Part I

The Finite Element Method for Elastostatics

Chapter 12

Variational formulation of linear elastostatics

In this chapter we derive the weak form for 3-D linear elasticity. Two approaches will be followed:

1. Derivation of the strong form using conservation of momentum, followed by application of the method of weighted residuals (MWR).

2. Derivation directly from the principle of minimum potential energy (PMPE).

The road map is shown in Fig. 12.1. The distinction between these two methods persists

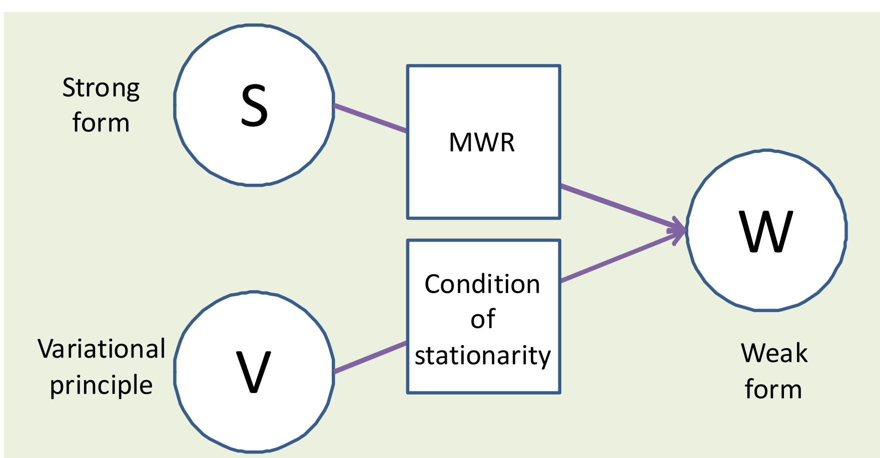

Figure 12.1: Road map for obtaining the variational equation for 3-D linear elasticity.

and this problem gives another opportunity to contrast them and compare the results.

12.1 Overview of the linear elasticity problem

Consider an elastic body occupying $\Omega \subset \mathbb{R}^3$ with boundary $\partial\Omega = \Gamma$. The body is subjected to various loads that cause the body to deform. Our goal is to predict the deformations as measured by the *displacement field vector* $\mathbf{u} : \Omega \to \mathbb{R}^3$ and to compute other quantities of interest, including the stress and strain fields. As shown in Fig. 12.2,

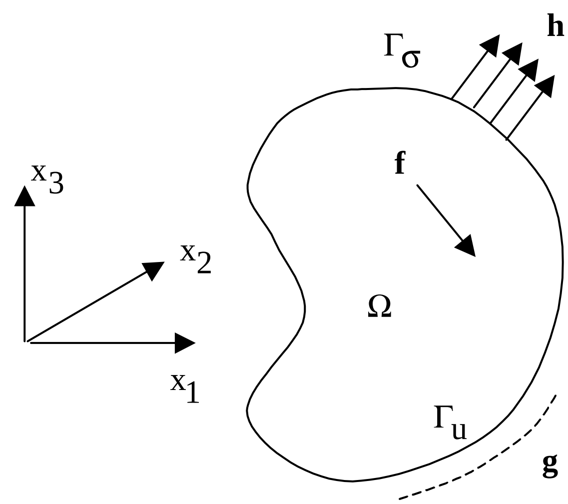

Figure 12.2: 3-D elasticity problem.

the boundary surface Γ is subdivided into two surfaces Γ_g and Γ_h for *each coordinate direction*, such that,

$$\Gamma_{h_i} \cup \Gamma_{g_i} = \Gamma, \quad i = 1, 2, 3$$

A boundary surface traction force (per unit area) with component $h_i(\mathbf{x})$ acts on Γ_{h_i}, and a prescribed boundary surface displacement component $g_i(\mathbf{x})$ is given on Γ_{g_i}. For each coordinate direction, we require and $\Gamma_{h_i} \cap \Gamma_{g_i} = \varnothing$ for $i = 1, 2$ and 3. The unknown displacement vector $\mathbf{u}(\mathbf{x})$ has components $u_i(\mathbf{x})$, $i = 1, 2$ and 3. The body is assumed to be linearly elastic and may be isotropic or anisotropic, and the loading is such that the deformations are small (in a sense that is provided later).The basic solid mechanics needed to frame the elasticity problem is given in this chapter.

12.2 Cauchy stress

Consider a deforrmable body in three dimensions. The body is cut into two parts and we can investigate the action of the forces applied by one region R_1 of the body on the remaining part R_2, with which it is in contact. For this purpose, consider the element of area Δa with unit normal vector $\mathbf{n}$ in the neighborhood of point P. If the resultant force on this area is $\Delta\mathbf{P}$, the traction vector $\mathbf{t}$ at P is defined as

$$\mathbf{t}(\mathbf{n}) = \lim_{\Delta a \to 0} \frac{\Delta \mathbf{P}}{\Delta a} \tag{12.1}$$

The notation $\mathbf{t}(\mathbf{n})$ reminds us that the traction vector $\mathbf{t}$ is defined on the surface with unit normal vector $\mathbf{n}$. Letting $\mathbf{f}$ be the body force per unit volume acting on the body at point P, the equilibrium of an elemental tetrahedron requires (see Fig. X),

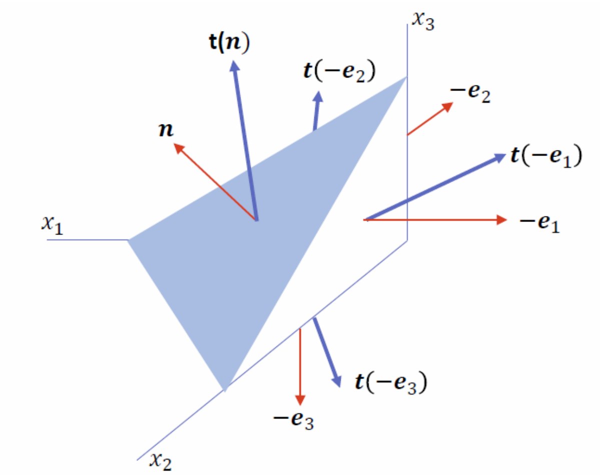

Figure 12.3: Elemental tetrahedron.

$$\mathbf{t}(\mathbf{n})\,\Delta a + \mathbf{t}(-\mathbf{e}_j)\,\Delta a_j + \mathbf{f}\,\Delta v = \mathbf{0} \tag{12.2}$$

with repeated indices implying summation and where

$$\Delta a_j = (\mathbf{n} \cdot \mathbf{e}_j)\,\Delta a = n_j \Delta a \tag{12.3}$$

is the projection of the area Δa onto the plane orthogonal to the Cartesian direction i and dv is the volume of the tetrahedron. The negative sign in the term $\mathbf{t}(-\mathbf{e}_j)$ results from the unit normal pointing in the negative coordinate direction. From Newton's third law of action and reaction, we must have $\mathbf{t}(-\mathbf{e}_j) = -\mathbf{t}(\mathbf{e}_j)$. Dividing by da and noting that $\lim_{da \to 0} \Delta v / \Delta a = 0$ gives,

$$\begin{aligned} \mathbf{t}(\mathbf{n}) &= -\mathbf{t}(-\mathbf{e}_j)\,\frac{\Delta a_j}{\Delta a} && (12.4) \\ &= \mathbf{t}(\mathbf{e}_j)\,n_j && (12.5) \end{aligned}$$

