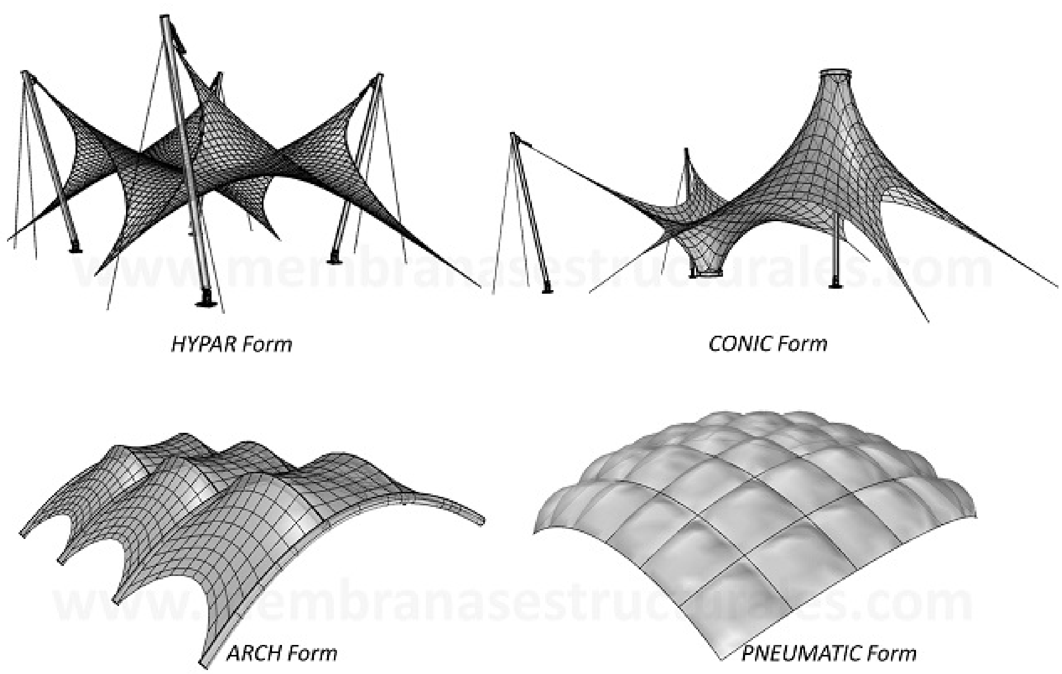
Chapter 7

Poisson equation in 2-D

7.1 Introduction

The Poisson equation govens many phenomena of interest, including deflection of membranes, steady heat conduction, seepage in porous media, diffusion in gasses, fluids and solids, elastic torsion, magnetostatics, and many others. We will use the analysis of membranes as our motivating application. Membranes are examples of tensile structures which carry only tension and no compression or bending. Membranes find diverse applications in engineering from MEMS through large roof structures such as those shown below.



We will consider the simple case of a two-dimensional membrane in the $x - y$ plane that is pretensioned to a uniform pretension T and loaded transversely to the membrane. The goal is to predict the transverse deflections of the membrane. We will make the strong simplifying assumptions that: (i) the pretension T does not vary with position over the membrane, and (ii) the pretension T has the same value in all directions. These assumptions allow us to develop an elementary theory of pretensioned elastic membranes. More realistic modeling, in which the tension is not uniform and varies with direction, can be developed by extending the presented theory.

We will obtain the weak form of the boundary value in two ways: (i) from the strong form, found by direct analysis of equilibrium, and use of the MWR, and (ii) by minimization of the membrane total potential energy, that is by using the principle of minimum potential energy (PMPE). Both approaches will lead to the same weak form and illustrate the alternative approaches. We will then proceed to develop the finite element formulation using 3-node triangles based on the Galerkin form.

7.2 Mathematical preliminaries

We will generally prefer to work with indicial notation, but vector notation will sometimes be used to give an alternative form of a result. Consider a function $u(x_1, x_2) \in C^2(\Omega)$, where x_1 and x_2 are coordinates of points in Ω. Partial derivatives of u are denoted,

$$u_{,1} = \frac{\partial u}{\partial x_1} \qquad u_{,2} = \frac{\partial u}{\partial x_2} \qquad u_{,11} = \frac{\partial^2 u}{\partial x_1^2} \qquad u_{,12} = \frac{\partial^2 u}{\partial x_1 \partial x_2} \qquad u_{,22} = \frac{\partial^2 u}{\partial x_2^2}$$

We define

$$\operatorname{grad} u = \nabla u = \left\{ \begin{array}{c} \partial u / \partial x_1 \\ \partial u / \partial x_2 \end{array} \right\}$$

with the $i-$th component $(\operatorname{grad} u)_i = u_{,i}$. The divergence of the gradient of u is

$$\operatorname{div}(\operatorname{grad} u) = \nabla \cdot \nabla u = \left\{ \begin{array}{c} \partial / \partial x_1 \\ \partial / \partial x_2 \end{array} \right\} \cdot \left\{ \begin{array}{c} \partial u / \partial x_1 \\ \partial u / \partial x_2 \end{array} \right\} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}$$

We can introduce the Laplacian operator ∇^2 and set,

$$\nabla^2 u = u_{,ii} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}$$

where repeated indices imply summation.

Generalizing in an obvious way, we say $u \in C^2(\Omega) \Leftrightarrow u$ and *all* first and second partial derivatives of u are continuous on Ω, and $u \in H^1(\Omega) \Leftrightarrow \int_\Omega \left[u^2 + (u_{,1})^2 + (u_{,2})^2 \right] dx < \infty$.

Two results from vector calculus will be important.

