Inverse Learning Notes

FEM notes: $-\nabla\cdot(c\otimes\nabla d) + ad = g$, $d=(u,v)$. $\nabla d = \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \hline \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{pmatrix}$

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{xy} \\ \sigma_{yy} \end{pmatrix} = \overset{\mathbb{C}}{\begin{bmatrix} \alpha & 0 & \beta \\ 0 & \omega & 0 \\ \beta & 0 & \gamma \end{bmatrix}} \begin{pmatrix} \tilde{e}_{xx} \\ \tilde{e}_{xy} \\ \tilde{e}_{yy} \end{pmatrix} = \mathbb{C}\left(\overset{A}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \nabla d - \overset{f(x_m)}{\begin{pmatrix} f_x(x_m) \\ 0 \\ f_y(x_m) \end{pmatrix}} \right)$$

$x_{\xi x}$ $x_{\xi xy}$

$$\mathbb{C}A = \begin{bmatrix} \alpha & 0 & 0 & \beta \\ 0 & \frac{\omega}{2} & \frac{\omega}{2} & 0 \\ \beta & 0 & 0 & \gamma \end{bmatrix}, \qquad \mathbb{C}f(x) = \begin{bmatrix} \alpha f_x(x_m) + \beta f_y(x_m) \\ 0 \\ \beta f_x(x_m) + \gamma f_y(x_m) \end{bmatrix}$$

① $-\frac{\partial}{\partial x_k}\left(C_{ijkl}\frac{\partial d_j}{\partial x_l}\right) + a_{ij}d_j = f_j$, $N=2$.

$r_1 - r_2$: $\nabla\cdot(\mathbb{C}A\nabla d - \mathbb{C}f(x_m)) = 0 \Rightarrow \nabla\cdot(\mathbb{C}A\nabla d) = \nabla\cdot\mathbb{C}f(x)$ 1-2

$r_2 - r_3$: $\nabla\cdot(\mathbb{C}A\nabla d - \mathbb{C}f(x_m)) = 0 \Rightarrow \nabla\cdot(\mathbb{C}A\nabla d) = \nabla\cdot\mathbb{C}f(x)$ 2-3.

$$c = \begin{bmatrix} \alpha & 0 & 0 & \beta \\ 0 & \frac{\omega}{2} & \frac{\omega}{2} & 0 \\ \hline 0 & \frac{\omega}{2} & \frac{\omega}{2} & 0 \\ \beta & 0 & 0 & \gamma \end{bmatrix}$$

$$\nabla d = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \hline \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{bmatrix}, \quad c = \begin{bmatrix} \alpha & 0 & 0 & \beta \\ 0 & \frac{\omega}{2} & \frac{\omega}{2} & 0 \\ 0 & \frac{\omega}{2} & \frac{\omega}{2} & 0 \\ \beta & 0 & 0 & \gamma \end{bmatrix}, \quad a = 0, \quad g = \begin{bmatrix} (\alpha f_x'(x_m) + \beta f_y'(x_m))\nabla_x x_m \\ (\beta f_x'(x_m) + \gamma f_y'(x_m))\nabla_y x_m \end{bmatrix}$$

Stage 0 : Assume $\nabla X_m = (0,0)$, then. the PDE is reduced to:

$$
- \nabla \cdot \left(\overset{C}{\begin{bmatrix} \alpha & 0 & 0 & \beta \\ 0 & \frac{w}{2} & \frac{w}{2} & 0 \\ 0 & \frac{w}{2} & \frac{w}{2} & 0 \\ \beta & 0 & 0 & \gamma \end{bmatrix}} \nabla d \right) = 0
$$

① B.C. $\hat{n} \cdot (C \nabla d) = 0$

② B.C.: $\hat{n} \cdot (C \nabla d) = 1$,

Forward Model:

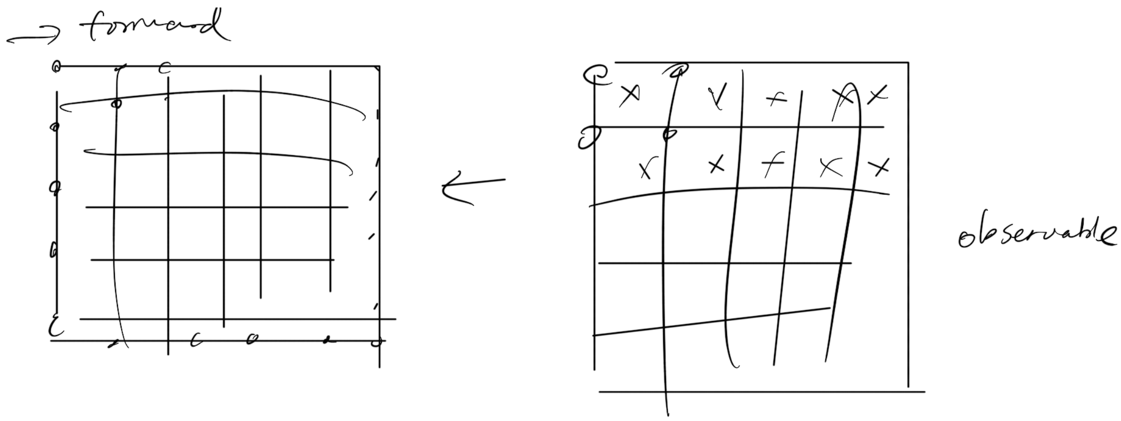
Given C, x, we can find $d = (u, v)$, ∇d. from FEM

Now given x, d, can we find C = ?

1°. Numerical gradient descent, initialize $C = C_0$, solve $e_{xx}(C_0)$, $e_{xy}(C_0)$, $e_{yy}(C_0)$

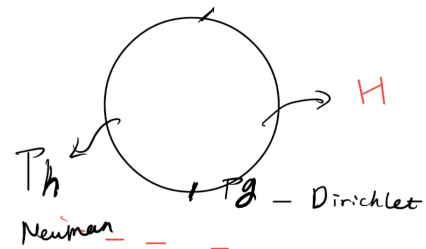
$$
F = \sum_{i,j} \| e_{xy} - e_{xy}(C_0) \|^2 + \| e_{xx} - e_{xx}(C_0) \|^2 + \| e_{yy} - e_{yy}(C_0) \|^2
$$

$$
\left. \frac{\partial F}{\partial C_0} \right|_{C_0} = \frac{F[C + \delta C] - F[C]}{\delta C}
$$



forward
observable

$u = B * Kc \backslash Fc + Ud$



$Kd = F_f + F_h + F_g$

$$m\frac{\partial^2 u}{\partial t^2} + d\frac{\partial u}{\partial t} - \nabla(c\cdot\nabla u) + au = f \qquad \hat{n}\cdot(c\nabla u) + qu = g$$

$M = 0$ $\quad K \quad | \quad A = 0 \quad F \qquad Q \quad G, \; R$

$$\mathbb{K}(\mathbb{C})\cdot d = F + G. \qquad Hu = R.$$

"1" observable

$$\min_{\mathbb{C}} f = \sum_{ij} \| \hat{e}_{xx} - e_{xx} \|^2 + \| \hat{e}_{xy} - e_{xy} \|^2 + \| \hat{e}_{yy} - e_{yy} \|^2$$

$$\text{s.t.} \quad e_{xx} = \frac{\partial u}{\partial x}, \quad e_{yy} = \frac{\partial v}{\partial y}, \quad e_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\nabla \cdot \sigma = 0, \quad \hat{n} \cdot \sigma = 0$$

$$\Leftrightarrow \quad \min_{\alpha,\beta,\gamma,\omega} f = \sum \| \hat{e}_{xx} - e_{xx} \|^2 + \| \hat{e}_{yy} - e_{yy} \|^2 + \| \hat{e}_{xy} - e_{xy} \|^2$$