Now define the components of the Cauchy stress σ_{ij} such that,

$$\mathbf{t}(\mathbf{e}_j) = \sigma_{ji}\mathbf{e}_i \tag{12.6}$$

with repeated indices implying summation; for example $\mathbf{t}(\mathbf{e}_1) = \sigma_{11}\mathbf{e}_1 + \sigma_{12}\mathbf{e}_2 + \sigma_{13}\mathbf{e}_3$; see Fig. 12.4. Then, we have,

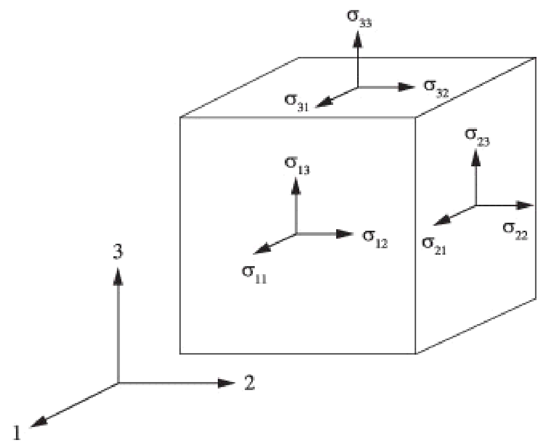

Figure 12.4: Nine components of the Cauchy stress tensor.

$$\mathbf{t}\left(\mathbf{n}\right) = \left(\sigma_{ji}\mathbf{e}_i\right) n_j = \sigma_{ji} n_j \mathbf{e}_i \tag{12.7}$$

But $\mathbf{t}\left(\mathbf{n}\right) = t_i\left(\mathbf{n}\right)\mathbf{e}_i$ which, by comparing with (12.7), implies[1]

$$t_i\left(\mathbf{n}\right) = \sigma_{ji} n_j$$

It is usual to drop the $\mathbf{n}$ and write $t_i = \sigma_{ji} n_j$. Noting that $\sigma_{ji} = \sigma_{ij}$, we can also write

$$t_i = \sigma_{ij} n_j \tag{12.8}$$

These components can be displayed in matrix form as $[\mathbf{t}] = [\boldsymbol{\sigma}][\mathbf{n}]$ or

$$\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \tag{12.9}$$

where the σ_{11}, σ_{22} and σ_{33} stress components are called *normal stresses* (because they act normal to the surface and produce stretching) and all the other components are called *shear stresses* (because they act parallel to the surface and produce shearing), see Fig. 12.4. The symmetry of $\boldsymbol{\sigma}$ implies that only six components are independent.

12.3 Equations of equilibrium

Consider now the equilibrium of a body occupying region $\Omega \subset \mathbb{R}^3$, with boundary Γ and assume that the body is acted upon by body forces $\mathbf{f} : \Omega \to \mathbb{R}^3$ per unit volume. See Fig.

[1]This development can be conducted more succinctly with direct notation and the dyadic product (see Appendix A):

$$\begin{aligned} \mathbf{t}\left(\mathbf{n}\right) &= \left(\sigma_{ji}\mathbf{e}_i\right)\left(\mathbf{e}_j \cdot \mathbf{n}\right) \\ &= \sigma_{ij}\left(\mathbf{e}_i \otimes \mathbf{e}_j\right)\mathbf{n} \\ &= \boldsymbol{\sigma}\mathbf{n} \end{aligned}$$

which implies $t_i = \sigma_{ij} n_j$.

12.2. Now let let $\mathcal{R} \subset \Omega$ be any region of the body and $\partial\mathcal{R}$ its boundary with traction forces $\mathbf{t} : \partial\mathcal{R} \to \mathbb{R}^3$ per unit area acting on the boundary $\partial\mathcal{R}$. Ignoring inertia forces, the global balance of linear momentum for $\mathcal{R}$ reduces to the balance of forces acting on the region $\mathcal{R}$ which is expressed by,

$$\begin{aligned} 0 &= \int_{\partial\mathcal{R}} \mathbf{t}\,(\mathbf{n})\; da + \int_{\mathcal{R}} \mathbf{f}\, dv \\ &= \int_{\partial\mathcal{R}} \boldsymbol{\sigma}\,\mathbf{n}\, da + \int_{\mathcal{R}} \mathbf{f}\, dv \\ &= \int_{\mathcal{R}} (\operatorname{div}\boldsymbol{\sigma} + \mathbf{f})\; dv \end{aligned} \tag{12.10}$$

where the last result follows from the divergence theorem. Since $\mathcal{R}$ is arbitrary, it can be reduced to a point, and (12.10) implies that,

$$\operatorname{div}\boldsymbol{\sigma} + \mathbf{f} = \mathbf{0} \qquad \mathbf{x} \in \Omega \tag{12.11}$$

This is the local equilibrium equation and in components has the form,

$$\sigma_{ij,j} + f_i = 0, \qquad \mathbf{x} \in \Omega, \quad i = 1, 2, 3 \tag{12.12}$$

where, as usual, we have used $\sigma_{ij,j}$ to mean

$$\sigma_{ij,j} = \frac{\partial \sigma_{ij}}{\partial x_j} = \frac{\partial \sigma_{i1}}{\partial x_1} + \frac{\partial \sigma_{i2}}{\partial x_2} + \frac{\partial \sigma_{i3}}{\partial x_3} \qquad i = 1, 2, 3$$

Written out in full, the three equations (12.12) read

$$\begin{aligned} \sigma_{11,1} + \sigma_{12,2} + \sigma_{13,3} + f_1 &= 0 \\ \sigma_{21,1} + \sigma_{22,2} + \sigma_{23,3} + f_2 &= 0 \\ \sigma_{31,1} + \sigma_{32,2} + \sigma_{33,3} + f_3 &= 0 \end{aligned}$$

12.4 Boundary Conditions

The displacement boundary is denoted Γ_{g_i}, $i = 1, 2, 3$ and the traction boundary is denoted Γ_{h_i}, $i - 1, 2, 3$ and these boundary regions satisfy,

$$\begin{aligned} \Gamma_{g_i} \cup \Gamma_{h_i} &= \Gamma \qquad i = 1, 2, 3 \\ \Gamma_{g_i} \cap \Gamma_{h_i} &= \emptyset \qquad i = 1, 2, 3 \end{aligned}$$

In general, $\Gamma_{g_1} \neq \Gamma_{g_2} \neq \Gamma_{g_3}$ and $\Gamma_{h_1} \neq \Gamma_{h_2} \neq \Gamma_{h_3}$.
The *displacement boundary conditions* are,

$$u_i = g_i \qquad \text{on } \Gamma_{g_i}$$

The *traction boundary conditions* are,

$$\sigma_{ij} n_j = h_i \qquad \text{on } \Gamma_{h_i}$$

where $g_i : \Gamma_{g_i} \to \mathbb{R}$ and $h_i : \Gamma_{h_i} \to \mathbb{R}$ are prescribed functions.