Theorem 1 *Divergence.*
Let $\Omega \subset \mathbb{R}^n$, $n = 1, 2,$ or 3, with boundary $\Gamma = \partial\Omega$, and let $\mathbf{F}(\mathbf{x})$ be a vector field defined on Ω, then

$$\int_\Omega \boldsymbol{\nabla} \cdot \mathbf{F} d\Omega = \int_\Gamma \mathbf{F} \cdot \mathbf{n} d\Gamma$$

where $\mathbf{n}$ *is the unit outward normal on* Γ*. Noting that* $\boldsymbol{\nabla}\cdot\mathbf{F} = \operatorname{div}\mathbf{F} = F_{i,i}$ *and* $\mathbf{F}\cdot\mathbf{n} = F_i n_i$*, we can also write the indicial form*

$$\int_\Omega F_{i,i}\, d\Omega = \int_\Gamma F_i n_i d\Gamma \tag{7.1}$$

where, in 2-D,

$$F_{i,i} = F_{1,1} + F_{2,2} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2}$$

and

$$F_i n_i = F_1 n_1 + F_2 n_2$$

Theorem 2 *Green-Gauss (integration by parts).*
Observe that if $\varphi : \Omega \to \mathbb{R}$*, then we have, by the product rule for differentiation,*

$$\left(\varphi F_i\right)_{,i} = \varphi_{,i} F_i + \varphi F_{i,i}$$

then, using the divergence theorem, it follows that

$$\int_\Omega \varphi F_{i,i}\, d\Omega = -\int_\Omega \varphi_{,i} F_i\, d\Gamma + \int_\Gamma \varphi F_i n_i\, d\Gamma \tag{7.2}$$

In direct notation, this reads

$$\int_\Omega \varphi \operatorname{div}\mathbf{F}\, d\Omega = -\int_\Omega \nabla\varphi\cdot\mathbf{F}\, d\Gamma + \int_\Gamma \varphi\mathbf{F}\cdot\mathbf{n}\, d\Gamma$$

Example 3 *Apply integration by parts to* $\int_\Omega \varphi u_{,ii} d\Omega$*. Let* $F_i = u_{,i}$*, then* $u_{,ii} = \left(u_{,i}\right)_{,i} = F_{i,i}$*. Applying (7.2), we need simply replace* $F_i \leftarrow u_{,i}$ *to obtain*

$$\int_\Omega \varphi u_{,ii}\, d\Omega = -\int_\Omega \varphi_{,i} u_{,i}\, d\Gamma + \int_\Gamma \varphi u_{,i} n_i\, d\Gamma$$

7.3 Weak form from the MWR

7.3.1 Direct derivation of the strong form

Consider an inextensible membrane with a specified tension $T(x,y)^1$ in every direction and at all points. Let the two-dimensional domain of the membrane be denoted $\Omega \subset \mathbb{R}^2$. The boundary Γ of the membrane is comprised of two sets Γ_g, on which the membrane position is given by the function $g(\mathbf{x})$, and Γ_h, on which a prescribed edge load intensity is given by the function $h(\mathbf{x})$. The boundaries satisfy $\Gamma_g \cup \Gamma_h = \Gamma$ and and $\Gamma_g \cup \Gamma_h = \emptyset$. In addition, the membrane is subject to a transverse load intensity $f(\mathbf{x})$. The transverse deflection is denoted $u(\mathbf{x}) : \Omega \to \mathbb{R}$. The equilibrium of the membrane can be

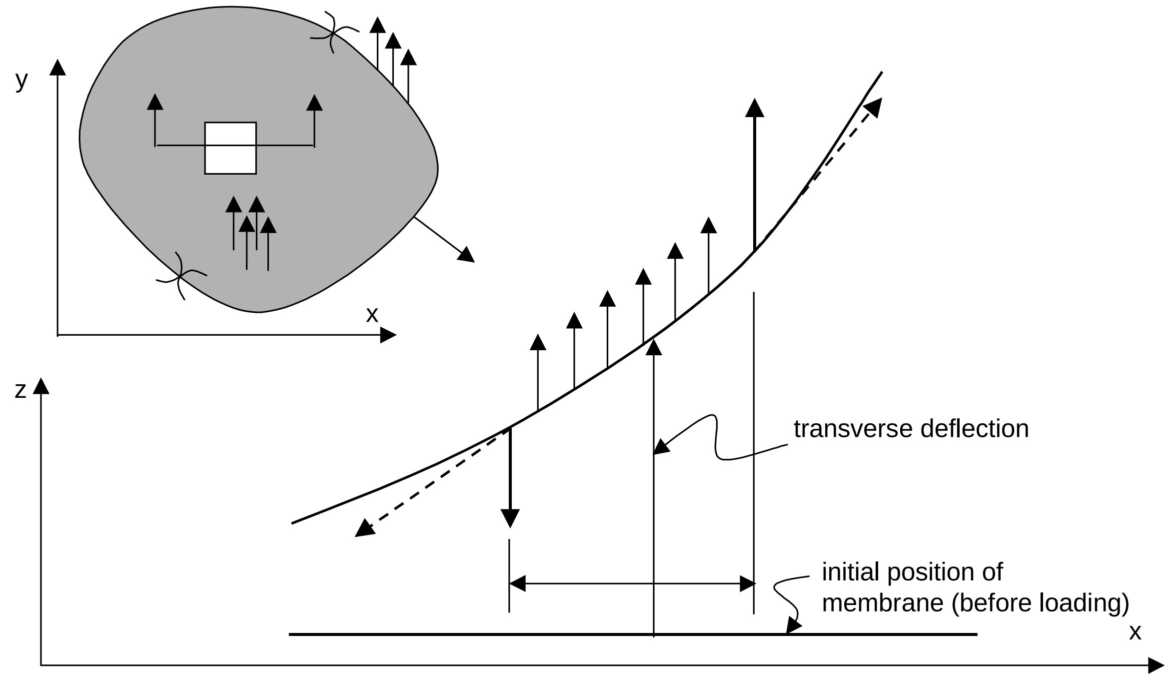


Figure 7.1: (a) Membrane domain and boundary conditions, (b) Equilibrium of an infinitesimal element.

determined by reference to Fig. 7.1. The governing equation can be found by considering the equilibrium of an infinitesimal element in the transverse direction. Similarly, the boundary condition on Γ_h can be related to the membrane tension and deflection. These considerations lead to the following strong form.

Strong form (S): Given functions $g : \Gamma_g \to \mathbb{R}$, $h : \Gamma_h \to \mathbb{R}$, and $f : \Omega \to \mathbb{R}$, find $u \in C^2(\Omega)$ such that

$$-Tu_{,ii} = f \quad \text{on } \Omega \tag{7.3}$$

$$u = g \quad \text{on } \Gamma_g \tag{7.4}$$

$$Tu_{,i}n_i = h \quad \text{on } \Gamma_h \tag{7.5}$$

7.3.2 Weak form

To obtain the weak form, we follow the (a) - (c) prescription out lined in Ch. 4 as follows.

