

Inverse Learning Notes

FEM notes: $-\nabla \cdot (C \otimes \nabla d) + ad = g$, $d = (u, v)$, $\nabla d = \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{pmatrix}$

$$\begin{pmatrix} G_{xx} \\ G_{xy} \\ G_{yy} \end{pmatrix} = C \begin{pmatrix} \alpha & 0 & \beta \\ 0 & w & 0 \\ \beta & 0 & r \end{pmatrix} \begin{pmatrix} \tilde{e}_{xx} \\ \tilde{e}_{xy} \\ \tilde{e}_{yy} \end{pmatrix} = C \left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \nabla d - \begin{pmatrix} f_x(x_m) \\ f_y(x_m) \\ 0 \\ f_y(x_m) \end{pmatrix} \right)$$

$$CA = \begin{pmatrix} \alpha & 0 & 0 & \beta \\ 0 & \frac{w}{2} & \frac{w}{2} & 0 \\ \beta & 0 & 0 & r \end{pmatrix}, \quad Cf(x) = \begin{pmatrix} \alpha f_x(x_m) + \beta f_y(x_m) \\ 0 \\ \beta f_x(x_m) + r f_y(x_m) \end{pmatrix}$$

$$\textcircled{1} \quad - \frac{\partial}{\partial x_k} \left(C_{ijk} \frac{\partial d_j}{\partial x_l} \right) + a_{ij} d_j = f_i, \quad N=2.$$

$$r_1-r_2: \quad \nabla \cdot (CA \nabla d - Cf(x)) = 0 \Rightarrow \nabla \cdot (CA \nabla d) = \nabla \cdot Cf(x), \quad 1-2$$

$$r_2-r_3: \quad \nabla \cdot (CA \nabla d - Cf(x)) = 0 \Rightarrow \nabla \cdot (CA \nabla d) = \nabla \cdot Cf(x) \quad 2-3.$$

$$C = \begin{pmatrix} \alpha & 0 & 0 & \beta \\ 0 & \frac{w}{2} & \frac{w}{2} & 0 \\ 0 & \frac{w}{2} & \frac{w}{2} & 0 \\ \beta & 0 & 0 & r \end{pmatrix}$$

$$\nabla d = \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ -\frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{pmatrix}, \quad C = \begin{pmatrix} \alpha & 0 & 0 & \beta \\ 0 & \frac{w}{2} & \frac{w}{2} & 0 \\ 0 & \frac{w}{2} & \frac{w}{2} & 0 \\ \beta & 0 & 0 & r \end{pmatrix}, \quad a = 0, \quad g = \begin{pmatrix} (f'_x(x_m) + \beta f'_y(x_m)) \nabla_x x_m \\ (\beta f'_x(x_m) + r f'_y(x_m)) \nabla_y x_m \end{pmatrix}$$

Stage 0 : Assume $\nabla \mathbf{x}_m = (0,0)$, then the PDE is reduced to:

$$-\nabla \cdot \begin{bmatrix} C \\ \alpha & 0 & 0 & \beta \\ 0 & \frac{\omega}{2} & \frac{\omega}{2} & 0 \\ -\frac{\omega}{2} & 0 & \frac{\omega}{2} & 0 \\ 0 & \frac{\omega}{2} & \frac{\omega}{2} & 0 \\ \beta & 0 & 0 & \gamma \end{bmatrix} \nabla d = 0$$

① B.C. $\hat{n} \cdot (C \nabla d) = 0$

② B.C. $\hat{n} \cdot (C \nabla d) = 1$

Forward Model:

Given C, x , we can find $d = (u, v)$, ∇d . from FEM

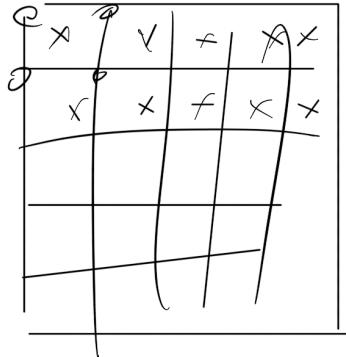
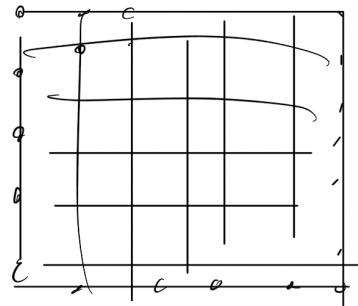
Now given x, d , can we find $C = ?$