12.5 Strain

The nine components of *linear strain* are defined by

$$\varepsilon_{ij}(\mathbf{u}) = u_{(i,j)} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{12.13}$$

The term $u_{(i,j)}$ is called the *symmetric gradient* of $u(\mathbf{x})$ this implies that $\varepsilon_{ij} = \varepsilon_{ji}$. The strain components can be displayed in a matrix,

$$[\boldsymbol{\varepsilon}] = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} \tag{12.14}$$

The diagonal entries, ε_{11}, ε_{22} and ε_{33} measure normal (stretching) strains in the coordinate directions and all other components measure shear strains which are related to changes in angle between coordinate lines. Clearly, from the definition of strain, the strain matrix is symmetric and has six independent components.

12.6 Elasticity tensor

We will consider a general statement of Hooke's law in the form

$$\begin{aligned} \sigma_{ij} &= C_{ij11}\varepsilon_{11} + C_{ij12}\varepsilon_{12} + C_{ij13}\varepsilon_{13} + C_{ij21}\varepsilon_{21} + \ldots \\ &= \sum_{k=1}^{3}\sum_{\ell=1}^{3} C_{ijk\ell}\varepsilon_{k\ell} \\ &= C_{ijk\ell}\varepsilon_{k\ell} \end{aligned} \tag{12.15}$$

In the last expression, i and j are *free indices* which appear just once on the left-hand side and right-hand side, whereas k and ℓ are *dummy indices* since they appear twice (repeated) on the right-hand side. The summation convention will always be applied to dummy (repeated) indices, and they are implicitly summed over their range of $1-3$ (as shown in the intermediate result above).

The quantity $C_{ijk\ell}$ represents the (i,j,k,ℓ) component of the *elasticity tensor* that relates the stress component σ_{ij} to strain component $\varepsilon_{k\ell}$. As such, $C_{ijk\ell}$ are simply elastic constants. We will review this in more detail later, but as an example, for isotropic elasticity there are only two independent elastic material parameters called λ and μ (which will be described later) and it is found that

$$C_{ijk\ell} = \lambda\delta_{ij}\delta_{k\ell} + \mu(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}) \tag{12.16}$$

where δ_{ij} is the Kronecker delta defined according to the *values* taken by the i and j indices

$$\delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases} \tag{12.17}$$

For example[2] $\delta_{33} = 1$ and $\delta_{23} = 0$ (but note that $\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3$). Notice also that $\delta_{ij} = \delta_{ji}$.

The elasticity tensor exhibits major and minor symmetries as follows:

1. Major symmetry
$$C_{ijk\ell} = C_{k\ell ij} \tag{12.18}$$

2. Minor symmetry
$$\begin{aligned} C_{ijk\ell} &= C_{jik\ell} \quad (12.19) \\ C_{ijk\ell} &= C_{ij\ell k} \quad (12.20) \end{aligned}$$

Major symmetry follows from the loading path-independence exhibited by elastic materials, and the minor symmetries derive from the symmetry of the stress and strain components.

Both the stress and strain matrices are symmetric and therefore only six of each are independent. We select six independent components and display them as *ordered vectors* using *Voight notation* such that,

$$[\boldsymbol{\sigma}] \ : = \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{Bmatrix} \tag{12.21}$$

$$[\boldsymbol{\varepsilon}(\mathbf{u})] \ : = \begin{Bmatrix} \varepsilon_{11}(\mathbf{u}) \\ \varepsilon_{22}(\mathbf{u}) \\ \varepsilon_{33}(\mathbf{u}) \\ 2\varepsilon_{12}(\mathbf{u}) \\ 2\varepsilon_{23}(\mathbf{u}) \\ 2\varepsilon_{31}(\mathbf{u}) \end{Bmatrix} = \begin{Bmatrix} u_{(1,1)} \\ u_{(2,2)} \\ u_{(3,3)} \\ 2u_{(1,2)} \\ 2u_{(2,3)} \\ 2u_{(3,1)} \end{Bmatrix} \tag{12.22}$$

Notice that a factor of 2 has been introduced for the off-diagonal entries in the shear strain vector. The term $2\varepsilon_{12}(\mathbf{u})$ is the engineering shear strain (that measures angle

[2] One useful property of the Kronecker delta that frequently appears is the substitution property. Consider the quantity $\sigma_{ij}\delta_{jm}$, then

$$\sigma_{ij}\delta_{jm} = \sigma_{i1}\delta_{1m} + \sigma_{i2}\delta_{2m} + \sigma_{i3}\delta_{3m} = \sigma_{im}.$$

in which index j on σ is replaced by m. Or consider

$$\sigma_{ij}\delta_{ji} = \sigma_{ii} = \sigma_{11} + \sigma_{22} + \sigma_{33}$$

which is called the trace of σ_{ij}.

change). Similarly for the 23 and 31 components. These vectors must be related by Hooke's law (12.15) such that,

$$\left\{\begin{array}{c}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{21}\\ \sigma_{23}\\ \sigma_{31}\end{array}\right\} = \left[\begin{array}{cccccc} C_{1111} & C_{1122} & C_{1133} & \frac{C_{1112}}{2} & \frac{C_{1123}}{2} & \frac{C_{1131}}{2}\\ C_{2211} & C_{2222} & C_{2233} & \frac{C_{2212}}{2} & \frac{C_{2223}}{2} & \frac{C_{2231}}{2}\\ C_{3311} & C_{3322} & C_{3333} & \frac{C_{3312}}{2} & \frac{C_{3323}}{2} & \frac{C_{3331}}{2}\\ C_{1211} & C_{1222} & C_{1233} & \frac{C_{1212}}{2} & \frac{C_{1223}}{2} & \frac{C_{1231}}{2}\\ C_{2311} & C_{2322} & C_{2333} & \frac{C_{2312}}{2} & \frac{C_{2323}}{2} & \frac{C_{2331}}{2}\\ C_{3111} & C_{3122} & C_{3133} & \frac{C_{3112}}{2} & \frac{C_{3123}}{2} & \frac{C_{3131}}{2} \end{array}\right] \left\{\begin{array}{c}\varepsilon_{11}(\mathbf{u})\\ \varepsilon_{22}(\mathbf{u})\\ \varepsilon_{33}(\mathbf{u})\\ 2\varepsilon_{12}(\mathbf{u})\\ 2\varepsilon_{23}(\mathbf{u})\\ 2\varepsilon_{31}(\mathbf{u})\end{array}\right\}$$

or

$$[\boldsymbol{\sigma}] = [\mathbf{C}]\,[\boldsymbol{\varepsilon}(\mathbf{u})] \tag{12.23}$$