(a) Classify the boundary conditions Determine m from the differential equation and classify the BCs according to Fig. 4.3. For the current problem $m = 1$ and

[1]For elastic membranes, it is possible to set up a problem in which the tension is determined by the deflection. However, this is a different situation to the current problem and would not lead to the Poisson equation.

$$\begin{aligned} u &= g \Rightarrow \text{essential BC} \\ Tu_{,i}n_i &= h \Rightarrow \text{natural BC} \end{aligned}$$

(b) Define trial solution and weighting function spaces

$$\begin{aligned} \mathcal{S} &= \left\{ u |\, u \in H^1(\Omega),\ u = g \text{ on } \Gamma_g \right\} \\ \mathcal{V} &= \left\{ w |\, w \in H^1(\Omega),\ w = 0 \text{ on } \Gamma_g \right\} \end{aligned}$$

(c) Obtain the variational equation from the MWR: (i) form the weighted residual, (ii) perform integration by parts to symmetrize the integrand with respect to u and w, and (iii) substitute natural boundary conditions, if any, and impose boundary restrictions, if any, on w.

(i) Weighted residual statement

$$0 = \int_\Omega w\left[Tu_{,ii} + f\right] d\Omega$$

(ii) Integration by parts

$$0 = \int_\Omega \left[-w_{,i}Tu_{,i} + wf\right] d\Omega + \int_\Gamma wTu_{,i}n_i d\Gamma$$

(iii) Enforce the NBC ($Tu_{,i}n_i = h$ on Γ_h) and $w \in \mathcal{V}$ ($w = 0$ on Γ_g)

$$0 = \int_\Omega \left[-w_{,i}Tu_{,i} + wf\right] d\Omega + \int_{\Gamma_h} whd\Gamma$$

and rearrange as

$$a(w,u) = (w,f) + (w,h)_{\Gamma_h}$$

with

$$a(w,u) = \int_\Omega w_{,i}Tu_{,i}d\Omega, \quad (w,f) = \int_\Omega wfd\Omega, \quad \text{and} \quad (w,h)_{\Gamma_h} = \int_{\Gamma_h} whd\Gamma$$

The weak form may now be stated.

Weak form (W) : Given functions $g : \Gamma_g \to \mathbb{R}$, $h : \Gamma_h \to \mathbb{R}$, and $f : \Omega \to \mathbb{R}$, find $u \in \mathcal{S} = \{u|\, u \in H^1(\Omega),\ u = g \text{ on } \Gamma_g\}$ such that,

$$a(w,u) = (w,f) + (w,h)_{\Gamma_h} \tag{7.6}$$

holds $\forall\ w \in \mathcal{V} = \{w|\, w \in H^1(\Omega),\ w = 0 \text{ on } \Gamma_g\}$, with

$$a(w,u) = \int_\Omega w_{,i}Tu_{,i}d\Omega, \quad (w,f) = \int_\Omega wfd\Omega, \quad \text{and} \quad (w,h)_{\Gamma_h} = \int_{\Gamma_h} whd\Gamma$$

7.4 Weak form from the PMPE

Omitted.

7.5 Galerkin form

Consider a subspace of trial functions $u^h \in \mathcal{S}^h \subset \mathcal{S}$ and weighting functions $w^h \in \mathcal{V}^h \subset \mathcal{V}$ where $\mathcal{S}^h$ and $\mathcal{V}^h$ are finite-dimensional function spaces. As in Ch. 5, the Galerkin form is found by *restricting* the weak form to the spaces $\mathcal{S}^h$ and $\mathcal{V}^h$. Furthermore, we can additively decompose u^h such that

$$\underbrace{u^h(x)}_{\in\mathcal{S}^h} = \underbrace{v^h(x)}_{\in\mathcal{V}^h} + \underbrace{g^h(x)}_{\in\mathcal{S}^h} \tag{7.7}$$

where g^h will be selected to satisfy the EBC on Γ_g. With this, the Galerkin form can be stated.

Galerkin form (G) : Given functions $g : \Gamma_g \to \mathbb{R}$, $h : \Gamma_h \to \mathbb{R}$, and $f : \Omega \to \mathbb{R}$, find $v^h \in \mathcal{V}^h = \left\{ v^h \middle| \, v^h \in H^1(\Omega),\ v^h = 0 \text{ on } \Gamma_g \right\}$ such that,

$$a\left(w^h, v^h\right) = \left(w^h, f\right) + \left(w^h, h\right)_{\Gamma_h} - a\left(w^h, g^h\right) \tag{7.8}$$

holds $\forall\ w^h \in \mathcal{V} = \left\{ w^h \middle| \, w^h \in H^1(\Omega),\ w^h = 0 \text{ on } \Gamma_g \right\}$, with

$$a\left(w^h, v^h\right) = \int_\Omega w^h_{,i} T v^h_{,i} d\Omega, \quad \left(w^h, f\right) = \int_\Omega w^h f d\Omega, \quad \text{and} \quad \left(w^h, h\right)_{\Gamma_h} = \int_{\Gamma_h} w^h h d\Gamma$$

We are now ready to introduce the finite element approximation for v^h and w^h.

7.6 Finite element approximation

Similar to the $1-D$ case, the finite element formulation proceeds by:

1. *Discretization.*
   Partition the domain into n elements

   $$\Omega = \bigcup_{e=1}^{n} \Omega^e$$

   in which case the Galerkin variational equation (7.8) becomes

   $$\sum_{e=1}^{n} a(w^h, v^h)_{\Omega^e} = \sum_{e=1}^{n} \left(w^h, f\right)_{\Omega^e} + \sum_{e=1}^{n} \left(w^h, h\right)_{\Gamma_h^e} - \sum_{e=1}^{n} a\left(w^h, g^h\right)_{\Omega^e} \tag{7.9}$$

2. *Interpolation.*
   Introduce approximations for $w^h(x)$ and $v^h(x)$ over each element based on interpolation of the element nodal values:

$$w^h(\mathbf{x}) = \sum_{a=1}^{n_{en}} N_a^e(\mathbf{x})\, c_a^e \tag{7.10}$$

$$v^h(\mathbf{x}) = \sum_{b=1}^{n_{en}} N_b^e(\mathbf{x})\, d_b^e \tag{7.11}$$

   where n_{en} is the number of element nodes. The shape functions N_a^e will be illustrated in the following section which introduces the very simple 3-node triangular element. The Galerkin variational equation (7.9) immediately leads to the following definition of the *element stiffness matrix*

$$k_{ab}^e = a\left(N_a^e, N_b^e\right)_{\Omega^e} \tag{7.12}$$

$$= \int_{\Omega^e} N_{a,i}^e T N_{b,i}^e d\Omega^e \tag{7.13}$$

$$= T\int_{\Omega^e} \left(N_{a,1}^e N_{b,1}^e + N_{a,2}^e N_{b,2}^e\right) d\Omega^e \tag{7.14}$$

