Chapter 13

Finite element formulation for 2-D elastostatics

Having obtained the weak form for the BVP in three dimensions, we proceed to specialize it for two important cases: plane stress and plane strain. These cases allow some problems to be modeled in 2-D, allowing considerable reduction in effort compared to 3-D elasticity.

13.1 Plane Stress

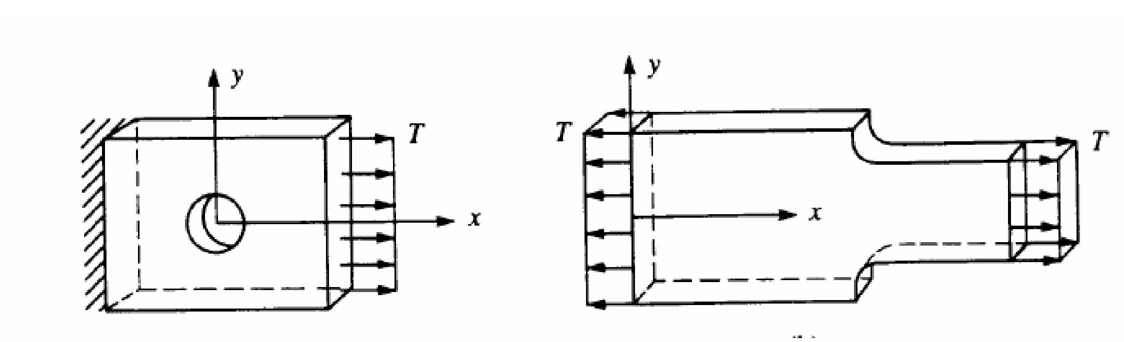


Figure 13.1: Examples of plane stress

Plane stress for a thin, plate-like body in the $x_1 - x_2$ plane satisfies the *loading conditions*:

$$\left.\begin{array}{l} f_3 = 0 \\ h_3 = 0 \end{array}\right\} \quad (13.1)$$

and *stress conditions*

$$\left.\begin{array}{l} \sigma_{13} = 0 \\ \sigma_{23} = 0 \\ \sigma_{33} = 0 \end{array}\right\} \quad (13.2)$$

everywhere. These conditions are illustrated in Fig. X. In practice, these conditions may not be exactly realized, but application of the theory may represent a reasonable approximation that allows modeling in 2-D instead of full 3-D analysis.

Introducing the reduced Voight vectors $[\boldsymbol{\varepsilon}(\mathbf{u})]$, $[\boldsymbol{\varepsilon}(\mathbf{w})]$ and $[\boldsymbol{\sigma}]$:

$$[\boldsymbol{\varepsilon}(\mathbf{u})] = \left\{ \begin{array}{c} u_{(11)} \\ u_{(22)} \\ 2u_{(12)} \end{array} \right\} \tag{13.3}$$

$$[\boldsymbol{\varepsilon}(\mathbf{w})] = \left\{ \begin{array}{c} w_{(11)} \\ w_{(22)} \\ 2w_{(12)} \end{array} \right\} \tag{13.4}$$

$$[\boldsymbol{\sigma}] = \left\{ \begin{array}{c} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{array} \right\} \tag{13.5}$$

The bilinear operator (stress version) for the 3-d case (X) is now reduced to,

$$\begin{aligned} a(\mathbf{w}, \boldsymbol{\sigma}) &= \int_\Omega [\boldsymbol{\varepsilon}(\mathbf{w})]^T [\boldsymbol{\sigma}] \, d\Omega \\ &= \int_\Omega \langle w_{(1,1)},\ w_{(2,2)},\ 2w_{(1,2)} \rangle \left\{ \begin{array}{c} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{array} \right\} d\Omega \end{aligned} \tag{13.6}$$

Likewise, the load terms () reduce to,

$$(\mathbf{w}, \mathbf{f}) = \int_\Omega \mathbf{w}^T \mathbf{f} \, d\Omega = \int_\Omega \langle w_1, w_2 \rangle \left\{ \begin{array}{c} f_1 \\ f_2 \end{array} \right\} d\Omega \tag{13.7}$$

$$(\mathbf{w}, \mathbf{h})_{\Gamma_h} = \sum_{i=1}^{n_{sd}} \int_{\Gamma_{h_i}} w_i h_i \, d\Gamma = \int_{\Gamma_{h_1}} w_1 h_1 \, d\Gamma + \int_{\Gamma_{h_2}} w_2 h_2 \, d\Gamma \tag{13.8}$$

Hooke's law for plane stress (derived from the plane stress conditions applied to the compliance matrix) is $[\boldsymbol{\sigma}] = [\mathbf{C}][\boldsymbol{\varepsilon}]$ or,

$$\left\{ \begin{array}{c} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{array} \right\} = \frac{E}{(1-\nu^2)} \left[\begin{array}{ccc} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{array} \right] \left\{ \begin{array}{c} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{array} \right\} \tag{13.9}$$