Because of the symmetry of $[\mathbf{C}]$, there can be at most 21 independent elastic constants. For *isotropic elastic materials* there are only two. In the case of isotropy, stress and strain can be directly related through the compliance matrix form,

$$\left\{\begin{array}{c}\varepsilon_{11}(\mathbf{u})\\ \varepsilon_{22}(\mathbf{u})\\ \varepsilon_{33}(\mathbf{u})\\ 2\varepsilon_{12}(\mathbf{u})\\ 2\varepsilon_{23}(\mathbf{u})\\ 2\varepsilon_{31}(\mathbf{u})\end{array}\right\} = \left[\begin{array}{cccccc} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0\\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0\\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} \end{array}\right] \left\{\begin{array}{c}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{21}\\ \sigma_{23}\\ \sigma_{31}\end{array}\right\} \tag{12.24}$$

where E, the Young's modulus, and ν, the Poisson ratio, have been taken as the independent constants. The shear modulus G is given by,

$$G = \frac{E}{2(1+\nu)} \tag{12.25}$$

Inverting this matrix form gives,

$$[\boldsymbol{\sigma}] = [\mathbf{C}]\,[\boldsymbol{\varepsilon}(\mathbf{u})] \tag{12.26}$$

where

$$[\mathbf{C}] = \frac{E}{(\nu+1)(1-2\nu)} \left[\begin{array}{cccccc} 1-\nu & \nu & \nu & 0 & 0 & 0\\ \nu & 1-\nu & \nu & 0 & 0 & 0\\ \nu & \nu & 1-\nu & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{array}\right]$$

Explicitly, Hooke's law in Voight notation for an isotropic elastic material is,

$$\left\{\begin{array}{c}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{21}\\ \sigma_{23}\\ \sigma_{31}\end{array}\right\} = \frac{E}{(\nu+1)(1-2\nu)} \left[\begin{array}{cccccc} 1-\nu & \nu & \nu & 0 & 0 & 0\\ \nu & 1-\nu & \nu & 0 & 0 & 0\\ \nu & \nu & 1-\nu & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{array}\right] \left\{\begin{array}{c}\varepsilon_{11}(\mathbf{u})\\ \varepsilon_{22}(\mathbf{u})\\ \varepsilon_{33}(\mathbf{u})\\ 2\varepsilon_{12}(\mathbf{u})\\ 2\varepsilon_{23}(\mathbf{u})\\ 2\varepsilon_{31}(\mathbf{u})\end{array}\right\} \tag{12.27}$$

12.7 Strong form

The *strong form* of the boundary value problem is stated as:

Strong form (S)**:** Given the body force (per unit volume) $\mathbf{f} : \Omega \to \mathbb{R}^3$, displacement boundary function $g_i : \Gamma_{g_i} \to \mathbb{R}$ and traction force component (per unit area) $h_i : \Gamma_{h_i} \to \mathbb{R}$, find $\mathbf{u} : \Omega \mapsto \mathbb{R}^3$, such that:

$$\begin{aligned} \sigma_{ij,j} + f_i &= 0 \quad \text{in } \Omega \quad i = 1,2,3 \qquad (12.28) \\ u_i &= g_i \quad \text{on } \Gamma_{g_i} \quad i = 1,2,3 \qquad (12.29) \\ \sigma_{ij} n_j &= h_i \quad \text{on } \Gamma_{h_i} \quad i = 1,2,3 \qquad (12.30) \end{aligned}$$

and where

$$\sigma_{ij} = C_{ijk\ell} u_{(k,\ell)} \qquad (12.31)$$

12.8 Variational equation from the MWR

Since the stress is expressed in terms of strains, we recognize that the order of the governing equation (12.28) is $2m = 2$ or $m = 1$. Then (12.29) and (12.30) can be identified as EBC and NBC, respectively. In general $\Gamma_{g_1} \neq \Gamma_{g_2} \neq \Gamma_{g_3}$ and therefore each component $u_i(\mathbf{x})$ of the vector displacement field $\mathbf{u}(\mathbf{x})$ must belong to a *different* trial function space, denoted $\mathcal{S}_i$, such that,

$$u_i \in \mathcal{S}_i = \left\{ u_i \in H^1(\Omega) \,|u_i = g_i \text{ on } \Gamma_{g_i} \right\}, \qquad i = 1,2,3 \qquad (12.32)$$

We will obtain the variational equation in two steps:

- Use the MWR on the *stress equilibrium equations* (12.28), and then,
- Introduce the elastic constitutive equations.

The variational equation is found from the MWR for each of the three equations in (12.28) as follows:

$$\begin{aligned} 0 &= \int_\Omega w_1 (\sigma_{1j,j} + f_1)\, d\Omega \quad \forall w_1 \in \mathcal{V}_1 \\ 0 &= \int_\Omega w_2 (\sigma_{2j,j} + f_2)\, d\Omega \quad \forall w_2 \in \mathcal{V}_2 \\ 0 &= \int_\Omega w_3 (\sigma_{3j,j} + f_3)\, d\Omega \quad \forall w_3 \in \mathcal{V}_3 \end{aligned}$$

Where the three functions $w_1(\mathbf{x})$, $w_2(\mathbf{x})$ and $w_3(\mathbf{x})$ are independent. In fact the particular method we are developing requires each function $w_i(\mathbf{x})$ to vanish where u_i is specified on Γ_{g_i} and therefore each must belong to a *different* weighting function space, denoted $\mathcal{V}_i$, such that,

$$w_i \in \mathcal{V}_i = \left\{ w_i \in H^1(\Omega) \,|\, w_i = 0 \text{ on } \Gamma_{g_i} \right\}, \qquad i = 1, 2, 3 \tag{12.33}$$

Using the repeated indices to imply summation, these equations can be written collectively as,

$$0 = \int_\Omega w_i \left(\sigma_{ij,j} + f_i \right) d\Omega \quad \forall w_i \in \mathcal{V}_i \tag{12.34}$$

We will think of the three independent weight functions $w_i(\mathbf{x})$ as defining a vector field $\mathbf{w}(\mathbf{x})$. Next we need to apply integration by parts (IBP) to (12.34) and this summarized in the following proposition.

Proposition 1 *Integration by parts,*

$$\int_\Omega w_i \sigma_{ij,j} \, d\Omega = -\int_\Omega w_{i,j} \sigma_{ij} \, d\Omega + \int_\Gamma w_i \sigma_{ij} n_j d\Gamma \tag{12.35}$$

Proof. This follows by noting from the product formula that,

$$w_i \sigma_{ij,j} = \left(w_i \sigma_{ij} \right)_{,j} - w_{i,j} \sigma_{ij} \tag{12.36}$$

Let $\upsilon_j = w_i \sigma_{ij} = w_1 \sigma_{1j} + w_2 \sigma_{2j} + w_3 \sigma_{3j}$ and recognize that $\left(w_i \sigma_{ij} \right)_{,j} = \upsilon_{j,j}$ is the divergence of $\boldsymbol{\upsilon}$. Integrating (12.36) over Ω and applying the divergence theorem leads to,

$$\begin{aligned} \int_\Omega w_i \sigma_{ij,j} \, d\Omega &= \int_\Omega \left[\upsilon_{j,j} - w_{i,j} \sigma_{ij} \right] d\Omega \\ &= -\int_\Omega w_{i,j} \sigma_{ij} \, d\Omega + \int_\Gamma \upsilon_j n_j d\Gamma \end{aligned}$$