$$= T\int_{\Omega^e} \left\langle \begin{array}{cc} N_{a,x}^e & N_{a,y}^e \end{array} \right\rangle \left\{ \begin{array}{c} N_{b,x}^e \\ N_{b,y}^e \end{array} \right\} d\Omega^e \tag{7.15}$$

   where, in the final equation, we have replaced $x \leftarrow x_1$ and $y \leftarrow x_2$ for convenience. The *element load vector* is defined by,

$$f_a^e = \left(N_a^e, f\right)_{\Omega^e} + \left(N_a^e, h\right)_{\Gamma_h^e} - \sum_{b=1}^{n_{en}} a(N_a^e, N_b^e)_{\Omega^e} g_b^e$$

$$= \underbrace{\int_{\Omega^e} N_a^e f(x,y)\, d\Omega^e}_{\left[f_f^e\right]_a} + \underbrace{\int_{\Gamma_h^e} N_a^e h\, d\Gamma^e}_{\left[f_h^e\right]_a} - \underbrace{\sum_{b=1}^{n_{en}} k_{ab}^e g_b^e}_{-\left[f_g^e\right]_a} \tag{7.16}$$

All of which can be assembled into the global system $\mathbf{Kd} = \mathbf{F}$ using the direct stiffness method.

7.7 The 3-node triangle

The 3-node triangle is the simplest possible element for analysis in two dimensions. The element shape functions are most conveniently expressed in terms of element natural coordinates (ξ, η) over a parent domain $\triangle$. The set-up is shown in Fig. 7.2.
Each element is assigned *local node numbers* $\{1, 2, 3\}$. Any node can be designated as local node 1. However, once local node number 1 has been assigned, the remaining local

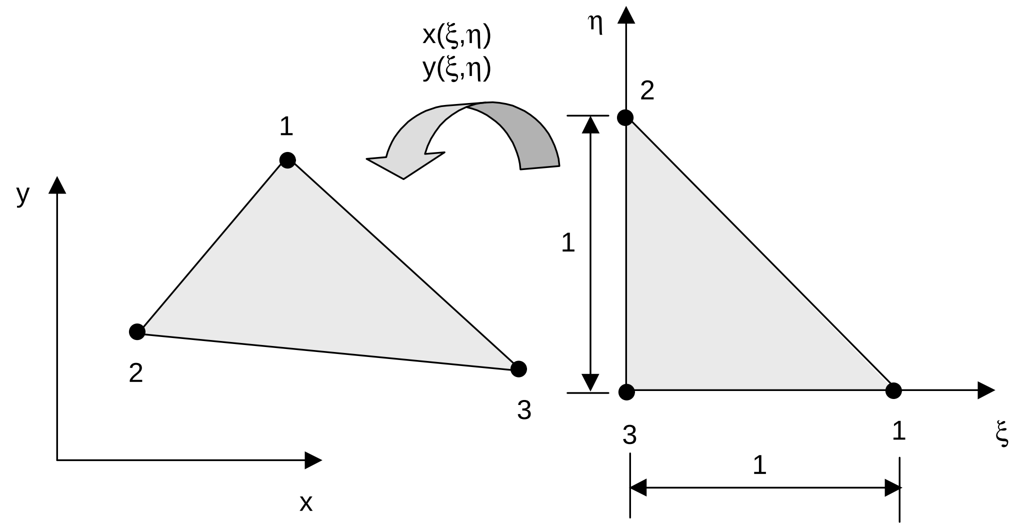


Figure 7.2: Triangular element natural coordinates. Physical domain on left, parent domain on right.

node numbers 2 and 3 must be assigned sequentially in a counter-clockwise orientation. This is necessary in order to guarantee that the element natural coordinate system is right-handed.
On the element parent domain we will agree that each of the nodes will be located at the following natural coordinate positions:

$$\begin{aligned} \text{Node 1} &: \quad (\xi,\eta) = (1,0) \\ \text{Node 2} &: \quad (\xi,\eta) = (0,1) \\ \text{Node 3} &: \quad (\xi,\eta) = (0,0) \end{aligned}$$

We think of the element domain in physical coordinate space Ω^e as derived from a mapping from the element parent domain $\triangle$ which is coordinate invariant for all elements (no superscript e).

7.7.1 Element Shape Functions

We start by constructing shape functions on the parent domain using the natural coordinates. We can introduce the following element shape functions by inspection:

$$N_1^e(\xi,\eta) = \xi \tag{7.17}$$
$$N_2^e(\xi,\eta) = \eta \tag{7.18}$$
$$N_3^e(\xi,\eta) = 1-\xi-\eta \tag{7.19}$$

and these are depicted graphically in Fig. 7.3. Observe that the shape functions satisfy the interpolation properties:

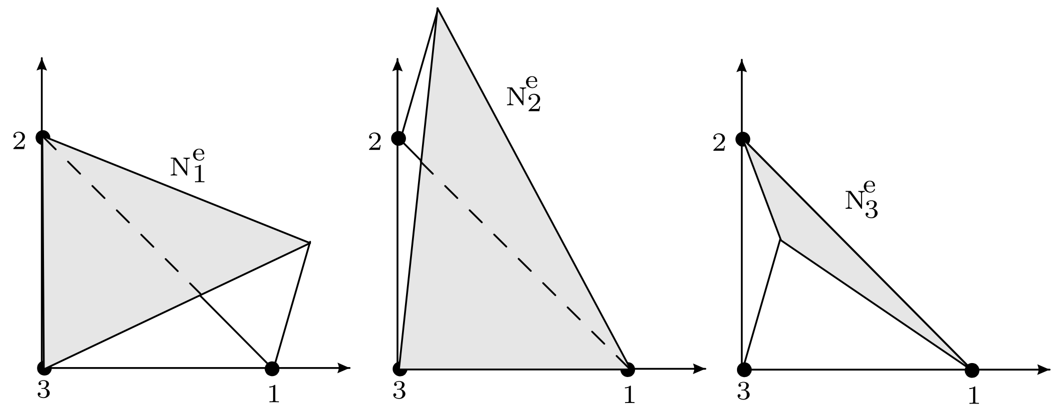


Figure 7.3: Triangular element shape functions on the natural coordinate domain.