The variational equation (displacement version) for plane stress becomes,

$$a(\mathbf{w}, \mathbf{u}) = (\mathbf{w}, \mathbf{f})_\Omega + (\mathbf{w}, \mathbf{h})_{\Gamma_h} \tag{13.10}$$

with,

$$a(\mathbf{w}, \mathbf{u}) = \int_\Omega \langle w_{(1,1)},\ w_{(2,2)},\ 2w_{(1,2)} \rangle \frac{E}{(1-\nu^2)} \left[\begin{array}{ccc} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{array} \right] \left\{ \begin{array}{c} u_{(11)} \\ u_{(22)} \\ 2u_{(12)} \end{array} \right\} d\Omega \tag{13.11}$$

$$(\mathbf{w},\mathbf{f})_\Omega = \int_\Omega \langle w_1, w_2\rangle \left\{ \begin{array}{c} f_1 \\ f_2 \end{array} \right\} d\Omega \tag{13.12}$$

$$(\mathbf{w},\mathbf{h})_{\Gamma_h} = \sum_{i=1}^{2} \int_{\Gamma_{h_i}} w_i h_i \, d\Gamma = \int_{\Gamma_{h1}} w_1 h_1 d\Gamma + \int_{\Gamma_{h_2}} w_2 h_2 d\Gamma \tag{13.13}$$

In general, $\Gamma_{h_1 \neq}\Gamma_{h_2}$.

13.2 Plane Strain

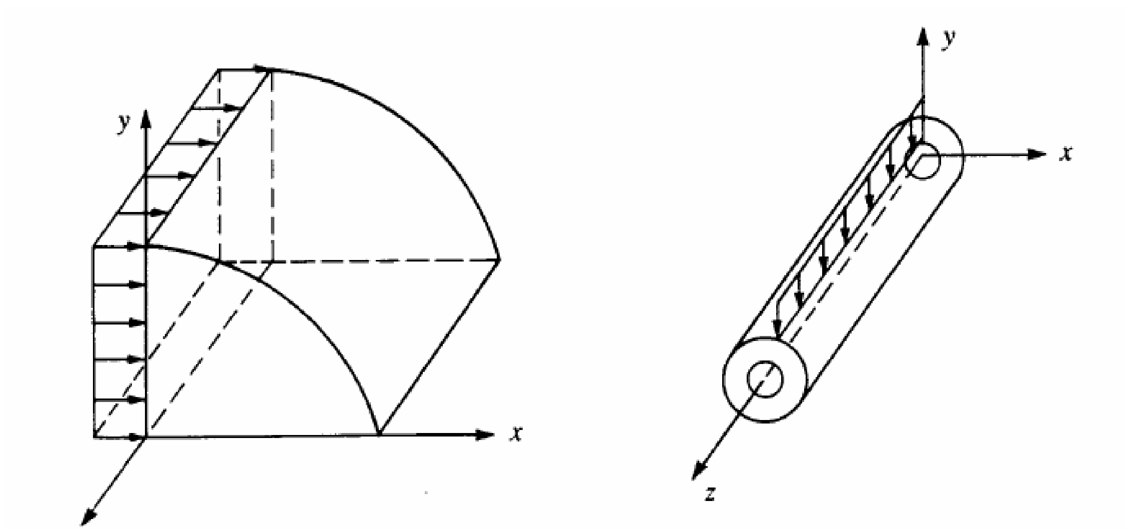


Figure 13.2: Examples of plane strain

Plane strain for a body with infinite extent in the x_3-direction satisfies the *loading conditions,*

$$\left. \begin{array}{r} f_3 = 0 \\ h_3 = 0 \end{array} \right\} \tag{13.14}$$

and *displacement conditions*:

$$\left. \begin{array}{r} u_3 = 0 \\ u_2\,(x_1, x_2) = 0 \\ u_2\,(x_1, x_2) = 0 \end{array} \right\} \tag{13.15}$$

which imply the *strain conditions*:

$$\left. \begin{array}{r} \varepsilon_{13}\,(\mathbf{u}) = u_{(1,3)} = 0 \\ \varepsilon_{23}\,(\mathbf{u}) = u_{(2,3)} = 0 \\ \varepsilon_{33}\,(\mathbf{u}) = u_{(3,3)} = 0 \end{array} \right\} \tag{13.16}$$

everywhere. These conditions are illustrated in Fig. X. In practice, these conditions may not be exactly realized, but application of the theory may represent a reasonable approximation that allows modeling in 2-D instead of full 3-D analysis.

It is remarked that the *weighting functions* must also satisfy the above kinematical constraints such that

$$\left.\begin{array}{r} w_3 = 0 \\ w_2\left(x_1, x_2\right) = 0 \\ w_2\left(x_1, x_2\right) = 0 \end{array}\right\} \tag{13.17}$$

which implies,

$$\left.\begin{array}{r} \varepsilon_{13}\left(\mathbf{w}\right) = w_{(1,3)} = 0 \\ \varepsilon_{23}\left(\mathbf{w}\right) = w_{(2,3)} = 0 \\ \varepsilon_{33}\left(\mathbf{w}\right) = w_{(3,3)} = 0 \end{array}\right\} \tag{13.18}$$

Introducing the reduced Voight vectors $[\boldsymbol{\varepsilon}\left(\mathbf{u}\right)]$, $[\boldsymbol{\varepsilon}\left(\mathbf{w}\right)]$ and $[\boldsymbol{\sigma}]$, given by (13.3), (13.4) and (13.5), into the bilinear operator for the 3-d case (X) gives exactly the same equation as for plane stress. Note, however, that differences between plane stress and strain will be apparent when constitutive equations are introduced to eliminate the stress.

