MATH 151B Homework 1

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1 Part II

Problem Population growth is described by an ODE of the form $y\prime(t) = ry(t)$, where r is the growth rate. In a typical populatio, the growth rate is not a constant, but is density dependent. For example, as the population grows, there might be less food available, and as a result the growth rate decreases. We consider the following *Logistics Equation*:

$$\begin{cases} y\prime(t) = r(1 - \frac{y}{K}y), 0 \leq t \leq 50; \\ y(0) = y_0 \end{cases}$$

where $0 < y_0 < K$. Then the exact solution is given by

$$y(t) = \frac{y_0 K}{y_0 + (K - y_0)e^{-rt}}$$

Solve IVP with $y_0 = 1000, r = 0.2, K = 4000$ numerically using Euler's method. Choose the step sizes $h = 10, 1, 0.1$, respectively.

a) Compare the solutions to the exact solution in plots of population vs. time. Compare the actual maximal error $\max_i |y(t_i) - w_i|$ with the error bound predicted in Theorem 5.9 (p.271).

b) Discuss the behavior of the solutions as a function of h. What happens for very large step size h?

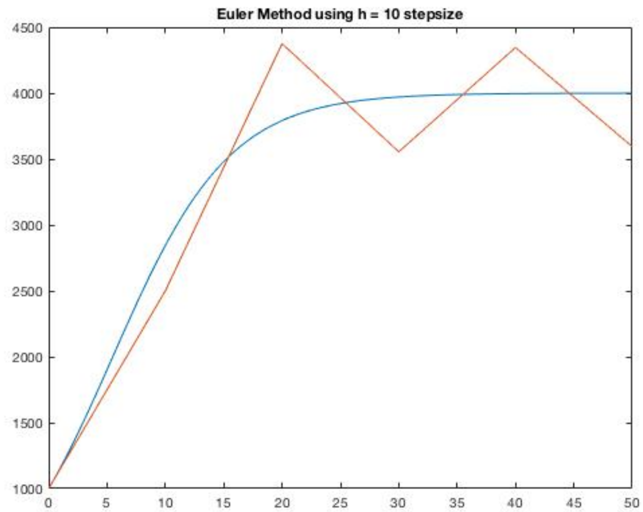
Solution. a) The plots of population vs. time for both exact and Euler's solutions when $h = 10, 1, 0.1$ are at the next page. According to the Matlab code, for $h = 10, 1, 0.1$, we have the actual maximal errors 583.34, 29.9747 and 2.8901, respectively. Now, in order to find the error bound predicted in Theorem 5.9, we first want to find L and M.
Because $f(t, y) = y\prime(t)$, we have $\left|\frac{\partial f(t,y)}{\partial y}\right| = \left|(\frac{1}{5} - \frac{y}{10000})\right|$. Since $t \in [0, 50]$, y is increasing and $y \in [1000, 3999.5]$. $\left|\frac{\partial f(t,y)}{\partial y}\right| \leq \left|\frac{1}{5} - \frac{3999.5}{10000}\right| = 0.1999.5$. Therefore, let $L = 0.2$. For this problem, the exact solution is $y(t) = \frac{y_0 K}{y_0 + (K - y_0)e^{-rt}}$, so

$$y''(t) = \frac{480000000000e^{-0.4t}(-e^{0.2t} + 3)}{(1000 + 3000e^{-0.2t})^3}$$

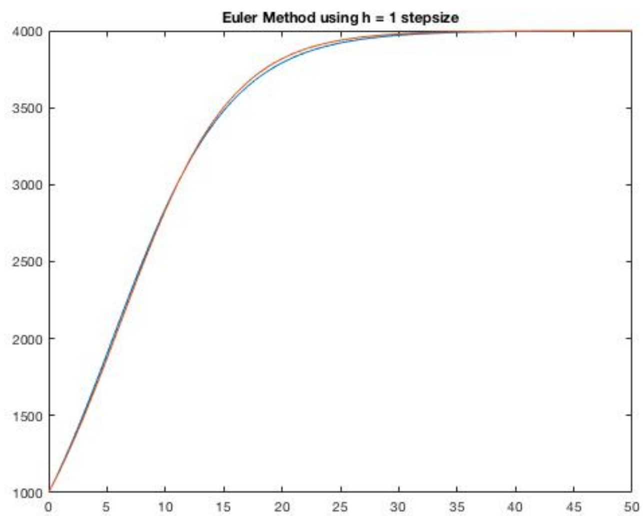
$$|y''(t)| \leq \frac{480000000000e^{0}(-e^{0} + 3)}{(1000 + 3000e^{0})^3} = 15$$

for all $t \in [0, 50]$. Using the inequality in the error bound for Euler's method with $h = 10, L = \frac{1}{5}$, and $M = 15$ gives $|y_i - w_i| \leq 100000(e^{0.2(t_i)} - 1)$ According to the Matlab code, we got the estimated error bound is 8259500. Similarly, for $h = 1$ and $h = 0.1$, we got the error bounds 825950 and 82595, respectively. The actual maximal error is in the predicted error bound but is much smaller than the error bound.

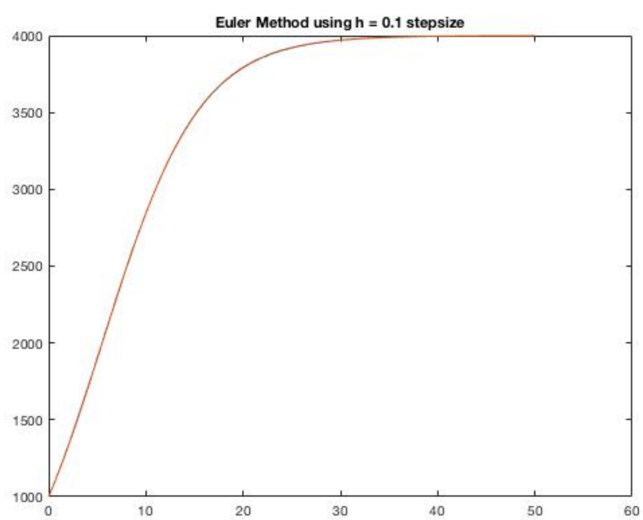
b) When h is too big (i.e. h=10), according to the graph (a), the Euler's solution is very different from the exact solution. When h is small enough (i.e. h=0.1), according to the graph (b), the Euler's solution is close to the exact solution. Thus, large step size h might cause a large error in Euler's solution.



(a) h=10



(b) h=1



(c) h=0.1