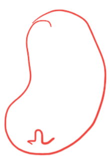
$$\text{s.t.} \quad \begin{bmatrix} e_{xx} \\ e_{xy} \\ e_{yy} \end{bmatrix} = A d$$

$$\mathbb{K}(\alpha,\beta,\gamma,\omega) \cdot \nabla d = F(x_m), \quad x_m \text{ Given.}$$

$$\Leftrightarrow \quad \min_{\alpha,\beta,\gamma,\omega} f = \sum \| \hat{e} - A \nabla d \|_2^2$$

$$\text{s.t.} \quad \mathbb{K}(\alpha,\beta,\gamma,\omega) \cdot \nabla d = F(x_m).$$

$$\mathbb{C} = \begin{bmatrix} \alpha & 0 & \beta \\ \beta & 0 & \omega \\ 0 & \gamma & 0 \end{bmatrix} \qquad C = \begin{bmatrix} \alpha & 0 & 0 & \beta \\ 0 & \frac{\omega}{2} & \frac{\omega}{2} & 0 \\ \beta & 0 & 0 & \gamma \end{bmatrix}$$



Convex problem.

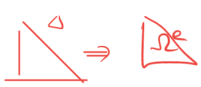
$$\text{Here:} \quad \nabla(C \nabla d) = \underset{G_\Omega}{g}, \quad \hat{n}(C \nabla d) = \underset{G_{\partial\Omega}}{g}, \quad g = (\nabla x_m)^T \begin{pmatrix} \alpha f_1'(x_m) + \beta f_2'(x_m) \\ \beta f_1'(x_m) + \gamma f_2'(x_m) \end{pmatrix}$$

$$\Rightarrow \quad \mathbb{K}(\alpha,\beta,\gamma,\omega) \nabla d = G_\Omega + G_{\partial\Omega}$$

$$\mathbb{K} = \mathop{\Large A}_{e=1}^{n_{elem}} K^e, \quad K^e = \int_{\Omega^e} [B_d^e]^T [C^e] [B_d^e] \, d\Omega^e = \int_{\Omega^e} [B_d^e] \, \mathbb{C} \, [B_d^e] \, d\Omega^e$$

for 3-node triangle:

$$\varepsilon(d^h) = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{bmatrix} = \sum_{b=1}^{3} \begin{bmatrix} N_{b,1}^e & 0 \\ 0 & N_{b,2}^e \\ N_{b,2}^e & N_{b,1}^e \end{bmatrix} \begin{bmatrix} d_{b_1}^e \\ d_{b_2}^e \end{bmatrix}$$



$$\varepsilon = \sum_{b=1}^{3} \begin{bmatrix} N_{b,1}^e & 0 \\ 0 & N_{b,2}^e \\ N_{b,2}^e & N_{b,1}^e \end{bmatrix} \begin{pmatrix} u_b^e \\ v_b^e \end{pmatrix}$$

here

$$\frac{\partial N_b^e}{\partial(\xi,\eta)} = [J^e]^T \frac{\partial N_b^e}{\partial(x,y)}$$

$$\begin{cases} N_1^e(\xi,\eta) = \xi \\ N_2^e(\xi,\eta) = \eta \\ N_3^e(\xi,\eta) = 1 - \xi - \eta \end{cases}$$

let: $\begin{cases} x = x(\xi,\eta) \\ y = y(\xi,\eta) \end{cases}$ then: $\dfrac{\partial N_b^e}{\partial(\xi,\eta)} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{bmatrix} \frac{\partial N_b^e}{\partial x} \\ \frac{\partial N_b^e}{\partial y} \end{bmatrix}$

$$[J^e] = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1^e & y_1^e \\ x_2^e & y_2^e \\ x_3^e & y_3^e \end{bmatrix} = \begin{bmatrix} x_1^e - x_3^e & y_1^e - y_3^e \\ x_2^e - x_3^e & y_2^e - y_3^e \end{bmatrix}$$