Hooke's law for plane strain (derived directly from the 3-D formulation of Hooke's law) is $[\boldsymbol{\sigma}] = [\mathbf{C}]\,[\boldsymbol{\varepsilon}]$ or,

$$\left\{\begin{array}{c} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{array}\right\} = \frac{E}{(1+\nu)(1-2\nu)}\left[\begin{array}{ccc} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1}{2}(1-2\nu) \end{array}\right]\left\{\begin{array}{c} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{array}\right\} \tag{13.19}$$

The variational equation (displacement version) for plane strain becomes,

$$a\left(\mathbf{w}, \mathbf{u}\right) = \left(\mathbf{w}, \mathbf{f}\right)_{\Omega} + \left(\mathbf{w}, \mathbf{h}\right)_{\Gamma_h} \tag{13.20}$$

with,

$$a\left(\mathbf{w}, \mathbf{u}\right) = \int_{\Omega} \langle w_{(1,1)},\ w_{(2,2)},\ 2w_{(1,2)}\rangle \frac{E}{(1+\nu)(1-2\nu)}\left[\begin{array}{ccc} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1}{2}(1-2\nu) \end{array}\right]\left\{\begin{array}{c} u_{(11)} \\ u_{(22)} \\ 2u_{(12)} \end{array}\right\} d\Omega \tag{13.21}$$

$$\left(\mathbf{w}, \mathbf{f}\right)_{\Omega} = \int_{\Omega} \langle w_1, w_2\rangle \left\{\begin{array}{c} f_1 \\ f_2 \end{array}\right\} d\Omega \tag{13.22}$$

$$\left(\mathbf{w}, \mathbf{h}\right)_{\Gamma_h} = \int_{\Gamma_h} \langle w_1, w_2\rangle \left\{\begin{array}{c} h_1 \\ h_2 \end{array}\right\} d\Gamma \tag{13.23}$$

13.3 Weak form for plane stress and plane strain

Because plane stress and plane strain have almost identical forms (differing only in the elasticity matrix) we can conveniently summary the weak form for both cases as follows.

Weak form (S) : Find $u_i \in \mathcal{S}_i = \left\{u_i \mid u_i \in H^1\left(\Omega\right),\ u_i = g_i \text{ on } \Gamma_{g_i}\right\}$ such that

$$a\left(\mathbf{w}, \mathbf{u}\right) = \left(\mathbf{w}, \mathbf{f}\right) + \left(\mathbf{w}, \mathbf{h}\right)_{\Gamma_h} \tag{13.24}$$

holds for all $w_i \in \mathcal{V}_i = \{ w_i \mid w_i \in H^1(\Omega), \ w_i = 0 \text{ on } \Gamma_{g_i} \}$, and where

$$a(\mathbf{w},\mathbf{u}) = \int_\Omega [\boldsymbol{\varepsilon}(\mathbf{w})]^T [\mathbf{C}] [\boldsymbol{\varepsilon}(\mathbf{u})] \, d\Omega \tag{13.25}$$

$$(\mathbf{w},\mathbf{b})_\Omega = \int_\Omega \mathbf{w}^T \mathbf{f} \, d\Omega \tag{13.26}$$

$$(\mathbf{w},\mathbf{h})_{\Gamma_h} = \sum_{i=1}^{2} \int_{\Gamma_{h_i}} w_i h_i \, d\Gamma \tag{13.27}$$

with

$$[\mathbf{C}]_{\text{plane-stress}} = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} \tag{13.28}$$

$$[\mathbf{C}]_{\text{plane-strain}} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1}{2}(1-2\nu) \end{bmatrix} \tag{13.29}$$

13.4 Galerkin form for 2-D elastostatics

To build in the essential boundary conditions we let $\mathcal{S}_i^h \subset \mathcal{S}_i$ and $\mathcal{V}_i^h \subset \mathcal{V}_i$ and set $u_i^h = v_i^h + g_i^h$, $i = 1,2$ with $v_i^h \in \mathcal{V}_i^h$ and $g_i^h \in \mathcal{S}_i^h$. Recalling the weak form of the 2-D (plane stress and plane strain) elastostatics problem from Ch. 7, the Galerkin form can be stated as follows.

Galerkin form (G) : Find $v_i^h \in \mathcal{V}_i^h = \{ v_i \mid v_i \in H^1(\Omega), \ v_i = 0 \text{ on } \Gamma_{g_i} \}$ such that

$$a\left(\mathbf{w}^h,\mathbf{v}^h\right) = \left(\mathbf{w}^h,\mathbf{b}\right) + \left(\mathbf{w}^h,\mathbf{h}\right)_{\Gamma_h} - a\left(\mathbf{w}^h,\mathbf{g}^h\right) \tag{13.30}$$

holds for all $w_i^h \in \mathcal{V}_i$ and where

$$a\left(\mathbf{w}^h,\mathbf{v}^h\right) = \int_\Omega \left[\varepsilon\left(\mathbf{w}^h\right)\right]^T [\mathbf{C}] \left[\varepsilon\left(\mathbf{v}^h\right)\right] \, d\Omega \tag{13.31}$$

$$\left(\mathbf{w}^h,\mathbf{b}\right)_\Omega = \int_\Omega \mathbf{w}^{hT} \mathbf{b} \, d\Omega \tag{13.32}$$

$$\left(\mathbf{w}^h,\mathbf{h}\right)_{\Gamma_h} = \sum_{i=1}^{2} \int_{\Gamma_{h_i}} w_i^h h_i \, d\Gamma \tag{13.33}$$

$$a\left(\mathbf{w}^h,\mathbf{g}^h\right) = \int_\Omega \left[\varepsilon\left(\mathbf{w}^h\right)\right]^T [\mathbf{C}] \left[\varepsilon\left(\mathbf{g}^h\right)\right] \, d\Omega \tag{13.34}$$

with

$$\mathbf{C}_{\text{plane-stress}} = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} \tag{13.35}$$

$$\mathbf{C}_{\text{plane-strain}} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1}{2}(1-2\nu) \end{bmatrix} \tag{13.36}$$