1°. Numerical gradient descent: initialize $C = C_0$, solve $\ell_{xx}(C_0), \ell_{xy}(C_0), \ell_{yy}(C_0)$

$$F = \sum_{i,j} \|\ell_{xy} - \ell_{xy}(C_0)\|^2 + \|\ell_{xx} - \ell_{xx}(C_0)\|^2 + \|\ell_{yy} - \ell_{yy}(C_0)\|^2$$

$$\frac{\partial F}{\partial C} \Big|_{C_0} = \frac{F[C_0 + \delta c] - F[C_0]}{\delta c}$$

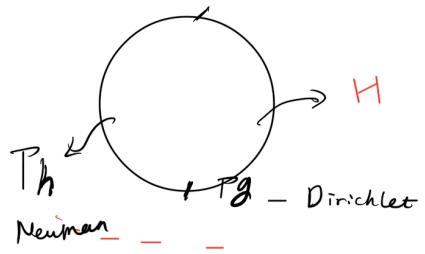
→ forward



observable

$$u = B * Kc \setminus Fc + u_d$$

$$Kd = F_f + F_h + F_g$$



$$\frac{m \frac{\partial^2 u}{\partial t^2} + d \frac{\partial u}{\partial t} - \nabla(c \cdot \nabla u) + au = f}{m.} \quad | \quad \begin{matrix} K \\ A \\ // \\ o \end{matrix} \quad | \quad \begin{matrix} F \\ // \\ o \end{matrix} \quad \hat{n} \cdot (c \nabla u) + gu = g \quad Q \quad G, R$$

$$K(C) \cdot d = F + G.$$

$$Hu = R.$$

"~" observable

$$\min f = \sum_b \|\hat{e}_{xx} - e_{xx}\|^2 + \|\hat{e}_{xy} - e_{xy}\|^2 + \|\hat{e}_{yy} - e_{yy}\|^2$$

C

$$\text{s.t. } e_{xx} = \frac{\partial u}{\partial x}, \quad e_{yy} = \frac{\partial v}{\partial y}, \quad e_{xy} = \frac{1}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)$$

$$\nabla \cdot \mathbf{G} = 0, \quad \hat{n} \cdot \mathbf{G} = 0$$

$$\Leftrightarrow \min_{\alpha, \beta, \gamma, \omega} f = \sum_b \|\hat{e}_{xx} - e_{xx}\|^2 + \|\hat{e}_{yy} - e_{yy}\|^2 + \|\hat{e}_{xy} - e_{xy}\|^2$$

$$\text{s.t. } \begin{bmatrix} e_{xx} \\ e_{xy} \\ e_{yy} \end{bmatrix} = Ad$$

$$K(\alpha, \beta, \gamma, \omega) \cdot \nabla d = F(x_m), \quad x_m \text{ Given.}$$

$$\Leftrightarrow \min_{\alpha, \beta, \gamma, \omega} f = \sum_b \|\hat{e} - Ad\|_2^2$$

$$C = \begin{bmatrix} \alpha & 0 & \beta \\ \beta & \omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} \alpha & 0 & 0 & \beta \\ 0 & \frac{\omega}{2} & \frac{\omega}{2} & 0 \\ 0 & 0 & 1 & 0 \\ \beta & 0 & 0 & \gamma \end{bmatrix}$$

$$\text{s.t. } K(\alpha, \beta, \gamma, \omega) \cdot \nabla d = F(x_m).$$

Convex problem.

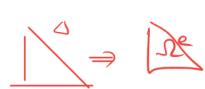


$$\text{Hence: } \underbrace{\nabla(C \nabla d)}_{= g}, \quad \hat{n}(C \nabla d) = \underbrace{g}_{= G_{22}}, \quad g = (\nabla x_m)^T \begin{pmatrix} \alpha f'_1(x_m) + \beta f'_2(x_m) \\ \beta f'_1(x_m) + \gamma f'_2(x_m) \end{pmatrix}$$

$$\Rightarrow K(\alpha, \beta, \gamma, \omega) \nabla d = G_{22} + G_{222}$$

$$K = \bigcup_{e=1}^{n_{\text{elem}}} K^e, \quad K^e = \int_{\Omega_e} [B_d^e] [C^e] [B_d^e] d\Omega^e = \int_{\Omega_e} [B_d^e] C [B_d^e] d\Omega^e$$