Substituting $\upsilon_j = w_i \sigma_{ij}$ in the last integral gives the result in (12.35) ■

Applying IBP to (12.34) then gives,

$$0 = \int_\Omega \left(-w_{i,j} \sigma_{ij} + w_i f_i \right) d\Omega + \int_\Gamma w_i \sigma_{ij} n_j d\Gamma \quad \forall w_i \in \mathcal{V}_i \tag{12.37}$$

Now let us deal with the boundary term by enforcing $w_i \in \mathcal{V}_i$ and the NBC (12.30). Noting that in general $\Gamma_{g_1} \neq \Gamma_{g_2} \neq \Gamma_{g_3}$ and $\Gamma_{h_1} \neq \Gamma_{h_2} \neq \Gamma_{h_3}$, we have,

$$\begin{aligned} \int_\Gamma w_i \sigma_{ij} n_j d\Gamma &= \int_{\Gamma_{g_1} \cup \Gamma_{h_1}} w_1 \sigma_{1j} n_j \, d\Gamma + \int_{\Gamma_{g_2} \cup \Gamma_{h_2}} w_2 \sigma_{2j} n_j \, d\Gamma + \int_{\Gamma_{g_3} \cup \Gamma_{h_3}} w_3 \sigma_{3j} n_j \, d\Gamma \\ &= \sum_{i=1}^{3} \int_{\Gamma_{g_i} \cup \Gamma_{h_i}} w_i \sigma_{ij} n_j \, d\Gamma \quad \text{summation convention voided} \\ &= \sum_{i=1}^{3} \int_{\Gamma_{h_i}} w_i \sigma_{ij} n_j \, d\Gamma \qquad \text{enforcing } \; w_i \in \mathcal{V}_i \\ &= \sum_{i=1}^{3} \int_{\Gamma_{h_i}} w_i h_i \, d\Gamma \qquad \text{enforcing NBC (12.30)} \end{aligned} \tag{12.38}$$

The summation sign must remain explicit because the summation convention is lost with greater than two repeated indices. Using this result in (12.37) gives,

$$\int_{\Omega} w_{i,j}\sigma_{ij}\, d\Omega = \int_{\Omega} w_i f_i\, d\Omega + \sum_{i=1}^{n_{sd}} \int_{\Gamma_{h_i}} w_i h_i\, d\Gamma \quad \forall w_i \in \mathcal{V}_i \tag{12.39}$$

which is the variational equation. One additional step that exploits the symmetry of $\boldsymbol{\sigma}$ uses the following result.

Proposition 2 *Since $\sigma_{ij} = \sigma_{ji}$ (symmetric), then $w_{i,j}\sigma_{ij} = w_{(i,j)}\sigma_{ij}$, where the symmetric gradient of $\mathbf{w}$ is given by $w_{(i,j)} = \frac{1}{2}\left(w_{i,j} + w_{j,i}\right)$.*

Proof. Defining the antisymmetric gradient $w_{[i,j]} = \frac{1}{2}(w_{i,j} - w_{j,i})$, it follows that $w_{i,j} = w_{(i,j)} + w_{[i,j]}$ and $w_{i,j}\sigma_{ij} = w_{(i,j)}\sigma_{ij} + w_{[i,j]}\sigma_{ij}$. But

$$\begin{aligned}
w_{[i,j]}\sigma_{ij} &= \frac{1}{2}(w_{i,j} - w_{j,i})\sigma_{ij} \\
&= \frac{1}{2}(w_{i,j}\sigma_{ij} - w_{j,i}\sigma_{ij}) \\
&= \frac{1}{2}(w_{i,j}\sigma_{ij} - w_{j,i}\sigma_{ji}) \quad \text{symmetry of } \sigma_{ij} \\
&= \frac{1}{2}(w_{i,j}\sigma_{ij} - w_{i,j}\sigma_{ij}) \quad \text{switching dummy indices} \\
&= 0
\end{aligned}$$

■

12.9 Weak form

The *weak form* of the elasticity boundary value problem is stated as:

Weak form: (W)**:** Find $u_i \in \mathcal{S}_i = \{u_i \in H^1(\Omega)\,|u_i = g_i \text{ on } \Gamma_{g_i}\}$ such that:

$$a(\mathbf{w}, \boldsymbol{\sigma}) = (\mathbf{w}, \mathbf{f})_{\Omega} + (\mathbf{w}, \mathbf{h})_{\Gamma_h} \tag{12.40}$$

holds for all $w_i \in \mathcal{V}_i = \{w_i \in H^1(\Omega)\,|w_i = 0 \text{ on } \Gamma_{g_i}\}$, where

$$a(\mathbf{w}, \boldsymbol{\sigma}) = \int_{\Omega} w_{(i,j)}\sigma_{ij}\, d\Omega \tag{12.41}$$

$$(\mathbf{w}, \mathbf{f})_{\Omega} = \int_{\Omega} w_i f_i\, d\Omega \tag{12.42}$$

$$(\mathbf{w}, \mathbf{h})_{\Gamma_h} = \sum_{i=1}^{n_{sd}} \int_{\Gamma_{h_i}} w_i h_i\, d\Gamma \tag{12.43}$$

It is recognized that the weak form is nothing more (or less) than the *principle of virtual displacements* (PVD).

Remark 3 *The variational equation (12.40) expresses equilibrium in terms of stress and is therefore applicable for* ***any*** *constitutive behavior.*

Remark 4 *For linear elasticity* $\sigma_{ij} = C_{ijk\ell}\varepsilon_{k\ell}(\mathbf{u})$. *Furthermore we can write,*

$$w_{(i,j)} = \frac{1}{2}(w_{i,j} + w_{j,i}) := \varepsilon_{ij}(\mathbf{w}) \tag{12.44}$$

Using these in (12.40) provides the displacement form of the variational equation, which reads,

$$\int_\Omega \varepsilon_{ij}(\mathbf{w})\, C_{ijk\ell}\varepsilon_{k\ell}(\mathbf{u})\; d\Omega = \int_\Omega w_i f_i \, d\Omega + \sum_{i=1}^{n_{sd}} \int_{\Gamma_{h_i}} w_i h_i \, d\Gamma \tag{12.45}$$

where $\varepsilon_{ij}(\mathbf{w}) = w_{(i,j)}$ *and* $\varepsilon_{k\ell}(\mathbf{u}) = u_{(k,\ell)}$. *The* $\mathbf{w}-\mathbf{u}$ *symmetry in (12.45) is now apparent.*

12.10 The variational equation in Voight notation

The goal is to replace the indicial notation with an equivalent matrix form which is convenient for finite element implementation. Return to the bilinear operator in stress form (12.41), which can be written as,

$$a(\mathbf{w}, \mathbf{u}) = \int_\Omega \varepsilon_{ij}(\mathbf{w})\, \sigma_{ij} d\Omega$$