A^e Area of element

$$\det(J^e) = 2A^e \qquad \begin{bmatrix} \frac{\partial N_b^e}{\partial x} \\ \frac{\partial N_b^e}{\partial y} \end{bmatrix} = \frac{1}{A^e} \begin{bmatrix} y_2^e - y_3^e & -(y_1^e - y_3^e) \\ -(x_2^e - x_3^e) & x_1^e - x_3^e \end{bmatrix} \begin{bmatrix} \frac{\partial N_b^e}{\partial \xi} \\ \frac{\partial N_b^e}{\partial \eta} \end{bmatrix}$$

$$\mathbb{K}^e = \int_{\Omega^e} B^{eT} \mathbb{C} B \, d\Omega^e = \frac{\partial N^e}{\partial(x^e, y^e)}^T \mathbb{C} \frac{\partial N^e}{\partial(x^e, y^e)} = \frac{1}{4A^e} [\,*\,]^T \mathbb{C} [\,*\,]$$

$$[\,*\,] = \begin{bmatrix} y_3^e - y_1^e & 0 \\ 0 & x_3^e - x_2^e \\ x_3^e - x_2^e & y_2^e - y_3^e \end{bmatrix} \sim \text{function of position.}$$

$\Rightarrow$ Each entry is only at most linear functions of $\mathbb{C}_{ijkl}$, a.k.a. $\alpha, \beta, \gamma, \omega$.

$$f = \min_{\alpha,\beta,\gamma,\omega / g(x_m)} \sum_{i,j} \| \hat{e} - Bd \|_2^2$$ B is a matrix independent of $\alpha, \beta, \gamma, \omega$

$$\mathbb{K}(\alpha,\beta,\gamma,\omega)\, d = G_\Omega + G_{\partial\Omega} = G$$

$$\mathbb{K}(\alpha,\beta,\gamma,\omega) \succ 0, \quad G = G(x,y)$$ only a function of position.

1° if we parameterize $g(x_m) = \sum_{n=1}^{10} a_n L_n(x_m)$, then we have:

Given $\alpha, \beta, \gamma, \omega$, $\mathbb{K}$ is known, $\Rightarrow d = \mathbb{K}^{-1} G$.

$$G = \mathop{\mathsf{A}}_{e=1}^{nelem} G^e_{\partial\Omega} + \mathop{\mathsf{A}}_{e=1}^{nelem} G^e_\Omega \qquad N^e = \{\underbrace{N_1^e(x,y)}_{\text{coln vector}}, N_2^e(x,y), N_3^e(x,y)\}$$

$$G^e_{\partial\Omega} = \int_{\partial\Omega^e} N^{eT} \begin{bmatrix} g_1^e \\ g_2^e \end{bmatrix} d\Omega^e = \int_{\partial\Omega^e} N^{eT} \begin{bmatrix} \sum_n a_n L_n(x_m) \\ \sum_n b_n L_n(x_m) \end{bmatrix} d\Omega^e \quad x_m = x_m(x,y) \text{ known.}$$

$$= \sum_n \begin{bmatrix} a_n \\ b_n \end{bmatrix} \int_{\partial\Omega^e} N^{eT} L_n(x_m)\, d\Omega^e$$

Similarly: $$G^e_\Omega = \int_{\Omega^e} N^{eT} \begin{bmatrix} g_1^e \\ g_2^e \end{bmatrix} d\Omega^e = \sum_n \begin{bmatrix} a_n \\ b_n \end{bmatrix} \underbrace{\int_{\Omega^e} N^{eT} L_n(x_m)\, d\Omega^e}_{D_n}$$

$$= \begin{bmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{bmatrix} \underbrace{\begin{bmatrix} D_1 \\ \vdots \\ D_n \end{bmatrix}}_{D}, \quad \text{Let } a = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}, b = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

$\Rightarrow$ $$\mathbb{K}(\alpha,\beta,\gamma,\omega)\, d = (a\ b)^T$$

$$\left.\begin{aligned} &\min_{a,b} \sum_{i,j} \| \hat{e} - Bd \|_2^2 \\ &\text{s.t. } d = \mathbb{K}^{-1} (a\ b)^T D = \mathbb{K}^{-1} D^T (a\ b) \end{aligned}\right\} \Rightarrow \min_{a,b} \sum_{i,j} \| \hat{e} - B \mathbb{K}^{-1} D^T (a, b) \|_2^2$$

non constrainted convex.