1. The $i-th$ shape function has unit value at the $i-th$ node, and is zero at the other two, i.e. $N_a^e\left(\xi_b, \eta_b\right) = \delta_{ab}$ and

2. The three shape functions sum to the value of one everywhere over the element, i.e.,

$$N_1^e(\xi, \eta) + N_1^e(\xi, \eta) + N_1^e(\xi, \eta) = 1$$

7.7.2 Finite Element Interpolation

Using the shape functions given by (7.17)-(7.19), we can now write the trial function v^h over each element as,

$$v^h(\xi, \eta) = \sum_{a=1}^{3} N_a^e(\xi, \eta) d_a^e = \mathbf{N}^e\left(\xi, \eta\right) \mathbf{d}^e \tag{7.20}$$

where

$$\mathbf{N}^e\left(\xi, \eta\right) = \left\langle\ N_1^e(\xi, \eta), \quad N_2^e(\xi, \eta), \quad N_3^e(\xi, \eta)\ \right\rangle, \qquad \mathbf{d}^e = \left\{ \begin{array}{c} d_1^e \\ d_2^e \\ d_3^e \end{array} \right\}$$

This nodal interpolation is illustrated in Fig. 7.4. Similarly, the weighting function w^h is expressed as

$$w^h(\xi, \eta) = \sum_{a=1}^{3} N_a^e(\xi, \eta) c_a^e = \mathbf{N}^e \mathbf{c}^e \tag{7.21}$$

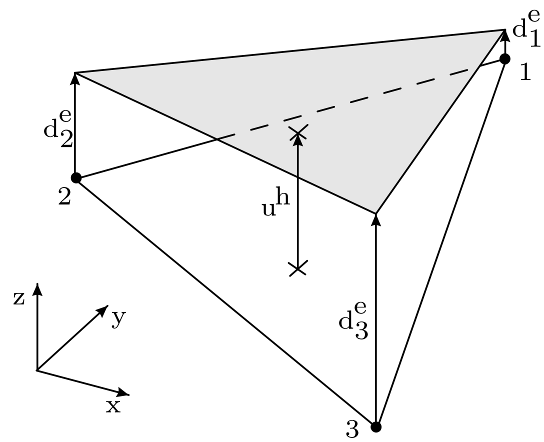


Figure 7.4: Finite element interpolation of nodal values over the element depicted in the physical domain.

7.7.3 Element Arrays

Differentiating v^h we find

$$\left\{ \begin{array}{c} v^h_{,x} \\ v^h_{,y} \end{array} \right\} = \left[\begin{array}{ccc} N^e_{1,x} & N^e_{2,x} & N^e_{3,x} \\ N^e_{1,y} & N^e_{2,y} & N^e_{3,y} \end{array} \right] \left\{ \begin{array}{c} d^e_1 \\ d^e_2 \\ d^e_3 \end{array} \right\} = \mathbf{B}^e \mathbf{d}^e$$

Likewise for the weighting functions,

$$\left\{ \begin{array}{c} w^h_{,x} \\ w^h_{,y} \end{array} \right\} = \left[\begin{array}{ccc} N^e_{1,x} & N^e_{2,x} & N^e_{3,x} \\ N^e_{1,y} & N^e_{2,y} & N^e_{3,y} \end{array} \right] \left\{ \begin{array}{c} c^e_1 \\ c^e_2 \\ c^e_3 \end{array} \right\} = \mathbf{B}^e \mathbf{c}^e$$

In matrix form we can write the 3×3 *element stiffness matrix* from (7.13) as,

$$\mathbf{k}^e = T \int_{\Omega^e} \mathbf{B}^{eT} \mathbf{B}^e dxdy \tag{7.22}$$

The 3×1 *element load vector* in matrix form is found from (7.16),

$$\begin{aligned} \mathbf{f}^e &= \int_{\Omega^e} \mathbf{N}^{eT} f \, d\Omega + \int_{\Gamma^e_h} \mathbf{N}^{eT} h \, d\Gamma - \mathbf{k}^e \mathbf{g}^e \\ &= \mathbf{f}^e_f + \mathbf{f}^e_h + \mathbf{f}^e_h \end{aligned} \tag{7.23}$$

7.7.4 Isoparametric Mapping for the 3-Node Triangle

We define the mapping $\mathbf{x} : \triangle \rightarrow \Omega^e$ by means of the following (isoparametric) interpolations:

$$\begin{aligned} x(\xi,\eta) &= \sum_{a=1}^{3} N_a^e(\xi,\eta)\, x_a^e = \mathbf{N}^e(\xi,\eta)\,\mathbf{x}^e \\ y(\xi,\eta) &= \sum_{a=1}^{3} N_a^e(\xi,\eta)\, y_a^e = \mathbf{N}^e(\xi,\eta)\,\mathbf{y}^e \end{aligned} \tag{7.24}$$

where, as defined above,

$$\mathbf{N}^e(\xi,\eta) = \left\langle \; N_1^e(\xi,\eta), \quad N_2^e(\xi,\eta), \quad N_3^e(\xi,\eta) \; \right\rangle \tag{7.25}$$

In order to compute the element strain-displacement matrix $\mathbf{B}^e$, consider an application of the chain rule,

$$\left\{ \begin{array}{c} \frac{\partial N_a^e}{\partial \xi} \\ \frac{\partial N_a^e}{\partial \eta} \end{array} \right\} = \left[\begin{array}{cc} \frac{\partial x(\xi,\eta)}{\partial \xi} & \frac{\partial y(\xi,\eta)}{\partial \xi} \\ \frac{\partial x(\xi,\eta)}{\partial \eta} & \frac{\partial y(\xi,\eta)}{\partial \eta} \end{array} \right] \left\{ \begin{array}{c} \frac{\partial N_a^e}{\partial x} \\ \frac{\partial N_a^e}{\partial y} \end{array} \right\} \tag{7.26}$$

Defining the *Jacobian matrix* $\mathbf{J}^e$ by,

$$\mathbf{J}^e(\xi,\eta) = \left[\begin{array}{cc} \frac{\partial x(\xi,\eta)}{\partial \xi} & \frac{\partial x(\xi,\eta)}{\partial \eta} \\ \frac{\partial y(\xi,\eta)}{\partial \xi} & \frac{\partial y(\xi,\eta)}{\partial \eta} \end{array} \right]$$

we have

$$\left\{ \begin{array}{c} \frac{\partial N_a^e}{\partial \xi} \\ \frac{\partial N_a^e}{\partial \eta} \end{array} \right\} = [\mathbf{J}^e]^T \left\{ \begin{array}{c} \frac{\partial N_a^e}{\partial x} \\ \frac{\partial N_a^e}{\partial y} \end{array} \right\}$$