Considering the symmetry of $\varepsilon_{ij}(\mathbf{w})$ and σ_{ij}, notice that

$$\begin{aligned}
\varepsilon_{ij}(\mathbf{w})\,\sigma_{ij} &= \varepsilon_{11}(\mathbf{w})\,\sigma_{11} + \varepsilon_{22}(\mathbf{w})\,\sigma_{22} + \varepsilon_{33}(\mathbf{w})\,\sigma_{33} \\
&\quad + 2\varepsilon_{12}(\mathbf{w})\,\sigma_{12} + 2\varepsilon_{23}(\mathbf{w})\,\sigma_{23} + 2\varepsilon_{31}(\mathbf{w})\,\sigma_{31} \\
&= [\boldsymbol{\varepsilon}(\mathbf{w})]^T [\boldsymbol{\sigma}]
\end{aligned} \tag{12.46}$$

where the Voight vectors $[\boldsymbol{\sigma}]$ and $[\boldsymbol{\varepsilon}(\mathbf{w})]$ are defined in (12.21) and 12.22), respectively. Note that,

$$[\boldsymbol{\varepsilon}(\mathbf{w})] = \left\{ \begin{array}{c} \varepsilon_{11}(\mathbf{w}) \\ \varepsilon_{22}(\mathbf{w}) \\ \varepsilon_{33}(\mathbf{w}) \\ 2\varepsilon_{12}(\mathbf{w}) \\ 2\varepsilon_{23}(\mathbf{w}) \\ 2\varepsilon_{31}(\mathbf{w}) \end{array} \right\} = \left\{ \begin{array}{c} w_{(1,1)} \\ w_{(2,2)} \\ w_{(3,3)} \\ 2w_{(1,2)} \\ 2w_{(2,3)} \\ 2w_{(3,1)} \end{array} \right\} \tag{12.47}$$

Using Hooke's law in Voight form (12.23) then allows the bilinear operator to be written in matrix form as,

$$a(\mathbf{w}, \mathbf{u}) = \int_\Omega [\boldsymbol{\varepsilon}(\mathbf{w})]^T [\mathbf{C}] [\boldsymbol{\varepsilon}(\mathbf{u})]\; d\Omega \tag{12.48}$$

Turning to the right hand side of (12.45), it follows that the matrix representation is,

$$
\begin{aligned}
(\mathbf{w}, \mathbf{f}) &= \int_\Omega \langle w_1,\, w_2,\, w_3 \rangle \left\{ \begin{array}{c} f_1 \\ f_2 \\ f_3 \end{array} \right\} d\Omega = \int_\Omega \mathbf{w}^T \mathbf{f}\, d\Omega \\
(\mathbf{w}, \mathbf{h})_{\Gamma_h} &= \sum_{i=1}^{n_{sd}} \int_{\Gamma_{h_i}} w_i h_i \, d\Gamma = \sum_{i=1}^{n_{sd}} \int_{\Gamma_{h_i}} \mathbf{w}^T \mathbf{h} \, d\Gamma
\end{aligned}
$$

In summary, the variational equation for 3-D elasticity has the form,

$$
a\,(\mathbf{w}, \mathbf{u}) = (\mathbf{w}, \mathbf{f})_\Omega + (\mathbf{w}, \mathbf{h})_{\Gamma_h} \tag{12.49}
$$

where

$$
\begin{aligned}
a\,(\mathbf{w}, \mathbf{u}) &= \int_\Omega [\boldsymbol{\varepsilon}\,(\mathbf{w})]^T\, [\mathbf{C}]\, [\boldsymbol{\varepsilon}\,(\mathbf{u})]\; d\Omega && (12.50) \\
(\mathbf{w}, \mathbf{f}) &= \int_\Omega \mathbf{w}^T \mathbf{f}\, d\Omega && (12.51) \\
(\mathbf{w}, \mathbf{h})_{\Gamma_h} &= \sum_{i=1}^{n_{sd}} \int_{\Gamma_{h_i}} \mathbf{w}^T \mathbf{h}\, d\Gamma && (12.52)
\end{aligned}
$$

This variational form is next derived directly from the PMPE as an alternative approach.

12.11 Weak form from the PMPE

Having directly derived the weak form based on the strong form using the MWR, we now turn to rederiving the weak form from the principle of minimum potential energy (PMPE).

12.12 Strain energy density

To develop the principle of minimum potential energy we need to characterize the strain energy density (per unit volume) of an elastic material. The strain energy density is the amount of work need to deform the solid into its strained state and which is retained in the body as stored energy (available to do work). In the general case this is given by

$$
u = \int_0^{\varepsilon_{ij}} \sigma_{ij} d\varepsilon_{ij} \tag{12.53}
$$

which is established in proposition 5 below. Note that summation convention is applied here so that $\sigma_{ij} d\varepsilon_{ij} = \sigma_{11} d\varepsilon_{11} + \sigma_{12} d\varepsilon_{12} + \ldots$. We require all elastic materials to be (loading) path-independent by definition and this implies that $\sigma_{ij} d\varepsilon_{ij}$ must be a perfect differential which we denote $du = \sigma_{ij} d\varepsilon_{ij}$. It then follows that u is a potential for the stress with

$$\sigma_{ij} = \frac{\partial u}{\partial \epsilon_{ij}} \tag{12.54}$$

Now if σ_{ij} is given by Hooke's law, i.e. $\sigma_{ij} = C_{ijk\ell}\varepsilon_{k\ell}$, and, simultaneously, u must satisfy (12.54), it follows that

$$u = \frac{1}{2} C_{ijk\ell}\varepsilon_{ij}\varepsilon_{k\ell} \tag{12.55}$$

To show this, set $u = \frac{1}{2} C_{mnpq}\varepsilon_{mn}\varepsilon_{pq}$ (all dummy indices) and consider

$$\begin{aligned}
\frac{\partial u}{\partial \epsilon_{ij}} &= \frac{1}{2}\left[C_{mnpq}\frac{\varepsilon_{mn}}{\partial \varepsilon_{ij}}\varepsilon_{pq} + C_{mnpq}\varepsilon_{mn}\frac{\varepsilon_{pq}}{\partial \varepsilon_{ij}}\right] \\
&= \frac{1}{2}\left[C_{mnpq}\delta_{mi}\delta_{nj}\varepsilon_{pq} + C_{mnpq}\varepsilon_{mn}\delta_{pi}\delta_{qj}\right] \\
&= \frac{1}{2}\left[C_{ijpq}\varepsilon_{pq} + C_{mnij}\varepsilon_{mn}\right] \\
&= \frac{1}{2}\left[C_{ijpq}\varepsilon_{pq} + C_{ijmn}\varepsilon_{mn}\right] \quad \text{(using major symmetry)} \\
&= C_{ijk\ell}\varepsilon_{k\ell} \quad \text{(since } p, q \text{ and } m, n \text{ are dummy indices)} \\
&= \sigma_{ij}
\end{aligned}$$

Note that major symmetry of the elasticity tensor may now be established; since $\sigma_{ij} = C_{ijk\ell}\varepsilon_{k\ell}$ it follows that

$$C_{ijk\ell} = \frac{\partial \sigma_{ij}}{\partial \varepsilon_{k\ell}} = \frac{\partial}{\partial \varepsilon_{k\ell}}\left(\frac{\partial u}{\partial \epsilon_{ij}}\right) = \frac{\partial^2 u}{\partial \varepsilon_{k\ell}\partial \epsilon_{ij}}$$

Similarly

$$C_{k\ell ij} = \frac{\partial^2 u}{\partial \varepsilon_{ij}\partial \epsilon_{k\ell}}$$

but $\frac{\partial^2 u}{\partial \varepsilon_{k\ell}\partial \epsilon_{ij}} = \frac{\partial^2 u}{\partial \varepsilon_{ij}\partial \epsilon_{k\ell}}$ and therefore $C_{ijk\ell} = C_{k\ell ij}$.