2° if we know don't know $\alpha, \beta, \gamma, \omega$. but know G. harder problem:

$$\min_{\alpha,\beta,\omega,\gamma} \|\hat{e} - Bd\|_2^2 \qquad d \in \mathbb{R}^{2n}, \quad n \text{ points}$$

$$\text{s.t. } \mathbb{K}(\alpha,\beta,\gamma,\omega)\, d = G$$

$$d = \mathbb{K}(\alpha,\beta,\gamma,\omega)^{-1} G. \qquad \mathbb{K}(\alpha,\beta,\gamma,\omega) \succ 0 \qquad \mathbb{K} \in \mathbb{R}^{2n\times 2n}$$

ple case: $K^e = \int_{\Omega^e} [B]_e^T \mathbb{C} [B]_e d\Omega^e \quad \mathbb{K} = \mathop{A}\limits_{e=1}^{n_{elem}}$, $\mathbb{K}(\alpha,\beta,\gamma,\omega)$ is linear in $\alpha,\beta,\gamma,\omega$, & $\mathbb{K} \succ 0$

* Convexity Analysis:

$$d = \mathbb{K}^{-1}(\alpha,\beta,\gamma,\omega)\, G$$

Assume 2 parameter sets: $\alpha_1, \beta_1, \gamma_1, \omega_1 \downarrow d_1$ & $\alpha_2, \beta_2, \gamma_2, \omega_2 \downarrow d_2$

Then $f_1 = \|\hat{e} - Bd_1\|_2^2$. $f_2 = \|\hat{e} - Bd_2\|^2$. $f = f(d)$ f is convex in d.

Let $\tilde{f} = \lambda f_1 + (1-\lambda) f_2$, with $\tilde{d}$. then:

$$\lambda f(d_1) + (1-\lambda) f(d_2) \geq f(\tilde{d}). \quad \tilde{d} = \lambda d_1 + (1-\lambda) d_2. \quad \forall \lambda.$$

$$\tilde{d} = \mathbb{K}(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\omega})^{-1} G = [\lambda \mathbb{K}_1^{-1} + (1-\lambda) \mathbb{K}_2^{-1}] G \geq \mathbb{K}^{-1}_{\lambda c_1 + (1-\lambda) c_2} G$$

$$d_1 = \mathbb{K}(\alpha_1, \beta_1, \gamma_1, \omega_1)^{-1} G = \mathbb{K}_{c_1}^{-1} G$$

$$d_2 = \mathbb{K}(\alpha_2, \beta_2, \gamma_2, \omega_2)^{-1} G = \mathbb{K}_{c_2}^{-1} G$$

We want to know when. $\lambda f(\underbrace{\alpha_1, \ldots, \omega_1}_{c_1}) + (1-\lambda) f(\underbrace{\alpha_2, \ldots, \omega_2}_{c_2}) \overset{?}{\geq} f(\lambda c_1 + (1-\lambda) c_2)$, $\forall \lambda$.

$$\mathbb{K}_{\cancel{c}} (\lambda d_1 + (1-\lambda) d_2) = G, \quad \mathbb{K}_{c_1}(\lambda d_1 + (1-\lambda) d_2) = \lambda G + \mathbb{K}_{c_1}(1-\lambda) d_2$$

$$\mathbb{K}_{c_2}(\lambda d_1 + (1-\lambda) d_2) = \mathbb{K}_{c_2} \lambda d_1 + (1-\lambda) G$$

$$\mathbb{K}_{c_1 + c_2}(\lambda d_1 + (1-\lambda) d_2) = G + \mathbb{K}_{c_1}(1-\lambda) d_2 + \mathbb{K}_{c_2} \lambda d_1$$