for 3-node triangle:



$$E = \sum_{b=1}^3 \begin{bmatrix} N_{b,1}^e & 0 \\ 0 & N_{b,2}^e \\ N_{b,2}^e & N_{b,1}^e \end{bmatrix} \begin{bmatrix} u_b^e \\ v_b^e \end{bmatrix}, \quad \text{here}$$

$$\begin{cases} N_1^e(\xi, \eta) = \xi \\ N_2^e(\xi, \eta) = \eta \\ N_3^e(\xi, \eta) = 1 - \xi - \eta \end{cases}$$

$$\text{let: } \begin{cases} x = x(\xi, \eta) \\ y = y(\xi, \eta) \end{cases} \quad \text{then: } \frac{\partial N_b^e}{\partial (\xi, \eta)} = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{pmatrix} \begin{pmatrix} \frac{\partial N_b^e}{\partial x} \\ \frac{\partial N_b^e}{\partial y} \end{pmatrix}$$

$$\det(J^e) \geq A^e \quad \begin{bmatrix} \frac{\partial N_b^e}{\partial x} \\ \frac{\partial N_b^e}{\partial y} \end{bmatrix} = \frac{1}{A^e} \begin{bmatrix} y_2 - y_3 & -(y_1 - y_3) \\ -(x_2 - x_3) & x_1 - x_3 \end{bmatrix} \begin{bmatrix} \frac{\partial N_b^e}{\partial \xi} \\ \frac{\partial N_b^e}{\partial \eta} \end{bmatrix}$$

$$E(d^h) = \begin{bmatrix} \frac{\partial u}{\partial \xi} & \frac{\partial u}{\partial \eta} \\ \frac{\partial v}{\partial \xi} & \frac{\partial v}{\partial \eta} \end{bmatrix} = \sum_{b=1}^3 \begin{bmatrix} N_{b,1}^e & 0 \\ 0 & N_{b,2}^e \\ N_{b,2}^e & N_{b,1}^e \end{bmatrix} \begin{bmatrix} d_{b,1}^e \\ d_{b,2}^e \end{bmatrix}$$

$$[J^e] = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1^e & y_1^e \\ x_2^e & y_2^e \\ x_3^e & y_3^e \end{bmatrix} = \begin{bmatrix} x_1^e - x_3^e & y_1^e - y_3^e \\ x_2^e - x_3^e & y_2^e - y_3^e \end{bmatrix}$$

$$K^e = \int_{\Omega^e} B^{eT} C B d\Omega^e = \frac{\partial N^e}{\partial (x, y)}^T C \frac{\partial N^e}{\partial (x, y)} = \frac{1}{4A^e} \begin{bmatrix} * \\ * \\ * \\ * \end{bmatrix}^T C \begin{bmatrix} * \\ * \\ * \\ * \end{bmatrix}$$

$$\begin{bmatrix} * \\ * \\ * \\ * \end{bmatrix} = \begin{bmatrix} y_3^e - y_1^e & 0 \\ 0 & x_3^e - x_1^e \\ x_3^e - x_2^e & y_3^e - y_2^e \\ y_2^e - y_1^e & 0 \end{bmatrix} \sim \text{function of position.}$$

\Rightarrow Each entry is only at most linear functions of $C_{ijk\ell}$, a.k.a. $\alpha, \beta, \gamma, \omega$.

$$f = \min_{\alpha, \beta, \gamma, \omega / g(x_m)} \sum_{i,j} \| \hat{e} - Bd \|^2_2 \quad B \text{ is a matrix independent of } \alpha, \beta, \gamma, \omega$$

$$K(\alpha, \beta, \gamma, \omega) d = G_{\alpha} + G_{\beta\gamma} = G$$

$K(\alpha, \beta, \gamma, \omega) > 0$, $G = G(x, y)$ only a function of position.

if we parameterize $g(x_m) = \sum_{n=1}^N a_n L_n(x_m)$, then we have:

Given $\alpha, \beta, \gamma, \omega$, K is known, $\Rightarrow d = K^{-1} G$.