The elements of $\mathbf{J}^e$ can easily be computed from the isoparametric map. For example,

$$\begin{aligned} \frac{\partial x}{\partial \xi} &= \frac{\partial}{\partial \xi}\mathbf{N}^e(\xi,\eta)\,\mathbf{x}^e \\ &= \left\langle \; N_{1,\xi}^e(\xi,\eta), \quad N_{2,\xi}^e(\xi,\eta), \quad N_{3,\xi}^e(\xi,\eta) \; \right\rangle \left\{ \begin{array}{c} x_1^e \\ x_2^e \\ x_3^e \end{array} \right\} \\ &= \left\langle \; 1, \quad 0, \quad -1 \; \right\rangle \left\{ \begin{array}{c} x_1^e \\ x_2^e \\ x_3^e \end{array} \right\} \\ &= x_1^e - x_3^e \end{aligned}$$

The other terms may be similarly computed, leading to,

$$\mathbf{J}^e = \left[\begin{array}{cc} x_1^e - x_3^e & x_2^e - x_3^e \\ y_1^e - y_3^e & y_2^e - y_3^e \end{array} \right] \tag{7.27}$$

It is observed that this Jacobian matrix is *constant* over the element. It is easy to show that the determinant of the Jacobian matrix is,

$$|\mathbf{J}^e| = 2A^e$$

where A^e is the area of the element (in the (x,y) domain).
Returning to (7.26), we now obtain,

$$\begin{aligned} \left\{ \begin{array}{c} \frac{\partial N_a^e}{\partial x} \\ \frac{\partial N_a^e}{\partial y} \end{array} \right\} &= \left[\begin{array}{cc} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{array} \right]^{-T} \left\{ \begin{array}{c} \frac{\partial N_a^e}{\partial \xi} \\ \frac{\partial N_a^e}{\partial \eta} \end{array} \right\} \\ &= [\mathbf{J}^e]^{-T} \left\{ \begin{array}{c} \frac{\partial N_a^e}{\partial \xi} \\ \frac{\partial N_a^e}{\partial \eta} \end{array} \right\} \end{aligned} \tag{7.28}$$

Introducing the notation,

$$\begin{aligned} x_{ab}^e &= x_a^e - x_b^e = -x_{ba}^e \\ y_{ab}^e &= y_a^e - y_b^e = -y_{ba}^e \end{aligned}$$

it follows from (7.27) that,

$$\mathbf{J}^e = \left[\begin{array}{cc} x_{13}^e & x_{23}^e \\ y_{13}^e & y_{23}^e \end{array} \right]$$

and,

$$\begin{aligned} \mathbf{J}^{e-1} &= \frac{1}{|J^e|} \left[\begin{array}{cc} y_{23}^e & -x_{23}^e \\ -y_{13}^e & x_{13}^e \end{array} \right] \\ &= \frac{1}{2A^e} \left[\begin{array}{cc} y_{23}^e & -x_{23}^e \\ -y_{13}^e & x_{13}^e \end{array} \right] \end{aligned} \tag{7.29}$$

7.7.5 The Element Stiffness Matrix

Recall that (7.22) provides the expression for the element stiffness matrix. Now, using (7.28) and (7.29), we can immediately write,

$$\begin{aligned} \left\{ \begin{array}{c} \frac{\partial N_1^e}{\partial x} \\ \frac{\partial N_1^e}{\partial y} \end{array} \right\} &= [\mathbf{J}^e]^{-T} \left\{ \begin{array}{c} \frac{\partial N_1^e}{\partial \xi} \\ \frac{\partial N_1^e}{\partial \eta} \end{array} \right\} = \frac{1}{2A^e} \left[\begin{array}{cc} y_{23}^e & -y_{13}^e \\ -x_{23}^e & x_{13}^e \end{array} \right] \left\{ \begin{array}{c} 1 \\ 0 \end{array} \right\} = \frac{1}{2A^e} \left\{ \begin{array}{c} y_{23}^e \\ x_{32}^e \end{array} \right\} \\ \left\{ \begin{array}{c} \frac{\partial N_2^e}{\partial x} \\ \frac{\partial N_2^e}{\partial y} \end{array} \right\} &= [\mathbf{J}^e]^{-T} \left\{ \begin{array}{c} \frac{\partial N_2^e}{\partial \xi} \\ \frac{\partial N_2^e}{\partial \eta} \end{array} \right\} = \frac{1}{2A^e} \left[\begin{array}{cc} y_{23}^e & -y_{13}^e \\ -x_{23}^e & x_{13}^e \end{array} \right] \left\{ \begin{array}{c} 0 \\ 1 \end{array} \right\} = \frac{1}{2A^e} \left\{ \begin{array}{c} y_{31}^e \\ x_{13}^e \end{array} \right\} \\ \left\{ \begin{array}{c} \frac{\partial N_3^e}{\partial x} \\ \frac{\partial N_3^e}{\partial y} \end{array} \right\} &= [\mathbf{J}^e]^{-T} \left\{ \begin{array}{c} \frac{\partial N_3^e}{\partial \xi} \\ \frac{\partial N_3^e}{\partial \eta} \end{array} \right\} = \frac{1}{2A^e} \left[\begin{array}{cc} y_{23}^e & -y_{13}^e \\ -x_{23}^e & x_{13}^e \end{array} \right] \left\{ \begin{array}{c} -1 \\ -1 \end{array} \right\} = \frac{1}{2A^e} \left\{ \begin{array}{c} y_{12}^e \\ x_{21}^e \end{array} \right\} \end{aligned}$$

and from which we obtain,

$$\mathbf{B}^e = \frac{1}{2A^e} \left[\begin{array}{ccc} y_{23}^e & y_{31}^e & y_{12}^e \\ x_{32}^e & x_{13}^e & x_{21}^e \end{array} \right] \tag{7.30}$$