13.5 Finite element approximation for plane stress/strain

Introducing the finite element approximation $\mathbf{u}^h = \mathbf{v}^h + \mathbf{g}^h$ (see Fig 2):

$$\begin{aligned} \mathbf{v}^h &= \begin{Bmatrix} v_1^h \\ v_2^h \end{Bmatrix} \\ &= \sum_{a=1}^{n_{en}} \begin{bmatrix} N_a^e(\xi,\eta) & 0 \\ 0 & N_a^e(\xi,\eta) \end{bmatrix} \begin{Bmatrix} d_{a1}^e \\ d_{a2}^e \end{Bmatrix} \\ &= \sum_{a=1}^{n_{en}} \mathbf{N}_a^e(\xi,\eta)\, \mathbf{d}_a^e \\ &= [\mathbf{N}^e(\xi,\eta)]\,[\mathbf{d}^e] \end{aligned}$$

where n_{en} is the number of nodes on element e, and (ξ,η) are a set of element natural coordinates defined on the parent domain. Similarly,

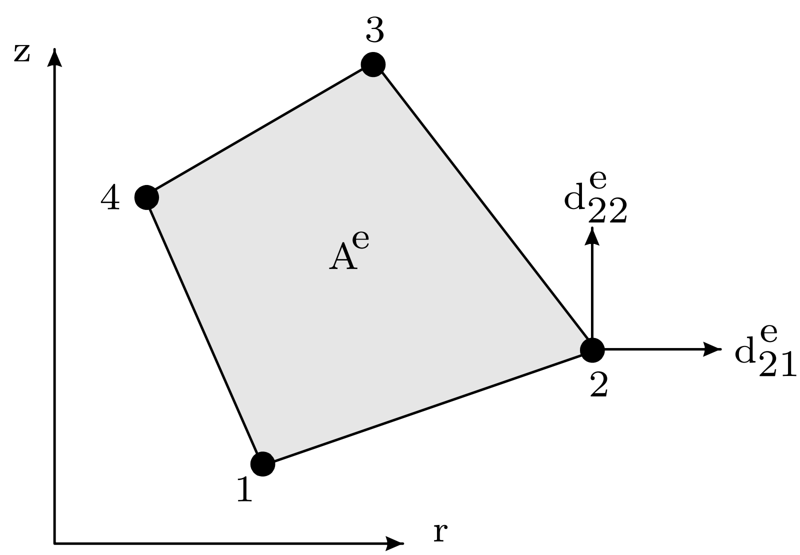


Figure 13.3: Nodal degrees of freedom for plane stress and plane strain

$$
\begin{aligned}
\mathbf{w}^h &= \left\{ \begin{array}{c} w_1^h \\ w_2^h \end{array} \right\} \\
&= \sum_{a=1}^{n_{en}} \left[\begin{array}{cc} N_a^e(\xi,\eta) & 0 \\ 0 & N_a^e(\xi,\eta) \end{array} \right] \left\{ \begin{array}{c} c_{a1}^e \\ c_{a2}^e \end{array} \right\} \\
&= \sum_{a=1}^{n_{en}} \mathbf{N}_a^e(\xi,\eta)\,\mathbf{c}_a^e \\
&= [\mathbf{N}^e(\xi,\eta)]\,[\mathbf{c}^e]
\end{aligned}
$$

The Galerkin equation for plane stress and plane strain follows directly from the variational equation (**??**), and remembering that $[\mathbf{C}]$ will be different for the two cases,

$$
\begin{aligned}
\int_\Omega \left[\boldsymbol{\varepsilon}\left(\mathbf{w}^h\right)\right]^T [\mathbf{C}] \left[\boldsymbol{\varepsilon}\left(\mathbf{v}^h\right)\right] d\Omega &= \int_\Omega \mathbf{w}^{hT}\mathbf{f}\, d\Omega + \sum_{i=1}^{2} \int_{\Gamma_{h_i}} w_i^h h_i \, d\Gamma \\
&\quad - \int_\Omega \left[\boldsymbol{\varepsilon}\left(\mathbf{w}^h\right)\right]^T [\mathbf{C}] \left[\boldsymbol{\varepsilon}\left(\mathbf{g}^h\right)\right] d\Omega \qquad (13.37)
\end{aligned}
$$

Discretization into n_{elem} finite elements such that $\Omega = \cup_{e=1}^{n_{el}} \Omega^e$ results as usual in $\int_\Omega (\cdot)\, d\Omega = \sum_{e=1}^{n_{el}} \int_{\Omega^e} (\cdot)\, d\Omega$.

