots & f_n \\ \hline x_1 & & & \\ x_2 & & & \\ \vdots & & & \\ x_n & & & \\ \hline \end{array} \quad \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

$C = C(\alpha)$

$$\min: \ f = (\hat{d} - d)^T(\hat{d} - d) = \hat{d}^2 - 2\hat{d}^T d + d^2$$

$C = \alpha$

$$\text{s.t.} \quad K(\alpha) \cdot d = G$$

$K = A\alpha.$

$$\frac{\partial f}{\partial \alpha} = -2\hat{d}^T \cdot \frac{\partial d}{\partial \alpha} + 2 d^T \frac{\partial d}{\partial \alpha}$$

$$\frac{\partial d}{\partial \alpha} = \frac{\partial K^{-1} G}{\partial \alpha} = \begin{bmatrix} 0 \\ \frac{\partial K_i^{-1}}{\partial \alpha} G_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -K_i^{-1} A K_i^{-1} G_1 \\ 0 \end{bmatrix}$$

$$\frac{\partial^2 d}{\partial \alpha^2} = \begin{bmatrix} 0 \\ 0 \\ * \\ 0 \end{bmatrix}.$$

$$* = -\frac{\partial}{\partial \alpha}\left[K_i^{-1} A K_i^{-1} G\right]$$
$$= -\frac{\partial}{\partial \alpha}(K_i^{-1}) \cdot A K_i^{-1} G - K_i^{-1} A \frac{\partial}{\partial \alpha}(K_i^{-1}) G$$

$$\frac{\partial K_i^{-1}}{\partial \alpha} = -K_i^{-1} A K_i^{-1} \Rightarrow \quad * = -K_i^{-1} A K_i^{-1} A K_i^{-1} G - K_i^{-1} A K_i^{-1} A K_i^{-1} G = -2 K_i^{-1} A K_i^{-1} A K_i^{-1} G$$

$$\frac{\partial^2 f}{\partial \alpha^2} = -2\hat{d}^T \frac{\partial^2 d}{\partial \alpha^2} + 2\left(\frac{\partial d}{\partial \alpha}\right)^T\left(\frac{\partial d}{\partial \alpha}\right) + 2 d^T \frac{\partial^2 d}{\partial \alpha^2}$$

When $\mathbb{C} = C(\alpha, \beta, \gamma, \omega)$

$$\min_{\mathbb{C}, \alpha, \beta, \gamma, \omega} f := \| \hat{d} - d \|^2 = \hat{d}^T \hat{d} - 2 \hat{d}^T d + d^T d$$

$$\text{s.t.} \quad \mathbb{K}(\alpha, \beta, \gamma, \omega) \cdot d = G$$

$$\frac{\partial f}{\partial \alpha} = -2 \hat{d}^T \frac{\partial d}{\partial \alpha} + 2 d^T \frac{\partial d}{\partial \alpha} \qquad \mathbb{K} = A\alpha + B\beta + C\gamma + D\omega$$

$$\frac{\partial d}{\partial \alpha} = \begin{pmatrix} 0 \\ 0 \\ \frac{\partial d_1}{\partial \alpha} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -K_1^{-1} A K_1^{-1} G \\ 0 \end{pmatrix} \qquad \frac{\partial d}{\partial \beta} = \begin{pmatrix} 0 \\ 0 \\ \frac{\partial d_1}{\partial \beta} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -K_1^{-1} B K_1^{-1} \\ 0 \end{pmatrix}$$

$$\frac{\partial d}{\partial \gamma} = \begin{pmatrix} 0 \\ 0 \\ \frac{\partial d_1}{\partial \gamma} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -K_1^{-1} C K_1^{-1} G_1 \\ 0 \end{pmatrix} \qquad \frac{\partial d}{\partial \omega} = \begin{pmatrix} 0 \\ 0 \\ \frac{\partial d_1}{\partial \omega} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -K_1^{-1} D K_1^{-1} G_1 \\ 0 \end{pmatrix}$$

$$\frac{\partial^2 d}{\partial \alpha^2} = \begin{pmatrix} 0 \\ 0 \\ 2K_1^{-1} A K_1^{-1} A K_1^{-1} G_1 \\ 0 \end{pmatrix}, \quad \frac{\partial^2 d}{\partial \beta^2} = \begin{pmatrix} 0 \\ 0 \\ 2K_1^{-1} B K_1^{-1} B K_1^{-1} G_1 \\ 0 \end{pmatrix}, \quad \frac{\partial^2 d}{\partial \gamma^2} = \begin{pmatrix} 0 \\ 0 \\ 2K_1^{-1} C K_1^{-1} C K_1^{-1} G_1 \\ 0 \end{pmatrix}, \quad \frac{\partial^2 d}{\partial \omega^2} = \begin{pmatrix} 0 \\ 0 \\ 2K_1^{-1} D K_1^{-1} D K_1^{-1} G_1 \\ 0 \end{pmatrix}$$

$$\frac{\partial^2 d}{\partial x_i \partial x_j} = \begin{pmatrix} 0 \\ 0 \\ \frac{\partial^2 d_1}{\partial x_i \partial x_j} \\ 0 \end{pmatrix}, \quad x_i \neq x_j \in (\alpha, \beta, \gamma, \omega), \quad \text{Let } H_{x_i} = \begin{cases} A, & x_i = \alpha \\ B, & x_i = \beta \\ C, & x_i = \gamma \\ D, & x_i = \omega \end{cases}$$

$$\frac{\partial^2 d_1}{\partial \alpha \partial \beta} = -K_1^{-1} A K_1^{-1} B K_1^{-1} G_1 - K_1^{-1} B K_1^{-1} A K_1^{-1} G_1$$

$$\frac{\partial^2 d_1}{\partial \alpha \partial \gamma} = -K_1^{-1} A K_1^{-1} C K_1^{-1} G_1 - K_1^{-1} C K_1^{-1} A K_1^{-1} G_1$$

...

$$\frac{\partial^2 d_1}{\partial x_i \partial x_j} = -K_1^{-1} H_{x_i} K_1^{-1} H_{x_j} K_1^{-1} G_1 - K_1^{-1} H_{x_j} K_1^{-1} H_{x_i} K_1^{-1} G_1$$