Since the $\mathbf{B}^e$ matrix is constant (i.e. is not a function of ξ or η over the element), the resulting element is commonly referred to as the *constant strain triangle.*
Returning to the element stiffness matrix, we observe,

$$
\begin{aligned}
\mathbf{k}^e &= T \int_{\Omega^e} \mathbf{B}^{eT} \mathbf{B}^e dxdy \\
&= T \mathbf{B}^{eT} \mathbf{B}^e \int_{\Omega^e} dxdy \\
&= \frac{T}{4\left(A^e\right)^2} \begin{bmatrix} y_{23}^e & x_{32}^e \\ y_{31}^e & x_{13}^e \\ y_{12}^e & x_{21}^e \end{bmatrix} \begin{bmatrix} y_{23}^e & y_{31}^e & y_{12}^e \\ x_{32}^e & x_{13}^e & x_{21}^e \end{bmatrix} \int_{\Omega^e} dxdy \\
&= \frac{T}{4A^e} \begin{bmatrix} y_{23}^e & x_{32}^e \\ y_{31}^e & x_{13}^e \\ y_{12}^e & x_{21}^e \end{bmatrix} \begin{bmatrix} y_{23}^e & y_{31}^e & y_{12}^e \\ x_{32}^e & x_{13}^e & x_{21}^e \end{bmatrix} \\
&= \frac{T}{4A^e} \begin{bmatrix} y_{23}^e y_{23}^e + x_{32}^e x_{32}^e & y_{23}^e y_{31}^e + x_{32}^e x_{13}^e & y_{23}^e y_{12}^e + x_{32}^e x_{21}^e \\ & y_{31}^e y_{31}^e + x_{13}^e x_{13}^e & y_{31}^e y_{12}^e + x_{13}^e x_{21}^e \\ \text{symmetric} & & y_{12}^e y_{12}^e + x_{21}^e x_{21}^e \end{bmatrix}
\end{aligned}
\tag{7.31}
$$

7.7.6 Element Load Vector

For the special case where the transverse load intensity $f(x, y)$ is constant over the element, we have,

$$
\int_{\Omega^e} \mathbf{N}^{eT} f \, d\Omega = (f) \int_{\Omega^e} \left\{ \begin{array}{c} N_1^e(\xi, \eta) \\ N_2^e(\xi, \eta) \\ N_3^e(\xi, \eta) \end{array} \right\} dxdy
$$

and it is obvious (volume of prism) that,

$$
\int_{\Omega^e} N_a^e(\xi, \eta) \, dxdy = \frac{A^e}{3}, \qquad a = 1, 2, 3
$$

so then

$$
\int_{\Omega^e} \mathbf{N}^{eT} f \, d\Omega = \left(\frac{f A^e}{3} \right) \left\{ \begin{array}{c} 1 \\ 1 \\ 1 \end{array} \right\} \tag{7.32}
$$

If $f(x, y)$ is not a constant function, numerical integration formulae may be used to evaluate the above integral.
For the integral on Γ_h we have

$$
\int_{\Gamma_h^e} \mathbf{N}^{eT} h \, d\Gamma = \int_{\Gamma_\sigma^e} \left\{ \begin{array}{c} N_1^e(\xi, \eta) \\ N_2^e(\xi, \eta) \\ N_3^e(\xi, \eta) \end{array} \right\} h(x, y) \, d\Gamma
$$

For the special case where the edge load intensity $h(x, y)$ is constant, we have,

$$
\mathbf{f}_h^e = (h) \int_{\Gamma_\sigma^e} \left\{ \begin{array}{c} N_1^e(\xi, \eta) \\ N_2^e(\xi, \eta) \\ N_3^e(\xi, \eta) \end{array} \right\} d\Gamma
$$

Suppose, for example, that Γ_σ^e is element edge 1-2, so that $\Gamma_\sigma^e = \ell_{1-2}$. Then,

$$\begin{aligned}
\mathbf{f}_h^e &= (h)\int_{\ell_{1-2}} \left\{ \begin{array}{c} N_1^e(\xi,\eta) \\ N_2^e(\xi,\eta) \\ 0 \end{array} \right\} d\ell_{1-2} \\
&= \frac{h\ell_{1-2}}{2} \left\{ \begin{array}{c} 1 \\ 1 \\ 0 \end{array} \right\}
\end{aligned}$$

Similarly, if $\Gamma_\sigma^e = \ell_{2-3}$, then,

$$\mathbf{f}_h^e = \frac{h\ell_{2-3}}{2} \left\{ \begin{array}{c} 0 \\ 1 \\ 1 \end{array} \right\}$$

and if $\Gamma_\sigma^e = \ell_{3-1}$, then,

$$\mathbf{f}_h^e = \frac{h\ell_{3-1}}{2} \left\{ \begin{array}{c} 1 \\ 0 \\ 1 \end{array} \right\}$$

Again, if the edge intensity function $h(x, y)$ is not constant, the above integrations become more involved and numerical quadrature may be used.