13.5.1 Strain-Displacement Matrix

The Voight vector of strains $\left[\boldsymbol{\varepsilon}\left(\mathbf{u}^h\right)\right]$ over Ω^e is given by:

$$
\begin{aligned}
\left[\boldsymbol{\varepsilon}\left(\mathbf{v}^h\right)\right] &= \left\{ \begin{array}{c} v_{(11)}^h \\ v_{(22)}^h \\ 2v_{(12)}^h \end{array} \right\} = \sum_{a=1}^{n_{en}} \left[\begin{array}{cc} N_{a,1}^e & 0 \\ 0 & N_{a,2}^e \\ N_{a,2}^e & N_{a,1}^e \end{array} \right] \left\{ \begin{array}{c} d_{a1}^e \\ d_{a2}^e \end{array} \right\} \\
&= \sum_{a=1}^{n_{en}} \mathbf{B}_a^e \mathbf{d}_a^e \\
&= [\mathbf{B}^e]\,[\mathbf{d}^e]
\end{aligned}
$$

where $\mathbf{B^e}$ is called the *strain-displacement matrix*. Similarly,

$$
\left[\boldsymbol{\varepsilon}\left(\mathbf{w}^h\right)\right] = \left\{ \begin{array}{c} w_{(11)}^h \\ w_{(22)}^h \\ 2w_{(12)}^h \end{array} \right\} = [\mathbf{B}^e]\,[\mathbf{c}^e]
$$

13.5.2 Element stiffness matrix

Using the above expressions in the left-hand side of (13.37) restricted to Ω^e gives,

$$
a\left(\mathbf{w}^h, \mathbf{v}^h\right)_{\Omega^e} = \mathbf{c}^{eT} \underbrace{\left(\int_{\Omega^e} \mathbf{B}^{eT} \mathbf{C}^e \mathbf{B}^e \, d\Omega^e \right)}_{\mathbf{k}^e} \mathbf{d}^e
$$

From which we obtain,

$$\mathbf{k}^e = \int_{\Omega^e} [\mathbf{B}^e]^T [\mathbf{C}^e] [\mathbf{B}^e] \, d\Omega^e$$

It is useful to note that the 2×2 nodal partition of $\mathbf{k}^e$ is given by,

$$\mathbf{k}^e_{ab} = \int_{\Omega^e} [\mathbf{B}^e_a]^T [\mathbf{C}^e] [\mathbf{B}^e_b] \, d\Omega^e$$

13.5.3 Element body load vector

The element (body force) load vector $\mathbf{f}^e_f$ is deduced from:

$$\int_\Omega \mathbf{w}^{hT} \mathbf{f} \, d\Omega = \int_{\Omega^e} \langle w^h_1, w^h_2 \rangle \left\{ \begin{array}{c} f^e_1 \\ f^e_2 \end{array} \right\} = \mathbf{c}^{eT} \underbrace{\left(\int_{\Omega^e} \mathbf{N}^{eT} \left\{ \begin{array}{c} f^e_1 \\ f^e_2 \end{array} \right\} d\Omega \right)}_{\mathbf{f}^e_f}$$

The element body force vector $\mathbf{f}^e_f$ is thus given by

$$\mathbf{f}^e_f = \int_{\Omega^e} \mathbf{N}^{eT} \left\{ \begin{array}{c} f^e_1 \\ f^e_2 \end{array} \right\} d\Omega$$

The 2×1 nodal partition of $\mathbf{f}^e_f$ is,

$$(\mathbf{f}^e_f)_a = \int_{\Omega^e} \mathbf{N}^{eT}_a \left\{ \begin{array}{c} f^e_1 \\ f^e_2 \end{array} \right\} d\Omega$$

13.5.4 Element traction load vector

The element (traction) load vector $\mathbf{f}^e_h$ is deduced from:

$$\sum_{i=1}^{2} \int_{\Gamma_{h_i}} w^h_i h_i \, d\Gamma \quad = \quad \mathbf{c}^{eT} \underbrace{\left(\int_{\Gamma^e_{h_1}} \mathbf{N}^{eT} \left\{ \begin{array}{c} h^e_1 \\ 0 \end{array} \right\} d\Gamma + \int_{\Gamma^e_{h_2}} \mathbf{N}^{eT} \left\{ \begin{array}{c} 0 \\ h^e_2 \end{array} \right\} d\Gamma \right)}_{\mathbf{f}^e_h}$$

The element traction force vector $\mathbf{f}^e_h$ is thus given by

$$\mathbf{f}^e_h = \int_{\Gamma^e_{h_1}} \mathbf{N}^{eT} \left\{ \begin{array}{c} h^e_1 \\ 0 \end{array} \right\} d\Gamma + \int_{\Gamma^e_{h_2}} \mathbf{N}^{eT} \left\{ \begin{array}{c} 0 \\ h^e_2 \end{array} \right\} d\Gamma$$

The 2×1 nodal partition of $\mathbf{f}^e_h$ is,

$$(\mathbf{f}^e_h)_a = \int_{\Gamma^e_{h_1}} \mathbf{N}^{eT}_a \left\{ \begin{array}{c} h^e_1 \\ 0 \end{array} \right\} d\Gamma + \int_{\Gamma^e_{h_2}} \mathbf{N}^{eT}_a \left\{ \begin{array}{c} 0 \\ h^e_2 \end{array} \right\} d\Gamma$$

13.5.5 Element boundary displacement load vector

The element load vector $\mathbf{f}_g^e$ from the essential boundary conditions is deduced from (13.37) as,

$$\begin{aligned} -a\left(\mathbf{w}^h, \mathbf{g}^h\right)_{\Omega^e} &= -\int_{\Omega^e} \left[\boldsymbol{\varepsilon}\left(\mathbf{w}^h\right)\right]^T [\mathbf{C}] \left[\boldsymbol{\varepsilon}\left(\mathbf{g}^h\right)\right] \, d\Omega \\ &= \mathbf{c}^{eT} \underbrace{\left(-\int_{\Omega^e} \mathbf{B}^{eT} \mathbf{C}^e \mathbf{B}^e \, d\Omega^e\right) \mathbf{g}^e}_{\mathbf{f}_g^e} \end{aligned}$$