This section is concluded by the proof of equation (12.53) which is given as follows.

Proposition 5 *The strain energy density in an elastic body is given by* $u = \int_0^{\varepsilon_{ij}} \sigma_{ij} d\varepsilon_{ij}$.
Proof. *Consider an infinitesimal rectangular parallelepiped of sides* dx_1, dx_2 *and* dx_3. *The normal forces acting on the faces are* $N_1 = \sigma_{11}dx_2dx_3$, $N_2 = \sigma_{22}dx_1dx_3$, *and* $N_3 = \sigma_{33}dx_2dx_3$. *The work done by the three normal forces resulting from incremental normal strains* $d\varepsilon_{11}$, $d\varepsilon_{22}$ *and* $d\varepsilon_{33}$ *is*

$$\begin{aligned}
dW_N &= N_1\left(d\varepsilon_{11}dx_1\right) + N_2\left(d\varepsilon_{22}dx_2\right) + N_3\left(d\varepsilon_{33}dx_3\right) \\
&= \left(\sigma_{11}d\varepsilon_{11} + \sigma_{22}d\varepsilon_{22} + \sigma_{33}d\varepsilon_{33}\right)dV
\end{aligned}$$

with $dV = dx_1dx_2dx_3$. *Similarly, the six shear forces acting on the three positive faces do work. Consider face* 1 *and shear forces* $S_{12} = \sigma_{12}dx_2dx_3$ *and* $S_{13} = \sigma_{13}dx_2dx_3$. *The* *work done by these shear forces resulting from incremental shear strains* $d\varepsilon_{12}$ *and* $d\varepsilon_{13}$ *is* $S_{12}\left(d\varepsilon_{12}dx_2\right)+S_{13}\left(d\varepsilon_{13}dx_3\right)=\left(\sigma_{12}d\varepsilon_{12}+\sigma_{13}d\varepsilon_{13}\right)dV.$ *The other two positive faces can also be accounted for giving*

$$dW_S=\left(\sigma_{12}d\varepsilon_{12}+\sigma_{13}d\varepsilon_{13}+\sigma_{21}d\varepsilon_{21}+\sigma_{23}d\varepsilon_{23}+\sigma_{31}d\varepsilon_{31}+\sigma_{32}d\varepsilon_{32}\right)dV$$

The net work done by all forces varying slowly from zero to their final value is given by

$$\begin{aligned} W_V &= \left(\int_0^{\varepsilon_{11}}\sigma_{11}d\varepsilon_{11}+\int_0^{\varepsilon_{12}}\sigma_{12}d\varepsilon_{12}+\cdots\int_0^{\varepsilon_{32}}\sigma_{32}d\varepsilon_{12}\right)dV \\ &= \left(\int_0^{\varepsilon_{ij}}\sigma_{ij}d\varepsilon_{ij}\right)dV \end{aligned}$$

The work done per unit volume, which is the strain energy density u, *is then*

$$u=\frac{W_V}{dV}=\int_0^{\varepsilon_{ij}}\sigma_{ij}d\varepsilon_{ij}$$

and which agrees with (12.53) ■

12.13 Principle of minimum potential energy

The total strain energy of the elastic system can now be stated as

$$\mathcal{U}\left(u_i\right)=\frac{1}{2}\int_{\Omega}C_{ijk\ell}\varepsilon_{ij}\varepsilon_{k\ell}d\Omega$$

and the potential of the applied loads is

$$\mathcal{V}\left(u_i\right)=-\int_{\Omega}f_iu_id\Omega-\sum_{i=1}^{3}\int_{\Gamma_{h_i}}t_iu_id\Gamma$$

with the total potential energy given by $\Pi\left(u_i\right)=\mathcal{U}\left(u_i\right)+\mathcal{V}\left(u_i\right).$ With this, the variational principle of minimum potential energy can be stated as follows:

Principle of minimum potential energy: Find $u_i\in\mathcal{S}_i=\left\{u_i\mid u_i\in H^1\left(\Omega\right),\ u_i=g_i \text{ on } \Gamma_{g_i}\right\}$ which minimizes

$$\Pi\left(u_i\right)=\frac{1}{2}\int_{\Omega}C_{ijk\ell}\varepsilon_{ij}\left(\mathbf{u}\right)\varepsilon_{k\ell}\left(\mathbf{u}\right)d\Omega-\int_{\Omega}f_iu_id\Omega-\sum_{i=1}^{3}\int_{\Gamma_{h_i}}h_iu_id\Gamma \tag{12.56}$$

where $\varepsilon_{ij}\left(\mathbf{u}\right)=\frac{1}{2}\left(u_{i,j}+u_{j,i}\right).$

Consider the conditions for a minimum.

1. **The condition of stationarity** is

$$0 = \delta\Pi = \frac{d}{d\epsilon}\Pi\left(u_i + \epsilon w_i\right)\Big|_{\epsilon=0} \tag{12.57}$$

with $w_i \in \mathcal{V}_i = \left\{ w_i \middle| \, w_i \in H^1\left(\Omega\right), \ w_i = 0 \text{ on } \Gamma_{g_i} \right\}$. Observe that

$$\begin{aligned} \frac{d}{d\epsilon}\varepsilon_{ij}\left(u_i + \epsilon w_i\right)\Big|_{\epsilon} &= \frac{1}{2}\frac{d}{d\epsilon}\Big(\left(u_i + \epsilon w_i\right)_{,j} + \left(u_j + \epsilon w_j\right)_{,i}\Big)\Big|_{\epsilon=0} \\ &= \frac{1}{2}\left(w_{i,j} + w_{j,i}\right) \\ &= \varepsilon_{ij}\left(\mathbf{w}\right) \end{aligned}$$

It then follows that

$$\begin{aligned} \frac{d}{d\epsilon}\Pi\left(u_i + \epsilon w_i\right)\Big|_{\epsilon=0} &= \frac{1}{2}\int_{\Omega} C_{ijk\ell}\,\varepsilon_{ij}\left(\mathbf{w}\right)\varepsilon_{k\ell}\left(\mathbf{u}\right)d\Omega + \frac{1}{2}\int_{\Omega} C_{ijk\ell}\varepsilon_{ij}\left(\mathbf{u}\right)\varepsilon_{k\ell}\left(\mathbf{w}\right)d\Omega \\ &\quad - \int_{\Omega} f_i w_i d\Omega - \sum_{i=1}^{3}\int_{\Gamma_{h_i}} t_i w_i d\Gamma \end{aligned}$$