$$G = \sum_{e=1}^{n_{\text{elem}}} G_{\alpha e} + \sum_{e=1}^{n_{\text{elem}}} G_{\beta\gamma e} \quad N^e = \begin{bmatrix} N_1^e(x, y) \\ N_2^e(x, y) \\ N_3^e(x, y) \end{bmatrix}$$

$$G_{\alpha e} = \int_{\Omega^e} N^e \begin{bmatrix} g_1^e \\ g_2^e \end{bmatrix} d\Omega^e = \int_{\Omega^e} N^e \begin{bmatrix} \sum_n a_n L_n(x_m) \\ \sum_n b_n L_n(x_m) \end{bmatrix} d\Omega^e \quad x_m = x_m(x, y) \text{ known.}$$

$$= \begin{bmatrix} a_n \\ b_n \end{bmatrix} \int_{\Omega^e} N^e L_n(x_m) d\Omega^e$$

Similarly: $G_{\beta\gamma e} = \int_{\Omega^e} N^e \begin{bmatrix} g_1^e \\ g_2^e \end{bmatrix} d\Omega^e = \sum_n \begin{bmatrix} a_n \\ b_n \end{bmatrix} \int_{\Omega^e} N^e L_n(x_m) d\Omega^e$

$$= \begin{pmatrix} D_1 & D_2 & \dots & D_N \end{pmatrix} \begin{bmatrix} a_1 \\ b_1 \\ a_2 \\ \vdots \\ a_n \\ b_n \end{bmatrix} = D \begin{bmatrix} a_1 \\ b_1 \\ a_2 \\ \vdots \\ a_n \\ b_n \end{bmatrix} \text{ let } c = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\Rightarrow K(\alpha, \beta, \gamma, \omega) d = (a, b)^T$$

$$\min_{a, b} \sum_{i,j} \| \hat{e} - Bd \|^2_2$$

$$\text{s.t. } d = K^{-1}$$

$$\left\{ \begin{array}{l} \min_{a, b} \sum_{i,j} \| \hat{e} - B K^{-1} D^T (a, b) \|^2_2 \\ \text{non constrained convex.} \end{array} \right.$$

2° if we know don't know α, β, γ, w . but know f . harder problem;

$$\min_{\alpha, \beta, \gamma, w} \| \hat{e} - Bd \|^2$$

s.t. $K(\alpha, \beta, \gamma, w) d = G$

$d \in \mathbb{R}^{2n}$, n points

$$d = K(\alpha, \beta, \gamma, w)^{-1} G. \quad K(\alpha, \beta, \gamma, w) \succ 0 \quad K \in \mathbb{R}^{2n \times 2n}$$

ple case:

$$K = \int_{\Omega^2} [B]_e^T C[B]_e d\Omega^2 \quad K = \begin{matrix} \text{matrix} \\ e=1 \end{matrix}, \quad K(\alpha, \beta, \gamma, w) \text{ is linear in } \alpha, \beta, \gamma, w, \text{ & } K \succ 0$$

* Convexity Analysis?

$$d = K^{-1}(\alpha, \beta, \gamma, w) G$$

Assume 2 parameter sets: $\alpha_1, \beta_1, \gamma_1, w_1$ & $\alpha_2, \beta_2, \gamma_2, w_2$

Then $f_1 = \| \hat{e} - Bd_1 \|^2$, $f_2 = \| \hat{e} - Bd_2 \|^2$, $f = f(d)$ f is convex in d .

let $\tilde{f} = \lambda f_1 + (1-\lambda) f_2$, with \tilde{d} . then:

$$\lambda f(d_1) + (1-\lambda) f(d_2) \geq f(\tilde{d}), \quad \tilde{d} = \lambda d_1 + (1-\lambda) d_2. \quad \forall x.$$

$$\tilde{d} = K(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{w})^{-1} G = [\lambda K_1^{-1} + (1-\lambda) K_2^{-1}] G = (\lambda K_{C_1}^{-1} + (1-\lambda) K_{C_2}^{-1}) G$$

$$d_1 = K(\alpha_1, \beta_1, \gamma_1, w_1)^{-1} G = K_{C_1}^{-1} G$$

$$d_2 = K(\alpha_2, \beta_2, \gamma_2, w_2)^{-1} G = K_{C_2}^{-1} G$$

We want to know when $\lambda f(d_1) + (1-\lambda) f(d_2) \geq f(\lambda d_1 + (1-\lambda) d_2)$, $\forall \lambda$.