The element traction force vector $\mathbf{f}_h^e$ is thus given by

$$\mathbf{f}_g^e = -\mathbf{k}^e \mathbf{g}^e$$

The element arrays defined above may be assembled using the direct stiffness method giving rise to the following matrix equation:

$$\mathbf{K}\mathbf{d} = \mathbf{F}_f + \mathbf{F}_h + \mathbf{F}_g \tag{13.38}$$

where

$$\mathbf{K} = \mathop{\mathbf{A}}_{e=1}^{n_{elem}} \mathbf{k}^e \tag{13.39}$$

and

$$\mathbf{F}_f = \mathop{\mathbf{A}}_{e=1}^{n_{elem}} \mathbf{f}_f^e \tag{13.40}$$

$$\mathbf{F}_h = \mathop{\mathbf{A}}_{e=1}^{n_{elem}} \mathbf{f}_h^e \tag{13.41}$$

$$\mathbf{F}_g = \mathop{\mathbf{A}}_{e=1}^{n_{elem}} \mathbf{f}_g^e \tag{13.42}$$

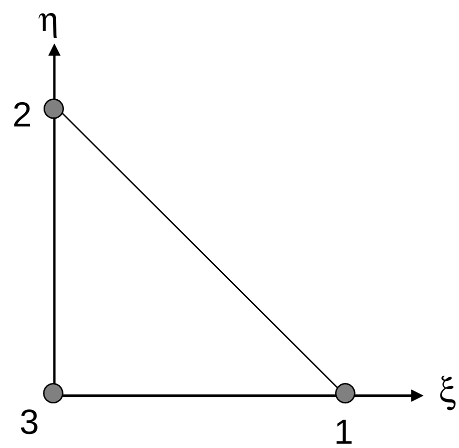
Each of the arrays above is assembled from element contributions mapped to the global array using the local destination array $\mathbf{L}$.

The following two sections, the details of the above arrays are reviewed for the 3-node triangle and the 4-node quadrilateral elements.

13.6 The 3-node triangle in plane stress/strain

- Natural coordinates on parent triangle

$$\begin{aligned} N_1^e(\xi,\eta) &= \xi \\ N_2^e(\xi,\eta) &= \eta \\ N_3^e(\xi,\eta) &= 1-\xi-\eta \end{aligned}$$



- Bilinear operator

$$
\begin{aligned}
a\left(\mathbf{w},\boldsymbol{\sigma}\right) &= \int_{\Omega} \left[\boldsymbol{\varepsilon}\left(\mathbf{w}\right)\right]^{T} \left[\mathbf{C}\right] \left[\boldsymbol{\varepsilon}\left(\mathbf{u}\right)\right]\, d\Omega \\
&= \int_{\Omega} \left\langle \begin{array}{ccc} w_{(11)} & w_{(22)} & 2w_{(12)} \end{array} \right\rangle \left[\mathbf{C}\right] \left\{ \begin{array}{c} u_{(11)} \\ u_{(22)} \\ 2u_{(12)} \end{array} \right\} d\Omega
\end{aligned}
$$

- Finite element approximation

$$
\begin{aligned}
\left. w_i^h \right|_{\Omega^e} &= \sum_{a=1}^{3} N_a^e c_{ai} \\
\left. u_j^h \right|_{\Omega^e} &= \sum_{b=1}^{3} N_b^e d_{bj}
\end{aligned}
$$

- Strain-displacement matrix $\mathbf{B}_a^e$

$$
\left[\boldsymbol{\varepsilon}\left(\mathbf{u}^h\right)\right] = \left\{ \begin{array}{c} u^h_{(11)} \\ u^h_{(22)} \\ 2u^h_{(12)} \end{array} \right\} = \sum_{b=1}^{3} \left[\begin{array}{cc} N^e_{b,1} & 0 \\ 0 & N^e_{b,2} \\ N^e_{b,2} & N^e_{b,1} \end{array} \right] \left\{ \begin{array}{c} d^e_{b1} \\ d^e_{b2} \end{array} \right\}
$$

- Strain-displacement matrix (constant $\rightarrow$ *constant strain triangle*)

$$
\begin{aligned}
\mathbf{B}_b^e &= \left[\begin{array}{cc} N^e_{b,1} & 0 \\ 0 & N^e_{b,2} \\ N^e_{b,2} & N^e_{b,1} \end{array} \right] \\
\mathbf{B}^e &= \left[\begin{array}{ccc} \mathbf{B}_1^e & \mathbf{B}_2^e & \mathbf{B}_3^e \end{array} \right] = \left[\begin{array}{cccccc} N^e_{1,1} & 0 & N^e_{2,1} & 0 & N^e_{3,1} & 0 \\ 0 & N^e_{1,2} & 0 & N^e_{2,2} & 0 & N^e_{3,2} \\ N^e_{1,2} & N^e_{1,1} & N^e_{2,2} & N^e_{2,1} & N^e_{3,2} & N^e_{3,1} \end{array} \right]
\end{aligned}
$$

- Stiffness matrix

$$\begin{aligned}
\mathbf{k}^e &= \int_{\Omega^e} \mathbf{B}^{eT}\mathbf{C}\mathbf{B}^e \, d\Omega \\
\mathbf{k}^e_{ab} &= \int_{\Omega^e} \mathbf{B}^{eT}_a\mathbf{C}\mathbf{B}^e_b \, d\Omega
\end{aligned}$$