Finally, using the major symmetry of $C_{ijk\ell}$, (12.57) can be written as

$$0 = \int_{\Omega}\varepsilon_{ij}\left(\mathbf{w}\right)C_{ijk\ell}\,\varepsilon_{k\ell}\left(\mathbf{u}\right)d\Omega - \int_{\Omega} f_i w_i d\Omega - \sum_{i=1}^{3}\int_{\Gamma_{h_i}} h_i w_i d\Gamma \tag{12.58}$$

or

$$a\left(\mathbf{w},\mathbf{u}\right) = \left(\mathbf{w},\mathbf{f}\right) + \left(\mathbf{w},\mathbf{h}\right)_{\Gamma_h} \tag{12.59}$$

where

$$\begin{aligned} a\left(\mathbf{w},\mathbf{u}\right) &= \int_{\Omega}\varepsilon_{ij}\left(\mathbf{w}\right)C_{ijk\ell}\varepsilon_{k\ell}\left(\mathbf{u}\right)d\Omega \\ \left(\mathbf{w},\mathbf{f}\right) &= \int_{\Omega} f_i w_i d\Omega \\ \left(\mathbf{w},\mathbf{h}\right)_{\Gamma_h} &= \sum_{i=1}^{3}\int_{\Gamma_{h_i}} t_i w_i d\Gamma \end{aligned}$$

This is the *variational equation* for elasticity.

2. **The positiveness condition** requires $\delta^2\Pi > 0$. To check this we compute

$$\frac{d}{d\epsilon}a\left(\mathbf{w},\mathbf{u}_{\epsilon}\right)\Big|_{\epsilon=0} = \int_{\Omega}\varepsilon_{ij}\left(\mathbf{w}\right)C_{ijk\ell}\,\varepsilon_{k\ell}\left(\mathbf{w}\right)d\Omega > 0$$

and confirm that it is a minimum principle.

The weak form can now be stated:
Weak form (W)**:** Find $u_i \in \mathcal{S}_i = \{u_i \in H^1(\Omega) \,|u_i = g_i \text{ on } \Gamma_{g_i}\}$ such that:

$$a(\mathbf{w}, \boldsymbol{\sigma}) = (\mathbf{w}, \mathbf{f})_\Omega + (\mathbf{w}, \mathbf{h})_{\Gamma_h} \tag{12.60}$$

holds for all $w_i \in \mathcal{V}_i = \{w_i \in H^1(\Omega) \,|w_i = 0 \text{ on } \Gamma_{g_i}\}$, where

$$a(\mathbf{w}, \boldsymbol{\sigma}) = \int_\Omega w_{(i,j)}\sigma_{ij}\, d\Omega \tag{12.61}$$

$$(\mathbf{w}, \mathbf{f})_\Omega = \int_\Omega w_i f_i \, d\Omega \tag{12.62}$$

$$(\mathbf{w}, \mathbf{h})_{\Gamma_h} = \sum_{i=1}^{n_{sd}} \int_{\Gamma_{h_i}} w_i h_i \, d\Gamma \tag{12.63}$$

This is identical to the weak form summarized in Sect. 12.9.

12.14 The Euler-Lagrange equations

Now we check that the Euler-Lagrange equations recover the strong form. The Euler-Lagrange equations are found by removing derivatives from w_i using the divergence theorem. Consider

$$\begin{aligned}
\varepsilon_{ij}(\mathbf{w})\, C_{ijk\ell}\, \varepsilon_{k\ell}(\mathbf{u}) &= \frac{1}{2}(w_{i,j} + w_{j,i})\,\sigma_{ij} \\
&= w_{i,j}\sigma_{ij} \qquad (\text{since } \sigma_{ij} = \sigma_{ji}) \\
&= (w_i\sigma_{ij})_{,j} - w_i\sigma_{ij,j}
\end{aligned}$$

Substituting in the variational equation (**??**), we find

$$\begin{aligned}
0 &= \int_\Omega \left[(w_i\sigma_{ij})_{,j} - w_i\sigma_{ij,j}\right] d\Omega - \int_\Omega f_i w_i d\Omega - \sum_{i=1}^{3}\int_{\Gamma_{h_i}} h_i w_i d\Gamma \\
&= \int_\Gamma w_i\sigma_{ij}n_j d\Gamma - \int_\Omega w_i\left[\sigma_{ij,j} + f_i\right] d\Omega - \sum_{i=1}^{3}\int_{\Gamma_{h_i}} h_i w_i d\Gamma
\end{aligned}$$

Recall that $w_i \in \mathcal{V}_i$, which implies that

$$\int_\Gamma w_i\sigma_{ij}n_j d\Gamma = \sum_{i=1}^{3}\int_{\Gamma_{g_i}\cup\Gamma_{h_i}} w_i\sigma_{ij}n_j d\Gamma = \sum_{i=1}^{3}\int_{\Gamma_{h_i}} w_i\sigma_{ij}n_j d\Gamma$$

Using in the above result gives

$$0 = -\int_\Omega w_i\left[\sigma_{ij,j} + f_i\right] d\Omega + \sum_{i=1}^{3}\int_{\Gamma_{h_i}} w_i\left[\sigma_{ij}n_j - h_i\right] d\Gamma$$

Since this must hold for all $w_i \in \mathcal{V}_i$, the fundamental lemma implies the following Euler-Lagrange equations

$$\begin{aligned} \sigma_{ij,j} + f_i &= 0 \quad \text{in } \Omega \\ \sigma_{ij} n_j &= h_i \quad \text{on } \Gamma_{h_i} \end{aligned}$$

Now, recalling that $w_i \in \mathcal{S}_i$, the strong form of the elasticity problem can be stated as follows

Strong form (S)**:** Given the body force per unit volume $f_i : \Omega \to \mathbb{R}$, the boundary traction force components $t_i : \Gamma_{h_i} \to \mathbb{R}$ and the boundary displacement components $g_i : \Gamma_{g_i} \to \mathbb{R}$, find $u_i \in C^2(\Omega)$ such that

$$\sigma_{ij,j} + f_i = 0 \quad \text{in } \Omega \quad i = 1,2,3 \tag{12.64}$$

$$u_i = g_i \quad \text{on } \Gamma_{g_i} \quad i = 1,2,3 \tag{12.65}$$

$$\sigma_{ij} n_j = h_i \quad \text{on } \Gamma_{h_i} \quad i = 1,2,3 \tag{12.66}$$

and where $\sigma_{ij} = C_{ijk\ell}\varepsilon_{k\ell}(\mathbf{u})$.

This, of course, agrees with the strong form summarized in Sect. 12.7.

12.15 Learning goals

The principle learning goals of this chapter are:

- To understand the definition of stress and strain in 3-D solids.
- To understand Hooke's law and its specialization for isotropic materials.
- To understand the derivation of the equations of equilibrium.
- To understand the strong form
- To understand the weak form and appreciate its development from the MWR.
- To appreciate the weak form using Voight (matrix) notation