$$K_C d_1 = G = K_{C_2} d_2 \quad K_{\lambda C_1 + (1-\lambda) C_2} d = G, \quad d$$

$$K_{C_1} \underline{\lambda d} + K_{C_2} \underline{(1-\lambda)d} = G \quad K(\alpha_1, \beta_1, \gamma_1, w_1) + K$$

Implementation:

N nodes, n^{th} order basis

$$\mathbf{a} = (a_1, \dots, a_n)$$

$$\mathbf{b} = (b_1, \dots, b_n)$$

$$\min_{\mathbf{a}, \mathbf{b}} \|\hat{\mathbf{e}} - \underline{\mathbf{e}_{\text{cal}}}\|_2^2$$

$$\mathbf{Kd} = \mathbf{G}_+ = \sum_{i=1}^n \mathbf{D}_i \begin{pmatrix} a_i \\ b_i \end{pmatrix} \quad \mathbf{D}_i \in \mathbb{R}^{2N \times 2}, \quad a_i \in \mathbb{R}$$

$$\mathbf{K} \in \mathbb{R}^{2N \times 2N}$$

$$\mathbf{d} \in \mathbb{R}^{2N \times 1}$$

$$\mathbf{d} = \sum_{i=1}^n \mathbf{K}^{-1} \mathbf{D}_i a_i$$

$$\underline{\mathbf{e}_{\text{cal}}} = \mathbf{B} \mathbf{d} = \sum_{i=1}^n \mathbf{B} \mathbf{K}^{-1} \mathbf{D}_i a_i = \mathbf{B} \mathbf{K}^{-1} \mathbf{D} \begin{pmatrix} a_1 \\ b_1 \\ a_2 \\ b_2 \\ \vdots \\ a_n \\ b_n \end{pmatrix}$$

$$\min_{a_1, b_1, \dots, a_n, b_n} \|\hat{\mathbf{e}} - \underbrace{\mathbf{B} \mathbf{K}^{-1} \mathbf{D}}_{\mathbf{E}} \begin{pmatrix} a_1 \\ b_1 \\ \vdots \\ a_n \\ b_n \end{pmatrix}\|_2^2 \quad \Rightarrow \quad \begin{pmatrix} a_1 \\ b_1 \\ \vdots \\ a_n \\ b_n \end{pmatrix} = \mathbf{E} \setminus \hat{\mathbf{e}} \\ = (\mathbf{E}^T \mathbf{E})^{-1} \mathbf{E}^T \hat{\mathbf{e}}$$

$$f = \|\hat{\mathbf{e}} - \underline{\mathbf{e}_{\text{cal}}}\|_2^2 \quad , \quad \mathbf{e}_{\text{cal}} = \mathbf{E} \mathbf{a} \quad \mathbf{a} \in \mathbb{R}^{2n}$$

$$\text{grad } f = 2 \sum \left[\hat{\mathbf{e}} - \underline{\mathbf{e}_{\text{cal}}} \right] \frac{\partial \mathbf{e}_{\text{cal}}}{\partial \mathbf{a}}$$

$$3N \times 1 \quad 3N \times 2n$$

get \mathbf{D} & \mathbf{B} .

$$\begin{pmatrix} C_{111} & C_{112} & C_{121} & C_{122} \\ 0 & C_{112} & C_{121} & C_{122} \\ 0 & 0 & C_{221} & C_{222} \\ 0 & 0 & 0 & C_{222} \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{pmatrix}$$

$$\begin{pmatrix} C_{11} & C_{12} \\ 0 & C_{22} \\ \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 0 & w \\ w & z \\ -w & -z \\ \beta & 0 \end{pmatrix}$$

$$\begin{pmatrix} G_{11} \\ G_{12} \\ G_{22} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ 2\epsilon_{12} \end{pmatrix}$$

$$C_{11} = (11 \ 13 \ 33) = \left[2 \ 0 \ \frac{w}{2} \right]$$

$$C_{12} = (13 \ 12 \ 33 \ 23) = \left[0 \ \beta \ \frac{w}{2} \ 0 \right]$$

$$C_{22} = (33 \ 23 \ 22) = \left[\frac{w}{2} \ 0 \ \gamma \right]$$

$$A \begin{pmatrix} u \\ v \end{pmatrix} = A \begin{pmatrix} u \\ v \end{pmatrix} \begin{pmatrix} d_{1p} \\ d_{2p} \end{pmatrix}$$

$$A = D \cdot U \setminus I$$

$$Kd = G$$

$$K = \alpha A_1 + A_0$$

forward model:

$$\begin{aligned}
 & \text{find } d: \quad \nabla \cdot G = 0, \quad \hat{n} \cdot G = 0 \quad \Rightarrow \quad d = (x, y) \in \mathbb{R}^2 \\
 & \quad K(\alpha) \quad d = \frac{G}{\alpha} \quad \leftarrow \quad G = \text{const} \\
 & \quad d = \begin{pmatrix} \vdots \\ \vdots \end{pmatrix} \quad K(\alpha) \geq 0 \\
 & \quad d = K \setminus G \\
 & \text{Discuss & Simplify: } K(\alpha) \geq 0
 \end{aligned}$$

$$d = K(\alpha)^{-1} G, \quad K(\alpha) \approx \alpha A + B.$$

Objective: $\min_{\alpha} \| \hat{d} - d(\alpha) \|_2^2$.
 A, B is const.
defined by geometry &
querypoints.

$$f = (\hat{d} - d(\alpha))^T (\hat{d} - d(\alpha)) =$$

$$f = \hat{d}^2 - 2 \hat{d} \cdot d(\alpha) + d(\alpha)^2$$

$$\frac{\partial f}{\partial \alpha} = -2 \hat{d} \cdot \frac{\partial d(\alpha)}{\partial \alpha} + 2 d(\alpha) \cdot \frac{\partial d(\alpha)}{\partial \alpha}$$

$$d = (\alpha A + B)^{-1} \underset{\alpha}{G} \quad \frac{\partial K^{-1}}{\partial p} = -K^{-1} \frac{\partial K}{\partial p} K^{-1}$$

$$\begin{aligned}
 \frac{\partial d}{\partial \alpha} &= \frac{\partial (\alpha A + B)^{-1}}{\partial \alpha} \cdot \underset{\alpha}{G} \\
 &= -(\alpha A + B)^{-1} \cdot \frac{\partial (\alpha A + B)}{\partial \alpha} \cdot (\alpha A + B)^{-1} \\
 &= -(\alpha A + B)^{-1} \cdot A \cdot (\alpha A + B)^{-1} \underset{\alpha}{G}
 \end{aligned}$$

$$\frac{\partial d(\omega)}{\partial \omega} = - \underbrace{K(\omega)^{-1}}_{\equiv} A \underbrace{K(\omega)^{-1}}_{\equiv} G \quad D^2 C(\omega) d = 0$$

\downarrow_{FEM}

$$K(\alpha) = \alpha A + B$$

$$\underline{K(\alpha)} \neq 0$$

$$\text{Let } \alpha = 0. \quad K(\alpha) = B - \Rightarrow \alpha = 0, \quad FEM. k_c = \tilde{B}, \quad B \stackrel{?}{=} \tilde{B} \begin{matrix} (3: end) \\ (3: end) \end{matrix}$$

$$\text{Let } \alpha = 1, \quad K(\alpha) = A + B,$$

2=1

$$FEM. K_C = \overset{\curvearrowleft}{A} + \overset{\curvearrowright}{B}$$

$$d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_{2m} \\ d_{2m+1}$$

$$2\left(\frac{G}{G_1}\right) = \begin{cases} - \\ - \\ G_1 \\ - \end{cases}$$

$$A+B = \begin{pmatrix} A & B \\ 0 & I_3 - A^{-1}B \end{pmatrix}$$

Script: PDE-constraint:

$$K_0 = \underbrace{FEM.K_C}_{\therefore K_0 \approx 0} \Rightarrow \frac{K_1}{\bar{s}} = FEM.K_C (3, end-1, 3, end-1)$$

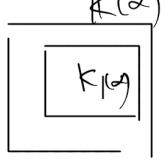
$$d = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ d(c(d)) \\ 0 \end{bmatrix}$$