- Isoparametric map $\mathbf{x} : \triangle \longrightarrow \Omega^e \subset \mathbb{R}^2$

$$\begin{aligned}
x(\xi,\eta) &= \sum_{a=1}^{3} N^e_a(\xi,\eta)\, x^e_a = \left\langle\; N^e_1 \quad N^e_2 \quad N^e_3 \;\right\rangle \left\{ \begin{array}{c} x^e_1 \\ x^e_2 \\ x^e_3 \end{array} \right\} = \mathbf{N}^e\mathbf{x}^e \\
y(\xi,\eta) &= \sum_{a=1}^{3} N^e_a(\xi,\eta)\, y^e_a = \left\langle\; N^e_1 \quad N^e_2 \quad N^e_3 \;\right\rangle \left\{ \begin{array}{c} y^e_1 \\ y^e_2 \\ y^e_3 \end{array} \right\} = \mathbf{N}^e\mathbf{y}^e
\end{aligned}$$

- Chain rule and Jacobian matrix $\mathbf{J}^e$

$$\left\{ \begin{array}{c} \frac{\partial N^e_a}{\partial \xi} \\ \frac{\partial N^e_a}{\partial \eta} \end{array} \right\} = \underbrace{\left[\begin{array}{cc} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{array} \right]}_{[\mathbf{J}^e]^T} \left\{ \begin{array}{c} \frac{\partial N^e_a}{\partial x} \\ \frac{\partial N^e_a}{\partial y} \end{array} \right\}$$

- Then

$$\begin{aligned}
[\mathbf{J}^e]^T &= \left[\begin{array}{cc} \sum_{a=1}^{3} N^e_{a,\xi}x^e_a & \sum_{a=1}^{3} N^e_{a,\xi}y^e_a \\ \sum_{a=1}^{3} N^e_{a,\eta}x^e_a & \sum_{a=1}^{3} N^e_{a,\eta}y^e_a \end{array} \right] \\
&= \left[\begin{array}{c} \mathbf{N}^e_{,\xi} \\ \mathbf{N}^e_{,\eta} \end{array} \right] \left[\; \mathbf{x}^e \quad \mathbf{y}^e \;\right]
\end{aligned}$$

- Elements of $\mathbf{J}^e$

$$\begin{aligned}
[\mathbf{J}^e]^T &= \left[\begin{array}{ccc} 1 & 0 & -1 \\ 0 & 1 & -1 \end{array} \right] \left[\; \mathbf{x}^e \quad \mathbf{y}^e \;\right] \\
&= \left[\begin{array}{cc} x^e_1 - x^e_3 & y^e_1 - y^e_3 \\ x^e_2 - x^e_3 & y^e_2 - y^e_3 \end{array} \right]
\end{aligned}$$

- Jacobian determinant j^e

$$j^e = \det \mathbf{J}^e = 2A^e$$

- Inverse Jacobian

$$\begin{aligned}
\left\{ \begin{array}{c} \frac{\partial N^e_a}{\partial x} \\ \frac{\partial N^e_a}{\partial y} \end{array} \right\} &= [\mathbf{J}^e]^{-T} \left\{ \begin{array}{c} \frac{\partial N^e_a}{\partial \xi} \\ \frac{\partial N^e_a}{\partial \eta} \end{array} \right\} \\
&= \frac{1}{2A^e} \left[\begin{array}{cc} y^e_2 - y^e_3 & -\left(y^e_1 - y^e_3\right) \\ -\left(x^e_2 - x^e_3\right) & x^e_1 - x^e_3 \end{array} \right] \left\{ \begin{array}{c} \frac{\partial N^e_a}{\partial \xi} \\ \frac{\partial N^e_a}{\partial \eta} \end{array} \right\}
\end{aligned}$$

$$\left\{ \begin{array}{c} \frac{\partial N_1^e}{\partial x} \\ \frac{\partial N_1^e}{\partial y} \end{array} \right\} = \frac{1}{2A^e} \left\{ \begin{array}{c} y_2^e - y_3^e \\ x_3^e - x_2^e \end{array} \right\}$$
$$\left\{ \begin{array}{c} \frac{\partial N_2^e}{\partial x} \\ \frac{\partial N_2^e}{\partial y} \end{array} \right\} = \frac{1}{2A^e} \left\{ \begin{array}{c} y_3^e - y_1^e \\ x_1^e - x_3^e \end{array} \right\}$$
$$\left\{ \begin{array}{c} \frac{\partial N_3^e}{\partial x} \\ \frac{\partial N_3^e}{\partial y} \end{array} \right\} = \frac{1}{2A^e} \left\{ \begin{array}{c} y_1^e - y_2^e \\ x_2^e - x_1^e \end{array} \right\}$$