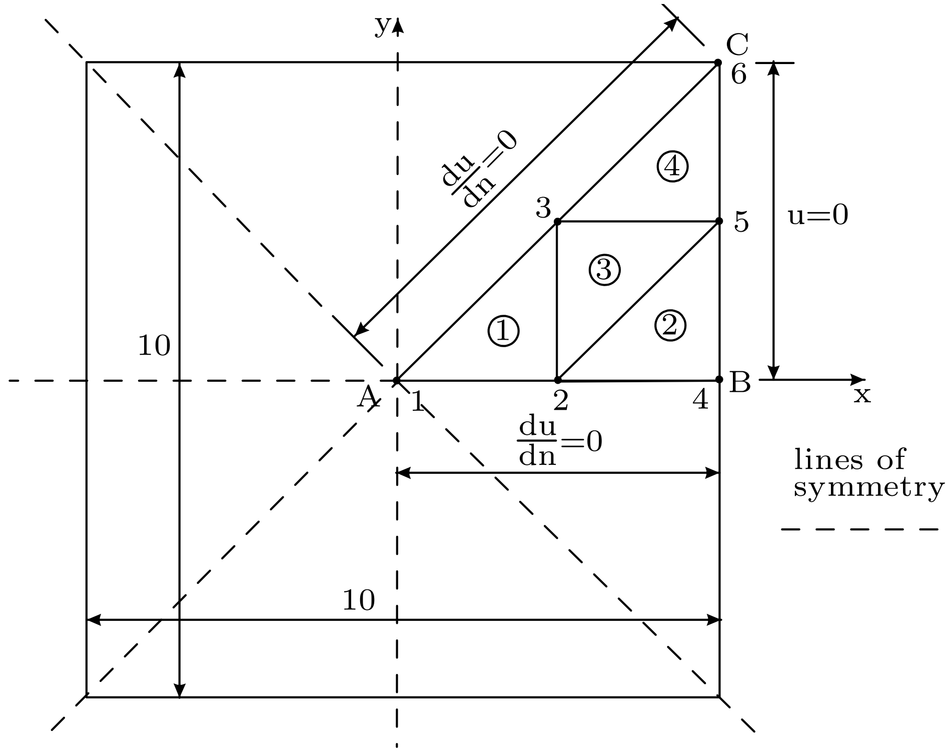
7.8 Example

Consider a square membrane of dimension 10×10, with a constant tension of $T = 2.0$, subject to a transverse load intensity of $f = -0.024$ (load per unit area). The membrane boundary is supported so that $u = 0$ everywhere around the boundary. Create a finite element model and determine the deflections of the membrane.The problem is shown in Fig. 7.8. Exploiting symmetry it is only necessary to discretize and analyze one eighth of the domain as shown in Fig. 7.8. Here we are using 4 constant strain triangles to analyze the membrane. Notice that along edge B-C, we have an *essential* boundary condition of $u = 0$, and along edges C-A and A-B we have *natural* boundary conditions of $\partial u/\partial n = 0$. For the mesh shown in Fig. 7.8, we can therefore identify edge B-C as belonging to Γ_g, and edges A-B and C-A as belonging to Γ_h. As required, we check that $\Gamma = \Gamma_g \cup \Gamma_h$ and $\Gamma_g \cap \Gamma_h = \emptyset$.
The nodal numbers are assigned arbitrarily and the system degrees of freedom are assigned to nodes accoding to essential boundary condition requirements as follows:

$$ID = \left[\ 1,\ \ 2,\ \ 3,\ \ 0,\ \ 0,\ \ 0\ \right]$$

The local destination array is,

$$L = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 2 & 0 & 0 & 0 \\ 3 & 0 & 3 & 0 \end{bmatrix}$$



Notice that the local destination array columns record the system *dof* numbers starting at *local* node 1 for each element. Therefore, the analyst needs to select which node of each element will be local node 1 – this is arbitrary. But having selected local node 1, local nodes 2 and 3 must be labelled through a counter-clockwise traversal of the element. For the present problem we infer that the selected element local node 1 are:

element	local node 1
1	1
2	2
3	2
4	3

Remember that once local node 1 has been selected (and this is arbitrary), the next two local node numbers are assigned by moving in a counter-clockwise direction. And this is reflected in the local destination array.

Assembly the system stiffness matrix

Element 1 (using Eq. (7.31)):

$A^1 = (1/2)\,(5/2)^2 = 25/8$

$$\begin{aligned}
\mathbf{k}^1 &= \frac{2.0}{(4)\,(25/8)} \begin{bmatrix} 25/4 & -25/4 & 0 \\ -25/4 & 25/2 & -25/4 \\ 0 & -25/4 & 25/4 \end{bmatrix} \\
&= \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}
\end{aligned}$$

$$\mathbf{l}^1 : \mathbf{k}^1 \rightarrow \mathbf{K}^1 = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Elements 2 and 4:
Notice that elements 2 and 4 are geometrically identical to element 1, therefore we can immediately set,

$$\mathbf{k}^2 = \mathbf{k}^4 = \mathbf{k}^1 = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\mathbf{l}^2 : \mathbf{k}^2 \rightarrow \mathbf{K}^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{l}^4 : \mathbf{k}^4 \rightarrow \mathbf{K}^4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Element 3 (using Eq. (7.31)):
$A^3 = (1/2)\,(5/2)^2 = 25/8$

$$\begin{aligned}
\mathbf{k}^3 &= \frac{2.0}{(4)\,(25/8)} \begin{bmatrix} 25/4 & 0 & -25/4 \\ 0 & 25/4 & -25/4 \\ -25/4 & -25/4 & 25/2 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix}
\end{aligned}$$

$$\mathbf{l}^3 : \mathbf{k}^3 \rightarrow \mathbf{K}^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

System stiffness matrix,

$$\mathbf{K} = \sum_{e=1}^{4} \mathbf{K}^e = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 4 & -2 \\ 0 & -2 & 4 \end{bmatrix}$$

Assembly the system load vectors
Element 1 (using Eq. (7.32)):
$A^1 = (1/2)\,(5/2)^2 = 25/8$

$$\mathbf{f}_f^1 = \frac{(-0.024)\,(25/8)}{3}\left\{\begin{array}{c}1\\1\\1\end{array}\right\} = (-0.025)\left\{\begin{array}{c}1\\1\\1\end{array}\right\}$$

$$\mathbf{l}^1 : \mathbf{f}_f^1 \rightarrow \mathbf{F}_f^1 = (-0.025)\left\{\begin{array}{c}1\\1\\1\end{array}\right\}$$

Elements 2,3 and 4:
These elements have the same area, so we can write,

$$\mathbf{f}_f^2 = \mathbf{f}_f^3 = \mathbf{f}_f^4 = \mathbf{f}_f^1 = (-0.025)\left\{\begin{array}{c}1\\1\\1\end{array}\right\}$$

$$\mathbf{l}^2 \;\;:\;\; \mathbf{f}_f^2 \rightarrow \mathbf{F}_f^2 = (-0.025)\left\{\begin{array}{c}0\\1\\0\end{array}\right\}$$

$$\mathbf{l}^3 \;\;:\;\; \mathbf{f}_f^3 \rightarrow \mathbf{F}_f^3 = (-0.025)\left\{\begin{array}{c}0\\1\\1\end{array}\right\}$$

$$\mathbf{l}^4 \;\;:\;\; \mathbf{f}_f^4 \rightarrow \mathbf{F}_f^4 = (-0.025)\left\{\begin{array}{c}0\\0\\1\end{array}\right\}$$

Structure load vector $\mathbf{F}_f$:

$$\mathbf{F}_f = \sum_{e=1}^{4} \mathbf{F}_f^e = (-0.025)\left\{\begin{array}{c}1\\3\\3\end{array}\right\}$$

Structure load vector $\mathbf{F}_h$:
For the boundary loads, notice that $\mathbf{f}_h^e = \mathbf{0}$ is zero for all elements, since the *natural* boundary condition is homogeneous, i.e. $T(x,y)\partial u/\partial n = h(x,y) = 0$, on the boundary Γ_h. Accordingly, the structure load vector $\mathbf{F}_h = \mathbf{0}$.
Summary of Equations

$$\begin{array}{ccc} \mathbf{Kd} & = & \mathbf{F}_f \\ \left[\begin{array}{rrr} 1 & -1 & 0 \\ -1 & 4 & -2 \\ 0 & -2 & 4 \end{array}\right]\left\{\begin{array}{c} d_1\\ d_2\\ d_3\end{array}\right\} & = & \left\{\begin{array}{c}-0.025\\-0.075\\-0.075\end{array}\right\}\end{array}$$

Solution of Equations

$$\begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} -0.0937 \\ -0.0687 \\ -0.0531 \end{Bmatrix}$$