→

$$K(\alpha) = K_1,$$

$$\frac{\partial d}{\partial \alpha} = \begin{pmatrix} b \\ 0 \\ \vdots \\ \frac{\partial d(\alpha)}{\partial \alpha} \\ 0 \end{pmatrix} \rightarrow \mathbb{K}_1(\alpha)^{-1} A \mathbb{K}_1(\alpha)^{-1} \cdot G_1$$

$$\Rightarrow \text{so: } \frac{\partial f}{\partial \alpha} = -2 \frac{\partial d(\alpha)}{\partial \alpha} \hat{d} + 2 d(\alpha) \cdot \frac{\partial d(\alpha)}{\partial \alpha}$$



$$d(\alpha) = \begin{pmatrix} 0 \\ 0 \\ d(\alpha) \\ 0 \end{pmatrix} \quad \frac{\partial d(\alpha)}{\partial \alpha} = \begin{pmatrix} 0 \\ 0 \\ \frac{\partial d}{\partial \alpha} \\ 0 \end{pmatrix}$$

$$\frac{\partial d_1(\alpha)}{\partial \alpha} = -K_1(\alpha)^{-1} A K_1(\alpha)^{-1} G_1$$

$$K(\alpha) = A\alpha + B$$

$$A = K(1) - K(0)$$

\hat{d} : known; A can be computed.

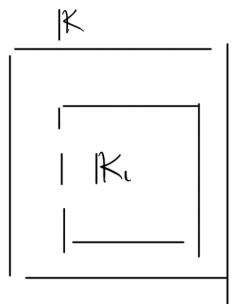
Implement this first. \square

Next step:

When \mathbf{C} is $C(\alpha, \beta, \gamma, \omega)$ $\mathbf{d} \in \mathbb{R}^{2N \times 1}$, $d_i \in \mathbb{R}^{2n-3}$

forward Model: $\nabla C(\alpha, \beta, \gamma, \omega) \nabla \mathbf{d} = 0$.

$$\hookrightarrow \text{PEM: } K(\alpha, \beta, \gamma, \omega) \cdot \mathbf{d} = 0 \quad \mathbf{d} = \begin{pmatrix} 0 \\ 0 \\ d_1 \\ 0 \end{pmatrix}$$



$$K \geq 0, \quad K_1 \geq 0 \quad K_1 = K(3:\text{end}-1, 3:\text{end}-1)$$

$$K_1 = \alpha A + \beta B + \gamma C + \omega D + F \quad K_0 = K_1(0, 0, 0, 0)$$

$$K_1(0, 0, 0, 0) = F \Rightarrow A = K_1(1, 0, 0, 0) - K_0$$

$$K_1(1, 0, 0, 0) = A + F \quad B = K_1(0, 1, 0, 0) - K_0$$

$$C = K_1(0, 0, 1, 0) - K_0$$

$$D = K_1(0, 0, 0, 1) - K_0$$

Backward: $\min f = (\hat{\mathbf{d}} - \mathbf{d})^2 = \hat{\mathbf{d}}^T - 2\hat{\mathbf{d}}^T \mathbf{d} + \mathbf{d}^T \mathbf{d}$
 $\mathbf{Q} = \alpha, \beta, \gamma, \omega$

\leftarrow s.t. $K(\alpha, \beta, \gamma, \omega) \mathbf{d} = \mathbf{G}$

$$\text{KKT: } \frac{\partial f}{\partial \alpha} = 0, \quad \frac{\partial f}{\partial \beta} = 0, \quad \frac{\partial f}{\partial \gamma} = 0, \quad \frac{\partial f}{\partial \omega} = 0 \quad \frac{\partial f}{\partial \alpha} = -2\hat{\mathbf{d}}^T \frac{\partial \mathbf{d}}{\partial \alpha} + 2\mathbf{d}^T \frac{\partial \mathbf{d}}{\partial \alpha}$$

$$\frac{\partial \mathbf{d}}{\partial \alpha} = \frac{\partial K^{-1} \mathbf{G}}{\partial \alpha} = \begin{pmatrix} 0 \\ 0 \\ \frac{\partial K_1^{-1} \mathbf{G}_1}{\partial \alpha} \\ 0 \end{pmatrix}$$

$$\frac{\partial (K_1^{-1} \mathbf{G}_1)}{\partial \alpha} = -K_1^{-1} A K_1^{-1} \mathbf{G}_1$$

$$\text{Similarly: } \frac{\partial K_1^{-1} \mathbf{G}_1}{\partial \beta} = -K_1^{-1} B K_1^{-1} \mathbf{G}_1$$

$$\frac{\partial K_1^{-1} \mathbf{G}_1}{\partial \gamma} = -K_1^{-1} C K_1^{-1} \mathbf{G}_1$$

$$\frac{\partial K_1^{-1} \mathbf{G}_1}{\partial \omega} = -K_1^{-1} D K_1^{-1} \mathbf{G}_1$$