- Stiffness matrix – consider *plane stress* and $\mathbf{k}_{12}^e$

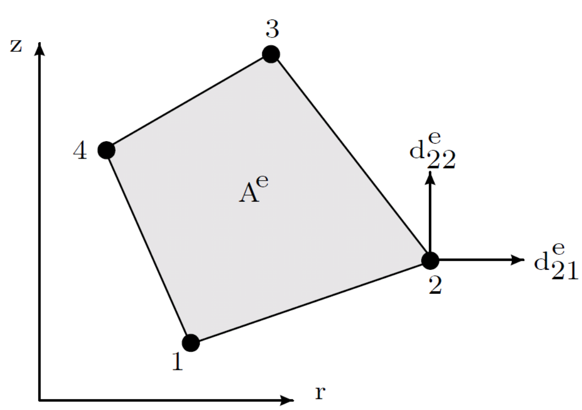
$$\mathbf{B}_1^e = \begin{bmatrix} N_{1,1}^e & 0 \\ 0 & N_{1,2}^e \\ N_{1,2}^e & N_{1,1}^e \end{bmatrix} = \frac{1}{2A^e} \begin{bmatrix} y_2^e - y_3^e & 0 \\ 0 & x_3^e - x_2^e \\ x_3^e - x_2^e & y_2^e - y_3^e \end{bmatrix}$$
$$\mathbf{B}_2^e = \begin{bmatrix} N_{2,1}^e & 0 \\ 0 & N_{2,2}^e \\ N_{2,2}^e & N_{2,1}^e \end{bmatrix} = \frac{1}{2A^e} \begin{bmatrix} y_3^e - y_1^e & 0 \\ 0 & x_1^e - x_3^e \\ x_1^e - x_3^e & y_3^e - y_1^e \end{bmatrix}$$

$$\mathbf{k}_{12}^e = \int_{\Omega^e} \mathbf{B}_1^{eT} \mathbf{C} \mathbf{B}_2^e \, d\Omega = A^e \mathbf{B}_1^{eT} \mathbf{C} \mathbf{B}_2^e$$

$$= \frac{1}{4A^e} \frac{E}{(1-\nu^2)} \begin{bmatrix} y_2^e - y_3^e & 0 \\ 0 & x_3^e - x_2^e \\ x_3^e - x_2^e & y_2^e - y_3^e \end{bmatrix}^T \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} y_3^e - y_1^e & 0 \\ 0 & x_1^e - x_3^e \\ x_1^e - x_3^e & y_3^e - y_1^e \end{bmatrix}$$

13.7 The 4-node quadrilateral in plane stress/strain

- Consider 4-node quadrilateral element $N_a^e(\xi, \eta) = \frac{1}{4}(1 + \xi_a \xi)(1 + \eta_a \eta)$



- Isoparametric map $\mathbf{x} : \square \longrightarrow \Omega^e \subset \mathbb{R}^2$

$$
\begin{aligned}
x(\xi,\eta) &= \sum_{a=1}^{3} N_a^e(\xi,\eta)\, x_a^e = \left\langle \begin{array}{cccc} N_1^e & N_2^e & N_3^e & N_4^e \end{array} \right\rangle \left\{ \begin{array}{c} x_1^e \\ x_2^e \\ x_3^e \\ x_4^e \end{array} \right\} = \mathbf{N}^e \mathbf{x}^e \\
y(\xi,\eta) &= \sum_{a=1}^{3} N_a^e(\xi,\eta)\, y_a^e = \mathbf{N}^e \mathbf{y}^e
\end{aligned}
$$

- Jacobian matrix $\mathbf{J}^e$

$$
\begin{aligned}
[\mathbf{J}^e]^T &= \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \\
&= \begin{bmatrix} \sum_{a=1}^4 N_{a,\xi}^e x_a^e & \sum_{a=1}^4 N_{a,\xi}^e y_a^e \\ \sum_{a=1}^4 N_{a,\eta}^e x_a^e & \sum_{a=1}^4 N_{a,\eta}^e y_a^e \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{N}_{,\xi}^e \\ \mathbf{N}_{,\eta}^e \end{bmatrix} \begin{bmatrix} \mathbf{x}^e & \mathbf{y}^e \end{bmatrix}
\end{aligned}
$$

$$
= \frac{1}{4} \begin{bmatrix} \xi_1(1+\eta_1\eta) & \xi_2(1+\eta_2\eta) & \xi_3(1+\eta_3\eta) & \xi_4(1+\eta_4\eta) \\ \eta_1(1+\xi_1\xi) & \eta_2(1+\xi_2\xi) & \eta_3(1+\xi_3\xi) & \eta_4(1+\xi_4\xi) \end{bmatrix} \begin{bmatrix} x_1^e & y_1^e \\ x_2^e & y_2^e \\ x_3^e & y_3^e \\ x_4^e & y_4^e \end{bmatrix}
$$

- Consider *plane stress* and $\mathbf{k}_{12}^e$

$$
\mathbf{B}_1^e = \begin{bmatrix} N_{1,1}^e & 0 \\ 0 & N_{1,2}^e \\ N_{1,2}^e & N_{1,1}^e \end{bmatrix} \qquad \mathbf{B}_2^e = \begin{bmatrix} N_{2,1}^e & 0 \\ 0 & N_{2,2}^e \\ N_{2,2}^e & N_{2,1}^e \end{bmatrix}
$$

$$
\begin{aligned}
\mathbf{k}_{12}^e &= \int_{\Omega^e} \mathbf{B}_1^{eT} \mathbf{C} \mathbf{B}_2^e \, d\Omega \\
&= \int_{\square} \mathbf{B}_1^{eT}(\xi,\eta)\, \mathbf{C} \mathbf{B}_2^e(\xi,\eta)\;\; j^e(\xi,\eta)\;\; d\xi d\eta \\
& \sum_{\ell=1}^{n_{int}} \mathbf{B}_1^{eT}(\xi_\ell,\eta_\ell)\, \mathbf{C} \mathbf{B}_2^e(\boldsymbol{\xi}_\ell,\boldsymbol{\eta}_\ell) j^e(\boldsymbol{\xi}_\ell,\boldsymbol{\eta}_\ell) W_\ell
\end{aligned}
$$